Analysis

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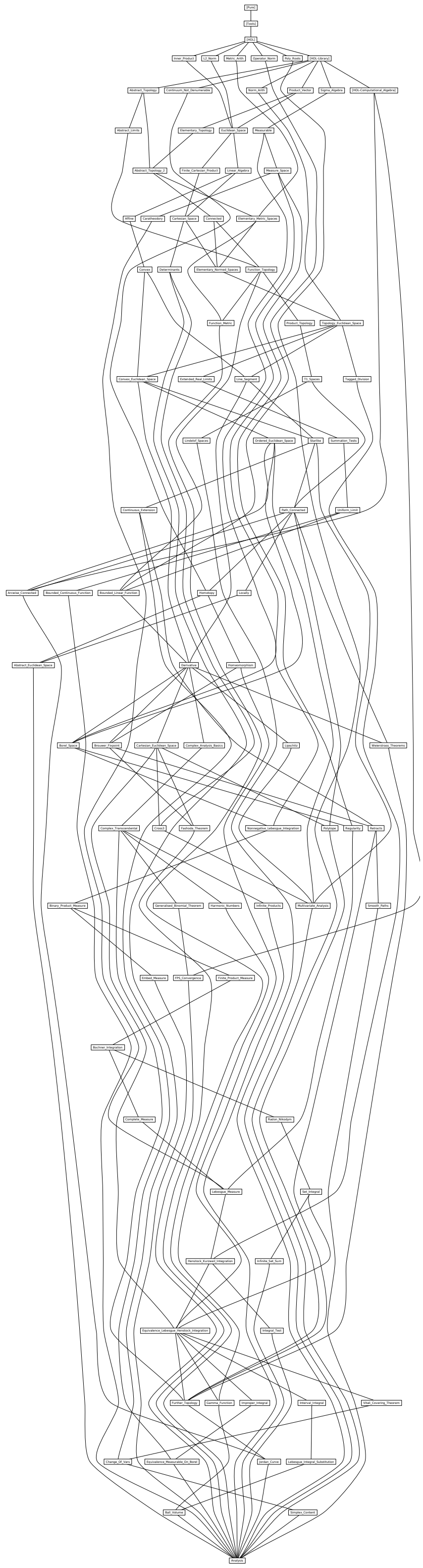
Contents



Chapter 1

Linear Algebra

theory *L2_Norm*
imports *Complex_Main*
begin

1.1 L2 Norm

definition *L2_set* :: $('a \Rightarrow real) \Rightarrow 'a\ set \Rightarrow real$ **where**
$L2\_set\ f\ A = sqrt\ (\sum i{\in}A.\ (f\ i)^2)$

lemma *L2_set_cong*:
$[\![A = B;\ \bigwedge x.\ x \in B \Longrightarrow f\ x = g\ x]\!] \Longrightarrow L2\_set\ f\ A = L2\_set\ g\ B$
unfolding *L2_set_def* **by** *simp*

lemma *L2_set_cong_simp*:
$[\![A = B;\ \bigwedge x.\ x \in B =simp{=}{>}\ f\ x = g\ x]\!] \Longrightarrow L2\_set\ f\ A = L2\_set\ g\ B$
unfolding *L2_set_def simp_implies_def* **by** *simp*

lemma *L2_set_infinite* [*simp*]: $\neg\ finite\ A \Longrightarrow L2\_set\ f\ A = 0$
unfolding *L2_set_def* **by** *simp*

lemma *L2_set_empty* [*simp*]: $L2\_set\ f\ \{\} = 0$
unfolding *L2_set_def* **by** *simp*

lemma *L2_set_insert* [*simp*]:
$[\![finite\ F;\ a \notin F]\!] \Longrightarrow$
$L2\_set\ f\ (insert\ a\ F) = sqrt\ ((f\ a)^2 + (L2\_set\ f\ F)^2)$
unfolding *L2_set_def* **by** (*simp add*: *sum_nonneg*)

lemma *L2_set_nonneg* [*simp*]: $0 \le L2\_set\ f\ A$
unfolding *L2_set_def* **by** (*simp add*: *sum_nonneg*)

lemma *L2_set_0'*: $\forall a{\in}A.\ f\ a = 0 \Longrightarrow L2\_set\ f\ A = 0$
unfolding *L2_set_def* **by** *simp*

```
lemma L2_set_constant: L2_set (λx. y) A = sqrt (of_nat (card A)) ∗ |y|
  unfolding L2_set_def by (simp add: real_sqrt_mult)

lemma L2_set_mono:
  assumes ⋀i. i ∈ K ⟹ f i ≤ g i
  assumes ⋀i. i ∈ K ⟹ 0 ≤ f i
  shows L2_set f K ≤ L2_set g K
  unfolding L2_set_def
  by (simp add: sum_nonneg sum_mono power_mono assms)

lemma L2_set_strict_mono:
  assumes finite K and K ≠ {}
  assumes ⋀i. i ∈ K ⟹ f i < g i
  assumes ⋀i. i ∈ K ⟹ 0 ≤ f i
  shows L2_set f K < L2_set g K
  unfolding L2_set_def
  by (simp add: sum_strict_mono power_strict_mono assms)

lemma L2_set_right_distrib:
  0 ≤ r ⟹ r ∗ L2_set f A = L2_set (λx. r ∗ f x) A
  unfolding L2_set_def
  apply (simp add: power_mult_distrib)
  apply (simp add: sum_distrib_left [symmetric])
  apply (simp add: real_sqrt_mult sum_nonneg)
  done

lemma L2_set_left_distrib:
  0 ≤ r ⟹ L2_set f A ∗ r = L2_set (λx. f x ∗ r) A
  unfolding L2_set_def
  apply (simp add: power_mult_distrib)
  apply (simp add: sum_distrib_right [symmetric])
  apply (simp add: real_sqrt_mult sum_nonneg)
  done

lemma L2_set_eq_0_iff: finite A ⟹ L2_set f A = 0 ⟷ (∀x∈A. f x = 0)
  unfolding L2_set_def
  by (simp add: sum_nonneg sum_nonneg_eq_0_iff)

proposition L2_set_triangle_ineq:
  L2_set (λi. f i + g i) A ≤ L2_set f A + L2_set g A
proof (cases finite A)
  case False
  thus ?thesis by simp
next
  case True
  thus ?thesis
  proof (induct set: finite)
    case empty
    show ?case by simp
```

```
  next
    case (insert x F)
    hence sqrt ((f x + g x)^2 + (L2_set (λi. f i + g i) F)^2) ≤
          sqrt ((f x + g x)^2 + (L2_set f F + L2_set g F)^2)
      by (intro real_sqrt_le_mono add_left_mono power_mono insert
               L2_set_nonneg add_increasing zero_le_power2)
    also have
      ... ≤ sqrt ((f x)^2 + (L2_set f F)^2) + sqrt ((g x)^2 + (L2_set g F)^2)
      by (rule real_sqrt_sum_squares_triangle_ineq)
    finally show ?case
      using insert by simp
  qed
qed

lemma L2_set_le_sum [rule_format]:
  (∀ i∈A. 0 ≤ f i) ⟶ L2_set f A ≤ sum f A
  apply (cases finite A)
  apply (induct set: finite)
  apply simp
  apply clarsimp
  apply (erule order_trans [OF sqrt_sum_squares_le_sum])
  apply simp
  apply simp
  apply simp
  done

lemma L2_set_le_sum_abs: L2_set f A ≤ (∑ i∈A. |f i|)
  apply (cases finite A)
  apply (induct set: finite)
  apply simp
  apply simp
  apply (rule order_trans [OF sqrt_sum_squares_le_sum_abs])
  apply simp
  apply simp
  done

lemma L2_set_mult_ineq: (∑ i∈A. |f i| * |g i|) ≤ L2_set f A * L2_set g A
  apply (cases finite A)
  apply (induct set: finite)
  apply simp
  apply (rule power2_le_imp_le, simp)
  apply (rule order_trans)
  apply (rule power_mono)
  apply (erule add_left_mono)
  apply (simp add: sum_nonneg)
  apply (simp add: power2_sum)
  apply (simp add: power_mult_distrib)
  apply (simp add: distrib_left distrib_right)
  apply (rule ord_le_eq_trans)
```

```
  apply (rule L2_set_mult_ineq_lemma)
  apply simp_all
  done

lemma member_le_L2_set: ⟦finite A; i ∈ A⟧ ⟹ f i ≤ L2_set f A
  unfolding L2_set_def
  by (auto intro!: member_le_sum real_le_rsqrt)

end
```

1.2 Inner Product Spaces and Gradient Derivative

```
theory Inner_Product
imports Complex_Main
begin
```

1.2.1 Real inner product spaces

Temporarily relax type constraints for *open*, *uniformity*, *dist*, and *norm*.

```
setup ‹Sign.add_const_constraint
  (const_name ‹open›, SOME typ ‹'a::open set ⇒ bool›)›

setup ‹Sign.add_const_constraint
  (const_name ‹dist›, SOME typ ‹'a::dist ⇒ 'a ⇒ real›)›

setup ‹Sign.add_const_constraint
  (const_name ‹uniformity›, SOME typ ‹('a::uniformity × 'a) filter›)›

setup ‹Sign.add_const_constraint
  (const_name ‹norm›, SOME typ ‹'a::norm ⇒ real›)›

class real_inner = real_vector + sgn_div_norm + dist_norm + uniformity_dist +
open_uniformity +
  fixes inner :: 'a ⇒ 'a ⇒ real
  assumes inner_commute: inner x y = inner y x
  and inner_add_left: inner (x + y) z = inner x z + inner y z
  and inner_scaleR_left [simp]: inner (scaleR r x) y = r * (inner x y)
  and inner_ge_zero [simp]: 0 ≤ inner x x
  and inner_eq_zero_iff [simp]: inner x x = 0 ⟷ x = 0
  and norm_eq_sqrt_inner: norm x = sqrt (inner x x)
begin

lemma inner_zero_left [simp]: inner 0 x = 0
  using inner_add_left [of 0 0 x] by simp

lemma inner_minus_left [simp]: inner (- x) y = - inner x y
  using inner_add_left [of x - x y] by simp
```

lemma *inner_diff_left*: *inner* (*x* − *y*) *z* = *inner x z* − *inner y z*
using *inner_add_left* [*of x* − *y z*] **by** *simp*

lemma *inner_sum_left*: *inner* (∑ *x*∈*A*. *f x*) *y* = (∑ *x*∈*A*. *inner* (*f x*) *y*)
by (*cases finite A*, *induct set*: *finite*, *simp_all add*: *inner_add_left*)

lemma *all_zero_iff* [*simp*]: (∀ *u*. *inner x u* = *0*) ⟷ (*x* = *0*)
by *auto* (*use inner_eq_zero_iff* **in** *blast*)

Transfer distributivity rules to right argument.

lemma *inner_add_right*: *inner x* (*y* + *z*) = *inner x y* + *inner x z*
using *inner_add_left* [*of y z x*] **by** (*simp only*: *inner_commute*)

lemma *inner_scaleR_right* [*simp*]: *inner x* (*scaleR r y*) = *r* ∗ (*inner x y*)
using *inner_scaleR_left* [*of r y x*] **by** (*simp only*: *inner_commute*)

lemma *inner_zero_right* [*simp*]: *inner x 0* = *0*
using *inner_zero_left* [*of x*] **by** (*simp only*: *inner_commute*)

lemma *inner_minus_right* [*simp*]: *inner x* (− *y*) = − *inner x y*
using *inner_minus_left* [*of y x*] **by** (*simp only*: *inner_commute*)

lemma *inner_diff_right*: *inner x* (*y* − *z*) = *inner x y* − *inner x z*
using *inner_diff_left* [*of y z x*] **by** (*simp only*: *inner_commute*)

lemma *inner_sum_right*: *inner x* (∑ *y*∈*A*. *f y*) = (∑ *y*∈*A*. *inner x* (*f y*))
using *inner_sum_left* [*of f A x*] **by** (*simp only*: *inner_commute*)

lemmas *inner_add* [*algebra_simps*] = *inner_add_left inner_add_right*
lemmas *inner_diff* [*algebra_simps*] = *inner_diff_left inner_diff_right*
lemmas *inner_scaleR* = *inner_scaleR_left inner_scaleR_right*

Legacy theorem names

lemmas *inner_left_distrib* = *inner_add_left*
lemmas *inner_right_distrib* = *inner_add_right*
lemmas *inner_distrib* = *inner_left_distrib inner_right_distrib*

lemma *inner_gt_zero_iff* [*simp*]: *0* < *inner x x* ⟷ *x* ≠ *0*
by (*simp add*: *order_less_le*)

lemma *power2_norm_eq_inner*: (*norm x*)^2 = *inner x x*
by (*simp add*: *norm_eq_sqrt_inner*)

Identities involving real multiplication and division.

lemma *inner_mult_left*: *inner* (*of_real m* ∗ *a*) *b* = *m* ∗ (*inner a b*)
by (*metis real_inner_class.inner_scaleR_left scaleR_conv_of_real*)

lemma *inner_mult_right*: *inner a* (*of_real m* ∗ *b*) = *m* ∗ (*inner a b*)
by (*metis real_inner_class.inner_scaleR_right scaleR_conv_of_real*)

```
lemma inner_mult_left': inner (a * of_real m) b = m * (inner a b)
  by (simp add: of_real_def)

lemma inner_mult_right': inner a (b * of_real m) = (inner a b) * m
  by (simp add: of_real_def real_inner_class.inner_scaleR_right)

lemma Cauchy_Schwarz_ineq:
  (inner x y)^2 ≤ inner x x * inner y y
proof (cases)
  assume y = 0
  thus ?thesis by simp
next
  assume y: y ≠ 0
  let ?r = inner x y / inner y y
  have 0 ≤ inner (x - scaleR ?r y) (x - scaleR ?r y)
    by (rule inner_ge_zero)
  also have ... = inner x x - inner y x * ?r
    by (simp add: inner_diff)
  also have ... = inner x x - (inner x y)^2 / inner y y
    by (simp add: power2_eq_square inner_commute)
  finally have 0 ≤ inner x x - (inner x y)^2 / inner y y .
  hence (inner x y)^2 / inner y y ≤ inner x x
    by (simp add: le_diff_eq)
  thus (inner x y)^2 ≤ inner x x * inner y y
    by (simp add: pos_divide_le_eq y)
qed

lemma Cauchy_Schwarz_ineq2:
  |inner x y| ≤ norm x * norm y
proof (rule power2_le_imp_le)
  have (inner x y)^2 ≤ inner x x * inner y y
    using Cauchy_Schwarz_ineq .
  thus |inner x y|^2 ≤ (norm x * norm y)^2
    by (simp add: power_mult_distrib power2_norm_eq_inner)
  show 0 ≤ norm x * norm y
    unfolding norm_eq_sqrt_inner
    by (intro mult_nonneg_nonneg real_sqrt_ge_zero inner_ge_zero)
qed

lemma norm_cauchy_schwarz: inner x y ≤ norm x * norm y
  using Cauchy_Schwarz_ineq2 [of x y] by auto

subclass real_normed_vector
proof
  fix a :: real and x y :: 'a
  show norm x = 0 ⟷ x = 0
    unfolding norm_eq_sqrt_inner by simp
  show norm (x + y) ≤ norm x + norm y
```

```
    proof (rule power2_le_imp_le)
      have inner x y ≤ norm x * norm y
        by (rule norm_cauchy_schwarz)
      thus (norm (x + y))^2 ≤ (norm x + norm y)^2
        unfolding power2_sum power2_norm_eq_inner
        by (simp add: inner_add inner_commute)
      show 0 ≤ norm x + norm y
        unfolding norm_eq_sqrt_inner by simp
    qed
  have sqrt (a^2 * inner x x) = |a| * sqrt (inner x x)
    by (simp add: real_sqrt_mult)
  then show norm (a *_R x) = |a| * norm x
    unfolding norm_eq_sqrt_inner
    by (simp add: power2_eq_square mult.assoc)
qed

end

lemma square_bound_lemma:
  fixes x :: real
  shows x < (1 + x) * (1 + x)
proof -
  have (x + 1/2)^2 + 3/4 > 0
    using zero_le_power2[of x+1/2] by arith
  then show ?thesis
    by (simp add: field_simps power2_eq_square)
qed

lemma square_continuous:
  fixes e :: real
  shows e > 0 ⟹ ∃ d. 0 < d ∧ (∀ y. |y − x| < d ⟶ |y * y − x * x| < e)
  using isCont_power[OF continuous_ident, of x, unfolded isCont_def LIM_eq, rule_format,
of e 2]
  by (force simp add: power2_eq_square)

lemma norm_le: norm x ≤ norm y ⟷ inner x x ≤ inner y y
  by (simp add: norm_eq_sqrt_inner)

lemma norm_lt: norm x < norm y ⟷ inner x x < inner y y
  by (simp add: norm_eq_sqrt_inner)

lemma norm_eq: norm x = norm y ⟷ inner x x = inner y y
  apply (subst order_eq_iff)
  apply (auto simp: norm_le)
  done

lemma norm_eq_1: norm x = 1 ⟷ inner x x = 1
  by (simp add: norm_eq_sqrt_inner)
```

lemma *inner_divide_left*:
fixes *a* :: *'a* :: {*real_inner*,*real_div_algebra*}
shows *inner* (*a* / *of_real m*) *b* = (*inner a b*) / *m*
by (*metis* (*no_types*) *divide_inverse inner_commute inner_scaleR_right mult.left_neutral mult.right_neutral mult_scaleR_right of_real_inverse scaleR_conv_of_real times_divide_eq_left*)

lemma *inner_divide_right*:
fixes *a* :: *'a* :: {*real_inner*,*real_div_algebra*}
shows *inner a* (*b* / *of_real m*) = (*inner a b*) / *m*
by (*metis inner_commute inner_divide_left*)

Re-enable constraints for *open*, *uniformity*, *dist*, and *norm*.

setup ‹*Sign.add_const_constraint*
(***const_name*** ‹*open*›, *SOME* ***typ*** ‹*'a*::*topological_space set* ⇒ *bool*›)›

setup ‹*Sign.add_const_constraint*
(***const_name*** ‹*uniformity*›, *SOME* ***typ*** ‹(*'a*::*uniform_space* × *'a*) *filter*›)›

setup ‹*Sign.add_const_constraint*
(***const_name*** ‹*dist*›, *SOME* ***typ*** ‹*'a*::*metric_space* ⇒ *'a* ⇒ *real*›)›

setup ‹*Sign.add_const_constraint*
(***const_name*** ‹*norm*›, *SOME* ***typ*** ‹*'a*::*real_normed_vector* ⇒ *real*›)›

lemma *bounded_bilinear_inner*:
bounded_bilinear (*inner*::*'a*::*real_inner* ⇒ *'a* ⇒ *real*)
proof
fix *x y z* :: *'a* **and** *r* :: *real*
show *inner* (*x* + *y*) *z* = *inner x z* + *inner y z*
by (*rule inner_add_left*)
show *inner x* (*y* + *z*) = *inner x y* + *inner x z*
by (*rule inner_add_right*)
show *inner* (*scaleR r x*) *y* = *scaleR r* (*inner x y*)
unfolding *real_scaleR_def* **by** (*rule inner_scaleR_left*)
show *inner x* (*scaleR r y*) = *scaleR r* (*inner x y*)
unfolding *real_scaleR_def* **by** (*rule inner_scaleR_right*)
show ∃ *K*. ∀ *x y*::*'a*. *norm* (*inner x y*) ≤ *norm x* ∗ *norm y* ∗ *K*
proof
show ∀ *x y*::*'a*. *norm* (*inner x y*) ≤ *norm x* ∗ *norm y* ∗ *1*
by (*simp add*: *Cauchy_Schwarz_ineq2*)
qed
qed

lemmas *tendsto_inner* [*tendsto_intros*] =
bounded_bilinear.tendsto [*OF bounded_bilinear_inner*]

lemmas *isCont_inner* [*simp*] =
bounded_bilinear.isCont [*OF bounded_bilinear_inner*]

lemmas *has_derivative_inner* [*derivative_intros*] =
bounded_bilinear.FDERIV [*OF bounded_bilinear_inner*]

lemmas *bounded_linear_inner_left* =
bounded_bilinear.bounded_linear_left [*OF bounded_bilinear_inner*]

lemmas *bounded_linear_inner_right* =
bounded_bilinear.bounded_linear_right [*OF bounded_bilinear_inner*]

lemmas *bounded_linear_inner_left_comp* = *bounded_linear_inner_left*[*THEN bounded_linear_compose*]

lemmas *bounded_linear_inner_right_comp* = *bounded_linear_inner_right*[*THEN bounded_linear_compose*]

lemmas *has_derivative_inner_right* [*derivative_intros*] =
bounded_linear.has_derivative [*OF bounded_linear_inner_right*]

lemmas *has_derivative_inner_left* [*derivative_intros*] =
bounded_linear.has_derivative [*OF bounded_linear_inner_left*]

lemma *differentiable_inner* [*simp*]:
f differentiable (*at x within s*) $\Longrightarrow$ *g differentiable at x within s* $\Longrightarrow$ (λx. *inner* (*f x*) (*g x*)) *differentiable at x within s*
unfolding *differentiable_def* **by** (*blast intro*: *has_derivative_inner*)

1.2.2 Class instances

instantiation *real* :: *real_inner*
begin

definition *inner_real_def* [*simp*]: *inner* = (∗)

instance
proof
fix *x y z r* :: *real*
show *inner x y* = *inner y x*
unfolding *inner_real_def* **by** (*rule mult.commute*)
show *inner* (*x* + *y*) *z* = *inner x z* + *inner y z*
unfolding *inner_real_def* **by** (*rule distrib_right*)
show *inner* (*scaleR r x*) *y* = *r* ∗ *inner x y*
unfolding *inner_real_def real_scaleR_def* **by** (*rule mult.assoc*)
show *0* ≤ *inner x x*
unfolding *inner_real_def* **by** *simp*
show *inner x x* = *0* ⟷ *x* = *0*
unfolding *inner_real_def* **by** *simp*
show *norm x* = *sqrt* (*inner x x*)
unfolding *inner_real_def* **by** *simp*
qed

end

```
lemma
  shows real_inner_1_left[simp]: inner 1 x = x
    and real_inner_1_right[simp]: inner x 1 = x
  by simp_all

instantiation complex :: real_inner
begin

definition inner_complex_def:
  inner x y = Re x * Re y + Im x * Im y

instance
proof
  fix x y z :: complex and r :: real
  show inner x y = inner y x
    unfolding inner_complex_def by (simp add: mult.commute)
  show inner (x + y) z = inner x z + inner y z
    unfolding inner_complex_def by (simp add: distrib_right)
  show inner (scaleR r x) y = r * inner x y
    unfolding inner_complex_def by (simp add: distrib_left)
  show 0 ≤ inner x x
    unfolding inner_complex_def by simp
  show inner x x = 0 ⟷ x = 0
    unfolding inner_complex_def
    by (simp add: add_nonneg_eq_0_iff complex_eq_iff)
  show norm x = sqrt (inner x x)
    unfolding inner_complex_def norm_complex_def
    by (simp add: power2_eq_square)
qed

end

lemma complex_inner_1 [simp]: inner 1 x = Re x
  unfolding inner_complex_def by simp

lemma complex_inner_1_right [simp]: inner x 1 = Re x
  unfolding inner_complex_def by simp

lemma complex_inner_i_left [simp]: inner i x = Im x
  unfolding inner_complex_def by simp

lemma complex_inner_i_right [simp]: inner x i = Im x
  unfolding inner_complex_def by simp


lemma dot_square_norm: inner x x = (norm x)^2
  by (simp only: power2_norm_eq_inner)
```

lemma *norm_eq_square*: *norm x* = *a* ⟷ *0* ≤ *a* ∧ *inner x x* = a^2
  by (*auto simp add*: *norm_eq_sqrt_inner*)

lemma *norm_le_square*: *norm x* ≤ *a* ⟷ *0* ≤ *a* ∧ *inner x x* ≤ a^2
  apply (*simp add*: *dot_square_norm abs_le_square_iff* [*symmetric*])
  using *norm_ge_zero*[*of x*]
  apply *arith*
  done

lemma *norm_ge_square*: *norm x* ≥ *a* ⟷ *a* ≤ *0* ∨ *inner x x* ≥ a^2
  apply (*simp add*: *dot_square_norm abs_le_square_iff* [*symmetric*])
  using *norm_ge_zero*[*of x*]
  apply *arith*
  done

lemma *norm_lt_square*: *norm x* < *a* ⟷ *0* < *a* ∧ *inner x x* < a^2
  by (*metis not_le norm_ge_square*)

lemma *norm_gt_square*: *norm x* > *a* ⟷ *a* < *0* ∨ *inner x x* > a^2
  by (*metis norm_le_square not_less*)

Dot product in terms of the norm rather than conversely.

lemmas *inner_simps* = *inner_add_left inner_add_right inner_diff_right inner_diff_left inner_scaleR_left inner_scaleR_right*

lemma *dot_norm*: *inner x y* = ((*norm* (*x* + *y*))^2 − (*norm x*)^2 − (*norm y*)^2) / *2*
  by (*simp only*: *power2_norm_eq_inner inner_simps inner_commute*) *auto*

lemma *dot_norm_neg*: *inner x y* = (((*norm x*)^2 + (*norm y*)^2) − (*norm* (*x* − *y*))^2) / *2*
  by (*simp only*: *power2_norm_eq_inner inner_simps inner_commute*)
    (*auto simp add*: *algebra_simps*)

lemma *of_real_inner_1* [*simp*]:
  inner (*of_real x*) (*1* :: ′*a* :: {*real_inner*, *real_normed_algebra_1*}) = *x*
  by (*simp add*: *of_real_def dot_square_norm*)

lemma *summable_of_real_iff*:
  summable (λ*x*. *of_real* (*f x*) :: ′*a* :: {*real_normed_algebra_1*,*real_inner*}) ⟷ *summable f*
proof
  assume ∗: *summable* (λ*x*. *of_real* (*f x*) :: ′*a*)
  interpret *bounded_linear* λ*x*::′*a*. *inner x 1*
    by (*rule bounded_linear_inner_left*)
  from *summable* [*OF* ∗] **show** *summable f* **by** *simp*
qed (*auto intro*: *summable_of_real*)

1.2.3 Gradient derivative

definition
gderiv ::
['*a*::*real_inner* ⇒ *real*, '*a*, '*a*] ⇒ *bool*
((*GDERIV* (_)/ (_)/ :> (_)) [*1000*, *1000*, *60*] *60*)
where
GDERIV f x :> *D* ⟷ *FDERIV f x* :> (λ*h*. *inner h D*)

lemma *gderiv_deriv* [*simp*]: *GDERIV f x* :> *D* ⟷ *DERIV f x* :> *D*
by (*simp only*: *gderiv_def has_field_derivative_def inner_real_def mult_commute_abs*)

lemma *GDERIV_DERIV_compose*:
⟦*GDERIV f x* :> *df*; *DERIV g* (*f x*) :> *dg*⟧
⟹ *GDERIV* (λ*x*. *g* (*f x*)) *x* :> *scaleR dg df*
unfolding *gderiv_def has_field_derivative_def*
apply (*drule* (*1*) *has_derivative_compose*)
apply (*simp add*: *ac_simps*)
done

lemma *has_derivative_subst*: ⟦*FDERIV f x* :> *df*; *df* = *d*⟧ ⟹ *FDERIV f x* :> *d*
by *simp*

lemma *GDERIV_subst*: ⟦*GDERIV f x* :> *df*; *df* = *d*⟧ ⟹ *GDERIV f x* :> *d*
by *simp*

lemma *GDERIV_const*: *GDERIV* (λ*x*. *k*) *x* :> *0*
unfolding *gderiv_def inner_zero_right* **by** (*rule has_derivative_const*)

lemma *GDERIV_add*:
⟦*GDERIV f x* :> *df*; *GDERIV g x* :> *dg*⟧
⟹ *GDERIV* (λ*x*. *f x* + *g x*) *x* :> *df* + *dg*
unfolding *gderiv_def inner_add_right* **by** (*rule has_derivative_add*)

lemma *GDERIV_minus*:
GDERIV f x :> *df* ⟹ *GDERIV* (λ*x*. − *f x*) *x* :> − *df*
unfolding *gderiv_def inner_minus_right* **by** (*rule has_derivative_minus*)

lemma *GDERIV_diff*:
⟦*GDERIV f x* :> *df*; *GDERIV g x* :> *dg*⟧
⟹ *GDERIV* (λ*x*. *f x* − *g x*) *x* :> *df* − *dg*
unfolding *gderiv_def inner_diff_right* **by** (*rule has_derivative_diff*)

lemma *GDERIV_scaleR*:
⟦*DERIV f x* :> *df*; *GDERIV g x* :> *dg*⟧
⟹ *GDERIV* (λ*x*. *scaleR* (*f x*) (*g x*)) *x*
:> (*scaleR* (*f x*) *dg* + *scaleR df* (*g x*))
unfolding *gderiv_def has_field_derivative_def inner_add_right inner_scaleR_right*
apply (*rule has_derivative_subst*)
apply (*erule* (*1*) *has_derivative_scaleR*)

```
  apply (simp add: ac_simps)
  done

lemma GDERIV_mult:
    ⟦GDERIV f x :> df; GDERIV g x :> dg⟧
     ⟹ GDERIV (λx. f x * g x) x :> scaleR (f x) dg + scaleR (g x) df
  unfolding gderiv_def
  apply (rule has_derivative_subst)
  apply (erule (1) has_derivative_mult)
  apply (simp add: inner_add ac_simps)
  done

lemma GDERIV_inverse:
    ⟦GDERIV f x :> df; f x ≠ 0⟧
     ⟹ GDERIV (λx. inverse (f x)) x :> − (inverse (f x))^2 *R df
  by (metis DERIV_inverse GDERIV_DERIV_compose numerals(2))

lemma GDERIV_norm:
  assumes x ≠ 0 shows GDERIV (λx. norm x) x :> sgn x
    unfolding gderiv_def norm_eq_sqrt_inner
     by (rule derivative_eq_intros | force simp add: inner_commute sgn_div_norm
norm_eq_sqrt_inner assms)+

lemmas has_derivative_norm = GDERIV_norm [unfolded gderiv_def]

bundle inner_syntax begin
notation inner (infix · 70)
end

bundle no_inner_syntax begin
no_notation inner (infix · 70)
end

end
```

1.3 Cartesian Products as Vector Spaces

```
theory Product_Vector
  imports
    Complex_Main
    HOL−Library.Product_Plus
begin

lemma Times_eq_image_sum:
  fixes S :: 'a :: comm_monoid_add set and T :: 'b :: comm_monoid_add set
  shows S × T = {u + v |u v. u ∈ (λx. (x, 0)) ‘ S ∧ v ∈ Pair 0 ‘ T}
  by force
```

1.3.1 Product is a Module

locale *module_prod* = *module_pair* **begin**

definition *scale* :: *'a* ⇒ *'b* × *'c* ⇒ *'b* × *'c*
where *scale a v* = (*s1 a* (*fst v*), *s2 a* (*snd v*))

lemma *scale_prod*: *scale x* (*a*, *b*) = (*s1 x a*, *s2 x b*)
by (*auto simp*: *scale_def*)

sublocale *p*: *module scale*
proof qed (*simp_all add*: *scale_def*
m1.scale_left_distrib m1.scale_right_distrib m2.scale_left_distrib m2.scale_right_distrib)

lemma *subspace_Times*: *m1.subspace A* ⟹ *m2.subspace B* ⟹ *p.subspace* (*A* × *B*)
unfolding *m1.subspace_def m2.subspace_def p.subspace_def*
by (*auto simp*: *zero_prod_def scale_def*)

lemma *module_hom_fst*: *module_hom scale s1 fst*
by *unfold_locales* (*auto simp*: *scale_def*)

lemma *module_hom_snd*: *module_hom scale s2 snd*
by *unfold_locales* (*auto simp*: *scale_def*)

end

locale *vector_space_prod* = *vector_space_pair* **begin**

sublocale *module_prod s1 s2*
rewrites *module_hom* = *Vector_Spaces.linear*
by *unfold_locales* (*fact module_hom_eq_linear*)

sublocale *p*: *vector_space scale* **by** *unfold_locales* (*auto simp*: *algebra_simps*)

lemmas *linear_fst* = *module_hom_fst*
and *linear_snd* = *module_hom_snd*

end

1.3.2 Product is a Real Vector Space

instantiation *prod* :: (*real_vector*, *real_vector*) *real_vector*
begin

definition *scaleR_prod_def*:
scaleR r A = (*scaleR r* (*fst A*), *scaleR r* (*snd A*))

lemma *fst_scaleR* [*simp*]: *fst* (*scaleR r A*) = *scaleR r* (*fst A*)
unfolding *scaleR_prod_def* **by** *simp*

lemma *snd_scaleR* [*simp*]: *snd* (*scaleR r A*) = *scaleR r* (*snd A*)
  unfolding *scaleR_prod_def* **by** *simp*

proposition *scaleR_Pair* [*simp*]: *scaleR r* (*a*, *b*) = (*scaleR r a*, *scaleR r b*)
  unfolding *scaleR_prod_def* **by** *simp*

instance
proof
  fix *a b* :: *real* **and** *x y* :: *'a* × *'b*
  show *scaleR a* (*x* + *y*) = *scaleR a x* + *scaleR a y*
    by (*simp add*: *prod_eq_iff scaleR_right_distrib*)
  show *scaleR* (*a* + *b*) *x* = *scaleR a x* + *scaleR b x*
    by (*simp add*: *prod_eq_iff scaleR_left_distrib*)
  show *scaleR a* (*scaleR b x*) = *scaleR* (*a* ∗ *b*) *x*
    by (*simp add*: *prod_eq_iff*)
  show *scaleR 1 x* = *x*
    by (*simp add*: *prod_eq_iff*)
qed

end

lemma *module_prod_scale_eq_scaleR*: *module_prod.scale* ($*_R$) ($*_R$) = *scaleR*
  apply (*rule ext*) **apply** (*rule ext*)
  apply (*subst module_prod.scale_def*)
  subgoal by *unfold_locales*
  by (*simp add*: *scaleR_prod_def*)

interpretation *real_vector?*: *vector_space_prod scaleR*::_⇒_⇒*'a*::*real_vector scaleR*::_⇒_⇒*'b*::*real_vector*
  rewrites *scale* = (($*_R$)::_⇒_⇒(*'a* × *'b*))
    and *module.dependent* ($*_R$) = *dependent*
    and *module.representation* ($*_R$) = *representation*
    and *module.subspace* ($*_R$) = *subspace*
    and *module.span* ($*_R$) = *span*
    and *vector_space.extend_basis* ($*_R$) = *extend_basis*
    and *vector_space.dim* ($*_R$) = *dim*
    and *Vector_Spaces.linear* ($*_R$) ($*_R$) = *linear*
  subgoal by *unfold_locales*
  subgoal by (*fact module_prod_scale_eq_scaleR*)
  unfolding *dependent_raw_def representation_raw_def subspace_raw_def span_raw_def*
    extend_basis_raw_def dim_raw_def linear_def
  by (*rule refl*)+

1.3.3 Product is a Metric Space

instantiation *prod* :: (*metric_space*, *metric_space*) *dist*
begin

definition *dist_prod_def* [*code del*]:

```
  dist x y = sqrt ((dist (fst x) (fst y))² + (dist (snd x) (snd y))²)

instance ..
end

instantiation prod :: (metric_space, metric_space) uniformity_dist
begin

definition [code del]:
  (uniformity :: (('a × 'b) × ('a × 'b)) filter) =
    (INF e∈{0 <..}. principal {(x, y). dist x y < e})

instance
  by standard (rule uniformity_prod_def)
end

declare uniformity_Abort[where 'a='a :: metric_space × 'b :: metric_space, code]

instantiation prod :: (metric_space, metric_space) metric_space
begin

proposition dist_Pair_Pair: dist (a, b) (c, d) = sqrt ((dist a c)² + (dist b d)²)
  unfolding dist_prod_def by simp

lemma dist_fst_le: dist (fst x) (fst y) ≤ dist x y
  unfolding dist_prod_def by (rule real_sqrt_sum_squares_ge1)

lemma dist_snd_le: dist (snd x) (snd y) ≤ dist x y
  unfolding dist_prod_def by (rule real_sqrt_sum_squares_ge2)

instance
proof
  fix x y :: 'a × 'b
  show dist x y = 0 ⟷ x = y
    unfolding dist_prod_def prod_eq_iff by simp
next
  fix x y z :: 'a × 'b
  show dist x y ≤ dist x z + dist y z
    unfolding dist_prod_def
    by (intro order_trans [OF _ real_sqrt_sum_squares_triangle_ineq]
        real_sqrt_le_mono add_mono power_mono dist_triangle2 zero_le_dist)
next
  fix S :: ('a × 'b) set
  have *: open S ⟷ (∀x∈S. ∃e>0. ∀y. dist y x < e ⟶ y ∈ S)
  proof
    assume open S show ∀x∈S. ∃e>0. ∀y. dist y x < e ⟶ y ∈ S
    proof
      fix x assume x ∈ S
      obtain A B where open A open B x ∈ A × B A × B ⊆ S
```

```
      using ‹open S› and ‹x ∈ S› by (rule open_prod_elim)
    obtain r where r: 0 < r ∀ y. dist y (fst x) < r ⟶ y ∈ A
      using ‹open A› and ‹x ∈ A × B› unfolding open_dist by auto
    obtain s where s: 0 < s ∀ y. dist y (snd x) < s ⟶ y ∈ B
      using ‹open B› and ‹x ∈ A × B› unfolding open_dist by auto
    let ?e = min r s
    have 0 < ?e ∧ (∀ y. dist y x < ?e ⟶ y ∈ S)
    proof (intro allI impI conjI)
      show 0 < min r s by (simp add: r(1) s(1))
    next
      fix y assume dist y x < min r s
      hence dist y x < r and dist y x < s
        by simp_all
      hence dist (fst y) (fst x) < r and dist (snd y) (snd x) < s
        by (auto intro: le_less_trans dist_fst_le dist_snd_le)
      hence fst y ∈ A and snd y ∈ B
        by (simp_all add: r(2) s(2))
      hence y ∈ A × B by (induct y, simp)
      with ‹A × B ⊆ S› show y ∈ S ..
    qed
    thus ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S ..
  qed
next
  assume *: ∀ x∈S. ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S show open S
  proof (rule open_prod_intro)
    fix x assume x ∈ S
    then obtain e where 0 < e and S: ∀ y. dist y x < e ⟶ y ∈ S
      using * by fast
    define r where r = e / sqrt 2
    define s where s = e / sqrt 2
    from ‹0 < e› have 0 < r and 0 < s
      unfolding r_def s_def by simp_all
    from ‹0 < e› have e = sqrt (r² + s²)
      unfolding r_def s_def by (simp add: power_divide)
    define A where A = {y. dist (fst x) y < r}
    define B where B = {y. dist (snd x) y < s}
    have open A and open B
      unfolding A_def B_def by (simp_all add: open_ball)
    moreover have x ∈ A × B
      unfolding A_def B_def mem_Times_iff
      using ‹0 < r› and ‹0 < s› by simp
    moreover have A × B ⊆ S
    proof (clarify)
      fix a b assume a ∈ A and b ∈ B
      hence dist a (fst x) < r and dist b (snd x) < s
        unfolding A_def B_def by (simp_all add: dist_commute)
      hence dist (a, b) x < e
        unfolding dist_prod_def ‹e = sqrt (r² + s²)›
        by (simp add: add_strict_mono power_strict_mono)
```

```
      thus (a, b) ∈ S
        by (simp add: S)
    qed
    ultimately show ∃A B. open A ∧ open B ∧ x ∈ A × B ∧ A × B ⊆ S by
fast
   qed
  qed
  show open S = (∀x∈S. ∀F (x', y) in uniformity. x' = x ⟶ y ∈ S)
    unfolding * eventually_uniformity_metric
    by (simp del: split_paired_All add: dist_prod_def dist_commute)
qed

end

declare [[code abort: dist::('a::metric_space*'b::metric_space)⇒('a*'b) ⇒ real]]

lemma Cauchy_fst: Cauchy X ⟹ Cauchy (λn. fst (X n))
  unfolding Cauchy_def by (fast elim: le_less_trans [OF dist_fst_le])

lemma Cauchy_snd: Cauchy X ⟹ Cauchy (λn. snd (X n))
  unfolding Cauchy_def by (fast elim: le_less_trans [OF dist_snd_le])

lemma Cauchy_Pair:
  assumes Cauchy X and Cauchy Y
  shows Cauchy (λn. (X n, Y n))
proof (rule metric_CauchyI)
  fix r :: real assume 0 < r
  hence 0 < r / sqrt 2 (is 0 < ?s) by simp
  obtain M where M: ∀m≥M. ∀n≥M. dist (X m) (X n) < ?s
    using metric_CauchyD [OF ‹Cauchy X› ‹0 < ?s›] ..
  obtain N where N: ∀m≥N. ∀n≥N. dist (Y m) (Y n) < ?s
    using metric_CauchyD [OF ‹Cauchy Y› ‹0 < ?s›] ..
  have ∀m≥max M N. ∀n≥max M N. dist (X m, Y m) (X n, Y n) < r
    using M N by (simp add: real_sqrt_sum_squares_less dist_Pair_Pair)
  then show ∃n0. ∀m≥n0. ∀n≥n0. dist (X m, Y m) (X n, Y n) < r ..
qed
```

1.3.4 Product is a Complete Metric Space

```
instance prod :: (complete_space, complete_space) complete_space
proof
  fix X :: nat ⇒ 'a × 'b assume Cauchy X
  have 1: (λn. fst (X n)) ⟶ lim (λn. fst (X n))
    using Cauchy_fst [OF ‹Cauchy X›]
    by (simp add: Cauchy_convergent_iff convergent_LIMSEQ_iff)
  have 2: (λn. snd (X n)) ⟶ lim (λn. snd (X n))
    using Cauchy_snd [OF ‹Cauchy X›]
    by (simp add: Cauchy_convergent_iff convergent_LIMSEQ_iff)
  have X ⟶ (lim (λn. fst (X n)), lim (λn. snd (X n)))
```

```
    using tendsto_Pair [OF 1 2] by simp
  then show convergent X
    by (rule convergentI)
qed
```

1.3.5 Product is a Normed Vector Space

```
instantiation prod :: (real_normed_vector, real_normed_vector) real_normed_vector
begin

definition norm_prod_def[code del]:
  norm x = sqrt ((norm (fst x))^2 + (norm (snd x))^2)

definition sgn_prod_def:
  sgn (x::'a × 'b) = scaleR (inverse (norm x)) x

proposition norm_Pair: norm (a, b) = sqrt ((norm a)^2 + (norm b)^2)
  unfolding norm_prod_def by simp

instance
proof
  fix r :: real and x y :: 'a × 'b
  show norm x = 0 ⟷ x = 0
    unfolding norm_prod_def
    by (simp add: prod_eq_iff)
  show norm (x + y) ≤ norm x + norm y
    unfolding norm_prod_def
    apply (rule order_trans [OF _ real_sqrt_sum_squares_triangle_ineq])
    apply (simp add: add_mono power_mono norm_triangle_ineq)
    done
  show norm (scaleR r x) = |r| * norm x
    unfolding norm_prod_def
    apply (simp add: power_mult_distrib)
    apply (simp add: distrib_left [symmetric])
    apply (simp add: real_sqrt_mult)
    done
  show sgn x = scaleR (inverse (norm x)) x
    by (rule sgn_prod_def)
  show dist x y = norm (x − y)
    unfolding dist_prod_def norm_prod_def
    by (simp add: dist_norm)
qed

end

declare [[code abort: norm::('a::real_normed_vector*'b::real_normed_vector) ⇒ real]]

instance prod :: (banach, banach) banach ..
```

Pair operations are linear

```
lemma bounded_linear_fst: bounded_linear fst
  using fst_add fst_scaleR
  by (rule bounded_linear_intro [where K=1], simp add: norm_prod_def)

lemma bounded_linear_snd: bounded_linear snd
  using snd_add snd_scaleR
  by (rule bounded_linear_intro [where K=1], simp add: norm_prod_def)

lemmas bounded_linear_fst_comp = bounded_linear_fst[THEN bounded_linear_compose]

lemmas bounded_linear_snd_comp = bounded_linear_snd[THEN bounded_linear_compose]

lemma bounded_linear_Pair:
  assumes f: bounded_linear f
  assumes g: bounded_linear g
  shows bounded_linear (λx. (f x, g x))
proof
  interpret f: bounded_linear f by fact
  interpret g: bounded_linear g by fact
  fix x y and r :: real
  show (f (x + y), g (x + y)) = (f x, g x) + (f y, g y)
    by (simp add: f.add g.add)
  show (f (r *R x), g (r *R x)) = r *R (f x, g x)
    by (simp add: f.scale g.scale)
  obtain Kf where 0 < Kf and norm_f: ⋀x. norm (f x) ≤ norm x * Kf
    using f.pos_bounded by fast
  obtain Kg where 0 < Kg and norm_g: ⋀x. norm (g x) ≤ norm x * Kg
    using g.pos_bounded by fast
  have ∀x. norm (f x, g x) ≤ norm x * (Kf + Kg)
    apply (rule allI)
    apply (simp add: norm_Pair)
    apply (rule order_trans [OF sqrt_add_le_add_sqrt], simp, simp)
    apply (simp add: distrib_left)
    apply (rule add_mono [OF norm_f norm_g])
    done
  then show ∃K. ∀x. norm (f x, g x) ≤ norm x * K ..
qed
```

Frechet derivatives involving pairs

```
proposition has_derivative_Pair [derivative_intros]:
  assumes f: (f has_derivative f') (at x within s)
    and g: (g has_derivative g') (at x within s)
  shows ((λx. (f x, g x)) has_derivative (λh. (f' h, g' h))) (at x within s)
proof (rule has_derivativeI_sandwich[of 1])
  show bounded_linear (λh. (f' h, g' h))
    using f g by (intro bounded_linear_Pair has_derivative_bounded_linear)
  let ?Rf = λy. f y − f x − f' (y − x)
```

let *?Rg* = λ*y*. *g y* − *g x* − *g′* (*y* − *x*)
let *?R* = λ*y*. ((*f y*, *g y*) − (*f x*, *g x*) − (*f′* (*y* − *x*), *g′* (*y* − *x*)))

show ((λ*y*. *norm* (*?Rf y*) / *norm* (*y* − *x*) + *norm* (*?Rg y*) / *norm* (*y* − *x*)) ⟶ *0*) (*at x within s*)
using *f g* **by** (*intro tendsto_add_zero*) (*auto simp*: *has_derivative_iff_norm*)

fix *y* :: *′a* **assume** *y* ≠ *x*
show *norm* (*?R y*) / *norm* (*y* − *x*) ≤ *norm* (*?Rf y*) / *norm* (*y* − *x*) + *norm* (*?Rg y*) / *norm* (*y* − *x*)
unfolding *add_divide_distrib* [*symmetric*]
by (*simp add*: *norm_Pair divide_right_mono order_trans* [*OF sqrt_add_le_add_sqrt*])
qed *simp*

lemma *differentiable_Pair* [*simp*, *derivative_intros*]:
f differentiable at x within s ⟹ *g differentiable at x within s* ⟹
(λ*x*. (*f x*, *g x*)) *differentiable at x within s*
unfolding *differentiable_def* **by** (*blast intro*: *has_derivative_Pair*)

lemmas *has_derivative_fst* [*derivative_intros*] = *bounded_linear.has_derivative* [*OF bounded_linear_fst*]
lemmas *has_derivative_snd* [*derivative_intros*] = *bounded_linear.has_derivative* [*OF bounded_linear_snd*]

lemma *has_derivative_split* [*derivative_intros*]:
((λ*p*. *f* (*fst p*) (*snd p*)) *has_derivative f′*) *F* ⟹ ((λ(*a*, *b*). *f a b*) *has_derivative f′*) *F*
unfolding *split_beta′* **.**

Vector derivatives involving pairs

lemma *has_vector_derivative_Pair*[*derivative_intros*]:
assumes (*f has_vector_derivative f′*) (*at x within s*)
(*g has_vector_derivative g′*) (*at x within s*)
shows ((λ*x*. (*f x*, *g x*)) *has_vector_derivative* (*f′*, *g′*)) (*at x within s*)
using *assms*
by (*auto simp*: *has_vector_derivative_def intro*!: *derivative_eq_intros*)

lemma
fixes *x* :: *′a*::*real_normed_vector*
shows *norm_Pair1* [*simp*]: *norm* (*0*,*x*) = *norm x*
and *norm_Pair2* [*simp*]: *norm* (*x*,*0*) = *norm x*
by (*auto simp*: *norm_Pair*)

lemma *norm_commute*: *norm* (*x*,*y*) = *norm* (*y*,*x*)
by (*simp add*: *norm_Pair*)

lemma *norm_fst_le*: *norm x* ≤ *norm* (*x*,*y*)
by (*metis dist_fst_le fst_conv fst_zero norm_conv_dist*)

lemma *norm_snd_le*: *norm y* ≤ *norm* (*x*,*y*)
by (*metis dist_snd_le snd_conv snd_zero norm_conv_dist*)

lemma *norm_Pair_le*:
shows *norm* (*x*, *y*) ≤ *norm x* + *norm y*
unfolding *norm_Pair*
by (*metis norm_ge_zero sqrt_sum_squares_le_sum*)

lemma (**in** *vector_space_prod*) *span_Times_sing1*: *p.span* ({*0*} × *B*) = {*0*} × *vs2.span B*
apply (*rule p.span_unique*)
subgoal by (*auto intro*!: *vs1.span_base vs2.span_base*)
subgoal using *vs1.subspace_single_0 vs2.subspace_span* **by** (*rule subspace_Times*)
subgoal for *T*
proof *safe*
fix *b*
assume *subset_T*: {*0*} × *B* ⊆ *T* **and** *subspace*: *p.subspace T* **and** *b_span*: *b* ∈ *vs2.span B*
then obtain *t r* **where** *b*: *b* = (∑ *a*∈*t*. *r a* ∗*b a*) **and** *t*: *finite t t* ⊆ *B*
by (*auto simp*: *vs2.span_explicit*)
have (*0*, *b*) = (∑ *b*∈*t*. *scale* (*r b*) (*0*, *b*))
unfolding *b scale_prod sum_prod*
by *simp*
also have ... ∈ *T*
using ‹*t* ⊆ *B*› *subset_T*
by (*auto intro*!: *p.subspace_sum p.subspace_scale subspace*)
finally show (*0*, *b*) ∈ *T* **.**
qed
done

lemma (**in** *vector_space_prod*) *span_Times_sing2*: *p.span* (*A* × {*0*}) = *vs1.span A* × {*0*}
apply (*rule p.span_unique*)
subgoal by (*auto intro*!: *vs1.span_base vs2.span_base*)
subgoal using *vs1.subspace_span vs2.subspace_single_0* **by** (*rule subspace_Times*)
subgoal for *T*
proof *safe*
fix *a*
assume *subset_T*: *A* × {*0*} ⊆ *T* **and** *subspace*: *p.subspace T* **and** *a_span*: *a* ∈ *vs1.span A*
then obtain *t r* **where** *a*: *a* = (∑ *a*∈*t*. *r a* ∗*a a*) **and** *t*: *finite t t* ⊆ *A*
by (*auto simp*: *vs1.span_explicit*)
have (*a*, *0*) = (∑ *a*∈*t*. *scale* (*r a*) (*a*, *0*))
unfolding *a scale_prod sum_prod*
by *simp*
also have ... ∈ *T*
using ‹*t* ⊆ *A*› *subset_T*
by (*auto intro*!: *p.subspace_sum p.subspace_scale subspace*)

```
    finally show (a, 0) ∈ T .
  qed
  done
```

1.3.6 Product is Finite Dimensional

```
lemma (in finite_dimensional_vector_space) zero_not_in_Basis[simp]: 0 ∉ Basis
  using dependent_zero local.independent_Basis by blast

locale finite_dimensional_vector_space_prod = vector_space_prod + finite_dimensional_vector_space_pair
begin

definition Basis_pair = B1 × {0} ∪ {0} × B2

sublocale p: finite_dimensional_vector_space scale Basis_pair
proof unfold_locales
  show finite Basis_pair
    by (auto intro!: finite_cartesian_product vs1.finite_Basis vs2.finite_Basis simp:
Basis_pair_def)
  show p.independent Basis_pair
    unfolding p.dependent_def Basis_pair_def
  proof safe
    fix a
    assume a: a ∈ B1
    assume (a, 0) ∈ p.span (B1 × {0} ∪ {0} × B2 − {(a, 0)})
    also have B1 × {0} ∪ {0} × B2 − {(a, 0)} = (B1 − {a}) × {0} ∪ {0} ×
B2
      by auto
    finally show False
      using a vs1.dependent_def vs1.independent_Basis
      by (auto simp: p.span_Un span_Times_sing1 span_Times_sing2)
  next
    fix b
    assume b: b ∈ B2
    assume (0, b) ∈ p.span (B1 × {0} ∪ {0} × B2 − {(0, b)})
    also have (B1 × {0} ∪ {0} × B2 − {(0, b)}) = B1 × {0} ∪ {0} × (B2 −
{b})
      by auto
    finally show False
      using b vs2.dependent_def vs2.independent_Basis
      by (auto simp: p.span_Un span_Times_sing1 span_Times_sing2)
  qed
  show p.span Basis_pair = UNIV
    by (auto simp: p.span_Un span_Times_sing2 span_Times_sing1 vs1.span_Basis
vs2.span_Basis
        Basis_pair_def)
qed

proposition dim_Times:
```

assumes *vs1.subspace S vs2.subspace T*
shows *p.dim*(*S* × *T*) = *vs1.dim S* + *vs2.dim T*
proof −
interpret *p1*: *Vector_Spaces.linear s1 scale* (λ*x*. (*x*, *0*))
by *unfold_locales* (*auto simp*: *scale_def*)
interpret *pair1*: *finite_dimensional_vector_space_pair* (∗*a*) *B1 scale Basis_pair*
by *unfold_locales*
interpret *p2*: *Vector_Spaces.linear s2 scale* (λ*x*. (*0*, *x*))
by *unfold_locales* (*auto simp*: *scale_def*)
interpret *pair2*: *finite_dimensional_vector_space_pair* (∗*b*) *B2 scale Basis_pair*
by *unfold_locales*
have *ss*: *p.subspace* ((λ*x*. (*x*, *0*)) ‘ *S*) *p.subspace* (*Pair 0* ‘ *T*)
by (*rule p1.subspace_image p2.subspace_image assms*)+
have *p.dim*(*S* × *T*) = *p.dim*({*u* + *v* |*u v*. *u* ∈ (λ*x*. (*x*, *0*)) ‘ *S* ∧ *v* ∈ *Pair 0* ‘ *T*})
by (*simp add*: *Times_eq_image_sum*)
moreover have *p.dim* ((λ*x*. (*x*, *0*::′*c*)) ‘ *S*) = *vs1.dim S p.dim* (*Pair* (*0*::′*b*) ‘ *T*) = *vs2.dim T*
by (*simp_all add*: *inj_on_def p1.linear_axioms pair1.dim_image_eq p2.linear_axioms pair2.dim_image_eq*)
moreover have *p.dim* ((λ*x*. (*x*, *0*)) ‘ *S* ∩ *Pair 0* ‘ *T*) = *0*
by (*subst p.dim_eq_0*) *auto*
ultimately show *?thesis*
using *p.dim_sums_Int* [*OF ss*] **by** *linarith*
qed

lemma *dimension_pair*: *p.dimension* = *vs1.dimension* + *vs2.dimension*
using *dim_Times*[*OF vs1.subspace_UNIV vs2.subspace_UNIV*]
by (*auto simp*: *p.dimension_def vs1.dimension_def vs2.dimension_def*)

end

end

1.4 Finite-Dimensional Inner Product Spaces

theory *Euclidean_Space*
imports
L2_Norm
Inner_Product
Product_Vector
begin

1.4.1 Interlude: Some properties of real sets

lemma *seq_mono_lemma*:
assumes ∀ (*n*::*nat*) ≥ *m*. (*d n* :: *real*) < *e n*
and ∀ *n* ≥ *m*. *e n* ≤ *e m*
shows ∀ *n* ≥ *m*. *d n* < *e m*

using *assms* **by** *force*

1.4.2 Type class of Euclidean spaces

class *euclidean_space* = *real_inner* +
  fixes *Basis* :: *'a set*
  assumes *nonempty_Basis* [*simp*]: *Basis* ≠ {}
  assumes *finite_Basis* [*simp*]: *finite Basis*
  assumes *inner_Basis*:
    ⟦*u* ∈ *Basis*; *v* ∈ *Basis*⟧ ⟹ *inner u v* = (*if u* = *v then 1 else 0*)
  assumes *euclidean_all_zero_iff*:
    (∀ *u*∈*Basis*. *inner x u* = *0*) ⟷ (*x* = *0*)

syntax *_type_dimension* :: *type* ⇒ *nat* ((*1DIM*/(*1*'(_'))))
translations *DIM*('*a*) ⇀ *CONST card* (*CONST Basis* :: '*a set*)
typed_print_translation ‹
  [(***const_syntax***‹*card*›,
    fn ctxt => *fn* _ => *fn* [*Const* (***const_syntax***‹*Basis*›, *Type* (***type_name***‹*set*›,
[*T*]))] =>
      Syntax.const ***syntax_const***‹*_type_dimension*› $ *Syntax_Phases.term_of_typ*
ctxt T)]
›

lemma (**in** *euclidean_space*) *norm_Basis*[*simp*]: *u* ∈ *Basis* ⟹ *norm u* = *1*
  unfolding *norm_eq_sqrt_inner* **by** (*simp add*: *inner_Basis*)

lemma (**in** *euclidean_space*) *inner_same_Basis*[*simp*]: *u* ∈ *Basis* ⟹ *inner u u* =
1
  by (*simp add*: *inner_Basis*)

lemma (**in** *euclidean_space*) *inner_not_same_Basis*: *u* ∈ *Basis* ⟹ *v* ∈ *Basis* ⟹
u ≠ *v* ⟹ *inner u v* = *0*
  by (*simp add*: *inner_Basis*)

lemma (**in** *euclidean_space*) *sgn_Basis*: *u* ∈ *Basis* ⟹ *sgn u* = *u*
  unfolding *sgn_div_norm* **by** (*simp add*: *scaleR_one*)

lemma (**in** *euclidean_space*) *Basis_zero* [*simp*]: *0* ∉ *Basis*
proof
  assume *0* ∈ *Basis* **thus** *False*
    using *inner_Basis* [*of 0 0*] **by** *simp*
qed

lemma (**in** *euclidean_space*) *nonzero_Basis*: *u* ∈ *Basis* ⟹ *u* ≠ *0*
  by *clarsimp*

lemma (**in** *euclidean_space*) *SOME_Basis*: (*SOME i*. *i* ∈ *Basis*) ∈ *Basis*
  by (*metis ex_in_conv nonempty_Basis someI_ex*)

```
lemma norm_some_Basis [simp]: norm (SOME i. i ∈ Basis) = 1
  by (simp add: SOME_Basis)

lemma (in euclidean_space) inner_sum_left_Basis[simp]:
    b ∈ Basis ⟹ inner (∑ i∈Basis. f i *R i) b = f b
 by (simp add: inner_sum_left inner_Basis if_distrib comm_monoid_add_class.sum.If_cases)

lemma (in euclidean_space) euclidean_eqI:
  assumes b: ⋀b. b ∈ Basis ⟹ inner x b = inner y b shows x = y
proof -
  from b have ∀ b∈Basis. inner (x - y) b = 0
    by (simp add: inner_diff_left)
  then show x = y
    by (simp add: euclidean_all_zero_iff)
qed

lemma (in euclidean_space) euclidean_eq_iff:
  x = y ⟷ (∀ b∈Basis. inner x b = inner y b)
  by (auto intro: euclidean_eqI)

lemma (in euclidean_space) euclidean_representation_sum:
  (∑ i∈Basis. f i *R i) = b ⟷ (∀ i∈Basis. f i = inner b i)
  by (subst euclidean_eq_iff) simp

lemma (in euclidean_space) euclidean_representation_sum':
  b = (∑ i∈Basis. f i *R i) ⟷ (∀ i∈Basis. f i = inner b i)
  by (auto simp add: euclidean_representation_sum[symmetric])

lemma (in euclidean_space) euclidean_representation: (∑ b∈Basis. inner x b *R
b) = x
  unfolding euclidean_representation_sum by simp

lemma (in euclidean_space) euclidean_inner: inner x y = (∑ b∈Basis. (inner x b)
* (inner y b))
  by (subst (1 2) euclidean_representation [symmetric])
    (simp add: inner_sum_right inner_Basis ac_simps)

lemma (in euclidean_space) choice_Basis_iff:
  fixes P :: 'a ⇒ real ⇒ bool
  shows (∀ i∈Basis. ∃ x. P i x) ⟷ (∃ x. ∀ i∈Basis. P i (inner x i))
  unfolding bchoice_iff
proof safe
  fix f assume ∀ i∈Basis. P i (f i)
  then show ∃ x. ∀ i∈Basis. P i (inner x i)
    by (auto intro!: exI[of _ ∑ i∈Basis. f i *R i])
qed auto

lemma (in euclidean_space) bchoice_Basis_iff:
  fixes P :: 'a ⇒ real ⇒ bool
```

```
 shows (∀ i∈Basis. ∃ x∈A. P i x) ⟷ (∃ x. ∀ i∈Basis. inner x i ∈ A ∧ P i (inner
x i))
by (simp add: choice_Basis_iff Bex_def)

lemma (in euclidean_space) euclidean_representation_sum_fun:
   (λx. ∑ b∈Basis. inner (f x) b *R b) = f
 by (rule ext) (simp add: euclidean_representation_sum)

lemma euclidean_isCont:
 assumes ⋀b. b ∈ Basis ⟹ isCont (λx. (inner (f x) b) *R b) x
   shows isCont f x
 apply (subst euclidean_representation_sum_fun [symmetric])
 apply (rule isCont_sum)
 apply (blast intro: assms)
 done

lemma DIM_positive [simp]: 0 < DIM('a::euclidean_space)
 by (simp add: card_gt_0_iff)

lemma DIM_ge_Suc0 [simp]: Suc 0 ≤ card Basis
 by (meson DIM_positive Suc_leI)


lemma sum_inner_Basis_scaleR [simp]:
 fixes f :: 'a::euclidean_space ⇒ 'b::real_vector
 assumes b ∈ Basis shows (∑ i∈Basis. (inner i b) *R f i) = f b
 by (simp add: comm_monoid_add_class.sum.remove [OF finite_Basis assms]
       assms inner_not_same_Basis comm_monoid_add_class.sum.neutral)

lemma sum_inner_Basis_eq [simp]:
 assumes b ∈ Basis shows (∑ i∈Basis. (inner i b) * f i) = f b
 by (simp add: comm_monoid_add_class.sum.remove [OF finite_Basis assms]
       assms inner_not_same_Basis comm_monoid_add_class.sum.neutral)

lemma sum_if_inner [simp]:
 assumes i ∈ Basis j ∈ Basis
   shows inner (∑ k∈Basis. if k = i then f i *R i else g k *R k) j = (if j=i then
f j else g j)
proof (cases i=j)
 case True
 with assms show ?thesis
    by (auto simp: inner_sum_left if_distrib [of λx. inner x j] inner_Basis cong:
if_cong)
next
 case False
 have (∑ k∈Basis. inner (if k = i then f i *R i else g k *R k) j) =
      (∑ k∈Basis. if k = j then g k else 0)
   apply (rule sum.cong)
   using False assms by (auto simp: inner_Basis)
```

```
  also have ... = g j
    using assms by auto
  finally show ?thesis
    using False by (auto simp: inner_sum_left)
qed

lemma norm_le_componentwise:
   (⋀b. b ∈ Basis ⟹ abs(inner x b) ≤ abs(inner y b)) ⟹ norm x ≤ norm y
 by (auto simp: norm_le euclidean_inner [of x x] euclidean_inner [of y y] abs_le_square_iff
power2_eq_square intro!: sum_mono)

lemma Basis_le_norm: b ∈ Basis ⟹ |inner x b| ≤ norm x
  by (rule order_trans [OF Cauchy_Schwarz_ineq2]) simp

lemma norm_bound_Basis_le: b ∈ Basis ⟹ norm x ≤ e ⟹ |inner x b| ≤ e
  by (metis Basis_le_norm order_trans)

lemma norm_bound_Basis_lt: b ∈ Basis ⟹ norm x < e ⟹ |inner x b| < e
  by (metis Basis_le_norm le_less_trans)

lemma norm_le_l1: norm x ≤ (∑ b∈Basis. |inner x b|)
  apply (subst euclidean_representation[of x, symmetric])
  apply (rule order_trans[OF norm_sum])
  apply (auto intro!: sum_mono)
  done

lemma sum_norm_allsubsets_bound:
  fixes f :: 'a ⇒ 'n::euclidean_space
  assumes fP: finite P
    and fPs: ⋀Q. Q ⊆ P ⟹ norm (sum f Q) ≤ e
  shows (∑ x∈P. norm (f x)) ≤ 2 * real DIM('n) * e
proof −
  have (∑ x∈P. norm (f x)) ≤ (∑ x∈P. ∑ b∈Basis. |inner (f x) b|)
    by (rule sum_mono) (rule norm_le_l1)
  also have (∑ x∈P. ∑ b∈Basis. |inner (f x) b|) = (∑ b∈Basis. ∑ x∈P. |inner
(f x) b|)
    by (rule sum.swap)
  also have . . . ≤ of_nat (card (Basis :: 'n set)) * (2 * e)
  proof (rule sum_bounded_above)
    fix i :: 'n
    assume i: i ∈ Basis
    have norm (∑ x∈P. |inner (f x) i|) ≤
       norm (inner (∑ x∈P ∩ − {x. inner (f x) i < 0}. f x) i) + norm (inner
(∑ x∈P ∩ {x. inner (f x) i < 0}. f x) i)
      by (simp add: abs_real_def sum.If_cases[OF fP] sum_negf norm_triangle_ineq4
inner_sum_left
        del: real_norm_def)
    also have . . . ≤ e + e
      unfolding real_norm_def
```

```
    by (intro add_mono norm_bound_Basis_le i fPs) auto
  finally show (∑x∈P. |inner (f x) i|) ≤ 2*e by simp
 qed
 also have ... = 2 * real DIM('n) * e by simp
 finally show ?thesis .
qed
```

1.4.3 Subclass relationships

```
instance euclidean_space ⊆ perfect_space
proof
 fix x :: 'a show ¬ open {x}
 proof
  assume open {x}
  then obtain e where 0 < e and e: ∀y. dist y x < e ⟶ y = x
    unfolding open_dist by fast
  define y where y = x + scaleR (e/2) (SOME b. b ∈ Basis)
  have [simp]: (SOME b. b ∈ Basis) ∈ Basis
    by (rule someI_ex) (auto simp: ex_in_conv)
  from ‹0 < e› have y ≠ x
    unfolding y_def by (auto intro!: nonzero_Basis)
  from ‹0 < e› have dist y x < e
    unfolding y_def by (simp add: dist_norm)
  from ‹y ≠ x› and ‹dist y x < e› show False
    using e by simp
 qed
qed
```

1.4.4 Class instances

Type *real*

```
instantiation real :: euclidean_space
begin

definition
 [simp]: Basis = {1::real}

instance
 by standard auto

end

lemma DIM_real[simp]: DIM(real) = 1
 by simp
```

Type *complex*

```
instantiation complex :: euclidean_space
begin
```

definition *Basis_complex_def*: *Basis* = {*1*, i}

instance
by *standard* (*auto simp add*: *Basis_complex_def intro*: *complex_eqI split*: *if_split_asm*)

end

lemma *DIM_complex*[*simp*]: *DIM*(*complex*) = *2*
unfolding *Basis_complex_def* **by** *simp*

lemma *complex_Basis_1* [*iff*]: (*1*::*complex*) ∈ *Basis*
by (*simp add*: *Basis_complex_def*)

lemma *complex_Basis_i* [*iff*]: i ∈ *Basis*
by (*simp add*: *Basis_complex_def*)

Type ′*a* × ′*b*

instantiation *prod* :: (*real_inner*, *real_inner*) *real_inner*
begin

definition *inner_prod_def*:
inner x y = *inner* (*fst x*) (*fst y*) + *inner* (*snd x*) (*snd y*)

lemma *inner_Pair* [*simp*]: *inner* (*a*, *b*) (*c*, *d*) = *inner a c* + *inner b d*
unfolding *inner_prod_def* **by** *simp*

instance
proof
fix *r* :: *real*
fix *x y z* :: ′*a*::*real_inner* × ′*b*::*real_inner*
show *inner x y* = *inner y x*
unfolding *inner_prod_def*
by (*simp add*: *inner_commute*)
show *inner* (*x* + *y*) *z* = *inner x z* + *inner y z*
unfolding *inner_prod_def*
by (*simp add*: *inner_add_left*)
show *inner* (*scaleR r x*) *y* = *r* ∗ *inner x y*
unfolding *inner_prod_def*
by (*simp add*: *distrib_left*)
show *0* ≤ *inner x x*
unfolding *inner_prod_def*
by (*intro add_nonneg_nonneg inner_ge_zero*)
show *inner x x* = *0* ⟷ *x* = *0*
unfolding *inner_prod_def prod_eq_iff*
by (*simp add*: *add_nonneg_eq_0_iff*)
show *norm x* = *sqrt* (*inner x x*)
unfolding *norm_prod_def inner_prod_def*

```
    by (simp add: power2_norm_eq_inner)
qed

end

lemma inner_Pair_0: inner x (0, b) = inner (snd x) b inner x (a, 0) = inner (fst
x) a
    by (cases x, simp)+

instantiation prod :: (euclidean_space, euclidean_space) euclidean_space
begin

definition
  Basis = (λu. (u, 0)) ‘ Basis ∪ (λv. (0, v)) ‘ Basis

lemma sum_Basis_prod_eq:
  fixes f::('a*'b)⇒('a*'b)
  shows sum f Basis = sum (λi. f (i, 0)) Basis + sum (λi. f (0, i)) Basis
proof −
  have inj_on (λu. (u::'a, 0::'b)) Basis inj_on (λu. (0::'a, u::'b)) Basis
    by (auto intro!: inj_onI Pair_inject)
  thus ?thesis
    unfolding Basis_prod_def
    by (subst sum.union_disjoint) (auto simp: Basis_prod_def sum.reindex)
qed

instance proof
  show (Basis :: ('a × 'b) set) ≠ {}
    unfolding Basis_prod_def by simp
next
  show finite (Basis :: ('a × 'b) set)
    unfolding Basis_prod_def by simp
next
  fix u v :: 'a × 'b
  assume u ∈ Basis and v ∈ Basis
  thus inner u v = (if u = v then 1 else 0)
    unfolding Basis_prod_def inner_prod_def
    by (auto simp add: inner_Basis split: if_split_asm)
next
  fix x :: 'a × 'b
  show (∀ u∈Basis. inner x u = 0) ⟷ x = 0
    unfolding Basis_prod_def ball_Un ball_simps
    by (simp add: inner_prod_def prod_eq_iff euclidean_all_zero_iff)
qed

lemma DIM_prod[simp]: DIM('a × 'b) = DIM('a) + DIM('b)
  unfolding Basis_prod_def
  by (subst card_Un_disjoint) (auto intro!: card_image arg_cong2[where f=(+)]
inj_onI)
```

end

1.4.5 Locale instances

lemma *finite_dimensional_vector_space_euclidean*:
  finite_dimensional_vector_space $(*_R)$ *Basis*
proof *unfold_locales*
  show *finite* (*Basis*::′*a set*) **by** (*metis finite_Basis*)
  show *real_vector*.*independent* (*Basis*::′*a set*)
    unfolding *dependent_def dependent_raw_def*[*symmetric*]
    apply (*subst span_finite*)
    apply *simp*
    apply *clarify*
    apply (*drule_tac f*=*inner a* **in** *arg_cong*)
    apply (*simp add*: *inner_Basis inner_sum_right eq_commute*)
    done
  show *module*.*span* $(*_R)$ *Basis* = *UNIV*
    unfolding *span_finite* [*OF finite_Basis*] *span_raw_def*[*symmetric*]
    by (*auto intro*!: *euclidean_representation*[*symmetric*])
qed

interpretation *eucl?*: *finite_dimensional_vector_space scaleR* :: *real* => ′*a* =>
′*a*::*euclidean_space Basis*
  rewrites *module*.*dependent* $(*_R)$ = *dependent*
    and *module*.*representation* $(*_R)$ = *representation*
    and *module*.*subspace* $(*_R)$ = *subspace*
    and *module*.*span* $(*_R)$ = *span*
    and *vector_space*.*extend_basis* $(*_R)$ = *extend_basis*
    and *vector_space*.*dim* $(*_R)$ = *dim*
    and *Vector_Spaces*.*linear* $(*_R)$ $(*_R)$ = *linear*
    and *Vector_Spaces*.*linear* $(*)$ $(*_R)$ = *linear*
    and *finite_dimensional_vector_space*.*dimension Basis* = *DIM*(′*a*)
    and *dimension* = *DIM*(′*a*)
  by (*auto simp add*: *dependent_raw_def representation_raw_def*
      subspace_raw_def span_raw_def extend_basis_raw_def dim_raw_def linear_def
      real_scaleR_def[*abs_def*]
      finite_dimensional_vector_space.*dimension_def*
      intro!: *finite_dimensional_vector_space*.*dimension_def*
      finite_dimensional_vector_space_euclidean)

interpretation *eucl?*: *finite_dimensional_vector_space_pair_1*
  scaleR::*real*⇒′*a*::*euclidean_space*⇒′*a Basis*
  scaleR::*real*⇒′*b*::*real_vector* ⇒ ′*b*
  by *unfold_locales*

interpretation *eucl?*: *finite_dimensional_vector_space_prod scaleR scaleR Basis Basis*
  rewrites *Basis_pair* = *Basis*
    and *module_prod*.*scale* $(*_R)$ $(*_R)$ = (*scaleR*::_⇒_⇒(′*a* × ′*b*))

```
proof –
  show finite_dimensional_vector_space_prod (*R) (*R) Basis Basis
    by unfold_locales
  interpret finite_dimensional_vector_space_prod (*R) (*R) Basis::'a set Basis::'b
set
    by fact
  show Basis_pair = Basis
    unfolding Basis_pair_def Basis_prod_def by auto
  show module_prod.scale (*R) (*R) = scaleR
    by (fact module_prod_scale_eq_scaleR)
qed

end
```

1.5 Elementary Linear Algebra on Euclidean Spaces

```
theory Linear_Algebra
imports
  Euclidean_Space
  HOL−Library.Infinite_Set
begin

lemma linear_simps:
  assumes bounded_linear f
  shows
    f (a + b) = f a + f b
    f (a − b) = f a − f b
    f 0 = 0
    f (− a) = − f a
    f (s *R v) = s *R (f v)
proof –
  interpret f: bounded_linear f by fact
  show f (a + b) = f a + f b by (rule f.add)
  show f (a − b) = f a − f b by (rule f.diff)
  show f 0 = 0 by (rule f.zero)
  show f (− a) = − f a by (rule f.neg)
  show f (s *R v) = s *R (f v) by (rule f.scale)
qed

lemma finite_Atleast_Atmost_nat[simp]: finite {f x |x. x ∈ (UNIV::'a::finite set)}
  using finite finite_image_set by blast

lemma substdbasis_expansion_unique:
  includes inner_syntax
  assumes d: d ⊆ Basis
  shows (∑i∈d. f i *R i) = (x::'a::euclidean_space) ⟷
    (∀i∈Basis. (i ∈ d ⟶ f i = x · i) ∧ (i ∉ d ⟶ x · i = 0))
proof –
  have *: ⋀x a b P. x * (if P then a else b) = (if P then x * a else x * b)
```

```
    by auto
  have **: finite d
    by (auto intro: finite_subset[OF assms])
  have ***: ⋀i. i ∈ Basis ⟹ (∑i∈d. f i *R i) · i = (∑x∈d. if x = i then f x
else 0)
    using d
    by (auto intro!: sum.cong simp: inner_Basis inner_sum_left)
  show ?thesis
     unfolding euclidean_eq_iff[where 'a='a] by (auto simp: sum.delta[OF **]
***)
qed

lemma independent_substdbasis: d ⊆ Basis ⟹ independent d
  by (rule independent_mono[OF independent_Basis])

lemma subset_translation_eq [simp]:
    fixes a :: 'a::real_vector shows (+) a ` s ⊆ (+) a ` t ⟷ s ⊆ t
  by auto

lemma translate_inj_on:
  fixes A :: 'a::ab_group_add set
  shows inj_on (λx. a + x) A
  unfolding inj_on_def by auto

lemma translation_assoc:
  fixes a b :: 'a::ab_group_add
  shows (λx. b + x) ` ((λx. a + x) ` S) = (λx. (a + b) + x) ` S
  by auto

lemma translation_invert:
  fixes a :: 'a::ab_group_add
  assumes (λx. a + x) ` A = (λx. a + x) ` B
  shows A = B
proof −
  have (λx. −a + x) ` ((λx. a + x) ` A) = (λx. − a + x) ` ((λx. a + x) ` B)
    using assms by auto
  then show ?thesis
    using translation_assoc[of −a a A] translation_assoc[of −a a B] by auto
qed

lemma translation_galois:
  fixes a :: 'a::ab_group_add
  shows T = ((λx. a + x) ` S) ⟷ S = ((λx. (− a) + x) ` T)
  using translation_assoc[of −a a S]
  apply auto
  using translation_assoc[of a −a T]
  apply auto
  done
```

```
lemma translation_inverse_subset:
  assumes ((λx. − a + x) ` V) ≤ (S :: 'n::ab_group_add set)
  shows V ≤ ((λx. a + x) ` S)
proof −
  {
    fix x
    assume x ∈ V
    then have x−a ∈ S using assms by auto
    then have x ∈ {a + v |v. v ∈ S}
      apply auto
      apply (rule exI[of _ x−a], simp)
      done
    then have x ∈ ((λx. a+x) ` S) by auto
  }
  then show ?thesis by auto
qed
```

1.5.1 More interesting properties of the norm

```
unbundle inner_syntax
```

Equality of vectors in terms of (·) products.

```
lemma linear_componentwise:
  fixes f:: 'a::euclidean_space ⇒ 'b::real_inner
  assumes lf: linear f
  shows (f x) · j = (∑ i∈Basis. (x·i) ∗ (f i·j)) (is ?lhs = ?rhs)
proof −
  interpret linear f by fact
  have ?rhs = (∑ i∈Basis. (x·i) ∗R (f i))·j
    by (simp add: inner_sum_left)
  then show ?thesis
    by (simp add: euclidean_representation sum[symmetric] scale[symmetric])
qed
```

```
lemma vector_eq: x = y ⟷ x · x = x · y ∧ y · y = x · x
  (is ?lhs ⟷ ?rhs)
proof
  assume ?lhs
  then show ?rhs by simp
next
  assume ?rhs
  then have x · x − x · y = 0 ∧ x · y − y · y = 0
    by simp
  then have x · (x − y) = 0 ∧ y · (x − y) = 0
    by (simp add: inner_diff inner_commute)
  then have (x − y) · (x − y) = 0
    by (simp add: field_simps inner_diff inner_commute)
  then show x = y by simp
qed
```

lemma *norm_triangle_half_r*:
norm $(y - x1) < e \, / \, 2 \Longrightarrow$ *norm* $(y - x2) < e \, / \, 2 \Longrightarrow$ *norm* $(x1 - x2) < e$
using *dist_triangle_half_r* **unfolding** *dist_norm*[*symmetric*] **by** *auto*

lemma *norm_triangle_half_l*:
assumes *norm* $(x - y) < e \, / \, 2$
and *norm* $(x' - y) < e \, / \, 2$
shows *norm* $(x - x') < e$
using *dist_triangle_half_l*[*OF assms*[*unfolded dist_norm*[*symmetric*]]]
unfolding *dist_norm*[*symmetric*] **.**

lemma *abs_triangle_half_r*:
fixes *y* :: *'a*::*linordered_field*
shows *abs* $(y - x1) < e \, / \, 2 \Longrightarrow$ *abs* $(y - x2) < e \, / \, 2 \Longrightarrow$ *abs* $(x1 - x2) < e$
by *linarith*

lemma *abs_triangle_half_l*:
fixes *y* :: *'a*::*linordered_field*
assumes *abs* $(x - y) < e \, / \, 2$
and *abs* $(x' - y) < e \, / \, 2$
shows *abs* $(x - x') < e$
using *assms* **by** *linarith*

lemma *sum_clauses*:
shows *sum f* $\{\} = 0$
and *finite S* $\Longrightarrow$ *sum f* (*insert x S*) $=$ (*if* $x \in S$ *then sum f S else f x* $+$ *sum f*
S)
by (*auto simp add*: *insert_absorb*)

lemma *vector_eq_ldot*: $(\forall x.\ x \cdot y = x \cdot z) \longleftrightarrow y = z$
proof
assume $\forall x.\ x \cdot y = x \cdot z$
then have $\forall x.\ x \cdot (y - z) = 0$
by (*simp add*: *inner_diff*)
then have $(y - z) \cdot (y - z) = 0$ **..**
then show $y = z$ **by** *simp*
qed *simp*

lemma *vector_eq_rdot*: $(\forall z.\ x \cdot z = y \cdot z) \longleftrightarrow x = y$
proof
assume $\forall z.\ x \cdot z = y \cdot z$
then have $\forall z.\ (x - y) \cdot z = 0$
by (*simp add*: *inner_diff*)
then have $(x - y) \cdot (x - y) = 0$ **..**
then show $x = y$ **by** *simp*
qed *simp*

1.5.2 Substandard Basis

```
lemma ex_card:
  assumes n ≤ card A
  shows ∃S⊆A. card S = n
proof (cases finite A)
  case True
  from ex_bij_betw_nat_finite[OF this] obtain f where f: bij_betw f {0..<card A}
A ..
  moreover from f ‹n ≤ card A› have {..< n} ⊆ {..< card A} inj_on f {..< n}
    by (auto simp: bij_betw_def intro: subset_inj_on)
  ultimately have f ‘ {..< n} ⊆ A card (f ‘ {..< n}) = n
    by (auto simp: bij_betw_def card_image)
  then show ?thesis by blast
next
  case False
  with ‹n ≤ card A› show ?thesis by force
qed

lemma subspace_substandard: subspace {x::'a::euclidean_space. (∀i∈Basis. P i ⟶
x·i = 0)}
  by (auto simp: subspace_def inner_add_left)

lemma dim_substandard:
  assumes d: d ⊆ Basis
  shows dim {x::'a::euclidean_space. ∀i∈Basis. i ∉ d ⟶ x·i = 0} = card d (is
dim ?A = _)
proof (rule dim_unique)
  from d show d ⊆ ?A
    by (auto simp: inner_Basis)
  from d show independent d
    by (rule independent_mono [OF independent_Basis])
  have x ∈ span d if ∀i∈Basis. i ∉ d ⟶ x · i = 0 for x
  proof −
    have finite d
      by (rule finite_subset [OF d finite_Basis])
    then have (∑i∈d. (x · i) *R i) ∈ span d
      by (simp add: span_sum span_clauses)
    also have (∑i∈d. (x · i) *R i) = (∑i∈Basis. (x · i) *R i)
      by (rule sum.mono_neutral_cong_left [OF finite_Basis d]) (auto simp: that)
    finally show x ∈ span d
      by (simp only: euclidean_representation)
  qed
  then show ?A ⊆ span d by auto
qed simp
```

1.5.3 Orthogonality

```
definition (in real_inner) orthogonal x y ⟷ x · y = 0
```

context *real_inner*
begin

lemma *orthogonal_self*: *orthogonal x x* $\longleftrightarrow$ *x* = *0*
  by (*simp add*: *orthogonal_def*)

lemma *orthogonal_clauses*:
  orthogonal a 0
  orthogonal a x $\Longrightarrow$ *orthogonal a* (*c* $*_R$ *x*)
  orthogonal a x $\Longrightarrow$ *orthogonal a* (− *x*)
  orthogonal a x $\Longrightarrow$ *orthogonal a y* $\Longrightarrow$ *orthogonal a* (*x* + *y*)
  orthogonal a x $\Longrightarrow$ *orthogonal a y* $\Longrightarrow$ *orthogonal a* (*x* − *y*)
  orthogonal 0 a
  orthogonal x a $\Longrightarrow$ *orthogonal* (*c* $*_R$ *x*) *a*
  orthogonal x a $\Longrightarrow$ *orthogonal* (− *x*) *a*
  orthogonal x a $\Longrightarrow$ *orthogonal y a* $\Longrightarrow$ *orthogonal* (*x* + *y*) *a*
  orthogonal x a $\Longrightarrow$ *orthogonal y a* $\Longrightarrow$ *orthogonal* (*x* − *y*) *a*
  unfolding *orthogonal_def inner_add inner_diff* **by** *auto*

end

lemma *orthogonal_commute*: *orthogonal x y* $\longleftrightarrow$ *orthogonal y x*
  by (*simp add*: *orthogonal_def inner_commute*)

lemma *orthogonal_scaleR* [*simp*]: *c* $\neq$ *0* $\Longrightarrow$ *orthogonal* (*c* $*_R$ *x*) = *orthogonal x*
  by (*rule ext*) (*simp add*: *orthogonal_def*)

lemma *pairwise_ortho_scaleR*:
    pairwise (λ*i j*. *orthogonal* (*f i*) (*g j*)) *B*
    $\Longrightarrow$ *pairwise* (λ*i j*. *orthogonal* (*a i* $*_R$ *f i*) (*a j* $*_R$ *g j*)) *B*
  by (*auto simp*: *pairwise_def orthogonal_clauses*)

lemma *orthogonal_rvsum*:
    [[*finite s*; $\bigwedge$*y*. *y* $\in$ *s* $\Longrightarrow$ *orthogonal x* (*f y*)]] $\Longrightarrow$ *orthogonal x* (*sum f s*)
  by (*induction s rule*: *finite_induct*) (*auto simp*: *orthogonal_clauses*)

lemma *orthogonal_lvsum*:
    [[*finite s*; $\bigwedge$*x*. *x* $\in$ *s* $\Longrightarrow$ *orthogonal* (*f x*) *y*]] $\Longrightarrow$ *orthogonal* (*sum f s*) *y*
  by (*induction s rule*: *finite_induct*) (*auto simp*: *orthogonal_clauses*)

lemma *norm_add_Pythagorean*:
  assumes *orthogonal a b*
    shows *norm*(*a* + *b*) ˆ *2* = *norm a* ˆ *2* + *norm b* ˆ *2*
proof −
  from *assms* **have** (*a* − (*0* − *b*)) · (*a* − (*0* − *b*)) = *a* · *a* − (*0* − *b* · *b*)
    by (*simp add*: *algebra_simps orthogonal_def inner_commute*)
  then show *?thesis*
    by (*simp add*: *power2_norm_eq_inner*)
qed

lemma *norm_sum_Pythagorean*:
  assumes *finite I pairwise* ($\lambda i\ j.$ *orthogonal* (*f i*) (*f j*)) *I*
    shows (*norm* (*sum f I*))^2 = ($\sum i{\in}I.$ (*norm* (*f i*))^2)
using *assms*
proof (*induction I rule*: *finite_induct*)
  case *empty* **then show** *?case* **by** *simp*
next
  case (*insert x I*)
  then have *orthogonal* (*f x*) (*sum f I*)
    by (*metis pairwise_insert orthogonal_rvsum*)
  with *insert* **show** *?case*
    by (*simp add*: *pairwise_insert norm_add_Pythagorean*)
qed

1.5.4 Orthogonality of a transformation

definition *orthogonal_transformation f* $\longleftrightarrow$ *linear f* $\wedge$ ($\forall$ *v w*. *f v* $\cdot$ *f w* = *v* $\cdot$ *w*)

lemma *orthogonal_transformation*:
  orthogonal_transformation f $\longleftrightarrow$ *linear f* $\wedge$ ($\forall$ *v*. *norm* (*f v*) = *norm v*)
  unfolding *orthogonal_transformation_def*
  apply *auto*
  apply (*erule_tac x=v* **in** *allE*)+
  apply (*simp add*: *norm_eq_sqrt_inner*)
  apply (*simp add*: *dot_norm linear_add*[*symmetric*])
  done

lemma *orthogonal_transformation_id* [*simp*]: *orthogonal_transformation* ($\lambda x.$ *x*)
  by (*simp add*: *linear_iff orthogonal_transformation_def*)

lemma *orthogonal_orthogonal_transformation*:
    orthogonal_transformation f $\Longrightarrow$ *orthogonal* (*f x*) (*f y*) $\longleftrightarrow$ *orthogonal x y*
  by (*simp add*: *orthogonal_def orthogonal_transformation_def*)

lemma *orthogonal_transformation_compose*:
  $[\![$*orthogonal_transformation f*; *orthogonal_transformation g*$]\!]$ $\Longrightarrow$ *orthogonal_transformation*(*f*
$\circ$ *g*)
  by (*auto simp*: *orthogonal_transformation_def linear_compose*)

lemma *orthogonal_transformation_neg*:
  orthogonal_transformation($\lambda x.$ −(*f x*)) $\longleftrightarrow$ *orthogonal_transformation f*
  by (*auto simp*: *orthogonal_transformation_def dest*: *linear_compose_neg*)

lemma *orthogonal_transformation_scaleR*: *orthogonal_transformation f* $\Longrightarrow$ *f* (*c*
$*_R$ *v*) = *c* $*_R$ *f v*
  by (*simp add*: *linear_iff orthogonal_transformation_def*)

lemma *orthogonal_transformation_linear*:

orthogonal_transformation f $\Longrightarrow$ *linear f*
by (*simp add*: *orthogonal_transformation_def*)

lemma *orthogonal_transformation_inj*:
orthogonal_transformation f $\Longrightarrow$ *inj f*
unfolding *orthogonal_transformation_def inj_on_def*
by (*metis vector_eq*)

lemma *orthogonal_transformation_surj*:
orthogonal_transformation f $\Longrightarrow$ *surj f*
for *f* :: *'a::euclidean_space* ⇒ *'a::euclidean_space*
by (*simp add*: *linear_injective_imp_surjective orthogonal_transformation_inj orthogonal_transformation_linear*)

lemma *orthogonal_transformation_bij*:
orthogonal_transformation f $\Longrightarrow$ *bij f*
for *f* :: *'a::euclidean_space* ⇒ *'a::euclidean_space*
by (*simp add*: *bij_def orthogonal_transformation_inj orthogonal_transformation_surj*)

lemma *orthogonal_transformation_inv*:
orthogonal_transformation f $\Longrightarrow$ *orthogonal_transformation* (*inv f*)
for *f* :: *'a::euclidean_space* ⇒ *'a::euclidean_space*
by (*metis* (*no_types*, *hide_lams*) *bijection.inv_right bijection_def inj_linear_imp_inv_linear orthogonal_transformation orthogonal_transformation_bij orthogonal_transformation_inj*)

lemma *orthogonal_transformation_norm*:
orthogonal_transformation f $\Longrightarrow$ *norm* (*f x*) = *norm x*
by (*metis orthogonal_transformation*)

1.5.5 Bilinear functions

definition
bilinear :: (*'a::real_vector* ⇒ *'b::real_vector* ⇒ *'c::real_vector*) ⇒ *bool* **where**
bilinear f ⟷ (∀ *x*. *linear* (λ*y*. *f x y*)) ∧ (∀ *y*. *linear* (λ*x*. *f x y*))

lemma *bilinear_ladd*: *bilinear h* $\Longrightarrow$ *h* (*x* + *y*) *z* = *h x z* + *h y z*
by (*simp add*: *bilinear_def linear_iff*)

lemma *bilinear_radd*: *bilinear h* $\Longrightarrow$ *h x* (*y* + *z*) = *h x y* + *h x z*
by (*simp add*: *bilinear_def linear_iff*)

lemma *bilinear_times*:
fixes *c::'a::real_algebra* **shows** *bilinear* (λ*x y::'a*. *x*∗*y*)
by (*auto simp*: *bilinear_def distrib_left distrib_right intro*!: *linearI*)

lemma *bilinear_lmul*: *bilinear h* $\Longrightarrow$ *h* (*c* $*_R$ *x*) *y* = *c* $*_R$ *h x y*
by (*simp add*: *bilinear_def linear_iff*)

lemma *bilinear_rmul*: *bilinear h* $\Longrightarrow$ *h x* (*c* $*_R$ *y*) = *c* $*_R$ *h x y*

by (*simp add*: *bilinear_def linear_iff*)

lemma *bilinear_lneg*: *bilinear h* ⟹ *h* (− *x*) *y* = − *h x y*
by (*drule bilinear_lmul* [*of* _ − *1*]) *simp*

lemma *bilinear_rneg*: *bilinear h* ⟹ *h x* (− *y*) = − *h x y*
by (*drule bilinear_rmul* [*of* _ _ − *1*]) *simp*

lemma (**in** *ab_group_add*) *eq_add_iff*: *x* = *x* + *y* ⟷ *y* = *0*
using *add_left_imp_eq*[*of x y 0*] **by** *auto*

lemma *bilinear_lzero*:
assumes *bilinear h*
shows *h 0 x* = *0*
using *bilinear_ladd* [*OF assms*, *of 0 0 x*] **by** (*simp add*: *eq_add_iff field_simps*)

lemma *bilinear_rzero*:
assumes *bilinear h*
shows *h x 0* = *0*
using *bilinear_radd* [*OF assms*, *of x 0 0*] **by** (*simp add*: *eq_add_iff field_simps*)

lemma *bilinear_lsub*: *bilinear h* ⟹ *h* (*x* − *y*) *z* = *h x z* − *h y z*
using *bilinear_ladd* [*of h x* − *y*] **by** (*simp add*: *bilinear_lneg*)

lemma *bilinear_rsub*: *bilinear h* ⟹ *h z* (*x* − *y*) = *h z x* − *h z y*
using *bilinear_radd* [*of h* _ *x* − *y*] **by** (*simp add*: *bilinear_rneg*)

lemma *bilinear_sum*:
assumes *bilinear h*
shows *h* (*sum f S*) (*sum g T*) = *sum* (λ(*i*,*j*). *h* (*f i*) (*g j*)) (*S* × *T*)
proof −
interpret *l*: *linear* λ*x*. *h x y* **for** *y* **using** *assms* **by** (*simp add*: *bilinear_def*)
interpret *r*: *linear* λ*y*. *h x y* **for** *x* **using** *assms* **by** (*simp add*: *bilinear_def*)
have *h* (*sum f S*) (*sum g T*) = *sum* (λ*x*. *h* (*f x*) (*sum g T*)) *S*
by (*simp add*: *l*.*sum*)
also have ... = *sum* (λ*x*. *sum* (λ*y*. *h* (*f x*) (*g y*)) *T*) *S*
by (*rule sum*.*cong*) (*simp_all add*: *r*.*sum*)
finally show *?thesis*
unfolding *sum*.*cartesian_product* **.**
qed

1.5.6 Adjoints

definition *adjoint* :: ((*'a*::*real_inner*) ⇒ (*'b*::*real_inner*)) ⇒ *'b* ⇒ *'a* **where**
adjoint f = (*SOME f'*. ∀ *x y*. *f x* • *y* = *x* • *f' y*)

lemma *adjoint_unique*:
assumes ∀ *x y*. *inner* (*f x*) *y* = *inner x* (*g y*)
shows *adjoint f* = *g*

```
  unfolding adjoint_def
proof (rule some_equality)
  show ∀ x y. inner (f x) y = inner x (g y)
    by (rule assms)
next
  fix h
  assume ∀ x y. inner (f x) y = inner x (h y)
  then have ∀ x y. inner x (g y) = inner x (h y)
    using assms by simp
  then have ∀ x y. inner x (g y − h y) = 0
    by (simp add: inner_diff_right)
  then have ∀ y. inner (g y − h y) (g y − h y) = 0
    by simp
  then have ∀ y. h y = g y
    by simp
  then show h = g by (simp add: ext)
qed
```

TODO: The following lemmas about adjoints should hold for any Hilbert space (i.e. complete inner product space). (see https://en.wikipedia.org/wiki/Hermitian_adjoint)

```
lemma adjoint_works:
  fixes f :: 'n::euclidean_space ⇒ 'm::euclidean_space
  assumes lf: linear f
  shows x · adjoint f y = f x · y
proof −
  interpret linear f by fact
  have ∀ y. ∃ w. ∀ x. f x · y = x · w
  proof (intro allI exI)
    fix y :: 'm and x
    let ?w = (∑ i∈Basis. (f i · y) *R i) :: 'n
    have f x · y = f (∑ i∈Basis. (x · i) *R i) · y
      by (simp add: euclidean_representation)
    also have ... = (∑ i∈Basis. (x · i) *R f i) · y
      by (simp add: sum scale)
    finally show f x · y = x · ?w
      by (simp add: inner_sum_left inner_sum_right mult.commute)
  qed
  then show ?thesis
    unfolding adjoint_def choice_iff
    by (intro someI2_ex[where Q=λf'. x · f' y = f x · y]) auto
qed

lemma adjoint_clauses:
  fixes f :: 'n::euclidean_space ⇒ 'm::euclidean_space
  assumes lf: linear f
  shows x · adjoint f y = f x · y
    and adjoint f y · x = y · f x
  by (simp_all add: adjoint_works[OF lf] inner_commute)
```

lemma *adjoint_linear*:
  fixes *f* :: *'n::euclidean_space* ⇒ *'m::euclidean_space*
  assumes *lf*: *linear f*
  shows *linear* (*adjoint f*)
  by (*simp add*: *lf linear_iff euclidean_eq_iff*[**where** *'a='n*] *euclidean_eq_iff*[**where** *'a='m*]
    adjoint_clauses[*OF lf*] *inner_distrib*)

lemma *adjoint_adjoint*:
  fixes *f* :: *'n::euclidean_space* ⇒ *'m::euclidean_space*
  assumes *lf*: *linear f*
  shows *adjoint* (*adjoint f*) = *f*
  by (*rule adjoint_unique*, *simp add*: *adjoint_clauses* [*OF lf*])

1.5.7 Euclidean Spaces as Typeclass

lemma *independent_Basis*: *independent Basis*
  by (*rule independent_Basis*)

lemma *span_Basis* [*simp*]: *span Basis* = *UNIV*
  by (*rule span_Basis*)

lemma *in_span_Basis*: *x* ∈ *span Basis*
  unfolding *span_Basis* **..**

1.5.8 Linearity and Bilinearity continued

lemma *linear_bounded*:
  fixes *f* :: *'a::euclidean_space* ⇒ *'b::real_normed_vector*
  assumes *lf*: *linear f*
  shows ∃ *B*. ∀ *x*. *norm* (*f x*) ≤ *B* ∗ *norm x*
proof
  interpret *linear f* **by** *fact*
  let *?B* = ∑ *b*∈*Basis*. *norm* (*f b*)
  show ∀ *x*. *norm* (*f x*) ≤ *?B* ∗ *norm x*
  proof
    fix *x* :: *'a*
    let *?g* = λ*b*. (*x* · *b*) $*_R$ *f b*
    have *norm* (*f x*) = *norm* (*f* (∑ *b*∈*Basis*. (*x* · *b*) $*_R$ *b*))
      unfolding *euclidean_representation* **..**
    also have ... = *norm* (*sum ?g Basis*)
      by (*simp add*: *sum scale*)
    finally have *th0*: *norm* (*f x*) = *norm* (*sum ?g Basis*) **.**
    have *th*: *norm* (*?g i*) ≤ *norm* (*f i*) ∗ *norm x* **if** *i* ∈ *Basis* **for** *i*
    proof −
      from *Basis_le_norm*[*OF that*, *of x*]
      show *norm* (*?g i*) ≤ *norm* (*f i*) ∗ *norm x*
      unfolding *norm_scaleR* **by** (*metis mult.commute mult_left_mono norm_ge_zero*)
    qed

```
    from sum_norm_le[of _ ?g, OF th]
    show norm (f x) ≤ ?B * norm x
      unfolding th0 sum_distrib_right by metis
  qed
qed

lemma linear_conv_bounded_linear:
  fixes f :: 'a::euclidean_space ⇒ 'b::real_normed_vector
  shows linear f ⟷ bounded_linear f
proof
  assume linear f
  then interpret f: linear f .
  show bounded_linear f
  proof
    have ∃B. ∀x. norm (f x) ≤ B * norm x
      using ‹linear f› by (rule linear_bounded)
    then show ∃K. ∀x. norm (f x) ≤ norm x * K
      by (simp add: mult.commute)
  qed
next
  assume bounded_linear f
  then interpret f: bounded_linear f .
  show linear f ..
qed

lemmas linear_linear = linear_conv_bounded_linear[symmetric]

lemma inj_linear_imp_inv_bounded_linear:
  fixes f::'a::euclidean_space ⇒ 'a
  shows ⟦bounded_linear f; inj f⟧ ⟹ bounded_linear (inv f)
  by (simp add: inj_linear_imp_inv_linear linear_linear)

lemma linear_bounded_pos:
  fixes f :: 'a::euclidean_space ⇒ 'b::real_normed_vector
  assumes lf: linear f
 obtains B where B > 0 ⋀x. norm (f x) ≤ B * norm x
proof -
  have ∃B > 0. ∀x. norm (f x) ≤ norm x * B
    using lf unfolding linear_conv_bounded_linear
    by (rule bounded_linear.pos_bounded)
  with that show ?thesis
    by (auto simp: mult.commute)
qed

lemma linear_invertible_bounded_below_pos:
  fixes f :: 'a::real_normed_vector ⇒ 'b::euclidean_space
  assumes linear f linear g g ∘ f = id
  obtains B where B > 0 ⋀x. B * norm x ≤ norm(f x)
proof -
```

obtain *B* **where** $B > 0$ **and** *B*: $\bigwedge x.\ norm\ (g\ x) \leq B * norm\ x$
  using *linear_bounded_pos* [*OF* ‹*linear g*›] **by** *blast*
show *thesis*
proof
  show $0 < 1/B$
    by (*simp add*: ‹$B > 0$›)
  show $1/B * norm\ x \leq norm\ (f\ x)$ **for** *x*
  proof −
    have $1/B * norm\ x = 1/B * norm\ (g\ (f\ x))$
      using *assms* **by** (*simp add*: *pointfree_idE*)
    also have ... $\leq norm\ (f\ x)$
      using *B* [*of f x*] **by** (*simp add*: ‹$B > 0$› *mult.commute pos_divide_le_eq*)
    finally show *?thesis* **.**
  qed
qed
qed

lemma *linear_inj_bounded_below_pos*:
  fixes *f* :: *'a::real_normed_vector* ⇒ *'b::euclidean_space*
  assumes *linear f inj f*
  obtains *B* **where** $B > 0$ $\bigwedge x.\ B * norm\ x \leq norm(f\ x)$
  using *linear_injective_left_inverse* [*OF assms*]
    linear_invertible_bounded_below_pos assms **by** *blast*

lemma *bounded_linearI'*:
  fixes *f* ::*'a::euclidean_space* ⇒ *'b::real_normed_vector*
  assumes $\bigwedge x\ y.\ f\ (x + y) = f\ x + f\ y$
    and $\bigwedge c\ x.\ f\ (c *_R x) = c *_R f\ x$
  shows *bounded_linear f*
  using *assms linearI linear_conv_bounded_linear* **by** *blast*

lemma *bilinear_bounded*:
  fixes *h* :: *'m::euclidean_space* ⇒ *'n::euclidean_space* ⇒ *'k::real_normed_vector*
  assumes *bh*: *bilinear h*
  shows $\exists B.\ \forall x\ y.\ norm\ (h\ x\ y) \leq B * norm\ x * norm\ y$
proof (*clarify intro*!: *exI*[*of* _ $\sum i{\in}Basis.\ \sum j{\in}Basis.\ norm\ (h\ i\ j)$])
  fix *x* :: *'m*
  fix *y* :: *'n*
  have $norm\ (h\ x\ y) = norm\ (h\ (sum\ (\lambda i.\ (x \cdot i) *_R i)\ Basis)\ (sum\ (\lambda i.\ (y \cdot i) *_R i)\ Basis))$
    by (*simp add*: *euclidean_representation*)
  also have ... $= norm\ (sum\ (\lambda\ (i,j).\ h\ ((x \cdot i) *_R i)\ ((y \cdot j) *_R j))\ (Basis \times Basis))$
    unfolding *bilinear_sum*[*OF bh*] **..**
  finally have *th*: $norm\ (h\ x\ y) = \ldots$ **.**
  have $\bigwedge i\ j.\ [\![i \in Basis;\ j \in Basis]\!]$
        $\Longrightarrow |x \cdot i| * (|y \cdot j| * norm\ (h\ i\ j)) \leq norm\ x * (norm\ y * norm\ (h\ i\ j))$
    by (*auto simp add*: *zero_le_mult_iff Basis_le_norm mult_mono*)
  then show $norm\ (h\ x\ y) \leq (\sum i{\in}Basis.\ \sum j{\in}Basis.\ norm\ (h\ i\ j)) * norm\ x *$

norm y
  unfolding *sum_distrib_right th sum.cartesian_product*
  by (*clarsimp simp add*: *bilinear_rmul*[*OF bh*] *bilinear_lmul*[*OF bh*]
   field_simps simp del: *scaleR_scaleR intro*!: *sum_norm_le*)
qed

lemma *bilinear_conv_bounded_bilinear*:
 fixes *h* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space* ⇒ *'c*::*real_normed_vector*
 shows *bilinear h* ⟷ *bounded_bilinear h*
proof
 assume *bilinear h*
 show *bounded_bilinear h*
 proof
  fix *x y z*
  show *h* (*x* + *y*) *z* = *h x z* + *h y z*
   using ⟨*bilinear h*⟩ **unfolding** *bilinear_def linear_iff* **by** *simp*
 next
  fix *x y z*
  show *h x* (*y* + *z*) = *h x y* + *h x z*
   using ⟨*bilinear h*⟩ **unfolding** *bilinear_def linear_iff* **by** *simp*
 next
  show *h* (*scaleR r x*) *y* = *scaleR r* (*h x y*) *h x* (*scaleR r y*) = *scaleR r* (*h x y*)
for *r x y*
   using ⟨*bilinear h*⟩ **unfolding** *bilinear_def linear_iff*
   by *simp_all*
 next
  have ∃ *B*. ∀ *x y*. *norm* (*h x y*) ≤ *B* ∗ *norm x* ∗ *norm y*
   using ⟨*bilinear h*⟩ **by** (*rule bilinear_bounded*)
  then show ∃ *K*. ∀ *x y*. *norm* (*h x y*) ≤ *norm x* ∗ *norm y* ∗ *K*
   by (*simp add*: *ac_simps*)
 qed
next
 assume *bounded_bilinear h*
 then interpret *h*: *bounded_bilinear h* **.**
 show *bilinear h*
  unfolding *bilinear_def linear_conv_bounded_linear*
  using *h.bounded_linear_left h.bounded_linear_right* **by** *simp*
qed

lemma *bilinear_bounded_pos*:
 fixes *h* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space* ⇒ *'c*::*real_normed_vector*
 assumes *bh*: *bilinear h*
 shows ∃ *B* > *0*. ∀ *x y*. *norm* (*h x y*) ≤ *B* ∗ *norm x* ∗ *norm y*
proof −
 have ∃ *B* > *0*. ∀ *x y*. *norm* (*h x y*) ≤ *norm x* ∗ *norm y* ∗ *B*
  using *bh* [*unfolded bilinear_conv_bounded_bilinear*]
  by (*rule bounded_bilinear.pos_bounded*)
 then show *?thesis*
  by (*simp only*: *ac_simps*)

qed

lemma *bounded_linear_imp_has_derivative*: *bounded_linear f* ⟹ (*f has_derivative f*) *net*
by (*auto simp add*: *has_derivative_def linear_diff linear_linear linear_def*
dest: *bounded_linear.linear*)

lemma *linear_imp_has_derivative*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::real_normed_vector*
shows *linear f* ⟹ (*f has_derivative f*) *net*
by (*simp add*: *bounded_linear_imp_has_derivative linear_conv_bounded_linear*)

lemma *bounded_linear_imp_differentiable*: *bounded_linear f* ⟹ *f differentiable net*
using *bounded_linear_imp_has_derivative differentiable_def* **by** *blast*

lemma *linear_imp_differentiable*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::real_normed_vector*
shows *linear f* ⟹ *f differentiable net*
by (*metis linear_imp_has_derivative differentiable_def*)

1.5.9 We continue

lemma *independent_bound*:
fixes *S* :: *'a::euclidean_space set*
shows *independent S* ⟹ *finite S* ∧ *card S* ≤ *DIM('a)*
by (*metis dim_subset_UNIV finiteI_independent dim_span_eq_card_independent*)

lemmas *independent_imp_finite* = *finiteI_independent*

corollary *independent_card_le*:
fixes *S* :: *'a::euclidean_space set*
assumes *independent S*
shows *card S* ≤ *DIM('a)*
using *assms independent_bound* **by** *auto*

lemma *dependent_biggerset*:
fixes *S* :: *'a::euclidean_space set*
shows (*finite S* ⟹ *card S* > *DIM('a)*) ⟹ *dependent S*
by (*metis independent_bound not_less*)

Picking an orthogonal replacement for a spanning set.

lemma *vector_sub_project_orthogonal*:
fixes *b x* :: *'a::euclidean_space*
shows $b \cdot (x - ((b \cdot x) / (b \cdot b)) *_R b) = 0$
unfolding *inner_simps* **by** *auto*

lemma *pairwise_orthogonal_insert*:
assumes *pairwise orthogonal S*
and ⋀*y*. *y* ∈ *S* ⟹ *orthogonal x y*

```
  shows pairwise orthogonal (insert x S)
  using assms unfolding pairwise_def
  by (auto simp add: orthogonal_commute)

lemma basis_orthogonal:
  fixes B :: 'a::real_inner set
  assumes fB: finite B
  shows ∃ C. finite C ∧ card C ≤ card B ∧ span C = span B ∧ pairwise orthogonal
C
  (is ∃ C. ?P B C)
  using fB
proof (induct rule: finite_induct)
  case empty
  then show ?case
    apply (rule exI[where x={}])
    apply (auto simp add: pairwise_def)
    done
next
  case (insert a B)
  note fB = ‹finite B› and aB = ‹a ∉ B›
  from ‹∃ C. finite C ∧ card C ≤ card B ∧ span C = span B ∧ pairwise orthogonal
C›
  obtain C where C: finite C card C ≤ card B
    span C = span B pairwise orthogonal C by blast
  let ?a = a − sum (λx. (x · a / (x · x)) *R x) C
  let ?C = insert ?a C
  from C(1) have fC: finite ?C
    by simp
  from fB aB C(1,2) have cC: card ?C ≤ card (insert a B)
    by (simp add: card_insert_if)
  {
    fix x k
    have th0: ⋀(a::'a) b c. a − (b − c) = c + (a − b)
      by (simp add: field_simps)
    have x − k *R (a − (∑x∈C. (x · a / (x · x)) *R x)) ∈ span C ⟷ x − k
*R a ∈ span C
      apply (simp only: scaleR_right_diff_distrib th0)
      apply (rule span_add_eq)
      apply (rule span_scale)
      apply (rule span_sum)
      apply (rule span_scale)
      apply (rule span_base)
      apply assumption
      done
  }
  then have SC: span ?C = span (insert a B)
    unfolding set_eq_iff span_breakdown_eq C(3)[symmetric] by auto
  {
    fix y
```

```
    assume yC: y ∈ C
    then have Cy: C = insert y (C − {y})
      by blast
    have fth: finite (C − {y})
      using C by simp
    have orthogonal ?a y
      unfolding orthogonal_def
      unfolding inner_diff inner_sum_left right_minus_eq
      unfolding sum.remove [OF ‹finite C› ‹y ∈ C›]
      apply (clarsimp simp add: inner_commute[of y a])
      apply (rule sum.neutral)
      apply clarsimp
      apply (rule C(4)[unfolded pairwise_def orthogonal_def, rule_format])
      using ‹y ∈ C› by auto
  }
  with ‹pairwise orthogonal C› have CPO: pairwise orthogonal ?C
    by (rule pairwise_orthogonal_insert)
  from fC cC SC CPO have ?P (insert a B) ?C
    by blast
  then show ?case by blast
qed

lemma orthogonal_basis_exists:
  fixes V :: ('a::euclidean_space) set
  shows ∃B. independent B ∧ B ⊆ span V ∧ V ⊆ span B ∧
  (card B = dim V) ∧ pairwise orthogonal B
proof −
  from basis_exists[of V] obtain B where
    B: B ⊆ V independent B V ⊆ span B card B = dim V
    by force
  from B have fB: finite B card B = dim V
    using independent_bound by auto
  from basis_orthogonal[OF fB(1)] obtain C where
    C: finite C card C ≤ card B span C = span B pairwise orthogonal C
    by blast
  from C B have CSV: C ⊆ span V
    by (metis span_superset span_mono subset_trans)
  from span_mono[OF B(3)] C have SVC: span V ⊆ span C
    by (simp add: span_span)
  from card_le_dim_spanning[OF CSV SVC C(1)] C(2,3) fB
  have iC: independent C
    by (simp)
  from C fB have card C ≤ dim V
    by simp
  moreover have dim V ≤ card C
    using span_card_ge_dim[OF CSV SVC C(1)]
    by simp
  ultimately have CdV: card C = dim V
    using C(1) by simp
```

```
  from C B CSV CdV iC show ?thesis
    by auto
qed
```

Low-dimensional subset is in a hyperplane (weak orthogonal complement).

```
lemma span_not_univ_orthogonal:
  fixes S :: 'a::euclidean_space set
  assumes sU: span S ≠ UNIV
  shows ∃ a::'a. a ≠ 0 ∧ (∀ x ∈ span S. a · x = 0)
proof −
  from sU obtain a where a: a ∉ span S
    by blast
  from orthogonal_basis_exists obtain B where
    B: independent B B ⊆ span S S ⊆ span B
    card B = dim S pairwise orthogonal B
    by blast
  from B have fB: finite B card B = dim S
    using independent_bound by auto
  from span_mono[OF B(2)] span_mono[OF B(3)]
  have sSB: span S = span B
    by (simp add: span_span)
  let ?a = a − sum (λb. (a · b / (b · b)) *R b) B
  have sum (λb. (a · b / (b · b)) *R b) B ∈ span S
    unfolding sSB
    apply (rule span_sum)
    apply (rule span_scale)
    apply (rule span_base)
    apply assumption
    done
  with a have a0:?a  ≠ 0
    by auto
  have ?a · x = 0 if x∈span B for x
  proof (rule span_induct [OF that])
    show subspace {x. ?a · x = 0}
      by (auto simp add: subspace_def inner_add)
  next
    {
      fix x
      assume x: x ∈ B
      from x have B': B = insert x (B − {x})
        by blast
      have fth: finite (B − {x})
        using fB by simp
      have ?a · x = 0
        apply (subst B')
        using fB fth
        unfolding sum_clauses(2)[OF fth]
        apply simp unfolding inner_simps
        apply (clarsimp simp add: inner_add inner_sum_left)
```

```
      apply (rule sum.neutral, rule ballI)
      apply (simp only: inner_commute)
      apply (auto simp add: x field_simps
        intro: B(5)[unfolded pairwise_def orthogonal_def, rule_format])
      done
    }
    then show ?a · x = 0 if x ∈ B for x
      using that by blast
    qed
  with a0 show ?thesis
    unfolding sSB by (auto intro: exI[where x=?a])
qed

lemma span_not_univ_subset_hyperplane:
  fixes S :: 'a::euclidean_space set
  assumes SU: span S ≠ UNIV
  shows ∃ a. a ≠0 ∧ span S ⊆ {x. a · x = 0}
  using span_not_univ_orthogonal[OF SU] by auto

lemma lowdim_subset_hyperplane:
  fixes S :: 'a::euclidean_space set
  assumes d: dim S < DIM('a)
  shows ∃ a::'a. a ≠ 0 ∧ span S ⊆ {x. a · x = 0}
proof −
  {
    assume span S = UNIV
    then have dim (span S) = dim (UNIV :: ('a) set)
      by simp
    then have dim S = DIM('a)
      by (metis Euclidean_Space.dim_UNIV dim_span)
    with d have False by arith
  }
  then have th: span S ≠ UNIV
    by blast
  from span_not_univ_subset_hyperplane[OF th] show ?thesis .
qed

lemma linear_eq_stdbasis:
  fixes f :: 'a::euclidean_space ⇒ _
  assumes lf: linear f
    and lg: linear g
    and fg: ⋀b. b ∈ Basis ⟹ f b = g b
  shows f = g
  using linear_eq_on_span[OF lf lg, of Basis] fg
  by auto
```

Similar results for bilinear functions.

```
lemma bilinear_eq:
  assumes bf: bilinear f
```

```
    and bg: bilinear g
    and SB: S ⊆ span B
    and TC: T ⊆ span C
    and x∈S y∈T
    and fg: ⋀x y. ⟦x ∈ B; y∈ C⟧ ⟹ f x y = g x y
  shows f x y = g x y
proof −
  let ?P = {x. ∀ y∈ span C. f x y = g x y}
  from bf bg have sp: subspace ?P
    unfolding bilinear_def linear_iff subspace_def bf bg
    by (auto simp add: span_zero bilinear_lzero[OF bf] bilinear_lzero[OF bg]
        span_add Ball_def
      intro: bilinear_ladd[OF bf])
  have sfg: ⋀x. x ∈ B ⟹ subspace {a. f x a = g x a}
    apply (auto simp add: subspace_def)
    using bf bg unfolding bilinear_def linear_iff
      apply (auto simp add: span_zero bilinear_rzero[OF bf] bilinear_rzero[OF bg]
        span_add Ball_def
      intro: bilinear_ladd[OF bf])
    done
  have ∀ y∈ span C. f x y = g x y if x ∈ span B for x
    apply (rule span_induct [OF that sp])
    using fg sfg span_induct by blast
  then show ?thesis
    using SB TC assms by auto
qed

lemma bilinear_eq_stdbasis:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space ⇒ _
  assumes bf: bilinear f
    and bg: bilinear g
    and fg: ⋀i j. i ∈ Basis ⟹ j ∈ Basis ⟹ f i j = g i j
  shows f = g
  using bilinear_eq[OF bf bg equalityD2[OF span_Basis] equalityD2[OF span_Basis]]
fg by blast
```

1.5.10 Infinity norm

```
definition infnorm (x::'a::euclidean_space) = Sup {|x · b| |b. b ∈ Basis}

lemma infnorm_set_image:
  fixes x :: 'a::euclidean_space
  shows {|x · i| |i. i ∈ Basis} = (λi. |x · i|) ‘ Basis
  by blast

lemma infnorm_Max:
  fixes x :: 'a::euclidean_space
  shows infnorm x = Max ((λi. |x · i|) ‘ Basis)
  by (simp add: infnorm_def infnorm_set_image cSup_eq_Max)
```

```
lemma infnorm_set_lemma:
  fixes x :: 'a::euclidean_space
  shows finite {|x · i| |i. i ∈ Basis}
    and {|x · i| |i. i ∈ Basis} ≠ {}
  unfolding infnorm_set_image
  by auto

lemma infnorm_pos_le:
  fixes x :: 'a::euclidean_space
  shows 0 ≤ infnorm x
  by (simp add: infnorm_Max Max_ge_iff ex_in_conv)

lemma infnorm_triangle:
  fixes x :: 'a::euclidean_space
  shows infnorm (x + y) ≤ infnorm x + infnorm y
proof −
  have *: ⋀a b c d :: real. |a| ≤ c ⟹ |b| ≤ d ⟹ |a + b| ≤ c + d
    by simp
  show ?thesis
    by (auto simp: infnorm_Max inner_add_left intro!: *)
qed

lemma infnorm_eq_0:
  fixes x :: 'a::euclidean_space
  shows infnorm x = 0 ⟷ x = 0
proof −
  have infnorm x ≤ 0 ⟷ x = 0
    unfolding infnorm_Max by (simp add: euclidean_all_zero_iff)
  then show ?thesis
    using infnorm_pos_le[of x] by simp
qed

lemma infnorm_0: infnorm 0 = 0
  by (simp add: infnorm_eq_0)

lemma infnorm_neg: infnorm (− x) = infnorm x
  unfolding infnorm_def by simp

lemma infnorm_sub: infnorm (x − y) = infnorm (y − x)
  by (metis infnorm_neg minus_diff_eq)

lemma absdiff_infnorm: |infnorm x − infnorm y| ≤ infnorm (x − y)
proof −
  have *: ⋀(nx::real) n ny. nx ≤ n + ny ⟹ ny ≤ n + nx ⟹ |nx − ny| ≤ n
    by arith
  show ?thesis
  proof (rule *)
    from infnorm_triangle[of x − y  y] infnorm_triangle[of x − y −x]
```

show *infnorm x* ≤ *infnorm* (*x* − *y*) + *infnorm y infnorm y* ≤ *infnorm* (*x* − *y*) + *infnorm x*
by (*simp_all add*: *field_simps infnorm_neg*)
qed
qed

lemma *real_abs_infnorm*: |*infnorm x*| = *infnorm x*
using *infnorm_pos_le*[*of x*] **by** *arith*

lemma *Basis_le_infnorm*:
fixes *x* :: ′*a*::*euclidean_space*
shows *b* ∈ *Basis* ⟹ |*x* • *b*| ≤ *infnorm x*
by (*simp add*: *infnorm_Max*)

lemma *infnorm_mul*: *infnorm* (*a* $*_R$ *x*) = |*a*| ∗ *infnorm x*
unfolding *infnorm_Max*
proof (*safe intro*!: *Max_eqI*)
let *?B* = (λ*i*. |*x* • *i*|) ‘ *Basis*
{ fix *b* :: ′*a*
assume *b* ∈ *Basis*
then show |*a* $*_R$ *x* • *b*| ≤ |*a*| ∗ *Max ?B*
by (*simp add*: *abs_mult mult_left_mono*)
next
from *Max_in*[*of ?B*] **obtain** *b* **where** *b* ∈ *Basis Max ?B* = |*x* • *b*|
by (*auto simp del*: *Max_in*)
then show |*a*| ∗ *Max* ((λ*i*. |*x* • *i*|) ‘ *Basis*) ∈ (λ*i*. |*a* $*_R$ *x* • *i*|) ‘ *Basis*
by (*intro image_eqI*[**where** *x*=*b*]) (*auto simp*: *abs_mult*)
}
qed *simp*

lemma *infnorm_mul_lemma*: *infnorm* (*a* $*_R$ *x*) ≤ |*a*| ∗ *infnorm x*
unfolding *infnorm_mul* **..**

lemma *infnorm_pos_lt*: *infnorm x* > *0* ⟷ *x* ≠ *0*
using *infnorm_pos_le*[*of x*] *infnorm_eq_0*[*of x*] **by** *arith*

Prove that it differs only up to a bound from Euclidean norm.

lemma *infnorm_le_norm*: *infnorm x* ≤ *norm x*
by (*simp add*: *Basis_le_norm infnorm_Max*)

lemma *norm_le_infnorm*:
fixes *x* :: ′*a*::*euclidean_space*
shows *norm x* ≤ *sqrt DIM*(′*a*) ∗ *infnorm x*
unfolding *norm_eq_sqrt_inner id_def*
proof (*rule real_le_lsqrt*[*OF inner_ge_zero*])
show *sqrt DIM*(′*a*) ∗ *infnorm x* ≥ *0*
by (*simp add*: *zero_le_mult_iff infnorm_pos_le*)
have *x* • *x* ≤ (∑ *b*∈*Basis*. *x* • *b* ∗ (*x* • *b*))
by (*metis euclidean_inner order_refl*)

also have ... $\leq$ *DIM('a)* $*$ $|$*infnorm x*$|^2$
by (*rule sum_bounded_above*) (*metis Basis_le_infnorm abs_le_square_iff power2_eq_square real_abs_infnorm*)
also have ... $\leq$ (*sqrt DIM('a)* $*$ *infnorm x*)^2
by (*simp add*: *power_mult_distrib*)
finally show $x \cdot x \leq$ (*sqrt DIM('a)* $*$ *infnorm x*)^2 **.**
qed

lemma *tendsto_infnorm* [*tendsto_intros*]:
assumes (*f* $\longrightarrow$ *a*) *F*
shows ((λx. *infnorm* (*f x*)) $\longrightarrow$ *infnorm a*) *F*
proof (*rule tendsto_compose* [*OF LIM_I assms*])
fix *r* :: *real*
assume $r > 0$
then show $\exists s>0. \forall x. x \neq a \wedge$ *norm* $(x - a) < s \longrightarrow$ *norm* (*infnorm x* $-$ *infnorm a*) $< r$
by (*metis real_norm_def le_less_trans absdiff_infnorm infnorm_le_norm*)
qed

Equality in Cauchy-Schwarz and triangle inequalities.

lemma *norm_cauchy_schwarz_eq*: $x \cdot y$ = *norm x* $*$ *norm y* $\longleftrightarrow$ *norm x* $*_R$ y = *norm y* $*_R$ x
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof (*cases x=0*)
case *True*
then show *?thesis*
by *auto*
next
case *False*
from *inner_eq_zero_iff*[*of norm y* $*_R$ x $-$ *norm x* $*_R$ y]
have *?rhs* $\longleftrightarrow$
(*norm y* $*$ (*norm y* $*$ *norm x* $*$ *norm x* $-$ *norm x* $*$ $(x \cdot y)$) $-$
norm x $*$ (*norm y* $*$ $(y \cdot x)$ $-$ *norm x* $*$ *norm y* $*$ *norm y*) = *0*)
using *False* **unfolding** *inner_simps*
by (*auto simp add*: *power2_norm_eq_inner*[*symmetric*] *power2_eq_square inner_commute field_simps*)
also have ... $\longleftrightarrow$ (*2* $*$ *norm x* $*$ *norm y* $*$ (*norm x* $*$ *norm y* $-$ $x \cdot y$) = *0*)
using *False* **by** (*simp add*: *field_simps inner_commute*)
also have ... $\longleftrightarrow$ *?lhs*
using *False* **by** *auto*
finally show *?thesis* **by** *metis*
qed

lemma *norm_cauchy_schwarz_abs_eq*:
$|x \cdot y|$ = *norm x* $*$ *norm y* $\longleftrightarrow$
norm x $*_R$ y = *norm y* $*_R$ x $\vee$ *norm x* $*_R$ y = $-$ *norm y* $*_R$ x
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof $-$
have *th*: $\bigwedge$(*x*::*real*) *a*. $a \geq 0 \Longrightarrow |x| = a \longleftrightarrow x = a \vee x = - a$

by *arith*
have *?rhs* ⟷ *norm x* $*_R$ *y* = *norm y* $*_R$ *x* ∨ *norm* (− *x*) $*_R$ *y* = *norm y* $*_R$ (− *x*)
by *simp*
also have ... ⟷ (*x* · *y* = *norm x* ∗ *norm y* ∨ (− *x*) · *y* = *norm x* ∗ *norm y*)
unfolding *norm_cauchy_schwarz_eq*[*symmetric*]
unfolding *norm_minus_cancel norm_scaleR* **..**
also have ... ⟷ *?lhs*
unfolding *th*[*OF mult_nonneg_nonneg*, *OF norm_ge_zero*[*of x*] *norm_ge_zero*[*of y*]] *inner_simps*
by *auto*
finally show *?thesis* **..**
qed

lemma *norm_triangle_eq*:
fixes *x y* :: ′*a*::*real_inner*
shows *norm* (*x* + *y*) = *norm x* + *norm y* ⟷ *norm x* $*_R$ *y* = *norm y* $*_R$ *x*
proof (*cases x* = *0* ∨ *y* = *0*)
case *True*
then show *?thesis*
by *force*
next
case *False*
then have *n*: *norm x* > *0 norm y* > *0*
by *auto*
have *norm* (*x* + *y*) = *norm x* + *norm y* ⟷ (*norm* (*x* + *y*))^2 = (*norm x* + *norm y*)^2
by *simp*
also have ... ⟷ *norm x* $*_R$ *y* = *norm y* $*_R$ *x*
unfolding *norm_cauchy_schwarz_eq*[*symmetric*]
unfolding *power2_norm_eq_inner inner_simps*
by (*simp add*: *power2_norm_eq_inner*[*symmetric*] *power2_eq_square inner_commute field_simps*)
finally show *?thesis* **.**
qed

1.5.11 Collinearity

definition *collinear* :: ′*a*::*real_vector set* ⇒ *bool*
where *collinear S* ⟷ (∃ *u*. ∀ *x* ∈ *S*. ∀ *y* ∈ *S*. ∃ *c*. *x* − *y* = *c* $*_R$ *u*)

lemma *collinear_alt*:
collinear S ⟷ (∃ *u v*. ∀ *x* ∈ *S*. ∃ *c*. *x* = *u* + *c* $*_R$ *v*) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
unfolding *collinear_def* **by** (*metis Groups.add_ac*(*2*) *diff_add_cancel*)
next
assume *?rhs*

```
  then obtain u v where *: ⋀x. x ∈ S ⟹ ∃c. x = u + c *R v
    by (auto simp: )
  have ∃c. x − y = c *R v if x ∈ S y ∈ S for x y
       by (metis *[OF ‹x ∈ S›] *[OF ‹y ∈ S›] scaleR_left.diff add_diff_cancel_left)
  then show ?lhs
    using collinear_def by blast
qed

lemma collinear:
  fixes S :: 'a::{perfect_space,real_vector} set
  shows collinear S ⟷ (∃u. u ≠ 0 ∧ (∀x ∈ S. ∀ y ∈ S. ∃c. x − y = c *R u))
proof −
  have ∃v. v ≠ 0 ∧ (∀x∈S. ∀y∈S. ∃c. x − y = c *R v)
    if ∀x∈S. ∀y∈S. ∃c. x − y = c *R u u=0 for u
  proof −
    have ∀x∈S. ∀y∈S. x = y
      using that by auto
    moreover
    obtain v::'a where v ≠ 0
      using UNIV_not_singleton [of 0] by auto
    ultimately have ∀x∈S. ∀y∈S. ∃c. x − y = c *R v
      by auto
    then show ?thesis
      using ‹v ≠ 0› by blast
  qed
  then show ?thesis
    apply (clarsimp simp: collinear_def)
    by (metis scaleR_zero_right vector_fraction_eq_iff)
qed

lemma collinear_subset: ⟦collinear T; S ⊆ T⟧ ⟹ collinear S
  by (meson collinear_def subsetCE)

lemma collinear_empty [iff]: collinear {}
  by (simp add: collinear_def)

lemma collinear_sing [iff]: collinear {x}
  by (simp add: collinear_def)

lemma collinear_2 [iff]: collinear {x, y}
  apply (simp add: collinear_def)
  apply (rule exI[where x=x − y])
  by (metis minus_diff_eq scaleR_left.minus scaleR_one)

lemma collinear_lemma: collinear {0, x, y} ⟷ x = 0 ∨ y = 0 ∨ (∃c. y = c
*R x)
  (is ?lhs ⟷ ?rhs)
proof (cases x = 0 ∨ y = 0)
  case True
```

```
  then show ?thesis
    by (auto simp: insert_commute)
next
  case False
  show ?thesis
  proof
    assume h: ?lhs
    then obtain u where u: ∀ x∈ {0,x,y}. ∀ y∈ {0,x,y}. ∃ c. x − y = c *R u
      unfolding collinear_def by blast
    from u[rule_format, of x 0] u[rule_format, of y 0]
    obtain cx and cy where
      cx: x = cx *R u and cy: y = cy *R u
      by auto
    from cx cy False have cx0: cx ≠ 0 and cy0: cy ≠ 0 by auto
    let ?d = cy / cx
    from cx cy cx0 have y = ?d *R x
      by simp
    then show ?rhs using False by blast
  next
    assume h: ?rhs
    then obtain c where c: y = c *R x
      using False by blast
    show ?lhs
      unfolding collinear_def c
      apply (rule exI[where x=x])
      apply auto
          apply (rule exI[where x=− 1], simp)
         apply (rule exI[where x= −c], simp)
        apply (rule exI[where x=1], simp)
       apply (rule exI[where x=1 − c], simp add: scaleR_left_diff_distrib)
      apply (rule exI[where x=c − 1], simp add: scaleR_left_diff_distrib)
      done
  qed
qed

lemma norm_cauchy_schwarz_equal: |x · y| = norm x * norm y ⟷ collinear {0,
x, y}
proof (cases x=0)
  case True
  then show ?thesis
    by (auto simp: insert_commute)
next
  case False
  then have nnz: norm x ≠ 0
    by auto
  show ?thesis
  proof
    assume |x · y| = norm x * norm y
    then show collinear {0, x, y}
```

```
    unfolding norm_cauchy_schwarz_abs_eq collinear_lemma
    by (meson eq_vector_fraction_iff nnz)
  next
   assume collinear {0, x, y}
   with False show |x · y| = norm x * norm y
      unfolding norm_cauchy_schwarz_abs_eq collinear_lemma  by (auto simp:
abs_if)
  qed
qed
```

1.5.12 Properties of special hyperplanes

```
lemma subspace_hyperplane: subspace {x. a · x = 0}
  by (simp add: subspace_def inner_right_distrib)

lemma subspace_hyperplane2: subspace {x. x · a = 0}
  by (simp add: inner_commute inner_right_distrib subspace_def)

lemma special_hyperplane_span:
  fixes S :: 'n::euclidean_space set
  assumes k ∈ Basis
  shows {x. k · x = 0} = span (Basis − {k})
proof −
  have *: x ∈ span (Basis − {k}) if k · x = 0 for x
  proof −
    have x = (∑ b∈Basis. (x · b) *R b)
      by (simp add: euclidean_representation)
    also have ... = (∑ b ∈ Basis − {k}. (x · b) *R b)
      by (auto simp: sum.remove [of _ k] inner_commute assms that)
    finally have x = (∑ b∈Basis − {k}. (x · b) *R b) .
    then show ?thesis
      by (simp add: span_finite)
  qed
  show ?thesis
    apply (rule span_subspace [symmetric])
    using assms
    apply (auto simp: inner_not_same_Basis intro: * subspace_hyperplane)
    done
qed

lemma dim_special_hyperplane:
  fixes k :: 'n::euclidean_space
  shows k ∈ Basis ⟹ dim {x. k · x = 0} = DIM('n) − 1
apply (simp add: special_hyperplane_span)
apply (rule dim_unique [OF subset_refl])
apply (auto simp: independent_substdbasis)
apply (metis member_remove remove_def span_base)
done
```

```
proposition dim_hyperplane:
  fixes a :: 'a::euclidean_space
  assumes a ≠ 0
    shows dim {x. a · x = 0} = DIM('a) − 1
proof −
  have span0: span {x. a · x = 0} = {x. a · x = 0}
    by (rule span_unique) (auto simp: subspace_hyperplane)
  then obtain B where independent B
            and Bsub: B ⊆ {x. a · x = 0}
            and subspB: {x. a · x = 0} ⊆ span B
            and card0: (card B = dim {x. a · x = 0})
            and ortho: pairwise orthogonal B
    using orthogonal_basis_exists by metis
  with assms have a ∉ span B
    by (metis (mono_tags, lifting) span_eq inner_eq_zero_iff mem_Collect_eq span0)
  then have ind: independent (insert a B)
    by (simp add: ‹independent B› independent_insert)
  have finite B
    using ‹independent B› independent_bound by blast
  have UNIV ⊆ span (insert a B)
  proof fix y::'a
    obtain r z where z: y = r *R a + z a · z = 0
      apply (rule_tac r=(a · y) / (a · a) and z = y − ((a · y) / (a · a)) *R a in
that)
       using assms
       by (auto simp: algebra_simps)
    show y ∈ span (insert a B)
       by (metis (mono_tags, lifting) z Bsub span_eq_iff
          add_diff_cancel_left' mem_Collect_eq span0 span_breakdown_eq span_subspace
subspB)
  qed
  then have dima: DIM('a) = dim(insert a B)
    by (metis independent_Basis span_Basis dim_eq_card top.extremum_uniqueI)
  then show ?thesis
    by (metis (mono_tags, lifting) Bsub Diff_insert_absorb ‹a ∉ span B› ind card0
        card_Diff_singleton dim_span indep_card_eq_dim_span insertI1 subsetCE
        subspB)
qed

lemma lowdim_eq_hyperplane:
  fixes S :: 'a::euclidean_space set
  assumes dim S = DIM('a) − 1
  obtains a where a ≠ 0 and span S = {x. a · x = 0}
proof −
  have dimS: dim S < DIM('a)
    by (simp add: assms)
  then obtain b where b: b ≠ 0 span S ⊆ {a. b · a = 0}
    using lowdim_subset_hyperplane [of S] by fastforce
  show ?thesis
```

```
    apply (rule that[OF b(1)])
    apply (rule subspace_dim_equal)
    by (auto simp: assms b dim_hyperplane subspace_hyperplane)
qed

lemma dim_eq_hyperplane:
  fixes S :: 'n::euclidean_space set
  shows dim S = DIM('n) − 1 ⟷ (∃ a. a ≠ 0 ∧ span S = {x. a · x = 0})
by (metis One_nat_def dim_hyperplane dim_span lowdim_eq_hyperplane)
```

1.5.13 Orthogonal bases and Gram-Schmidt process

```
lemma pairwise_orthogonal_independent:
  assumes pairwise orthogonal S and 0 ∉ S
    shows independent S
proof −
  have 0: ⋀x y. ⟦x ≠ y; x ∈ S; y ∈ S⟧ ⟹ x · y = 0
    using assms by (simp add: pairwise_def orthogonal_def)
  have False if a ∈ S and a: a ∈ span (S − {a}) for a
  proof −
    obtain T U where T ⊆ S − {a} a = (∑ v∈T. U v *R v)
      using a by (force simp: span_explicit)
    then have a · a = a · (∑ v∈T. U v *R v)
      by simp
    also have ... = 0
      apply (simp add: inner_sum_right)
      apply (rule comm_monoid_add_class.sum.neutral)
      by (metis 0 DiffE ‹T ⊆ S − {a}› mult_not_zero singletonI subsetCE ‹a ∈ S›)
    finally show ?thesis
      using ‹0 ∉ S› ‹a ∈ S› by auto
  qed
  then show ?thesis
    by (force simp: dependent_def)
qed

lemma pairwise_orthogonal_imp_finite:
  fixes S :: 'a::euclidean_space set
  assumes pairwise orthogonal S
    shows finite S
proof −
  have independent (S − {0})
    apply (rule pairwise_orthogonal_independent)
     apply (metis Diff_iff assms pairwise_def)
    by blast
  then show ?thesis
    by (meson independent_imp_finite infinite_remove)
qed

lemma subspace_orthogonal_to_vector: subspace {y. orthogonal x y}
```

```
  by (simp add: subspace_def orthogonal_clauses)

lemma subspace_orthogonal_to_vectors: subspace {y. ∀ x ∈ S. orthogonal x y}
  by (simp add: subspace_def orthogonal_clauses)

lemma orthogonal_to_span:
  assumes a: a ∈ span S and x: ⋀y. y ∈ S ⟹ orthogonal x y
    shows orthogonal x a
  by (metis a orthogonal_clauses(1,2,4)
      span_induct_alt x)

proposition Gram_Schmidt_step:
  fixes S :: 'a::euclidean_space set
  assumes S: pairwise orthogonal S and x: x ∈ span S
    shows orthogonal x (a − (∑ b∈S. (b · a / (b · b)) *R b))
proof −
  have finite S
    by (simp add: S pairwise_orthogonal_imp_finite)
  have orthogonal (a − (∑ b∈S. (b · a / (b · b)) *R b)) x
       if x ∈ S for x
  proof −
    have a · x = (∑ y∈S. if y = x then y · a else 0)
      by (simp add: ‹finite S› inner_commute that)
    also have ... =  (∑ b∈S. b · a * (b · x) / (b · b))
      apply (rule sum.cong [OF refl], simp)
      by (meson S orthogonal_def pairwise_def that)
   finally show ?thesis
     by (simp add: orthogonal_def algebra_simps inner_sum_left)
  qed
  then show ?thesis
    using orthogonal_to_span orthogonal_commute x by blast
qed


lemma orthogonal_extension_aux:
  fixes S :: 'a::euclidean_space set
  assumes finite T finite S pairwise orthogonal S
    shows ∃ U. pairwise orthogonal (S ∪ U) ∧ span (S ∪ U) = span (S ∪ T)
using assms
proof (induction arbitrary: S)
  case empty then show ?case
    by simp (metis sup_bot_right)
next
  case (insert a T)
  have 0: ⋀x y. ⟦x ≠ y; x ∈ S; y ∈ S⟧ ⟹ x · y = 0
    using insert by (simp add: pairwise_def orthogonal_def)
  define a' where a' = a − (∑ b∈S. (b · a / (b · b)) *R b)
  obtain U where orthU: pairwise orthogonal (S ∪ insert a' U)
           and spanU: span (insert a' S ∪ U) = span (insert a' S ∪ T)
```

```
    by (rule exE [OF insert.IH [of insert a' S]])
      (auto simp: Gram_Schmidt_step a'_def insert.prems orthogonal_commute
        pairwise_orthogonal_insert span_clauses)
  have orthS: ⋀x. x ∈ S ⟹ a' · x = 0
    apply (simp add: a'_def)
    using Gram_Schmidt_step [OF ‹pairwise orthogonal S›]
     apply (force simp: orthogonal_def inner_commute span_superset [THEN sub-
setD])
    done
  have span (S ∪ insert a' U) = span (insert a' (S ∪ T))
    using spanU by simp
  also have ... = span (insert a (S ∪ T))
    apply (rule eq_span_insert_eq)
    apply (simp add: a'_def span_neg span_sum span_base span_mul)
    done
  also have ... = span (S ∪ insert a T)
    by simp
  finally show ?case
    by (rule_tac x=insert a' U in exI) (use orthU in auto)
qed


proposition orthogonal_extension:
  fixes S :: 'a::euclidean_space set
  assumes S: pairwise orthogonal S
  obtains U where pairwise orthogonal (S ∪ U) span (S ∪ U) = span (S ∪ T)
proof -
  obtain B where finite B span B = span T
    using basis_subspace_exists [of span T] subspace_span by metis
  with orthogonal_extension_aux [of B S]
  obtain U where pairwise orthogonal (S ∪ U) span (S ∪ U) = span (S ∪ B)
    using assms pairwise_orthogonal_imp_finite by auto
  with ‹span B = span T› show ?thesis
    by (rule_tac U=U in that) (auto simp: span_Un)
qed

corollary orthogonal_extension_strong:
  fixes S :: 'a::euclidean_space set
  assumes S: pairwise orthogonal S
  obtains U where U ∩ (insert 0 S) = {} pairwise orthogonal (S ∪ U)
                  span (S ∪ U) = span (S ∪ T)
proof -
  obtain U where pairwise orthogonal (S ∪ U) span (S ∪ U) = span (S ∪ T)
    using orthogonal_extension assms by blast
  then show ?thesis
    apply (rule_tac U = U - (insert 0 S) in that)
      apply blast
     apply (force simp: pairwise_def)
    apply (metis Un_Diff_cancel Un_insert_left span_redundant span_zero)
```

```
  done
qed
```

1.5.14 Decomposing a vector into parts in orthogonal subspaces

existence of orthonormal basis for a subspace.

```
lemma orthogonal_spanningset_subspace:
 fixes S :: 'a :: euclidean_space set
 assumes subspace S
 obtains B where B ⊆ S pairwise orthogonal B span B = S
proof −
 obtain B where B ⊆ S independent B S ⊆ span B card B = dim S
   using basis_exists by blast
 with orthogonal_extension [of {} B]
 show ?thesis
  by (metis Un_empty_left assms pairwise_empty span_superset span_subspace that)
qed

lemma orthogonal_basis_subspace:
 fixes S :: 'a :: euclidean_space set
 assumes subspace S
 obtains B where 0 ∉ B B ⊆ S pairwise orthogonal B independent B
               card B = dim S span B = S
proof −
 obtain B where B ⊆ S pairwise orthogonal B span B = S
   using assms orthogonal_spanningset_subspace by blast
 then show ?thesis
   apply (rule_tac B = B − {0} in that)
  apply (auto simp: indep_card_eq_dim_span pairwise_subset pairwise_orthogonal_independent
elim: pairwise_subset)
   done
qed

proposition orthonormal_basis_subspace:
 fixes S :: 'a :: euclidean_space set
 assumes subspace S
 obtains B where B ⊆ S pairwise orthogonal B
             and ⋀x. x ∈ B ⟹ norm x = 1
             and independent B card B = dim S span B = S
proof −
 obtain B where 0 ∉ B B ⊆ S
            and orth: pairwise orthogonal B
            and independent B card B = dim S span B = S
   by (blast intro: orthogonal_basis_subspace [OF assms])
 have 1: (λx. x /R norm x) ` B ⊆ S
   using ‹span B = S› span_superset span_mul by fastforce
 have 2: pairwise orthogonal ((λx. x /R norm x) ` B)
   using orth by (force simp: pairwise_def orthogonal_clauses)
```

have *3*: $\bigwedge x.\ x \in (\lambda x.\ x /_R \ norm\ x)$ ‘ $B \Longrightarrow norm\ x = 1$
by (*metis* (*no_types*, *lifting*) ‹*0* ∉ *B*› *image_iff norm_sgn sgn_div_norm*)
have *4*: *independent* ((λ*x*. *x* $/_R$ *norm x*) ‘ *B*)
by (*metis 2 3 norm_zero pairwise_orthogonal_independent zero_neq_one*)
have *inj_on* (λ*x*. *x* $/_R$ *norm x*) *B*
proof
fix *x y*
assume *x* ∈ *B* *y* ∈ *B* *x* $/_R$ *norm x* = *y* $/_R$ *norm y*
moreover have ⋀*i*. *i* ∈ *B* ⟹ *norm* (*i* $/_R$ *norm i*) = *1*
using *3* **by** *blast*
ultimately show *x* = *y*
by (*metis norm_eq_1 orth orthogonal_clauses*(*7*) *orthogonal_commute orthogonal_def pairwise_def zero_neq_one*)
qed
then have *5*: *card* ((λ*x*. *x* $/_R$ *norm x*) ‘ *B*) = *dim S*
by (*metis* ‹*card B* = *dim S*› *card_image*)
have *6*: *span* ((λ*x*. *x* $/_R$ *norm x*) ‘ *B*) = *S*
by (*metis 1 4 5 assms card_eq_dim independent_imp_finite span_subspace*)
show *?thesis*
by (*rule that* [*OF 1 2 3 4 5 6*])
qed

proposition *orthogonal_to_subspace_exists_gen*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *span S* ⊂ *span T*
obtains *x* **where** *x* ≠ *0* *x* ∈ *span T* ⋀*y*. *y* ∈ *span S* ⟹ *orthogonal x y*
proof −
obtain *B* **where** *B* ⊆ *span S* **and** *orthB*: *pairwise orthogonal B*
and ⋀*x*. *x* ∈ *B* ⟹ *norm x* = *1*
and *independent B card B* = *dim S span B* = *span S*
by (*rule orthonormal_basis_subspace* [*of span S*, *OF subspace_span*]) (*auto*)
with *assms* **obtain** *u* **where** *spanBT*: *span B* ⊆ *span T* **and** *u* ∉ *span B* *u* ∈ *span T*
by *auto*
obtain *C* **where** *orthBC*: *pairwise orthogonal* (*B* ∪ *C*) **and** *spanBC*: *span* (*B* ∪ *C*) = *span* (*B* ∪ {*u*})
by (*blast intro*: *orthogonal_extension* [*OF orthB*])
show *thesis*
proof (*cases C* ⊆ *insert 0 B*)
case *True*
then have *C* ⊆ *span B*
using *span_eq*
by (*metis span_insert_0 subset_trans*)
moreover have *u* ∈ *span* (*B* ∪ *C*)
using ‹*span* (*B* ∪ *C*) = *span* (*B* ∪ {*u*})› *span_superset* **by** *force*
ultimately show *?thesis*
using *True* ‹*u* ∉ *span B*›
by (*metis Un_insert_left span_insert_0 sup.orderE*)

```
  next
    case False
    then obtain x where x ∈ C x ≠ 0 x ∉ B
      by blast
    then have x ∈ span T
     by (metis (no_types, lifting) Un_insert_right Un_upper2 ‹u ∈ span T› spanBT
spanBC
          ‹u ∈ span T› insert_subset span_superset span_mono
          span_span subsetCE subset_trans sup_bot.comm_neutral)
    moreover have orthogonal x y if y ∈ span B for y
      using that
    proof (rule span_induct)
      show subspace {a. orthogonal x a}
        by (simp add: subspace_orthogonal_to_vector)
      show ⋀b. b ∈ B ⟹ orthogonal x b
        by (metis Un_iff ‹x ∈ C› ‹x ∉ B› orthBC pairwise_def)
    qed
    ultimately show ?thesis
      using ‹x ≠ 0› that ‹span B = span S› by auto
  qed
qed

corollary orthogonal_to_subspace_exists:
  fixes S :: 'a :: euclidean_space set
  assumes dim S < DIM('a)
  obtains x where x ≠ 0 ⋀y. y ∈ span S ⟹ orthogonal x y
proof -
  have span S ⊂ UNIV
 by (metis (mono_tags) UNIV_I assms inner_eq_zero_iff less_le lowdim_subset_hyperplane
      mem_Collect_eq top.extremum_strict top.not_eq_extremum)
  with orthogonal_to_subspace_exists_gen [of S UNIV] that show ?thesis
    by (auto)
qed

corollary orthogonal_to_vector_exists:
  fixes x :: 'a :: euclidean_space
  assumes 2 ≤ DIM('a)
  obtains y where y ≠ 0 orthogonal x y
proof -
  have dim {x} < DIM('a)
    using assms by auto
  then show thesis
   by (rule orthogonal_to_subspace_exists) (simp add: orthogonal_commute span_base
that)
qed

proposition orthogonal_subspace_decomp_exists:
  fixes S :: 'a :: euclidean_space set
  obtains y z
```

```
  where y ∈ span S
    and ⋀w. w ∈ span S ⟹ orthogonal z w
    and x = y + z
proof -
  obtain T where 0 ∉ T T ⊆ span S pairwise orthogonal T independent T
    card T = dim (span S) span T = span S
    using orthogonal_basis_subspace subspace_span by blast
  let ?a = ∑b∈T. (b · x / (b · b)) *R b
  have orth: orthogonal (x - ?a) w if w ∈ span S for w
    by (simp add: Gram_Schmidt_step ‹pairwise orthogonal T› ‹span T = span S›
        orthogonal_commute that)
  show ?thesis
    apply (rule_tac y = ?a and z = x - ?a in that)
      apply (meson ‹T ⊆ span S› span_scale span_sum subsetCE)
     apply (fact orth, simp)
    done
qed

lemma orthogonal_subspace_decomp_unique:
  fixes S :: 'a :: euclidean_space set
  assumes x + y = x' + y'
      and ST: x ∈ span S x' ∈ span S y ∈ span T y' ∈ span T
      and orth: ⋀a b. ⟦a ∈ S; b ∈ T⟧ ⟹ orthogonal a b
  shows x = x' ∧ y = y'
proof -
  have x + y - y' = x'
    by (simp add: assms)
  moreover have ⋀a b. ⟦a ∈ span S; b ∈ span T⟧ ⟹ orthogonal a b
    by (meson orth orthogonal_commute orthogonal_to_span)
  ultimately have 0 = x' - x
   by (metis (full_types) add_diff_cancel_left' ST diff_right_commute orthogonal_clauses(10)
orthogonal_clauses(5) orthogonal_self)
  with assms show ?thesis by auto
qed

lemma vector_in_orthogonal_spanningset:
  fixes a :: 'a::euclidean_space
  obtains S where a ∈ S pairwise orthogonal S span S = UNIV
 by (metis UNIV_I Un_iff empty_iff insert_subset orthogonal_extension pairwise_def
      pairwise_orthogonal_insert span_UNIV subsetI subset_antisym)

lemma vector_in_orthogonal_basis:
  fixes a :: 'a::euclidean_space
  assumes a ≠ 0
  obtains S where a ∈ S 0 ∉ S pairwise orthogonal S independent S finite S
                    span S = UNIV card S = DIM('a)
proof -
  obtain S where S: a ∈ S pairwise orthogonal S span S = UNIV
    using vector_in_orthogonal_spanningset .
```

```
  show thesis
  proof
    show pairwise orthogonal (S − {0})
      using pairwise_mono S(2) by blast
    show independent (S − {0})
     by (simp add: ‹pairwise orthogonal (S − {0})› pairwise_orthogonal_independent)
    show finite (S − {0})
      using ‹independent (S − {0})› independent_imp_finite by blast
    show card (S − {0}) = DIM('a)
      using span_delete_0 [of S] S
      by (simp add: ‹independent (S − {0})› indep_card_eq_dim_span)
  qed (use S ‹a ≠ 0› in auto)
qed

lemma vector_in_orthonormal_basis:
  fixes a :: 'a::euclidean_space
  assumes norm a = 1
  obtains S where a ∈ S pairwise orthogonal S ⋀x. x ∈ S ⟹ norm x = 1
    independent S card S = DIM('a) span S = UNIV
proof −
  have a ≠ 0
    using assms by auto
  then obtain S where a ∈ S 0 ∉ S finite S
          and S: pairwise orthogonal S independent S span S = UNIV card S =
DIM('a)
    by (metis vector_in_orthogonal_basis)
  let ?S = (λx. x /R norm x) ` S
  show thesis
  proof
    show a ∈ ?S
      using ‹a ∈ S› assms image_iff by fastforce
  next
    show pairwise orthogonal ?S
      using ‹pairwise orthogonal S› by (auto simp: pairwise_def orthogonal_def)
    show ⋀x. x ∈ (λx. x /R norm x) ` S ⟹ norm x = 1
      using ‹0 ∉ S› by (auto simp: field_split_simps)
    then show independent ?S
       by (metis ‹pairwise orthogonal ((λx. x /R norm x) ` S)› norm_zero pair-
wise_orthogonal_independent zero_neq_one)
    have inj_on (λx. x /R norm x) S
      unfolding inj_on_def
      by (metis (full_types) S(1) ‹0 ∉ S› inverse_nonzero_iff_nonzero norm_eq_zero
orthogonal_scaleR orthogonal_self pairwise_def)
    then show card ?S = DIM('a)
      by (simp add: card_image S)
    show span ?S = UNIV
      by (metis (no_types) ‹0 ∉ S› ‹finite S› ‹span S = UNIV›
          field_class.field_inverse_zero inverse_inverse_eq less_irrefl span_image_scale
          zero_less_norm_iff)
```

```
  qed
qed

proposition dim_orthogonal_sum:
  fixes A :: 'a::euclidean_space set
  assumes ⋀x y. ⟦x ∈ A; y ∈ B⟧ ⟹ x · y = 0
    shows dim(A ∪ B) = dim A + dim B
proof -
  have 1: ⋀x y. ⟦x ∈ span A; y ∈ B⟧ ⟹ x · y = 0
    by (erule span_induct [OF _ subspace_hyperplane2]; simp add: assms)
  have ⋀x y. ⟦x ∈ span A; y ∈ span B⟧ ⟹ x · y = 0
    using 1 by (simp add: span_induct [OF _ subspace_hyperplane])
  then have 0: ⋀x y. ⟦x ∈ span A; y ∈ span B⟧ ⟹ x · y = 0
    by simp
  have dim(A ∪ B) = dim (span (A ∪ B))
    by (simp)
  also have span (A ∪ B) = ((λ(a, b). a + b) ` (span A × span B))
    by (auto simp add: span_Un image_def)
  also have dim ... = dim {x + y |x y. x ∈ span A ∧ y ∈ span B}
    by (auto intro!: arg_cong [where f=dim])
  also have ... = dim {x + y |x y. x ∈ span A ∧ y ∈ span B} + dim(span A ∩
span B)
    by (auto simp: dest: 0)
  also have ... = dim (span A) + dim (span B)
    by (rule dim_sums_Int) (auto)
  also have ... = dim A + dim B
    by (simp)
  finally show ?thesis .
qed

lemma dim_subspace_orthogonal_to_vectors:
  fixes A :: 'a::euclidean_space set
  assumes subspace A subspace B A ⊆ B
    shows dim {y ∈ B. ∀x ∈ A. orthogonal x y} + dim A = dim B
proof -
  have dim (span ({y ∈ B. ∀x∈A. orthogonal x y} ∪ A)) = dim (span B)
  proof (rule arg_cong [where f=dim, OF subset_antisym])
    show span ({y ∈ B. ∀x∈A. orthogonal x y} ∪ A) ⊆ span B
      by (simp add: ‹A ⊆ B› Collect_restrict span_mono)
  next
    have *: x ∈ span ({y ∈ B. ∀x∈A. orthogonal x y} ∪ A)
         if x ∈ B for x
    proof -
      obtain y z where x = y + z y ∈ span A and orth: ⋀w. w ∈ span A ⟹
orthogonal z w
        using orthogonal_subspace_decomp_exists [of A x] that by auto
      have y ∈ span B
        using ‹y ∈ span A› assms(3) span_mono by blast
      then have z ∈ {a ∈ B. ∀x. x ∈ A ⟶ orthogonal x a}
```

```
      apply simp
     using ‹x = y + z› assms(1) assms(2) orth orthogonal_commute span_add_eq
       span_eq_iff that by blast
    then have z: z ∈ span {y ∈ B. ∀x∈A. orthogonal x y}
      by (meson span_superset subset_iff)
    then show ?thesis
      apply (auto simp: span_Un image_def ‹x = y + z› ‹y ∈ span A›)
      using ‹y ∈ span A› add.commute by blast
  qed
  show span B ⊆ span ({y ∈ B. ∀x∈A. orthogonal x y} ∪ A)
    by (rule span_minimal) (auto intro: ∗ span_minimal)
 qed
 then show ?thesis
  by (metis (no_types, lifting) dim_orthogonal_sum dim_span mem_Collect_eq
      orthogonal_commute orthogonal_def)
qed
```

1.5.15 Linear functions are (uniformly) continuous on any set

1.5.16 Topological properties of linear functions

```
lemma linear_lim_0:
 assumes bounded_linear f
 shows (f ⟶ 0) (at (0))
proof −
 interpret f: bounded_linear f by fact
 have (f ⟶ f 0) (at 0)
  using tendsto_ident_at by (rule f.tendsto)
 then show ?thesis unfolding f.zero .
qed

lemma linear_continuous_at:
 assumes bounded_linear f
 shows continuous (at a) f
 unfolding continuous_at using assms
 apply (rule bounded_linear.tendsto)
 apply (rule tendsto_ident_at)
 done

lemma linear_continuous_within:
 bounded_linear f ⟹ continuous (at x within s) f
 using continuous_at_imp_continuous_at_within linear_continuous_at by blast

lemma linear_continuous_on:
 bounded_linear f ⟹ continuous_on s f
 using continuous_at_imp_continuous_on[of s f] using linear_continuous_at[of f]
by auto

lemma Lim_linear:
```

```
 fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space and h :: 'b ⇒ 'c::real_normed_vector
  assumes (f ⟶ l) F linear h
  shows ((λx. h(f x)) ⟶ h l) F
proof -
  obtain B where B: B > 0 ⋀x. norm (h x) ≤ B * norm x
    using linear_bounded_pos [OF ‹linear h›] by blast
  show ?thesis
    unfolding tendsto_iff
  proof (intro allI impI)
    show ∀F x in F. dist (h (f x)) (h l) < e if e > 0 for e
    proof -
      have ∀F x in F. dist (f x) l < e/B
        by (simp add: ‹0 < B› assms(1) tendstoD that)
      then show ?thesis
        unfolding dist_norm
      proof (rule eventually_mono)
        show norm (h (f x) - h l) < e if norm (f x - l) < e / B for x
          using that B
          apply (simp add: field_split_simps)
          by (metis ‹linear h› le_less_trans linear_diff)
      qed
    qed
  qed
qed

lemma linear_continuous_compose:
 fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space and g :: 'b ⇒ 'c::real_normed_vector
  assumes continuous F f linear g
  shows continuous F (λx. g(f x))
  using assms unfolding continuous_def by (rule Lim_linear)

lemma linear_continuous_on_compose:
 fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space and g :: 'b ⇒ 'c::real_normed_vector
  assumes continuous_on S f linear g
  shows continuous_on S (λx. g(f x))
 using assms by (simp add: continuous_on_eq_continuous_within linear_continuous_compose)
```

Also bilinear functions, in composition form

```
lemma bilinear_continuous_compose:
  fixes h :: 'a::euclidean_space ⇒ 'b::euclidean_space ⇒ 'c::real_normed_vector
  assumes continuous F f continuous F g bilinear h
  shows continuous F (λx. h (f x) (g x))
  using assms bilinear_conv_bounded_bilinear bounded_bilinear.continuous by blast

lemma bilinear_continuous_on_compose:
  fixes h :: 'a::euclidean_space ⇒ 'b::euclidean_space ⇒ 'c::real_normed_vector
    and f :: 'd::t2_space ⇒ 'a
  assumes continuous_on S f continuous_on S g bilinear h
  shows continuous_on S (λx. h (f x) (g x))
```

using *assms* **by** (*simp add*: *continuous_on_eq_continuous_within bilinear_continuous_compose*)

end

1.6 Affine Sets

theory *Affine*
imports *Linear_Algebra*
begin

lemma *if_smult*: (*if P then x else* (*y*::*real*)) $*_R$ *v* = (*if P then x* $*_R$ *v else y* $*_R$ *v*)
by (*fact if_distrib*)

lemma *sum_delta_notmem*:
assumes *x* ∉ *s*
shows *sum* (λ*y*. *if* (*y* = *x*) *then P x else Q y*) *s* = *sum Q s*
and *sum* (λ*y*. *if* (*x* = *y*) *then P x else Q y*) *s* = *sum Q s*
and *sum* (λ*y*. *if* (*y* = *x*) *then P y else Q y*) *s* = *sum Q s*
and *sum* (λ*y*. *if* (*x* = *y*) *then P y else Q y*) *s* = *sum Q s*
apply (*rule_tac* [!] *sum.cong*)
using *assms*
apply *auto*
done

lemmas *independent_finite* = *independent_imp_finite*

lemma *span_substd_basis*:
assumes *d*: *d* ⊆ *Basis*
shows *span d* = {*x*. ∀ *i*∈*Basis*. *i* ∉ *d* ⟶ *x*·*i* = *0*}
(**is** _ = *?B*)
proof −
have *d* ⊆ *?B*
using *d* **by** (*auto simp*: *inner_Basis*)
moreover have *s*: *subspace ?B*
using *subspace_substandard*[*of* λ*i*. *i* ∉ *d*] **.**
ultimately have *span d* ⊆ *?B*
using *span_mono*[*of d ?B*] *span_eq_iff*[*of ?B*] **by** *blast*
moreover have ∗: *card d* ≤ *dim* (*span d*)
using *independent_card_le_dim*[*of d span d*] *independent_substdbasis*[*OF assms*]
span_superset[*of d*]
by *auto*
moreover from ∗ **have** *dim ?B* ≤ *dim* (*span d*)
using *dim_substandard*[*OF assms*] **by** *auto*
ultimately show *?thesis*
using *s subspace_dim_equal*[*of span d ?B*] *subspace_span*[*of d*] **by** *auto*
qed

lemma *basis_to_substdbasis_subspace_isomorphism*:

fixes B :: $'a$::*euclidean_space set*
assumes *independent B*
shows $\exists f\ d$::$'a$ *set*. *card* d = *card* $B \wedge$ *linear* $f \wedge f$ ‘ $B = d \wedge$
f ‘ *span* $B = \{x.\ \forall i{\in}Basis.\ i \notin d \longrightarrow x \cdot i = 0\} \wedge$ *inj_on* f (*span* B) $\wedge d \subseteq$ *Basis*
proof –
have B: *card* B = *dim* B
using *dim_unique*[*of B B card B*] *assms span_superset*[*of B*] **by** *auto*
have *dim* $B \leq$ *card* (*Basis* :: $'a$ *set*)
using *dim_subset_UNIV*[*of B*] **by** *simp*
from *ex_card*[*OF this*] **obtain** d :: $'a$ *set* **where** d: $d \subseteq$ *Basis* **and** t: *card* d = *dim* B
by *auto*
let *?t* = $\{x$::$'a$::*euclidean_space*. $\forall i{\in}Basis.\ i \notin d \longrightarrow x{\cdot}i = 0\}$
have $\exists f$. *linear* $f \wedge f$ ‘ $B = d \wedge f$ ‘ *span* B = *?t* $\wedge$ *inj_on* f (*span* B)
proof (*intro basis_to_basis_subspace_isomorphism subspace_span subspace_substandard span_superset*)
show $d \subseteq \{x.\ \forall i{\in}Basis.\ i \notin d \longrightarrow x \cdot i = 0\}$
using *d inner_not_same_Basis* **by** *blast*
qed (*auto simp*: *span_substd_basis independent_substdbasis dim_substandard d t B assms*)
with *t* ‹*card B = dim B*› *d* **show** *?thesis* **by** *auto*
qed

1.6.1 Affine set and affine hull

definition *affine* :: $'a$::*real_vector set* $\Rightarrow$ *bool*
where *affine* $s \longleftrightarrow (\forall x{\in}s.\ \forall y{\in}s.\ \forall u\ v.\ u + v = 1 \longrightarrow u *_R x + v *_R y \in s)$

lemma *affine_alt*: *affine* $s \longleftrightarrow (\forall x{\in}s.\ \forall y{\in}s.\ \forall u{::}real.\ (1 - u) *_R x + u *_R y \in s)$
unfolding *affine_def* **by** (*metis eq_diff_eq'*)

lemma *affine_empty* [*iff*]: *affine* {}
unfolding *affine_def* **by** *auto*

lemma *affine_sing* [*iff*]: *affine* $\{x\}$
unfolding *affine_alt* **by** (*auto simp*: *scaleR_left_distrib* [*symmetric*])

lemma *affine_UNIV* [*iff*]: *affine UNIV*
unfolding *affine_def* **by** *auto*

lemma *affine_Inter* [*intro*]: $(\bigwedge s.\ s{\in}f \Longrightarrow$ *affine* $s) \Longrightarrow$ *affine* $(\bigcap f)$
unfolding *affine_def* **by** *auto*

lemma *affine_Int*[*intro*]: *affine* $s \Longrightarrow$ *affine* $t \Longrightarrow$ *affine* $(s \cap t)$
unfolding *affine_def* **by** *auto*

lemma *affine_scaling*: *affine* $s \Longrightarrow$ *affine* (*image* $(\lambda x.\ c *_R x)$ s)

apply (*clarsimp simp add*: *affine_def*)
apply (*rule_tac x*=$u *_R x + v *_R y$ **in** *image_eqI*)
apply (*auto simp*: *algebra_simps*)
done

lemma *affine_affine_hull* [*simp*]: *affine*(*affine hull s*)
unfolding *hull_def*
using *affine_Inter*[*of* $\{t.$ *affine* $t \wedge s \subseteq t\}$] **by** *auto*

lemma *affine_hull_eq*[*simp*]: (*affine hull* $s = s$) $\longleftrightarrow$ *affine s*
by (*metis affine_affine_hull hull_same*)

lemma *affine_hyperplane*: *affine* $\{x.\ a \cdot x = b\}$
by (*simp add*: *affine_def algebra_simps*) (*metis distrib_right mult.left_neutral*)

Some explicit formulations

Formalized by Lars Schewe.

lemma *affine*:
fixes V::$'a$::*real_vector set*
shows *affine* $V \longleftrightarrow$
($\forall S\ u.$ *finite* $S \wedge S \neq \{\} \wedge S \subseteq V \wedge$ *sum* $u\ S = 1 \longrightarrow (\sum x{\in}S.\ u\ x *_R x) \in V$)
proof −
have $u *_R x + v *_R y \in V$ **if** $x \in V\ y \in V\ u + v = (1::real)$
and $*$: $\bigwedge S\ u.$ ⟦*finite* S; $S \neq \{\}$; $S \subseteq V$; *sum* $u\ S = 1$⟧ $\Longrightarrow (\sum x{\in}S.\ u\ x *_R x) \in V$ **for** $x\ y\ u\ v$
proof (*cases* $x = y$)
case *True*
then show *?thesis*
using *that* **by** (*metis scaleR_add_left scaleR_one*)
next
case *False*
then show *?thesis*
using *that* $*$[*of* $\{x,y\}$ $\lambda w.$ *if* $w = x$ *then* u *else* v] **by** *auto*
qed
moreover have $(\sum x{\in}S.\ u\ x *_R x) \in V$
if $*$: $\bigwedge x\ y\ u\ v.$ ⟦$x{\in}V$; $y{\in}V$; $u + v = 1$⟧ $\Longrightarrow u *_R x + v *_R y \in V$
and *finite* S $S \neq \{\}$ $S \subseteq V$ *sum* $u\ S = 1$ **for** $S\ u$
proof −
define n **where** $n = card\ S$
consider *card* $S = 0$ | *card* $S = 1$ | *card* $S = 2$ | *card* $S > 2$ **by** *linarith*
then show $(\sum x{\in}S.\ u\ x *_R x) \in V$
proof *cases*
assume *card* $S = 1$
then obtain a **where** $S = \{a\}$
by (*auto simp*: *card_Suc_eq*)
then show *?thesis*
using *that* **by** *simp*

```
  next
    assume card S = 2
    then obtain a b where S = {a, b}
      by (metis Suc_1 card_1_singletonE card_Suc_eq)
    then show ?thesis
      using *[of a b] that
      by (auto simp: sum_clauses(2))
  next
    assume card S > 2
    then show ?thesis using that n_def
    proof (induct n arbitrary: u S)
      case 0
      then show ?case by auto
    next
      case (Suc n u S)
      have sum u S = card S if ¬ (∃x∈S. u x ≠ 1)
        using that unfolding card_eq_sum by auto
      with Suc.prems obtain x where x ∈ S and x: u x ≠ 1 by force
      have c: card (S − {x}) = card S − 1
        by (simp add: Suc.prems(3) ‹x ∈ S›)
      have sum u (S − {x}) = 1 − u x
        by (simp add: Suc.prems sum_diff1 ‹x ∈ S›)
      with x have eq1: inverse (1 − u x) * sum u (S − {x}) = 1
        by auto
      have inV: (∑y∈S − {x}. (inverse (1 − u x) * u y) *R y) ∈ V
      proof (cases card (S − {x}) > 2)
        case True
        then have S: S − {x} ≠ {} card (S − {x}) = n
          using Suc.prems c by force+
        show ?thesis
        proof (rule Suc.hyps)
          show (∑a∈S − {x}. inverse (1 − u x) * u a) = 1
            by (auto simp: eq1 sum_distrib_left[symmetric])
        qed (use S Suc.prems True in auto)
      next
        case False
        then have card (S − {x}) = Suc (Suc 0)
          using Suc.prems c by auto
        then obtain a b where ab: (S − {x}) = {a, b} a≠b
          unfolding card_Suc_eq by auto
        then show ?thesis
          using eq1 ‹S ⊆ V›
          by (auto simp: sum_distrib_left distrib_left intro!: Suc.prems(2)[of a b])
      qed
      have u x + (1 − u x) = 1 ⟹
        u x *R x + (1 − u x) *R ((∑y∈S − {x}. u y *R y) /R (1 − u x)) ∈ V
     by (rule Suc.prems) (use ‹x ∈ S› Suc.prems inV in ‹auto simp: scaleR_right.sum›)
     moreover have (∑a∈S. u a *R a) = u x *R x + (∑a∈S − {x}. u a *R
a)
```

```
          by (meson Suc.prems(3) sum.remove ‹x ∈ S›)
        ultimately show (∑x∈S. u x *R x) ∈ V
          by (simp add: x)
      qed
    qed (use ‹S≠{}› ‹finite S› in auto)
  qed
  ultimately show ?thesis
    unfolding affine_def by meson
qed


lemma affine_hull_explicit:
  affine hull p = {y. ∃S u. finite S ∧ S ≠ {} ∧ S ⊆ p ∧ sum u S = 1 ∧ sum (λv.
u v *R v) S = y}
  (is _ = ?rhs)
proof (rule hull_unique)
  show p ⊆ ?rhs
  proof (intro subsetI CollectI exI conjI)
    show ⋀x. sum (λz. 1) {x} = 1
      by auto
  qed auto
  show ?rhs ⊆ T if p ⊆ T affine T for T
    using that unfolding affine by blast
  show affine ?rhs
    unfolding affine_def
  proof clarify
    fix u v :: real and sx ux sy uy
    assume uv: u + v = 1
      and x: finite sx sx ≠ {} sx ⊆ p sum ux sx = (1::real)
      and y: finite sy sy ≠ {} sy ⊆ p sum uy sy = (1::real)
    have **: (sx ∪ sy) ∩ sx = sx (sx ∪ sy) ∩ sy = sy
      by auto
    show ∃S w. finite S ∧ S ≠ {} ∧ S ⊆ p ∧
        sum w S = 1 ∧ (∑v∈S. w v *R v) = u *R (∑v∈sx. ux v *R v) + v *R
(∑v∈sy. uy v *R v)
    proof (intro exI conjI)
      show finite (sx ∪ sy)
        using x y by auto
      show sum (λi. (if i∈sx then u * ux i else 0) + (if i∈sy then v * uy i else 0))
(sx ∪ sy) = 1
        using x y uv
      by (simp add: sum_Un sum.distrib sum.inter_restrict[symmetric] sum_distrib_left
[symmetric] **)
      have (∑i∈sx ∪ sy. ((if i ∈ sx then u * ux i else 0) + (if i ∈ sy then v * uy
i else 0)) *R i)
          = (∑i∈sx. (u * ux i) *R i) + (∑i∈sy. (v * uy i) *R i)
        using x y
        unfolding scaleR_left_distrib scaleR_zero_left if_smult
        by (simp add: sum_Un sum.distrib sum.inter_restrict[symmetric] **)
```

also have ... = $u *_R (\sum v{\in}sx.\ ux\ v *_R v) + v *_R (\sum v{\in}sy.\ uy\ v *_R v)$
unfolding *scaleR_scaleR*[*symmetric*] *scaleR_right.sum* [*symmetric*] **by** *blast*
finally show $(\sum i{\in}sx \cup sy.\ ((if\ i \in sx\ then\ u * ux\ i\ else\ 0) + (if\ i \in sy\ then\ v * uy\ i\ else\ 0)) *_R i)$
$= u *_R (\sum v{\in}sx.\ ux\ v *_R v) + v *_R (\sum v{\in}sy.\ uy\ v *_R v)$.
qed (*use x y* **in** *auto*)
qed
qed

lemma *affine_hull_finite*:
assumes *finite S*
shows *affine hull S* = $\{y.\ \exists u.\ sum\ u\ S = 1 \wedge sum\ (\lambda v.\ u\ v *_R v)\ S = y\}$
proof –
have $*$: $\exists h.\ sum\ h\ S = 1 \wedge (\sum v{\in}S.\ h\ v *_R v) = x$
if $F \subseteq S$ *finite F* $F \neq \{\}$ **and** *sum*: *sum u F* = *1* **and** *x*: $(\sum v{\in}F.\ u\ v *_R v) = x$ **for** *x F u*
proof –
have $S \cap F = F$
using *that* **by** *auto*
show *?thesis*
proof (*intro exI conjI*)
show $(\sum x{\in}S.\ if\ x \in F\ then\ u\ x\ else\ 0) = 1$
by (*metis* (*mono_tags*, *lifting*) ‹$S \cap F = F$› *assms sum.inter_restrict sum*)
show $(\sum v{\in}S.\ (if\ v \in F\ then\ u\ v\ else\ 0) *_R v) = x$
by (*simp add*: *if_smult cong*: *if_cong*) (*metis* (*no_types*) ‹$S \cap F = F$› *assms sum.inter_restrict x*)
qed
qed
show *?thesis*
unfolding *affine_hull_explicit* **using** *assms*
by (*fastforce dest*: $*$)
qed

Stepping theorems and hence small special cases

lemma *affine_hull_empty*[*simp*]: *affine hull* {} = {}
by *simp*

lemma *affine_hull_finite_step*:
fixes *y* :: *'a::real_vector*
shows *finite S* $\Longrightarrow$
$(\exists u.\ sum\ u\ (insert\ a\ S) = w \wedge sum\ (\lambda x.\ u\ x *_R x)\ (insert\ a\ S) = y) \longleftrightarrow$
$(\exists v\ u.\ sum\ u\ S = w - v \wedge sum\ (\lambda x.\ u\ x *_R x)\ S = y - v *_R a)$ (**is** _ $\Longrightarrow$ *?lhs* = *?rhs*)
proof –
assume *fin*: *finite S*
show *?lhs* = *?rhs*
proof
assume *?lhs*

then obtain *u* **where** *u*: *sum u* (*insert a S*) = *w* ∧ (∑*x*∈*insert a S*. *u x* $*_R$ *x*) = *y*
  by *auto*
show *?rhs*
proof (*cases a* ∈ *S*)
  case *True*
  then show *?thesis*
  using *u* **by** (*simp add*: *insert_absorb*) (*metis diff_zero real_vector.scale_zero_left*)
next
  case *False*
  show *?thesis*
    by (*rule exI* [**where** *x*=*u a*]) (*use u fin False* **in** *auto*)
qed
next
assume *?rhs*
then obtain *v u* **where** *vu*: *sum u S* = *w* − *v* (∑*x*∈*S*. *u x* $*_R$ *x*) = *y* − *v* $*_R$ *a*
  by *auto*
have ∗: ⋀*x M*. (*if x* = *a then v else M*) $*_R$ *x* = (*if x* = *a then v* $*_R$ *x else M* $*_R$ *x*)
  by *auto*
show *?lhs*
proof (*cases a* ∈ *S*)
  case *True*
  show *?thesis*
    by (*rule exI* [**where** *x*=λ*x*. (*if x*=*a then v else 0*) + *u x*])
      (*simp add*: *True scaleR_left_distrib sum.distrib sum_clauses fin vu* ∗ *cong*: *if_cong*)
next
  case *False*
  then show *?thesis*
    apply (*rule_tac x*=λ*x*. *if x*=*a then v else u x* **in** *exI*)
    apply (*simp add*: *vu sum_clauses*(*2*)[*OF fin*] ∗)
    by (*simp add*: *sum_delta_notmem*(*3*) *vu*)
qed
qed
qed

lemma *affine_hull_2*:
fixes *a b* :: ′*a*::*real_vector*
shows *affine hull* {*a*,*b*} = {*u* $*_R$ *a* + *v* $*_R$ *b*| *u v*. (*u* + *v* = *1*)}
(**is** *?lhs* = *?rhs*)
proof −
have ∗:
  ⋀*x y z*. *z* = *x* − *y* ⟷ *y* + *z* = (*x*::*real*)
  ⋀*x y z*. *z* = *x* − *y* ⟷ *y* + *z* = (*x*::′*a*) **by** *auto*
have *?lhs* = {*y*. ∃ *u*. *sum u* {*a*, *b*} = *1* ∧ (∑*v*∈{*a*, *b*}. *u v* $*_R$ *v*) = *y*}
  using *affine_hull_finite*[*of* {*a*,*b*}] **by** *auto*
also have . . . = {*y*. ∃ *v u*. *u b* = *1* − *v* ∧ *u b* $*_R$ *b* = *y* − *v* $*_R$ *a*}

```
    by (simp add: affine_hull_finite_step[of {b} a])
  also have ... = ?rhs unfolding * by auto
  finally show ?thesis by auto
qed

lemma affine_hull_3:
  fixes a b c :: 'a::real_vector
  shows affine hull {a,b,c} = { u *R a + v *R b + w *R c| u v w. u + v + w =
1}
proof -
  have *:
    ⋀x y z. z = x - y ⟷ y + z = (x::real)
    ⋀x y z. z = x - y ⟷ y + z = (x::'a) by auto
  show ?thesis
    apply (simp add: affine_hull_finite affine_hull_finite_step)
    unfolding *
    apply safe
     apply (metis add.assoc)
    apply (rule_tac x=u in exI, force)
    done
qed

lemma mem_affine:
  assumes affine S x ∈ S y ∈ S u + v = 1
  shows u *R x + v *R y ∈ S
  using assms affine_def[of S] by auto

lemma mem_affine_3:
  assumes affine S x ∈ S y ∈ S z ∈ S u + v + w = 1
  shows u *R x + v *R y + w *R z ∈ S
proof -
  have u *R x + v *R y + w *R z ∈ affine hull {x, y, z}
    using affine_hull_3[of x y z] assms by auto
  moreover
  have affine hull {x, y, z} ⊆ affine hull S
    using hull_mono[of {x, y, z} S] assms by auto
  moreover
  have affine hull S = S
    using assms affine_hull_eq[of S] by auto
  ultimately show ?thesis by auto
qed

lemma mem_affine_3_minus:
  assumes affine S x ∈ S y ∈ S z ∈ S
  shows x + v *R (y-z) ∈ S
  using mem_affine_3[of S x y z 1 v -v] assms
  by (simp add: algebra_simps)

corollary mem_affine_3_minus2:
```

⟦*affine S*; $x \in S$; $y \in S$; $z \in S$⟧ $\Longrightarrow x - v *_R (y-z) \in S$
by (*metis add_uminus_conv_diff mem_affine_3_minus real_vector.scale_minus_left*)

Some relations between affine hull and subspaces

lemma *affine_hull_insert_subset_span*:
affine hull (*insert a S*) $\subseteq$ {$a + v$| v . $v \in$ *span* {$x - a$ | x . $x \in S$}}
proof −
have $\exists\, v\ T\ u.\ x = a + v \wedge$ (*finite T* $\wedge$ $T \subseteq \{x - a\ |x.\ x \in S\} \wedge (\sum v{\in}T.\ u\ v *_R v) = v$)
if *finite F* $F \neq \{\}$ $F \subseteq$ *insert a S sum u F* = *1* ($\sum v{\in}F.\ u\ v *_R v) = x$
for $x\ F\ u$
proof −
have ∗: ($\lambda x.\ x - a$) ‘ ($F - \{a\}$) $\subseteq \{x - a\ |x.\ x \in S\}$
using *that* **by** *auto*
show *?thesis*
proof (*intro exI conjI*)
show *finite* (($\lambda x.\ x - a$) ‘ ($F - \{a\}$))
by (*simp add*: *that*(*1*))
show ($\sum v{\in}(\lambda x.\ x - a)$ ‘ ($F - \{a\}$). $u(v+a) *_R v) = x-a$
by (*simp add*: *sum.reindex*[*unfolded inj_on_def*] *algebra_simps*
sum_subtractf scaleR_left.sum[*symmetric*] *sum_diff1 that*)
qed (*use* ‹$F \subseteq$ *insert a S*› **in** *auto*)
qed
then show *?thesis*
unfolding *affine_hull_explicit span_explicit* **by** *fast*
qed

lemma *affine_hull_insert_span*:
assumes $a \notin S$
shows *affine hull* (*insert a S*) = {$a + v$ | v . $v \in$ *span* {$x - a$ | x. $x \in S$}}
proof −
have ∗: $\exists\, G\ u.$ *finite G* $\wedge G \neq \{\} \wedge G \subseteq$ *insert a S* $\wedge$ *sum u G* = *1* $\wedge (\sum v{\in}G.\ u\ v *_R v) = y$
if $v \in$ *span* $\{x - a\ |x.\ x \in S\}$ $y = a + v$ **for** $y\ v$
proof −
from *that*
obtain $T\ u$ **where** u: *finite T* $T \subseteq \{x - a\ |x.\ x \in S\}$ $a + (\sum v{\in}T.\ u\ v *_R v) = y$
unfolding *span_explicit* **by** *auto*
define F **where** $F = (\lambda x.\ x + a)$ ‘ T
have F: *finite F* $F \subseteq S$ ($\sum v{\in}F.\ u\ (v - a) *_R (v - a)) = y - a$
unfolding *F_def* **using** u **by** (*auto simp*: *sum.reindex*[*unfolded inj_on_def*])
have ∗: $F \cap \{a\} = \{\}$ $F \cap - \{a\} = F$
using F *assms* **by** *auto*
show $\exists\, G\ u.$ *finite G* $\wedge G \neq \{\} \wedge G \subseteq$ *insert a S* $\wedge$ *sum u G* = *1* $\wedge (\sum v{\in}G.\ u\ v *_R v) = y$
apply (*rule_tac x* = *insert a F* **in** *exI*)
apply (*rule_tac x* = $\lambda x.$ *if x=a then 1* − *sum* ($\lambda x.\ u\ (x - a)$) *F else u* ($x -$

```
a) in exI)
      using assms F
    apply (auto simp: sum_clauses sum.If_cases if_smult sum_subtractf scaleR_left.sum
algebra_simps ∗)
      done
  qed
  show ?thesis
   by (intro subset_antisym affine_hull_insert_subset_span) (auto simp: affine_hull_explicit
dest!: ∗)
qed

lemma affine_hull_span:
  assumes a ∈ S
  shows affine hull S = {a + v | v. v ∈ span {x − a | x. x ∈ S − {a}}}
   using affine_hull_insert_span[of a S − {a}, unfolded insert_Diff [OF assms]] by
auto
```

Parallel affine sets

```
definition affine_parallel :: 'a::real_vector set ⇒ 'a::real_vector set ⇒ bool
  where affine_parallel S T ⟷ (∃ a. T = (λx. a + x) ‘ S)

lemma affine_parallel_expl_aux:
  fixes S T :: 'a::real_vector set
  assumes ⋀x. x ∈ S ⟷ a + x ∈ T
  shows T = (λx. a + x) ‘ S
proof −
  have x ∈ ((λx. a + x) ‘ S) if x ∈ T for x
    using that
    by (simp add: image_iff) (metis add.commute diff_add_cancel assms)
  moreover have T ≥ (λx. a + x) ‘ S
    using assms by auto
  ultimately show ?thesis by auto
qed

lemma affine_parallel_expl: affine_parallel S T ⟷ (∃ a. ∀ x. x ∈ S ⟷ a + x ∈
T)
  by (auto simp add: affine_parallel_def)
    (use affine_parallel_expl_aux [of S _ T] in blast)

lemma affine_parallel_reflex: affine_parallel S S
  unfolding affine_parallel_def
  using image_add_0 by blast

lemma affine_parallel_commut:
  assumes affine_parallel A B
  shows affine_parallel B A
proof −
  from assms obtain a where B: B = (λx. a + x) ‘ A
```

```
    unfolding affine_parallel_def by auto
  have [simp]: (λx. x − a) = plus (− a) by (simp add: fun_eq_iff)
  from B show ?thesis
    using translation_galois [of B a A]
    unfolding affine_parallel_def by blast
qed

lemma affine_parallel_assoc:
  assumes affine_parallel A B
    and affine_parallel B C
  shows affine_parallel A C
proof −
  from assms obtain ab where B = (λx. ab + x) ‘ A
    unfolding affine_parallel_def by auto
  moreover
  from assms obtain bc where C = (λx. bc + x) ‘ B
    unfolding affine_parallel_def by auto
  ultimately show ?thesis
    using translation_assoc[of bc ab A] unfolding affine_parallel_def by auto
qed

lemma affine_translation_aux:
  fixes a :: 'a::real_vector
  assumes affine ((λx. a + x) ‘ S)
  shows affine S
proof −
  {
    fix x y u v
    assume xy: x ∈ S y ∈ S (u :: real) + v = 1
    then have (a + x) ∈ ((λx. a + x) ‘ S) (a + y) ∈ ((λx. a + x) ‘ S)
      by auto
    then have h1: u *R  (a + x) + v *R (a + y) ∈ (λx. a + x) ‘ S
      using xy assms unfolding affine_def by auto
    have u *R (a + x) + v *R (a + y) = (u + v) *R a + (u *R x + v *R y)
      by (simp add: algebra_simps)
    also have ... = a + (u *R x + v *R y)
      using ‹u + v = 1› by auto
    ultimately have a + (u *R x + v *R y) ∈ (λx. a + x) ‘ S
      using h1 by auto
    then have u *R x + v *R y ∈ S by auto
  }
  then show ?thesis unfolding affine_def by auto
qed

lemma affine_translation:
  affine S ⟷ affine ((+) a ‘ S) for a :: 'a::real_vector
proof
  show affine ((+) a ‘ S) if affine S
    using that translation_assoc [of − a a S]
```

```
    by (auto intro: affine_translation_aux [of − a ((+) a ‘ S)])
  show affine S if affine ((+) a ‘ S)
    using that by (rule affine_translation_aux)
qed

lemma parallel_is_affine:
  fixes S T :: 'a::real_vector set
  assumes affine S affine_parallel S T
  shows affine T
proof −
  from assms obtain a where T = (λx. a + x) ‘ S
    unfolding affine_parallel_def by auto
  then show ?thesis
    using affine_translation assms by auto
qed

lemma subspace_imp_affine: subspace s ⟹ affine s
  unfolding subspace_def affine_def by auto

lemma affine_hull_subset_span: (affine hull s) ⊆ (span s)
  by (metis hull_minimal span_superset subspace_imp_affine subspace_span)
```

Subspace parallel to an affine set

```
lemma subspace_affine: subspace S ⟷ affine S ∧ 0 ∈ S
proof −
  have h0: subspace S ⟹ affine S ∧ 0 ∈ S
    using subspace_imp_affine[of S] subspace_0 by auto
  {
    assume assm: affine S ∧ 0 ∈ S
    {
      fix c :: real
      fix x
      assume x: x ∈ S
      have c *R x = (1−c) *R 0 + c *R x by auto
      moreover
      have (1 − c) *R 0 + c *R x ∈ S
        using affine_alt[of S] assm x by auto
      ultimately have c *R x ∈ S by auto
    }
    then have h1: ∀ c. ∀ x ∈ S. c *R x ∈ S by auto

    {
      fix x y
      assume xy: x ∈ S y ∈ S
      define u where u = (1 :: real)/2
      have (1/2) *R (x+y) = (1/2) *R (x+y)
        by auto
      moreover
```

```
      have (1/2) *R (x+y)=(1/2) *R x + (1−(1/2)) *R y
        by (simp add: algebra_simps)
      moreover
      have (1 − u) *R x + u *R y ∈ S
        using affine_alt[of S] assm xy by auto
      ultimately
      have (1/2) *R (x+y) ∈ S
        using u_def by auto
      moreover
      have x + y = 2 *R ((1/2) *R (x+y))
        by auto
      ultimately
      have x + y ∈ S
        using h1[rule_format, of (1/2) *R (x+y) 2] by auto
    }
    then have ∀ x ∈ S. ∀ y ∈ S. x + y ∈ S
      by auto
    then have subspace S
      using h1 assm unfolding subspace_def by auto
  }
  then show ?thesis using h0 by metis
qed

lemma affine_diffs_subspace:
  assumes affine S a ∈ S
  shows subspace ((λx. (−a)+x) ‘ S)
proof −
  have [simp]: (λx. x − a) = plus (− a) by (simp add: fun_eq_iff)
  have affine ((λx. (−a)+x) ‘ S)
    using affine_translation assms by blast
  moreover have 0 ∈ ((λx. (−a)+x) ‘ S)
    using assms exI[of (λx. x∈S ∧ −a+x = 0) a] by auto
  ultimately show ?thesis using subspace_affine by auto
qed

lemma affine_diffs_subspace_subtract:
  subspace ((λx. x − a) ‘ S) if affine S a ∈ S
  using that affine_diffs_subspace [of _ a] by simp

lemma parallel_subspace_explicit:
  assumes affine S
    and a ∈ S
  assumes L ≡ {y. ∃ x ∈ S. (−a) + x = y}
  shows subspace L ∧ affine_parallel S L
proof −
  from assms have L = plus (− a) ‘ S by auto
  then have par: affine_parallel S L
    unfolding affine_parallel_def ..
  then have affine L using assms parallel_is_affine by auto
```

```
  moreover have 0 ∈ L
    using assms by auto
  ultimately show ?thesis
    using subspace_affine par by auto
qed

lemma parallel_subspace_aux:
  assumes subspace A
    and subspace B
    and affine_parallel A B
  shows A ⊇ B
proof -
  from assms obtain a where a: ∀x. x ∈ A ⟷ a + x ∈ B
    using affine_parallel_expl[of A B] by auto
  then have -a ∈ A
    using assms subspace_0[of B] by auto
  then have a ∈ A
    using assms subspace_neg[of A -a] by auto
  then show ?thesis
    using assms a unfolding subspace_def by auto
qed

lemma parallel_subspace:
  assumes subspace A
    and subspace B
    and affine_parallel A B
  shows A = B
proof
  show A ⊇ B
    using assms parallel_subspace_aux by auto
  show A ⊆ B
    using assms parallel_subspace_aux[of B A] affine_parallel_commut by auto
qed

lemma affine_parallel_subspace:
  assumes affine S S ≠ {}
  shows ∃!L. subspace L ∧ affine_parallel S L
proof -
  have ex: ∃L. subspace L ∧ affine_parallel S L
    using assms parallel_subspace_explicit by auto
  {
    fix L1 L2
    assume ass: subspace L1 ∧ affine_parallel S L1 subspace L2 ∧ affine_parallel S
L2
    then have affine_parallel L1 L2
        using affine_parallel_commut[of S L1] affine_parallel_assoc[of L1 S L2] by
auto
    then have L1 = L2
      using ass parallel_subspace by auto
```

}
then show *?thesis* **using** *ex* **by** *auto*
qed

1.6.2 Affine Dependence

Formalized by Lars Schewe.

definition *affine_dependent* :: *'a::real_vector set* ⇒ *bool*
where *affine_dependent s* ⟷ (∃*x*∈*s*. *x* ∈ *affine hull* (*s* − {*x*}))

lemma *affine_dependent_imp_dependent*: *affine_dependent s* ⟹ *dependent s*
unfolding *affine_dependent_def dependent_def*
using *affine_hull_subset_span* **by** *auto*

lemma *affine_dependent_subset*:
⟦*affine_dependent s*; *s* ⊆ *t*⟧ ⟹ *affine_dependent t*
apply (*simp add*: *affine_dependent_def Bex_def*)
apply (*blast dest*: *hull_mono* [*OF Diff_mono* [*OF _ subset_refl*]])
done

lemma *affine_independent_subset*:
shows ⟦¬ *affine_dependent t*; *s* ⊆ *t*⟧ ⟹ ¬ *affine_dependent s*
by (*metis affine_dependent_subset*)

lemma *affine_independent_Diff*:
¬ *affine_dependent s* ⟹ ¬ *affine_dependent*(*s* − *t*)
by (*meson Diff_subset affine_dependent_subset*)

proposition *affine_dependent_explicit*:
affine_dependent p ⟷
(∃*S u*. *finite S* ∧ *S* ⊆ *p* ∧ *sum u S* = *0* ∧ (∃*v*∈*S*. *u v* ≠ *0*) ∧ *sum* (λ*v*. *u v* $*_R$ *v*) *S* = *0*)
proof −
have ∃*S u*. *finite S* ∧ *S* ⊆ *p* ∧ *sum u S* = *0* ∧ (∃*v*∈*S*. *u v* ≠ *0*) ∧ (∑*w*∈*S*. *u w* $*_R$ *w*) = *0*
if (∑*w*∈*S*. *u w* $*_R$ *w*) = *x x* ∈ *p finite S S* ≠ {} *S* ⊆ *p* − {*x*} *sum u S* = *1* **for** *x S u*
proof (*intro exI conjI*)
have *x* ∉ *S*
using *that* **by** *auto*
then show (∑ *v* ∈ *insert x S*. *if v* = *x then* − *1 else u v*) = *0*
using *that* **by** (*simp add*: *sum_delta_notmem*)
show (∑ *w* ∈ *insert x S*. (*if w* = *x then* − *1 else u w*) $*_R$ *w*) = *0*
using *that* ‹*x* ∉ *S*› **by** (*simp add*: *if_smult sum_delta_notmem cong*: *if_cong*)
qed (*use that* **in** *auto*)
moreover have ∃*x*∈*p*. ∃*S u*. *finite S* ∧ *S* ≠ {} ∧ *S* ⊆ *p* − {*x*} ∧ *sum u S* = *1* ∧ (∑*v*∈*S*. *u v* $*_R$ *v*) = *x*
if (∑*v*∈*S*. *u v* $*_R$ *v*) = *0 finite S S* ⊆ *p sum u S* = *0 v* ∈ *S u v* ≠ *0* **for** *S u v*
proof (*intro bexI exI conjI*)

```
    have S ≠ {v}
      using that by auto
    then show S − {v} ≠ {}
      using that by auto
    show (∑x ∈ S − {v}. − (1 / u v) ∗ u x) = 1
     unfolding sum_distrib_left[symmetric] sum_diff1[OF ‹finite S›] by (simp add:
that)
    show (∑x∈S − {v}. (− (1 / u v) ∗ u x) ∗R x) = v
      unfolding sum_distrib_left [symmetric] scaleR_scaleR[symmetric]
              scaleR_right.sum [symmetric] sum_diff1[OF ‹finite S›]
      using that by auto
    show S − {v} ⊆ p − {v}
      using that by auto
  qed (use that in auto)
  ultimately show ?thesis
    unfolding affine_dependent_def affine_hull_explicit by auto
qed

lemma affine_dependent_explicit_finite:
  fixes S :: 'a::real_vector set
  assumes finite S
  shows affine_dependent S ⟷
    (∃u. sum u S = 0 ∧ (∃v∈S. u v ≠ 0) ∧ sum (λv. u v ∗R v) S = 0)
  (is ?lhs = ?rhs)
proof
  have ∗: ⋀vt u v. (if vt then u v else 0) ∗R v = (if vt then (u v) ∗R v else 0::'a)
    by auto
  assume ?lhs
  then obtain t u v where
    finite t t ⊆ S sum u t = 0 v∈t u v ≠ 0 (∑v∈t. u v ∗R v) = 0
    unfolding affine_dependent_explicit by auto
  then show ?rhs
    apply (rule_tac x=λx. if x∈t then u x else 0 in exI)
    apply (auto simp: ∗ sum.inter_restrict[OF assms, symmetric] Int_absorb1[OF
‹t⊆S›])
    done
next
  assume ?rhs
  then obtain u v where sum u S = 0 v∈S u v ≠ 0 (∑v∈S. u v ∗R v) = 0
    by auto
  then show ?lhs unfolding affine_dependent_explicit
    using assms by auto
qed

lemma dependent_imp_affine_dependent:
  assumes dependent {x − a| x . x ∈ s}
    and a ∉ s
  shows affine_dependent (insert a s)
proof −
```

from *assms*(*1*)[*unfolded dependent_explicit*] **obtain** *S u v*
where *obt*: *finite S S* ⊆ {*x* − *a* |*x*. *x* ∈ *s*} *v*∈*S u v* ≠ *0* (∑ *v*∈*S*. *u v* ∗*R* *v*) = *0*
by *auto*
define *t* **where** *t* = (λ*x*. *x* + *a*) ‘ *S*

have *inj*: *inj_on* (λ*x*. *x* + *a*) *S*
unfolding *inj_on_def* **by** *auto*
have *0* ∉ *S*
using *obt*(*2*) *assms*(*2*) **unfolding** *subset_eq* **by** *auto*
have *fin*: *finite t* **and** *t* ⊆ *s*
unfolding *t_def* **using** *obt*(*1*,*2*) **by** *auto*
then have *finite* (*insert a t*) **and** *insert a t* ⊆ *insert a s*
by *auto*
moreover have ∗: ⋀*P Q*. (∑ *x*∈*t*. (*if x* = *a then P x else Q x*)) = (∑ *x*∈*t*. *Q x*)
apply (*rule sum.cong*)
using ‹*a*∉*s*› ‹*t*⊆*s*›
apply *auto*
done
have (∑ *x*∈*insert a t*. *if x* = *a then* − (∑ *x*∈*t*. *u* (*x* − *a*)) *else u* (*x* − *a*)) = *0*
unfolding *sum_clauses*(*2*)[*OF fin*] ∗ **using** ‹*a*∉*s*› ‹*t*⊆*s*› **by** *auto*
moreover have ∃ *v*∈*insert a t*. (*if v* = *a then* − (∑ *x*∈*t*. *u* (*x* − *a*)) *else u* (*v* − *a*)) ≠ *0*
using *obt*(*3*,*4*) ‹*0*∉*S*›
by (*rule_tac x*=*v* + *a* **in** *bexI*) (*auto simp*: *t_def*)
moreover have ∗: ⋀*P Q*. (∑ *x*∈*t*. (*if x* = *a then P x else Q x*) ∗*R* *x*) = (∑ *x*∈*t*. *Q x* ∗*R* *x*)
using ‹*a*∉*s*› ‹*t*⊆*s*› **by** (*auto intro*!: *sum.cong*)
have (∑ *x*∈*t*. *u* (*x* − *a*)) ∗*R* *a* = (∑ *v*∈*t*. *u* (*v* − *a*) ∗*R* *v*)
unfolding *scaleR_left.sum*
unfolding *t_def* **and** *sum.reindex*[*OF inj*] **and** *o_def*
using *obt*(*5*)
by (*auto simp*: *sum.distrib scaleR_right_distrib*)
then have (∑ *v*∈*insert a t*. (*if v* = *a then* − (∑ *x*∈*t*. *u* (*x* − *a*)) *else u* (*v* − *a*)) ∗*R* *v*) = *0*
unfolding *sum_clauses*(*2*)[*OF fin*]
using ‹*a*∉*s*› ‹*t*⊆*s*›
by (*auto simp*: ∗)
ultimately show *?thesis*
unfolding *affine_dependent_explicit*
apply (*rule_tac x*=*insert a t* **in** *exI*, *auto*)
done
qed

lemma *affine_dependent_biggerset*:
fixes *s* :: ′*a*::*euclidean_space set*
assumes *finite s card s* ≥ *DIM*(′*a*) + *2*
shows *affine_dependent s*

```
proof –
  have s ≠ {} using assms by auto
  then obtain a where a∈s by auto
  have *: {x − a |x. x ∈ s − {a}} = (λx. x − a) ‘ (s − {a})
    by auto
  have card {x − a |x. x ∈ s − {a}} = card (s − {a})
    unfolding * by (simp add: card_image inj_on_def)
  also have . . . > DIM('a) using assms(2)
    unfolding card_Diff_singleton[OF assms(1) ‹a∈s›] by auto
  finally show ?thesis
    apply (subst insert_Diff[OF ‹a∈s›, symmetric])
    apply (rule dependent_imp_affine_dependent)
    apply (rule dependent_biggerset, auto)
    done
qed

lemma affine_dependent_biggerset_general:
  assumes finite (S :: 'a::euclidean_space set)
    and card S ≥ dim S + 2
  shows affine_dependent S
proof –
  from assms(2) have S ≠ {} by auto
  then obtain a where a∈S by auto
  have *: {x − a |x. x ∈ S − {a}} = (λx. x − a) ‘ (S − {a})
    by auto
  have **: card {x − a |x. x ∈ S − {a}} = card (S − {a})
    by (metis (no_types, lifting) * card_image diff_add_cancel inj_on_def)
  have dim {x − a |x. x ∈ S − {a}} ≤ dim S
    using ‹a∈S› by (auto simp: span_base span_diff intro: subset_le_dim)
  also have . . . < dim S + 1 by auto
  also have . . . ≤ card (S − {a})
    using assms
    using card_Diff_singleton[OF assms(1) ‹a∈S›]
    by auto
  finally show ?thesis
    apply (subst insert_Diff[OF ‹a∈S›, symmetric])
    apply (rule dependent_imp_affine_dependent)
    apply (rule dependent_biggerset_general)
    unfolding **
    apply auto
    done
qed
```

1.6.3 Some Properties of Affine Dependent Sets

```
lemma affine_independent_0 [simp]: ¬ affine_dependent {}
  by (simp add: affine_dependent_def)

lemma affine_independent_1 [simp]: ¬ affine_dependent {a}
```

by (*simp add*: *affine_dependent_def*)

lemma *affine_independent_2* [*simp*]: ¬ *affine_dependent* {*a*,*b*}
by (*simp add*: *affine_dependent_def insert_Diff_if hull_same*)

lemma *affine_hull_translation*: *affine hull* ((λ*x*. *a* + *x*) ‘ *S*) = (λ*x*. *a* + *x*) ‘ (*affine hull S*)
proof −
have *affine* ((λ*x*. *a* + *x*) ‘ (*affine hull S*))
using *affine_translation affine_affine_hull* **by** *blast*
moreover have (λ*x*. *a* + *x*) ‘ *S* ⊆ (λ*x*. *a* + *x*) ‘ (*affine hull S*)
using *hull_subset*[*of S*] **by** *auto*
ultimately have *h1*: *affine hull* ((λ*x*. *a* + *x*) ‘ *S*) ⊆ (λ*x*. *a* + *x*) ‘ (*affine hull S*)
by (*metis hull_minimal*)
have *affine*((λ*x*. −*a* + *x*) ‘ (*affine hull* ((λ*x*. *a* + *x*) ‘ *S*)))
using *affine_translation affine_affine_hull* **by** *blast*
moreover have (λ*x*. −*a* + *x*) ‘ (λ*x*. *a* + *x*) ‘ *S* ⊆ (λ*x*. −*a* + *x*) ‘ (*affine hull* ((λ*x*. *a* + *x*) ‘ *S*))
using *hull_subset*[*of* (λ*x*. *a* + *x*) ‘ *S*] **by** *auto*
moreover have *S* = (λ*x*. −*a* + *x*) ‘ (λ*x*. *a* + *x*) ‘ *S*
using *translation_assoc*[*of* −*a a*] **by** *auto*
ultimately have (λ*x*. −*a* + *x*) ‘ (*affine hull* ((λ*x*. *a* + *x*) ‘ *S*)) >= (*affine hull S*)
by (*metis hull_minimal*)
then have *affine hull* ((λ*x*. *a* + *x*) ‘ *S*) >= (λ*x*. *a* + *x*) ‘ (*affine hull S*)
by *auto*
then show *?thesis* **using** *h1* **by** *auto*
qed

lemma *affine_dependent_translation*:
assumes *affine_dependent S*
shows *affine_dependent* ((λ*x*. *a* + *x*) ‘ *S*)
proof −
obtain *x* **where** *x*: *x* ∈ *S* ∧ *x* ∈ *affine hull* (*S* − {*x*})
using *assms affine_dependent_def* **by** *auto*
have (+) *a* ‘ (*S* − {*x*}) = (+) *a* ‘ *S* − {*a* + *x*}
by *auto*
then have *a* + *x* ∈ *affine hull* ((λ*x*. *a* + *x*) ‘ *S* − {*a* + *x*})
using *affine_hull_translation*[*of a S* − {*x*}] *x* **by** *auto*
moreover have *a* + *x* ∈ (λ*x*. *a* + *x*) ‘ *S*
using *x* **by** *auto*
ultimately show *?thesis*
unfolding *affine_dependent_def* **by** *auto*
qed

lemma *affine_dependent_translation_eq*:
affine_dependent S ⟷ *affine_dependent* ((λ*x*. *a* + *x*) ‘ *S*)
proof −

```
  {
    assume affine_dependent ((λx. a + x) ` S)
    then have affine_dependent S
     using affine_dependent_translation[of ((λx. a + x) ` S) −a] translation_assoc[of
−a a]
      by auto
  }
  then show ?thesis
    using affine_dependent_translation by auto
qed

lemma affine_hull_0_dependent:
  assumes 0 ∈ affine hull S
  shows dependent S
proof −
  obtain s u where s_u: finite s ∧ s ≠ {} ∧ s ⊆ S ∧ sum u s = 1 ∧ (∑ v∈s. u
v *R v) = 0
    using assms affine_hull_explicit[of S] by auto
  then have ∃ v∈s. u v ≠ 0 by auto
  then have finite s ∧ s ⊆ S ∧ (∃ v∈s. u v ≠ 0 ∧ (∑ v∈s. u v *R v) = 0)
    using s_u by auto
  then show ?thesis
    unfolding dependent_explicit[of S] by auto
qed

lemma affine_dependent_imp_dependent2:
  assumes affine_dependent (insert 0 S)
  shows dependent S
proof −
  obtain x where x: x ∈ insert 0 S ∧ x ∈ affine hull (insert 0 S − {x})
    using affine_dependent_def[of (insert 0 S)] assms by blast
  then have x ∈ span (insert 0 S − {x})
    using affine_hull_subset_span by auto
  moreover have span (insert 0 S − {x}) = span (S − {x})
    using insert_Diff_if[of 0 S {x}] span_insert_0[of S−{x}] by auto
  ultimately have x ∈ span (S − {x}) by auto
  then have x ≠ 0 ⟹ dependent S
    using x dependent_def by auto
  moreover
  {
    assume x = 0
    then have 0 ∈ affine hull S
      using x hull_mono[of S − {0} S] by auto
    then have dependent S
      using affine_hull_0_dependent by auto
  }
  ultimately show ?thesis by auto
qed
```

lemma *affine_dependent_iff_dependent*:
  assumes $a \notin S$
  shows *affine_dependent* (*insert a S*) $\longleftrightarrow$ *dependent* $((\lambda x. -a + x)$ ‘ $S)$
proof −
  have $((+)\ (-\ a)$ ‘ $S) = \{x - a|\ x\ .\ x \in S\}$ **by** *auto*
  then show *?thesis*
    using *affine_dependent_translation_eq*[*of* (*insert a S*) −*a*]
      affine_dependent_imp_dependent2 assms
      dependent_imp_affine_dependent[*of a S*]
    by (*auto simp del*: *uminus_add_conv_diff*)
qed

lemma *affine_dependent_iff_dependent2*:
  assumes $a \in S$
  shows *affine_dependent S* $\longleftrightarrow$ *dependent* $((\lambda x. -a + x)$ ‘ $(S-\{a\}))$
proof −
  have *insert a* $(S - \{a\}) = S$
    using *assms* **by** *auto*
  then show *?thesis*
    using *assms affine_dependent_iff_dependent*[*of a* $S-\{a\}$] **by** *auto*
qed

lemma *affine_hull_insert_span_gen*:
  affine hull (*insert a s*) $= (\lambda x.\ a + x)$ ‘ *span* $((\lambda x. -\ a + x)$ ‘ $s)$
proof −
  have *h1*: $\{x - a\ |x.\ x \in s\} = ((\lambda x. -a+x)$ ‘ $s)$
    by *auto*
  {
    assume $a \notin s$
    then have *?thesis*
      using *affine_hull_insert_span*[*of a s*] *h1* **by** *auto*
  }
  moreover
  {
    assume *a1*: $a \in s$
    have $\exists x.\ x \in s \wedge -a+x=0$
      apply (*rule exI*[*of* _ *a*])
      using *a1*
      apply *auto*
      done
    then have *insert 0* $((\lambda x. -a+x)$ ‘ $(s - \{a\})) = (\lambda x. -a+x)$ ‘ s
      by *auto*
    then have *span* $((\lambda x. -a+x)$ ‘ $(s - \{a\}))$=*span* $((\lambda x. -a+x)$ ‘ $s)$
        using *span_insert_0*[*of* $(+)\ (-\ a)$ ‘ $(s - \{a\})$] **by** (*auto simp del*: *uminus_add_conv_diff*)
    moreover have $\{x - a\ |x.\ x \in (s - \{a\})\} = ((\lambda x. -a+x)$ ‘ $(s - \{a\}))$
      by *auto*
    moreover have *insert a* $(s - \{a\})$ = *insert a s*
      by *auto*

```
    ultimately have ?thesis
      using affine_hull_insert_span[of a s−{a}] by auto
  }
  ultimately show ?thesis by auto
qed

lemma affine_hull_span2:
  assumes a ∈ s
  shows affine hull s = (λx. a+x) ` span ((λx. −a+x) ` (s−{a}))
  using affine_hull_insert_span_gen[of a s − {a}, unfolded insert_Diff[OF assms]]
  by auto

lemma affine_hull_span_gen:
  assumes a ∈ affine hull s
  shows affine hull s = (λx. a+x) ` span ((λx. −a+x) ` s)
proof −
  have affine hull (insert a s) = affine hull s
    using hull_redundant[of a affine s] assms by auto
  then show ?thesis
    using affine_hull_insert_span_gen[of a s] by auto
qed

lemma affine_hull_span_0:
  assumes 0 ∈ affine hull S
  shows affine hull S = span S
  using affine_hull_span_gen[of 0 S] assms by auto

lemma extend_to_affine_basis_nonempty:
  fixes S V :: 'n::real_vector set
  assumes ¬ affine_dependent S S ⊆ V S ≠ {}
  shows ∃ T. ¬ affine_dependent T ∧ S ⊆ T ∧ T ⊆ V ∧ affine hull T = affine
hull V
proof −
  obtain a where a: a ∈ S
    using assms by auto
  then have h0: independent ((λx. −a + x) ` (S−{a}))
    using affine_dependent_iff_dependent2 assms by auto
  obtain B where B:
   (λx. −a+x) ` (S − {a}) ⊆ B ∧ B ⊆ (λx. −a+x) ` V ∧ independent B ∧ (λx.
−a+x) ` V ⊆ span B
    using assms
    by (blast intro: maximal_independent_subset_extend[OF _ h0, of (λx. −a + x)
` V])
  define T where T = (λx. a+x) ` insert 0 B
  then have T = insert a ((λx. a+x) ` B)
    by auto
  then have affine hull T = (λx. a+x) ` span B
   using affine_hull_insert_span_gen[of a ((λx. a+x) ` B)] translation_assoc[of −a
a B]
```

```
    by auto
  then have V ⊆ affine hull T
    using B assms translation_inverse_subset[of a V span B]
    by auto
  moreover have T ⊆ V
    using T_def B a assms by auto
  ultimately have affine hull T = affine hull V
    by (metis Int_absorb1 Int_absorb2 hull_hull hull_mono)
  moreover have S ⊆ T
    using T_def B translation_inverse_subset[of a S−{a} B]
    by auto
  moreover have ¬ affine_dependent T
    using T_def affine_dependent_translation_eq[of insert 0 B]
      affine_dependent_imp_dependent2 B
    by auto
  ultimately show ?thesis using ‹T ⊆ V› by auto
qed

lemma affine_basis_exists:
  fixes V :: 'n::real_vector set
  shows ∃ B. B ⊆ V ∧ ¬ affine_dependent B ∧ affine hull V = affine hull B
proof (cases V = {})
  case True
  then show ?thesis
    using affine_independent_0 by auto
next
  case False
  then obtain x where x ∈ V by auto
  then show ?thesis
    using affine_dependent_def[of {x}] extend_to_affine_basis_nonempty[of {x} V]
    by auto
qed

proposition extend_to_affine_basis:
  fixes S V :: 'n::real_vector set
  assumes ¬ affine_dependent S S ⊆ V
  obtains T where ¬ affine_dependent T S ⊆ T T ⊆ V affine hull T = affine
hull V
proof (cases S = {})
  case True then show ?thesis
    using affine_basis_exists by (metis empty_subsetI that)
next
  case False
  then show ?thesis by (metis assms extend_to_affine_basis_nonempty that)
qed
```

1.6.4 Affine Dimension of a Set

```
definition aff_dim :: ('a::euclidean_space) set ⇒ int
```

```
  where aff_dim V =
  (SOME d :: int.
    ∃B. affine hull B = affine hull V ∧ ¬ affine_dependent B ∧ of_nat (card B) =
d + 1)

lemma aff_dim_basis_exists:
  fixes V :: ('n::euclidean_space) set
  shows ∃B. affine hull B = affine hull V ∧ ¬ affine_dependent B ∧ of_nat (card
B) = aff_dim V + 1
proof -
  obtain B where ¬ affine_dependent B ∧ affine hull B = affine hull V
    using affine_basis_exists[of V] by auto
  then show ?thesis
    unfolding aff_dim_def
      some_eq_ex[of λd. ∃B. affine hull B = affine hull V ∧ ¬ affine_dependent B
∧ of_nat (card B) = d + 1]
    apply auto
    apply (rule exI[of _ int (card B) − (1 :: int)])
    apply (rule exI[of _ B], auto)
    done
qed

lemma affine_hull_eq_empty [simp]: affine hull S = {} ⟷ S = {}
by (metis affine_empty subset_empty subset_hull)

lemma empty_eq_affine_hull[simp]: {} = affine hull S ⟷ S = {}
by (metis affine_hull_eq_empty)

lemma aff_dim_parallel_subspace_aux:
  fixes B :: 'n::euclidean_space set
  assumes ¬ affine_dependent B a ∈ B
  shows finite B ∧ ((card B) − 1 = dim (span ((λx. −a+x) ‘ (B−{a}))))
proof -
  have independent ((λx. −a + x) ‘ (B−{a}))
    using affine_dependent_iff_dependent2 assms by auto
  then have fin: dim (span ((λx. −a+x) ‘ (B−{a}))) = card ((λx. −a + x) ‘
(B−{a}))
    finite ((λx. −a + x) ‘ (B − {a}))
    using indep_card_eq_dim_span[of (λx. −a+x) ‘ (B−{a})] by auto
  show ?thesis
  proof (cases (λx. −a + x) ‘ (B − {a}) = {})
    case True
    have B = insert a ((λx. a + x) ‘ (λx. −a + x) ‘ (B − {a}))
      using translation_assoc[of a −a (B − {a})] assms by auto
    then have B = {a} using True by auto
    then show ?thesis using assms fin by auto
  next
    case False
    then have card ((λx. −a + x) ‘ (B − {a})) > 0
```

```
    using fin by auto
  moreover have h1: card ((λx. −a + x) ` (B−{a})) = card (B−{a})
    by (rule card_image) (use translate_inj_on in blast)
  ultimately have card (B−{a}) > 0 by auto
  then have ∗: finite (B − {a})
    using card_gt_0_iff[of (B − {a})] by auto
  then have card (B − {a}) = card B − 1
    using card_Diff_singleton assms by auto
  with ∗ show ?thesis using fin h1 by auto
 qed
qed

lemma aff_dim_parallel_subspace:
 fixes V L :: 'n::euclidean_space set
 assumes V ≠ {}
   and subspace L
   and affine_parallel (affine hull V) L
 shows aff_dim V = int (dim L)
proof −
 obtain B where
    B: affine hull B = affine hull V ∧ ¬ affine_dependent B ∧ int (card B) = aff_dim V + 1
   using aff_dim_basis_exists by auto
 then have B ≠ {}
   using assms B
   by auto
 then obtain a where a: a ∈ B by auto
 define Lb where Lb = span ((λx. −a+x) ` (B−{a}))
 moreover have affine_parallel (affine hull B) Lb
   using Lb_def B assms affine_hull_span2[of a B] a
     affine_parallel_commut[of Lb (affine hull B)]
   unfolding affine_parallel_def
   by auto
 moreover have subspace Lb
   using Lb_def subspace_span by auto
 moreover have affine hull B ≠ {}
   using assms B by auto
 ultimately have L = Lb
   using assms affine_parallel_subspace[of affine hull B] affine_affine_hull[of B] B
   by auto
 then have dim L = dim Lb
   by auto
 moreover have card B − 1 = dim Lb and finite B
   using Lb_def aff_dim_parallel_subspace_aux a B by auto
 ultimately show ?thesis
   using B ‹B ≠ {}› card_gt_0_iff[of B] by auto
qed

lemma aff_independent_finite:
```

fixes *B* :: *'n::euclidean_space set*
assumes ¬ *affine_dependent B*
shows *finite B*
proof −
{
assume *B* ≠ {}
then obtain *a* **where** *a* ∈ *B* **by** *auto*
then have *?thesis*
using *aff_dim_parallel_subspace_aux assms* **by** *auto*
}
then show *?thesis* **by** *auto*
qed

lemma *aff_dim_empty*:
fixes *S* :: *'n::euclidean_space set*
shows *S* = {} ⟷ *aff_dim S* = −*1*
proof −
obtain *B* **where** ∗: *affine hull B* = *affine hull S*
and ¬ *affine_dependent B*
and *int* (*card B*) = *aff_dim S* + *1*
using *aff_dim_basis_exists* **by** *auto*
moreover
from ∗ **have** *S* = {} ⟷ *B* = {}
by *auto*
ultimately show *?thesis*
using *aff_independent_finite*[*of B*] *card_gt_0_iff*[*of B*] **by** *auto*
qed

lemma *aff_dim_empty_eq* [*simp*]: *aff_dim* ({}::*'a::euclidean_space set*) = −*1*
by (*simp add*: *aff_dim_empty* [*symmetric*])

lemma *aff_dim_affine_hull* [*simp*]: *aff_dim* (*affine hull S*) = *aff_dim S*
unfolding *aff_dim_def* **using** *hull_hull*[*of _ S*] **by** *auto*

lemma *aff_dim_affine_hull2*:
assumes *affine hull S* = *affine hull T*
shows *aff_dim S* = *aff_dim T*
unfolding *aff_dim_def* **using** *assms* **by** *auto*

lemma *aff_dim_unique*:
fixes *B V* :: *'n::euclidean_space set*
assumes *affine hull B* = *affine hull V* ∧ ¬ *affine_dependent B*
shows *of_nat* (*card B*) = *aff_dim V* + *1*
proof (*cases B* = {})
case *True*
then have *V* = {}
using *assms*
by *auto*

```
  then have aff_dim V = (−1::int)
    using aff_dim_empty by auto
  then show ?thesis
    using ‹B = {}› by auto
next
  case False
  then obtain a where a: a ∈ B by auto
  define Lb where Lb = span ((λx. −a+x) ‘ (B−{a}))
  have affine_parallel (affine hull B) Lb
    using Lb_def affine_hull_span2[of a B] a
      affine_parallel_commut[of Lb (affine hull B)]
    unfolding affine_parallel_def by auto
  moreover have subspace Lb
    using Lb_def subspace_span by auto
  ultimately have aff_dim B = int(dim Lb)
    using aff_dim_parallel_subspace[of B Lb] ‹B ≠ {}› by auto
  moreover have (card B) − 1 = dim Lb finite B
    using Lb_def aff_dim_parallel_subspace_aux a assms by auto
  ultimately have of_nat (card B) = aff_dim B + 1
    using ‹B ≠ {}› card_gt_0_iff[of B] by auto
  then show ?thesis
    using aff_dim_affine_hull2 assms by auto
qed

lemma aff_dim_affine_independent:
  fixes B :: 'n::euclidean_space set
  assumes ¬ affine_dependent B
  shows of_nat (card B) = aff_dim B + 1
  using aff_dim_unique[of B B] assms by auto

lemma affine_independent_iff_card:
    fixes s :: 'a::euclidean_space set
    shows ¬ affine_dependent s ⟷ finite s ∧ aff_dim s = int(card s) − 1
  apply (rule iffI)
  apply (simp add: aff_dim_affine_independent aff_independent_finite)
   by (metis affine_basis_exists [of s] aff_dim_unique card_subset_eq diff_add_cancel
of_nat_eq_iff)

lemma aff_dim_sing [simp]:
  fixes a :: 'n::euclidean_space
  shows aff_dim {a} = 0
  using aff_dim_affine_independent[of {a}] affine_independent_1 by auto

lemma aff_dim_2 [simp]:
  fixes a :: 'n::euclidean_space
  shows aff_dim {a,b} = (if a = b then 0 else 1)
proof (clarsimp)
  assume a ≠ b
  then have aff_dim{a,b} = card{a,b} − 1
```

```
    using affine_independent_2 [of a b] aff_dim_affine_independent by fastforce
  also have ... = 1
    using ‹a ≠ b› by simp
  finally show aff_dim {a, b} = 1 .
qed

lemma aff_dim_inner_basis_exists:
  fixes V :: ('n::euclidean_space) set
  shows ∃B. B ⊆ V ∧ affine hull B = affine hull V ∧
    ¬ affine_dependent B ∧ of_nat (card B) = aff_dim V + 1
proof –
  obtain B where B: ¬ affine_dependent B B ⊆ V affine hull B = affine hull V
    using affine_basis_exists[of V] by auto
  then have of_nat(card B) = aff_dim V+1 using aff_dim_unique by auto
  with B show ?thesis by auto
qed

lemma aff_dim_le_card:
  fixes V :: 'n::euclidean_space set
  assumes finite V
  shows aff_dim V ≤ of_nat (card V) – 1
proof –
  obtain B where B: B ⊆ V of_nat (card B) = aff_dim V + 1
    using aff_dim_inner_basis_exists[of V] by auto
  then have card B ≤ card V
    using assms card_mono by auto
  with B show ?thesis by auto
qed

lemma aff_dim_parallel_eq:
  fixes S T :: 'n::euclidean_space set
  assumes affine_parallel (affine hull S) (affine hull T)
  shows aff_dim S = aff_dim T
proof –
  {
    assume T ≠ {} S ≠ {}
    then obtain L where L: subspace L ∧ affine_parallel (affine hull T) L
      using affine_parallel_subspace[of affine hull T]
        affine_affine_hull[of T]
      by auto
    then have aff_dim T = int (dim L)
      using aff_dim_parallel_subspace ‹T ≠ {}› by auto
    moreover have ∗: subspace L ∧ affine_parallel (affine hull S) L
       using L affine_parallel_assoc[of affine hull S affine hull T L] assms by auto
    moreover from ∗ have aff_dim S = int (dim L)
      using aff_dim_parallel_subspace ‹S ≠ {}› by auto
    ultimately have ?thesis by auto
  }
  moreover
```

```
  {
    assume S = {}
    then have S = {} and T = {}
      using assms
      unfolding affine_parallel_def
      by auto
    then have ?thesis using aff_dim_empty by auto
  }
  moreover
  {
    assume T = {}
    then have S = {} and T = {}
      using assms
      unfolding affine_parallel_def
      by auto
    then have ?thesis
      using aff_dim_empty by auto
  }
  ultimately show ?thesis by blast
qed

lemma aff_dim_translation_eq:
  aff_dim ((+) a ` S) = aff_dim S for a :: 'n::euclidean_space
proof -
  have affine_parallel (affine hull S) (affine hull ((λx. a + x) ` S))
    unfolding affine_parallel_def
    apply (rule exI[of _ a])
    using affine_hull_translation[of a S]
    apply auto
    done
  then show ?thesis
    using aff_dim_parallel_eq[of S (λx. a + x) ` S] by auto
qed

lemma aff_dim_translation_eq_subtract:
  aff_dim ((λx. x - a) ` S) = aff_dim S for a :: 'n::euclidean_space
  using aff_dim_translation_eq [of - a] by (simp cong: image_cong_simp)

lemma aff_dim_affine:
  fixes S L :: 'n::euclidean_space set
  assumes S ≠ {}
    and affine S
    and subspace L
    and affine_parallel S L
  shows aff_dim S = int (dim L)
proof -
  have *: affine hull S = S
    using assms affine_hull_eq[of S] by auto
  then have affine_parallel (affine hull S) L
```

```
    using assms by (simp add: ∗)
  then show ?thesis
    using assms aff_dim_parallel_subspace[of S L] by blast
qed

lemma dim_affine_hull:
  fixes S :: 'n::euclidean_space set
  shows dim (affine hull S) = dim S
proof −
  have dim (affine hull S) ≥ dim S
    using dim_subset by auto
  moreover have dim (span S) ≥ dim (affine hull S)
    using dim_subset affine_hull_subset_span by blast
  moreover have dim (span S) = dim S
    using dim_span by auto
  ultimately show ?thesis by auto
qed

lemma aff_dim_subspace:
  fixes S :: 'n::euclidean_space set
  assumes subspace S
  shows aff_dim S = int (dim S)
proof (cases S={})
  case True with assms show ?thesis
    by (simp add: subspace_affine)
next
  case False
  with aff_dim_affine[of S S] assms subspace_imp_affine[of S] affine_parallel_reflex[of
S] subspace_affine
  show ?thesis by auto
qed

lemma aff_dim_zero:
  fixes S :: 'n::euclidean_space set
  assumes 0 ∈ affine hull S
  shows aff_dim S = int (dim S)
proof −
  have subspace (affine hull S)
    using subspace_affine[of affine hull S] affine_affine_hull assms
    by auto
  then have aff_dim (affine hull S) = int (dim (affine hull S))
    using assms aff_dim_subspace[of affine hull S] by auto
  then show ?thesis
    using aff_dim_affine_hull[of S] dim_affine_hull[of S]
    by auto
qed

lemma aff_dim_eq_dim:
  aff_dim S = int (dim ((+) (− a) ‘ S)) if a ∈ affine hull S
```

```
    for S :: 'n::euclidean_space set
proof -
  have 0 ∈ affine hull (+) (- a) ` S
    unfolding affine_hull_translation
    using that by (simp add: ac_simps)
  with aff_dim_zero show ?thesis
    by (metis aff_dim_translation_eq)
qed

lemma aff_dim_eq_dim_subtract:
  aff_dim S = int (dim ((λx. x - a) ` S)) if a ∈ affine hull S
    for S :: 'n::euclidean_space set
  using aff_dim_eq_dim [of a] that by (simp cong: image_cong_simp)

lemma aff_dim_UNIV [simp]: aff_dim (UNIV :: 'n::euclidean_space set) = int(DIM('n))
  using aff_dim_subspace[of (UNIV :: 'n::euclidean_space set)]
    dim_UNIV[where 'a='n::euclidean_space]
  by auto

lemma aff_dim_geq:
  fixes V :: 'n::euclidean_space set
  shows aff_dim V ≥ -1
proof -
  obtain B where affine hull B = affine hull V
    and ¬ affine_dependent B
    and int (card B) = aff_dim V + 1
    using aff_dim_basis_exists by auto
  then show ?thesis by auto
qed

lemma aff_dim_negative_iff [simp]:
  fixes S :: 'n::euclidean_space set
  shows aff_dim S < 0 ⟷ S = {}
by (metis aff_dim_empty aff_dim_geq diff_0 eq_iff zle_diff1_eq)

lemma aff_lowdim_subset_hyperplane:
  fixes S :: 'a::euclidean_space set
  assumes aff_dim S < DIM('a)
  obtains a b where a ≠ 0 S ⊆ {x. a · x = b}
proof (cases S={})
  case True
  moreover
  have (SOME b. b ∈ Basis) ≠ 0
    by (metis norm_some_Basis norm_zero zero_neq_one)
  ultimately show ?thesis
    using that by blast
next
  case False
  then obtain c S' where c ∉ S' S = insert c S'
```

```
    by (meson equals0I mk_disjoint_insert)
  have dim ((+) (-c) ` S) < DIM('a)
   by (metis ‹S = insert c S'› aff_dim_eq_dim assms hull_inc insertI1 of_nat_less_imp_less)
  then obtain a where a ≠ 0 span ((+) (-c) ` S) ⊆ {x. a · x = 0}
    using lowdim_subset_hyperplane by blast
  moreover
  have a · w = a · c if span ((+) (- c) ` S) ⊆ {x. a · x = 0} w ∈ S for w
  proof -
    have w-c ∈ span ((+) (- c) ` S)
      by (simp add: span_base ‹w ∈ S›)
    with that have w-c ∈ {x. a · x = 0}
      by blast
    then show ?thesis
      by (auto simp: algebra_simps)
  qed
  ultimately have S ⊆ {x. a · x = a · c}
    by blast
  then show ?thesis
    by (rule that[OF ‹a ≠ 0›])
qed

lemma affine_independent_card_dim_diffs:
  fixes S :: 'a :: euclidean_space set
  assumes ¬ affine_dependent S a ∈ S
    shows card S = dim ((λx. x - a) ` S) + 1
proof -
  have non: ¬ affine_dependent (insert a S)
    by (simp add: assms insert_absorb)
  have finite S
    by (meson assms aff_independent_finite)
  with ‹a ∈ S› have card S ≠ 0 by auto
  moreover have dim ((λx. x - a) ` S) = card S - 1
   using aff_dim_eq_dim_subtract aff_dim_unique ‹a ∈ S› hull_inc insert_absorb non
by fastforce
  ultimately show ?thesis
    by auto
qed

lemma independent_card_le_aff_dim:
  fixes B :: 'n::euclidean_space set
  assumes B ⊆ V
  assumes ¬ affine_dependent B
  shows int (card B) ≤ aff_dim V + 1
proof -
  obtain T where T: ¬ affine_dependent T ∧ B ⊆ T ∧ T ⊆ V ∧ affine hull T
= affine hull V
    by (metis assms extend_to_affine_basis[of B V])
  then have of_nat (card T) = aff_dim V + 1
    using aff_dim_unique by auto
```

then show *?thesis*
using *T card_mono*[*of T B*] *aff_independent_finite*[*of T*] **by** *auto*
qed

lemma *aff_dim_subset*:
fixes *S T* :: *'n::euclidean_space set*
assumes $S \subseteq T$
shows *aff_dim S* ≤ *aff_dim T*
proof −
obtain *B* **where** *B*: ¬ *affine_dependent B B* ⊆ *S affine hull B* = *affine hull S*
of_nat (*card B*) = *aff_dim S* + *1*
using *aff_dim_inner_basis_exists*[*of S*] **by** *auto*
then have *int* (*card B*) ≤ *aff_dim T* + *1*
using *assms independent_card_le_aff_dim*[*of B T*] **by** *auto*
with *B* **show** *?thesis* **by** *auto*
qed

lemma *aff_dim_le_DIM*:
fixes *S* :: *'n::euclidean_space set*
shows *aff_dim S* ≤ *int* (*DIM*(*'n*))
proof −
have *aff_dim* (*UNIV* :: *'n::euclidean_space set*) = *int*(*DIM*(*'n*))
using *aff_dim_UNIV* **by** *auto*
then show *aff_dim* (*S*:: *'n::euclidean_space set*) ≤ *int*(*DIM*(*'n*))
using *aff_dim_subset*[*of S* (*UNIV* :: (*'n::euclidean_space*) *set*)] *subset_UNIV* **by** *auto*
qed

lemma *affine_dim_equal*:
fixes *S* :: *'n::euclidean_space set*
assumes *affine S affine T S* ≠ {} *S* ⊆ *T aff_dim S* = *aff_dim T*
shows *S* = *T*
proof −
obtain *a* **where** *a* ∈ *S* **using** *assms* **by** *auto*
then have *a* ∈ *T* **using** *assms* **by** *auto*
define *LS* **where** *LS* = {*y*. ∃ *x* ∈ *S*. (−*a*) + *x* = *y*}
then have *ls*: *subspace LS affine_parallel S LS*
using *assms parallel_subspace_explicit*[*of S a LS*] ‹*a* ∈ *S*› **by** *auto*
then have *h1*: *int*(*dim LS*) = *aff_dim S*
using *assms aff_dim_affine*[*of S LS*] **by** *auto*
have *T* ≠ {} **using** *assms* **by** *auto*
define *LT* **where** *LT* = {*y*. ∃ *x* ∈ *T*. (−*a*) + *x* = *y*}
then have *lt*: *subspace LT* ∧ *affine_parallel T LT*
using *assms parallel_subspace_explicit*[*of T a LT*] ‹*a* ∈ *T*› **by** *auto*
then have *int*(*dim LT*) = *aff_dim T*
using *assms aff_dim_affine*[*of T LT*] ‹*T* ≠ {}› **by** *auto*
then have *dim LS* = *dim LT*
using *h1 assms* **by** *auto*
moreover have *LS* ≤ *LT*

```
    using LS_def LT_def assms by auto
  ultimately have LS = LT
    using subspace_dim_equal[of LS LT] ls lt by auto
  moreover have S = {x. ∃ y ∈ LS. a+y=x}
    using LS_def by auto
  moreover have T = {x. ∃ y ∈ LT. a+y=x}
    using LT_def by auto
  ultimately show ?thesis by auto
qed

lemma aff_dim_eq_0:
  fixes S :: 'a::euclidean_space set
  shows aff_dim S = 0 ⟷ (∃ a. S = {a})
proof (cases S = {})
  case True
  then show ?thesis
    by auto
next
  case False
  then obtain a where a ∈ S by auto
  show ?thesis
  proof safe
    assume 0: aff_dim S = 0
    have ¬ {a,b} ⊆ S if b ≠ a for b
      by (metis 0 aff_dim_2 aff_dim_subset not_one_le_zero that)
    then show ∃ a. S = {a}
      using ‹a ∈ S› by blast
  qed auto
qed

lemma affine_hull_UNIV:
  fixes S :: 'n::euclidean_space set
  assumes aff_dim S = int(DIM('n))
  shows affine hull S = (UNIV :: ('n::euclidean_space) set)
proof -
  have S ≠ {}
    using assms aff_dim_empty[of S] by auto
  have h0: S ⊆ affine hull S
    using hull_subset[of S _] by auto
  have h1: aff_dim (UNIV :: ('n::euclidean_space) set) = aff_dim S
    using aff_dim_UNIV assms by auto
  then have h2: aff_dim (affine hull S) ≤ aff_dim (UNIV :: ('n::euclidean_space)
set)
    using aff_dim_le_DIM[of affine hull S] assms h0 by auto
  have h3: aff_dim S ≤ aff_dim (affine hull S)
    using h0 aff_dim_subset[of S affine hull S] assms by auto
  then have h4: aff_dim (affine hull S) = aff_dim (UNIV :: ('n::euclidean_space)
set)
    using h0 h1 h2 by auto
```

```
  then show ?thesis
    using affine_dim_equal[of affine hull S (UNIV :: ('n::euclidean_space) set)]
      affine_affine_hull[of S] affine_UNIV assms h4 h0 ‹S ≠ {}›
    by auto
qed

lemma disjoint_affine_hull:
  fixes s :: 'n::euclidean_space set
  assumes ¬ affine_dependent s t ⊆ s u ⊆ s t ∩ u = {}
    shows (affine hull t) ∩ (affine hull u) = {}
proof −
  have finite s using assms by (simp add: aff_independent_finite)
  then have finite t finite u using assms finite_subset by blast+
  { fix y
    assume yt: y ∈ affine hull t and yu: y ∈ affine hull u
    then obtain a b
           where a1 [simp]: sum a t = 1 and [simp]: sum (λv. a v *R v) t = y
             and [simp]: sum b u = 1 sum (λv. b v *R v) u = y
      by (auto simp: affine_hull_finite ‹finite t› ‹finite u›)
    define c where c x = (if x ∈ t then a x else if x ∈ u then −(b x) else 0) for x
    have [simp]: s ∩ t = t s ∩ − t ∩ u = u using assms by auto
    have sum c s = 0
     by (simp add: c_def comm_monoid_add_class.sum.If_cases ‹finite s› sum_negf)
    moreover have ¬ (∀ v∈s. c v = 0)
      by (metis (no_types) IntD1 ‹s ∩ t = t› a1 c_def sum.neutral zero_neq_one)
    moreover have (∑ v∈s. c v *R v) = 0
      by (simp add: c_def if_smult sum_negf
           comm_monoid_add_class.sum.If_cases ‹finite s›)
    ultimately have False
      using assms ‹finite s› by (auto simp: affine_dependent_explicit)
  }
  then show ?thesis by blast
qed

end
```

1.7 Convex Sets and Functions

```
theory Convex
imports
  Affine
  HOL−Library.Set_Algebras
begin
```

1.7.1 Convex Sets

```
definition convex :: 'a::real_vector set ⇒ bool
  where convex s ⟷ (∀ x∈s. ∀ y∈s. ∀ u≥0. ∀ v≥0. u + v = 1 ⟶ u *R x + v
*R y ∈ s)
```

```
lemma convexI:
  assumes ⋀x y u v. x ∈ s ⟹ y ∈ s ⟹ 0 ≤ u ⟹ 0 ≤ v ⟹ u + v = 1 ⟹
u *R x + v *R y ∈ s
  shows convex s
  using assms unfolding convex_def by fast

lemma convexD:
  assumes convex s and x ∈ s and y ∈ s and 0 ≤ u and 0 ≤ v and u + v = 1
  shows u *R x + v *R y ∈ s
  using assms unfolding convex_def by fast

lemma convex_alt: convex s ⟷ (∀ x∈s. ∀ y∈s. ∀ u. 0 ≤ u ∧ u ≤ 1 ⟶ ((1 −
u) *R x + u *R y) ∈ s)
  (is _ ⟷ ?alt)
proof
  show convex s if alt: ?alt
  proof −
    {
      fix x y and u v :: real
      assume mem: x ∈ s y ∈ s
      assume 0 ≤ u 0 ≤ v
      moreover
      assume u + v = 1
      then have u = 1 − v by auto
      ultimately have u *R x + v *R y ∈ s
        using alt [rule_format, OF mem] by auto
    }
    then show ?thesis
      unfolding convex_def by auto
  qed
  show ?alt if convex s
    using that by (auto simp: convex_def)
qed

lemma convexD_alt:
  assumes convex s a ∈ s b ∈ s 0 ≤ u u ≤ 1
  shows ((1 − u) *R a + u *R b) ∈ s
  using assms unfolding convex_alt by auto

lemma mem_convex_alt:
  assumes convex S x ∈ S y ∈ S u ≥ 0 v ≥ 0 u + v > 0
  shows ((u/(u+v)) *R x + (v/(u+v)) *R y) ∈ S
  using assms
  by (simp add: convex_def zero_le_divide_iff add_divide_distrib [symmetric])

lemma convex_empty[intro,simp]: convex {}
  unfolding convex_def by simp
```

lemma *convex_singleton*[*intro*,*simp*]: *convex* {*a*}
unfolding *convex_def* **by** (*auto simp*: *scaleR_left_distrib*[*symmetric*])

lemma *convex_UNIV*[*intro*,*simp*]: *convex UNIV*
unfolding *convex_def* **by** *auto*

lemma *convex_Inter*: ($\bigwedge$*s*. *s*∈*f* ⟹ *convex s*) ⟹ *convex*($\bigcap$*f*)
unfolding *convex_def* **by** *auto*

lemma *convex_Int*: *convex s* ⟹ *convex t* ⟹ *convex* (*s* ∩ *t*)
unfolding *convex_def* **by** *auto*

lemma *convex_INT*: ($\bigwedge$*i*. *i* ∈ *A* ⟹ *convex* (*B i*)) ⟹ *convex* ($\bigcap$*i*∈*A*. *B i*)
unfolding *convex_def* **by** *auto*

lemma *convex_Times*: *convex s* ⟹ *convex t* ⟹ *convex* (*s* × *t*)
unfolding *convex_def* **by** *auto*

lemma *convex_halfspace_le*: *convex* {*x*. *inner a x* ≤ *b*}
unfolding *convex_def*
by (*auto simp*: *inner_add intro*!: *convex_bound_le*)

lemma *convex_halfspace_ge*: *convex* {*x*. *inner a x* ≥ *b*}
proof −
have ∗: {*x*. *inner a x* ≥ *b*} = {*x*. *inner* (−*a*) *x* ≤ −*b*}
by *auto*
show *?thesis*
unfolding ∗ **using** *convex_halfspace_le*[*of* −*a* −*b*] **by** *auto*
qed

lemma *convex_halfspace_abs_le*: *convex* {*x*. |*inner a x*| ≤ *b*}
proof −
have ∗: {*x*. |*inner a x*| ≤ *b*} = {*x*. *inner a x* ≤ *b*} ∩ {*x*. −*b* ≤ *inner a x*}
by *auto*
show *?thesis*
unfolding ∗ **by** (*simp add*: *convex_Int convex_halfspace_ge convex_halfspace_le*)
qed

lemma *convex_hyperplane*: *convex* {*x*. *inner a x* = *b*}
proof −
have ∗: {*x*. *inner a x* = *b*} = {*x*. *inner a x* ≤ *b*} ∩ {*x*. *inner a x* ≥ *b*}
by *auto*
show *?thesis* **using** *convex_halfspace_le convex_halfspace_ge*
by (*auto intro*!: *convex_Int simp*: ∗)
qed

lemma *convex_halfspace_lt*: *convex* {*x*. *inner a x* < *b*}
unfolding *convex_def*
by (*auto simp*: *convex_bound_lt inner_add*)

```
lemma convex_halfspace_gt: convex {x. inner a x > b}
  using convex_halfspace_lt[of -a -b] by auto

lemma convex_halfspace_Re_ge: convex {x. Re x ≥ b}
  using convex_halfspace_ge[of b 1::complex] by simp

lemma convex_halfspace_Re_le: convex {x. Re x ≤ b}
  using convex_halfspace_le[of 1::complex b] by simp

lemma convex_halfspace_Im_ge: convex {x. Im x ≥ b}
  using convex_halfspace_ge[of b i] by simp

lemma convex_halfspace_Im_le: convex {x. Im x ≤ b}
  using convex_halfspace_le[of i b] by simp

lemma convex_halfspace_Re_gt: convex {x. Re x > b}
  using convex_halfspace_gt[of b 1::complex] by simp

lemma convex_halfspace_Re_lt: convex {x. Re x < b}
  using convex_halfspace_lt[of 1::complex b] by simp

lemma convex_halfspace_Im_gt: convex {x. Im x > b}
  using convex_halfspace_gt[of b i] by simp

lemma convex_halfspace_Im_lt: convex {x. Im x < b}
  using convex_halfspace_lt[of i b] by simp

lemma convex_real_interval [iff]:
  fixes a b :: real
  shows convex {a..} and convex {..b}
    and convex {a<..} and convex {..<b}
    and convex {a..b} and convex {a<..b}
    and convex {a..<b} and convex {a<..<b}
proof -
  have {a..} = {x. a ≤ inner 1 x}
    by auto
  then show 1: convex {a..}
    by (simp only: convex_halfspace_ge)
  have {..b} = {x. inner 1 x ≤ b}
    by auto
  then show 2: convex {..b}
    by (simp only: convex_halfspace_le)
  have {a<..} = {x. a < inner 1 x}
    by auto
  then show 3: convex {a<..}
    by (simp only: convex_halfspace_gt)
  have {..<b} = {x. inner 1 x < b}
    by auto
```

```
  then show 4: convex {..<b}
    by (simp only: convex_halfspace_lt)
  have {a..b} = {a..} ∩ {..b}
    by auto
  then show convex {a..b}
    by (simp only: convex_Int 1 2)
  have {a<..b} = {a<..} ∩ {..b}
    by auto
  then show convex {a<..b}
    by (simp only: convex_Int 3 2)
  have {a..<b} = {a..} ∩ {..<b}
    by auto
  then show convex {a..<b}
    by (simp only: convex_Int 1 4)
  have {a<..<b} = {a<..} ∩ {..<b}
    by auto
  then show convex {a<..<b}
    by (simp only: convex_Int 3 4)
qed

lemma convex_Reals: convex ℝ
  by (simp add: convex_def scaleR_conv_of_real)
```

1.7.2 Explicit expressions for convexity in terms of arbitrary sums

```
lemma convex_sum:
  fixes C :: 'a::real_vector set
  assumes finite S
    and convex C
    and (∑ i ∈ S. a i) = 1
  assumes ⋀i. i ∈ S ⟹ a i ≥ 0
    and ⋀i. i ∈ S ⟹ y i ∈ C
  shows (∑ j ∈ S. a j *R y j) ∈ C
  using assms(1,3,4,5)
proof (induct arbitrary: a set: finite)
  case empty
  then show ?case by simp
next
  case (insert i S) note IH = this(3)
  have a i + sum a S = 1
    and 0 ≤ a i
    and ∀j∈S. 0 ≤ a j
    and y i ∈ C
    and ∀j∈S. y j ∈ C
    using insert.hyps(1,2) insert.prems by simp_all
  then have 0 ≤ sum a S
    by (simp add: sum_nonneg)
  have a i *R y i + (∑j∈S. a j *R y j) ∈ C
```

```
  proof (cases sum a S = 0)
    case True
    with ‹a i + sum a S = 1› have a i = 1
      by simp
    from sum_nonneg_0 [OF ‹finite S› _ True] ‹∀j∈S. 0 ≤ a j› have ∀j∈S. a j =
0
      by simp
    show ?thesis using ‹a i = 1› and ‹∀j∈S. a j = 0› and ‹y i ∈ C›
      by simp
  next
    case False
    with ‹0 ≤ sum a S› have 0 < sum a S
      by simp
    then have (∑j∈S. (a j / sum a S) *R y j) ∈ C
      using ‹∀j∈S. 0 ≤ a j› and ‹∀j∈S. y j ∈ C›
      by (simp add: IH sum_divide_distrib [symmetric])
    from ‹convex C› and ‹y i ∈ C› and this and ‹0 ≤ a i›
      and ‹0 ≤ sum a S› and ‹a i + sum a S = 1›
    have a i *R y i + sum a S *R (∑j∈S. (a j / sum a S) *R y j) ∈ C
      by (rule convexD)
    then show ?thesis
      by (simp add: scaleR_sum_right False)
  qed
  then show ?case using ‹finite S› and ‹i ∉ S›
    by simp
qed

lemma convex:
  convex S ⟷ (∀(k::nat) u x. (∀i. 1≤i ∧ i≤k ⟶ 0 ≤ u i ∧ x i ∈S) ∧ (sum u
{1..k} = 1)
      ⟶ sum (λi. u i *R x i) {1..k} ∈ S)
proof safe
  fix k :: nat
  fix u :: nat ⇒ real
  fix x
  assume convex S
    ∀i. 1 ≤ i ∧ i ≤ k ⟶ 0 ≤ u i ∧ x i ∈ S
    sum u {1..k} = 1
  with convex_sum[of {1 .. k} S] show (∑j∈{1 .. k}. u j *R x j) ∈ S
    by auto
next
  assume *: ∀k u x. (∀ i :: nat. 1 ≤ i ∧ i ≤ k ⟶ 0 ≤ u i ∧ x i ∈ S) ∧ sum u
{1..k} = 1
    ⟶ (∑i = 1..k. u i *R (x i :: 'a)) ∈ S
  {
    fix μ :: real
    fix x y :: 'a
    assume xy: x ∈ S y ∈ S
    assume mu: μ ≥ 0 μ ≤ 1
```

let *?u* = *λi. if (i :: nat) = 1 then μ else 1 − μ*
let *?x* = *λi. if (i :: nat) = 1 then x else y*
have {*1 :: nat .. 2*} ∩ − {*x. x = 1*} = {*2*}
by *auto*
then have *card*: *card* ({*1 :: nat .. 2*} ∩ − {*x. x = 1*}) = *1*
by *simp*
then have *sum ?u* {*1 .. 2*} = *1*
using *sum.If_cases*[*of* {(*1 :: nat*) .. *2*} *λ x. x = 1 λ x. μ λ x. 1 − μ*]
by *auto*
with ∗[*rule_format*, *of 2 ?u ?x*] **have** *S*: ($\sum$ *j* ∈ {*1..2*}. *?u j* $*_R$ *?x j*) ∈ *S*
using *mu xy* **by** *auto*
have *grarr*: ($\sum$ *j* ∈ {*Suc* (*Suc 0*)..*2*}. *?u j* $*_R$ *?x j*) = (*1* − *μ*) $*_R$ *y*
using *sum.atLeast_Suc_atMost*[*of Suc* (*Suc 0*) *2 λ j.* (*1* − *μ*) $*_R$ *y*] **by** *auto*
from *sum.atLeast_Suc_atMost*[*of Suc 0 2 λ j. ?u j* $*_R$ *?x j*, *simplified this*]
have ($\sum$ *j* ∈ {*1..2*}. *?u j* $*_R$ *?x j*) = *μ* $*_R$ *x* + (*1* − *μ*) $*_R$ *y*
by *auto*
then have (*1* − *μ*) $*_R$ *y* + *μ* $*_R$ *x* ∈ *S*
using *S* **by** (*auto simp*: *add.commute*)
}
then show *convex S*
unfolding *convex_alt* **by** *auto*
qed

lemma *convex_explicit*:
fixes *S* :: *'a::real_vector set*
shows *convex S* ⟷
(∀ *t u. finite t* ∧ *t* ⊆ *S* ∧ (∀ *x*∈*t. 0* ≤ *u x*) ∧ *sum u t* = *1* ⟶ *sum* (*λx. u x* $*_R$ *x*) *t* ∈ *S*)
proof *safe*
fix *t*
fix *u* :: *'a* ⇒ *real*
assume *convex S*
and *finite t*
and *t* ⊆ *S* ∀ *x*∈*t. 0* ≤ *u x sum u t* = *1*
then show ($\sum$ *x*∈*t. u x* $*_R$ *x*) ∈ *S*
using *convex_sum*[*of t S u λ x. x*] **by** *auto*
next
assume ∗: ∀ *t.* ∀ *u. finite t* ∧ *t* ⊆ *S* ∧ (∀ *x*∈*t. 0* ≤ *u x*) ∧
sum u t = *1* ⟶ ($\sum$ *x*∈*t. u x* $*_R$ *x*) ∈ *S*
show *convex S*
unfolding *convex_alt*
proof *safe*
fix *x y*
fix *μ* :: *real*
assume ∗∗: *x* ∈ *S y* ∈ *S 0* ≤ *μ μ* ≤ *1*
show (*1* − *μ*) $*_R$ *x* + *μ* $*_R$ *y* ∈ *S*
proof (*cases x* = *y*)
case *False*

```
      then show ?thesis
        using *[rule_format, of {x, y} λ z. if z = x then 1 − μ else μ] **
        by auto
    next
      case True
      then show ?thesis
        using *[rule_format, of {x, y} λ z. 1] **
        by (auto simp: field_simps real_vector.scale_left_diff_distrib)
    qed
  qed
qed

lemma convex_finite:
  assumes finite S
  shows convex S ⟷ (∀ u. (∀ x∈S. 0 ≤ u x) ∧ sum u S = 1 ⟶ sum (λx. u x
*R x) S ∈ S)
      (is ?lhs = ?rhs)
proof
  { have if_distrib_arg: ⋀P f g x. (if P then f else g) x = (if P then f x else g x)
      by simp
    fix T :: 'a set and u :: 'a ⇒ real
    assume sum: ∀ u. (∀ x∈S. 0 ≤ u x) ∧ sum u S = 1 ⟶ (∑ x∈S. u x *R x)
∈ S
    assume *: ∀ x∈T. 0 ≤ u x sum u T = 1
    assume T ⊆ S
    then have S ∩ T = T by auto
    with sum[THEN spec[where x=λx. if x∈T then u x else 0]] * have (∑ x∈T.
u x *R x) ∈ S
      by (auto simp: assms sum.If_cases if_distrib if_distrib_arg) }
  moreover assume ?rhs
  ultimately show ?lhs
    unfolding convex_explicit by auto
qed (auto simp: convex_explicit assms)
```

1.7.3 Convex Functions on a Set

```
definition convex_on :: 'a::real_vector set ⇒ ('a ⇒ real) ⇒ bool
  where convex_on S f ⟷
    (∀ x∈S. ∀ y∈S. ∀ u≥0. ∀ v≥0. u + v = 1 ⟶ f (u *R x + v *R y) ≤ u * f x
+ v * f y)

lemma convex_onI [intro?]:
  assumes ⋀t x y. t > 0 ⟹ t < 1 ⟹ x ∈ A ⟹ y ∈ A ⟹
    f ((1 − t) *R x + t *R y) ≤ (1 − t) * f x + t * f y
  shows convex_on A f
  unfolding convex_on_def
proof clarify
  fix x y
  fix u v :: real
```

assume *A*: $x \in A$ $y \in A$ $u \geq 0$ $v \geq 0$ $u + v = 1$
from *A*(*5*) **have** [*simp*]: $v = 1 - u$
by (*simp add*: *algebra_simps*)
from *A*(*1*−*4*) **show** $f\ (u *_R x + v *_R y) \leq u * f\, x + v * f\, y$
using *assms*[*of u y x*]
by (*cases* $u = 0 \lor u = 1$) (*auto simp*: *algebra_simps*)
qed

lemma *convex_on_linorderI* [*intro?*]:
fixes *A* :: (*'a*::{*linorder*,*real_vector*}) *set*
assumes $\bigwedge t\ x\ y.\ t > 0 \Longrightarrow t < 1 \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow x < y \Longrightarrow$
$f\ ((1 - t) *_R x + t *_R y) \leq (1 - t) * f\, x + t * f\, y$
shows *convex_on A f*
proof
fix *x y*
fix *t* :: *real*
assume *A*: $x \in A$ $y \in A$ $t > 0$ $t < 1$
with *assms* [*of t x y*] *assms* [*of* $1 - t$ *y x*]
show $f\ ((1 - t) *_R x + t *_R y) \leq (1 - t) * f\, x + t * f\, y$
by (*cases x y rule*: *linorder_cases*) (*auto simp*: *algebra_simps*)
qed

lemma *convex_onD*:
assumes *convex_on A f*
shows $\bigwedge t\ x\ y.\ t \geq 0 \Longrightarrow t \leq 1 \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow$
$f\ ((1 - t) *_R x + t *_R y) \leq (1 - t) * f\, x + t * f\, y$
using *assms* **by** (*auto simp*: *convex_on_def*)

lemma *convex_onD_Icc*:
assumes *convex_on* {*x*..*y*} *f* $x \leq$ (*y* :: _ :: {*real_vector*,*preorder*})
shows $\bigwedge t.\ t \geq 0 \Longrightarrow t \leq 1 \Longrightarrow$
$f\ ((1 - t) *_R x + t *_R y) \leq (1 - t) * f\, x + t * f\, y$
using *assms*(*2*) **by** (*intro convex_onD* [*OF assms*(*1*)]) *simp_all*

lemma *convex_on_subset*: *convex_on t f* $\Longrightarrow S \subseteq t \Longrightarrow$ *convex_on S f*
unfolding *convex_on_def* **by** *auto*

lemma *convex_on_add* [*intro*]:
assumes *convex_on S f*
and *convex_on S g*
shows *convex_on S* ($\lambda x.\ f\, x + g\, x$)
proof −
{
fix *x y*
assume $x \in S$ $y \in S$
moreover
fix *u v* :: *real*
assume $0 \leq u$ $0 \leq v$ $u + v = 1$
ultimately

```
    have f (u *R x + v *R y) + g (u *R x + v *R y) ≤ (u * f x + v * f y) + (u
* g x + v * g y)
      using assms unfolding convex_on_def by (auto simp: add_mono)
    then have f (u *R x + v *R y) + g (u *R x + v *R y) ≤ u * (f x + g x) +
v * (f y + g y)
      by (simp add: field_simps)
  }
  then show ?thesis
    unfolding convex_on_def by auto
qed

lemma convex_on_cmul [intro]:
  fixes c :: real
  assumes 0 ≤ c
    and convex_on S f
  shows convex_on S (λx. c * f x)
proof −
  have *: u * (c * fx) + v * (c * fy) = c * (u * fx + v * fy)
    for u c fx v fy :: real
    by (simp add: field_simps)
  show ?thesis using assms(2) and mult_left_mono [OF _ assms(1)]
    unfolding convex_on_def and * by auto
qed

lemma convex_lower:
  assumes convex_on S f
    and x ∈ S
    and y ∈ S
    and 0 ≤ u
    and 0 ≤ v
    and u + v = 1
  shows f (u *R x + v *R y) ≤ max (f x) (f y)
proof −
  let ?m = max (f x) (f y)
  have u * f x + v * f y ≤ u * max (f x) (f y) + v * max (f x) (f y)
    using assms(4,5) by (auto simp: mult_left_mono add_mono)
  also have . . . = max (f x) (f y)
    using assms(6) by (simp add: distrib_right [symmetric])
  finally show ?thesis
    using assms unfolding convex_on_def by fastforce
qed

lemma convex_on_dist [intro]:
  fixes S :: 'a::real_normed_vector set
  shows convex_on S (λx. dist a x)
proof (auto simp: convex_on_def dist_norm)
  fix x y
  assume x ∈ S y ∈ S
  fix u v :: real
```

assume *0* $\leq$ *u*
assume *0* $\leq$ *v*
assume *u* + *v* = *1*
have *a* = *u* $*_R$ *a* + *v* $*_R$ *a*
unfolding *scaleR_left_distrib*[*symmetric*] **and** ‹*u* + *v* = *1*› **by** *simp*
then have ∗: *a* − (*u* $*_R$ *x* + *v* $*_R$ *y*) = (*u* $*_R$ (*a* − *x*)) + (*v* $*_R$ (*a* − *y*))
by (*auto simp*: *algebra_simps*)
show *norm* (*a* − (*u* $*_R$ *x* + *v* $*_R$ *y*)) ≤ *u* ∗ *norm* (*a* − *x*) + *v* ∗ *norm* (*a* − *y*)
unfolding ∗ **using** *norm_triangle_ineq*[*of u* $*_R$ (*a* − *x*) *v* $*_R$ (*a* − *y*)]
using ‹*0* ≤ *u*› ‹*0* ≤ *v*› **by** *auto*
qed

1.7.4 Arithmetic operations on sets preserve convexity

lemma *convex_linear_image*:
assumes *linear f*
and *convex S*
shows *convex* (*f* ‘ *S*)
proof −
interpret *f*: *linear f* **by** *fact*
from ‹*convex S*› **show** *convex* (*f* ‘ *S*)
by (*simp add*: *convex_def f.scaleR* [*symmetric*] *f.add* [*symmetric*])
qed

lemma *convex_linear_vimage*:
assumes *linear f*
and *convex S*
shows *convex* (*f* −‘ *S*)
proof −
interpret *f*: *linear f* **by** *fact*
from ‹*convex S*› **show** *convex* (*f* −‘ *S*)
by (*simp add*: *convex_def f.add f.scaleR*)
qed

lemma *convex_scaling*:
assumes *convex S*
shows *convex* ((λ*x*. *c* $*_R$ *x*) ‘ *S*)
proof −
have *linear* (λ*x*. *c* $*_R$ *x*)
by (*simp add*: *linearI scaleR_add_right*)
then show *?thesis*
using ‹*convex S*› **by** (*rule convex_linear_image*)
qed

lemma *convex_scaled*:
assumes *convex S*
shows *convex* ((λ*x*. *x* $*_R$ *c*) ‘ *S*)
proof −
have *linear* (λ*x*. *x* $*_R$ *c*)

```
    by (simp add: linearI scaleR_add_left)
  then show ?thesis
    using ‹convex S› by (rule convex_linear_image)
qed

lemma convex_negations:
  assumes convex S
  shows convex ((λx. − x) ‘ S)
proof −
  have linear (λx. − x)
    by (simp add: linearI)
  then show ?thesis
    using ‹convex S› by (rule convex_linear_image)
qed

lemma convex_sums:
  assumes convex S
    and convex T
  shows convex (⋃x∈ S. ⋃y ∈ T. {x + y})
proof −
  have linear (λ(x, y). x + y)
    by (auto intro: linearI simp: scaleR_add_right)
  with assms have convex ((λ(x, y). x + y) ‘ (S × T))
    by (intro convex_linear_image convex_Times)
  also have ((λ(x, y). x + y) ‘ (S × T)) = (⋃x∈ S. ⋃y ∈ T. {x + y})
    by auto
  finally show ?thesis .
qed

lemma convex_differences:
  assumes convex S convex T
  shows convex (⋃x∈ S. ⋃y ∈ T. {x − y})
proof −
  have {x − y| x y. x ∈ S ∧ y ∈ T} = {x + y |x y. x ∈ S ∧ y ∈ uminus ‘ T}
    by (auto simp: diff_conv_add_uminus simp del: add_uminus_conv_diff)
  then show ?thesis
    using convex_sums[OF assms(1) convex_negations[OF assms(2)]] by auto
qed

lemma convex_translation:
  convex ((+) a ‘ S) if convex S
proof −
  have (⋃ x∈ {a}. ⋃y ∈ S. {x + y}) = (+) a ‘ S
    by auto
  then show ?thesis
    using convex_sums [OF convex_singleton [of a] that] by auto
qed

lemma convex_translation_subtract:
```

convex ((λ*b*. *b* − *a*) ‘ *S*) **if** *convex S*
using *convex_translation* [*of S* − *a*] *that* **by** (*simp cong*: *image_cong_simp*)

lemma *convex_affinity*:
assumes *convex S*
shows *convex* ((λ*x*. *a* + *c* $*_R$ *x*) ‘ *S*)
proof −
have (λ*x*. *a* + *c* $*_R$ *x*) ‘ *S* = (+) *a* ‘ ($*_R$) *c* ‘ *S*
by *auto*
then show *?thesis*
using *convex_translation*[*OF convex_scaling*[*OF assms*], *of a c*] **by** *auto*
qed

lemma *convex_on_sum*:
fixes *a* :: ′*a* ⇒ *real*
and *y* :: ′*a* ⇒ ′*b*::*real_vector*
and *f* :: ′*b* ⇒ *real*
assumes *finite s s* ≠ {}
and *convex_on C f*
and *convex C*
and (∑ *i* ∈ *s*. *a i*) = *1*
and ⋀*i*. *i* ∈ *s* ⟹ *a i* ≥ *0*
and ⋀*i*. *i* ∈ *s* ⟹ *y i* ∈ *C*
shows *f* (∑ *i* ∈ *s*. *a i* $*_R$ *y i*) ≤ (∑ *i* ∈ *s*. *a i* ∗ *f* (*y i*))
using *assms*
proof (*induct s arbitrary*: *a rule*: *finite_ne_induct*)
case (*singleton i*)
then have *ai*: *a i* = *1*
by *auto*
then show *?case*
by *auto*
next
case (*insert i s*)
then have *convex_on C f*
by *simp*
from *this*[*unfolded convex_on_def*, *rule_format*]
have *conv*: ⋀*x y μ*. *x* ∈ *C* ⟹ *y* ∈ *C* ⟹ *0* ≤ *μ* ⟹ *μ* ≤ *1* ⟹
f (*μ* $*_R$ *x* + (*1* − *μ*) $*_R$ *y*) ≤ *μ* ∗ *f x* + (*1* − *μ*) ∗ *f y*
by *simp*
show *?case*
proof (*cases a i* = *1*)
case *True*
then have (∑ *j* ∈ *s*. *a j*) = *0*
using *insert* **by** *auto*
then have ⋀*j*. *j* ∈ *s* ⟹ *a j* = *0*
using *insert* **by** (*fastforce simp*: *sum_nonneg_eq_0_iff*)
then show *?thesis*
using *insert* **by** *auto*
next

```
  case False
  from insert have yai: y i ∈ C a i ≥ 0
    by auto
  have fis: finite (insert i s)
    using insert by auto
  then have ai1: a i ≤ 1
    using sum_nonneg_leq_bound[of insert i s a] insert by simp
  then have a i < 1
    using False by auto
  then have i0: 1 − a i > 0
    by auto
  let ?a = λj. a j / (1 − a i)
  have a_nonneg: ?a j ≥ 0 if j ∈ s for j
    using i0 insert that by fastforce
  have (∑ j ∈ insert i s. a j) = 1
    using insert by auto
  then have (∑ j ∈ s. a j) = 1 − a i
    using sum.insert insert by fastforce
  then have (∑ j ∈ s. a j) / (1 − a i) = 1
    using i0 by auto
  then have a1: (∑ j ∈ s. ?a j) = 1
    unfolding sum_divide_distrib by simp
  have convex C using insert by auto
  then have asum: (∑ j ∈ s. ?a j *R y j) ∈ C
    using insert convex_sum [OF ‹finite s› ‹convex C› a1 a_nonneg] by auto
  have asum_le: f (∑ j ∈ s. ?a j *R y j) ≤ (∑ j ∈ s. ?a j * f (y j))
    using a_nonneg a1 insert by blast
  have f (∑ j ∈ insert i s. a j *R y j) = f ((∑ j ∈ s. a j *R y j) + a i *R y i)
    using sum.insert[of s i λ j. a j *R y j, OF ‹finite s› ‹i ∉ s›] insert
    by (auto simp only: add.commute)
  also have ... = f (((1 − a i) * inverse (1 − a i)) *R (∑ j ∈ s. a j *R y j)
+ a i *R y i)
    using i0 by auto
  also have ... = f ((1 − a i) *R (∑ j ∈ s. (a j * inverse (1 − a i)) *R y j)
+ a i *R y i)
    using scaleR_right.sum[of inverse (1 − a i) λ j. a j *R y j s, symmetric]
    by (auto simp: algebra_simps)
  also have ... = f ((1 − a i) *R (∑ j ∈ s. ?a j *R y j) + a i *R y i)
    by (auto simp: divide_inverse)
  also have ... ≤ (1 − a i) *R f ((∑ j ∈ s. ?a j *R y j)) + a i * f (y i)
    using conv[of y i (∑ j ∈ s. ?a j *R y j) a i, OF yai(1) asum yai(2) ai1]
    by (auto simp: add.commute)
  also have ... ≤ (1 − a i) * (∑ j ∈ s. ?a j * f (y j)) + a i * f (y i)
    using add_right_mono [OF mult_left_mono [of _ _ 1 − a i,
          OF asum_le less_imp_le[OF i0]], of a i * f (y i)]
    by simp
  also have ... = (∑ j ∈ s. (1 − a i) * ?a j * f (y j)) + a i * f (y i)
    unfolding sum_distrib_left[of 1 − a i λ j. ?a j * f (y j)]
    using i0 by auto
```

```
    also have ... = (∑ j ∈ s. a j * f (y j)) + a i * f (y i)
      using i0 by auto
    also have ... = (∑ j ∈ insert i s. a j * f (y j))
      using insert by auto
    finally show ?thesis
      by simp
  qed
qed

lemma convex_on_alt:
  fixes C :: 'a::real_vector set
  shows convex_on C f ⟷
    (∀ x ∈ C. ∀ y ∈ C. ∀ μ :: real. μ ≥ 0 ∧ μ ≤ 1 ⟶
      f (μ *R x + (1 − μ) *R y) ≤ μ * f x + (1 − μ) * f y)
proof safe
  fix x y
  fix μ :: real
  assume *: convex_on C f x ∈ C y ∈ C 0 ≤ μ μ ≤ 1
  from this[unfolded convex_on_def, rule_format]
  have 0 ≤ u ⟹ 0 ≤ v ⟹ u + v = 1 ⟹ f (u *R x + v *R y) ≤ u * f x + v
* f y for u v
    by auto
  from this [of μ 1 − μ, simplified] *
  show f (μ *R x + (1 − μ) *R y) ≤ μ * f x + (1 − μ) * f y
    by auto
next
  assume *: ∀ x∈C. ∀ y∈C. ∀ μ. 0 ≤ μ ∧ μ ≤ 1 ⟶
    f (μ *R x + (1 − μ) *R y) ≤ μ * f x + (1 − μ) * f y
  {
    fix x y
    fix u v :: real
    assume **: x ∈ C y ∈ C u ≥ 0 v ≥ 0 u + v = 1
    then have[simp]: 1 − u = v by auto
    from *[rule_format, of x y u]
    have f (u *R x + v *R y) ≤ u * f x + v * f y
      using ** by auto
  }
  then show convex_on C f
    unfolding convex_on_def by auto
qed

lemma convex_on_diff:
  fixes f :: real ⇒ real
  assumes f: convex_on I f
    and I: x ∈ I y ∈ I
    and t: x < t t < y
  shows (f x − f t) / (x − t) ≤ (f x − f y) / (x − y)
    and (f x − f y) / (x − y) ≤ (f t − f y) / (t − y)
proof −
```

```
  define a where a ≡ (t − y) / (x − y)
  with t have 0 ≤ a 0 ≤ 1 − a
    by (auto simp: field_simps)
  with f ‹x ∈ I› ‹y ∈ I› have cvx: f (a ∗ x + (1 − a) ∗ y) ≤ a ∗ f x + (1 − a)
∗ f y
    by (auto simp: convex_on_def)
  have a ∗ x + (1 − a) ∗ y = a ∗ (x − y) + y
    by (simp add: field_simps)
  also have ... = t
    unfolding a_def using ‹x < t› ‹t < y› by simp
  finally have f t ≤ a ∗ f x + (1 − a) ∗ f y
    using cvx by simp
  also have ... = a ∗ (f x − f y) + f y
    by (simp add: field_simps)
  finally have f t − f y ≤ a ∗ (f x − f y)
    by simp
  with t show (f x − f t) / (x − t) ≤ (f x − f y) / (x − y)
    by (simp add: le_divide_eq divide_le_eq field_simps a_def)
  with t show (f x − f y) / (x − y) ≤ (f t − f y) / (t − y)
    by (simp add: le_divide_eq divide_le_eq field_simps)
qed

lemma pos_convex_function:
  fixes f :: real ⇒ real
  assumes convex C
    and leq: ⋀x y. x ∈ C ⟹ y ∈ C ⟹ f′ x ∗ (y − x) ≤ f y − f x
  shows convex_on C f
  unfolding convex_on_alt
  using assms
proof safe
  fix x y μ :: real
  let ?x = μ ∗R x + (1 − μ) ∗R y
  assume ∗: convex C x ∈ C y ∈ C μ ≥ 0 μ ≤ 1
  then have 1 − μ ≥ 0 by auto
  then have xpos: ?x ∈ C
    using ∗ unfolding convex_alt by fastforce
  have geq: μ ∗ (f x − f ?x) + (1 − μ) ∗ (f y − f ?x) ≥
      μ ∗ f′ ?x ∗ (x − ?x) + (1 − μ) ∗ f′ ?x ∗ (y − ?x)
    using add_mono [OF mult_left_mono [OF leq [OF xpos ∗(2)] ‹μ ≥ 0›]
        mult_left_mono [OF leq [OF xpos ∗(3)] ‹1 − μ ≥ 0›]]
    by auto
  then have μ ∗ f x + (1 − μ) ∗ f y − f ?x ≥ 0
    by (auto simp: field_simps)
  then show f (μ ∗R x + (1 − μ) ∗R y) ≤ μ ∗ f x + (1 − μ) ∗ f y
    by auto
qed

lemma atMostAtLeast_subset_convex:
  fixes C :: real set
```

  assumes *convex C*
    and $x \in C$ $y \in C$ $x < y$
  shows $\{x .. y\} \subseteq C$
proof *safe*
  fix z **assume** z: $z \in \{x .. y\}$
  have *less*: $z \in C$ **if** $*$: $x < z$ $z < y$
  proof −
    let $?\mu = (y - z) / (y - x)$
    have $0 \leq ?\mu$ $?\mu \leq 1$
      using *assms* $*$ **by** (*auto simp*: *field_simps*)
    then have *comb*: $?\mu * x + (1 - ?\mu) * y \in C$
      using *assms iffD1*[*OF convex_alt*, *rule_format*, *of C y x ?μ*]
      by (*simp add*: *algebra_simps*)
    have $?\mu * x + (1 - ?\mu) * y = (y - z) * x / (y - x) + (1 - (y - z) / (y -$
$x)) * y$
      by (*auto simp*: *field_simps*)
    also have $\ldots = ((y - z) * x + (y - x - (y - z)) * y) / (y - x)$
      using *assms* **by** (*simp only*: *add_divide_distrib*) (*auto simp*: *field_simps*)
    also have $\ldots = z$
      using *assms* **by** (*auto simp*: *field_simps*)
    finally show *?thesis*
      using *comb* **by** *auto*
  qed
  show $z \in C$
    using *z less assms* **by** (*auto simp*: *le_less*)
qed

lemma *f''_imp_f'*:
  fixes f :: *real* ⇒ *real*
  assumes *convex C*
    and f': $\bigwedge x.\ x \in C \Longrightarrow$ *DERIV f x* :> $(f'\ x)$
    and f'': $\bigwedge x.\ x \in C \Longrightarrow$ *DERIV f' x* :> $(f''\ x)$
    and *pos*: $\bigwedge x.\ x \in C \Longrightarrow f''\ x \geq 0$
    and x: $x \in C$
    and y: $y \in C$
  shows $f'\ x * (y - x) \leq f\ y - f\ x$
  using *assms*
proof −
  have *less_imp*: $f\ y - f\ x \geq f'\ x * (y - x)$ $f'\ y * (x - y) \leq f\ x - f\ y$
    if $*$: $x \in C$ $y \in C$ $y > x$ **for** x y :: *real*
  proof −
    from $*$ **have** *ge*: $y - x > 0$ $y - x \geq 0$
      by *auto*
    from $*$ **have** *le*: $x - y < 0$ $x - y \leq 0$
      by *auto*
    then obtain *z1* **where** *z1*: $z1 > x$ $z1 < y$ $f\ y - f\ x = (y - x) * f'\ z1$
      using *subsetD*[*OF atMostAtLeast_subset_convex*[*OF* ‹*convex C*› ‹$x \in C$› ‹$y \in$
C› ‹$x < y$›],
          THEN f', *THEN MVT2*[*OF* ‹$x < y$›, *rule_format*, *unfolded atLeastAt-*

Most_iff[*symmetric*]]]
  by *auto*
 then have *z1* ∈ *C*
  using *atMostAtLeast_subset_convex* ‹*convex C*› ‹*x* ∈ *C*› ‹*y* ∈ *C*› ‹*x* < *y*›
  by *fastforce*
 from *z1* **have** *z1'*: *f x* − *f y* = (*x* − *y*) ∗ *f' z1*
  by (*simp add*: *field_simps*)
 obtain *z2* **where** *z2*: *z2* > *x z2* < *z1 f' z1* − *f' x* = (*z1* − *x*) ∗ *f'' z2*
  using *subsetD*[*OF atMostAtLeast_subset_convex*[*OF* ‹*convex C*› ‹*x* ∈ *C*› ‹*z1* ∈ *C*› ‹*x* < *z1*›],
   THEN f'', *THEN MVT2*[*OF* ‹*x* < *z1*›, *rule_format*, *unfolded atLeastAtMost_iff*[*symmetric*]]] *z1*
  by *auto*
 obtain *z3* **where** *z3*: *z3* > *z1 z3* < *y f' y* − *f' z1* = (*y* − *z1*) ∗ *f'' z3*
  using *subsetD*[*OF atMostAtLeast_subset_convex*[*OF* ‹*convex C*› ‹*z1* ∈ *C*› ‹*y* ∈ *C*› ‹*z1* < *y*›],
   THEN f'', *THEN MVT2*[*OF* ‹*z1* < *y*›, *rule_format*, *unfolded atLeastAtMost_iff*[*symmetric*]]] *z1*
  by *auto*
 have *f' y* − (*f x* − *f y*) / (*x* − *y*) = *f' y* − *f' z1*
  using ∗ *z1'* **by** *auto*
 also have ... = (*y* − *z1*) ∗ *f'' z3*
  using *z3* **by** *auto*
 finally have *cool'*: *f' y* − (*f x* − *f y*) / (*x* − *y*) = (*y* − *z1*) ∗ *f'' z3*
  by *simp*
 have *A'*: *y* − *z1* ≥ *0*
  using *z1* **by** *auto*
 have *z3* ∈ *C*
  using *z3* ∗ *atMostAtLeast_subset_convex* ‹*convex C*› ‹*x* ∈ *C*› ‹*z1* ∈ *C*› ‹*x* < *z1*›
  by *fastforce*
 then have *B'*: *f'' z3* ≥ *0*
  using *assms* **by** *auto*
 from *A' B'* **have** (*y* − *z1*) ∗ *f'' z3* ≥ *0*
  by *auto*
 from *cool' this* **have** *f' y* − (*f x* − *f y*) / (*x* − *y*) ≥ *0*
  by *auto*
 from *mult_right_mono_neg*[*OF this le*(*2*)]
 have *f' y* ∗ (*x* − *y*) − (*f x* − *f y*) / (*x* − *y*) ∗ (*x* − *y*) ≤ *0* ∗ (*x* − *y*)
  by (*simp add*: *algebra_simps*)
 then have *f' y* ∗ (*x* − *y*) − (*f x* − *f y*) ≤ *0*
  using *le* **by** *auto*
 then have *res*: *f' y* ∗ (*x* − *y*) ≤ *f x* − *f y*
  by *auto*
 have (*f y* − *f x*) / (*y* − *x*) − *f' x* = *f' z1* − *f' x*
  using ∗ *z1* **by** *auto*
 also have ... = (*z1* − *x*) ∗ *f'' z2*
  using *z2* **by** *auto*
 finally have *cool*: (*f y* − *f x*) / (*y* − *x*) − *f' x* = (*z1* − *x*) ∗ *f'' z2*

```
      by simp
    have A: z1 − x ≥ 0
      using z1 by auto
    have z2 ∈ C
      using z2 z1 ∗ atMostAtLeast_subset_convex ‹convex C› ‹z1 ∈ C› ‹y ∈ C› ‹z1
< y›
      by fastforce
    then have B: f'' z2 ≥ 0
      using assms by auto
    from A B have (z1 − x) ∗ f'' z2 ≥ 0
      by auto
    with cool have (f y − f x) / (y − x) − f' x ≥ 0
      by auto
    from mult_right_mono[OF this ge(2)]
    have (f y − f x) / (y − x) ∗ (y − x) − f' x ∗ (y − x) ≥ 0 ∗ (y − x)
      by (simp add: algebra_simps)
    then have f y − f x − f' x ∗ (y − x) ≥ 0
      using ge by auto
    then show f y − f x ≥ f' x ∗ (y − x) f' y ∗ (x − y) ≤ f x − f y
      using res by auto
  qed
  show ?thesis
  proof (cases x = y)
    case True
    with x y show ?thesis by auto
  next
    case False
    with less_imp x y show ?thesis
      by (auto simp: neq_iff)
  qed
qed

lemma f''_ge0_imp_convex:
  fixes f :: real ⇒ real
  assumes conv: convex C
    and f': ⋀x. x ∈ C ⟹ DERIV f x :> (f' x)
    and f'': ⋀x. x ∈ C ⟹ DERIV f' x :> (f'' x)
    and pos: ⋀x. x ∈ C ⟹ f'' x ≥ 0
  shows convex_on C f
  using f''_imp_f'[OF conv f' f'' pos] assms pos_convex_function
  by fastforce

lemma minus_log_convex:
  fixes b :: real
  assumes b > 1
  shows convex_on {0 <..} (λ x. − log b x)
proof −
  have ⋀z. z > 0 ⟹ DERIV (log b) z :> 1 / (ln b ∗ z)
    using DERIV_log by auto
```

then have f': $\bigwedge z.\ z > 0 \Longrightarrow DERIV\ (\lambda\ z.\ -\ log\ b\ z)\ z :> -\ 1\ /\ (ln\ b * z)$
  by (*auto simp*: *DERIV_minus*)
  have $\bigwedge z::real.\ z > 0 \Longrightarrow DERIV\ inverse\ z :> -\ (inverse\ z\ \hat{}\ Suc\ (Suc\ 0))$
    using *less_imp_neq*[*THEN not_sym*, *THEN DERIV_inverse*] **by** *auto*
  from *this*[*THEN DERIV_cmult*, *of* _ $-\ 1\ /\ ln\ b$]
  have $\bigwedge z::real.\ z > 0 \Longrightarrow$
    $DERIV\ (\lambda\ z.\ (-\ 1\ /\ ln\ b) * inverse\ z)\ z :> (-\ 1\ /\ ln\ b) * (-\ (inverse\ z\ \hat{}\ Suc\ (Suc\ 0)))$
    by *auto*
  then have $f''0$: $\bigwedge z::real.\ z > 0 \Longrightarrow$
    $DERIV\ (\lambda\ z.\ -\ 1\ /\ (ln\ b * z))\ z :> 1\ /\ (ln\ b * z * z)$
    unfolding *inverse_eq_divide* **by** (*auto simp*: *mult.assoc*)
  have f''_*ge0*: $\bigwedge z::real.\ z > 0 \Longrightarrow 1\ /\ (ln\ b * z * z) \geq 0$
    using ‹$b > 1$› **by** (*auto intro*!: *less_imp_le*)
  from f''_*ge0_imp_convex*[*OF convex_real_interval*(*3*), *unfolded greaterThan_iff*, *OF* f' $f''0$ f''_*ge0*]
  show *?thesis*
    by *auto*
qed

1.7.5 Convexity of real functions

lemma *convex_on_realI*:
  assumes *connected A*
    and $\bigwedge x.\ x \in A \Longrightarrow (f\ has\_real\_derivative\ f'\ x)\ (at\ x)$
    and $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow x \leq y \Longrightarrow f'\ x \leq f'\ y$
  shows *convex_on A f*
proof (*rule convex_on_linorderI*)
  fix *t x y* :: *real*
  assume *t*: $t > 0$ $t < 1$
  assume *xy*: $x \in A$ $y \in A$ $x < y$
  define *z* **where** $z = (1\ -\ t) * x + t * y$
  with ‹*connected A*› **and** *xy* **have** *ivl*: $\{x..y\} \subseteq A$
    using *connected_contains_Icc* **by** *blast*

  from *xy t* **have** *xz*: $z > x$
    by (*simp add*: *z_def algebra_simps*)
  have $y\ -\ z = (1\ -\ t) * (y\ -\ x)$
    by (*simp add*: *z_def algebra_simps*)
  also from *xy t* **have** $\ldots > 0$
    by (*intro mult_pos_pos*) *simp_all*
  finally have *yz*: $z < y$
    by *simp*

  from *assms xz yz ivl t* **have** $\exists\xi.\ \xi > x \land \xi < z \land f\ z\ -\ f\ x = (z\ -\ x) * f'\ \xi$
    by (*intro MVT2*) (*auto intro*!: *assms*(*2*))
  then obtain ξ **where** ξ: $\xi > x$ $\xi < z$ $f'\ \xi = (f\ z\ -\ f\ x)\ /\ (z\ -\ x)$
    by *auto*
  from *assms xz yz ivl t* **have** $\exists\eta.\ \eta > z \land \eta < y \land f\ y\ -\ f\ z = (y\ -\ z) * f'\ \eta$

by (*intro MVT2*) (*auto intro*!: *assms*(*2*))
then obtain η **where** η: $\eta > z$ $\eta < y$ $f' \eta = (f y - f z) / (y - z)$
by *auto*

from η(*3*) **have** $(f y - f z) / (y - z) = f' \eta$ **..**
also from ξ η *ivl* **have** $\xi \in A$ $\eta \in A$
by *auto*
with ξ η **have** $f' \eta \geq f' \xi$
by (*intro assms*(*3*)) *auto*
also from ξ(*3*) **have** $f' \xi = (f z - f x) / (z - x)$ **.**
finally have $(f y - f z) * (z - x) \geq (f z - f x) * (y - z)$
using *xz yz* **by** (*simp add*: *field_simps*)
also have $z - x = t * (y - x)$
by (*simp add*: *z_def algebra_simps*)
also have $y - z = (1 - t) * (y - x)$
by (*simp add*: *z_def algebra_simps*)
finally have $(f y - f z) * t \geq (f z - f x) * (1 - t)$
using *xy* **by** *simp*
then show $(1 - t) * f x + t * f y \geq f ((1 - t) *_R x + t *_R y)$
by (*simp add*: *z_def algebra_simps*)
qed

lemma *convex_on_inverse*:
assumes $A \subseteq \{0<..\}$
shows *convex_on A* (*inverse* :: *real* $\Rightarrow$ *real*)
proof (*rule convex_on_subset*[*OF _ assms*], *intro convex_on_realI*[*of _ _* $\lambda x. -$*inverse* (xˆ*2*)])
fix *u v* :: *real*
assume $u \in \{0<..\}$ $v \in \{0<..\}$ $u \leq v$
with *assms* **show** $-$*inverse* (uˆ*2*) $\leq -$*inverse* (vˆ*2*)
by (*intro le_imp_neg_le le_imp_inverse_le power_mono*) (*simp_all*)
qed (*insert assms*, *auto intro*!: *derivative_eq_intros simp*: *field_split_simps power2_eq_square*)

lemma *convex_onD_Icc'*:
assumes *convex_on* $\{x..y\}$ f $c \in \{x..y\}$
defines $d \equiv y - x$
shows $f c \leq (f y - f x) / d * (c - x) + f x$
proof (*cases x y rule*: *linorder_cases*)
case *less*
then have d: $d > 0$
by (*simp add*: *d_def*)
from *assms*(*2*) *less* **have** A: $0 \leq (c - x) / d$ $(c - x) / d \leq 1$
by (*simp_all add*: *d_def field_split_simps*)
have $f c = f (x + (c - x) * 1)$
by *simp*
also from *less* **have** $1 = ((y - x) / d)$
by (*simp add*: *d_def*)
also from d **have** $x + (c - x) * \ldots = (1 - (c - x) / d) *_R x + ((c - x) / d) *_R y$

```
    by (simp add: field_simps)
  also have f ... ≤ (1 − (c − x) / d) ∗ f x + (c − x) / d ∗ f y
    using assms less by (intro convex_onD_Icc) simp_all
  also from d have ... = (f y − f x) / d ∗ (c − x) + f x
    by (simp add: field_simps)
  finally show ?thesis .
qed (insert assms(2), simp_all)

lemma convex_onD_Icc'':
  assumes convex_on {x..y} f c ∈ {x..y}
  defines d ≡ y − x
  shows f c ≤ (f x − f y) / d ∗ (y − c) + f y
proof (cases x y rule: linorder_cases)
  case less
  then have d: d > 0
    by (simp add: d_def)
  from assms(2) less have A: 0 ≤ (y − c) / d (y − c) / d ≤ 1
    by (simp_all add: d_def field_split_simps)
  have f c = f (y − (y − c) ∗ 1)
    by simp
  also from less have 1 = ((y − x) / d)
    by (simp add: d_def)
  also from d have y − (y − c) ∗ ... = (1 − (1 − (y − c) / d)) ∗R x + (1 −
(y − c) / d) ∗R y
    by (simp add: field_simps)
  also have f ... ≤ (1 − (1 − (y − c) / d)) ∗ f x + (1 − (y − c) / d) ∗ f y
    using assms less by (intro convex_onD_Icc) (simp_all add: field_simps)
  also from d have ... = (f x − f y) / d ∗ (y − c) + f y
    by (simp add: field_simps)
  finally show ?thesis .
qed (insert assms(2), simp_all)

lemma convex_translation_eq [simp]:
  convex ((+) a ‘ s) ⟷ convex s
  by (metis convex_translation translation_galois)

lemma convex_translation_subtract_eq [simp]:
  convex ((λb. b − a) ‘ s) ⟷ convex s
  using convex_translation_eq [of − a] by (simp cong: image_cong_simp)

lemma convex_linear_image_eq [simp]:
    fixes f :: 'a::real_vector ⇒ 'b::real_vector
    shows ⟦linear f; inj f⟧ ⟹ convex (f ‘ s) ⟷ convex s
   by (metis (no_types) convex_linear_image convex_linear_vimage inj_vimage_image_eq)

lemma fst_snd_linear: linear (λ(x,y). x + y)
  unfolding linear_iff by (simp add: algebra_simps)

lemma vector_choose_size:
```

assumes *0 ≤ c*
obtains *x* :: *'a*::{*real_normed_vector*, *perfect_space*} **where** *norm x* = *c*
proof −
obtain *a*::*'a* **where** *a* ≠ *0*
using *UNIV_not_singleton UNIV_eq_I set_zero singletonI* **by** *fastforce*
then show *?thesis*
by (*rule_tac x*=*scaleR* (*c* / *norm a*) *a* **in** *that*) (*simp add*: *assms*)
qed

lemma *vector_choose_dist*:
assumes *0 ≤ c*
obtains *y* :: *'a*::{*real_normed_vector*, *perfect_space*} **where** *dist x y* = *c*
by (*metis add_diff_cancel_left' assms dist_commute dist_norm vector_choose_size*)

lemma *sum_delta''*:
fixes *s*::*'a*::*real_vector set*
assumes *finite s*
shows (∑ *x*∈*s*. (*if y* = *x then f x else 0*) $*_R$ *x*) = (*if y*∈*s then* (*f y*) $*_R$ *y else 0*)
proof −
have ∗: ⋀*x y*. (*if y* = *x then f x else* (*0*::*real*)) $*_R$ *x* = (*if x*=*y then* (*f x*) $*_R$ *x else 0*)
by *auto*
show *?thesis*
unfolding ∗ **using** *sum.delta*[*OF assms*, *of y* λ*x*. *f x* $*_R$ *x*] **by** *auto*
qed

lemma *dist_triangle_eq*:
fixes *x y z* :: *'a*::*real_inner*
shows *dist x z* = *dist x y* + *dist y z* ⟷
norm (*x* − *y*) $*_R$ (*y* − *z*) = *norm* (*y* − *z*) $*_R$ (*x* − *y*)
proof −
have ∗: *x* − *y* + (*y* − *z*) = *x* − *z* **by** *auto*
show *?thesis* **unfolding** *dist_norm norm_triangle_eq*[*of x* − *y y* − *z*, *unfolded* ∗]
by (*auto simp*:*norm_minus_commute*)
qed

1.7.6 Cones

definition *cone* :: *'a*::*real_vector set* ⇒ *bool*
where *cone s* ⟷ (∀ *x*∈*s*. ∀ *c*≥*0*. *c* $*_R$ *x* ∈ *s*)

lemma *cone_empty*[*intro*, *simp*]: *cone* {}
unfolding *cone_def* **by** *auto*

lemma *cone_univ*[*intro*, *simp*]: *cone UNIV*
unfolding *cone_def* **by** *auto*

lemma *cone_Inter*[*intro*]: ∀ *s*∈*f*. *cone s* ⟹ *cone* (⋂ *f*)
unfolding *cone_def* **by** *auto*

```
lemma subspace_imp_cone: subspace S ⟹ cone S
  by (simp add: cone_def subspace_scale)
```

Conic hull

```
lemma cone_cone_hull: cone (cone hull S)
  unfolding hull_def by auto

lemma cone_hull_eq: cone hull S = S ⟷ cone S
  by (metis cone_cone_hull hull_same)

lemma mem_cone:
  assumes cone S x ∈ S c ≥ 0
  shows c *R x ∈ S
  using assms cone_def[of S] by auto

lemma cone_contains_0:
  assumes cone S
  shows S ≠ {} ⟷ 0 ∈ S
  using assms mem_cone by fastforce

lemma cone_0: cone {0}
  unfolding cone_def by auto

lemma cone_Union[intro]: (∀ s∈f. cone s) ⟶ cone (⋃ f)
  unfolding cone_def by blast

lemma cone_iff:
  assumes S ≠ {}
  shows cone S ⟷ 0 ∈ S ∧ (∀ c. c > 0 ⟶ ((*R) c) ‘ S = S)
proof -
  {
    assume cone S
    {
      fix c :: real
      assume c > 0
      {
        fix x
        assume x ∈ S
        then have x ∈ ((*R) c) ‘ S
          unfolding image_def
          using ‹cone S› ‹c>0› mem_cone[of S x 1/c]
            exI[of (λt. t ∈ S ∧ x = c *R t) (1 / c) *R x]
          by auto
      }
      moreover
      {
        fix x
```

assume $x \in ((*_R)\ c)$ ‘ S
then have $x \in S$
using ‹$0 < c$› ‹*cone S*› *mem_cone* **by** *fastforce*
}
ultimately have $((*_R)\ c)$ ‘ $S = S$ **by** *blast*
}
then have $0 \in S \wedge (\forall c.\ c > 0 \longrightarrow ((*_R)\ c)$ ‘ $S = S)$
using ‹*cone S*› *cone_contains_0*[*of S*] *assms* **by** *auto*
}
moreover
{
assume *a*: $0 \in S \wedge (\forall c.\ c > 0 \longrightarrow ((*_R)\ c)$ ‘ $S = S)$
{
fix x
assume $x \in S$
fix *c1* :: *real*
assume $c1 \geq 0$
then have $c1 = 0 \vee c1 > 0$ **by** *auto*
then have $c1 *_R x \in S$ **using** *a* ‹$x \in S$› **by** *auto*
}
then have *cone S* **unfolding** *cone_def* **by** *auto*
}
ultimately show *?thesis* **by** *blast*
qed

lemma *cone_hull_empty*: *cone hull* {} = {}
by (*metis cone_empty cone_hull_eq*)

lemma *cone_hull_empty_iff*: $S = \{\} \longleftrightarrow$ *cone hull* $S = \{\}$
by (*metis bot_least cone_hull_empty hull_subset xtrans*(*5*))

lemma *cone_hull_contains_0*: $S \neq \{\} \longleftrightarrow 0 \in$ *cone hull S*
using *cone_cone_hull*[*of S*] *cone_contains_0*[*of cone hull S*] *cone_hull_empty_iff*[*of S*]
by *auto*

lemma *mem_cone_hull*:
assumes $x \in S$ $c \geq 0$
shows $c *_R x \in$ *cone hull S*
by (*metis assms cone_cone_hull hull_inc mem_cone*)

proposition *cone_hull_expl*: *cone hull* $S = \{c *_R x \mid c\ x.\ c \geq 0 \wedge x \in S\}$
(**is** *?lhs* = *?rhs*)
proof −
{
fix x
assume $x \in$ *?rhs*
then obtain *cx* :: *real* **and** *xx* **where** *x*: $x = cx *_R xx$ $cx \geq 0$ $xx \in S$
by *auto*

```
    fix c :: real
    assume c: c ≥ 0
    then have c *R x = (c * cx) *R xx
      using x by (simp add: algebra_simps)
    moreover
    have c * cx ≥ 0 using c x by auto
    ultimately
    have c *R x ∈ ?rhs using x by auto
  }
  then have cone ?rhs
    unfolding cone_def by auto
  then have ?rhs ∈ Collect cone
    unfolding mem_Collect_eq by auto
  {
    fix x
    assume x ∈ S
    then have 1 *R x ∈ ?rhs
      using zero_le_one by blast
    then have x ∈ ?rhs by auto
  }
  then have S ⊆ ?rhs by auto
  then have ?lhs ⊆ ?rhs
    using ‹?rhs ∈ Collect cone› hull_minimal[of S ?rhs cone] by auto
  moreover
  {
    fix x
    assume x ∈ ?rhs
    then obtain cx :: real and xx where x: x = cx *R xx cx ≥ 0 xx ∈ S
      by auto
    then have xx ∈ cone hull S
      using hull_subset[of S] by auto
    then have x ∈ ?lhs
      using x cone_cone_hull[of S] cone_def[of cone hull S] by auto
  }
  ultimately show ?thesis by auto
qed

lemma convex_cone:
  convex s ∧ cone s ⟷ (∀x∈s. ∀y∈s. (x + y) ∈ s) ∧ (∀x∈s. ∀c≥0. (c *R x) ∈
s)
  (is ?lhs = ?rhs)
proof –
  {
    fix x y
    assume x∈s y∈s and ?lhs
    then have 2 *R x ∈s 2 *R y ∈ s
      unfolding cone_def by auto
    then have x + y ∈ s
      using ‹?lhs›[unfolded convex_def, THEN conjunct1]
```

```
    apply (erule_tac x=2*R x in ballE)
    apply (erule_tac x=2*R y in ballE)
    apply (erule_tac x=1/2 in allE, simp)
    apply (erule_tac x=1/2 in allE, auto)
    done
  }
  then show ?thesis
    unfolding convex_def cone_def by blast
qed
```

1.7.7 Connectedness of convex sets

```
lemma convex_connected:
  fixes S :: 'a::real_normed_vector set
  assumes convex S
  shows connected S
proof (rule connectedI)
  fix A B
  assume open A open B A ∩ B ∩ S = {} S ⊆ A ∪ B
  moreover
  assume A ∩ S ≠ {} B ∩ S ≠ {}
  then obtain a b where a: a ∈ A a ∈ S and b: b ∈ B b ∈ S by auto
  define f where [abs_def]: f u = u *R a + (1 − u) *R b for u
  then have continuous_on {0 .. 1} f
    by (auto intro!: continuous_intros)
  then have connected (f ‘ {0 .. 1})
    by (auto intro!: connected_continuous_image)
  note connectedD[OF this, of A B]
  moreover have a ∈ A ∩ f ‘ {0 .. 1}
    using a by (auto intro!: image_eqI[of _ _ 1] simp: f_def)
  moreover have b ∈ B ∩ f ‘ {0 .. 1}
    using b by (auto intro!: image_eqI[of _ _ 0] simp: f_def)
  moreover have f ‘ {0 .. 1} ⊆ S
    using ‹convex S› a b unfolding convex_def f_def by auto
  ultimately show False by auto
qed

corollary connected_UNIV[intro]: connected (UNIV :: 'a::real_normed_vector set)
by (simp add: convex_connected)

lemma convex_prod:
  assumes ⋀i. i ∈ Basis ⟹ convex {x. P i x}
  shows convex {x. ∀ i∈Basis. P i (x•i)}
  using assms unfolding convex_def
  by (auto simp: inner_add_left)

lemma convex_positive_orthant: convex {x::'a::euclidean_space. (∀ i∈Basis. 0 ≤
x•i)}
by (rule convex_prod) (simp flip: atLeast_def)
```

1.7.8 Convex hull

lemma *convex_convex_hull* [*iff*]: *convex* (*convex hull s*)
unfolding *hull_def*
using *convex_Inter*[*of* {*t*. *convex t* ∧ *s* ⊆ *t*}]
by *auto*

lemma *convex_hull_subset*:
s ⊆ *convex hull t* ⟹ *convex hull s* ⊆ *convex hull t*
by (*simp add*: *subset_hull*)

lemma *convex_hull_eq*: *convex hull s* = *s* ⟷ *convex s*
by (*metis convex_convex_hull hull_same*)

Convex hull is "preserved" by a linear function

lemma *convex_hull_linear_image*:
assumes *f*: *linear f*
shows *f* ' (*convex hull s*) = *convex hull* (*f* ' *s*)
proof
show *convex hull* (*f* ' *s*) ⊆ *f* ' (*convex hull s*)
by (*intro hull_minimal image_mono hull_subset convex_linear_image assms convex_convex_hull*)
show *f* ' (*convex hull s*) ⊆ *convex hull* (*f* ' *s*)
proof (*unfold image_subset_iff_subset_vimage*, *rule hull_minimal*)
show *s* ⊆ *f* −' (*convex hull* (*f* ' *s*))
by (*fast intro*: *hull_inc*)
show *convex* (*f* −' (*convex hull* (*f* ' *s*)))
by (*intro convex_linear_vimage* [*OF f*] *convex_convex_hull*)
qed
qed

lemma *in_convex_hull_linear_image*:
assumes *linear f*
and *x* ∈ *convex hull s*
shows *f x* ∈ *convex hull* (*f* ' *s*)
using *convex_hull_linear_image*[*OF assms*(*1*)] *assms*(*2*) **by** *auto*

lemma *convex_hull_Times*:
convex hull (*s* × *t*) = (*convex hull s*) × (*convex hull t*)
proof
show *convex hull* (*s* × *t*) ⊆ (*convex hull s*) × (*convex hull t*)
by (*intro hull_minimal Sigma_mono hull_subset convex_Times convex_convex_hull*)
have (*x*, *y*) ∈ *convex hull* (*s* × *t*) **if** *x*: *x* ∈ *convex hull s* **and** *y*: *y* ∈ *convex hull t* **for** *x y*
proof (*rule hull_induct* [*OF x*], *rule hull_induct* [*OF y*])
fix *x y* **assume** *x* ∈ *s* **and** *y* ∈ *t*
then show (*x*, *y*) ∈ *convex hull* (*s* × *t*)
by (*simp add*: *hull_inc*)
next

fix *x* **let** *?S* = ((λ*y*. (*0*, *y*)) −' (λ*p*. (− *x*, *0*) + *p*) ' (*convex hull s* × *t*))
have *convex ?S*
by (*intro convex_linear_vimage convex_translation convex_convex_hull*, *simp add*: *linear_iff*)
also have *?S* = {*y*. (*x*, *y*) ∈ *convex hull* (*s* × *t*)}
by (*auto simp*: *image_def Bex_def*)
finally show *convex* {*y*. (*x*, *y*) ∈ *convex hull* (*s* × *t*)} **.**
next
show *convex* {*x*. (*x*, *y*) ∈ *convex hull s* × *t*}
proof −
fix *y* **let** *?S* = ((λ*x*. (*x*, *0*)) −' (λ*p*. (*0*, − *y*) + *p*) ' (*convex hull s* × *t*))
have *convex ?S*
by (*intro convex_linear_vimage convex_translation convex_convex_hull*, *simp add*: *linear_iff*)
also have *?S* = {*x*. (*x*, *y*) ∈ *convex hull* (*s* × *t*)}
by (*auto simp*: *image_def Bex_def*)
finally show *convex* {*x*. (*x*, *y*) ∈ *convex hull* (*s* × *t*)} **.**
qed
qed
then show (*convex hull s*) × (*convex hull t*) ⊆ *convex hull* (*s* × *t*)
unfolding *subset_eq split_paired_Ball_Sigma* **by** *blast*
qed

Stepping theorems for convex hulls of finite sets

lemma *convex_hull_empty*[*simp*]: *convex hull* {} = {}
by (*rule hull_unique*) *auto*

lemma *convex_hull_singleton*[*simp*]: *convex hull* {*a*} = {*a*}
by (*rule hull_unique*) *auto*

lemma *convex_hull_insert*:
fixes *S* :: *'a*::*real_vector set*
assumes *S* ≠ {}
shows *convex hull* (*insert a S*) =
{*x*. ∃ *u*≥*0*. ∃ *v*≥*0*. ∃ *b*. (*u* + *v* = *1*) ∧ *b* ∈ (*convex hull S*) ∧ (*x* = *u* $*_R$ *a* + *v* $*_R$ *b*)}
(**is** _ = *?hull*)
proof (*intro equalityI hull_minimal subsetI*)
fix *x*
assume *x* ∈ *insert a S*
then have ∃ *u*≥*0*. ∃ *v*≥*0*. *u* + *v* = *1* ∧ (∃ *b*. *b* ∈ *convex hull S* ∧ *x* = *u* $*_R$ *a* + *v* $*_R$ *b*)
unfolding *insert_iff*
proof
assume *x* = *a*
then show *?thesis*
by (*rule_tac x=1* **in** *exI*) (*use assms hull_subset* **in** *fastforce*)
next

```
    assume x ∈ S
    with hull_subset[of S convex] show ?thesis
      by force
  qed
  then show x ∈ ?hull
    by simp
next
  fix x
  assume x ∈ ?hull
  then obtain u v b where obt: u≥0 v≥0 u + v = 1 b ∈ convex hull S x = u *R
a + v *R b
    by auto
  have a ∈ convex hull insert a S b ∈ convex hull insert a S
    using hull_mono[of S insert a S convex] hull_mono[of {a} insert a S convex]
and obt(4)
    by auto
  then show x ∈ convex hull insert a S
    unfolding obt(5) using obt(1−3)
    by (rule convexD [OF convex_convex_hull])
next
  show convex ?hull
  proof (rule convexI)
    fix x y u v
    assume as: (0::real) ≤ u 0 ≤ v u + v = 1 and x: x ∈ ?hull and y: y ∈ ?hull
    from x obtain u1 v1 b1 where
      obt1: u1≥0 v1≥0 u1 + v1 = 1 b1 ∈ convex hull S and xeq: x = u1 *R a +
v1 *R b1
      by auto
    from y obtain u2 v2 b2 where
      obt2: u2≥0 v2≥0 u2 + v2 = 1 b2 ∈ convex hull S and yeq: y = u2 *R a +
v2 *R b2
      by auto
    have *: ⋀(x::'a) s1 s2. x − s1 *R x − s2 *R x = ((1::real) − (s1 + s2)) *R x
      by (auto simp: algebra_simps)
    have ∃b ∈ convex hull S. u *R x + v *R y =
      (u * u1) *R a + (v * u2) *R a + (b − (u * u1) *R b − (v * u2) *R b)
    proof (cases u * v1 + v * v2 = 0)
      case True
      have *: ⋀(x::'a) s1 s2. x − s1 *R x − s2 *R x = ((1::real) − (s1 + s2))
*R x
        by (auto simp: algebra_simps)
      have eq0: u * v1 = 0 v * v2 = 0
       using True mult_nonneg_nonneg[OF ‹u≥0› ‹v1≥0›] mult_nonneg_nonneg[OF
‹v≥0› ‹v2≥0›]
        by arith+
      then have u * u1 + v * u2 = 1
        using as(3) obt1(3) obt2(3) by auto
      then show ?thesis
        using * eq0 as obt1(4) xeq yeq by auto
```

```
next
  case False
  have 1 − (u ∗ u1 + v ∗ u2) = (u + v) − (u ∗ u1 + v ∗ u2)
    using as(3) obt1(3) obt2(3) by (auto simp: field_simps)
  also have . . . = u ∗ (v1 + u1 − u1) + v ∗ (v2 + u2 − u2)
    using as(3) obt1(3) obt2(3) by (auto simp: field_simps)
  also have . . . = u ∗ v1 + v ∗ v2
    by simp
  finally have ∗∗:1 − (u ∗ u1 + v ∗ u2) = u ∗ v1 + v ∗ v2 by auto
  let ?b = ((u ∗ v1) / (u ∗ v1 + v ∗ v2)) ∗R b1 + ((v ∗ v2) / (u ∗ v1 + v ∗
v2)) ∗R b2
  have zeroes: 0 ≤ u ∗ v1 + v ∗ v2 0 ≤ u ∗ v1 0 ≤ u ∗ v1 + v ∗ v2 0 ≤ v ∗
v2
    using as(1,2) obt1(1,2) obt2(1,2) by auto
  show ?thesis
  proof
     show u ∗R x + v ∗R y = (u ∗ u1) ∗R a + (v ∗ u2) ∗R a + (?b − (u ∗
u1) ∗R ?b − (v ∗ u2) ∗R ?b)
      unfolding xeq yeq ∗ ∗∗
      using False by (auto simp: scaleR_left_distrib scaleR_right_distrib)
    show ?b ∈ convex hull S
      using False zeroes obt1(4) obt2(4)
          by (auto simp: convexD [OF convex_convex_hull] scaleR_left_distrib
scaleR_right_distrib add_divide_distrib[symmetric] zero_le_divide_iff)
  qed
qed
then obtain b where b: b ∈ convex hull S
   u ∗R x + v ∗R y = (u ∗ u1) ∗R a + (v ∗ u2) ∗R a + (b − (u ∗ u1) ∗R b
− (v ∗ u2) ∗R b) ..

have u1: u1 ≤ 1
  unfolding obt1(3)[symmetric] and not_le using obt1(2) by auto
have u2: u2 ≤ 1
  unfolding obt2(3)[symmetric] and not_le using obt2(2) by auto
have u1 ∗ u + u2 ∗ v ≤ max u1 u2 ∗ u + max u1 u2 ∗ v
proof (rule add_mono)
  show u1 ∗ u ≤ max u1 u2 ∗ u u2 ∗ v ≤ max u1 u2 ∗ v
    by (simp_all add: as mult_right_mono)
qed
also have . . . ≤ 1
  unfolding distrib_left[symmetric] and as(3) using u1 u2 by auto
finally have le1: u1 ∗ u + u2 ∗ v ≤ 1 .
show u ∗R x + v ∗R y ∈ ?hull
proof (intro CollectI exI conjI)
  show 0 ≤ u ∗ u1 + v ∗ u2
    by (simp add: as(1) as(2) obt1(1) obt2(1))
  show 0 ≤ 1 − u ∗ u1 − v ∗ u2
    by (simp add: le1 diff_diff_add mult.commute)
qed (use b in ‹auto simp: algebra_simps›)
```

```
  qed
qed

lemma convex_hull_insert_alt:
   convex hull (insert a S) =
     (if S = {} then {a}
      else {(1 − u) *R a + u *R x |x u. 0 ≤ u ∧ u ≤ 1 ∧ x ∈ convex hull S})
  apply (auto simp: convex_hull_insert)
  using diff_eq_eq apply fastforce
  using diff_add_cancel diff_ge_0_iff_ge by blast
```

Explicit expression for convex hull

```
proposition convex_hull_indexed:
  fixes S :: 'a::real_vector set
  shows convex hull S =
    {y. ∃ k u x. (∀ i∈{1::nat .. k}. 0 ≤ u i ∧ x i ∈ S) ∧
                (sum u {1..k} = 1) ∧ (∑ i = 1..k. u i *R x i) = y}
    (is ?xyz = ?hull)
proof (rule hull_unique [OF _ convexI])
  show S ⊆ ?hull
    by (clarsimp, rule_tac x=1 in exI, rule_tac x=λx. 1 in exI, auto)
next
  fix T
  assume S ⊆ T convex T
  then show ?hull ⊆ T
    by (blast intro: convex_sum)
next
  fix x y u v
  assume uv: 0 ≤ u 0 ≤ v u + v = (1::real)
  assume xy: x ∈ ?hull y ∈ ?hull
  from xy obtain k1 u1 x1 where
    x [rule_format]: ∀ i∈{1::nat..k1}. 0≤u1 i ∧ x1 i ∈ S
                      sum u1 {Suc 0..k1} = 1 (∑ i = Suc 0..k1. u1 i *R x1 i) = x
    by auto
  from xy obtain k2 u2 x2 where
    y [rule_format]: ∀ i∈{1::nat..k2}. 0≤u2 i ∧ x2 i ∈ S
                     sum u2 {Suc 0..k2} = 1 (∑ i = Suc 0..k2. u2 i *R x2 i) = y
    by auto
  have *: ⋀P (x::'a) y s t i. (if P i then s else t) *R (if P i then x else y) = (if P
i then s *R x else t *R y)
          {1..k1 + k2} ∩ {1..k1} = {1..k1} {1..k1 + k2} ∩ − {1..k1} = (λi. i +
k1) ` {1..k2}
    by auto
  have inj: inj_on (λi. i + k1) {1..k2}
    unfolding inj_on_def by auto
  let ?uu = λi. if i ∈ {1..k1} then u * u1 i else v * u2 (i − k1)
  let ?xx = λi. if i ∈ {1..k1} then x1 i else x2 (i − k1)
  show u *R x + v *R y ∈ ?hull
```

```
  proof (intro CollectI exI conjI ballI)
    show 0 ≤ ?uu i ?xx i ∈ S if i ∈ {1..k1+k2} for i
      using that by (auto simp add: le_diff_conv uv(1) x(1) uv(2) y(1))
    show (∑ i = 1..k1 + k2. ?uu i) = 1  (∑ i = 1..k1 + k2. ?uu i *R ?xx i) =
u *R x + v *R y
      unfolding * sum.If_cases[OF finite_atLeastAtMost[of 1 k1 + k2]]
        sum.reindex[OF inj] Collect_mem_eq o_def
     unfolding scaleR_scaleR[symmetric] scaleR_right.sum [symmetric] sum_distrib_left[symmetric]
      by (simp_all add: sum_distrib_left[symmetric]  x(2,3) y(2,3) uv(3))
  qed
qed

lemma convex_hull_finite:
  fixes S :: 'a::real_vector set
  assumes finite S
  shows convex hull S = {y. ∃ u. (∀ x∈S. 0 ≤ u x) ∧ sum u S = 1 ∧ sum (λx. u
x *R x) S = y}
  (is ?HULL = _)
proof (rule hull_unique [OF _ convexI]; clarify)
  fix x
  assume x ∈ S
  then show ∃ u. (∀ x∈S. 0 ≤ u x) ∧ sum u S = 1 ∧ (∑ x∈S. u x *R x) = x
     by (rule_tac x=λy. if x=y then 1 else 0 in exI) (auto simp: sum.delta'[OF
assms] sum_delta''[OF assms])
next
  fix u v :: real
  assume uv: 0 ≤ u 0 ≤ v u + v = 1
  fix ux assume ux [rule_format]: ∀ x∈S. 0 ≤ ux x sum ux S = (1::real)
  fix uy assume uy [rule_format]: ∀ x∈S. 0 ≤ uy x sum uy S = (1::real)
  have 0 ≤ u * ux x + v * uy x if x∈S for x
    by (simp add: that uv ux(1) uy(1))
  moreover
  have (∑ x∈S. u * ux x + v * uy x) = 1
    unfolding sum.distrib and sum_distrib_left[symmetric] ux(2) uy(2)
    using uv(3) by auto
  moreover
  have (∑ x∈S. (u * ux x + v * uy x) *R x) = u *R (∑ x∈S. ux x *R x) + v *R
(∑ x∈S. uy x *R x)
   unfolding scaleR_left_distrib sum.distrib scaleR_scaleR[symmetric] scaleR_right.sum
[symmetric]
    by auto
  ultimately
  show ∃ uc. (∀ x∈S. 0 ≤ uc x) ∧ sum uc S = 1 ∧
          (∑ x∈S. uc x *R x) = u *R (∑ x∈S. ux x *R x) + v *R (∑ x∈S. uy x
*R x)
    by (rule_tac x=λx. u * ux x + v * uy x in exI, auto)
qed (use assms in ‹auto simp: convex_explicit›)
```

Another formulation

Formalized by Lars Schewe.

lemma *convex_hull_explicit*:
  fixes *p* :: *'a::real_vector set*
  shows *convex hull p* =
    {*y*. ∃ *S u*. *finite S* ∧ *S* ⊆ *p* ∧ (∀ *x*∈*S*. *0* ≤ *u x*) ∧ *sum u S* = *1* ∧ *sum* (λ*v*. *u v* $*_R$ *v*) *S* = *y*}
  (**is** *?lhs* = *?rhs*)
proof −
  {
    fix *x*
    assume *x*∈*?lhs*
    then obtain *k u y* **where**
        obt: ∀ *i*∈{*1::nat..k*}. *0* ≤ *u i* ∧ *y i* ∈ *p sum u* {*1..k*} = *1* (∑ *i* = *1..k*. *u i* $*_R$ *y i*) = *x*
      unfolding *convex_hull_indexed* **by** *auto*

    have *fin*: *finite* {*1..k*} **by** *auto*
    have *fin'*: ⋀*v*. *finite* {*i* ∈ {*1..k*}. *y i* = *v*} **by** *auto*
    {
      fix *j*
      assume *j*∈{*1..k*}
      then have *y j* ∈ *p* ∧ *0* ≤ *sum u* {*i*. *Suc 0* ≤ *i* ∧ *i* ≤ *k* ∧ *y i* = *y j*}
        using *obt*(*1*)[*THEN bspec*[**where** *x*=*j*]] **and** *obt*(*2*)
        by (*metis* (*no_types*, *lifting*) *One_nat_def atLeastAtMost_iff mem_Collect_eq obt*(*1*) *sum_nonneg*)
    }
    moreover
    have (∑ *v*∈*y* ‘ {*1..k*}. *sum u* {*i* ∈ {*1..k*}. *y i* = *v*}) = *1*
      unfolding *sum.image_gen*[*OF fin*, *symmetric*] **using** *obt*(*2*) **by** *auto*
    moreover have (∑ *v*∈*y* ‘ {*1..k*}. *sum u* {*i* ∈ {*1..k*}. *y i* = *v*} $*_R$ *v*) = *x*
      using *sum.image_gen*[*OF fin*, *of* λ*i*. *u i* $*_R$ *y i y*, *symmetric*]
      unfolding *scaleR_left.sum* **using** *obt*(*3*) **by** *auto*
    ultimately
    have ∃ *S u*. *finite S* ∧ *S* ⊆ *p* ∧ (∀ *x*∈*S*. *0* ≤ *u x*) ∧ *sum u S* = *1* ∧ (∑ *v*∈*S*. *u v* $*_R$ *v*) = *x*
      apply (*rule_tac x*=*y* ‘ {*1..k*} **in** *exI*)
      apply (*rule_tac x*=λ*v*. *sum u* {*i*∈{*1..k*}. *y i* = *v*} **in** *exI*, *auto*)
      done
    then have *x*∈*?rhs* **by** *auto*
  }
  moreover
  {
    fix *y*
    assume *y*∈*?rhs*
    then obtain *S u* **where**
      obt: *finite S S* ⊆ *p* ∀ *x*∈*S*. *0* ≤ *u x sum u S* = *1* (∑ *v*∈*S*. *u v* $*_R$ *v*) = *y*
      by *auto*

obtain *f* **where** *f*: *inj_on f* {*1..card S*} *f* ' {*1..card S*} = *S*
using *ex_bij_betw_nat_finite_1*[*OF obt*(*1*)] **unfolding** *bij_betw_def* **by** *auto*
{
fix *i* :: *nat*
assume *i*∈{*1..card S*}
then have *f i* ∈ *S*
using *f*(*2*) **by** *blast*
then have *0* ≤ *u* (*f i*) *f i* ∈ *p* **using** *obt*(*2*,*3*) **by** *auto*
}
moreover have ∗: *finite* {*1..card S*} **by** *auto*
{
fix *y*
assume *y*∈*S*
then obtain *i* **where** *i*∈{*1..card S*} *f i* = *y*
using *f* **using** *image_iff*[*of y f* {*1..card S*}]
by *auto*
then have {*x*. *Suc 0* ≤ *x* ∧ *x* ≤ *card S* ∧ *f x* = *y*} = {*i*}
using *f*(*1*) *inj_onD* **by** *fastforce*
then have *card* {*x*. *Suc 0* ≤ *x* ∧ *x* ≤ *card S* ∧ *f x* = *y*} = *1* **by** *auto*
then have (∑ *x*∈{*x* ∈ {*1..card S*}. *f x* = *y*}. *u* (*f x*)) = *u y*
(∑ *x*∈{*x* ∈ {*1..card S*}. *f x* = *y*}. *u* (*f x*) $*_R$ *f x*) = *u y* $*_R$ *y*
by (*auto simp*: *sum_constant_scaleR*)
}
then have (∑ *x* = *1..card S*. *u* (*f x*)) = *1* (∑ *i* = *1..card S*. *u* (*f i*) $*_R$ *f i*) =
y
unfolding *sum.image_gen*[*OF* ∗(*1*), *of* λ*x*. *u* (*f x*) $*_R$ *f x f*]
and *sum.image_gen*[*OF* ∗(*1*), *of* λ*x*. *u* (*f x*) *f*]
unfolding *f*
using *sum.cong* [*of S S* λ*y*. (∑ *x*∈{*x* ∈ {*1..card S*}. *f x* = *y*}. *u* (*f x*) $*_R$ *f*
x) λ*v*. *u v* $*_R$ *v*]
using *sum.cong* [*of S S* λ*y*. (∑ *x*∈{*x* ∈ {*1..card S*}. *f x* = *y*}. *u* (*f x*)) *u*]
unfolding *obt*(*4*,*5*)
by *auto*
ultimately
have ∃ *k u x*. (∀ *i*∈{*1..k*}. *0* ≤ *u i* ∧ *x i* ∈ *p*) ∧ *sum u* {*1..k*} = *1* ∧
(∑ *i*::*nat* = *1..k*. *u i* $*_R$ *x i*) = *y*
apply (*rule_tac x*=*card S* **in** *exI*)
apply (*rule_tac x*=*u* ∘ *f* **in** *exI*)
apply (*rule_tac x*=*f* **in** *exI*, *fastforce*)
done
then have *y* ∈ *?lhs*
unfolding *convex_hull_indexed* **by** *auto*
}
ultimately show *?thesis*
unfolding *set_eq_iff* **by** *blast*
qed

A stepping theorem for that expansion

```
lemma convex_hull_finite_step:
  fixes S :: 'a::real_vector set
  assumes finite S
  shows
    (∃u. (∀x∈insert a S. 0 ≤ u x) ∧ sum u (insert a S) = w ∧ sum (λx. u x *R
x) (insert a S) = y)
      ⟷ (∃v≥0. ∃u. (∀x∈S. 0 ≤ u x) ∧ sum u S = w − v ∧ sum (λx. u x *R
x) S = y − v *R a)
  (is ?lhs = ?rhs)
proof (cases a ∈ S)
  case True
  then have *: insert a S = S by auto
  show ?thesis
  proof
    assume ?lhs
    then show ?rhs
      unfolding * by force
  next
    have fin: finite (insert a S) using assms by auto
    assume ?rhs
    then obtain v u where uv: v≥0 ∀x∈S. 0 ≤ u x sum u S = w − v (∑x∈S.
u x *R x) = y − v *R a
      by auto
    then show ?lhs
      using uv True assms
      apply (rule_tac x = λx. (if a = x then v else 0) + u x in exI)
      apply (auto simp: sum_clauses scaleR_left_distrib sum.distrib sum_delta''[OF
fin])
      done
  qed
next
  case False
  show ?thesis
  proof
    assume ?lhs
     then obtain u where u: ∀x∈insert a S. 0 ≤ u x sum u (insert a S) = w
(∑x∈insert a S. u x *R x) = y
      by auto
    then show ?rhs
      using u ‹a∉S› by (rule_tac x=u a in exI) (auto simp: sum_clauses assms)
  next
    assume ?rhs
    then obtain v u where uv: v≥0 ∀x∈S. 0 ≤ u x sum u S = w − v (∑x∈S.
u x *R x) = y − v *R a
      by auto
    moreover
    have (∑x∈S. if a = x then v else u x) = sum u S  (∑x∈S. (if a = x then v
else u x) *R x) = (∑x∈S. u x *R x)
```

using *False* **by** (*auto intro*!: *sum.cong*)
ultimately show *?lhs*
using *False* **by** (*rule_tac x=λx. if a = x then v else u x* **in** *exI*) (*auto simp*: *sum_clauses*(*2*)[*OF assms*])
qed
qed

Hence some special cases

lemma *convex_hull_2*: *convex hull* {*a*,*b*} = {*u* $*_R$ *a* + *v* $*_R$ *b* | *u v*. *0* ≤ *u* ∧ *0* ≤ *v* ∧ *u* + *v* = *1*}
(**is** *?lhs* = *?rhs*)
proof −
have ∗∗: *finite* {*b*} **by** *auto*
have ⋀*x v u*. ⟦*0* ≤ *v*; *v* ≤ *1*; (*1* − *v*) $*_R$ *b* = *x* − *v* $*_R$ *a*⟧
⟹ ∃ *u v*. *x* = *u* $*_R$ *a* + *v* $*_R$ *b* ∧ *0* ≤ *u* ∧ *0* ≤ *v* ∧ *u* + *v* = *1*
by (*metis add.commute diff_add_cancel diff_ge_0_iff_ge*)
moreover
have ⋀*u v*. ⟦*0* ≤ *u*; *0* ≤ *v*; *u* + *v* = *1*⟧
⟹ ∃ *p*≥*0*. ∃ *q*. *0* ≤ *q b* ∧ *q b* = *1* − *p* ∧ *q b* $*_R$ *b* = *u* $*_R$ *a* + *v* $*_R$ *b* − *p* $*_R$ *a*
apply (*rule_tac x=u* **in** *exI*, *simp*)
apply (*rule_tac x=λx. v* **in** *exI*, *simp*)
done
ultimately show *?thesis*
using *convex_hull_finite_step*[*OF* ∗∗, *of a 1*]
by (*auto simp add*: *convex_hull_finite*)
qed

lemma *convex_hull_2_alt*: *convex hull* {*a*,*b*} = {*a* + *u* $*_R$ (*b* − *a*) | *u*. *0* ≤ *u* ∧ *u* ≤ *1*}
unfolding *convex_hull_2*
proof (*rule Collect_cong*)
have ∗: ⋀*x y* ::*real*. *x* + *y* = *1* ⟷ *x* = *1* − *y*
by *auto*
fix *x*
show (∃ *v u*. *x* = *v* $*_R$ *a* + *u* $*_R$ *b* ∧ *0* ≤ *v* ∧ *0* ≤ *u* ∧ *v* + *u* = *1*) ⟷
(∃ *u*. *x* = *a* + *u* $*_R$ (*b* − *a*) ∧ *0* ≤ *u* ∧ *u* ≤ *1*)
apply (*simp add*: ∗)
by (*rule ex_cong1*) (*auto simp*: *algebra_simps*)
qed

lemma *convex_hull_3*:
convex hull {*a*,*b*,*c*} = { *u* $*_R$ *a* + *v* $*_R$ *b* + *w* $*_R$ *c* | *u v w*. *0* ≤ *u* ∧ *0* ≤ *v* ∧ *0* ≤ *w* ∧ *u* + *v* + *w* = *1*}
proof −
have *fin*: *finite* {*a*,*b*,*c*} *finite* {*b*,*c*} *finite* {*c*}
by *auto*
have ∗: ⋀*x y z* ::*real*. *x* + *y* + *z* = *1* ⟷ *x* = *1* − *y* − *z*

```
    by (auto simp: field_simps)
  show ?thesis
    unfolding convex_hull_finite[OF fin(1)] and convex_hull_finite_step[OF fin(2)]
and *
    unfolding convex_hull_finite_step[OF fin(3)]
    apply (rule Collect_cong, simp)
    apply auto
    apply (rule_tac x=va in exI)
    apply (rule_tac x=u c in exI, simp)
    apply (rule_tac x=1 − v − w in exI, simp)
    apply (rule_tac x=v in exI, simp)
    apply (rule_tac x=λx. w in exI, simp)
    done
qed

lemma convex_hull_3_alt:
  convex hull {a,b,c} = {a + u *R (b − a) + v *R (c − a) | u v.  0 ≤ u ∧ 0 ≤
v ∧ u + v ≤ 1}
proof −
  have *: ⋀x y z ::real. x + y + z = 1 ⟷ x = 1 − y − z
    by auto
  show ?thesis
    unfolding convex_hull_3
    apply (auto simp: *)
    apply (rule_tac x=v in exI)
    apply (rule_tac x=w in exI)
    apply (simp add: algebra_simps)
    apply (rule_tac x=u in exI)
    apply (rule_tac x=v in exI)
    apply (simp add: algebra_simps)
    done
qed
```

1.7.9 Relations among closure notions and corresponding hulls

```
lemma affine_imp_convex: affine s ⟹ convex s
  unfolding affine_def convex_def by auto

lemma convex_affine_hull [simp]: convex (affine hull S)
  by (simp add: affine_imp_convex)

lemma subspace_imp_convex: subspace s ⟹ convex s
  using subspace_imp_affine affine_imp_convex by auto

lemma convex_hull_subset_span: (convex hull s) ⊆ (span s)
  by (metis hull_minimal span_superset subspace_imp_convex subspace_span)

lemma convex_hull_subset_affine_hull: (convex hull s) ⊆ (affine hull s)
```

by (*metis affine_affine_hull affine_imp_convex hull_minimal hull_subset*)

lemma *aff_dim_convex_hull*:
fixes *S* :: *'n::euclidean_space set*
shows *aff_dim* (*convex hull S*) = *aff_dim S*
using *aff_dim_affine_hull*[*of S*] *convex_hull_subset_affine_hull*[*of S*]
hull_subset[*of S convex*] *aff_dim_subset*[*of S convex hull S*]
aff_dim_subset[*of convex hull S affine hull S*]
by *auto*

1.7.10 Caratheodory's theorem

lemma *convex_hull_caratheodory_aff_dim*:
fixes *p* :: (*'a::euclidean_space*) *set*
shows *convex hull p* =
{*y*. ∃ *S u*. *finite S* ∧ *S* ⊆ *p* ∧ *card S* ≤ *aff_dim p* + *1* ∧
(∀ *x*∈*S*. *0* ≤ *u x*) ∧ *sum u S* = *1* ∧ *sum* (λ*v*. *u v* $*_R$ *v*) *S* = *y*}
unfolding *convex_hull_explicit set_eq_iff mem_Collect_eq*
proof (*intro allI iffI*)
fix *y*
let *?P* = λ*n*. ∃ *S u*. *finite S* ∧ *card S* = *n* ∧ *S* ⊆ *p* ∧ (∀ *x*∈*S*. *0* ≤ *u x*) ∧
sum u S = *1* ∧ (∑ *v*∈*S*. *u v* $*_R$ *v*) = *y*
assume ∃ *S u*. *finite S* ∧ *S* ⊆ *p* ∧ (∀ *x*∈*S*. *0* ≤ *u x*) ∧ *sum u S* = *1* ∧ (∑ *v*∈*S*. *u v* $*_R$ *v*) = *y*
then obtain *N* **where** *?P N* **by** *auto*
then have ∃ *n*≤*N*. (∀ *k*<*n*. ¬ *?P k*) ∧ *?P n*
by (*rule_tac ex_least_nat_le*, *auto*)
then obtain *n* **where** *?P n* **and** *smallest*: ∀ *k*<*n*. ¬ *?P k*
by *blast*
then obtain *S u* **where** *obt*: *finite S card S* = *n S*⊆*p* ∀ *x*∈*S*. *0* ≤ *u x*
sum u S = *1* (∑ *v*∈*S*. *u v* $*_R$ *v*) = *y* **by** *auto*

have *card S* ≤ *aff_dim p* + *1*
proof (*rule ccontr*, *simp only*: *not_le*)
assume *aff_dim p* + *1* < *card S*
then have *affine_dependent S*
using *affine_dependent_biggerset*[*OF obt*(*1*)] *independent_card_le_aff_dim not_less obt*(*3*)
by *blast*
then obtain *w v* **where** *wv*: *sum w S* = *0 v*∈*S w v* ≠ *0* (∑ *v*∈*S*. *w v* $*_R$ *v*) = *0*
using *affine_dependent_explicit_finite*[*OF obt*(*1*)] **by** *auto*
define *i* **where** *i* = (λ*v*. (*u v*) / (− *w v*)) ‘ {*v*∈*S*. *w v* < *0*}
define *t* **where** *t* = *Min i*
have ∃ *x*∈*S*. *w x* < *0*
proof (*rule ccontr*, *simp add*: *not_less*)
assume *as*:∀ *x*∈*S*. *0* ≤ *w x*
then have *sum w* (*S* − {*v*}) ≥ *0*
by (*meson Diff_iff sum_nonneg*)

```
    then have sum w S > 0
      using as obt(1) sum_nonneg_eq_0_iff wv by blast
    then show False using wv(1) by auto
  qed
  then have i ≠ {} unfolding i_def by auto
  then have t ≥ 0
    using Min_ge_iff[of i 0] and obt(1)
    unfolding t_def i_def
    using obt(4)[unfolded le_less]
    by (auto simp: divide_le_0_iff)
  have t: ∀ v∈S. u v + t * w v ≥ 0
  proof
    fix v
    assume v ∈ S
    then have v: 0 ≤ u v
      using obt(4)[THEN bspec[where x=v]] by auto
    show 0 ≤ u v + t * w v
    proof (cases w v < 0)
      case False
      thus ?thesis using v ‹t≥0› by auto
    next
      case True
      then have t ≤ u v / (− w v)
        using ‹v∈S› obt unfolding t_def i_def by (auto intro: Min_le)
      then show ?thesis
        unfolding real_0_le_add_iff
        using True neg_le_minus_divide_eq by auto
    qed
  qed
  obtain a where a ∈ S and t = (λv. (u v) / (− w v)) a and w a < 0
    using Min_in[OF _ ‹i≠{}›] and obt(1) unfolding i_def t_def by auto
  then have a: a ∈ S u a + t * w a = 0 by auto
  have *: ⋀f. sum f (S − {a}) = sum f S − ((f a)::'b::ab_group_add)
    unfolding sum.remove[OF obt(1) ‹a∈S›] by auto
  have (∑ v∈S. u v + t * w v) = 1
    unfolding sum.distrib wv(1) sum_distrib_left[symmetric] obt(5) by auto
  moreover have (∑ v∈S. u v *R v + (t * w v) *R v) − (u a *R a + (t * w
a) *R a) = y
     unfolding sum.distrib obt(6) scaleR_scaleR[symmetric] scaleR_right.sum
[symmetric] wv(4)
    using a(2) [THEN eq_neg_iff_add_eq_0 [THEN iffD2]] by simp
  ultimately have ?P (n − 1)
    apply (rule_tac x=(S − {a}) in exI)
    apply (rule_tac x=λv. u v + t * w v in exI)
    using obt(1−3) and t and a
    apply (auto simp: * scaleR_left_distrib)
    done
  then show False
    using smallest[THEN spec[where x=n − 1]] by auto
```

qed
then show $\exists S\ u.$ *finite S* $\wedge$ $S \subseteq p$ $\wedge$ *card S* $\leq$ *aff_dim p* + *1* $\wedge$
($\forall x{\in}S.$ *0* $\leq$ *u x*) $\wedge$ *sum u S* = *1* $\wedge$ ($\sum v{\in}S.\ u\ v *_R v$) = *y*
using *obt* **by** *auto*
qed *auto*

lemma *caratheodory_aff_dim*:
fixes *p* :: ('*a*::*euclidean_space*) *set*
shows *convex hull p* = {*x*. $\exists S$. *finite S* $\wedge$ $S \subseteq p$ $\wedge$ *card S* $\leq$ *aff_dim p* + *1* $\wedge$ *x* $\in$ *convex hull S*}
(**is** *?lhs* = *?rhs*)
proof
have $\bigwedge x\ S\ u.$ ⟦*finite S*; $S \subseteq p$; *int* (*card S*) $\leq$ *aff_dim p* + *1*; $\forall x{\in}S.$ *0* $\leq$ *u x*; *sum u S* = *1*⟧
$\Longrightarrow$ ($\sum v{\in}S.\ u\ v *_R v$) $\in$ *convex hull S*
by (*simp add*: *hull_subset convex_explicit* [*THEN iffD1*, *OF convex_convex_hull*])
then show *?lhs* $\subseteq$ *?rhs*
by (*subst convex_hull_caratheodory_aff_dim*, *auto*)
qed (*use hull_mono* **in** *auto*)

lemma *convex_hull_caratheodory*:
fixes *p* :: ('*a*::*euclidean_space*) *set*
shows *convex hull p* =
{*y*. $\exists S\ u.$ *finite S* $\wedge$ $S \subseteq p$ $\wedge$ *card S* $\leq$ *DIM*('*a*) + *1* $\wedge$
($\forall x{\in}S.$ *0* $\leq$ *u x*) $\wedge$ *sum u S* = *1* $\wedge$ *sum* ($\lambda v.\ u\ v *_R v$) *S* = *y*}
(**is** *?lhs* = *?rhs*)
proof (*intro set_eqI iffI*)
fix *x*
assume *x* $\in$ *?lhs* **then show** *x* $\in$ *?rhs*
unfolding *convex_hull_caratheodory_aff_dim*
using *aff_dim_le_DIM* [*of p*] **by** *fastforce*
qed (*auto simp*: *convex_hull_explicit*)

theorem *caratheodory*:
convex hull p =
{*x*::'*a*::*euclidean_space*. $\exists S$. *finite S* $\wedge$ $S \subseteq p$ $\wedge$ *card S* $\leq$ *DIM*('*a*) + *1* $\wedge$ *x* $\in$ *convex hull S*}
proof *safe*
fix *x*
assume *x* $\in$ *convex hull p*
then obtain *S u* **where** *finite S* $S \subseteq p$ *card S* $\leq$ *DIM*('*a*) + *1*
$\forall x{\in}S.$ *0* $\leq$ *u x sum u S* = *1* ($\sum v{\in}S.\ u\ v *_R v$) = *x*
unfolding *convex_hull_caratheodory* **by** *auto*
then show $\exists S$. *finite S* $\wedge$ $S \subseteq p$ $\wedge$ *card S* $\leq$ *DIM*('*a*) + *1* $\wedge$ *x* $\in$ *convex hull S*
using *convex_hull_finite* **by** *fastforce*
qed (*use hull_mono* **in** *force*)

1.7.11 Some Properties of subset of standard basis

lemma *affine_hull_substd_basis*:
  assumes *d* ⊆ *Basis*
  shows *affine hull* (*insert 0 d*) = {*x*::'*a*::*euclidean_space*. ∀ *i*∈*Basis*. *i* ∉ *d* ⟶ *x*•*i* = *0*}
  (**is** *affine hull* (*insert 0 ?A*) = *?B*)
proof −
  have ∗: ⋀*A*. (+) (*0*::'*a*) ' *A* = *A* ⋀*A*. (+) (− (*0*::'*a*)) ' *A* = *A*
    by *auto*
  show *?thesis*
    unfolding *affine_hull_insert_span_gen span_substd_basis*[*OF assms*,*symmetric*] ∗ **..**
qed

lemma *affine_hull_convex_hull* [*simp*]: *affine hull* (*convex hull S*) = *affine hull S*
  by (*metis Int_absorb1 Int_absorb2 convex_hull_subset_affine_hull hull_hull hull_mono hull_subset*)

1.7.12 Moving and scaling convex hulls

lemma *convex_hull_set_plus*:
  convex hull (*S* + *T*) = *convex hull S* + *convex hull T*
  unfolding *set_plus_image*
  apply (*subst convex_hull_linear_image* [*symmetric*])
  apply (*simp add*: *linear_iff scaleR_right_distrib*)
  apply (*simp add*: *convex_hull_Times*)
  done

lemma *translation_eq_singleton_plus*: (λ*x*. *a* + *x*) ' *T* = {*a*} + *T*
  unfolding *set_plus_def* **by** *auto*

lemma *convex_hull_translation*:
  convex hull ((λ*x*. *a* + *x*) ' *S*) = (λ*x*. *a* + *x*) ' (*convex hull S*)
  unfolding *translation_eq_singleton_plus*
  by (*simp only*: *convex_hull_set_plus convex_hull_singleton*)

lemma *convex_hull_scaling*:
  convex hull ((λ*x*. *c* $*_R$ *x*) ' *S*) = (λ*x*. *c* $*_R$ *x*) ' (*convex hull S*)
  using *linear_scaleR* **by** (*rule convex_hull_linear_image* [*symmetric*])

lemma *convex_hull_affinity*:
  convex hull ((λ*x*. *a* + *c* $*_R$ *x*) ' *S*) = (λ*x*. *a* + *c* $*_R$ *x*) ' (*convex hull S*)
  by (*metis convex_hull_scaling convex_hull_translation image_image*)

1.7.13 Convexity of cone hulls

lemma *convex_cone_hull*:
  assumes *convex S*
  shows *convex* (*cone hull S*)

proof (*rule convexI*)
  fix *x y*
  assume *xy*: $x \in$ *cone hull S* $y \in$ *cone hull S*
  then have $S \neq \{\}$
    using *cone_hull_empty_iff*[*of S*] **by** *auto*
  fix *u v* :: *real*
  assume *uv*: $u \geq 0$ $v \geq 0$ $u + v = 1$
  then have $*$: $u *_R x \in$ *cone hull S* $v *_R y \in$ *cone hull S*
    using *cone_cone_hull*[*of S*] *xy cone_def*[*of cone hull S*] **by** *auto*
  from $*$ **obtain** *cx* :: *real* **and** *xx* **where** *x*: $u *_R x = cx *_R xx$ $cx \geq 0$ $xx \in S$
    using *cone_hull_expl*[*of S*] **by** *auto*
  from $*$ **obtain** *cy* :: *real* **and** *yy* **where** *y*: $v *_R y = cy *_R yy$ $cy \geq 0$ $yy \in S$
    using *cone_hull_expl*[*of S*] **by** *auto*
  {
    assume $cx + cy \leq 0$
    then have $u *_R x = 0$ **and** $v *_R y = 0$
      using *x y* **by** *auto*
    then have $u *_R x + v *_R y = 0$
      by *auto*
    then have $u *_R x + v *_R y \in$ *cone hull S*
      using *cone_hull_contains_0*[*of S*] ‹$S \neq \{\}$› **by** *auto*
  }
  moreover
  {
    assume $cx + cy > 0$
    then have $(cx / (cx + cy)) *_R xx + (cy / (cx + cy)) *_R yy \in S$
      using *assms mem_convex_alt*[*of S xx yy cx cy*] *x y* **by** *auto*
    then have $cx *_R xx + cy *_R yy \in$ *cone hull S*
      using *mem_cone_hull*[*of* $(cx/(cx+cy)) *_R xx + (cy/(cx+cy)) *_R yy$ *S* $cx+cy$]
‹$cx+cy>0$›
      by (*auto simp*: *scaleR_right_distrib*)
    then have $u *_R x + v *_R y \in$ *cone hull S*
      using *x y* **by** *auto*
  }
  moreover have $cx + cy \leq 0 \vee cx + cy > 0$ **by** *auto*
  ultimately show $u *_R x + v *_R y \in$ *cone hull S* **by** *blast*
qed

lemma *cone_convex_hull*:
  assumes *cone S*
  shows *cone* (*convex hull S*)
proof (*cases* $S = \{\}$)
  case *True*
  then show *?thesis* **by** *auto*
next
  case *False*
  then have $*$: $0 \in S \wedge (\forall c.\ c > 0 \longrightarrow (*_R)\ c$ ‘ $S = S)$
    using *cone_iff*[*of S*] *assms* **by** *auto*
  {

```
    fix c :: real
    assume c > 0
    then have (*R) c ` (convex hull S) = convex hull ((*R) c ` S)
      using convex_hull_scaling[of _ S] by auto
    also have ... = convex hull S
      using * ‹c > 0› by auto
    finally have (*R) c ` (convex hull S) = convex hull S
      by auto
  }
  then have 0 ∈ convex hull S ⋀c. c > 0 ⟹ ((*R) c ` (convex hull S)) = (convex
hull S)
    using * hull_subset[of S convex] by auto
  then show ?thesis
    using ‹S ≠ {}› cone_iff[of convex hull S] by auto
qed
```

1.7.14 Radon's theorem

Formalized by Lars Schewe.

```
lemma Radon_ex_lemma:
  assumes finite c affine_dependent c
  shows ∃u. sum u c = 0 ∧ (∃v∈c. u v ≠ 0) ∧ sum (λv. u v *R v) c = 0
proof -
  from assms(2)[unfolded affine_dependent_explicit]
  obtain S u where
      finite S S ⊆ c sum u S = 0 ∃v∈S. u v ≠ 0 (∑v∈S. u v *R v) = 0
    by blast
  then show ?thesis
    apply (rule_tac x=λv. if v∈S then u v else 0 in exI)
    unfolding if_smult scaleR_zero_left
    by (auto simp: Int_absorb1 sum.inter_restrict[OF ‹finite c›, symmetric])
qed

lemma Radon_s_lemma:
  assumes finite S
    and sum f S = (0::real)
  shows sum f {x∈S. 0 < f x} = - sum f {x∈S. f x < 0}
proof -
  have *: ⋀x. (if f x < 0 then f x else 0) + (if 0 < f x then f x else 0) = f x
    by auto
  show ?thesis
    unfolding add_eq_0_iff[symmetric] and sum.inter_filter[OF assms(1)]
      and sum.distrib[symmetric] and *
    using assms(2)
    by assumption
qed

lemma Radon_v_lemma:
  assumes finite S
```

```
    and sum f S = 0
    and ∀ x. g x = (0::real) ⟶ f x = (0::'a::euclidean_space)
  shows (sum f {x∈S. 0 < g x}) = − sum f {x∈S. g x < 0}
proof −
  have ∗: ⋀x. (if 0 < g x then f x else 0) + (if g x < 0 then f x else 0) = f x
    using assms(3) by auto
  show ?thesis
    unfolding eq_neg_iff_add_eq_0 and sum.inter_filter[OF assms(1)]
      and sum.distrib[symmetric] and ∗
    using assms(2)
    apply assumption
    done
qed

lemma Radon_partition:
  assumes finite C affine_dependent C
  shows ∃ m p. m ∩ p = {} ∧ m ∪ p = C ∧ (convex hull m) ∩ (convex hull p) ≠
{}
proof −
  obtain u v where uv: sum u C = 0 v∈C u v ≠ 0 (∑ v∈C. u v ∗R v) = 0
    using Radon_ex_lemma[OF assms] by auto
  have fin: finite {x ∈ C. 0 < u x} finite {x ∈ C. 0 > u x}
    using assms(1) by auto
  define z where z = inverse (sum u {x∈C. u x > 0}) ∗R sum (λx. u x ∗R x)
{x∈C. u x > 0}
  have sum u {x ∈ C. 0 < u x} ≠ 0
  proof (cases u v ≥ 0)
    case False
    then have u v < 0 by auto
    then show ?thesis
    proof (cases ∃ w∈{x ∈ C. 0 < u x}. u w > 0)
      case True
      then show ?thesis
        using sum_nonneg_eq_0_iff[of _ u, OF fin(1)] by auto
    next
      case False
      then have sum u C ≤ sum (λx. if x=v then u v else 0) C
        by (rule_tac sum_mono, auto)
      then show ?thesis
        unfolding sum.delta[OF assms(1)] using uv(2) and ‹u v < 0› and uv(1)
by auto
    qed
  qed (insert sum_nonneg_eq_0_iff[of _ u, OF fin(1)] uv(2−3), auto)

  then have ∗: sum u {x∈C. u x > 0} > 0
    unfolding less_le by (metis (no_types, lifting) mem_Collect_eq sum_nonneg)
  moreover have sum u ({x ∈ C. 0 < u x} ∪ {x ∈ C. u x < 0}) = sum u C
    (∑ x∈{x ∈ C. 0 < u x} ∪ {x ∈ C. u x < 0}. u x ∗R x) = (∑ x∈C. u x ∗R x)
    using assms(1)
```

```
    by (rule_tac[!] sum.mono_neutral_left, auto)
  then have sum u {x ∈ C. 0 < u x} = − sum u {x ∈ C. 0 > u x}
    (∑ x∈{x ∈ C. 0 < u x}. u x *R x) = − (∑ x∈{x ∈ C. 0 > u x}. u x *R x)
    unfolding eq_neg_iff_add_eq_0
    using uv(1,4)
    by (auto simp: sum.union_inter_neutral[OF fin, symmetric])
  moreover have ∀ x∈{v ∈ C. u v < 0}. 0 ≤ inverse (sum u {x ∈ C. 0 < u x})
* − u x
    using * by (fastforce intro: mult_nonneg_nonneg)
  ultimately have z ∈ convex hull {v ∈ C. u v ≤ 0}
    unfolding convex_hull_explicit mem_Collect_eq
    apply (rule_tac x={v ∈ C. u v < 0} in exI)
    apply (rule_tac x=λy. inverse (sum u {x∈C. u x > 0}) * − u y in exI)
   using assms(1) unfolding scaleR_scaleR[symmetric] scaleR_right.sum [symmetric]

    by (auto simp: z_def sum_negf sum_distrib_left[symmetric])
  moreover have ∀ x∈{v ∈ C. 0 < u v}. 0 ≤ inverse (sum u {x ∈ C. 0 < u x})
* u x
    using * by (fastforce intro: mult_nonneg_nonneg)
  then have z ∈ convex hull {v ∈ C. u v > 0}
    unfolding convex_hull_explicit mem_Collect_eq
    apply (rule_tac x={v ∈ C. 0 < u v} in exI)
    apply (rule_tac x=λy. inverse (sum u {x∈C. u x > 0}) * u y in exI)
    using assms(1)
    unfolding scaleR_scaleR[symmetric] scaleR_right.sum [symmetric]
    using * by (auto simp: z_def sum_negf sum_distrib_left[symmetric])
  ultimately show ?thesis
    apply (rule_tac x={v∈C. u v ≤ 0} in exI)
    apply (rule_tac x={v∈C. u v > 0} in exI, auto)
    done
qed

theorem Radon:
  assumes affine_dependent c
  obtains m p where m ⊆ c p ⊆ c m ∩ p = {} (convex hull m) ∩ (convex hull
p) ≠ {}
proof −
  from assms[unfolded affine_dependent_explicit]
  obtain S u where
      finite S S ⊆ c sum u S = 0 ∃ v∈S. u v ≠ 0 (∑ v∈S. u v *R v) = 0
    by blast
  then have *: finite S affine_dependent S and S: S ⊆ c
    unfolding affine_dependent_explicit by auto
  from Radon_partition[OF *]
  obtain m p where m ∩ p = {} m ∪ p = S convex hull m ∩ convex hull p ≠ {}
    by blast
  with S show ?thesis
    by (force intro: that[of p m])
qed
```

1.7.15 Helly's theorem

lemma *Helly_induct*:
  fixes f :: *'a::euclidean_space set set*
  assumes $card\ f = n$
    and $n \geq DIM('a) + 1$
    and $\forall s{\in}f.\ convex\ s\ \forall t{\subseteq}f.\ card\ t = DIM('a) + 1 \longrightarrow \bigcap t \neq \{\}$
  shows $\bigcap f \neq \{\}$
  using *assms*
proof (*induction n arbitrary*: *f*)
  case *0*
  then show *?case* **by** *auto*
next
  case (*Suc n*)
  have *finite f*
    using ‹*card f = Suc n*› **by** (*auto intro*: *card_ge_0_finite*)
  show $\bigcap f \neq \{\}$
  proof (*cases* $n = DIM('a)$)
    case *True*
    then show *?thesis*
      by (*simp add*: *Suc.prems*(*1*) *Suc.prems*(*4*))
  next
    case *False*
    have $\bigcap(f - \{s\}) \neq \{\}$ **if** $s \in f$ **for** *s*
    proof (*rule Suc.IH*[*rule_format*])
      show $card\ (f - \{s\}) = n$
        by (*simp add*: *Suc.prems*(*1*) ‹*finite f*› *that*)
      show $DIM('a) + 1 \leq n$
        using *False Suc.prems*(*2*) **by** *linarith*
      show $\bigwedge t.\ [\![t \subseteq f - \{s\};\ card\ t = DIM('a) + 1]\!] \Longrightarrow \bigcap t \neq \{\}$
        by (*simp add*: *Suc.prems*(*4*) *subset_Diff_insert*)
    qed (*use Suc* **in** *auto*)
    then have $\forall s{\in}f.\ \exists x.\ x \in \bigcap(f - \{s\})$
      by *blast*
    then obtain *X* **where** *X*: $\bigwedge s.\ s{\in}f \Longrightarrow X\ s \in \bigcap(f - \{s\})$
      by *metis*
    show *?thesis*
    proof (*cases inj_on X f*)
      case *False*
      then obtain *s t* **where** $s{\neq}t$ **and** *st*: $s{\in}f\ t{\in}f\ X\ s = X\ t$
        unfolding *inj_on_def* **by** *auto*
      then have $*$: $\bigcap f = \bigcap(f - \{s\}) \cap \bigcap(f - \{t\})$ **by** *auto*
      show *?thesis*
        by (*metis* $*$ *X disjoint_iff_not_equal st*)
    next
      case *True*
      then obtain *m p* **where** *mp*: $m \cap p = \{\}\ m \cup p = X\ `\ f$ *convex hull* $m \cap$ *convex hull* $p \neq \{\}$
        using *Radon_partition*[*of X ‘ f*] **and** *affine_dependent_biggerset*[*of X ‘ f*]
        unfolding *card_image*[*OF True*] **and** ‹*card f = Suc n*›

```
        using Suc(3) ‹finite f› and False
        by auto
      have m ⊆ X ` f p ⊆ X ` f
        using mp(2) by auto
      then obtain g h where gh:m = X ` g p = X ` h g ⊆ f h ⊆ f
        unfolding subset_image_iff by auto
      then have f ∪ (g ∪ h) = f by auto
      then have f: f = g ∪ h
        using inj_on_Un_image_eq_iff[of X f g ∪ h] and True
        unfolding mp(2)[unfolded image_Un[symmetric] gh]
        by auto
      have *: g ∩ h = {}
        using gh(1) gh(2) local.mp(1) by blast
      have convex hull (X ` h) ⊆ ⋂g convex hull (X ` g) ⊆ ⋂h
      by (rule hull_minimal; use X * f in ‹auto simp: Suc.prems(3) convex_Inter›)+
      then show ?thesis
        unfolding f using mp(3)[unfolded gh] by blast
    qed
  qed
qed

theorem Helly:
  fixes f :: 'a::euclidean_space set set
  assumes card f ≥ DIM('a) + 1 ∀ s∈f. convex s
    and ⋀t. ⟦t⊆f; card t = DIM('a) + 1⟧ ⟹ ⋂t ≠ {}
  shows ⋂f ≠ {}
  using Helly_induct assms by blast
```

1.7.16 Epigraphs of convex functions

```
definition epigraph S (f :: _ ⇒ real) = {xy. fst xy ∈ S ∧ f (fst xy) ≤ snd xy}

lemma mem_epigraph: (x, y) ∈ epigraph S f ⟷ x ∈ S ∧ f x ≤ y
  unfolding epigraph_def by auto

lemma convex_epigraph: convex (epigraph S f) ⟷ convex_on S f ∧ convex S
proof safe
  assume L: convex (epigraph S f)
  then show convex_on S f
    by (auto simp: convex_def convex_on_def epigraph_def)
  show convex S
    using L by (fastforce simp: convex_def convex_on_def epigraph_def)
next
  assume convex_on S f convex S
  then show convex (epigraph S f)
    unfolding convex_def convex_on_def epigraph_def
    apply safe
     apply (rule_tac [2] y=u * f a + v * f aa in order_trans)
      apply (auto intro!:mult_left_mono add_mono)
```

```
  done
qed

lemma convex_epigraphI: convex_on S f ⟹ convex S ⟹ convex (epigraph S f)
  unfolding convex_epigraph by auto

lemma convex_epigraph_convex: convex S ⟹ convex_on S f ⟷ convex(epigraph
S f)
  by (simp add: convex_epigraph)
```

Use this to derive general bound property of convex function

```
lemma convex_on:
  assumes convex S
  shows convex_on S f ⟷
    (∀ k u x. (∀ i∈{1..k::nat}. 0 ≤ u i ∧ x i ∈ S) ∧ sum u {1..k} = 1 ⟶
      f (sum (λi. u i *R x i) {1..k}) ≤ sum (λi. u i * f(x i)) {1..k})
  (is ?lhs = (∀ k u x. ?rhs k u x))
proof
  assume ?lhs
  then have §: convex {xy. fst xy ∈ S ∧ f (fst xy) ≤ snd xy}
    by (metis assms convex_epigraph epigraph_def)
  show ∀ k u x. ?rhs k u x
  proof (intro allI)
    fix k u x
    show ?rhs k u x
      using §
      unfolding convex mem_Collect_eq fst_sum snd_sum
      apply safe
      apply (drule_tac x=k in spec)
      apply (drule_tac x=u in spec)
      apply (drule_tac x=λi. (x i, f (x i)) in spec)
      apply simp
      done
  qed
next
  assume ∀ k u x. ?rhs k u x
  then show ?lhs
 unfolding convex_epigraph_convex[OF assms] convex epigraph_def Ball_def mem_Collect_eq
fst_sum snd_sum
  using assms[unfolded convex] apply clarsimp
  apply (rule_tac y=∑ i = 1..k. u i * f (fst (x i)) in order_trans)
  by (auto simp add: mult_left_mono intro: sum_mono)
qed
```

1.7.17 A bound within a convex hull

```
lemma convex_on_convex_hull_bound:
  assumes convex_on (convex hull S) f
    and ∀ x∈S. f x ≤ b
```

shows $\forall x \in$ *convex hull S*. $f\ x \le b$
proof
fix x
assume $x \in$ *convex hull S*
then obtain $k\ u\ v$ **where**
u: $\forall i \in \{1..k::nat\}.\ 0 \le u\ i \wedge v\ i \in S$ $sum\ u\ \{1..k\} = 1$ $(\sum i = 1..k.\ u\ i *_R v\ i) = x$
unfolding *convex_hull_indexed mem_Collect_eq* **by** *auto*
have $(\sum i = 1..k.\ u\ i * f\ (v\ i)) \le b$
using *sum_mono*[*of* $\{1..k\}$ $\lambda i.\ u\ i * f\ (v\ i)$ $\lambda i.\ u\ i * b$]
unfolding *sum_distrib_right*[*symmetric*] *u*(*2*) *mult_1*
using *assms*(*2*) *mult_left_mono u*(*1*) **by** *blast*
then show $f\ x \le b$
using *assms*(*1*)[*unfolded convex_on*[*OF convex_convex_hull*], *rule_format*, *of k u v*]
using *hull_inc u* **by** *fastforce*
qed

lemma *inner_sum_Basis*[*simp*]: $i \in Basis \Longrightarrow (\sum Basis) \cdot i = 1$
by (*simp add*: *inner_sum_left sum.If_cases inner_Basis*)

lemma *convex_set_plus*:
assumes *convex S* **and** *convex T* **shows** *convex* $(S + T)$
proof −
have *convex* $(\bigcup x \in S.\ \bigcup y \in T.\ \{x + y\})$
using *assms* **by** (*rule convex_sums*)
moreover have $(\bigcup x \in S.\ \bigcup y \in T.\ \{x + y\}) = S + T$
unfolding *set_plus_def* **by** *auto*
finally show *convex* $(S + T)$ **.**
qed

lemma *convex_set_sum*:
assumes $\bigwedge i.\ i \in A \Longrightarrow$ *convex* $(B\ i)$
shows *convex* $(\sum i \in A.\ B\ i)$
proof (*cases finite A*)
case *True* **then show** *?thesis* **using** *assms*
by *induct* (*auto simp*: *convex_set_plus*)
qed *auto*

lemma *finite_set_sum*:
assumes *finite A* **and** $\forall i \in A.$ *finite* $(B\ i)$ **shows** *finite* $(\sum i \in A.\ B\ i)$
using *assms* **by** (*induct set*: *finite*, *simp*, *simp add*: *finite_set_plus*)

lemma *box_eq_set_sum_Basis*:
$\{x.\ \forall i \in Basis.\ x \cdot i \in B\ i\} = (\sum i \in Basis.\ (\lambda x.\ x *_R i)\ `\ (B\ i))$ (**is** *?lhs* = *?rhs*)
proof −
have $\bigwedge x.\ \forall i \in Basis.\ x \cdot i \in B\ i \Longrightarrow$
$\exists s.\ x = sum\ s\ Basis \wedge (\forall i \in Basis.\ s\ i \in (\lambda x.\ x *_R i)\ `\ B\ i)$
by (*metis* (*mono_tags*, *lifting*) *euclidean_representation image_iff*)

moreover
have *sum f Basis* · *i* ∈ *B i* **if** *i* ∈ *Basis* **and** *f*: ∀ *i*∈*Basis*. *f i* ∈ (λ*x*. *x* $*_R$ *i*) ‘ *B i* **for** *i f*
proof −
have (∑ *x*∈*Basis* − {*i*}. *f x* · *i*) = *0*
proof (*rule sum.neutral*, *intro strip*)
show *f x* · *i* = *0* **if** *x* ∈ *Basis* − {*i*} **for** *x*
using *that f* ‹*i* ∈ *Basis*› *inner_Basis that* **by** *fastforce*
qed
then have (∑ *x*∈*Basis*. *f x* · *i*) = *f i* · *i*
by (*metis* (*no_types*) ‹*i* ∈ *Basis*› *add.right_neutral sum.remove* [*OF finite_Basis*])
then have (∑ *x*∈*Basis*. *f x* · *i*) ∈ *B i*
using *f that*(*1*) **by** *auto*
then show *?thesis*
by (*simp add*: *inner_sum_left*)
qed
ultimately show *?thesis*
by (*subst set_sum_alt* [*OF finite_Basis*]) *auto*
qed

lemma *convex_hull_set_sum*:
convex hull (∑ *i*∈*A*. *B i*) = (∑ *i*∈*A*. *convex hull* (*B i*))
proof (*cases finite A*)
assume *finite A* **then show** *?thesis*
by (*induct set*: *finite*, *simp*, *simp add*: *convex_hull_set_plus*)
qed *simp*

end

1.8 Definition of Finite Cartesian Product Type

theory *Finite_Cartesian_Product*
imports
Euclidean_Space
L2_Norm
HOL−Library.Numeral_Type
HOL−Library.Countable_Set
HOL−Library.FuncSet
begin

1.8.1 Finite Cartesian products, with indexing and lambdas

typedef (′*a*, ′*b*) *vec* = *UNIV* :: (′*b*::*finite* ⇒ ′*a*) *set*
morphisms *vec_nth vec_lambda* **..**

declare *vec_lambda_inject* [*simplified*, *simp*]

```
bundle vec_syntax begin
notation
  vec_nth (infixl $ 90) and
  vec_lambda (binder χ 10)
end

bundle no_vec_syntax begin
no_notation
  vec_nth (infixl $ 90) and
  vec_lambda (binder χ 10)
end

unbundle vec_syntax
```

Concrete syntax for ('a, 'b) vec:

- 'a^'b becomes ('a, 'b::finite) vec
- 'a^'b::_ becomes ('a, 'b) vec without extra sort-constraint

```
syntax _vec_type :: type ⇒ type ⇒ type (infixl ^ 15)
parse_translation ‹
  let
    fun vec t u = Syntax.const type_syntax‹vec› $ t $ u;
    fun finite_vec_tr [t, u] =
      (case Term_Position.strip_positions u of
        v as Free (x, _) =>
          if Lexicon.is_tid x then
            vec t (Syntax.const syntax_const‹_ofsort› $ v $
              Syntax.const class_syntax‹finite›)
          else vec t u
      | _ => vec t u)
  in
    [(syntax_const‹_vec_type›, K finite_vec_tr)]
  end
›

lemma vec_eq_iff: (x = y) ⟷ (∀ i. x$i = y$i)
  by (simp add: vec_nth_inject [symmetric] fun_eq_iff)

lemma vec_lambda_beta [simp]: vec_lambda g $ i = g i
  by (simp add: vec_lambda_inverse)

lemma vec_lambda_unique: (∀ i. f$i = g i) ⟷ vec_lambda g = f
  by (auto simp add: vec_eq_iff)

lemma vec_lambda_eta [simp]: (χ i. (g$i)) = g
  by (simp add: vec_eq_iff)
```

1.8.2 Cardinality of vectors

```
instance vec :: (finite, finite) finite
proof
  show finite (UNIV :: ('a, 'b) vec set)
  proof (subst bij_betw_finite)
    show bij_betw vec_nth UNIV (Pi (UNIV :: 'b set) (λ_. UNIV :: 'a set))
      by (intro bij_betwI[of _ _ _ vec_lambda]) (auto simp: vec_eq_iff)
    have finite (PiE (UNIV :: 'b set) (λ_. UNIV :: 'a set))
      by (intro finite_PiE) auto
    also have (PiE (UNIV :: 'b set) (λ_. UNIV :: 'a set)) = Pi UNIV (λ_. UNIV)
      by auto
    finally show finite ... .
  qed
qed

lemma countable_PiE:
  finite I ⟹ (⋀i. i ∈ I ⟹ countable (F i)) ⟹ countable (Pi_E I F)
  by (induct I arbitrary: F rule: finite_induct) (auto simp: PiE_insert_eq)

instance vec :: (countable, finite) countable
proof
  have countable (UNIV :: ('a, 'b) vec set)
  proof (rule countableI_bij2)
    show bij_betw vec_nth UNIV (Pi (UNIV :: 'b set) (λ_. UNIV :: 'a set))
      by (intro bij_betwI[of _ _ _ vec_lambda]) (auto simp: vec_eq_iff)
    have countable (PiE (UNIV :: 'b set) (λ_. UNIV :: 'a set))
      by (intro countable_PiE) auto
    also have (PiE (UNIV :: 'b set) (λ_. UNIV :: 'a set)) = Pi UNIV (λ_. UNIV)
      by auto
    finally show countable ... .
  qed
  thus ∃ t::('a, 'b) vec ⇒ nat. inj t
    by (auto elim!: countableE)
qed

lemma infinite_UNIV_vec:
  assumes infinite (UNIV :: 'a set)
  shows   infinite (UNIV :: ('a^'b) set)
proof (subst bij_betw_finite)
  show bij_betw vec_nth UNIV (Pi (UNIV :: 'b set) (λ_. UNIV :: 'a set))
    by (intro bij_betwI[of _ _ _ vec_lambda]) (auto simp: vec_eq_iff)
  have infinite (PiE (UNIV :: 'b set) (λ_. UNIV :: 'a set)) (is infinite ?A)
  proof
    assume finite ?A
    hence finite ((λf. f undefined) ` ?A)
      by (rule finite_imageI)
    also have (λf. f undefined) ` ?A = UNIV
      by auto
    finally show False
```

```
      using ‹infinite (UNIV :: 'a set)› by contradiction
  qed
  also have ?A = Pi UNIV (λ_. UNIV)
    by auto
  finally show infinite (Pi (UNIV :: 'b set) (λ_. UNIV :: 'a set)) .
qed

proposition CARD_vec [simp]:
  CARD('a^'b) = CARD('a) ^ CARD('b)
proof (cases finite (UNIV :: 'a set))
  case True
  show ?thesis
  proof (subst bij_betw_same_card)
    show bij_betw vec_nth UNIV (Pi (UNIV :: 'b set) (λ_. UNIV :: 'a set))
      by (intro bij_betwI[of _ _ _ vec_lambda]) (auto simp: vec_eq_iff)
    have CARD('a) ^ CARD('b) = card (PiE (UNIV :: 'b set) (λ_. UNIV :: 'a
set))
      (is _ = card ?A)
      by (subst card_PiE) (auto)
    also have ?A = Pi UNIV (λ_. UNIV)
      by auto
    finally show card ... = CARD('a) ^ CARD('b) ..
  qed
qed (simp_all add: infinite_UNIV_vec)

lemma countable_vector:
  fixes B:: 'n::finite ⇒ 'a set
  assumes ⋀i. countable (B i)
  shows countable {V. ∀i::'n::finite. V $ i ∈ B i}
proof -
  have f ∈ ($) ` {V. ∀i. V $ i ∈ B i} if f ∈ Pi_E UNIV B for f
  proof -
    have ∃W. (∀i. W $ i ∈ B i) ∧ ($) W = f
      by (metis that PiE_iff UNIV_I vec_lambda_inverse)
    then show f ∈ ($) ` {v. ∀i. v $ i ∈ B i}
      by blast
  qed
  then have Pi_E UNIV B = vec_nth ` {V. ∀i::'n. V $ i ∈ B i}
    by blast
  then have countable (vec_nth ` {V. ∀i. V $ i ∈ B i})
    by (metis finite_class.finite_UNIV countable_PiE assms)
  then have countable (vec_lambda ` vec_nth ` {V. ∀i. V $ i ∈ B i})
    by auto
  then show ?thesis
    by (simp add: image_comp o_def vec_nth_inverse)
qed
```

1.8.3 Group operations and class instances

instantiation *vec* :: (*zero*, *finite*) *zero*
begin
  definition *0* ≡ (χ *i*. *0*)
  instance ..
end

instantiation *vec* :: (*plus*, *finite*) *plus*
begin
  definition (+) ≡ (λ *x y*. (χ *i*. *x*$*i* + *y*$*i*))
  instance ..
end

instantiation *vec* :: (*minus*, *finite*) *minus*
begin
  definition (−) ≡ (λ *x y*. (χ *i*. *x*$*i* − *y*$*i*))
  instance ..
end

instantiation *vec* :: (*uminus*, *finite*) *uminus*
begin
  definition *uminus* ≡ (λ *x*. (χ *i*. − (*x*$*i*)))
  instance ..
end

lemma *zero_index* [*simp*]: *0* $ *i* = *0*
  unfolding *zero_vec_def* **by** *simp*

lemma *vector_add_component* [*simp*]: (*x* + *y*)$*i* = *x*$*i* + *y*$*i*
  unfolding *plus_vec_def* **by** *simp*

lemma *vector_minus_component* [*simp*]: (*x* − *y*)$*i* = *x*$*i* − *y*$*i*
  unfolding *minus_vec_def* **by** *simp*

lemma *vector_uminus_component* [*simp*]: (− *x*)$*i* = − (*x*$*i*)
  unfolding *uminus_vec_def* **by** *simp*

instance *vec* :: (*semigroup_add*, *finite*) *semigroup_add*
  by *standard* (*simp add*: *vec_eq_iff add.assoc*)

instance *vec* :: (*ab_semigroup_add*, *finite*) *ab_semigroup_add*
  by *standard* (*simp add*: *vec_eq_iff add.commute*)

instance *vec* :: (*monoid_add*, *finite*) *monoid_add*
  by *standard* (*simp_all add*: *vec_eq_iff*)

instance *vec* :: (*comm_monoid_add*, *finite*) *comm_monoid_add*
  by *standard* (*simp add*: *vec_eq_iff*)

instance *vec* :: (*cancel_semigroup_add*, *finite*) *cancel_semigroup_add*
  by *standard* (*simp_all add*: *vec_eq_iff*)

instance *vec* :: (*cancel_ab_semigroup_add*, *finite*) *cancel_ab_semigroup_add*
  by *standard* (*simp_all add*: *vec_eq_iff diff_diff_eq*)

instance *vec* :: (*cancel_comm_monoid_add*, *finite*) *cancel_comm_monoid_add* **..**

instance *vec* :: (*group_add*, *finite*) *group_add*
  by *standard* (*simp_all add*: *vec_eq_iff*)

instance *vec* :: (*ab_group_add*, *finite*) *ab_group_add*
  by *standard* (*simp_all add*: *vec_eq_iff*)

1.8.4 Basic componentwise operations on vectors

instantiation *vec* :: (*times*, *finite*) *times*
begin

definition $(*) \equiv (\lambda\ x\ y.\ \ (\chi\ i.\ (x\$i) * (y\$i)))$
instance ..

end

instantiation *vec* :: (*one*, *finite*) *one*
begin

definition $1 \equiv (\chi\ i.\ 1)$
instance ..

end

instantiation *vec* :: (*ord*, *finite*) *ord*
begin

definition $x \le y \longleftrightarrow (\forall\, i.\ x\$i \le y\$i)$
definition $x < (y{::}'a\hat{\ }'b) \longleftrightarrow x \le y \wedge \neg\ y \le x$
instance ..

end

The ordering on one-dimensional vectors is linear.

instance *vec*:: (*order*, *finite*) *order*
  by *standard* (*auto simp*: *less_eq_vec_def less_vec_def vec_eq_iff*
    intro: *order.trans order.antisym order.strict_implies_order*)

instance *vec* :: (*linorder*, *CARD_1*) *linorder*
proof
  obtain $a :: 'b$ **where** *all*: $\bigwedge P.\ (\forall\, i.\ P\ i) \longleftrightarrow P\ a$

```
  proof -
    have CARD ('b) = 1 by (rule CARD_1)
    then obtain b :: 'b where UNIV = {b} by (auto iff: card_Suc_eq)
    then have ⋀P. (∀i∈UNIV. P i) ⟷ P b by auto
    then show thesis by (auto intro: that)
  qed
  fix x y :: 'a^'b::CARD_1
  note [simp] = less_eq_vec_def less_vec_def all vec_eq_iff field_simps
  show x ≤ y ∨ y ≤ x by auto
qed
```

Constant Vectors

```
definition vec x = (χ i. x)
```

Also the scalar-vector multiplication.

```
definition vector_scalar_mult:: 'a::times ⇒ 'a ^ 'n ⇒ 'a ^ 'n (infixl *s 70)
  where c *s x = (χ i. c * (x$i))
```

scalar product

```
definition scalar_product :: 'a :: semiring_1 ^ 'n ⇒ 'a ^ 'n ⇒ 'a where
  scalar_product v w = (∑ i ∈ UNIV. v $ i * w $ i)
```

1.8.5 Real vector space

```
instantiation vec :: (real_vector, finite) real_vector
begin

definition scaleR ≡ (λ r x. (χ i. scaleR r (x$i)))

lemma vector_scaleR_component [simp]: (scaleR r x)$i = scaleR r (x$i)
  unfolding scaleR_vec_def by simp

instance
  by standard (simp_all add: vec_eq_iff scaleR_left_distrib scaleR_right_distrib)

end
```

1.8.6 Topological space

```
instantiation vec :: (topological_space, finite) topological_space
begin

definition [code del]:
  open (S :: ('a ^ 'b) set) ⟷
    (∀x∈S. ∃A. (∀i. open (A i) ∧ x$i ∈ A i) ∧
      (∀y. (∀i. y$i ∈ A i) ⟶ y ∈ S))

instance proof
  show open (UNIV :: ('a ^ 'b) set)
```

```
    unfolding open_vec_def by auto
next
  fix S T :: ('a ^ 'b) set
  assume open S open T thus open (S ∩ T)
    unfolding open_vec_def
    apply clarify
    apply (drule (1) bspec)+
    apply (clarify, rename_tac Sa Ta)
    apply (rule_tac x=λi. Sa i ∩ Ta i in exI)
    apply (simp add: open_Int)
    done
next
  fix K :: ('a ^ 'b) set set
  assume ∀S∈K. open S thus open (⋃K)
    unfolding open_vec_def
    apply clarify
    apply (drule (1) bspec)
    apply (drule (1) bspec)
    apply clarify
    apply (rule_tac x=A in exI)
    apply fast
    done
qed

end

lemma open_vector_box: ∀i. open (S i) ⟹ open {x. ∀i. x $ i ∈ S i}
  unfolding open_vec_def by auto

lemma open_vimage_vec_nth: open S ⟹ open ((λx. x $ i) −` S)
  unfolding open_vec_def
  apply clarify
  apply (rule_tac x=λk. if k = i then S else UNIV in exI, simp)
  done

lemma closed_vimage_vec_nth: closed S ⟹ closed ((λx. x $ i) −` S)
  unfolding closed_open vimage_Compl [symmetric]
  by (rule open_vimage_vec_nth)

lemma closed_vector_box: ∀i. closed (S i) ⟹ closed {x. ∀i. x $ i ∈ S i}
proof −
  have {x. ∀i. x $ i ∈ S i} = (⋂i. (λx. x $ i) −` S i) by auto
  thus ∀i. closed (S i) ⟹ closed {x. ∀i. x $ i ∈ S i}
    by (simp add: closed_INT closed_vimage_vec_nth)
qed

lemma tendsto_vec_nth [tendsto_intros]:
  assumes ((λx. f x) ⟶ a) net
  shows ((λx. f x $ i) ⟶ a $ i) net
```

proof (*rule topological_tendstoI*)
  fix *S* **assume** *open S a* $ *i* ∈ *S*
  then have *open* ((λ*y*. *y* $ *i*) −' *S*) *a* ∈ ((λ*y*. *y* $ *i*) −' *S*)
    by (*simp_all add*: *open_vimage_vec_nth*)
  with *assms* **have** *eventually* (λ*x*. *f x* ∈ (λ*y*. *y* $ *i*) −' *S*) *net*
    by (*rule topological_tendstoD*)
  then show *eventually* (λ*x*. *f x* $ *i* ∈ *S*) *net*
    by *simp*
qed

lemma *isCont_vec_nth* [*simp*]: *isCont f a* ⟹ *isCont* (λ*x*. *f x* $ *i*) *a*
  unfolding *isCont_def* **by** (*rule tendsto_vec_nth*)

lemma *vec_tendstoI*:
  assumes ⋀*i*. ((λ*x*. *f x* $ *i*) ⟶ *a* $ *i*) *net*
  shows ((λ*x*. *f x*) ⟶ *a*) *net*
proof (*rule topological_tendstoI*)
  fix *S* **assume** *open S* **and** *a* ∈ *S*
  then obtain *A* **where** *A*: ⋀*i*. *open* (*A i*) ⋀*i*. *a* $ *i* ∈ *A i*
    and *S*: ⋀*y*. ∀ *i*. *y* $ *i* ∈ *A i* ⟹ *y* ∈ *S*
    unfolding *open_vec_def* **by** *metis*
  have ⋀*i*. *eventually* (λ*x*. *f x* $ *i* ∈ *A i*) *net*
    using *assms A* **by** (*rule topological_tendstoD*)
  hence *eventually* (λ*x*. ∀ *i*. *f x* $ *i* ∈ *A i*) *net*
    by (*rule eventually_all_finite*)
  thus *eventually* (λ*x*. *f x* ∈ *S*) *net*
    by (*rule eventually_mono*, *simp add*: *S*)
qed

lemma *tendsto_vec_lambda* [*tendsto_intros*]:
  assumes ⋀*i*. ((λ*x*. *f x i*) ⟶ *a i*) *net*
  shows ((λ*x*. χ *i*. *f x i*) ⟶ (χ *i*. *a i*)) *net*
  using *assms* **by** (*simp add*: *vec_tendstoI*)

lemma *open_image_vec_nth*: **assumes** *open S* **shows** *open* ((λ*x*. *x* $ *i*) ' *S*)
proof (*rule openI*)
  fix *a* **assume** *a* ∈ (λ*x*. *x* $ *i*) ' *S*
  then obtain *z* **where** *a* = *z* $ *i* **and** *z* ∈ *S* **..**
  then obtain *A* **where** *A*: ∀ *i*. *open* (*A i*) ∧ *z* $ *i* ∈ *A i*
    and *S*: ∀ *y*. (∀ *i*. *y* $ *i* ∈ *A i*) ⟶ *y* ∈ *S*
    using ‹*open S*› **unfolding** *open_vec_def* **by** *auto*
  hence *A i* ⊆ (λ*x*. *x* $ *i*) ' *S*
    by (*clarsimp*, *rule_tac x*=χ *j*. *if j* = *i then x else z* $ *j* **in** *image_eqI*,
      simp_all)
  hence *open* (*A i*) ∧ *a* ∈ *A i* ∧ *A i* ⊆ (λ*x*. *x* $ *i*) ' *S*
    using *A* ‹*a* = *z* $ *i*› **by** *simp*
  then show ∃ *T*. *open T* ∧ *a* ∈ *T* ∧ *T* ⊆ (λ*x*. *x* $ *i*) ' *S* **by** − (*rule exI*)
qed

```
instance vec :: (perfect_space, finite) perfect_space
proof
  fix x :: 'a ^ 'b show ¬ open {x}
  proof
    assume open {x}
    hence ∀ i. open ((λx. x $ i) ` {x}) by (fast intro: open_image_vec_nth)
    hence ∀ i. open {x $ i} by simp
    thus False by (simp add: not_open_singleton)
  qed
qed
```

1.8.7 Metric space

```
instantiation vec :: (metric_space, finite) dist
begin

definition
  dist x y = L2_set (λi. dist (x$i) (y$i)) UNIV

instance ..
end

instantiation vec :: (metric_space, finite) uniformity_dist
begin

definition [code del]:
  (uniformity :: (('a^'b::_) × ('a^'b::_)) filter) =
    (INF e∈{0 <..}. principal {(x, y). dist x y < e})

instance
  by standard (rule uniformity_vec_def)
end

declare uniformity_Abort[where 'a='a :: metric_space ^ 'b :: finite, code]

instantiation vec :: (metric_space, finite) metric_space
begin

proposition dist_vec_nth_le: dist (x $ i) (y $ i) ≤ dist x y
  unfolding dist_vec_def by (rule member_le_L2_set) simp_all

instance proof
  fix x y :: 'a ^ 'b
  show dist x y = 0 ⟷ x = y
    unfolding dist_vec_def
    by (simp add: L2_set_eq_0_iff vec_eq_iff)
next
  fix x y z :: 'a ^ 'b
  show dist x y ≤ dist x z + dist y z
```

```
    unfolding dist_vec_def
    apply (rule order_trans [OF _ L2_set_triangle_ineq])
    apply (simp add: L2_set_mono dist_triangle2)
    done
next
  fix S :: ('a ^ 'b) set
  have *: open S ⟷ (∀x∈S. ∃e>0. ∀y. dist y x < e ⟶ y ∈ S)
  proof
    assume open S show ∀x∈S. ∃e>0. ∀y. dist y x < e ⟶ y ∈ S
    proof
      fix x assume x ∈ S
      obtain A where A: ∀i. open (A i) ∀i. x $ i ∈ A i
        and S: ∀y. (∀i. y $ i ∈ A i) ⟶ y ∈ S
        using ‹open S› and ‹x ∈ S› unfolding open_vec_def by metis
      have ∀i∈UNIV. ∃r>0. ∀y. dist y (x $ i) < r ⟶ y ∈ A i
        using A unfolding open_dist by simp
      hence ∃r. ∀i∈UNIV. 0 < r i ∧ (∀y. dist y (x $ i) < r i ⟶ y ∈ A i)
        by (rule finite_set_choice [OF finite])
      then obtain r where r1: ∀i. 0 < r i
        and r2: ∀i y. dist y (x $ i) < r i ⟶ y ∈ A i by fast
      have 0 < Min (range r) ∧ (∀y. dist y x < Min (range r) ⟶ y ∈ S)
        by (simp add: r1 r2 S le_less_trans [OF dist_vec_nth_le])
      thus ∃e>0. ∀y. dist y x < e ⟶ y ∈ S ..
    qed
  next
    assume *: ∀x∈S. ∃e>0. ∀y. dist y x < e ⟶ y ∈ S show open S
    proof (unfold open_vec_def, rule)
      fix x assume x ∈ S
      then obtain e where 0 < e and S: ∀y. dist y x < e ⟶ y ∈ S
        using * by fast
      define r where [abs_def]: r i = e / sqrt (of_nat CARD('b)) for i :: 'b
      from ‹0 < e› have r: ∀i. 0 < r i
        unfolding r_def by simp_all
      from ‹0 < e› have e: e = L2_set r UNIV
        unfolding r_def by (simp add: L2_set_constant)
      define A where A i = {y. dist (x $ i) y < r i} for i
      have ∀i. open (A i) ∧ x $ i ∈ A i
        unfolding A_def by (simp add: open_ball r)
      moreover have ∀y. (∀i. y $ i ∈ A i) ⟶ y ∈ S
        by (simp add: A_def S dist_vec_def e L2_set_strict_mono dist_commute)
      ultimately show ∃A. (∀i. open (A i) ∧ x $ i ∈ A i) ∧
        (∀y. (∀i. y $ i ∈ A i) ⟶ y ∈ S) by metis
    qed
  qed
  show open S = (∀x∈S. ∀F (x', y) in uniformity. x' = x ⟶ y ∈ S)
    unfolding * eventually_uniformity_metric
    by (simp del: split_paired_All add: dist_vec_def dist_commute)
qed
```

end

lemma *Cauchy_vec_nth*:
Cauchy (λ*n*. *X n*) ⟹ *Cauchy* (λ*n*. *X n* $ *i*)
unfolding *Cauchy_def* **by** (*fast intro*: *le_less_trans* [*OF dist_vec_nth_le*])

lemma *vec_CauchyI*:
fixes *X* :: *nat* ⇒ ′*a*::*metric_space* ^ ′*n*
assumes *X*: ⋀*i*. *Cauchy* (λ*n*. *X n* $ *i*)
shows *Cauchy* (λ*n*. *X n*)
proof (*rule metric_CauchyI*)
fix *r* :: *real* **assume** *0* < *r*
hence *0* < *r* / *of_nat CARD*(′*n*) (**is** *0* < *?s*) **by** *simp*
define *N* **where** *N i* = (*LEAST N*. ∀ *m*≥*N*. ∀ *n*≥*N*. *dist* (*X m* $ *i*) (*X n* $ *i*) < *?s*) **for** *i*
define *M* **where** *M* = *Max* (*range N*)
have ⋀*i*. ∃ *N*. ∀ *m*≥*N*. ∀ *n*≥*N*. *dist* (*X m* $ *i*) (*X n* $ *i*) < *?s*
using *X* ‹*0* < *?s*› **by** (*rule metric_CauchyD*)
hence ⋀*i*. ∀ *m*≥*N i*. ∀ *n*≥*N i*. *dist* (*X m* $ *i*) (*X n* $ *i*) < *?s*
unfolding *N_def* **by** (*rule LeastI_ex*)
hence *M*: ⋀*i*. ∀ *m*≥*M*. ∀ *n*≥*M*. *dist* (*X m* $ *i*) (*X n* $ *i*) < *?s*
unfolding *M_def* **by** *simp*
{
fix *m n* :: *nat*
assume *M* ≤ *m M* ≤ *n*
have *dist* (*X m*) (*X n*) = *L2_set* (λ*i*. *dist* (*X m* $ *i*) (*X n* $ *i*)) *UNIV*
unfolding *dist_vec_def* **..**
also have ... ≤ *sum* (λ*i*. *dist* (*X m* $ *i*) (*X n* $ *i*)) *UNIV*
by (*rule L2_set_le_sum* [*OF zero_le_dist*])
also have ... < *sum* (λ*i*::′*n*. *?s*) *UNIV*
by (*rule sum_strict_mono*, *simp_all add*: *M* ‹*M* ≤ *m*› ‹*M* ≤ *n*›)
also have ... = *r*
by *simp*
finally have *dist* (*X m*) (*X n*) < *r* **.**
}
hence ∀ *m*≥*M*. ∀ *n*≥*M*. *dist* (*X m*) (*X n*) < *r*
by *simp*
then show ∃ *M*. ∀ *m*≥*M*. ∀ *n*≥*M*. *dist* (*X m*) (*X n*) < *r* **..**
qed

instance *vec* :: (*complete_space*, *finite*) *complete_space*
proof
fix *X* :: *nat* ⇒ ′*a* ^ ′*b* **assume** *Cauchy X*
have ⋀*i*. (λ*n*. *X n* $ *i*) ⟶ *lim* (λ*n*. *X n* $ *i*)
using *Cauchy_vec_nth* [*OF* ‹*Cauchy X*›]
by (*simp add*: *Cauchy_convergent_iff convergent_LIMSEQ_iff*)
hence *X* ⟶ *vec_lambda* (λ*i*. *lim* (λ*n*. *X n* $ *i*))
by (*simp add*: *vec_tendstoI*)
then show *convergent X*

by (*rule convergentI*)
qed

1.8.8 Normed vector space

instantiation *vec* :: (*real_normed_vector*, *finite*) *real_normed_vector*
begin

definition *norm x* = *L2_set* (λ*i*. *norm* (*x*$*i*)) *UNIV*

definition *sgn* (*x*::'*a*^'*b*) = *scaleR* (*inverse* (*norm x*)) *x*

instance proof
  fix *a* :: *real* **and** *x y* :: '*a* ^ '*b*
  show *norm x* = *0* ⟷ *x* = *0*
    unfolding *norm_vec_def*
    by (*simp add*: *L2_set_eq_0_iff vec_eq_iff*)
  show *norm* (*x* + *y*) ≤ *norm x* + *norm y*
    unfolding *norm_vec_def*
    apply (*rule order_trans* [*OF* _ *L2_set_triangle_ineq*])
    apply (*simp add*: *L2_set_mono norm_triangle_ineq*)
    done
  show *norm* (*scaleR a x*) = |*a*| ∗ *norm x*
    unfolding *norm_vec_def*
    by (*simp add*: *L2_set_right_distrib*)
  show *sgn x* = *scaleR* (*inverse* (*norm x*)) *x*
    by (*rule sgn_vec_def*)
  show *dist x y* = *norm* (*x* − *y*)
    unfolding *dist_vec_def norm_vec_def*
    by (*simp add*: *dist_norm*)
qed

end

lemma *norm_nth_le*: *norm* (*x* $ *i*) ≤ *norm x*
unfolding *norm_vec_def*
by (*rule member_le_L2_set*) *simp_all*

lemma *norm_le_componentwise_cart*:
  fixes *x* :: '*a*::*real_normed_vector*^'*n*
  assumes ⋀*i*. *norm*(*x*$*i*) ≤ *norm*(*y*$*i*)
  shows *norm x* ≤ *norm y*
  unfolding *norm_vec_def*
  by (*rule L2_set_mono*) (*auto simp*: *assms*)

lemma *component_le_norm_cart*: |*x*$*i*| ≤ *norm x*
  apply (*simp add*: *norm_vec_def*)
  apply (*rule member_le_L2_set*, *simp_all*)
  done

```
lemma norm_bound_component_le_cart: norm x ≤ e ==> |x$i| ≤ e
  by (metis component_le_norm_cart order_trans)

lemma norm_bound_component_lt_cart: norm x < e ==> |x$i| < e
  by (metis component_le_norm_cart le_less_trans)

lemma norm_le_l1_cart: norm x ≤ sum(λi. |x$i|) UNIV
  by (simp add: norm_vec_def L2_set_le_sum)

lemma bounded_linear_vec_nth[intro]: bounded_linear (λx. x $ i)
apply standard
apply (rule vector_add_component)
apply (rule vector_scaleR_component)
apply (rule_tac x=1 in exI, simp add: norm_nth_le)
done

instance vec :: (banach, finite) banach ..
```

1.8.9 Inner product space

```
instantiation vec :: (real_inner, finite) real_inner
begin

definition inner x y = sum (λi. inner (x$i) (y$i)) UNIV

instance proof
  fix r :: real and x y z :: 'a ^ 'b
  show inner x y = inner y x
    unfolding inner_vec_def
    by (simp add: inner_commute)
  show inner (x + y) z = inner x z + inner y z
    unfolding inner_vec_def
    by (simp add: inner_add_left sum.distrib)
  show inner (scaleR r x) y = r * inner x y
    unfolding inner_vec_def
    by (simp add: sum_distrib_left)
  show 0 ≤ inner x x
    unfolding inner_vec_def
    by (simp add: sum_nonneg)
  show inner x x = 0 ⟷ x = 0
    unfolding inner_vec_def
    by (simp add: vec_eq_iff sum_nonneg_eq_0_iff)
  show norm x = sqrt (inner x x)
    unfolding inner_vec_def norm_vec_def L2_set_def
    by (simp add: power2_norm_eq_inner)
qed

end
```

1.8.10 Euclidean space

Vectors pointing along a single axis.

```
definition axis k x = (χ i. if i = k then x else 0)

lemma axis_nth [simp]: axis i x $ i = x
  unfolding axis_def by simp

lemma axis_eq_axis: axis i x = axis j y ⟷ x = y ∧ i = j ∨ x = 0 ∧ y = 0
  unfolding axis_def vec_eq_iff by auto

lemma inner_axis_axis:
  inner (axis i x) (axis j y) = (if i = j then inner x y else 0)
  unfolding inner_vec_def
  apply (cases i = j)
  apply clarsimp
  apply (subst sum.remove [of _ j], simp_all)
  apply (rule sum.neutral, simp add: axis_def)
  apply (rule sum.neutral, simp add: axis_def)
  done

lemma inner_axis: inner x (axis i y) = inner (x $ i) y
  by (simp add: inner_vec_def axis_def sum.remove [where x=i])

lemma inner_axis': inner(axis i y) x = inner y (x $ i)
  by (simp add: inner_axis inner_commute)

instantiation vec :: (euclidean_space, finite) euclidean_space
begin

definition Basis = (⋃ i. ⋃ u∈Basis. {axis i u})

instance proof
  show (Basis :: ('a ^ 'b) set) ≠ {}
    unfolding Basis_vec_def by simp
next
  show finite (Basis :: ('a ^ 'b) set)
    unfolding Basis_vec_def by simp
next
  fix u v :: 'a ^ 'b
  assume u ∈ Basis and v ∈ Basis
  thus inner u v = (if u = v then 1 else 0)
    unfolding Basis_vec_def
    by (auto simp add: inner_axis_axis axis_eq_axis inner_Basis)
next
  fix x :: 'a ^ 'b
  show (∀ u∈Basis. inner x u = 0) ⟷ x = 0
    unfolding Basis_vec_def
    by (simp add: inner_axis euclidean_all_zero_iff vec_eq_iff)
```

qed

proposition *DIM_cart* [*simp*]: *DIM*('*a* ^'*b*) = *CARD*('*b*) ∗ *DIM*('*a*)
proof −
have *card* (⋃ *i*::'*b*. ⋃ *u*::'*a*∈*Basis*. {*axis i u*}) = (∑ *i*::'*b*∈*UNIV*. *card* (⋃ *u*::'*a*∈*Basis*. {*axis i u*}))
by (*rule card_UN_disjoint*) (*auto simp*: *axis_eq_axis*)
also have ... = *CARD*('*b*) ∗ *DIM*('*a*)
by (*subst card_UN_disjoint*) (*auto simp*: *axis_eq_axis*)
finally show *?thesis*
by (*simp add*: *Basis_vec_def*)
qed

end

lemma *norm_axis_1* [*simp*]: *norm* (*axis m* (*1*::*real*)) = *1*
by (*simp add*: *inner_axis' norm_eq_1*)

lemma *sum_norm_allsubsets_bound_cart*:
fixes *f*:: '*a* ⇒ *real* ^'*n*
assumes *fP*: *finite P* **and** *fPs*: ⋀*Q*. *Q* ⊆ *P* ⟹ *norm* (*sum f Q*) ≤ *e*
shows *sum* (λ*x*. *norm* (*f x*)) *P* ≤ *2* ∗ *real CARD*('*n*) ∗ *e*
using *sum_norm_allsubsets_bound*[*OF assms*]
by *simp*

lemma *cart_eq_inner_axis*: *a* \$ *i* = *inner a* (*axis i 1*)
by (*simp add*: *inner_axis*)

lemma *axis_eq_0_iff* [*simp*]:
shows *axis m x* = *0* ⟷ *x* = *0*
by (*simp add*: *axis_def vec_eq_iff*)

lemma *axis_in_Basis_iff* [*simp*]: *axis i a* ∈ *Basis* ⟷ *a* ∈ *Basis*
by (*auto simp*: *Basis_vec_def axis_eq_axis*)

Mapping each basis element to the corresponding finite index

definition *axis_index* :: ('*a*::*comm_ring_1*)^'*n* ⇒ '*n* **where** *axis_index v* ≡ *SOME i*. *v* = *axis i 1*

lemma *axis_inverse*:
fixes *v* :: *real*^'*n*
assumes *v* ∈ *Basis*
shows ∃ *i*. *v* = *axis i 1*
proof −
have *v* ∈ (⋃ *n*. ⋃ *r*∈*Basis*. {*axis n r*})
using *assms Basis_vec_def* **by** *blast*
then show *?thesis*
by (*force simp add*: *vec_eq_iff*)
qed

lemma *axis_index*:
fixes *v* :: *real^'n*
assumes *v* ∈ *Basis*
shows *v* = *axis* (*axis_index v*) *1*
by (*metis* (*mono_tags*) *assms axis_inverse axis_index_def someI_ex*)

lemma *axis_index_axis* [*simp*]:
fixes *UU* :: *real^'n*
shows (*axis_index* (*axis u 1* :: *real^'n*)) = (*u*::*'n*)
by (*simp add*: *axis_eq_axis axis_index_def*)

1.8.11 A naive proof procedure to lift really trivial arithmetic stuff from the basis of the vector space

lemma *sum_cong_aux*:
(⋀*x*. *x* ∈ *A* ⟹ *f x* = *g x*) ⟹ *sum f A* = *sum g A*
by (*auto intro*: *sum.cong*)

hide_fact (**open**) *sum_cong_aux*

method_setup *vector* = ‹

```
let
  val ss1 =
    simpset_of (put_simpset HOL_basic_ss context
      addsimps [@{thm sum.distrib} RS sym,
      @{thm sum_subtractf} RS sym, @{thm sum_distrib_left},
      @{thm sum_distrib_right}, @{thm sum_negf} RS sym])
  val ss2 =
    simpset_of (context addsimps
            [@{thm plus_vec_def}, @{thm times_vec_def},
             @{thm minus_vec_def}, @{thm uminus_vec_def},
             @{thm one_vec_def}, @{thm zero_vec_def}, @{thm vec_def},
             @{thm scaleR_vec_def}, @{thm vector_scalar_mult_def}])
  fun vector_arith_tac ctxt ths =
    simp_tac (put_simpset ss1 ctxt)
   THEN' (fn i => resolve_tac ctxt @{thms Finite_Cartesian_Product.sum_cong_aux}
i
        ORELSE resolve_tac ctxt @{thms sum.neutral} i
            ORELSE simp_tac (put_simpset HOL_basic_ss ctxt addsimps [@{thm
vec_eq_iff}]) i)
    (* THEN' TRY o clarify_tac HOL_cs  THEN' (TRY o rtac @{thm iffI}) *)
    THEN' asm_full_simp_tac (put_simpset ss2 ctxt addsimps ths)
in
  Attrib.thms >> (fn ths => fn ctxt => SIMPLE_METHOD' (vector_arith_tac
ctxt ths))
end
```

› *lift trivial vector statements to real arith statements*

lemma *vec_0*[*simp*]: *vec 0* = *0* **by** *vector*
lemma *vec_1*[*simp*]: *vec 1* = *1* **by** *vector*

lemma *vec_inj*[*simp*]: *vec x* = *vec y* ⟷ *x* = *y* **by** *vector*

lemma *vec_in_image_vec*: *vec x* ∈ (*vec* ‘ *S*) ⟷ *x* ∈ *S* **by** *auto*

lemma *vec_add*: *vec*(*x* + *y*) = *vec x* + *vec y* **by** *vector*
lemma *vec_sub*: *vec*(*x* − *y*) = *vec x* − *vec y* **by** *vector*
lemma *vec_cmul*: *vec*(*c* ∗ *x*) = *c* ∗*s vec x* **by** *vector*
lemma *vec_neg*: *vec*(− *x*) = − *vec x* **by** *vector*

lemma *vec_scaleR*: *vec*(*c* ∗ *x*) = *c* $*_R$ *vec x*
  by *vector*

lemma *vec_sum*:
  assumes *finite S*
  shows *vec*(*sum f S*) = *sum* (*vec* ∘ *f*) *S*
  using *assms*
proof *induct*
  case *empty*
  then show *?case* **by** *simp*
next
  case *insert*
  then show *?case* **by** (*auto simp add*: *vec_add*)
qed

Obvious ”component-pushing”.

lemma *vec_component* [*simp*]: *vec x* $ *i* = *x*
  by *vector*

lemma *vector_mult_component* [*simp*]: (*x* ∗ *y*)$*i* = *x*$*i* ∗ *y*$*i*
  by *vector*

lemma *vector_smult_component* [*simp*]: (*c* ∗*s y*)$*i* = *c* ∗ (*y*$*i*)
  by *vector*

lemma *cond_component*: (*if b then x else y*)$*i* = (*if b then x*$*i* *else y*$*i*) **by** *vector*

lemmas *vector_component* =
  vec_component vector_add_component vector_mult_component
  vector_smult_component vector_minus_component vector_uminus_component
  vector_scaleR_component cond_component

1.8.12 Some frequently useful arithmetic lemmas over vectors

instance *vec* :: (*semigroup_mult*, *finite*) *semigroup_mult*
  by *standard* (*vector mult.assoc*)

instance *vec* :: (*monoid_mult*, *finite*) *monoid_mult*
  by *standard vector*+

instance *vec* :: (*ab_semigroup_mult*, *finite*) *ab_semigroup_mult*
  by *standard* (*vector mult.commute*)

instance *vec* :: (*comm_monoid_mult*, *finite*) *comm_monoid_mult*
  by *standard vector*

instance *vec* :: (*semiring*, *finite*) *semiring*
  by *standard* (*vector field_simps*)+

instance *vec* :: (*semiring_0*, *finite*) *semiring_0*
  by *standard* (*vector field_simps*)+
instance *vec* :: (*semiring_1*, *finite*) *semiring_1*
  by *standard vector*
instance *vec* :: (*comm_semiring*, *finite*) *comm_semiring*
  by *standard* (*vector field_simps*)+

instance *vec* :: (*comm_semiring_0*, *finite*) *comm_semiring_0* **..**
instance *vec* :: (*semiring_0_cancel*, *finite*) *semiring_0_cancel* **..**
instance *vec* :: (*comm_semiring_0_cancel*, *finite*) *comm_semiring_0_cancel* **..**
instance *vec* :: (*ring*, *finite*) *ring* **..**
instance *vec* :: (*semiring_1_cancel*, *finite*) *semiring_1_cancel* **..**
instance *vec* :: (*comm_semiring_1*, *finite*) *comm_semiring_1* **..**

instance *vec* :: (*ring_1*, *finite*) *ring_1* **..**

instance *vec* :: (*real_algebra*, *finite*) *real_algebra*
  by *standard* (*simp_all add*: *vec_eq_iff*)

instance *vec* :: (*real_algebra_1*, *finite*) *real_algebra_1* **..**

lemma *of_nat_index*: (*of_nat n* :: *'a*::*semiring_1* ^*'n*)$*i* = *of_nat n*
proof (*induct n*)
  case *0*
  then show *?case* **by** *vector*
next
  case *Suc*
  then show *?case* **by** *vector*
qed

lemma *one_index* [*simp*]: (*1* :: *'a* :: *one* ^ *'n*) $ *i* = *1*
  by *vector*

lemma *neg_one_index* [*simp*]: (− *1* :: *'a* :: {*one*, *uminus*} ^ *'n*) $ *i* = − *1*
  by *vector*

instance *vec* :: (*semiring_char_0*, *finite*) *semiring_char_0*
proof
  fix *m n* :: *nat*
  show *inj* (*of_nat* :: *nat* ⇒ *'a* ^ *'b*)
    by (*auto intro*!: *injI simp add*: *vec_eq_iff of_nat_index*)
qed

instance *vec* :: (*numeral*, *finite*) *numeral* ..
instance *vec* :: (*semiring_numeral*, *finite*) *semiring_numeral* ..

lemma *numeral_index* [*simp*]: *numeral w* $ *i* = *numeral w*
  by (*induct w*) (*simp_all only*: *numeral.simps vector_add_component one_index*)

lemma *neg_numeral_index* [*simp*]: − *numeral w* $ *i* = − *numeral w*
  by (*simp only*: *vector_uminus_component numeral_index*)

instance *vec* :: (*comm_ring_1*, *finite*) *comm_ring_1* ..
instance *vec* :: (*ring_char_0*, *finite*) *ring_char_0* ..

lemma *vector_smult_assoc*: *a* ∗*s* (*b* ∗*s* *x*) = ((*a*::*'a*::*semigroup_mult*) ∗ *b*) ∗*s* *x*
  by (*vector mult.assoc*)
lemma *vector_sadd_rdistrib*: ((*a*::*'a*::*semiring*) + *b*) ∗*s* *x* = *a* ∗*s* *x* + *b* ∗*s* *x*
  by (*vector field_simps*)
lemma *vector_add_ldistrib*: (*c*::*'a*::*semiring*) ∗*s* (*x* + *y*) = *c* ∗*s* *x* + *c* ∗*s* *y*
  by (*vector field_simps*)
lemma *vector_smult_lzero*[*simp*]: (*0*::*'a*::*mult_zero*) ∗*s* *x* = *0* **by** *vector*
lemma *vector_smult_lid*[*simp*]: (*1*::*'a*::*monoid_mult*) ∗*s* *x* = *x* **by** *vector*
lemma *vector_ssub_ldistrib*: (*c*::*'a*::*ring*) ∗*s* (*x* − *y*) = *c* ∗*s* *x* − *c* ∗*s* *y*
  by (*vector field_simps*)
lemma *vector_smult_rneg*: (*c*::*'a*::*ring*) ∗*s* −*x* = −(*c* ∗*s* *x*) **by** *vector*
lemma *vector_smult_lneg*: − (*c*::*'a*::*ring*) ∗*s* *x* = −(*c* ∗*s* *x*) **by** *vector*
lemma *vector_sneg_minus1*: −*x* = (−*1*::*'a*::*ring_1*) ∗*s* *x* **by** *vector*
lemma *vector_smult_rzero*[*simp*]: *c* ∗*s* *0* = (*0*::*'a*::*mult_zero* ^ *'n*) **by** *vector*
lemma *vector_sub_rdistrib*: ((*a*::*'a*::*ring*) − *b*) ∗*s* *x* = *a* ∗*s* *x* − *b* ∗*s* *x*
  by (*vector field_simps*)

lemma *vec_eq*[*simp*]: (*vec m* = *vec n*) ⟷ (*m* = *n*)
  by (*simp add*: *vec_eq_iff*)

lemma *Vector_Spaces_linear_vec* [*simp*]: *Vector_Spaces.linear* (∗) *vector_scalar_mult vec*
  by *unfold_locales* (*vector algebra_simps*)+

lemma *vector_mul_eq_0*[*simp*]: (*a* ∗*s* *x* = *0*) ⟷ *a* = (*0*::*'a*::*idom*) ∨ *x* = *0*
  by *vector*

lemma *vector_mul_lcancel*[*simp*]: *a* ∗*s* *x* = *a* ∗*s* *y* ⟷ *a* = (*0*::*'a*::*field*) ∨ *x* = *y*
  by (*metis eq_iff_diff_eq_0 vector_mul_eq_0 vector_ssub_ldistrib*)

lemma *vector_mul_rcancel*[*simp*]: $a *s\ x = b *s\ x \longleftrightarrow (a{::}'a{::}field) = b \lor x = 0$
by (*subst eq_iff_diff_eq_0*, *subst vector_sub_rdistrib* [*symmetric*]) *simp*

lemma *scalar_mult_eq_scaleR* [*abs_def*]: $c *s\ x = c *_R x$
unfolding *scaleR_vec_def vector_scalar_mult_def* **by** *simp*

lemma *dist_mul*[*simp*]: *dist* $(c *s\ x)\ (c *s\ y) = |c| * dist\ x\ y$
unfolding *dist_norm scalar_mult_eq_scaleR*
unfolding *scaleR_right_diff_distrib*[*symmetric*] **by** *simp*

lemma *sum_component* [*simp*]:
fixes $f{::}\ 'a \Rightarrow ('b{::}comm\_monoid\_add)\ \hat{}'n$
shows $(sum\ f\ S)\$i = sum\ (\lambda x.\ (f\ x)\$i)\ S$
proof (*cases finite S*)
case *True*
then show *?thesis* **by** *induct simp_all*
next
case *False*
then show *?thesis* **by** *simp*
qed

lemma *sum_eq*: $sum\ f\ S = (\chi\ i.\ sum\ (\lambda x.\ (f\ x)\$i\ )\ S)$
by (*simp add*: *vec_eq_iff*)

lemma *sum_cmul*:
fixes $f{::}\ 'c \Rightarrow ('a{::}semiring\_1)\hat{}'n$
shows $sum\ (\lambda x.\ c *s\ f\ x)\ S = c *s\ sum\ f\ S$
by (*simp add*: *vec_eq_iff sum_distrib_left*)

lemma *linear_vec* [*simp*]: *linear vec*
using *Vector_Spaces_linear_vec*
apply (*auto*)
by *unfold_locales* (*vector algebra_simps*)+

1.8.13 Matrix operations

Matrix notation. NB: an MxN matrix is of type $(('a, 'n)\ vec, 'm)\ vec$, not $(('a, 'm)\ vec, 'n)\ vec$

definition *map_matrix*::$('a \Rightarrow 'b) \Rightarrow (('a, 'i{::}finite)vec, 'j{::}finite)\ vec \Rightarrow (('b, 'i)vec, 'j)\ vec$ **where**
map_matrix $f\ x = (\chi\ i\ j.\ f\ (x\ \$\ i\ \$\ j))$

lemma *nth_map_matrix*[*simp*]: *map_matrix* $f\ x\ \$\ i\ \$\ j = f\ (x\ \$\ i\ \$\ j)$
by (*simp add*: *map_matrix_def*)

definition *matrix_matrix_mult* :: $('a{::}semiring\_1)\ \hat{}'n\hat{}'m \Rightarrow 'a\ \hat{}'p\hat{}'n \Rightarrow 'a\ \hat{}\ 'p\ \hat{}'m$
(**infixl** $**$ *70*)
where $m ** m' == (\chi\ i\ j.\ sum\ (\lambda k.\ ((m\$i)\$k) * ((m'\$k)\$j))\ (UNIV :: 'n\ set))$

```
::'a ^ 'p ^'m

definition matrix_vector_mult :: ('a::semiring_1) ^'n^'m ⇒ 'a ^'n ⇒ 'a ^ 'm
    (infixl *v 70)
  where m *v x ≡ (χ i. sum (λj. ((m$i)$j) * (x$j)) (UNIV ::'n set)) :: 'a^'m

definition vector_matrix_mult :: 'a ^ 'm ⇒ ('a::semiring_1) ^'n^'m ⇒ 'a ^'n
    (infixl v* 70)
  where v v* m == (χ j. sum (λi. ((m$i)$j) * (v$i)) (UNIV :: 'm set)) :: 'a^'n

definition (mat::'a::zero => 'a ^'n^'n) k = (χ i j. if i = j then k else 0)
definition transpose where
  (transpose::'a^'n^'m ⇒ 'a^'m^'n) A = (χ i j. ((A$j)$i))
definition (row::'m => 'a ^'n^'m ⇒ 'a ^'n) i A = (χ j. ((A$i)$j))
definition (column::'n =>'a^'n^'m =>'a^'m) j A = (χ i. ((A$i)$j))
definition rows(A::'a^'n^'m) = { row i A | i. i ∈ (UNIV :: 'm set)}
definition columns(A::'a^'n^'m) = { column i A | i. i ∈ (UNIV :: 'n set)}

lemma times0_left [simp]: (0::'a::semiring_1^'n^'m) ** (A::'a ^'p^'n) = 0
  by (simp add: matrix_matrix_mult_def zero_vec_def)

lemma times0_right [simp]: (A::'a::semiring_1^'n^'m) ** (0::'a ^'p^'n) = 0
  by (simp add: matrix_matrix_mult_def zero_vec_def)

lemma mat_0[simp]: mat 0 = 0 by (vector mat_def)
lemma matrix_add_ldistrib: (A ** (B + C)) = (A ** B) + (A ** C)
  by (vector matrix_matrix_mult_def sum.distrib[symmetric] field_simps)

lemma matrix_mul_lid [simp]:
  fixes A :: 'a::semiring_1 ^ 'm ^ 'n
  shows mat 1 ** A = A
  apply (simp add: matrix_matrix_mult_def mat_def)
  apply vector
  apply (auto simp only: if_distrib if_distribR sum.delta'[OF finite]
    mult_1_left mult_zero_left if_True UNIV_I)
  done

lemma matrix_mul_rid [simp]:
  fixes A :: 'a::semiring_1 ^ 'm ^ 'n
  shows A ** mat 1 = A
  apply (simp add: matrix_matrix_mult_def mat_def)
  apply vector
  apply (auto simp only: if_distrib if_distribR sum.delta[OF finite]
    mult_1_right mult_zero_right if_True UNIV_I cong: if_cong)
  done

proposition matrix_mul_assoc: A ** (B ** C) = (A ** B) ** C
 apply (vector matrix_matrix_mult_def sum_distrib_left sum_distrib_right mult.assoc)
  apply (subst sum.swap)
```

```
  apply simp
  done

proposition matrix_vector_mul_assoc: A *v (B *v x) = (A ** B) *v x
  apply (vector matrix_matrix_mult_def matrix_vector_mult_def
    sum_distrib_left sum_distrib_right mult.assoc)
  apply (subst sum.swap)
  apply simp
  done

proposition scalar_matrix_assoc:
  fixes A :: ('a::real_algebra_1)^'m^'n
  shows k *R (A ** B) = (k *R A) ** B
 by (simp add: matrix_matrix_mult_def sum_distrib_left mult_ac vec_eq_iff scaleR_sum_right)

proposition matrix_scalar_ac:
  fixes A :: ('a::real_algebra_1)^'m^'n
  shows A ** (k *R B) = k *R A ** B
  by (simp add: matrix_matrix_mult_def sum_distrib_left mult_ac vec_eq_iff)

lemma matrix_vector_mul_lid [simp]: mat 1 *v x = (x::'a::semiring_1 ^ 'n)
  apply (vector matrix_vector_mult_def mat_def)
  apply (simp add: if_distrib if_distribR cong del: if_weak_cong)
  done

lemma matrix_transpose_mul:
    transpose(A ** B) = transpose B ** transpose (A::'a::comm_semiring_1^_^_)
  by (simp add: matrix_matrix_mult_def transpose_def vec_eq_iff mult.commute)

lemma matrix_mult_transpose_dot_column:
  shows transpose A ** A = (χ i j. inner (column i A) (column j A))
   by (simp add: matrix_matrix_mult_def vec_eq_iff transpose_def column_def inner_vec_def)

lemma matrix_mult_transpose_dot_row:
  shows A ** transpose A = (χ i j. inner (row i A) (row j A))
 by (simp add: matrix_matrix_mult_def vec_eq_iff transpose_def row_def inner_vec_def)

lemma matrix_eq:
  fixes A B :: 'a::semiring_1 ^ 'n ^ 'm
  shows A = B ⟷ (∀ x. A *v x = B *v x) (is ?lhs ⟷ ?rhs)
  apply auto
  apply (subst vec_eq_iff)
  apply clarify
  apply (clarsimp simp add: matrix_vector_mult_def if_distrib if_distribR vec_eq_iff
cong del: if_weak_cong)
  apply (erule_tac x=axis ia 1 in allE)
  apply (erule_tac x=i in allE)
  apply (auto simp add: if_distrib if_distribR axis_def
```

sum.delta[*OF finite*] *cong del*: *if_weak_cong*)
done

lemma *matrix_vector_mul_component*: $(A *v\ x)\$k = inner\ (A\$k)\ x$
by (*simp add*: *matrix_vector_mult_def inner_vec_def*)

lemma *dot_lmul_matrix*: $inner\ ((x::real\ \hat{}\ \_)\ v*\ A)\ y = inner\ x\ (A *v\ y)$
apply (*simp add*: *inner_vec_def matrix_vector_mult_def vector_matrix_mult_def sum_distrib_right sum_distrib_left ac_simps*)
apply (*subst sum.swap*)
apply *simp*
done

lemma *transpose_mat* [*simp*]: *transpose* (*mat n*) = *mat n*
by (*vector transpose_def mat_def*)

lemma *transpose_transpose* [*simp*]: *transpose*(*transpose A*) = *A*
by (*vector transpose_def*)

lemma *row_transpose* [*simp*]: *row i* (*transpose A*) = *column i A*
by (*simp add*: *row_def column_def transpose_def vec_eq_iff*)

lemma *column_transpose* [*simp*]: *column i* (*transpose A*) = *row i A*
by (*simp add*: *row_def column_def transpose_def vec_eq_iff*)

lemma *rows_transpose* [*simp*]: *rows*(*transpose A*) = *columns A*
by (*auto simp add*: *rows_def columns_def intro*: *set_eqI*)

lemma *columns_transpose* [*simp*]: *columns*(*transpose A*) = *rows A*
by (*metis transpose_transpose rows_transpose*)

lemma *transpose_scalar*: $transpose\ (k *_R A) = k *_R transpose\ A$
unfolding *transpose_def*
by (*simp add*: *vec_eq_iff*)

lemma *transpose_iff* [*iff*]: $transpose\ A = transpose\ B \longleftrightarrow A = B$
by (*metis transpose_transpose*)

lemma *matrix_mult_sum*:
$(A::'a::comm\_semiring\_1\hat{}'n\hat{}'m) *v\ x = sum\ (\lambda i.\ (x\$i) *s\ column\ i\ A)\ (UNIV::'n\ set)$
by (*simp add*: *matrix_vector_mult_def vec_eq_iff column_def mult.commute*)

lemma *vector_componentwise*:
$(x::'a::ring\_1\hat{}'n) = (\chi\ j.\ \sum i\in UNIV.\ (x\$i) * (axis\ i\ 1 :: 'a\hat{}'n)\ \$\ j)$
by (*simp add*: *axis_def if_distrib sum.If_cases vec_eq_iff*)

lemma *basis_expansion*: $sum\ (\lambda i.\ (x\$i) *s\ axis\ i\ 1)\ UNIV = (x::('a::ring\_1)\ \hat{}'n)$
by (*auto simp add*: *axis_def vec_eq_iff if_distrib sum.If_cases cong del*: *if_weak_cong*)

Correspondence between matrices and linear operators.

definition *matrix* :: ('*a*::{*plus*,*times*, *one*, *zero*} ^'*m* ⇒ '*a* ^ '*n*) ⇒ '*a* ^'*m* ^'*n*
where *matrix f* = (χ *i j*. (*f*(*axis j 1*))$*i*)

lemma *matrix_id_mat_1*: *matrix id* = *mat 1*
by (*simp add*: *mat_def matrix_def axis_def*)

lemma *matrix_scaleR*: (*matrix* ((*$_R$) *r*)) = *mat r*
by (*simp add*: *mat_def matrix_def axis_def if_distrib cong*: *if_cong*)

lemma *matrix_vector_mul_linear*[*intro*, *simp*]: *linear* (λ*x*. *A* *v (*x*::'*a*::*real_algebra_1* ^ _))
by (*simp add*: *linear_iff matrix_vector_mult_def vec_eq_iff field_simps sum_distrib_left sum.distrib scaleR_right.sum*)

lemma *vector_matrix_left_distrib* [*algebra_simps*]:
shows (*x* + *y*) *v** *A* = *x v** *A* + *y v** *A*
unfolding *vector_matrix_mult_def*
by (*simp add*: *algebra_simps sum.distrib vec_eq_iff*)

lemma *matrix_vector_right_distrib* [*algebra_simps*]:
A *v (*x* + *y*) = *A* *v *x* + *A* *v *y*
by (*vector matrix_vector_mult_def sum.distrib distrib_left*)

lemma *matrix_vector_mult_diff_distrib* [*algebra_simps*]:
fixes *A* :: '*a*::*ring_1* ^'*n* ^'*m*
shows *A* *v (*x* − *y*) = *A* *v *x* − *A* *v *y*
by (*vector matrix_vector_mult_def sum_subtractf right_diff_distrib*)

lemma *matrix_vector_mult_scaleR*[*algebra_simps*]:
fixes *A* :: *real* ^'*n* ^'*m*
shows *A* *v (*c* *$_R$ *x*) = *c* *$_R$ (*A* *v *x*)
using *linear_iff matrix_vector_mul_linear* **by** *blast*

lemma *matrix_vector_mult_0_right* [*simp*]: *A* *v *0* = *0*
by (*simp add*: *matrix_vector_mult_def vec_eq_iff*)

lemma *matrix_vector_mult_0* [*simp*]: *0* *v *w* = *0*
by (*simp add*: *matrix_vector_mult_def vec_eq_iff*)

lemma *matrix_vector_mult_add_rdistrib* [*algebra_simps*]:
(*A* + *B*) *v *x* = (*A* *v *x*) + (*B* *v *x*)
by (*vector matrix_vector_mult_def sum.distrib distrib_right*)

lemma *matrix_vector_mult_diff_rdistrib* [*algebra_simps*]:
fixes *A* :: '*a* :: *ring_1* ^'*n* ^'*m*
shows (*A* − *B*) *v *x* = (*A* *v *x*) − (*B* *v *x*)
by (*vector matrix_vector_mult_def sum_subtractf left_diff_distrib*)

lemma *matrix_vector_column*:
(*A*::'*a*::*comm_semiring_1*^'*n*^_) *v *x* = *sum* (λ*i*. (*x*$*i*) *s ((*transpose* *A*)$*i*)) (*UNIV*:: '*n set*)
by (*simp add*: *matrix_vector_mult_def transpose_def vec_eq_iff mult.commute*)

1.8.14 Inverse matrices (not necessarily square)

definition
invertible(*A*::'*a*::*semiring_1*^'*n*^'*m*) ⟷ (∃ *A*'::'*a*^'*m*^'*n*. *A* ** *A*' = *mat 1* ∧ *A*' ** *A* = *mat 1*)

definition
matrix_inv(*A*:: '*a*::*semiring_1*^'*n*^'*m*) =
(*SOME A*'::'*a*^'*m*^'*n*. *A* ** *A*' = *mat 1* ∧ *A*' ** *A* = *mat 1*)

lemma *inj_matrix_vector_mult*:
fixes *A*::'*a*::*field*^'*n*^'*m*
assumes *invertible A*
shows *inj* ((*v) *A*)
by (*metis assms inj_on_inverseI invertible_def matrix_vector_mul_assoc matrix_vector_mul_lid*)

lemma *scalar_invertible*:
fixes *A* :: ('*a*::*real_algebra_1*)^'*m*^'*n*
assumes *k* ≠ *0* **and** *invertible A*
shows *invertible* (*k* $*_R$ *A*)
proof −
obtain *A*' **where** *A* ** *A*' = *mat 1* **and** *A*' ** *A* = *mat 1*
using *assms* **unfolding** *invertible_def* **by** *auto*
with ‹*k* ≠ *0*›
have (*k* $*_R$ *A*) ** ((*1*/*k*) $*_R$ *A*') = *mat 1* ((*1*/*k*) $*_R$ *A*') ** (*k* $*_R$ *A*) = *mat 1*
by (*simp_all add*: *assms matrix_scalar_ac*)
thus *invertible* (*k* $*_R$ *A*)
unfolding *invertible_def* **by** *auto*
qed

proposition *scalar_invertible_iff*:
fixes *A* :: ('*a*::*real_algebra_1*)^'*m*^'*n*
assumes *k* ≠ *0* **and** *invertible A*
shows *invertible* (*k* $*_R$ *A*) ⟷ *k* ≠ *0* ∧ *invertible A*
by (*simp add*: *assms scalar_invertible*)

lemma *vector_transpose_matrix* [*simp*]: *x* v* *transpose A* = *A* *v *x*
unfolding *transpose_def vector_matrix_mult_def matrix_vector_mult_def*
by *simp*

lemma *transpose_matrix_vector* [*simp*]: *transpose A* *v *x* = *x* v* *A*
unfolding *transpose_def vector_matrix_mult_def matrix_vector_mult_def*
by *simp*

lemma *vector_scalar_commute*:
fixes A :: $'a$::{*field*} ^$'m$ ^$'n$
shows $A *v (c *s x) = c *s (A *v x)$
by (*simp add*: *vector_scalar_mult_def matrix_vector_mult_def mult_ac sum_distrib_left*)

lemma *scalar_vector_matrix_assoc*:
fixes k :: $'a$::{*field*} **and** x :: $'a$::{*field*} ^$'n$ **and** A :: $'a$ ^$'m$ ^$'n$
shows $(k *s x)\ v*\ A = k *s (x\ v*\ A)$
by (*metis transpose_matrix_vector vector_scalar_commute*)

lemma *vector_matrix_mult_0* [*simp*]: *0* $v*$ A = *0*
unfolding *vector_matrix_mult_def* **by** (*simp add*: *zero_vec_def*)

lemma *vector_matrix_mult_0_right* [*simp*]: x $v*$ *0* = *0*
unfolding *vector_matrix_mult_def* **by** (*simp add*: *zero_vec_def*)

lemma *vector_matrix_mul_rid* [*simp*]:
fixes v :: ($'a$::*semiring_1*) ^$'n$
shows $v\ v*$ *mat 1* = v
by (*metis matrix_vector_mul_lid transpose_mat vector_transpose_matrix*)

lemma *scaleR_vector_matrix_assoc*:
fixes k :: *real* **and** x :: *real*^$'n$ **and** A :: *real*^$'m$ ^$'n$
shows $(k *_R x)\ v*\ A = k *_R (x\ v*\ A)$
by (*metis matrix_vector_mult_scaleR transpose_matrix_vector*)

proposition *vector_scaleR_matrix_ac*:
fixes k :: *real* **and** x :: *real*^$'n$ **and** A :: *real*^$'m$ ^$'n$
shows $x\ v*\ (k *_R A) = k *_R (x\ v*\ A)$
proof −
have $x\ v*\ (k *_R A) = (k *_R x)\ v*\ A$
unfolding *vector_matrix_mult_def*
by (*simp add*: *algebra_simps*)
with *scaleR_vector_matrix_assoc*
show $x\ v*\ (k *_R A) = k *_R (x\ v*\ A)$
by *auto*
qed

end

1.9 Linear Algebra on Finite Cartesian Products

theory *Cartesian_Space*
imports
Finite_Cartesian_Product Linear_Algebra
begin

1.9.1 Type ('a, 'n) vec and fields as vector spaces

```
definition cart_basis = {axis i 1 | i. i∈UNIV}

lemma finite_cart_basis: finite (cart_basis) unfolding cart_basis_def
  using finite_Atleast_Atmost_nat by fastforce

lemma card_cart_basis: card (cart_basis::('a::zero_neq_one^'i) set) = CARD('i)
  unfolding cart_basis_def Setcompr_eq_image
  by (rule card_image) (auto simp: inj_on_def axis_eq_axis)

interpretation vec: vector_space (*s)
  by unfold_locales (vector algebra_simps)+

lemma independent_cart_basis:
  vec.independent (cart_basis)
proof (rule vec.independent_if_scalars_zero)
  show finite (cart_basis) using finite_cart_basis .
  fix f::('a, 'b) vec ⇒ 'a and x::('a, 'b) vec
  assume eq_0: (∑x∈cart_basis. f x *s x) = 0 and x_in: x ∈ cart_basis
  obtain i where x: x = axis i 1 using x_in unfolding cart_basis_def by auto
  have sum_eq_0: (∑x∈(cart_basis) − {x}. f x * (x $ i)) = 0
  proof (rule sum.neutral, rule ballI)
    fix xa assume xa: xa ∈ cart_basis − {x}
    obtain a where a: xa = axis a 1 and a_not_i: a ≠ i
      using xa x unfolding cart_basis_def by auto
    have xa $ i = 0 unfolding a axis_def using a_not_i by auto
    thus f xa * xa $ i = 0 by simp
  qed
  have 0 = (∑x∈cart_basis. f x *s x) $ i using eq_0 by simp
  also have ... = (∑x∈cart_basis. (f x *s x) $ i) unfolding sum_component ..
 also have ... = (∑x∈cart_basis. f x * (x $ i)) unfolding vector_smult_component
..
  also have ... = f x * (x $ i) + (∑x∈(cart_basis) − {x}. f x * (x $ i))
    by (rule sum.remove[OF finite_cart_basis x_in])
  also have ... =  f x * (x $ i) unfolding sum_eq_0 by simp
  also have ... = f x unfolding x axis_def by auto
  finally show f x = 0 ..
qed

lemma span_cart_basis:
  vec.span (cart_basis) = UNIV
proof (auto)
  fix x::('a, 'b) vec
  let ?f=λv. x $ (THE i. v = axis i 1)
  show x ∈ vec.span (cart_basis)
    apply (unfold vec.span_finite[OF finite_cart_basis])
    apply (rule image_eqI[of _ _ ?f])
     apply (subst  vec_eq_iff)
     apply clarify
```

```
  proof -
    fix i::'b
    let ?w = axis i (1::'a)
    have the_eq_i: (THE a. ?w = axis a 1) = i
      by (rule the_equality, auto simp: axis_eq_axis)
    have sum_eq_0: (∑ v∈(cart_basis) - {?w}. x $ (THE i. v = axis i 1) * v $ i)
= 0
    proof (rule sum.neutral, rule ballI)
      fix xa::('a, 'b) vec
      assume xa: xa ∈ cart_basis - {?w}
       obtain j where j: xa = axis j 1 and i_not_j: i ≠ j using xa unfolding
cart_basis_def by auto
      have the_eq_j: (THE i. xa = axis i 1) = j
      proof (rule the_equality)
        show xa = axis j 1 using j .
        show ⋀i. xa = axis i 1 ⟹ i = j by (metis axis_eq_axis j zero_neq_one)
      qed
      show x $ (THE i. xa = axis i 1) * xa $ i = 0
        apply (subst (2) j)
        unfolding the_eq_j unfolding axis_def using i_not_j by simp
    qed
    have (∑ v∈cart_basis. x $ (THE i. v = axis i 1) *s v) $ i =
  (∑ v∈cart_basis. (x $ (THE i. v = axis i 1) *s v) $ i) unfolding sum_component
..
    also have ... = (∑ v∈cart_basis. x $ (THE i. v = axis i 1) * v $ i)
      unfolding vector_smult_component ..
    also have ... = x $ (THE a. ?w = axis a 1) * ?w $ i + (∑ v∈(cart_basis) -
{?w}. x $ (THE i. v = axis i 1) * v $ i)
      by (rule sum.remove[OF finite_cart_basis], auto simp add: cart_basis_def)
    also have ... = x $ (THE a. ?w = axis a 1) * ?w $ i unfolding sum_eq_0 by
simp
    also have ... = x $ i unfolding the_eq_i unfolding axis_def by auto
    finally show x $ i = (∑ v∈cart_basis. x $ (THE i. v = axis i 1) *s v) $ i by
simp
  qed simp
qed


interpretation vec: finite_dimensional_vector_space (*s) cart_basis
  by (unfold_locales, auto simp add: finite_cart_basis independent_cart_basis span_cart_basis)

lemma matrix_vector_mul_linear_gen[intro, simp]:
  Vector_Spaces.linear (*s) (*s) ((*v) A)
  by unfold_locales
    (vector matrix_vector_mult_def sum.distrib algebra_simps)+

lemma span_vec_eq: vec.span X = span X
  and dim_vec_eq: vec.dim X = dim X
  and dependent_vec_eq: vec.dependent X = dependent X
```

```
  and subspace_vec_eq: vec.subspace X = subspace X
  for X::(real^'n) set
  unfolding span_raw_def dim_raw_def dependent_raw_def subspace_raw_def
  by (auto simp: scalar_mult_eq_scaleR)

lemma linear_componentwise:
  fixes f:: 'a::field ^'m ⇒ 'a ^ 'n
  assumes lf: Vector_Spaces.linear (*s) (*s) f
  shows (f x)$j = sum (λi. (x$i) * (f (axis i 1)$j)) (UNIV :: 'm set) (is ?lhs =
?rhs)
proof -
  interpret lf: Vector_Spaces.linear (*s) (*s) f
    using lf .
  let ?M = (UNIV :: 'm set)
  let ?N = (UNIV :: 'n set)
  have fM: finite ?M by simp
  have ?rhs = (sum (λi. (x$i) *s (f (axis i 1))) ?M)$j
    unfolding sum_component by simp
  then show ?thesis
    unfolding lf.sum[symmetric] lf.scale[symmetric]
    unfolding basis_expansion by auto
qed

interpretation vec: Vector_Spaces.linear (*s) (*s) (*v) A
  using matrix_vector_mul_linear_gen.

interpretation vec: finite_dimensional_vector_space_pair (*s) cart_basis (*s) cart_basis
..

lemma matrix_works:
  assumes lf: Vector_Spaces.linear (*s) (*s) f
  shows matrix f *v x = f (x::'a::field ^ 'n)
  apply (simp add: matrix_def matrix_vector_mult_def vec_eq_iff mult.commute)
  apply clarify
  apply (rule linear_componentwise[OF lf, symmetric])
  done

lemma matrix_of_matrix_vector_mul[simp]: matrix(λx. A *v (x :: 'a::field ^ 'n))
= A
  by (simp add: matrix_eq matrix_works)

lemma matrix_compose_gen:
  assumes lf: Vector_Spaces.linear (*s) (*s) (f::'a::{field}^'n ⇒ 'a^'m)
    and lg: Vector_Spaces.linear (*s) (*s) (g::'a^'m ⇒ 'a^_)
  shows matrix (g o f) = matrix g ** matrix f
 using lf lg Vector_Spaces.linear_compose[OF lf lg] matrix_works[OF Vector_Spaces.linear_compose[OF
lf lg]]
 by (simp add: matrix_eq matrix_works matrix_vector_mul_assoc[symmetric] o_def)
```

lemma *matrix_compose*:
assumes *linear* (*f*::*real^'n* ⇒ *real^'m*) *linear* (*g*::*real^'m* ⇒ *real^_*)
shows *matrix* (*g o f*) = *matrix g* ** *matrix f*
using *matrix_compose_gen*[*of f g*] *assms*
by (*simp add*: *linear_def scalar_mult_eq_scaleR*)

lemma *left_invertible_transpose*:
(∃(*B*). *B* ** *transpose* (*A*) = *mat* (*1*::'*a*::*comm_semiring_1*)) ⟷ (∃(*B*). *A* ** *B* = *mat 1*)
by (*metis matrix_transpose_mul transpose_mat transpose_transpose*)

lemma *right_invertible_transpose*:
(∃(*B*). *transpose* (*A*) ** *B* = *mat* (*1*::'*a*::*comm_semiring_1*)) ⟷ (∃(*B*). *B* ** *A* = *mat 1*)
by (*metis matrix_transpose_mul transpose_mat transpose_transpose*)

lemma *linear_matrix_vector_mul_eq*:
Vector_Spaces.linear (*s) (*s) *f* ⟷ *linear* (*f* :: *real^'n* ⇒ *real ^'m*)
by (*simp add*: *scalar_mult_eq_scaleR linear_def*)

lemma *matrix_vector_mul*[*simp*]:
Vector_Spaces.linear (*s) (*s) *g* ⟹ (λ*y*. *matrix g* *v *y*) = *g*
linear f ⟹ (λ*x*. *matrix f* *v *x*) = *f*
bounded_linear f ⟹ (λ*x*. *matrix f* *v *x*) = *f*
for *f* :: *real^'n* ⇒ *real ^'m*
by (*simp_all add*: *ext matrix_works linear_matrix_vector_mul_eq linear_linear*)

lemma *matrix_left_invertible_injective*:
fixes *A* :: '*a*::*field^'n^'m*
shows (∃ *B*. *B* ** *A* = *mat 1*) ⟷ *inj* ((*v) *A*)
proof *safe*
fix *B*
assume *B*: *B* ** *A* = *mat 1*
show *inj* ((*v) *A*)
unfolding *inj_on_def*
by (*metis B matrix_vector_mul_assoc matrix_vector_mul_lid*)
next
assume *inj* ((*v) *A*)
from *vec.linear_injective_left_inverse*[*OF matrix_vector_mul_linear_gen this*]
obtain *g* **where** *Vector_Spaces.linear* (*s) (*s) *g* **and** *g*: *g* ∘ (*v) *A* = *id*
by *blast*
have *matrix g* ** *A* = *mat 1*
by (*metis matrix_vector_mul_linear_gen* ‹*Vector_Spaces.linear* (*s) (*s) *g*› *g* *matrix_compose_gen*
matrix_eq matrix_id_mat_1 matrix_vector_mul(*1*))
then show ∃ *B*. *B* ** *A* = *mat 1*
by *metis*
qed

```
lemma matrix_left_invertible_ker:
  (∃ B. (B::'a::{field} ^'m^'n) ** (A::'a::{field}^'n^'m) = mat 1) ⟷ (∀ x. A *v
x = 0 ⟶ x = 0)
  unfolding matrix_left_invertible_injective
  using vec.inj_on_iff_eq_0[OF vec.subspace_UNIV, of A]
  by (simp add: inj_on_def)

lemma matrix_right_invertible_surjective:
  (∃ B. (A::'a::field^'n^'m) ** (B::'a::field^'m^'n) = mat 1) ⟷ surj (λx. A *v x)
proof -
  { fix B :: 'a ^'m^'n
    assume AB: A ** B = mat 1
    { fix x :: 'a ^ 'm
      have A *v (B *v x) = x
        by (simp add: matrix_vector_mul_assoc AB) }
    hence surj ((*v) A) unfolding surj_def by metis }
  moreover
  { assume sf: surj ((*v) A)
    from vec.linear_surjective_right_inverse[OF _ this]
    obtain g:: 'a ^'m ⇒ 'a ^'n where g: Vector_Spaces.linear (*s) (*s) g (*v) A
∘ g = id
      by blast

    have A ** (matrix g) = mat 1
      unfolding matrix_eq matrix_vector_mul_lid
        matrix_vector_mul_assoc[symmetric] matrix_works[OF g(1)]
      using g(2) unfolding o_def fun_eq_iff id_def
      .
    hence ∃ B. A ** (B::'a^'m^'n) = mat 1 by blast
  }
  ultimately show ?thesis unfolding surj_def by blast
qed

lemma matrix_left_invertible_independent_columns:
  fixes A :: 'a::{field}^'n^'m
  shows (∃(B::'a ^'m^'n). B ** A = mat 1) ⟷
      (∀ c. sum (λi. c i *s column i A) (UNIV :: 'n set) = 0 ⟶ (∀ i. c i = 0))
    (is ?lhs ⟷ ?rhs)
proof -
  let ?U = UNIV :: 'n set
  { assume k: ∀ x. A *v x = 0 ⟶ x = 0
    { fix c i
      assume c: sum (λi. c i *s column i A) ?U = 0 and i: i ∈ ?U
      let ?x = χ i. c i
      have th0:A *v ?x = 0
        using c
        by (vector matrix_mult_sum)
      from k[rule_format, OF th0] i
      have c i = 0 by (vector vec_eq_iff)}
```

```
    hence ?rhs by blast }
  moreover
  { assume H: ?rhs
    { fix x assume x: A *v x = 0
      let ?c = λi. ((x$i ):: 'a)
      from H[rule_format, of ?c, unfolded matrix_mult_sum[symmetric], OF x]
      have x = 0 by vector }
  }
  ultimately show ?thesis unfolding matrix_left_invertible_ker by auto
qed

lemma matrix_right_invertible_independent_rows:
  fixes A :: 'a::{field}^'n^'m
  shows (∃(B::'a^'m^'n). A ** B = mat 1) ⟷
    (∀c. sum (λi. c i *s row i A) (UNIV :: 'm set) = 0 ⟶ (∀i. c i = 0))
  unfolding left_invertible_transpose[symmetric]
    matrix_left_invertible_independent_columns
  by (simp add:)

lemma matrix_right_invertible_span_columns:
  (∃(B::'a::field ^'n^'m). (A::'a ^'m^'n) ** B = mat 1) ⟷
    vec.span (columns A) = UNIV (is ?lhs = ?rhs)
proof −
  let ?U = UNIV :: 'm set
  have fU: finite ?U by simp
  have lhseq: ?lhs ⟷ (∀y. ∃(x::'a^'m). sum (λi. (x$i) *s column i A) ?U = y)
    unfolding matrix_right_invertible_surjective matrix_mult_sum surj_def
    by (simp add: eq_commute)
  have rhseq: ?rhs ⟷ (∀x. x ∈ vec.span (columns A)) by blast
  { assume h: ?lhs
    { fix x:: 'a ^'n
      from h[unfolded lhseq, rule_format, of x] obtain y :: 'a ^'m
        where y: sum (λi. (y$i) *s column i A) ?U = x by blast
      have x ∈ vec.span (columns A)
        unfolding y[symmetric] scalar_mult_eq_scaleR
      proof (rule vec.span_sum [OF vec.span_scale])
        show column i A ∈ vec.span (columns A) for i
          using columns_def vec.span_superset by auto
      qed
    }
    then have ?rhs unfolding rhseq by blast }
  moreover
  { assume h:?rhs
    let ?P = λ(y::'a ^'n). ∃(x::'a^'m). sum (λi. (x$i) *s column i A) ?U = y
    { fix y
      have y ∈ vec.span (columns A)
        unfolding h by blast
      then have ?P y
      proof (induction rule: vec.span_induct_alt)
```

```
        case base
        then show ?case
          by (metis (full_types) matrix_mult_sum matrix_vector_mult_0_right)
      next
        case (step c y1 y2)
        from step obtain i where i: i ∈ ?U y1 = column i A
          unfolding columns_def by blast
        obtain x:: 'a ^'m where x: sum (λi. (x$i) *s column i A) ?U = y2
          using step by blast
        let ?x = (χ j. if j = i then c + (x$i) else (x$j))::'a^'m
        show ?case
          proof (rule exI[where x= ?x], vector, auto simp add: i x[symmetric]
if_distrib distrib_left if_distribR cong del: if_weak_cong)
          fix j
          have th: ∀ xa ∈ ?U. (if xa = i then (c + (x$i)) * ((column xa A)$j)
             else (x$xa) * ((column xa A$j))) = (if xa = i then c * ((column i A)$j)
else 0) + ((x$xa) * ((column xa A)$j))
            using i(1) by (simp add: field_simps)
          have sum (λxa. if xa = i then (c + (x$i)) * ((column xa A)$j)
              else (x$xa) * ((column xa A$j))) ?U = sum (λxa. (if xa = i then c *
((column i A)$j) else 0) + ((x$xa) * ((column xa A)$j))) ?U
            by (rule sum.cong[OF refl]) (use th in blast)
          also have ... = sum (λxa. if xa = i then c * ((column i A)$j) else 0) ?U
+ sum (λxa. ((x$xa) * ((column xa A)$j))) ?U
            by (simp add: sum.distrib)
          also have ... = c * ((column i A)$j) + sum (λxa. ((x$xa) * ((column xa
A)$j))) ?U
            unfolding sum.delta[OF fU]
            using i(1) by simp
          finally show sum (λxa. if xa = i then (c + (x$i)) * ((column xa A)$j)
              else (x$xa) * ((column xa A$j))) ?U = c * ((column i A)$j) + sum
(λxa. ((x$xa) * ((column xa A)$j))) ?U .
        qed
      qed
    }
    then have ?lhs unfolding lhseq ..
  }
  ultimately show ?thesis by blast
qed

lemma matrix_left_invertible_span_rows_gen:
  (∃(B::'a^'m^'n). B ** (A::'a::field^'n^'m) = mat 1) ⟷ vec.span (rows A) =
UNIV
  unfolding right_invertible_transpose[symmetric]
  unfolding columns_transpose[symmetric]
  unfolding matrix_right_invertible_span_columns
  ..

lemma matrix_left_invertible_span_rows:
```

(∃(B::real^'m^'n). B ** (A::real^'n^'m) = mat 1) ⟷ span (rows A) = UNIV
using *matrix_left_invertible_span_rows_gen*[*of A*] **by** (*simp add*: *span_vec_eq*)

lemma *matrix_left_right_inverse*:
fixes A A' :: 'a::{field} ^'n^'n
shows A ** A' = mat 1 ⟷ A' ** A = mat 1
proof −
{ **fix** A A' :: 'a ^'n^'n
assume AA': A ** A' = mat 1
have sA: surj ((*v) A)
using AA' *matrix_right_invertible_surjective* **by** *auto*
from *vec.linear_surjective_isomorphism*[*OF matrix_vector_mul_linear_gen sA*]
obtain f' :: 'a ^'n ⇒ 'a ^'n
where f': Vector_Spaces.linear (*s) (*s) f' ∀x. f' (A *v x) = x ∀x. A *v f'
x = x **by** *blast*
have th: matrix f' ** A = mat 1
by (*simp add*: *matrix_eq matrix_works*[*OF f'(1)*]
matrix_vector_mul_assoc[*symmetric*] *f'(2)*[*rule_format*])
hence (matrix f' ** A) ** A' = mat 1 ** A' **by** *simp*
hence matrix f' = A'
by (*simp add*: *matrix_mul_assoc*[*symmetric*] *AA'*)
hence matrix f' ** A = A' ** A **by** *simp*
hence A' ** A = mat 1 **by** (*simp add*: *th*)
}
then show *?thesis* **by** *blast*
qed

lemma *invertible_left_inverse*:
fixes A :: 'a::{field} ^'n^'n
shows invertible A ⟷ (∃(B::'a^'n^'n). B ** A = mat 1)
by (*metis invertible_def matrix_left_right_inverse*)

lemma *invertible_right_inverse*:
fixes A :: 'a::{field} ^'n^'n
shows invertible A ⟷ (∃(B::'a^'n^'n). A** B = mat 1)
by (*metis invertible_def matrix_left_right_inverse*)

lemma *invertible_mult*:
assumes inv_A: invertible A
and inv_B: invertible B
shows invertible (A**B)
proof −
obtain A' **where** AA': A ** A' = mat 1 **and** A'A: A' ** A = mat 1
using *inv_A* **unfolding** *invertible_def* **by** *blast*
obtain B' **where** BB': B ** B' = mat 1 **and** B'B: B' ** B = mat 1
using *inv_B* **unfolding** *invertible_def* **by** *blast*
show *?thesis*
proof (*unfold invertible_def*, *rule exI*[*of _ B'**A'*], *rule conjI*)
have A ** B ** (B' ** A') = A ** (B ** (B' ** A'))

```
      using matrix_mul_assoc[of A B (B' ** A'), symmetric] .
    also have ... = A ** (B ** B' ** A') unfolding matrix_mul_assoc[of B B' A']
..
    also have ... = A ** (mat 1 ** A') unfolding BB' ..
    also have ... = A ** A' unfolding matrix_mul_lid ..
    also have ... = mat 1 unfolding AA' ..
    finally show A ** B ** (B' ** A') = mat (1::'a) .
  have B' ** A' ** (A ** B) = B' ** (A' ** (A ** B)) using matrix_mul_assoc[of
B' A' (A ** B), symmetric] .
    also have ... =  B' ** (A' ** A ** B) unfolding matrix_mul_assoc[of A' A B]
..
    also have ... =  B' ** (mat 1 ** B) unfolding A'A ..
    also have ... = B' ** B  unfolding matrix_mul_lid ..
    also have ... = mat 1 unfolding B'B ..
    finally show B' ** A' ** (A ** B) = mat 1 .
  qed
qed

lemma transpose_invertible:
  fixes A :: real^'n^'n
  assumes invertible A
  shows invertible (transpose A)
 by (meson assms invertible_def matrix_left_right_inverse right_invertible_transpose)

lemma vector_matrix_mul_assoc:
  fixes v :: ('a::comm_semiring_1)^'n
  shows (v v* M) v* N = v v* (M ** N)
proof -
  from matrix_vector_mul_assoc
  have transpose N *v (transpose M *v v) = (transpose N ** transpose M) *v v
by fast
  thus (v v* M) v* N = v v* (M ** N)
    by (simp add: matrix_transpose_mul [symmetric])
qed

lemma matrix_scaleR_vector_ac:
  fixes A :: real^('m::finite)^'n
  shows A *v (k *R v) = k *R A *v v
 by (metis matrix_vector_mult_scaleR transpose_scalar vector_scaleR_matrix_ac vec-
tor_transpose_matrix)

lemma scaleR_matrix_vector_assoc:
  fixes A :: real^('m::finite)^'n
  shows k *R (A *v v) = k *R A *v v
  by (metis matrix_scaleR_vector_ac matrix_vector_mult_scaleR)


locale linear_first_finite_dimensional_vector_space =
```

l?: *Vector_Spaces.linear scaleB scaleC f* +
B?: *finite_dimensional_vector_space scaleB BasisB*
for *scaleB* :: (*'a::field* => *'b::ab_group_add* => *'b*) (**infixr** *∗b 75*)
and *scaleC* :: (*'a* => *'c::ab_group_add* => *'c*) (**infixr** *∗c 75*)
and *BasisB* :: (*'b set*)
and *f* :: (*'b=>'c*)

lemma *vec_dim_card*: *vec.dim* (*UNIV*::(*'a*::{*field*} ˆ*'n*) *set*) = *CARD* (*'n*)
proof −
  let *?f*=λ*i*::*'n*. *axis i* (*1*::*'a*)
  have *vec.dim* (*UNIV*::(*'a*::{*field*} ˆ*'n*) *set*) = *card* (*cart_basis*::(*'a*ˆ*'n*) *set*)
    unfolding *vec.dim_UNIV* **..**
  also have ... = *card* ({*i*. *i*∈ *UNIV*}::(*'n*) *set*)
    proof (*rule bij_betw_same_card*[*of ?f*, *symmetric*], *unfold bij_betw_def*, *auto*)
      show *inj* (λ*i*::*'n*. *axis i* (*1*::*'a*)) **by** (*simp add*: *inj_on_def axis_eq_axis*)
      fix *i*::*'n*
      show *axis i 1* ∈ *cart_basis* **unfolding** *cart_basis_def* **by** *auto*
      fix *x*::*'a*ˆ*'n*
      assume *x* ∈ *cart_basis*
      thus *x* ∈ *range* (λ*i*. *axis i 1*) **unfolding** *cart_basis_def* **by** *auto*
    qed
  also have ... = *CARD*(*'n*) **by** *auto*
  finally show *?thesis* **.**
qed

interpretation *vector_space_over_itself*: *vector_space* (∗) :: *'a::field* ⇒ *'a* ⇒ *'a*
by *unfold_locales* (*simp_all add*: *algebra_simps*)

lemmas [*simp del*] = *vector_space_over_itself.scale_scale*

interpretation *vector_space_over_itself*: *finite_dimensional_vector_space*
  (∗) :: *'a::field* => *'a* => *'a* {*1*}
  by *unfold_locales* (*auto simp*: *vector_space_over_itself.span_singleton*)

lemma *dimension_eq_1*[*code_unfold*]: *vector_space_over_itself.dimension TYPE*(*'a::field*)=
1
  unfolding *vector_space_over_itself.dimension_def* **by** *simp*

lemma *dim_subset_UNIV_cart_gen*:
  fixes *S* :: (*'a::field*ˆ*'n*) *set*
  shows *vec.dim S* ≤ *CARD*(*'n*)
  by (*metis vec.dim_eq_full vec.dim_subset_UNIV vec.span_UNIV vec_dim_card*)

lemma *dim_subset_UNIV_cart*:
  fixes *S* :: (*real*ˆ*'n*) *set*
  shows *dim S* ≤ *CARD*(*'n*)
  using *dim_subset_UNIV_cart_gen*[*of S*] **by** (*simp add*: *dim_vec_eq*)

Two sometimes fruitful ways of looking at matrix-vector multiplication.

lemma *matrix_mult_dot*: $A *v\ x = (\chi\ i.\ inner\ (A\$i)\ x)$
  by (*simp add*: *matrix_vector_mult_def inner_vec_def*)

lemma *adjoint_matrix*: *adjoint*($\lambda x.$ (A::*real*^'*n*^'*m*) $*v\ x$) = ($\lambda x.$ *transpose* $A *v$ x)
  apply (*rule adjoint_unique*)
  apply (*simp add*: *transpose_def inner_vec_def matrix_vector_mult_def*
    sum_distrib_right sum_distrib_left)
  apply (*subst sum.swap*)
  apply (*simp add*: *ac_simps*)
  done

lemma *matrix_adjoint*: **assumes** *lf*: *linear* (f :: *real*^'*n* ⇒ *real* ^'*m*)
  shows *matrix*(*adjoint f*) = *transpose*(*matrix f*)
proof −
  have *matrix*(*adjoint f*) = *matrix*(*adjoint* ((∗*v*) (*matrix f*)))
    by (*simp add*: *lf*)
  also have ... = *transpose*(*matrix f*)
    unfolding *adjoint_matrix matrix_of_matrix_vector_mul*
    apply *rule*
    done
  finally show *?thesis* **.**
qed

1.9.2 Rank of a matrix

Equivalence of row and column rank is taken from George Mackiw's paper, Mathematics Magazine 1995, p. 285.

lemma *matrix_vector_mult_in_columnspace_gen*:
  fixes A :: '*a*::*field*^'*n*^'*m*
  shows ($A *v\ x$) ∈ *vec.span*(*columns* A)
  apply (*simp add*: *matrix_vector_column columns_def transpose_def column_def*)
  apply (*intro vec.span_sum vec.span_scale*)
  apply (*force intro*: *vec.span_base*)
  done

lemma *matrix_vector_mult_in_columnspace*:
  fixes A :: *real*^'*n*^'*m*
  shows ($A *v\ x$) ∈ *span*(*columns* A)
  using *matrix_vector_mult_in_columnspace_gen*[*of A x*] **by** (*simp add*: *span_vec_eq*)

lemma *subspace_orthogonal_to_vector*: *subspace* {*y*. *orthogonal x y*}
  by (*simp add*: *subspace_def orthogonal_clauses*)

lemma *orthogonal_nullspace_rowspace*:
  fixes A :: *real*^'*n*^'*m*
  assumes *0*: $A *v\ x = 0$ **and** *y*: y ∈ *span*(*rows* A)
  shows *orthogonal x y*
  using *y*

proof (*induction rule*: *span_induct*)
  case *base*
  then show *?case*
    by (*simp add*: *subspace_orthogonal_to_vector*)
next
  case (*step v*)
  then obtain *i* **where** *v* = *row i A*
    by (*auto simp*: *rows_def*)
  with *0* **show** *?case*
    unfolding *orthogonal_def inner_vec_def matrix_vector_mult_def row_def*
    by (*simp add*: *mult.commute*) (*metis* (*no_types*) *vec_lambda_beta zero_index*)
qed

lemma *nullspace_inter_rowspace*:
  fixes *A* :: *real^'n^'m*
  shows *A* ∗*v x* = *0* ∧ *x* ∈ *span*(*rows A*) ⟷ *x* = *0*
  using *orthogonal_nullspace_rowspace orthogonal_self span_zero matrix_vector_mult_0_right*
  by *blast*

lemma *matrix_vector_mul_injective_on_rowspace*:
  fixes *A* :: *real^'n^'m*
  shows ⟦*A* ∗*v x* = *A* ∗*v y*; *x* ∈ *span*(*rows A*); *y* ∈ *span*(*rows A*)⟧ ⟹ *x* = *y*
  using *nullspace_inter_rowspace* [*of A x*−*y*]
  by (*metis diff_eq_diff_eq diff_self matrix_vector_mult_diff_distrib span_diff*)

definition *rank* :: *'a::field^'n^'m=>nat*
  where *row_rank_def_gen*: *rank A* ≡ *vec.dim*(*rows A*)

lemma *row_rank_def*: *rank A* = *dim* (*rows A*) **for** *A::real^'n^'m*
  by (*auto simp*: *row_rank_def_gen dim_vec_eq*)

lemma *dim_rows_le_dim_columns*:
  fixes *A* :: *real^'n^'m*
  shows *dim*(*rows A*) ≤ *dim*(*columns A*)
proof −
  have *dim* (*span* (*rows A*)) ≤ *dim* (*span* (*columns A*))
  proof −
    obtain *B* **where** *independent B span*(*rows A*) ⊆ *span B*
          and *B*: *B* ⊆ *span*(*rows A*)*card B* = *dim* (*span*(*rows A*))
      using *basis_exists* [*of span*(*rows A*)] **by** *metis*
    with *span_subspace* **have** *eq*: *span B* = *span*(*rows A*)
      by *auto*
    then have *inj*: *inj_on* ((∗*v*) *A*) (*span B*)
      by (*simp add*: *inj_on_def matrix_vector_mul_injective_on_rowspace*)
    then have *ind*: *independent* ((∗*v*) *A* ‘ *B*)
    by (*rule linear_independent_injective_image* [*OF Finite_Cartesian_Product.matrix_vector_mul_linear* ‹*independent B*›])
    have *dim* (*span* (*rows A*)) ≤ *card* ((∗*v*) *A* ‘ *B*)
      unfolding *B*(*2*)[*symmetric*]

```
      using inj
      by (auto simp: card_image inj_on_subset span_superset)
    also have ... ≤ dim (span (columns A))
      using _ ind
     by (rule independent_card_le_dim) (auto intro!: matrix_vector_mult_in_columnspace)
    finally show ?thesis .
  qed
  then show ?thesis
    by (simp)
qed

lemma column_rank_def:
  fixes A :: real^'n^'m
  shows rank A = dim(columns A)
  unfolding row_rank_def
 by (metis columns_transpose dim_rows_le_dim_columns le_antisym rows_transpose)

lemma rank_transpose:
  fixes A :: real^'n^'m
  shows rank(transpose A) = rank A
  by (metis column_rank_def row_rank_def rows_transpose)

lemma matrix_vector_mult_basis:
  fixes A :: real^'n^'m
  shows A *v (axis k 1) = column k A
  by (simp add: cart_eq_inner_axis column_def matrix_mult_dot)

lemma columns_image_basis:
  fixes A :: real^'n^'m
  shows columns A = (*v) A ` (range (λi. axis i 1))
  by (force simp: columns_def matrix_vector_mult_basis [symmetric])

lemma rank_dim_range:
  fixes A :: real^'n^'m
  shows rank A = dim(range (λx. A *v x))
  unfolding column_rank_def
proof (rule span_eq_dim)
  have span (columns A) ⊆ span (range ((*v) A)) (is ?l ⊆ ?r)
    by (simp add: columns_image_basis image_subsetI span_mono)
  then show ?l = ?r
   by (metis (no_types, lifting) image_subset_iff matrix_vector_mult_in_columnspace
        span_eq span_span)
qed

lemma rank_bound:
  fixes A :: real^'n^'m
  shows rank A ≤ min CARD('m) (CARD('n))
  by (metis (mono_tags, lifting) dim_subset_UNIV_cart min.bounded_iff
      column_rank_def row_rank_def)
```

lemma *full_rank_injective*:
  fixes *A* :: *real^'n^'m*
  shows *rank A = CARD('n) ⟷ inj ((*v) A)*
 by (*simp add*: *matrix_left_invertible_injective* [*symmetric*] *matrix_left_invertible_span_rows row_rank_def*
    dim_eq_full [*symmetric*] *card_cart_basis vec.dimension_def*)

lemma *full_rank_surjective*:
  fixes *A* :: *real^'n^'m*
  shows *rank A = CARD('m) ⟷ surj ((*v) A)*
 by (*simp add*: *matrix_right_invertible_surjective* [*symmetric*] *left_invertible_transpose* [*symmetric*]
    matrix_left_invertible_injective full_rank_injective [*symmetric*] *rank_transpose*)

lemma *rank_I*: *rank(mat 1::real^'n^'n) = CARD('n)*
  by (*simp add*: *full_rank_injective inj_on_def*)

lemma *less_rank_noninjective*:
  fixes *A* :: *real^'n^'m*
  shows *rank A < CARD('n) ⟷ ¬ inj ((*v) A)*
using *less_le rank_bound* **by** (*auto simp*: *full_rank_injective* [*symmetric*])

lemma *matrix_nonfull_linear_equations_eq*:
  fixes *A* :: *real^'n^'m*
  shows (∃ *x*. (*x ≠ 0*) ∧ *A *v x = 0*) ⟷ *rank A ≠ CARD('n)*
 by (*meson matrix_left_invertible_injective full_rank_injective matrix_left_invertible_ker*)

lemma *rank_eq_0*: *rank A = 0 ⟷ A = 0* **and** *rank_0* [*simp*]: *rank (0::real^'n^'m) = 0*
  for *A* :: *real^'n^'m*
  by (*auto simp*: *rank_dim_range matrix_eq*)

lemma *rank_mul_le_right*:
  fixes *A* :: *real^'n^'m* **and** *B* :: *real^'p^'n*
  shows *rank(A ** B) ≤ rank B*
proof −
  have *rank(A ** B) ≤ dim ((*v) A ‘ range ((*v) B))*
    by (*auto simp*: *rank_dim_range image_comp o_def matrix_vector_mul_assoc*)
  also have ... ≤ *rank B*
    by (*simp add*: *rank_dim_range dim_image_le*)
  finally show *?thesis* **.**
qed

lemma *rank_mul_le_left*:
  fixes *A* :: *real^'n^'m* **and** *B* :: *real^'p^'n*
  shows *rank(A ** B) ≤ rank A*
  by (*metis matrix_transpose_mul rank_mul_le_right rank_transpose*)

1.9.3 Lemmas for working on *real^1/2/3/4*

lemma *exhaust_2*:
  fixes x :: *2*
  shows $x = 1 \lor x = 2$
proof (*induct* x)
  case (*of_int* z)
  then have $0 \leq z$ **and** $z < 2$ **by** *simp_all*
  then have $z = 0 \mid z = 1$ **by** *arith*
  then show *?case* **by** *auto*
qed

lemma *forall_2*: $(\forall i{::}2.\ P\ i) \longleftrightarrow P\ 1 \land P\ 2$
  by (*metis exhaust_2*)

lemma *exhaust_3*:
  fixes x :: *3*
  shows $x = 1 \lor x = 2 \lor x = 3$
proof (*induct* x)
  case (*of_int* z)
  then have $0 \leq z$ **and** $z < 3$ **by** *simp_all*
  then have $z = 0 \lor z = 1 \lor z = 2$ **by** *arith*
  then show *?case* **by** *auto*
qed

lemma *forall_3*: $(\forall i{::}3.\ P\ i) \longleftrightarrow P\ 1 \land P\ 2 \land P\ 3$
  by (*metis exhaust_3*)

lemma *exhaust_4*:
  fixes x :: *4*
  shows $x = 1 \lor x = 2 \lor x = 3 \lor x = 4$
proof (*induct* x)
  case (*of_int* z)
  then have $0 \leq z$ **and** $z < 4$ **by** *simp_all*
  then have $z = 0 \lor z = 1 \lor z = 2 \lor z = 3$ **by** *arith*
  then show *?case* **by** *auto*
qed

lemma *forall_4*: $(\forall i{::}4.\ P\ i) \longleftrightarrow P\ 1 \land P\ 2 \land P\ 3 \land P\ 4$
  by (*metis exhaust_4*)

lemma *UNIV_1* [*simp*]: *UNIV* = $\{1{::}1\}$
  by (*auto simp add*: *num1_eq_iff*)

lemma *UNIV_2*: *UNIV* = $\{1{::}2,\ 2{::}2\}$
  using *exhaust_2* **by** *auto*

lemma *UNIV_3*: *UNIV* = $\{1{::}3,\ 2{::}3,\ 3{::}3\}$
  using *exhaust_3* **by** *auto*

lemma *UNIV_4*: *UNIV* = {*1*::*4*, *2*::*4*, *3*::*4*, *4*::*4*}
using *exhaust_4* **by** *auto*

lemma *sum_1*: *sum f* (*UNIV*::*1 set*) = *f 1*
unfolding *UNIV_1* **by** *simp*

lemma *sum_2*: *sum f* (*UNIV*::*2 set*) = *f 1* + *f 2*
unfolding *UNIV_2* **by** *simp*

lemma *sum_3*: *sum f* (*UNIV*::*3 set*) = *f 1* + *f 2* + *f 3*
unfolding *UNIV_3* **by** (*simp add*: *ac_simps*)

lemma *sum_4*: *sum f* (*UNIV*::*4 set*) = *f 1* + *f 2* + *f 3* + *f 4*
unfolding *UNIV_4* **by** (*simp add*: *ac_simps*)

1.9.4 The collapse of the general concepts to dimension one

lemma *vector_one*: (*x*::*'a* ^*1*) = (χ *i*. (*x*$*1*))
by (*simp add*: *vec_eq_iff*)

lemma *forall_one*: (∀ (*x*::*'a* ^*1*). *P x*) ⟷ (∀ *x*. *P*(χ *i*. *x*))
apply *auto*
apply (*erule_tac x*= *x*$*1* **in** *allE*)
apply (*simp only*: *vector_one*[*symmetric*])
done

lemma *norm_vector_1*: *norm* (*x* :: _^*1*) = *norm* (*x*$*1*)
by (*simp add*: *norm_vec_def*)

lemma *dist_vector_1*:
fixes *x* :: *'a*::*real_normed_vector*^*1*
shows *dist x y* = *dist* (*x*$*1*) (*y*$*1*)
by (*simp add*: *dist_norm norm_vector_1*)

lemma *norm_real*: *norm*(*x*::*real* ^ *1*) = |*x*$*1*|
by (*simp add*: *norm_vector_1*)

lemma *dist_real*: *dist*(*x*::*real* ^ *1*) *y* = |(*x*$*1*) − (*y*$*1*)|
by (*auto simp add*: *norm_real dist_norm*)

1.9.5 Routine results connecting the types (*real*, *1*) *vec* and *real*

lemma *vector_one_nth* [*simp*]:
fixes *x* :: *'a*^*1* **shows** *vec* (*x* $ *1*) = *x*
by (*metis vec_def vector_one*)

lemma *tendsto_at_within_vector_1*:
fixes *S* :: *'a* :: *metric_space set*

assumes $(f \longrightarrow fx)$ (*at x within S*)
shows $((\lambda y::'a\hat{}1.\ \chi\ i.\ f\ (y\ \$\ 1)) \longrightarrow (vec\ fx::'a\hat{}1))$ (*at* (*vec x*) *within vec* ‘ *S*)
proof (*rule topological_tendstoI*)
fix $T :: ('a\hat{}1)$ *set*
assume *open T vec fx* $\in T$
have $\forall_F\ x$ *in at x within S*. $f\ x \in (\lambda x.\ x\ \$\ 1)$ ‘ T
using ‹*open T*› ‹*vec fx* $\in$ *T*› *assms open_image_vec_nth tendsto_def* **by** *fastforce*
then show $\forall_F\ x::'a\hat{}1$ *in at* (*vec x*) *within vec* ‘ *S*. $(\chi\ i.\ f\ (x\ \$\ 1)) \in T$
unfolding *eventually_at dist_norm* [*symmetric*]
by (*rule ex_forward*)
(*use* ‹*open T*› **in**
‹*fastforce simp*: *dist_norm dist_vec_def L2_set_def image_iff vector_one open_vec_def*›)
qed

lemma *has_derivative_vector_1*:
assumes *der_g*: (*g has_derivative* $(\lambda x.\ x * g'\ a)$) (*at a within S*)
shows $((\lambda x.\ vec\ (g\ (x\ \$\ 1)))$ *has_derivative* $(*_R)\ (g'\ a))$
(*at* ((*vec a*)::*real*ˆ*1*) *within vec* ‘ *S*)
using *der_g*
apply (*auto simp*: *Deriv.has_derivative_within bounded_linear_scaleR_right norm_vector_1*)
apply (*drule tendsto_at_within_vector_1*, *vector*)
apply (*auto simp*: *algebra_simps eventually_at tendsto_def*)
done

1.9.6 Explicit vector construction from lists

definition *vector* $l = (\chi\ i.\ foldr\ (\lambda x\ f\ n.\ fun\_upd\ (f\ (n+1))\ n\ x)\ l\ (\lambda n\ x.\ 0)\ 1\ i)$

lemma *vector_1* [*simp*]: $(vector[x])\ \$1 = x$
unfolding *vector_def* **by** *simp*

lemma *vector_2* [*simp*]: $(vector[x,y])\ \$1 = x\ (vector[x,y] :: 'a\hat{}2)\$2 = (y::'a::zero)$
unfolding *vector_def* **by** *simp_all*

lemma *vector_3* [*simp*]:
$(vector\ [x,y,z] ::('a::zero)\hat{}3)\$1 = x$
$(vector\ [x,y,z] ::('a::zero)\hat{}3)\$2 = y$
$(vector\ [x,y,z] ::('a::zero)\hat{}3)\$3 = z$
unfolding *vector_def* **by** *simp_all*

lemma *forall_vector_1*: $(\forall v::'a::zero\hat{}1.\ P\ v) \longleftrightarrow (\forall x.\ P(vector[x]))$
by (*metis vector_1 vector_one*)

lemma *forall_vector_2*: $(\forall v::'a::zero\hat{}2.\ P\ v) \longleftrightarrow (\forall x\ y.\ P(vector[x,\ y]))$
apply *auto*
apply (*erule_tac x=v$1* **in** *allE*)
apply (*erule_tac x=v$2* **in** *allE*)

apply (*subgoal_tac vector* [*v*\$*1*, *v*\$*2*] = *v*)
apply *simp*
apply (*vector vector_def*)
apply (*simp add*: *forall_2*)
done

lemma *forall_vector_3*: (∀ *v*::′*a*::*zero*ˆ*3*. *P v*) ⟷ (∀ *x y z*. *P*(*vector*[*x*, *y*, *z*]))
apply *auto*
apply (*erule_tac x*=*v*\$*1* **in** *allE*)
apply (*erule_tac x*=*v*\$*2* **in** *allE*)
apply (*erule_tac x*=*v*\$*3* **in** *allE*)
apply (*subgoal_tac vector* [*v*\$*1*, *v*\$*2*, *v*\$*3*] = *v*)
apply *simp*
apply (*vector vector_def*)
apply (*simp add*: *forall_3*)
done

1.9.7 lambda skolemization on cartesian products

lemma *lambda_skolem*: (∀ *i*. ∃ *x*. *P i x*) ⟷
(∃ *x*::′*a* ˆ ′*n*. ∀ *i*. *P i* (*x* \$ *i*)) (**is** *?lhs* ⟷ *?rhs*)
proof −
let *?S* = (*UNIV* :: ′*n set*)
{ assume *H*: *?rhs*
then have *?lhs* **by** *auto* **}**
moreover
{ assume *H*: *?lhs*
then obtain *f* **where** *f*:∀ *i*. *P i* (*f i*) **unfolding** *choice_iff* **by** *metis*
let *?x* = (χ *i*. (*f i*)) :: ′*a* ˆ ′*n*
{ fix *i*
from *f* **have** *P i* (*f i*) **by** *metis*
then have *P i* (*?x* \$ *i*) **by** *auto*
}
hence ∀ *i*. *P i* (*?x*\$*i*) **by** *metis*
hence *?rhs* **by** *metis* **}**
ultimately show *?thesis* **by** *metis*
qed

The same result in terms of square matrices.

Considering an n-element vector as an n-by-1 or 1-by-n matrix.

definition *rowvector v* = (χ *i j*. (*v*\$*j*))

definition *columnvector v* = (χ *i j*. (*v*\$*i*))

lemma *transpose_columnvector*: *transpose*(*columnvector v*) = *rowvector v*
by (*simp add*: *transpose_def rowvector_def columnvector_def vec_eq_iff*)

lemma *transpose_rowvector*: *transpose*(*rowvector v*) = *columnvector v*
by (*simp add*: *transpose_def columnvector_def rowvector_def vec_eq_iff*)

lemma *dot_rowvector_columnvector*: *columnvector* (*A* ∗*v v*) = *A* ∗∗ *columnvector* *v*
  by (*vector columnvector_def matrix_matrix_mult_def matrix_vector_mult_def*)

lemma *dot_matrix_product*:
  (*x*::*real*^′*n*) · *y* = (((*rowvector x* ::*real*^′*n*^*1*) ∗∗ (*columnvector y* :: *real*^*1*^′*n*))$*1*)$*1*
  by (*vector matrix_matrix_mult_def rowvector_def columnvector_def inner_vec_def*)

lemma *dot_matrix_vector_mul*:
  fixes *A B* :: *real* ^′*n* ^′*n* **and** *x y* :: *real* ^′*n*
  shows (*A* ∗*v x*) · (*B* ∗*v y*) =
      (((*rowvector x* :: *real*^′*n*^*1*) ∗∗ ((*transpose A* ∗∗ *B*) ∗∗ (*columnvector y* :: *real* ^*1*^′*n*)))$*1*)$*1*
  unfolding *dot_matrix_product transpose_columnvector*[*symmetric*]
    dot_rowvector_columnvector matrix_transpose_mul matrix_mul_assoc **..**

lemma *dim_substandard_cart*: *vec.dim* {*x*::′*a*::*field*^′*n*. ∀ *i*. *i* ∉ *d* ⟶ *x*$*i* = *0*} = *card d*
  (**is** *vec.dim ?A* = _)
proof (*rule vec.dim_unique*)
  let *?B* = ((λ*x*. *axis x 1*) ‘ *d*)
  have *subset_basis*: *?B* ⊆ *cart_basis*
    by (*auto simp*: *cart_basis_def*)
  show *?B* ⊆ *?A*
    by (*auto simp*: *axis_def*)
  show *vec.independent* ((λ*x*. *axis x 1*) ‘ *d*)
    using *subset_basis*
    by (*rule vec.independent_mono*[*OF vec.independent_Basis*])
  have *x* ∈ *vec.span ?B* **if** ∀ *i*. *i* ∉ *d* ⟶ *x* $ *i* = *0* **for** *x*::′*a*^′*n*
  proof −
    have *finite ?B*
      using *subset_basis finite_cart_basis*
      by (*rule finite_subset*)
    have *x* = (∑ *i*∈*UNIV*. *x* $ *i* ∗*s axis i 1*)
      by (*rule basis_expansion*[*symmetric*])
    also have . . . = (∑ *i*∈*d*. (*x* $ *i*) ∗*s axis i 1*)
      by (*rule sum.mono_neutral_cong_right*) (*auto simp*: *that*)
    also have . . . ∈ *vec.span ?B*
      by (*simp add*: *vec.span_sum vec.span_clauses*)
    finally show *x* ∈ *vec.span ?B* **.**
  qed
  then show *?A* ⊆ *vec.span ?B* **by** *auto*
qed (*simp add*: *card_image inj_on_def axis_eq_axis*)

lemma *affinity_inverses*:
  assumes *m0*: *m* ≠ (*0*::′*a*::*field*)
  shows (λ*x*. *m* ∗*s x* + *c*) ∘ (λ*x*. *inverse*(*m*) ∗*s x* + (−(*inverse*(*m*) ∗*s c*))) = *id*

(λx. inverse(m) ∗s x + (−(inverse(m) ∗s c))) ∘ (λx. m ∗s x + c) = id
  using m0
  by (auto simp add: fun_eq_iff vector_add_ldistrib diff_conv_add_uminus simp del: add_uminus_conv_diff)

lemma vector_affinity_eq:
  assumes m0: (m::'a::field) ≠ 0
  shows m ∗s x + c = y ⟷ x = inverse m ∗s y + −(inverse m ∗s c)
proof
  assume h: m ∗s x + c = y
  hence m ∗s x = y − c **by** (simp add: field_simps)
  hence inverse m ∗s (m ∗s x) = inverse m ∗s (y − c) **by** simp
  then show x = inverse m ∗s y + − (inverse m ∗s c)
    using m0 **by** (simp add: vector_smult_assoc vector_ssub_ldistrib)
next
  assume h: x = inverse m ∗s y + − (inverse m ∗s c)
  show m ∗s x + c = y **unfolding** h
    using m0 **by** (simp add: vector_smult_assoc vector_ssub_ldistrib)
qed

lemma vector_eq_affinity:
  (m::'a::field) ≠ 0 ==> (y = m ∗s x + c ⟷ inverse(m) ∗s y + −(inverse(m) ∗s c) = x)
  using vector_affinity_eq[**where** m=m **and** x=x **and** y=y **and** c=c]
  by metis

lemma vector_cart:
  fixes f :: real^'n ⇒ real
  shows (χ i. f (axis i 1)) = (∑i∈Basis. f i ∗R i)
  unfolding euclidean_eq_iff[**where** 'a=real^'n]
  by simp (simp add: Basis_vec_def inner_axis)

lemma const_vector_cart:((χ i. d)::real^'n) = (∑i∈Basis. d ∗R i)
  by (rule vector_cart)

1.9.8 Explicit formulas for low dimensions

lemma prod_neutral_const: prod f {(1::nat)..1} = f 1
  by simp

lemma prod_2: prod f {(1::nat)..2} = f 1 ∗ f 2
  by (simp add: eval_nat_numeral atLeastAtMostSuc_conv mult.commute)

lemma prod_3: prod f {(1::nat)..3} = f 1 ∗ f 2 ∗ f 3
  by (simp add: eval_nat_numeral atLeastAtMostSuc_conv mult.commute)

1.9.9 Orthogonality of a matrix

definition orthogonal_matrix (Q::'a::semiring_1^'n^'n) ⟷
  transpose Q ∗∗ Q = mat 1 ∧ Q ∗∗ transpose Q = mat 1

lemma *orthogonal_matrix*: *orthogonal_matrix* (*Q*:: *real* ^'*n*^'*n*) ⟷ *transpose Q* ** *Q* = *mat 1*
by (*metis matrix_left_right_inverse orthogonal_matrix_def*)

lemma *orthogonal_matrix_id*: *orthogonal_matrix* (*mat 1* :: _^'*n*^'*n*)
by (*simp add*: *orthogonal_matrix_def*)

proposition *orthogonal_matrix_mul*:
fixes *A* :: *real* ^'*n*^'*n*
assumes *orthogonal_matrix A orthogonal_matrix B*
shows *orthogonal_matrix*(*A* ** *B*)
using *assms*
by (*simp add*: *orthogonal_matrix matrix_transpose_mul matrix_left_right_inverse matrix_mul_assoc*)

proposition *orthogonal_transformation_matrix*:
fixes *f*:: *real*^'*n* ⇒ *real*^'*n*
shows *orthogonal_transformation f* ⟷ *linear f* ∧ *orthogonal_matrix*(*matrix f*)
(**is** *?lhs* ⟷ *?rhs*)
proof −
let *?mf* = *matrix f*
let *?ot* = *orthogonal_transformation f*
let *?U* = *UNIV* :: '*n set*
have *fU*: *finite ?U* **by** *simp*
let *?m1* = *mat 1* :: *real* ^'*n*^'*n*
{
assume *ot*: *?ot*
from *ot* **have** *lf*: *Vector_Spaces.linear* (*s) (*s) *f* **and** *fd*: ⋀*v w*. *f v* • *f w* = *v* • *w*
unfolding *orthogonal_transformation_def orthogonal_matrix linear_def scalar_mult_eq_scaleR*
by *blast+*
{
fix *i j*
let *?A* = *transpose ?mf* ** *?mf*
have *th0*: ⋀*b* (*x*::'*a*::*comm_ring_1*). (*if b then 1 else 0*)**x* = (*if b then x else 0*)
⋀*b* (*x*::'*a*::*comm_ring_1*). *x**(*if b then 1 else 0*) = (*if b then x else 0*)
by *simp_all*
from *fd*[*of axis i 1 axis j 1*,
simplified matrix_works[*OF lf*, *symmetric*] *dot_matrix_vector_mul*]
have *?A*\$*i*\$*j* = *?m1* \$ *i* \$ *j*
by (*simp add*: *inner_vec_def matrix_matrix_mult_def columnvector_def rowvector_def*
th0 sum.delta[*OF fU*] *mat_def axis_def*)
}
then have *orthogonal_matrix ?mf*
unfolding *orthogonal_matrix*
by *vector*

with *lf* **have** *?rhs*
unfolding *linear_def scalar_mult_eq_scaleR*
by *blast*
}
moreover
{
assume *lf*: *Vector_Spaces.linear* (∗*s*) (∗*s*) *f* **and** *om*: *orthogonal_matrix ?mf*
from *lf om* **have** *?lhs*
unfolding *orthogonal_matrix_def norm_eq orthogonal_transformation*
apply (*simp only*: *matrix_works*[*OF lf*, *symmetric*] *dot_matrix_vector_mul*)
apply (*simp add*: *dot_matrix_product linear_def scalar_mult_eq_scaleR*)
done
}
ultimately show *?thesis*
by (*auto simp*: *linear_def scalar_mult_eq_scaleR*)
qed

1.9.10 Finding an Orthogonal Matrix

We can find an orthogonal matrix taking any unit vector to any other.

lemma *orthogonal_matrix_transpose* [*simp*]:
orthogonal_matrix(*transpose A*) ⟷ *orthogonal_matrix A*
by (*auto simp*: *orthogonal_matrix_def*)

lemma *orthogonal_matrix_orthonormal_columns*:
fixes *A* :: *real*^'*n*^'*n*
shows *orthogonal_matrix A* ⟷
(∀ *i*. *norm*(*column i A*) = *1*) ∧
(∀ *i j*. *i* ≠ *j* ⟶ *orthogonal* (*column i A*) (*column j A*))
by (*auto simp*: *orthogonal_matrix matrix_mult_transpose_dot_column vec_eq_iff mat_def norm_eq_1 orthogonal_def*)

lemma *orthogonal_matrix_orthonormal_rows*:
fixes *A* :: *real*^'*n*^'*n*
shows *orthogonal_matrix A* ⟷
(∀ *i*. *norm*(*row i A*) = *1*) ∧
(∀ *i j*. *i* ≠ *j* ⟶ *orthogonal* (*row i A*) (*row j A*))
using *orthogonal_matrix_orthonormal_columns* [*of transpose A*] **by** *simp*

proposition *orthogonal_matrix_exists_basis*:
fixes *a* :: *real*^'*n*
assumes *norm a* = *1*
obtains *A* **where** *orthogonal_matrix A A* ∗*v* (*axis k 1*) = *a*
proof −
obtain *S* **where** *a* ∈ *S pairwise orthogonal S* **and** *noS*: ⋀*x*. *x* ∈ *S* ⟹ *norm x* = *1*
and *independent S card S* = *CARD*('*n*) *span S* = *UNIV*
using *vector_in_orthonormal_basis assms* **by** *force*
then obtain *f0* **where** *bij_betw f0* (*UNIV*::'*n set*) *S*

```
    by (metis finite_class.finite_UNIV finite_same_card_bij finiteI_independent)
  then obtain f where f: bij_betw f (UNIV::'n set) S and a: a = f k
    using bij_swap_iff [of k inv f0 a f0]
   by (metis UNIV_I ‹a ∈ S› bij_betw_inv_into_right bij_betw_swap_iff swap_apply(1))
  show thesis
  proof
    have [simp]: ⋀i. norm (f i) = 1
      using bij_betwE [OF ‹bij_betw f UNIV S›] by (blast intro: noS)
    have [simp]: ⋀i j. i ≠ j ⟹ orthogonal (f i) (f j)
      using ‹pairwise orthogonal S› ‹bij_betw f UNIV S›
      by (auto simp: pairwise_def bij_betw_def inj_on_def)
    show orthogonal_matrix (χ i j. f j $ i)
      by (simp add: orthogonal_matrix_orthonormal_columns column_def)
    show (χ i j. f j $ i) *v axis k 1 = a
      by (simp add: matrix_vector_mult_def axis_def a if_distrib cong: if_cong)
  qed
qed

lemma orthogonal_transformation_exists_1:
  fixes a b :: real^'n
  assumes norm a = 1 norm b = 1
  obtains f where orthogonal_transformation f f a = b
proof –
  obtain k::'n where True
    by simp
  obtain A B where AB: orthogonal_matrix A orthogonal_matrix B and eq: A *v
(axis k 1) = a B *v (axis k 1) = b
    using orthogonal_matrix_exists_basis assms by metis
  let ?f = λx. (B ** transpose A) *v x
  show thesis
  proof
    show orthogonal_transformation ?f
      by (subst orthogonal_transformation_matrix)
        (auto simp: AB orthogonal_matrix_mul)
  next
    show ?f a = b
      using ‹orthogonal_matrix A› unfolding orthogonal_matrix_def
      by (metis eq matrix_mul_rid matrix_vector_mul_assoc)
  qed
qed

proposition orthogonal_transformation_exists:
  fixes a b :: real^'n
  assumes norm a = norm b
  obtains f where orthogonal_transformation f f a = b
proof (cases a = 0 ∨ b = 0)
  case True
  with assms show ?thesis
    using that by force
```

```
next
  case False
  then obtain f where f: orthogonal_transformation f and eq: f (a /R norm a)
= (b /R norm b)
    by (auto intro: orthogonal_transformation_exists_1 [of a /R norm a b /R norm
b])
  show ?thesis
  proof
    interpret linear f
      using f by (simp add: orthogonal_transformation_linear)
    have f a /R norm a = f (a /R norm a)
      by (simp add: scale)
    also have ... = b /R norm a
      by (simp add: eq assms [symmetric])
    finally show f a = b
      using False by auto
  qed (use f in auto)
qed
```

1.9.11 Scaling and isometry

```
proposition scaling_linear:
  fixes f :: 'a::real_inner ⇒ 'a::real_inner
  assumes f0: f 0 = 0
    and fd: ∀ x y. dist (f x) (f y) = c * dist x y
  shows linear f
proof –
  {
    fix v w
    have norm (f x) = c * norm x for x
      by (metis dist_0_norm f0 fd)
    then have f v · f w = c² * (v · w)
      unfolding dot_norm_neg dist_norm[symmetric]
      by (simp add: fd power2_eq_square field_simps)
  }
  then show ?thesis
    unfolding linear_iff vector_eq[where 'a='a] scalar_mult_eq_scaleR
    by (simp add: inner_add field_simps)
qed

lemma isometry_linear:
  f (0::'a::real_inner) = (0::'a) =⇒ ∀ x y. dist(f x) (f y) = dist x y =⇒ linear f
  by (rule scaling_linear[where c=1]) simp_all
```

Hence another formulation of orthogonal transformation

```
proposition orthogonal_transformation_isometry:
  orthogonal_transformation f ←→ f(0::'a::real_inner) = (0::'a) ∧ (∀ x y. dist(f x)
(f y) = dist x y)
  unfolding orthogonal_transformation
```

```
  apply (auto simp: linear_0 isometry_linear)
   apply (metis (no_types, hide_lams) dist_norm linear_diff)
  by (metis dist_0_norm)
```

Can extend an isometry from unit sphere:

```
lemma isometry_sphere_extend:
  fixes f:: 'a::real_inner ⇒ 'a
  assumes f1: ⋀x. norm x = 1 ⟹ norm (f x) = 1
    and fd1: ⋀x y. ⟦norm x = 1; norm y = 1⟧ ⟹ dist (f x) (f y) = dist x y
  shows ∃g. orthogonal_transformation g ∧ (∀x. norm x = 1 ⟶ g x = f x)
proof -
  {
    fix x y x' y' u v u' v' :: 'a
    assume H: x = norm x *R u y = norm y *R v
              x' = norm x *R u' y' = norm y *R v'
      and J: norm u = 1 norm u' = 1 norm v = 1 norm v' = 1 norm(u' − v') =
norm(u − v)
    then have *: u • v = u' • v' + v' • u' − v • u
      by (simp add: norm_eq norm_eq_1 inner_add inner_diff)
    have norm (norm x *R u' − norm y *R v') = norm (norm x *R u − norm y
*R v)
      using J by (simp add: norm_eq norm_eq_1 inner_diff * field_simps)
    then have norm(x' − y') = norm(x − y)
      using H by metis
  }
  note norm_eq = this
  let ?g = λx. if x = 0 then 0 else norm x *R f (x /R norm x)
  have thfg: ?g x = f x if norm x = 1 for x
    using that by auto
  have thd: dist (?g x) (?g y) = dist x y for x y
  proof (cases x=0 ∨ y=0)
    case False
    show dist (?g x) (?g y) = dist x y
      unfolding dist_norm
    proof (rule norm_eq)
      show x = norm x *R (x /R norm x) y = norm y *R (y /R norm y)
           norm (f (x /R norm x)) = 1 norm (f (y /R norm y)) = 1
        using False f1 by auto
    qed (use False in ‹auto simp: field_simps intro: f1 fd1[unfolded dist_norm]›)
  qed (auto simp: f1)
  show ?thesis
    unfolding orthogonal_transformation_isometry
    by (rule exI[where x= ?g]) (metis thfg thd)
qed
```

1.9.12 Induction on matrix row operations

```
lemma induct_matrix_row_operations:
  fixes P :: real^'n^'n ⇒ bool
```

```
  assumes zero_row: ⋀A i. row i A = 0 ⟹ P A
    and diagonal: ⋀A. (⋀i j. i ≠ j ⟹ A$i$j = 0) ⟹ P A
    and swap_cols: ⋀A m n. ⟦P A; m ≠ n⟧ ⟹ P(χ i j. A $ i $ Fun.swap m n id
j)
    and row_op: ⋀A m n c. ⟦P A; m ≠ n⟧
                ⟹ P(χ i. if i = m then row m A + c *R row n A else row i A)
  shows P A
proof -
  have P A if (⋀i j. ⟦j ∈ -K;  i ≠ j⟧ ⟹ A$i$j = 0) for A K
  proof -
    have finite K
      by simp
    then show ?thesis using that
    proof (induction arbitrary: A rule: finite_induct)
      case empty
      with diagonal show ?case
        by simp
    next
      case (insert k K)
      note insertK = insert
      have P A if kk: A$k$k ≠ 0
        and 0: ⋀i j. ⟦j ∈ - insert k K; i ≠ j⟧ ⟹ A$i$j = 0
            ⋀i. ⟦i ∈ -L; i ≠ k⟧ ⟹ A$i$k = 0 for A L
      proof -
        have finite L
          by simp
        then show ?thesis using 0 kk
        proof (induction arbitrary: A rule: finite_induct)
          case (empty B)
          show ?case
          proof (rule insertK)
            fix i j
            assume i ∈ - K j ≠ i
            show B $ j $ i = 0
              using ‹j ≠ i› ‹i ∈ - K› empty
                by (metis ComplD ComplI Compl_eq_Diff_UNIV Diff_empty UNIV_I
insert_iff)
          qed
        next
          case (insert l L B)
          show ?case
          proof (cases k = l)
            case True
            with insert show ?thesis
              by auto
          next
            case False
            let ?C = χ i. if i = l then row l B - (B $ l $ k / B $ k $ k) *R row k
B else row i B
```

```
          have 1: ⟦j ∈ − insert k K; i ≠ j⟧ ⟹ ?C $ i $ j = 0 for j i
            by (auto simp: insert.prems(1) row_def)
          have 2: ?C $ i $ k = 0
            if i ∈ − L i ≠ k for i
          proof (cases i=l)
            case True
            with that insert.prems show ?thesis
              by (simp add: row_def)
          next
            case False
            with that show ?thesis
              by (simp add: insert.prems(2) row_def)
          qed
          have 3: ?C $ k $ k ≠ 0
            by (auto simp: insert.prems row_def ‹k ≠ l›)
          have PC: P ?C
            using insert.IH [OF 1 2 3] by auto
          have eqB: (χ i. if i = l then row l ?C + (B $ l $ k / B $ k $ k) *R row
k ?C else row i ?C) = B
            using ‹k ≠ l› by (simp add: vec_eq_iff row_def)
          show ?thesis
            using row_op [OF PC, of l k, where c = B$l$k / B$k$k] eqB ‹k ≠ l›
            by (simp add: cong: if_cong)
        qed
      qed
    qed
    then have nonzero_hyp: P A
      if kk: A$k$k ≠ 0 and zeroes: ⋀i j. j ∈ − insert k K ∧ i≠j ⟹ A$i$j =
0 for A
      by (auto simp: intro!: kk zeroes)
    show ?case
    proof (cases row k A = 0)
      case True
      with zero_row show ?thesis by auto
    next
      case False
      then obtain l where l: A$k$l ≠ 0
        by (auto simp: row_def zero_vec_def vec_eq_iff)
      show ?thesis
      proof (cases k = l)
        case True
        with l nonzero_hyp insert.prems show ?thesis
          by blast
      next
        case False
        have *: A $ i $ Fun.swap k l id j = 0 if j ≠ k j ∉ K i ≠ j for i j
          using False l insert.prems that
          by (auto simp: swap_def insert split: if_split_asm)
        have P (χ i j. (χ i j. A $ i $ Fun.swap k l id j) $ i $ Fun.swap k l id j)
```

```
            by (rule swap_cols [OF nonzero_hyp False]) (auto simp: l *)
          moreover
          have (χ i j. (χ i j. A $ i $ Fun.swap k l id j) $ i $ Fun.swap k l id j) = A
            by (vector Fun.swap_def)
          ultimately show ?thesis
            by simp
        qed
      qed
    qed
  qed
  then show ?thesis
    by blast
qed

lemma induct_matrix_elementary:
  fixes P :: real^'n^'n ⇒ bool
  assumes mult: ⋀A B. ⟦P A; P B⟧ ⟹ P(A ** B)
    and zero_row: ⋀A i. row i A = 0 ⟹ P A
    and diagonal: ⋀A. (⋀i j. i ≠ j ⟹ A$i$j = 0) ⟹ P A
    and swap1: ⋀m n. m ≠ n ⟹ P(χ i j. mat 1 $ i $ Fun.swap m n id j)
    and idplus: ⋀m n c. m ≠ n ⟹ P(χ i j. if i = m ∧ j = n then c else of_bool
(i = j))
  shows P A
proof −
  have swap: P (χ i j. A $ i $ Fun.swap m n id j)  (is P ?C)
    if P A m ≠ n for A m n
  proof −
    have A ** (χ i j. mat 1 $ i $ Fun.swap m n id j) = ?C
     by (simp add: matrix_matrix_mult_def mat_def vec_eq_iff if_distrib sum.delta_remove)
    then show ?thesis
      using mult swap1 that by metis
  qed
  have row: P (χ i. if i = m then row m A + c *R row n A else row i A)  (is P
?C)
    if P A m ≠ n for A m n c
  proof −
    let ?B = χ i j. if i = m ∧ j = n then c else of_bool (i = j)
    have ?B ** A = ?C
      using ‹m ≠ n› unfolding matrix_matrix_mult_def row_def of_bool_def
       by (auto simp: vec_eq_iff if_distrib [of λx. x * y for y] sum.remove cong:
if_cong)
    then show ?thesis
      by (rule subst) (auto simp: that mult idplus)
  qed
  show ?thesis
    by (rule induct_matrix_row_operations [OF zero_row diagonal swap row])
qed

lemma induct_matrix_elementary_alt:
```

```
  fixes P :: real^'n^'n ⇒ bool
  assumes mult: ⋀A B. ⟦P A; P B⟧ ⟹ P(A ** B)
    and zero_row: ⋀A i. row i A = 0 ⟹ P A
    and diagonal: ⋀A. (⋀i j. i ≠ j ⟹ A$i$j = 0) ⟹ P A
    and swap1: ⋀m n. m ≠ n ⟹ P(χ i j. mat 1 $ i $ Fun.swap m n id j)
    and idplus: ⋀m n. m ≠ n ⟹ P(χ i j. of_bool (i = m ∧ j = n ∨ i = j))
  shows P A
proof −
  have *: P (χ i j. if i = m ∧ j = n then c else of_bool (i = j))
    if m ≠ n for m n c
  proof (cases c = 0)
    case True
    with diagonal show ?thesis by auto
  next
    case False
    then have eq: (χ i j. if i = m ∧ j = n then c else of_bool (i = j)) =
                  (χ i j. if i = j then (if j = n then inverse c else 1) else 0) **
                  (χ i j. of_bool (i = m ∧ j = n ∨ i = j)) **
                  (χ i j. if i = j then if j = n then c else 1 else 0)
      using ‹m ≠ n›
       apply (simp add: matrix_matrix_mult_def vec_eq_iff of_bool_def if_distrib [of
λx. y * x for y] cong: if_cong)
      apply (simp add: if_if_eq_conj sum.neutral conj_commute cong: conj_cong)
      done
    show ?thesis
      apply (subst eq)
      apply (intro mult idplus that)
       apply (auto intro: diagonal)
      done
  qed
  show ?thesis
    by (rule induct_matrix_elementary) (auto intro: assms *)
qed

lemma matrix_vector_mult_matrix_matrix_mult_compose:
  (*v) (A ** B) = (*v) A ∘ (*v) B
  by (auto simp: matrix_vector_mul_assoc)

lemma induct_linear_elementary:
  fixes f :: real^'n ⇒ real^'n
  assumes linear f
    and comp: ⋀f g. ⟦linear f; linear g; P f; P g⟧ ⟹ P(f ∘ g)
    and zeroes: ⋀f i. ⟦linear f; ⋀x. (f x) $ i = 0⟧ ⟹ P f
    and const: ⋀c. P(λx. χ i. c i * x$i)
    and swap: ⋀m n::'n. m ≠ n ⟹ P(λx. χ i. x $ Fun.swap m n id i)
     and idplus: ⋀m n::'n. m ≠ n ⟹ P(λx. χ i. if i = m then x$m + x$n else
x$i)
  shows P f
proof −
```

```
  have P ((*v) A) for A
  proof (rule induct_matrix_elementary_alt)
    fix A B
    assume P ((*v) A) and P ((*v) B)
    then show P ((*v) (A ** B))
        by (auto simp add: matrix_vector_mult_matrix_matrix_mult_compose intro!:
comp)
  next
    fix A :: real^'n^'n and i
    assume row i A = 0
    show P ((*v) A)
      using matrix_vector_mul_linear
      by (rule zeroes[where i=i])
        (metis ‹row i A = 0› inner_zero_left matrix_vector_mul_component row_def
vec_lambda_eta)
  next
    fix A :: real^'n^'n
    assume 0: ⋀i j. i ≠ j ⟹ A $ i $ j = 0
    have A $ i $ i * x $ i = (∑j∈UNIV. A $ i $ j * x $ j) for x and i :: 'n
      by (simp add: 0 comm_monoid_add_class.sum.remove [where x=i])
    then have (λx. χ i. A $ i $ i * x $ i) = ((*v) A)
      by (auto simp: 0 matrix_vector_mult_def)
    then show P ((*v) A)
      using const [of λi. A $ i $ i] by simp
  next
    fix m n :: 'n
    assume m ≠ n
    have eq: (∑j∈UNIV. if i = Fun.swap m n id j then x $ j else 0) =
            (∑j∈UNIV. if j = Fun.swap m n id i then x $ j else 0)
      for i and x :: real^'n
      unfolding swap_def by (rule sum.cong) auto
    have (λx::real^'n. χ i. x $ Fun.swap m n id i) = ((*v) (χ i j. if i = Fun.swap
m n id j then 1 else 0))
      by (auto simp: mat_def matrix_vector_mult_def eq if_distrib [of λx. x * y for
y] cong: if_cong)
    with swap [OF ‹m ≠ n›] show P ((*v) (χ i j. mat 1 $ i $ Fun.swap m n id
j))
      by (simp add: mat_def matrix_vector_mult_def)
  next
    fix m n :: 'n
    assume m ≠ n
    then have x $ m + x $ n = (∑j∈UNIV. of_bool (j = n ∨ m = j) * x $ j)
for x :: real^'n
      by (auto simp: of_bool_def if_distrib [of λx. x * y for y] sum.remove cong:
if_cong)
    then have (λx::real^'n. χ i. if i = m then x $ m + x $ n else x $ i) =
              ((*v) (χ i j. of_bool (i = m ∧ j = n ∨ i = j)))
      unfolding matrix_vector_mult_def of_bool_def
      by (auto simp: vec_eq_iff if_distrib [of λx. x * y for y] cong: if_cong)
```

```
    then show P ((*v) (χ i j. of_bool (i = m ∧ j = n ∨ i = j)))
      using idplus [OF ‹m ≠ n›] by simp
  qed
  then show ?thesis
    by (metis ‹linear f› matrix_vector_mul(2))
qed

end
```

1.10 Traces and Determinants of Square Matrices

```
theory Determinants
imports
  Cartesian_Space
  HOL−Library.Permutations
begin
```

1.10.1 Trace

```
definition trace :: 'a::semiring_1^'n^'n ⇒ 'a
  where trace A = sum (λi. ((A$i)$i)) (UNIV::'n set)

lemma trace_0: trace (mat 0) = 0
  by (simp add: trace_def mat_def)

lemma trace_I: trace (mat 1 :: 'a::semiring_1^'n^'n) = of_nat(CARD('n))
  by (simp add: trace_def mat_def)

lemma trace_add: trace ((A::'a::comm_semiring_1^'n^'n) + B) = trace A + trace
B
  by (simp add: trace_def sum.distrib)

lemma trace_sub: trace ((A::'a::comm_ring_1^'n^'n) − B) = trace A − trace B
  by (simp add: trace_def sum_subtractf)

lemma trace_mul_sym: trace ((A::'a::comm_semiring_1^'n^'m) ** B) = trace (B**A)
  apply (simp add: trace_def matrix_matrix_mult_def)
  apply (subst sum.swap)
  apply (simp add: mult.commute)
  done
```

Definition of determinant

```
definition det:: 'a::comm_ring_1^'n^'n ⇒ 'a where
  det A =
    sum (λp. of_int (sign p) * prod (λi. A$i$p i) (UNIV :: 'n set))
      {p. p permutes (UNIV :: 'n set)}
```

Basic determinant properties

lemma *det_transpose* [*simp*]: *det* (*transpose A*) = *det* (*A*::'*a*::*comm_ring_1* ^'*n*^'*n*)
proof −
  let *?di* = λ*A i j*. *A*\$*i*\$*j*
  let *?U* = (*UNIV* :: '*n set*)
  have *fU*: *finite ?U* **by** *simp*
  {
    fix *p*
    assume *p*: *p* ∈ {*p*. *p permutes ?U*}
    from *p* **have** *pU*: *p permutes ?U*
      by *blast*
    have *sth*: *sign* (*inv p*) = *sign p*
      by (*metis sign_inverse fU p mem_Collect_eq permutation_permutes*)
    from *permutes_inj*[*OF pU*]
    have *pi*: *inj_on p ?U*
      by (*blast intro*: *subset_inj_on*)
    from *permutes_image*[*OF pU*]
    have *prod* (λ*i*. *?di* (*transpose A*) *i* (*inv p i*)) *?U* =
      prod (λ*i*. *?di* (*transpose A*) *i* (*inv p i*)) (*p* ‘ *?U*)
      by *simp*
    also have … = *prod* ((λ*i*. *?di* (*transpose A*) *i* (*inv p i*)) ∘ *p*) *?U*
      unfolding *prod.reindex*[*OF pi*] **..**
    also have … = *prod* (λ*i*. *?di A i* (*p i*)) *?U*
    proof −
      have ((λ*i*. *?di* (*transpose A*) *i* (*inv p i*)) ∘ *p*) *i* = *?di A i* (*p i*) **if** *i* ∈ *?U* **for**
i
        using *that permutes_inv_o*[*OF pU*] *permutes_in_image*[*OF pU*]
        unfolding *transpose_def* **by** (*simp add*: *fun_eq_iff*)
      then show *prod* ((λ*i*. *?di* (*transpose A*) *i* (*inv p i*)) ∘ *p*) *?U* = *prod* (λ*i*. *?di*
A i (*p i*)) *?U*
        by (*auto intro*: *prod.cong*)
    qed
    finally have *of_int* (*sign* (*inv p*)) ∗ (*prod* (λ*i*. *?di* (*transpose A*) *i* (*inv p i*))
?U) =
      of_int (*sign p*) ∗ (*prod* (λ*i*. *?di A i* (*p i*)) *?U*)
      using *sth* **by** *simp*
  }
  then show *?thesis*
    unfolding *det_def*
    by (*subst sum_permutations_inverse*) (*blast intro*: *sum.cong*)
qed

lemma *det_lowerdiagonal*:
  fixes *A* :: '*a*::*comm_ring_1*^('*n*::{*finite*,*wellorder*})^('*n*::{*finite*,*wellorder*})
  assumes *ld*: ⋀*i j*. *i* < *j* ⟹ *A*\$*i*\$*j* = *0*
  shows *det A* = *prod* (λ*i*. *A*\$*i*\$*i*) (*UNIV*:: '*n set*)
proof −
  let *?U* = *UNIV*:: '*n set*
  let *?PU* = {*p*. *p permutes ?U*}
  let *?pp* = λ*p*. *of_int* (*sign p*) ∗ *prod* (λ*i*. *A*\$*i*\$*p i*) (*UNIV* :: '*n set*)

```
  have fU: finite ?U
    by simp
  have id0: {id} ⊆ ?PU
    by (auto simp: permutes_id)
  have p0: ∀ p ∈ ?PU − {id}. ?pp p = 0
  proof
    fix p
    assume p ∈ ?PU − {id}
    then obtain i where i: p i > i
      by clarify (meson leI permutes_natset_le)
    from ld[OF i] have ∃ i ∈ ?U. A$i$p i = 0
      by blast
    with prod_zero[OF fU] show ?pp p = 0
      by force
  qed
  from sum.mono_neutral_cong_left[OF finite_permutations[OF fU] id0 p0] show
?thesis
    unfolding det_def by (simp add: sign_id)
qed

lemma det_upperdiagonal:
  fixes A :: 'a::comm_ring_1^'n::{finite,wellorder}^'n::{finite,wellorder}
  assumes ld: ⋀i j. i > j ⟹ A$i$j = 0
  shows det A = prod (λi. A$i$i) (UNIV:: 'n set)
proof −
  let ?U = UNIV:: 'n set
  let ?PU = {p. p permutes ?U}
  let ?pp = (λp. of_int (sign p) ∗ prod (λi. A$i$p i) (UNIV :: 'n set))
  have fU: finite ?U
    by simp
  have id0: {id} ⊆ ?PU
    by (auto simp: permutes_id)
  have p0: ∀ p ∈ ?PU −{id}. ?pp p = 0
  proof
    fix p
    assume p: p ∈ ?PU − {id}
    then obtain i where i: p i < i
      by clarify (meson leI permutes_natset_ge)
    from ld[OF i] have ∃ i ∈ ?U. A$i$p i = 0
      by blast
    with prod_zero[OF fU]  show ?pp p = 0
      by force
  qed
  from sum.mono_neutral_cong_left[OF finite_permutations[OF fU] id0 p0] show
?thesis
    unfolding det_def by (simp add: sign_id)
qed

proposition det_diagonal:
```

fixes A :: $'a::comm\_ring\_1$ ^$'n$^$'n$
assumes ld: $\bigwedge i\ j.\ i \neq j \Longrightarrow A\$i\$j = 0$
shows $det\ A = prod\ (\lambda i.\ A\$i\$i)\ (UNIV::'n\ set)$
proof −
let $?U = UNIV::\ 'n\ set$
let $?PU = \{p.\ p\ permutes\ ?U\}$
let $?pp = \lambda p.\ of\_int\ (sign\ p) * prod\ (\lambda i.\ A\$i\$p\ i)\ (UNIV\ ::\ 'n\ set)$
have fU: *finite* $?U$ **by** *simp*
from $finite\_permutations[OF\ fU]$ **have** fPU: *finite* $?PU$ **.**
have $id0$: $\{id\} \subseteq ?PU$
by (*auto simp*: *permutes_id*)
have $p0$: $\forall p \in ?PU - \{id\}.\ ?pp\ p = 0$
proof
fix p
assume p: $p \in ?PU - \{id\}$
then obtain i **where** i: $p\ i \neq i$
by *fastforce*
with ld **have** $\exists i \in ?U.\ A\$i\$p\ i = 0$
by (*metis UNIV_I*)
with $prod\_zero\ [OF\ fU]$ **show** $?pp\ p = 0$
by *force*
qed
from $sum.mono\_neutral\_cong\_left[OF\ fPU\ id0\ p0]$ **show** *?thesis*
unfolding *det_def* **by** (*simp add*: *sign_id*)
qed

lemma *det_I* [*simp*]: $det\ (mat\ 1\ ::\ 'a::comm\_ring\_1$^$'n$^$'n) = 1$
by (*simp add*: *det_diagonal mat_def*)

lemma *det_0* [*simp*]: $det\ (mat\ 0\ ::\ 'a::comm\_ring\_1$^$'n$^$'n) = 0$
by (*simp add*: *det_def prod_zero power_0_left*)

lemma *det_permute_rows*:
fixes A :: $'a::comm\_ring\_1$^$'n$^$'n$
assumes p: $p\ permutes\ (UNIV\ ::\ 'n::finite\ set)$
shows $det\ (\chi\ i.\ A\$p\ i\ ::\ 'a$^$'n$^$'n) = of\_int\ (sign\ p) * det\ A$
proof −
let $?U = UNIV\ ::\ 'n\ set$
let $?PU = \{p.\ p\ permutes\ ?U\}$
have $*$: $(\sum q{\in}?PU.\ of\_int\ (sign\ (q \circ p)) * (\prod i{\in}?U.\ A\ \$\ p\ i\ \$\ (q \circ p)\ i)) =$
$(\sum n{\in}?PU.\ of\_int\ (sign\ p) * of\_int\ (sign\ n) * (\prod i{\in}?U.\ A\ \$\ i\ \$\ n\ i))$
proof (*rule sum.cong*)
fix q
assume qPU: $q \in ?PU$
have fU: *finite* $?U$
by *simp*
from qPU **have** q: $q\ permutes\ ?U$
by *blast*
have $prod\ (\lambda i.\ A\$p\ i\$\ (q \circ p)\ i)\ ?U = prod\ ((\lambda i.\ A\$p\ i\$(q \circ p)\ i) \circ inv\ p)\ ?U$

```
      by (simp only: prod.permute[OF permutes_inv[OF p], symmetric])
    also have ... = prod (λi. A $ (p ∘ inv p) i $ (q ∘ (p ∘ inv p)) i) ?U
      by (simp only: o_def)
    also have ... = prod (λi. A$i$q i) ?U
      by (simp only: o_def permutes_inverses[OF p])
    finally have thp: prod (λi. A$p i$ (q ∘ p) i) ?U = prod (λi. A$i$q i) ?U
      by blast
    from p q have pp: permutation p and qp: permutation q
      by (metis fU permutation_permutes)+
    show of_int (sign (q ∘ p)) * prod (λi. A$ p i$ (q ∘ p) i) ?U =
        of_int (sign p) * of_int (sign q) * prod (λi. A$i$q i) ?U
      by (simp only: thp sign_compose[OF qp pp] mult.commute of_int_mult)
  qed auto
  show ?thesis
    apply (simp add: det_def sum_distrib_left mult.assoc[symmetric])
    apply (subst sum_permutations_compose_right[OF p])
    apply (rule *)
    done
qed

lemma det_permute_columns:
  fixes A :: 'a::comm_ring_1^'n^'n
  assumes p: p permutes (UNIV :: 'n set)
  shows det(χ i j. A$i$ p j :: 'a^'n^'n) = of_int (sign p) * det A
proof -
  let ?Ap = χ i j. A$i$ p j :: 'a^'n^'n
  let ?At = transpose A
  have of_int (sign p) * det A = det (transpose (χ i. transpose A $ p i))
    unfolding det_permute_rows[OF p, of ?At] det_transpose ..
  moreover
  have ?Ap = transpose (χ i. transpose A $ p i)
    by (simp add: transpose_def vec_eq_iff)
  ultimately show ?thesis
    by simp
qed

lemma det_identical_columns:
  fixes A :: 'a::comm_ring_1^'n^'n
  assumes jk: j ≠ k
    and r: column j A = column k A
  shows det A = 0
proof -
  let ?U=UNIV::'n set
  let ?t_jk=Fun.swap j k id
  let ?PU={p. p permutes ?U}
  let ?S1={p. p∈?PU ∧ evenperm p}
  let ?S2={(?t_jk ∘ p) |p. p ∈?S1}
  let ?f=λp. of_int (sign p) * (∏ i∈UNIV. A $ i $ p i)
  let ?g=λp. ?t_jk ∘ p
```

have *g_S1*: *?S2* = *?g* ‘ *?S1* **by** *auto*
have *inj_g*: *inj_on ?g ?S1*
proof (*unfold inj_on_def*, *auto*)
fix *x y* **assume** *x*: *x permutes ?U* **and** *even_x*: *evenperm x*
and *y*: *y permutes ?U* **and** *even_y*: *evenperm y* **and** *eq*: *?t_jk* ∘ *x* = *?t_jk* ∘ *y*
show *x* = *y* **by** (*metis* (*hide_lams*, *no_types*) *comp_assoc eq id_comp swap_id_idempotent*)
qed
have *tjk_permutes*: *?t_jk permutes ?U* **unfolding** *permutes_def swap_id_eq* **by** (*auto*,*metis*)
have *tjk_eq*: ∀ *i l*. *A* \$ *i* \$ *?t_jk l* = *A* \$ *i* \$ *l*
using *r jk*
unfolding *column_def vec_eq_iff swap_id_eq* **by** *fastforce*
have *sign_tjk*: *sign ?t_jk* = −*1* **using** *sign_swap_id*[*of j k*] *jk* **by** *auto*
{**fix** *x*
assume *x*: *x*∈ *?S1*
have *sign* (*?t_jk* ∘ *x*) = *sign* (*?t_jk*) ∗ *sign x*
by (*metis* (*lifting*) *finite_class.finite_UNIV mem_Collect_eq*
permutation_permutes permutation_swap_id sign_compose x)
also have . . . = − *sign x* **using** *sign_tjk* **by** *simp*
also have . . . ≠ *sign x* **unfolding** *sign_def* **by** *simp*
finally have *sign* (*?t_jk* ∘ *x*) ≠ *sign x* **and** (*?t_jk* ∘ *x*) ∈ *?S2*
using *x* **by** *force*+
}
hence *disjoint*: *?S1* ∩ *?S2* = {}
by (*force simp*: *sign_def*)
have *PU_decomposition*: *?PU* = *?S1* ∪ *?S2*
proof (*auto*)
fix *x*
assume *x*: *x permutes ?U* **and** ∀ *p*. *p permutes ?U* ⟶ *x* = *Fun.swap j k id* ∘ *p* ⟶ ¬ *evenperm p*
then obtain *p* **where** *p*: *p permutes UNIV* **and** *x_eq*: *x* = *Fun.swap j k id* ∘ *p*
and *odd_p*: ¬ *evenperm p*
by (*metis* (*mono_tags*) *id_o o_assoc permutes_compose swap_id_idempotent tjk_permutes*)
thus *evenperm x*
by (*meson evenperm_comp evenperm_swap finite_class.finite_UNIV*
jk permutation_permutes permutation_swap_id)
next
fix *p* **assume** *p*: *p permutes ?U*
show *Fun.swap j k id* ∘ *p permutes UNIV* **by** (*metis p permutes_compose tjk_permutes*)
qed
have *sum ?f ?S2* = *sum* ((λ*p*. *of_int* (*sign p*) ∗ (∏ *i*∈*UNIV*. *A* \$ *i* \$ *p i*))
∘ (∘) (*Fun.swap j k id*)) {*p* ∈ {*p*. *p permutes UNIV*}. *evenperm p*}
unfolding *g_S1* **by** (*rule sum.reindex*[*OF inj_g*])
also have . . . = *sum* (λ*p*. *of_int* (*sign* (*?t_jk* ∘ *p*)) ∗ (∏ *i*∈*UNIV*. *A* \$ *i* \$ *p i*)) *?S1*
unfolding *o_def* **by** (*rule sum.cong*, *auto simp*: *tjk_eq*)
also have . . . = *sum* (λ*p*. − *?f p*) *?S1*

```
  proof (rule sum.cong, auto)
    fix x assume x: x permutes ?U
      and even_x: evenperm x
    hence perm_x: permutation x and perm_tjk: permutation ?t_jk
        using permutation_permutes[of x] permutation_permutes[of ?t_jk] permuta-
tion_swap_id
      by (metis finite_code)+
    have (sign (?t_jk ∘ x)) = − (sign x)
      unfolding sign_compose[OF perm_tjk perm_x] sign_tjk by auto
    thus of_int (sign (?t_jk ∘ x)) ∗ (∏ i∈UNIV. A $ i $ x i)
      = − (of_int (sign x) ∗ (∏ i∈UNIV. A $ i $ x i))
      by auto
  qed
  also have ... = − sum ?f ?S1 unfolding sum_negf ..
  finally have ∗: sum ?f ?S2 = − sum ?f ?S1 .
  have det A = (∑ p | p permutes UNIV. of_int (sign p) ∗ (∏ i∈UNIV. A $ i $
p i))
    unfolding det_def ..
  also have ... = sum ?f ?S1 + sum ?f ?S2
    by (subst PU_decomposition, rule sum.union_disjoint[OF _ _ disjoint], auto)
  also have ... = sum ?f ?S1 − sum ?f ?S1 unfolding ∗ by auto
  also have ... = 0 by simp
  finally show det A = 0 by simp
qed

lemma det_identical_rows:
  fixes A :: 'a::comm_ring_1^'n^'n
  assumes ij: i ≠ j and r: row i A = row j A
  shows det A = 0
  by (metis column_transpose det_identical_columns det_transpose ij r)

lemma det_zero_row:
  fixes A :: 'a::{idom, ring_char_0}^'n^'n and F :: 'b::{field}^'m^'m
  shows row i A = 0 ⟹ det A = 0 and row j F = 0 ⟹ det F = 0
  by (force simp: row_def det_def vec_eq_iff sign_nz intro!: sum.neutral)+

lemma det_zero_column:
  fixes A :: 'a::{idom, ring_char_0}^'n^'n and F :: 'b::{field}^'m^'m
  shows column i A = 0 ⟹ det A = 0 and column j F = 0 ⟹ det F = 0
  unfolding atomize_conj atomize_imp
  by (metis det_transpose det_zero_row row_transpose)

lemma det_row_add:
  fixes a b c :: 'n::finite ⇒ _ ^ 'n
  shows det((χ i. if i = k then a i + b i else c i)::'a::comm_ring_1^'n^'n) =
    det((χ i. if i = k then a i else c i)::'a::comm_ring_1^'n^'n) +
    det((χ i. if i = k then b i else c i)::'a::comm_ring_1^'n^'n)
  unfolding det_def vec_lambda_beta sum.distrib[symmetric]
proof (rule sum.cong)
```

let *?U* = *UNIV* :: *'n set*
let *?pU* = {*p. p permutes ?U*}
let *?f* = (λ*i. if i* = *k then a i* + *b i else c i*)::*'n* ⇒ *'a*::*comm_ring_1*ˆ*'n*
let *?g* = (λ *i. if i* = *k then a i else c i*)::*'n* ⇒ *'a*::*comm_ring_1*ˆ*'n*
let *?h* = (λ *i. if i* = *k then b i else c i*)::*'n* ⇒ *'a*::*comm_ring_1*ˆ*'n*
fix *p*
assume *p*: *p* ∈ *?pU*
let *?Uk* = *?U* − {*k*}
from *p* **have** *pU*: *p permutes ?U*
by *blast*
have *kU*: *?U* = *insert k ?Uk*
by *blast*
have *eq*: *prod* (λ*i. ?f i* $ *p i*) *?Uk* = *prod* (λ*i. ?g i* $ *p i*) *?Uk*
prod (λ*i. ?f i* $ *p i*) *?Uk* = *prod* (λ*i. ?h i* $ *p i*) *?Uk*
by *auto*
have *Uk*: *finite ?Uk k* ∉ *?Uk*
by *auto*
have *prod* (λ*i. ?f i* $ *p i*) *?U* = *prod* (λ*i. ?f i* $ *p i*) (*insert k ?Uk*)
unfolding *kU*[*symmetric*] **..**
also have ... = *?f k* $ *p k* ∗ *prod* (λ*i. ?f i* $ *p i*) *?Uk*
by (*rule prod.insert*) *auto*
also have ... = (*a k* $ *p k* ∗ *prod* (λ*i. ?f i* $ *p i*) *?Uk*) + (*b k*$ *p k* ∗ *prod* (λ*i. ?f i* $ *p i*) *?Uk*)
by (*simp add*: *field_simps*)
also have ... = (*a k* $ *p k* ∗ *prod* (λ*i. ?g i* $ *p i*) *?Uk*) + (*b k*$ *p k* ∗ *prod* (λ*i. ?h i* $ *p i*) *?Uk*)
by (*metis eq*)
also have ... = *prod* (λ*i. ?g i* $ *p i*) (*insert k ?Uk*) + *prod* (λ*i. ?h i* $ *p i*) (*insert k ?Uk*)
unfolding *prod.insert*[*OF Uk*] **by** *simp*
finally have *prod* (λ*i. ?f i* $ *p i*) *?U* = *prod* (λ*i. ?g i* $ *p i*) *?U* + *prod* (λ*i. ?h i* $ *p i*) *?U*
unfolding *kU*[*symmetric*] **.**
then show *of_int* (*sign p*) ∗ *prod* (λ*i. ?f i* $ *p i*) *?U* =
of_int (*sign p*) ∗ *prod* (λ*i. ?g i* $ *p i*) *?U* + *of_int* (*sign p*) ∗ *prod* (λ*i. ?h i* $ *p i*) *?U*
by (*simp add*: *field_simps*)
qed *auto*

lemma *det_row_mul*:
fixes *a b* :: *'n*::*finite* ⇒ _ ˆ *'n*
shows *det*((χ *i. if i* = *k then c* ∗*s a i else b i*)::*'a*::*comm_ring_1*ˆ*'n*ˆ*'n*) =
c ∗ *det*((χ *i. if i* = *k then a i else b i*)::*'a*::*comm_ring_1*ˆ*'n*ˆ*'n*)
unfolding *det_def vec_lambda_beta sum_distrib_left*
proof (*rule sum.cong*)
let *?U* = *UNIV* :: *'n set*
let *?pU* = {*p. p permutes ?U*}
let *?f* = (λ*i. if i* = *k then c*∗*s a i else b i*)::*'n* ⇒ *'a*::*comm_ring_1*ˆ*'n*
let *?g* = (λ *i. if i* = *k then a i else b i*)::*'n* ⇒ *'a*::*comm_ring_1*ˆ*'n*

```
  fix p
  assume p: p ∈ ?pU
  let ?Uk = ?U − {k}
  from p have pU: p permutes ?U
    by blast
  have kU: ?U = insert k ?Uk
    by blast
  have eq: prod (λi. ?f i $ p i) ?Uk = prod (λi. ?g i $ p i) ?Uk
    by auto
  have Uk: finite ?Uk k ∉ ?Uk
    by auto
  have prod (λi. ?f i $ p i) ?U = prod (λi. ?f i $ p i) (insert k ?Uk)
    unfolding kU[symmetric] ..
  also have ... = ?f k $ p k * prod (λi. ?f i $ p i) ?Uk
    by (rule prod.insert) auto
  also have ... = (c*s a k) $ p k * prod (λi. ?f i $ p i) ?Uk
    by (simp add: field_simps)
  also have ... = c* (a k $ p k * prod (λi. ?g i $ p i) ?Uk)
    unfolding eq by (simp add: ac_simps)
  also have ... = c* (prod (λi. ?g i $ p i) (insert k ?Uk))
    unfolding prod.insert[OF Uk] by simp
  finally have prod (λi. ?f i $ p i) ?U = c* (prod (λi. ?g i $ p i) ?U)
    unfolding kU[symmetric] .
  then show of_int (sign p) * prod (λi. ?f i $ p i) ?U = c * (of_int (sign p) *
prod (λi. ?g i $ p i) ?U)
    by (simp add: field_simps)
qed auto

lemma det_row_0:
  fixes b :: 'n::finite ⇒ _ ^ 'n
  shows det((χ i. if i = k then 0 else b i)::'a::comm_ring_1^'n^'n) = 0
  using det_row_mul[of k 0 λi. 1 b]
  apply simp
  apply (simp only: vector_smult_lzero)
  done

lemma det_row_operation:
  fixes A :: 'a::{comm_ring_1}^'n^'n
  assumes ij: i ≠ j
  shows det (χ k. if k = i then row i A + c *s row j A else row k A) = det A
proof −
  let ?Z = (χ k. if k = i then row j A else row k A) :: 'a ^'n^'n
  have th: row i ?Z = row j ?Z by (vector row_def)
  have th2: ((χ k. if k = i then row i A else row k A) :: 'a^'n^'n) = A
    by (vector row_def)
  show ?thesis
    unfolding det_row_add [of i] det_row_mul[of i] det_identical_rows[OF ij th] th2
    by simp
qed
```

```
lemma det_row_span:
  fixes A :: 'a::{field}^'n^'n
  assumes x: x ∈ vec.span {row j A |j. j ≠ i}
  shows det (χ k. if k = i then row i A + x else row k A) = det A
  using x
proof (induction rule: vec.span_induct_alt)
  case base
  have (if k = i then row i A + 0 else row k A) = row k A for k
    by simp
  then show ?case
    by (simp add: row_def)
next
  case (step c z y)
  then obtain j where j: z = row j A i ≠ j
    by blast
  let ?w = row i A + y
  have th0: row i A + (c*s z + y) = ?w + c*s z
    by vector
  let ?d = λx. det (χ k. if k = i then x else row k A)
  have thz: ?d z = 0
    apply (rule det_identical_rows[OF j(2)])
    using j
    apply (vector row_def)
    done
  have ?d (row i A + (c*s z + y)) = ?d (?w + c*s z)
    unfolding th0 ..
  then have ?d (row i A + (c*s z + y)) = det A
    unfolding thz step.IH det_row_mul[of i] det_row_add[of i] by simp
  then show ?case
    unfolding scalar_mult_eq_scaleR .
qed

lemma matrix_id [simp]: det (matrix id) = 1
  by (simp add: matrix_id_mat_1)

proposition det_matrix_scaleR [simp]: det (matrix (((*R) r)) :: real^'n^'n) = r
^ CARD('n::finite)
  apply (subst det_diagonal)
   apply (auto simp: matrix_def mat_def)
  apply (simp add: cart_eq_inner_axis inner_axis_axis)
  done
```

May as well do this, though it's a bit unsatisfactory since it ignores exact duplicates by considering the rows/columns as a set.

```
lemma det_dependent_rows:
  fixes A:: 'a::{field}^'n^'n
  assumes d: vec.dependent (rows A)
  shows det A = 0
```

```
proof –
  let ?U = UNIV :: 'n set
  from d obtain i where i: row i A ∈ vec.span (rows A − {row i A})
    unfolding vec.dependent_def rows_def by blast
  show ?thesis
  proof (cases ∀ i j. i ≠ j ⟶ row i A ≠ row j A)
    case True
    with i have vec.span (rows A − {row i A}) ⊆ vec.span {row j A |j. j ≠ i}
      by (auto simp: rows_def intro!: vec.span_mono)
    then have − row i A ∈ vec.span {row j A|j. j ≠ i}
      by (meson i subsetCE vec.span_neg)
    from det_row_span[OF this]
    have det A = det (χ k. if k = i then 0 *s 1 else row k A)
      unfolding right_minus vector_smult_lzero ..
    with det_row_mul[of i 0 λi. 1]
    show ?thesis by simp
  next
    case False
    then obtain j k where jk: j ≠ k row j A = row k A
      by auto
    from det_identical_rows[OF jk] show ?thesis .
  qed
qed

lemma det_dependent_columns:
  assumes d: vec.dependent (columns (A::real^'n^'n))
  shows det A = 0
  by (metis d det_dependent_rows rows_transpose det_transpose)
```

Multilinearity and the multiplication formula

```
lemma Cart_lambda_cong: (⋀x. f x = g x) ⟹ (vec_lambda f::'a^'n) = (vec_lambda
g :: 'a^'n)
  by auto

lemma det_linear_row_sum:
  assumes fS: finite S
  shows det ((χ i. if i = k then sum (a i) S else c i)::'a::comm_ring_1^'n^'n) =
    sum (λj. det ((χ i. if i = k then a  i j else c i)::'a^'n^'n)) S
  using fS  by (induct rule: finite_induct; simp add: det_row_0 det_row_add cong:
if_cong)

lemma finite_bounded_functions:
  assumes fS: finite S
  shows finite {f. (∀ i ∈ {1.. (k::nat)}. f i ∈ S) ∧ (∀ i. i ∉ {1 .. k} ⟶ f i = i)}
proof (induct k)
  case 0
  have *: {f. ∀ i. f i = i} = {id}
    by auto
  show ?case
```

by (*auto simp*: ∗)
next
case (*Suc k*)
let *?f* = λ(*y*::*nat*,*g*) *i*. *if i* = *Suc k then y else g i*
let *?S* = *?f* ‘ (*S* × {*f*. (∀ *i*∈{*1*..*k*}. *f i* ∈ *S*) ∧ (∀ *i*. *i* ∉ {*1*..*k*} ⟶ *f i* = *i*)})
have *?S* = {*f*. (∀ *i*∈{*1*.. *Suc k*}. *f i* ∈ *S*) ∧ (∀ *i*. *i* ∉ {*1*.. *Suc k*} ⟶ *f i* = *i*)}
apply (*auto simp*: *image_iff*)
apply (*rename_tac f*)
apply (*rule_tac x*=*f* (*Suc k*) **in** *bexI*)
apply (*rule_tac x* = λ*i*. *if i* = *Suc k then i else f i* **in** *exI*, *auto*)
done
with *finite_imageI*[*OF finite_cartesian_product*[*OF fS Suc*.*hyps*(*1*)], *of ?f*]
show *?case*
by *metis*
qed

lemma *det_linear_rows_sum_lemma*:
assumes *fS*: *finite S*
and *fT*: *finite T*
shows *det* ((χ *i*. *if i* ∈ *T then sum* (*a i*) *S else c i*):: ′*a*::*comm_ring_1*ˆ′*n*ˆ′*n*) =
sum (λ*f*. *det*((χ *i*. *if i* ∈ *T then a i* (*f i*) *else c i*)::′*a*ˆ′*n*ˆ′*n*))
{*f*. (∀ *i* ∈ *T*. *f i* ∈ *S*) ∧ (∀ *i*. *i* ∉ *T* ⟶ *f i* = *i*)}
using *fT*
proof (*induct T arbitrary*: *a c set*: *finite*)
case *empty*
have *th0*: ⋀*x y*. (χ *i*. *if i* ∈ {} *then x i else y i*) = (χ *i*. *y i*)
by *vector*
from *empty*.*prems* **show** *?case*
unfolding *th0* **by** (*simp add*: *eq_id_iff*)
next
case (*insert z T a c*)
let *?F* = λ*T*. {*f*. (∀ *i* ∈ *T*. *f i* ∈ *S*) ∧ (∀ *i*. *i* ∉ *T* ⟶ *f i* = *i*)}
let *?h* = λ(*y*,*g*) *i*. *if i* = *z then y else g i*
let *?k* = λ*h*. (*h*(*z*),(λ*i*. *if i* = *z then i else h i*))
let *?s* = λ *k a c f*. *det*((χ *i*. *if i* ∈ *T then a i* (*f i*) *else c i*)::′*a*ˆ′*n*ˆ′*n*)
let *?c* = λ*j i*. *if i* = *z then a i j else c i*
have *thif*: ⋀*a b c d*. (*if a* ∨ *b then c else d*) = (*if a then c else if b then c else d*)
by *simp*
have *thif2*: ⋀*a b c d e*. (*if a then b else if c then d else e*) =
(*if c then* (*if a then b else d*) *else* (*if a then b else e*))
by *simp*
from ‹*z* ∉ *T*› **have** *nz*: ⋀*i*. *i* ∈ *T* ⟹ *i* ≠ *z*
by *auto*
have *det* (χ *i*. *if i* ∈ *insert z T then sum* (*a i*) *S else c i*) =
det (χ *i*. *if i* = *z then sum* (*a i*) *S else if i* ∈ *T then sum* (*a i*) *S else c i*)
unfolding *insert_iff thif* **..**
also have ... = (∑*j*∈*S*. *det* (χ *i*. *if i* ∈ *T then sum* (*a i*) *S else if i* = *z then a i j else c i*))

```
    unfolding det_linear_row_sum[OF fS]
    by (subst thif2) (simp add: nz cong: if_cong)
  finally have tha:
    det (χ i. if i ∈ insert z T then sum (a i) S else c i) =
     (∑(j, f)∈S × ?F T. det (χ i. if i ∈ T then a i (f i)
                                  else if i = z then a i j
                                  else c i))
    unfolding insert.hyps unfolding sum.cartesian_product by blast
  show ?case unfolding tha
    using ‹z ∉ T›
    by (intro sum.reindex_bij_witness[where i=?k and j=?h])
       (auto intro!: cong[OF refl[of det]] simp: vec_eq_iff)
qed

lemma det_linear_rows_sum:
  fixes S :: 'n::finite set
  assumes fS: finite S
  shows det (χ i. sum (a i) S) =
    sum (λf. det (χ i. a i (f i) :: 'a::comm_ring_1 ^ 'n^'n)) {f. ∀ i. f i ∈ S}
proof -
  have th0: ⋀x y. ((χ i. if i ∈ (UNIV:: 'n set) then x i else y i) :: 'a^'n^'n) = (χ
i. x i)
    by vector
  from det_linear_rows_sum_lemma[OF fS, of UNIV :: 'n set a, unfolded th0, OF
finite]
  show ?thesis by simp
qed

lemma matrix_mul_sum_alt:
  fixes A B :: 'a::comm_ring_1^'n^'n
  shows A ** B = (χ i. sum (λk. A$i$k *s B $ k) (UNIV :: 'n set))
  by (vector matrix_matrix_mult_def sum_component)

lemma det_rows_mul:
  det((χ i. c i *s a i)::'a::comm_ring_1^'n^'n) =
    prod (λi. c i) (UNIV:: 'n set) * det((χ i. a i)::'a^'n^'n)
proof (simp add: det_def sum_distrib_left cong add: prod.cong, rule sum.cong)
  let ?U = UNIV :: 'n set
  let ?PU = {p. p permutes ?U}
  fix p
  assume pU: p ∈ ?PU
  let ?s = of_int (sign p)
  from pU have p: p permutes ?U
    by blast
  have prod (λi. c i * a i $ p i) ?U = prod c ?U * prod (λi. a i $ p i) ?U
    unfolding prod.distrib ..
  then show ?s * (∏xa∈?U. c xa * a xa $ p xa) =
    prod c ?U * (?s* (∏xa∈?U. a xa $ p xa))
    by (simp add: field_simps)
```

qed *rule*

proposition *det_mul*:
  fixes *A B* :: *'a::comm_ring_1^'n^'n*
  shows *det* (*A* ** *B*) = *det A* * *det B*
proof –
  let *?U* = *UNIV* :: *'n set*
  let *?F* = {*f*. (∀ *i* ∈ *?U*. *f i* ∈ *?U*) ∧ (∀ *i*. *i* ∉ *?U* ⟶ *f i* = *i*)}
  let *?PU* = {*p*. *p permutes ?U*}
  have *p* ∈ *?F* **if** *p permutes ?U* **for** *p*
    by *simp*
  then have *PUF*: *?PU* ⊆ *?F* **by** *blast*
  {
    fix *f*
    assume *fPU*: *f* ∈ *?F* − *?PU*
    have *fUU*: *f* ‘ *?U* ⊆ *?U*
      using *fPU* **by** *auto*
    from *fPU* **have** *f*: ∀ *i* ∈ *?U*. *f i* ∈ *?U* ∀ *i*. *i* ∉ *?U* ⟶ *f i* = *i* ¬(∀ *y*. ∃!*x*. *f x* = *y*)
      unfolding *permutes_def* **by** *auto*

    let *?A* = (χ *i*. *Aif i* *s *B$f i*) :: *'a^'n^'n*
    let *?B* = (χ *i*. *B$f i*) :: *'a^'n^'n*
    {
      assume *fni*: ¬ *inj_on f ?U*
      then obtain *i j* **where** *ij*: *f i* = *f j i* ≠ *j*
        unfolding *inj_on_def* **by** *blast*
      then have *row i ?B* = *row j ?B*
        by (*vector row_def*)
      with *det_identical_rows*[*OF ij*(*2*)]
      have *det* (χ *i*. *Aif i* *s *B$f i*) = *0*
        unfolding *det_rows_mul* **by** *force*
    }
    moreover
    {
      assume *fi*: *inj_on f ?U*
      from *f fi* **have** *fith*: ⋀*i j*. *f i* = *f j* ⟹ *i* = *j*
        unfolding *inj_on_def* **by** *metis*
        note *fs* = *fi*[*unfolded surjective_iff_injective_gen*[*OF finite finite refl fUU*, *symmetric*]]
      have ∃!*x*. *f x* = *y* **for** *y*
        using *fith fs* **by** *blast*
      with *f*(*3*) **have** *det* (χ *i*. *Aif i* *s *B$f i*) = *0*
        by *blast*
    }
    ultimately have *det* (χ *i*. *Aif i* *s *B$f i*) = *0*
      by *blast*
  }
  then have *zth*: ∀ *f*∈ *?F* − *?PU*. *det* (χ *i*. *Aif i* *s *B$f i*) = *0*

```
    by simp
  {
    fix p
    assume pU: p ∈ ?PU
    from pU have p: p permutes ?U
      by blast
    let ?s = λp. of_int (sign p)
    let ?f = λq. ?s p * (∏i∈ ?U. A $ i $ p i) * (?s q * (∏i∈ ?U. B $ i $ q i))
    have (sum (λq. ?s q *
        (∏i∈ ?U. (χ i. A $ i $ p i *s B $ p i :: 'a^'n^'n) $ i $ q i)) ?PU) =
      (sum (λq. ?s p * (∏i∈ ?U. A $ i $ p i) * (?s q * (∏i∈ ?U. B $ i $ q i)))
?PU)
      unfolding sum_permutations_compose_right[OF permutes_inv[OF p], of ?f]
    proof (rule sum.cong)
      fix q
      assume qU: q ∈ ?PU
      then have q: q permutes ?U
        by blast
      from p q have pp: permutation p and pq: permutation q
        unfolding permutation_permutes by auto
      have th00: of_int (sign p) * of_int (sign p) = (1::'a)
        ⋀a. of_int (sign p) * (of_int (sign p) * a) = a
        unfolding mult.assoc[symmetric]
        unfolding of_int_mult[symmetric]
        by (simp_all add: sign_idempotent)
      have ths: ?s q = ?s p * ?s (q ∘ inv p)
        using pp pq permutation_inverse[OF pp] sign_inverse[OF pp]
        by (simp add: th00 ac_simps sign_idempotent sign_compose)
     have th001: prod (λi. B$i$ q (inv p i)) ?U = prod ((λi. B$i$ q (inv p i)) ∘
p) ?U
        by (rule prod.permute[OF p])
      have thp: prod (λi. (χ i. A$i$p i *s B$p i :: 'a^'n^'n) $i $ q i) ?U =
        prod (λi. A$i$p i) ?U * prod (λi. B$i$ q (inv p i)) ?U
        unfolding th001 prod.distrib[symmetric] o_def permutes_inverses[OF p]
        apply (rule prod.cong[OF refl])
        using permutes_in_image[OF q]
        apply vector
        done
      show ?s q * prod (λi. (((χ i. A$i$p i *s B$p i) :: 'a^'n^'n)$i$q i)) ?U =
         ?s p * (prod (λi. A$i$p i) ?U) * (?s (q ∘ inv p) * prod (λi. B$i$(q ∘ inv
p) i) ?U)
        using ths thp pp pq permutation_inverse[OF pp] sign_inverse[OF pp]
        by (simp add: sign_nz th00 field_simps sign_idempotent sign_compose)
    qed rule
  }
  then have th2: sum (λf. det (χ i. A$i$f i *s B$f i)) ?PU = det A * det B
    unfolding det_def sum_product
    by (rule sum.cong [OF refl])
  have det (A**B) = sum (λf.  det (χ i. A $ i $ f i *s B $ f i)) ?F
```

unfolding *matrix_mul_sum_alt det_linear_rows_sum*[*OF finite*]
by *simp*
also have ... = *sum* (λ*f*. *det* (χ *i*. *A*\$*i*\$*f i* ∗*s B*\$*f i*)) *?PU*
using *sum.mono_neutral_cong_left*[*OF finite PUF zth*, *symmetric*]
unfolding *det_rows_mul* **by** *auto*
finally show *?thesis* **unfolding** *th2* **.**
qed

1.10.2 Relation to invertibility

proposition *invertible_det_nz*:
fixes *A*::′*a*::{*field*}ˆ′*n*ˆ′*n*
shows *invertible A* ⟷ *det A* ≠ *0*
proof (*cases invertible A*)
case *True*
then obtain *B* :: ′*a*ˆ′*n*ˆ′*n* **where** *B*: *A* ∗∗ *B* = *mat 1*
unfolding *invertible_right_inverse* **by** *blast*
then have *det* (*A* ∗∗ *B*) = *det* (*mat 1* :: ′*a*ˆ′*n*ˆ′*n*)
by *simp*
then show *?thesis*
by (*metis True det_I det_mul mult_zero_left one_neq_zero*)
next
case *False*
let *?U* = *UNIV* :: ′*n set*
have *fU*: *finite ?U*
by *simp*
from *False* **obtain** *c i* **where** *c*: *sum* (λ*i*. *c i* ∗*s row i A*) *?U* = *0* **and** *iU*: *i* ∈ *?U* **and** *ci*: *c i* ≠ *0*
unfolding *invertible_right_inverse matrix_right_invertible_independent_rows*
by *blast*
have *thr0*: − *row i A* = *sum* (λ*j*. (*1*/ *c i*) ∗*s* (*c j* ∗*s row j A*)) (*?U* − {*i*})
unfolding *sum_cmul* **using** *c ci*
by (*auto simp*: *sum.remove*[*OF fU iU*] *eq_vector_fraction_iff add_eq_0_iff*)
have *thr*: − *row i A* ∈ *vec.span* {*row j A*| *j*. *j* ≠ *i*}
unfolding *thr0* **by** (*auto intro*: *vec.span_base vec.span_scale vec.span_sum*)
let *?B* = (χ *k*. *if k* = *i then 0 else row k A*) :: ′*a*ˆ′*n*ˆ′*n*
have *thrb*: *row i ?B* = *0* **using** *iU* **by** (*vector row_def*)
have *det A* = *0*
unfolding *det_row_span*[*OF thr*, *symmetric*] *right_minus*
unfolding *det_zero_row*(*2*)[*OF thrb*] **..**
then show *?thesis*
by (*simp add*: *False*)
qed

lemma *det_nz_iff_inj_gen*:
fixes *f* :: ′*a*::*field*ˆ′*n* ⇒ ′*a*::*field*ˆ′*n*
assumes *Vector_Spaces.linear* (∗*s*) (∗*s*) *f*
shows *det* (*matrix f*) ≠ *0* ⟷ *inj f*

proof
  assume *det* (*matrix f*) ≠ *0*
  then show *inj f*
    using *assms invertible_det_nz inj_matrix_vector_mult* **by** *force*
next
  assume *inj f*
  show *det* (*matrix f*) ≠ *0*
    using *vec.linear_injective_left_inverse* [*OF assms* ‹*inj f*›]
   by (*metis assms invertible_det_nz invertible_left_inverse matrix_compose_gen matrix_id_mat_1*)
qed

lemma *det_nz_iff_inj*:
  fixes *f* :: *real*^'*n* ⇒ *real*^'*n*
  assumes *linear f*
  shows *det* (*matrix f*) ≠ *0* ⟷ *inj f*
  using *det_nz_iff_inj_gen*[*of f*] *assms*
  unfolding *linear_matrix_vector_mul_eq* **.**

lemma *det_eq_0_rank*:
  fixes *A* :: *real*^'*n*^'*n*
  shows *det A* = *0* ⟷ *rank A* < *CARD*('*n*)
  using *invertible_det_nz* [*of A*]
 by (*auto simp*: *matrix_left_invertible_injective invertible_left_inverse less_rank_noninjective*)

Invertibility of matrices and corresponding linear functions

lemma *matrix_left_invertible_gen*:
  fixes *f* :: '*a*::*field*^'*m* ⇒ '*a*::*field*^'*n*
  assumes *Vector_Spaces.linear* (∗*s*) (∗*s*) *f*
  shows ((∃ *B*. *B* ∗∗ *matrix f* = *mat 1*) ⟷ (∃ *g*. *Vector_Spaces.linear* (∗*s*) (∗*s*) *g* ∧ *g* ∘ *f* = *id*))
proof *safe*
  fix *B*
  assume *1*: *B* ∗∗ *matrix f* = *mat 1*
  show ∃ *g*. *Vector_Spaces.linear* (∗*s*) (∗*s*) *g* ∧ *g* ∘ *f* = *id*
  proof (*intro exI conjI*)
    show *Vector_Spaces.linear* (∗*s*) (∗*s*) (λ*y*. *B* ∗*v y*)
      by *simp*
    show ((∗*v*) *B*) ∘ *f* = *id*
      unfolding *o_def*
        by (*metis assms 1 eq_id_iff matrix_vector_mul*(*1*) *matrix_vector_mul_assoc matrix_vector_mul_lid*)
  qed
next
  fix *g*
  assume *Vector_Spaces.linear* (∗*s*) (∗*s*) *g g* ∘ *f* = *id*
  then have *matrix g* ∗∗ *matrix f* = *mat 1*
    by (*metis assms matrix_compose_gen matrix_id_mat_1*)

then show ∃ *B*. *B* ∗∗ *matrix f* = *mat 1* **..**
qed

lemma *matrix_left_invertible*:
linear f ⟹ ((∃ *B*. *B* ∗∗ *matrix f* = *mat 1*) ⟷ (∃ *g*. *linear g* ∧ *g* ∘ *f* = *id*))
for *f*::*real*^'*m* ⇒ *real*^'*n*
using *matrix_left_invertible_gen*[*of f*]
by (*auto simp*: *linear_matrix_vector_mul_eq*)

lemma *matrix_right_invertible_gen*:
fixes *f* :: '*a*::*field*^'*m* ⇒ '*a*^'*n*
assumes *Vector_Spaces.linear* (∗*s*) (∗*s*) *f*
shows ((∃ *B*. *matrix f* ∗∗ *B* = *mat 1*) ⟷ (∃ *g*. *Vector_Spaces.linear* (∗*s*) (∗*s*) *g* ∧ *f* ∘ *g* = *id*))
proof *safe*
fix *B*
assume *1*: *matrix f* ∗∗ *B* = *mat 1*
show ∃ *g*. *Vector_Spaces.linear* (∗*s*) (∗*s*) *g* ∧ *f* ∘ *g* = *id*
proof (*intro exI conjI*)
show *Vector_Spaces.linear* (∗*s*) (∗*s*) ((∗*v*) *B*)
by *simp*
show *f* ∘ (∗*v*) *B* = *id*
using *1 assms comp_apply eq_id_iff vec.linear_id matrix_id_mat_1 matrix_vector_mul_assoc matrix_works*
by (*metis* (*no_types*, *hide_lams*))
qed
next
fix *g*
assume *Vector_Spaces.linear* (∗*s*) (∗*s*) *g* **and** *f* ∘ *g* = *id*
then have *matrix f* ∗∗ *matrix g* = *mat 1*
by (*metis assms matrix_compose_gen matrix_id_mat_1*)
then show ∃ *B*. *matrix f* ∗∗ *B* = *mat 1* **..**
qed

lemma *matrix_right_invertible*:
linear f ⟹ ((∃ *B*. *matrix f* ∗∗ *B* = *mat 1*) ⟷ (∃ *g*. *linear g* ∧ *f* ∘ *g* = *id*))
for *f*::*real*^'*m* ⇒ *real*^'*n*
using *matrix_right_invertible_gen*[*of f*]
by (*auto simp*: *linear_matrix_vector_mul_eq*)

lemma *matrix_invertible_gen*:
fixes *f* :: '*a*::*field*^'*m* ⇒ '*a*::*field*^'*n*
assumes *Vector_Spaces.linear* (∗*s*) (∗*s*) *f*
shows *invertible* (*matrix f*) ⟷ (∃ *g*. *Vector_Spaces.linear* (∗*s*) (∗*s*) *g* ∧ *f* ∘ *g* = *id* ∧ *g* ∘ *f* = *id*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*metis assms invertible_def left_right_inverse_eq matrix_left_invertible_gen*

matrix_right_invertible_gen)
next
  assume *?rhs* **then show** *?lhs*
    by (*metis assms invertible_def matrix_compose_gen matrix_id_mat_1*)
qed

lemma *matrix_invertible*:
  linear f $\Longrightarrow$ *invertible* (*matrix f*) $\longleftrightarrow$ ($\exists$ *g*. *linear g* $\wedge$ *f* $\circ$ *g* = *id* $\wedge$ *g* $\circ$ *f* = *id*)
  for *f*::*real^'m* $\Rightarrow$ *real^'n*
  using *matrix_invertible_gen*[*of f*]
  by (*auto simp*: *linear_matrix_vector_mul_eq*)

lemma *invertible_eq_bij*:
  fixes *m* :: *'a::field^'m^'n*
  shows *invertible m* $\longleftrightarrow$ *bij* ((*v) *m*)
  using *matrix_invertible_gen*[*OF matrix_vector_mul_linear_gen*, *of m*, *simplified matrix_of_matrix_vector_mul*]
  by (*metis bij_betw_def left_right_inverse_eq matrix_vector_mul_linear_gen o_bij vec.linear_injective_left_inverse vec.linear_surjective_right_inverse*)

1.10.3 Cramer's rule

lemma *cramer_lemma_transpose*:
  fixes *A*:: *'a::{field}^'n^'n*
    and *x* :: *'a::{field}^'n*
  shows *det* ((χ *i*. *if i* = *k then sum* (λ*i*. *x*\$*i* **s row i A*) (*UNIV*::*'n set*)
                    else row i A)::*'a::{field}^'n^'n*) = *x*\$*k* * *det A*
  (**is** *?lhs* = *?rhs*)
proof –
  let *?U* = *UNIV* :: *'n set*
  let *?Uk* = *?U* – {*k*}
  have *U*: *?U* = *insert k ?Uk*
    by *blast*
  have *kUk*: *k* $\notin$ *?Uk*
    by *simp*
  have *th00*: $\bigwedge$*k s*. *x*\$*k* **s row k A* + *s* = (*x*\$*k* – *1*) **s row k A* + *row k A* + *s*
    by (*vector field_simps*)
  have *th001*: $\bigwedge$*f k* . (λ*x*. *if x* = *k then f k else f x*) = *f*
    by *auto*
  have (χ *i*. *row i A*) = *A* **by** (*vector row_def*)
  then have *thd1*: *det* (χ *i*. *row i A*) = *det A*
    by *simp*
  have *thd0*: *det* (χ *i*. *if i* = *k then row k A* + ($\sum$ *i* $\in$ *?Uk*. *x* \$ *i* **s row i A*) *else row i A*) = *det A*
    by (*force intro*: *det_row_span vec.span_sum vec.span_scale vec.span_base*)
  show *?lhs* = *x*\$*k* * *det A*
    apply (*subst U*)
    unfolding *sum.insert*[*OF finite kUk*]
    apply (*subst th00*)

unfolding *add.assoc*
apply (*subst det_row_add*)
unfolding *thd0*
unfolding *det_row_mul*
unfolding *th001*[*of k λi. row i A*]
unfolding *thd1*
apply (*simp add*: *field_simps*)
done
qed

proposition *cramer_lemma*:
fixes *A* :: *'a*::{*field*} ^*'n*^*'n*
shows *det*((*χ i j. if j = k then* (*A* ∗*v x*)\$*i else A*\$*i*\$*j*):: *'a*::{*field*} ^*'n*^*'n*) = *x*\$*k* ∗ *det A*
proof −
let *?U* = *UNIV* :: *'n set*
have ∗: ⋀*c. sum* (*λi. c i* ∗*s row i* (*transpose A*)) *?U* = *sum* (*λi. c i* ∗*s column i A*) *?U*
by (*auto intro*: *sum.cong*)
show *?thesis*
unfolding *matrix_mult_sum*
unfolding *cramer_lemma_transpose*[*of k x transpose A*, *unfolded det_transpose*, *symmetric*]
unfolding ∗[*of λi. x*\$*i*]
apply (*subst det_transpose*[*symmetric*])
apply (*rule cong*[*OF refl*[*of det*]])
apply (*vector transpose_def column_def row_def*)
done
qed

proposition *cramer*:
fixes *A* ::*'a*::{*field*} ^*'n*^*'n*
assumes *d0*: *det A* ≠ *0*
shows *A* ∗*v x* = *b* ⟷ *x* = (*χ k. det*(*χ i j. if j=k then b*\$*i else A*\$*i*\$*j*) / *det A*)
proof −
from *d0* **obtain** *B* **where** *B*: *A* ∗∗ *B* = *mat 1 B* ∗∗ *A* = *mat 1*
unfolding *invertible_det_nz*[*symmetric*] *invertible_def*
by *blast*
have (*A* ∗∗ *B*) ∗*v b* = *b*
by (*simp add*: *B*)
then have *A* ∗*v* (*B* ∗*v b*) = *b*
by (*simp add*: *matrix_vector_mul_assoc*)
then have *xe*: ∃*x. A* ∗*v x* = *b*
by *blast*
{
fix *x*
assume *x*: *A* ∗*v x* = *b*
have *x* = (*χ k. det*(*χ i j. if j=k then b*\$*i else A*\$*i*\$*j*) / *det A*)
unfolding *x*[*symmetric*]

```
    using d0 by (simp add: vec_eq_iff cramer_lemma field_simps)
  }
  with xe show ?thesis
    by auto
qed

lemma det_1: det (A::'a::comm_ring_1^1^1) = A$1$1
  by (simp add: det_def sign_id)

lemma det_2: det (A::'a::comm_ring_1^2^2) = A$1$1 * A$2$2 - A$1$2 *
A$2$1
proof -
  have f12: finite {2::2} 1 ∉ {2::2} by auto
  show ?thesis
    unfolding det_def UNIV_2
    unfolding sum_over_permutations_insert[OF f12]
    unfolding permutes_sing
    by (simp add: sign_swap_id sign_id swap_id_eq)
qed

lemma det_3:
  det (A::'a::comm_ring_1^3^3) =
    A$1$1 * A$2$2 * A$3$3 +
    A$1$2 * A$2$3 * A$3$1 +
    A$1$3 * A$2$1 * A$3$2 -
    A$1$1 * A$2$3 * A$3$2 -
    A$1$2 * A$2$1 * A$3$3 -
    A$1$3 * A$2$2 * A$3$1
proof -
  have f123: finite {2::3, 3} 1 ∉ {2::3, 3}
    by auto
  have f23: finite {3::3} 2 ∉ {3::3}
    by auto

  show ?thesis
    unfolding det_def UNIV_3
    unfolding sum_over_permutations_insert[OF f123]
    unfolding sum_over_permutations_insert[OF f23]
    unfolding permutes_sing
   by (simp add: sign_swap_id permutation_swap_id sign_compose sign_id swap_id_eq)
qed

proposition det_orthogonal_matrix:
  fixes Q:: 'a::linordered_idom^'n^'n
  assumes oQ: orthogonal_matrix Q
  shows det Q = 1 ∨ det Q = - 1
proof -
  have Q ** transpose Q = mat 1
    by (metis oQ orthogonal_matrix_def)
```

then have *det* (*Q* ∗∗ *transpose Q*) = *det* (*mat 1*:: ′*a*ˆ′*n*ˆ′*n*)
  by *simp*
then have *det Q* ∗ *det Q* = *1*
  by (*simp add*: *det_mul*)
then show *?thesis*
  by (*simp add*: *square_eq_1_iff*)
qed

proposition *orthogonal_transformation_det* [*simp*]:
  fixes *f* :: *real*ˆ′*n* ⇒ *real*ˆ′*n*
  shows *orthogonal_transformation f* ⟹ |*det* (*matrix f*)| = *1*
  using *det_orthogonal_matrix orthogonal_transformation_matrix* **by** *fastforce*

1.10.4 Rotation, reflection, rotoinversion

definition *rotation_matrix Q* ⟷ *orthogonal_matrix Q* ∧ *det Q* = *1*
definition *rotoinversion_matrix Q* ⟷ *orthogonal_matrix Q* ∧ *det Q* = − *1*

lemma *orthogonal_rotation_or_rotoinversion*:
  fixes *Q* :: ′*a*::*linordered_idom*ˆ′*n*ˆ′*n*
  shows *orthogonal_matrix Q* ⟷ *rotation_matrix Q* ∨ *rotoinversion_matrix Q*
  by (*metis rotoinversion_matrix_def rotation_matrix_def det_orthogonal_matrix*)

Slightly stronger results giving rotation, but only in two or more dimensions

lemma *rotation_matrix_exists_basis*:
  fixes *a* :: *real*ˆ′*n*
  assumes *2*: *2* ≤ *CARD*(′*n*) **and** *norm a* = *1*
  obtains *A* **where** *rotation_matrix A A* ∗*v* (*axis k 1*) = *a*
proof −
  obtain *A* **where** *orthogonal_matrix A* **and** *A*: *A* ∗*v* (*axis k 1*) = *a*
    using *orthogonal_matrix_exists_basis assms* **by** *metis*
  with *orthogonal_rotation_or_rotoinversion*
  consider *rotation_matrix A* | *rotoinversion_matrix A*
    by *metis*
  then show *thesis*
  proof *cases*
    assume *rotation_matrix A*
    then show *?thesis*
      using ‹*A* ∗*v axis k 1* = *a*› *that* **by** *auto*
  next
    from *ex_card*[*OF 2*] **obtain** *h i*::′*n* **where** *h* ≠ *i*
      by (*auto simp add*: *eval_nat_numeral card_Suc_eq*)
    then obtain *j* **where** *j* ≠ *k*
      by (*metis* (*full_types*))
    let *?TA* = *transpose A*
    let *?A* = χ *i*. *if i* = *j then* − *1* $*_R$ (*?TA* $ *i*) *else ?TA* $*i*
    assume *rotoinversion_matrix A*
    then have [*simp*]: *det A* = −*1*
      by (*simp add*: *rotoinversion_matrix_def*)

```
    show ?thesis
    proof
      have [simp]: row i (χ i. if i = j then − 1 *R ?TA $ i else ?TA $ i) = (if i
= j then − row i ?TA else row i ?TA) for i
        by (auto simp: row_def)
      have orthogonal_matrix ?A
        unfolding orthogonal_matrix_orthonormal_rows
      using ‹orthogonal_matrix A› by (auto simp: orthogonal_matrix_orthonormal_columns
orthogonal_clauses)
      then show rotation_matrix (transpose ?A)
        unfolding rotation_matrix_def
       by (simp add: det_row_mul[of j _ λi. ?TA $ i, unfolded scalar_mult_eq_scaleR])
      show transpose ?A *v axis k 1 = a
      using ‹j ≠ k› A by (simp add: matrix_vector_column axis_def scalar_mult_eq_scaleR
if_distrib [of λz. z *R c for c] cong: if_cong)
    qed
  qed
qed

lemma rotation_exists_1:
  fixes a :: real^'n
  assumes 2 ≤ CARD('n) norm a = 1 norm b = 1
  obtains f where orthogonal_transformation f det(matrix f) = 1 f a = b
proof −
  obtain k::'n where True
    by simp
  obtain A B where AB: rotation_matrix A rotation_matrix B
              and eq: A *v (axis k 1) = a B *v (axis k 1) = b
    using rotation_matrix_exists_basis assms by metis
  let ?f = λx. (B ** transpose A) *v x
  show thesis
  proof
    show orthogonal_transformation ?f
     using AB orthogonal_matrix_mul orthogonal_transformation_matrix rotation_matrix_def
matrix_vector_mul_linear by force
    show det (matrix ?f) = 1
      using AB by (auto simp: det_mul rotation_matrix_def)
    show ?f a = b
      using AB unfolding orthogonal_matrix_def rotation_matrix_def
      by (metis eq matrix_mul_rid matrix_vector_mul_assoc)
  qed
qed

lemma rotation_exists:
  fixes a :: real^'n
  assumes 2: 2 ≤ CARD('n) and eq: norm a = norm b
  obtains f where orthogonal_transformation f det(matrix f) = 1 f a = b
proof (cases a = 0 ∨ b = 0)
  case True
```

```
  with assms have a = 0 b = 0
    by auto
  then show ?thesis
    by (metis eq_id_iff matrix_id orthogonal_transformation_id that)
next
  case False
  then obtain f where f: orthogonal_transformation f det (matrix f) = 1
    and f': f (a /R norm a) = b /R norm b
    using rotation_exists_1 [of a /R norm a b /R norm b, OF 2] by auto
  then interpret linear f by (simp add: orthogonal_transformation)
  have f a = b
    using f' False
    by (simp add: eq scale)
  with f show thesis ..
qed

lemma rotation_rightward_line:
  fixes a :: real^'n
  obtains f where orthogonal_transformation f 2 ≤ CARD('n) ⟹ det(matrix f)
= 1
                f(norm a *R axis k 1) = a
proof (cases CARD('n) = 1)
  case True
  obtain f where orthogonal_transformation f f (norm a *R axis k (1::real)) = a
  proof (rule orthogonal_transformation_exists)
    show norm (norm a *R axis k (1::real)) = norm a
      by simp
  qed auto
  then show thesis
    using True that by auto
next
  case False
  obtain f where orthogonal_transformation f det(matrix f) = 1 f (norm a *R
axis k 1) = a
  proof (rule rotation_exists)
    show 2 ≤ CARD('n)
      using False one_le_card_finite [where 'a='n] by linarith
    show norm (norm a *R axis k (1::real)) = norm a
      by simp
  qed auto
  then show thesis
    using that by blast
qed

end
```

Chapter 2

Topology

theory *Elementary_Topology*
imports
  HOL−Library.Set_Idioms
  HOL−Library.Disjoint_Sets
  Product_Vector
begin

2.1 Elementary Topology

Affine transformations of intervals

lemma *real_affinity_le*: $0 < m \Longrightarrow m * x + c \leq y \longleftrightarrow x \leq inverse\ m * y + - (c / m)$
  for m :: $'a$::*linordered_field*
  by (*simp add*: *field_simps*)

lemma *real_le_affinity*: $0 < m \Longrightarrow y \leq m * x + c \longleftrightarrow inverse\ m * y + - (c / m) \leq x$
  for m :: $'a$::*linordered_field*
  by (*simp add*: *field_simps*)

lemma *real_affinity_lt*: $0 < m \Longrightarrow m * x + c < y \longleftrightarrow x < inverse\ m * y + - (c / m)$
  for m :: $'a$::*linordered_field*
  by (*simp add*: *field_simps*)

lemma *real_lt_affinity*: $0 < m \Longrightarrow y < m * x + c \longleftrightarrow inverse\ m * y + - (c / m) < x$
  for m :: $'a$::*linordered_field*
  by (*simp add*: *field_simps*)

lemma *real_affinity_eq*: $m \neq 0 \Longrightarrow m * x + c = y \longleftrightarrow x = inverse\ m * y + - (c / m)$
  for m :: $'a$::*linordered_field*

```
  by (simp add: field_simps)

lemma real_eq_affinity: m ≠ 0 ⟹ y = m * x + c ⟷ inverse m * y + − (c /
m) = x
  for m :: 'a::linordered_field
  by (simp add: field_simps)
```

2.1.1 Topological Basis

```
context topological_space
begin

definition topological_basis B ⟷
  (∀ b∈B. open b) ∧ (∀ x. open x ⟶ (∃ B'. B' ⊆ B ∧ ⋃ B' = x))

lemma topological_basis:
  topological_basis B ⟷ (∀ x. open x ⟷ (∃ B'. B' ⊆ B ∧ ⋃ B' = x))
  unfolding topological_basis_def
  apply safe
     apply fastforce
    apply fastforce
   apply (erule_tac x=x in allE, simp)
   apply (rule_tac x={x} in exI, auto)
  done

lemma topological_basis_iff:
  assumes ⋀B'. B' ∈ B ⟹ open B'
  shows topological_basis B ⟷ (∀ O'. open O' ⟶ (∀ x∈O'. ∃ B'∈B. x ∈ B' ∧
B' ⊆ O'))
    (is _ ⟷ ?rhs)
proof safe
  fix O' and x::'a
  assume H: topological_basis B open O' x ∈ O'
  then have (∃ B'⊆B. ⋃ B' = O') by (simp add: topological_basis_def)
  then obtain B' where B' ⊆ B O' = ⋃ B' by auto
  then show ∃ B'∈B. x ∈ B' ∧ B' ⊆ O' using H by auto
next
  assume H: ?rhs
  show topological_basis B
    using assms unfolding topological_basis_def
  proof safe
    fix O' :: 'a set
    assume open O'
    with H obtain f where ∀ x∈O'. f x ∈ B ∧ x ∈ f x ∧ f x ⊆ O'
      by (force intro: bchoice simp: Bex_def)
    then show ∃ B'⊆B. ⋃ B' = O'
      by (auto intro: exI[where x={f x |x. x ∈ O'}])
  qed
qed
```

lemma *topological_basisI*:
  assumes $\bigwedge B'.\ B' \in B \Longrightarrow open\ B'$
    and $\bigwedge O'\ x.\ open\ O' \Longrightarrow x \in O' \Longrightarrow \exists B' \in B.\ x \in B' \wedge B' \subseteq O'$
  shows *topological_basis B*
  using *assms* **by** (*subst topological_basis_iff*) *auto*

lemma *topological_basisE*:
  fixes O'
  assumes *topological_basis B*
    and *open* O'
    and $x \in O'$
  obtains B' **where** $B' \in B\ x \in B'\ B' \subseteq O'$
proof *atomize_elim*
  from *assms* **have** $\bigwedge B'.\ B' \in B \Longrightarrow open\ B'$
    by (*simp add*: *topological_basis_def*)
  with *topological_basis_iff assms*
  show $\exists B'.\ B' \in B \wedge x \in B' \wedge B' \subseteq O'$
    using *assms* **by** (*simp add*: *Bex_def*)
qed

lemma *topological_basis_open*:
  assumes *topological_basis B*
    and $X \in B$
  shows *open X*
  using *assms* **by** (*simp add*: *topological_basis_def*)

lemma *topological_basis_imp_subbasis*:
  assumes *B*: *topological_basis B*
  shows *open* = *generate_topology B*
proof (*intro ext iffI*)
  fix *S* :: *'a set*
  assume *open S*
  with *B* **obtain** B' **where** $B' \subseteq B\ S = \bigcup B'$
    unfolding *topological_basis_def* **by** *blast*
  then show *generate_topology B S*
    by (*auto intro*: *generate_topology.intros dest*: *topological_basis_open*)
next
  fix *S* :: *'a set*
  assume *generate_topology B S*
  then show *open S*
    by *induct* (*auto dest*: *topological_basis_open*[*OF B*])
qed

lemma *basis_dense*:
  fixes *B* :: *'a set set*
    and *f* :: *'a set* ⇒ *'a*
  assumes *topological_basis B*
    and *choosefrom_basis*: $\bigwedge B'.\ B' \neq \{\} \Longrightarrow f\ B' \in B'$

shows $\forall X.$ *open* $X \longrightarrow X \neq \{\} \longrightarrow (\exists B' \in B.\ f\ B' \in X)$
proof (*intro allI impI*)
fix X :: *'a set*
assume *open* X **and** $X \neq \{\}$
from *topological_basisE*[*OF* ‹*topological_basis B*› ‹*open X*› *choosefrom_basis*[*OF* ‹$X \neq \{\}$›]]
obtain B' **where** $B' \in B$ $f\ X \in B'$ $B' \subseteq X$ **.**
then show $\exists B' \in B.\ f\ B' \in X$
by (*auto intro*!: *choosefrom_basis*)
qed

end

lemma *topological_basis_prod*:
assumes *A*: *topological_basis A*
and *B*: *topological_basis B*
shows *topological_basis* $((\lambda(a, b).\ a \times b)$ ‘ $(A \times B))$
unfolding *topological_basis_def*
proof (*safe*, *simp_all del*: *ex_simps add*: *subset_image_iff ex_simps*(*1*)[*symmetric*])
fix S :: $('a \times 'b)$ *set*
assume *open S*
then show $\exists X \subseteq A \times B.\ (\bigcup (a,b) \in X.\ a \times b) = S$
proof (*safe intro*!: *exI*[*of* _ $\{x \in A \times B.$ *fst* $x \times$ *snd* $x \subseteq S\}$])
fix *x y*
assume $(x, y) \in S$
from *open_prod_elim*[*OF* ‹*open S*› *this*]
obtain *a b* **where** *a*: *open ax* $\in a$ **and** *b*: *open b* $y \in b$ **and** $a \times b \subseteq S$
by (*metis mem_Sigma_iff*)
moreover
from *A a* **obtain** *A0* **where** $A0 \in A$ $x \in A0$ $A0 \subseteq a$
by (*rule topological_basisE*)
moreover
from *B b* **obtain** *B0* **where** $B0 \in B$ $y \in B0$ $B0 \subseteq b$
by (*rule topological_basisE*)
ultimately show $(x, y) \in (\bigcup (a, b) \in \{X \in A \times B.$ *fst* $X \times$ *snd* $X \subseteq S\}.\ a \times b)$
by (*intro UN_I*[*of* (*A0*, *B0*)]) *auto*
qed *auto*
qed (*metis A B topological_basis_open open_Times*)

2.1.2 Countable Basis

locale *countable_basis* = *topological_space p* **for** *p*::*'a set* $\Rightarrow$ *bool* +
fixes *B* :: *'a set set*
assumes *is_basis*: *topological_basis B*
and *countable_basis*: *countable B*
begin

lemma *open_countable_basis_ex*:

```
  assumes p X
  shows ∃B' ⊆ B. X = ⋃B'
  using assms countable_basis is_basis
  unfolding topological_basis_def by blast

lemma open_countable_basisE:
  assumes p X
  obtains B' where B' ⊆ B X = ⋃B'
  using assms open_countable_basis_ex
  by atomize_elim simp

lemma countable_dense_exists:
  ∃D::'a set. countable D ∧ (∀X. p X ⟶ X ≠ {} ⟶ (∃d ∈ D. d ∈ X))
proof –
  let ?f = (λB'. SOME x. x ∈ B')
  have countable (?f ` B) using countable_basis by simp
  with basis_dense[OF is_basis, of ?f] show ?thesis
    by (intro exI[where x=?f ` B]) (metis (mono_tags) all_not_in_conv imageI
someI)
qed

lemma countable_dense_setE:
  obtains D :: 'a set
  where countable D ⋀X. p X ⟹ X ≠ {} ⟹ ∃d ∈ D. d ∈ X
  using countable_dense_exists by blast

end

lemma countable_basis_openI: countable_basis open B
  if countable B topological_basis B
  using that
  by unfold_locales
   (simp_all add: topological_basis topological_space.topological_basis topological_space_axioms)

lemma (in first_countable_topology) first_countable_basisE:
  fixes x :: 'a
  obtains 𝒜 where countable 𝒜 ⋀A. A ∈ 𝒜 ⟹ x ∈ A ⋀A. A ∈ 𝒜 ⟹ open A
    ⋀S. open S ⟹ x ∈ S ⟹ (∃A∈𝒜. A ⊆ S)
proof –
  obtain 𝒜 where 𝒜: (∀i::nat. x ∈ 𝒜 i ∧ open (𝒜 i)) (∀S. open S ∧ x ∈ S ⟶
(∃i. 𝒜 i ⊆ S))
    using first_countable_basis[of x] by metis
  show thesis
  proof
    show countable (range 𝒜)
      by simp
  qed (use 𝒜 in auto)
qed
```

lemma (**in** *first_countable_topology*) *first_countable_basis_Int_stableE*:
obtains $\mathcal{A}$ **where** *countable* $\mathcal{A}$ $\bigwedge A.\ A \in \mathcal{A} \Longrightarrow x \in A$ $\bigwedge A.\ A \in \mathcal{A} \Longrightarrow$ *open* A
$\bigwedge S.$ *open* $S \Longrightarrow x \in S \Longrightarrow (\exists A\in\mathcal{A}.\ A \subseteq S)$
$\bigwedge A\ B.\ A \in \mathcal{A} \Longrightarrow B \in \mathcal{A} \Longrightarrow A \cap B \in \mathcal{A}$
proof *atomize_elim*
obtain $\mathcal{B}$ **where** $\mathcal{B}$:
countable $\mathcal{B}$
$\bigwedge B.\ B \in \mathcal{B} \Longrightarrow x \in B$
$\bigwedge B.\ B \in \mathcal{B} \Longrightarrow$ *open* B
$\bigwedge S.$ *open* $S \Longrightarrow x \in S \Longrightarrow \exists B\in\mathcal{B}.\ B \subseteq S$
by (*rule first_countable_basisE*) *blast*
define $\mathcal{A}$ **where** [*abs_def*]:
$\mathcal{A} = (\lambda N. \bigcap((\lambda n.$ *from_nat_into* $\mathcal{B}$ $n)$ ‘ $N))$ ‘ (*Collect finite*::*nat set set*)
then show $\exists\mathcal{A}.$ *countable* $\mathcal{A} \wedge (\forall A.\ A \in \mathcal{A} \longrightarrow x \in A) \wedge (\forall A.\ A \in \mathcal{A} \longrightarrow$ *open* $A) \wedge$
$(\forall S.$ *open* $S \longrightarrow x \in S \longrightarrow (\exists A\in\mathcal{A}.\ A \subseteq S)) \wedge (\forall A\ B.\ A \in \mathcal{A} \longrightarrow B \in \mathcal{A} \longrightarrow A \cap B \in \mathcal{A})$
proof (*safe intro*!: *exI*[**where** $x=\mathcal{A}$])
show *countable* $\mathcal{A}$
unfolding $\mathcal{A}$*_def* **by** (*intro countable_image countable_Collect_finite*)
fix A
assume $A \in \mathcal{A}$
then show $x \in A$ *open* A
using $\mathcal{B}$(*4*)[*OF open_UNIV*] **by** (*auto simp*: $\mathcal{A}$*_def intro*: $\mathcal{B}$ *from_nat_into*)
next
let *?int* $= \lambda N. \bigcap($*from_nat_into* $\mathcal{B}$ ‘ $N)$
fix $A\ B$
assume $A \in \mathcal{A}$ $B \in \mathcal{A}$
then obtain $N\ M$ **where** $A =$ *?int* N $B =$ *?int* M *finite* $(N \cup M)$
by (*auto simp*: $\mathcal{A}$*_def*)
then show $A \cap B \in \mathcal{A}$
by (*auto simp*: $\mathcal{A}$*_def intro*!: *image_eqI*[**where** $x=N \cup M$])
next
fix S
assume *open* S $x \in S$
then obtain a **where** a: $a\in\mathcal{B}$ $a \subseteq S$ **using** $\mathcal{B}$ **by** *blast*
then show $\exists a\in\mathcal{A}.\ a \subseteq S$ **using** a $\mathcal{B}$
by (*intro bexI*[**where** $x=a$]) (*auto simp*: $\mathcal{A}$*_def intro*: *image_eqI*[**where** $x=\{$*to_nat_on* $\mathcal{B}$ $a\}$])
qed
qed

lemma (**in** *topological_space*) *first_countableI*:
assumes *countable* $\mathcal{A}$
and *1*: $\bigwedge A.\ A \in \mathcal{A} \Longrightarrow x \in A$ $\bigwedge A.\ A \in \mathcal{A} \Longrightarrow$ *open* A
and *2*: $\bigwedge S.$ *open* $S \Longrightarrow x \in S \Longrightarrow \exists A\in\mathcal{A}.\ A \subseteq S$
shows $\exists\mathcal{A}::nat \Rightarrow 'a\ set.\ (\forall i.\ x \in \mathcal{A}\ i \wedge$ *open* $(\mathcal{A}\ i)) \wedge (\forall S.$ *open* $S \wedge x \in S \longrightarrow (\exists i.\ \mathcal{A}\ i \subseteq S))$
proof (*safe intro*!: *exI*[*of _ from_nat_into* $\mathcal{A}$])

```
  fix i
  have 𝒜 ≠ {} using 2[of UNIV] by auto
  show x ∈ from_nat_into 𝒜 i open (from_nat_into 𝒜 i)
    using range_from_nat_into_subset[OF ‹𝒜 ≠ {}›] 1 by auto
next
  fix S
  assume open S x∈S from 2[OF this]
  show ∃ i. from_nat_into 𝒜 i ⊆ S
    using subset_range_from_nat_into[OF ‹countable 𝒜›] by auto
qed

instance prod :: (first_countable_topology, first_countable_topology) first_countable_topology
proof
  fix x :: 'a × 'b
  obtain 𝒜 where 𝒜:
      countable 𝒜
      ⋀a. a ∈ 𝒜 ⟹ fst x ∈ a
      ⋀a. a ∈ 𝒜 ⟹ open a
      ⋀S. open S ⟹ fst x ∈ S ⟹ ∃ a∈𝒜. a ⊆ S
    by (rule first_countable_basisE[of fst x]) blast
  obtain B where B:
      countable B
      ⋀a. a ∈ B ⟹ snd x ∈ a
      ⋀a. a ∈ B ⟹ open a
      ⋀S. open S ⟹ snd x ∈ S ⟹ ∃ a∈B. a ⊆ S
    by (rule first_countable_basisE[of snd x]) blast
  show ∃ 𝒜::nat ⇒ ('a × 'b) set.
    (∀ i. x ∈ 𝒜 i ∧ open (𝒜 i)) ∧ (∀ S. open S ∧ x ∈ S ⟶ (∃ i. 𝒜 i ⊆ S))
  proof (rule first_countableI[of (λ(a, b). a × b) ` (𝒜 × B)], safe)
    fix a b
    assume x: a ∈ 𝒜 b ∈ B
    show x ∈ a × b
      by (simp add: 𝒜(2) B(2) mem_Times_iff x)
    show open (a × b)
      by (simp add: 𝒜(3) B(3) open_Times x)
  next
    fix S
    assume open S x ∈ S
    then obtain a' b' where a'b': open a' open b' x ∈ a' × b' a' × b' ⊆ S
      by (rule open_prod_elim)
    moreover
    from a'b' 𝒜(4)[of a'] B(4)[of b']
    obtain a b where a ∈ 𝒜 a ⊆ a' b ∈ B b ⊆ b'
      by auto
    ultimately
    show ∃ a∈(λ(a, b). a × b) ` (𝒜 × B). a ⊆ S
      by (auto intro!: bexI[of _ a × b] bexI[of _ a] bexI[of _ b])
  qed (simp add: 𝒜 B)
qed
```

```
class second_countable_topology = topological_space +
  assumes ex_countable_subbasis:
    ∃ B::'a set set. countable B ∧ open = generate_topology B
begin

lemma ex_countable_basis: ∃ B::'a set set. countable B ∧ topological_basis B
proof –
   from ex_countable_subbasis obtain B where B: countable B open = generate_topology B
    by blast
  let ?B = Inter ‘ {b. finite b ∧ b ⊆ B }

  show ?thesis
  proof (intro exI conjI)
    show countable ?B
      by (intro countable_image countable_Collect_finite_subset B)
    {
      fix S
      assume open S
      then have ∃ B'⊆{b. finite b ∧ b ⊆ B}. (⋃ b∈B'. ⋂ b) = S
        unfolding B
      proof induct
        case UNIV
        show ?case by (intro exI[of _ {{}}]) simp
      next
        case (Int a b)
       then obtain x y where x: a = ⋃(Inter ‘ x) ⋀i. i ∈ x ⟹ finite i ∧ i ⊆ B
          and y: b = ⋃(Inter ‘ y) ⋀i. i ∈ y ⟹ finite i ∧ i ⊆ B
          by blast
        show ?case
          unfolding x y Int_UN_distrib2
          by (intro exI[of _ {i ∪ j| i j.  i ∈ x ∧ j ∈ y}]) (auto dest: x(2) y(2))
      next
        case (UN K)
       then have ∀ k∈K. ∃ B'⊆{b. finite b ∧ b ⊆ B}. ⋃ (Inter ‘ B') = k by auto
        then obtain k where
            ∀ ka∈K. k ka ⊆ {b. finite b ∧ b ⊆ B} ∧ ⋃(Inter ‘ (k ka)) = ka
          unfolding bchoice_iff ..
        then show ∃ B'⊆{b. finite b ∧ b ⊆ B}. ⋃ (Inter ‘ B') = ⋃K
          by (intro exI[of _ ⋃(k ‘ K)]) auto
      next
        case (Basis S)
        then show ?case
          by (intro exI[of _ {{S}}]) auto
      qed
      then have (∃ B'⊆Inter ‘ {b. finite b ∧ b ⊆ B}. ⋃B' = S)
        unfolding subset_image_iff by blast }
    then show topological_basis ?B
```

```
    unfolding topological_basis_def
    by (safe intro!: open_Inter)
      (simp_all add: B generate_topology.Basis subset_eq)
  qed
qed


end

lemma univ_second_countable:
  obtains 𝓑 :: 'a::second_countable_topology set set
  where countable 𝓑 ⋀C. C ∈ 𝓑 ⟹ open C
      ⋀S. open S ⟹ ∃ U. U ⊆ 𝓑 ∧ S = ⋃ U
by (metis ex_countable_basis topological_basis_def)

proposition Lindelof:
  fixes 𝓕 :: 'a::second_countable_topology set set
  assumes 𝓕: ⋀S. S ∈ 𝓕 ⟹ open S
  obtains 𝓕' where 𝓕' ⊆ 𝓕 countable 𝓕' ⋃𝓕' = ⋃𝓕
proof –
  obtain 𝓑 :: 'a set set
    where countable 𝓑 ⋀C. C ∈ 𝓑 ⟹ open C
      and 𝓑: ⋀S. open S ⟹ ∃ U. U ⊆ 𝓑 ∧ S = ⋃ U
    using univ_second_countable by blast
  define 𝓓 where 𝓓 ≡ {S. S ∈ 𝓑 ∧ (∃ U. U ∈ 𝓕 ∧ S ⊆ U)}
  have countable 𝓓
    apply (rule countable_subset [OF _ ‹countable 𝓑›])
    apply (force simp: 𝓓_def)
    done
  have ⋀S. ∃ U. S ∈ 𝓓 ⟶ U ∈ 𝓕 ∧ S ⊆ U
    by (simp add: 𝓓_def)
  then obtain G where G: ⋀S. S ∈ 𝓓 ⟶ G S ∈ 𝓕 ∧ S ⊆ G S
    by metis
  have ⋃𝓕 ⊆ ⋃𝓓
    unfolding 𝓓_def by (blast dest: 𝓕 𝓑)
  moreover have ⋃𝓓 ⊆ ⋃𝓕
    using 𝓓_def by blast
  ultimately have eq1: ⋃𝓕 = ⋃𝓓 ..
  have eq2: ⋃𝓓 = ⋃ (G ` 𝓓)
    using G eq1 by auto
  show ?thesis
    apply (rule_tac 𝓕' = G ` 𝓓 in that)
    using G ‹countable 𝓓›
    by (auto simp: eq1 eq2)
qed

lemma countable_disjoint_open_subsets:
  fixes 𝓕 :: 'a::second_countable_topology set set
  assumes ⋀S. S ∈ 𝓕 ⟹ open S and pw: pairwise disjnt 𝓕
```

shows *countable* $\mathcal{F}$
proof −
obtain $\mathcal{F}'$ **where** $\mathcal{F}' \subseteq \mathcal{F}$ *countable* $\mathcal{F}'$ $\bigcup \mathcal{F}' = \bigcup \mathcal{F}$
by (*meson assms Lindelof*)
with *pw* **have** $\mathcal{F} \subseteq$ *insert* {} $\mathcal{F}'$
by (*fastforce simp add*: *pairwise_def disjnt_iff*)
then show *?thesis*
by (*simp add*: ‹*countable* $\mathcal{F}'$› *countable_subset*)
qed

sublocale *second_countable_topology* <
countable_basis open SOME B. countable B ∧ *topological_basis B*
using *someI_ex*[*OF ex_countable_basis*]
by *unfold_locales safe*

instance *prod* :: (*second_countable_topology*, *second_countable_topology*) *second_countable_topology*
proof
obtain *A* :: *'a set set* **where** *countable A topological_basis A*
using *ex_countable_basis* **by** *auto*
moreover
obtain *B* :: *'b set set* **where** *countable B topological_basis B*
using *ex_countable_basis* **by** *auto*
ultimately show ∃ *B*::(*'a* × *'b*) *set set*. *countable B* ∧ *open* = *generate_topology*
B
by (*auto intro*!: *exI*[*of* _ (λ(*a*, *b*). *a* × *b*) ‘ (*A* × *B*)] *topological_basis_prod*
topological_basis_imp_subbasis)
qed

instance *second_countable_topology* ⊆ *first_countable_topology*
proof
fix *x* :: *'a*
define *B* :: *'a set set* **where** *B* = (*SOME B*. *countable B* ∧ *topological_basis B*)
then have *B*: *countable B topological_basis B*
using *countable_basis is_basis*
by (*auto simp*: *countable_basis is_basis*)
then show ∃ *A*::*nat* ⇒ *'a set*.
(∀ *i*. *x* ∈ *A i* ∧ *open* (*A i*)) ∧ (∀ *S*. *open S* ∧ *x* ∈ *S* ⟶ (∃ *i*. *A i* ⊆ *S*))
by (*intro first_countableI*[*of* {*b*∈*B*. *x* ∈ *b*}])
(*fastforce simp*: *topological_space_class.topological_basis_def*)+
qed

instance *nat* :: *second_countable_topology*
proof
show ∃ *B*::*nat set set*. *countable B* ∧ *open* = *generate_topology B*
by (*intro exI*[*of* _ *range lessThan* ∪ *range greaterThan*]) (*auto simp*: *open_nat_def*)
qed

lemma *countable_separating_set_linorder1*:

shows ∃ *B*::(*'a*::{*linorder_topology*, *second_countable_topology*} *set*). *countable B* ∧ (∀ *x y*. *x* < *y* ⟶ (∃ *b* ∈ *B*. *x* < *b* ∧ *b* ≤ *y*))
proof –
obtain *A*::*'a set set* **where** *countable A topological_basis A* **using** *ex_countable_basis* **by** *auto*
define *B1* **where** *B1* = {(*LEAST x*. *x* ∈ *U*)| *U*. *U* ∈ *A*}
then have *countable B1* **using** ‹*countable A*› **by** (*simp add*: *Setcompr_eq_image*)
define *B2* **where** *B2* = {(*SOME x*. *x* ∈ *U*)| *U*. *U* ∈ *A*}
then have *countable B2* **using** ‹*countable A*› **by** (*simp add*: *Setcompr_eq_image*)
have ∃ *b* ∈ *B1* ∪ *B2*. *x* < *b* ∧ *b* ≤ *y* **if** *x* < *y* **for** *x y*
proof (*cases*)
assume ∃ *z*. *x* < *z* ∧ *z* < *y*
then obtain *z* **where** *z*: *x* < *z* ∧ *z* < *y* **by** *auto*
define *U* **where** *U* = {*x*<..<*y*}
then have *open U* **by** *simp*
moreover have *z* ∈ *U* **using** *z U_def* **by** *simp*
ultimately obtain *V* **where** *V* ∈ *A z* ∈ *V V* ⊆ *U*
using *topological_basisE*[*OF* ‹*topological_basis A*›] **by** *auto*
define *w* **where** *w* = (*SOME x*. *x* ∈ *V*)
then have *w* ∈ *V* **using** ‹*z* ∈ *V*› **by** (*metis someI2*)
then have *x* < *w* ∧ *w* ≤ *y* **using** ‹*w* ∈ *V*› ‹*V* ⊆ *U*› *U_def* **by** *fastforce*
moreover have *w* ∈ *B1* ∪ *B2* **using** *w_def B2_def* ‹*V* ∈ *A*› **by** *auto*
ultimately show *?thesis* **by** *auto*
next
assume ¬(∃ *z*. *x* < *z* ∧ *z* < *y*)
then have ∗: ⋀*z*. *z* > *x* ⟹ *z* ≥ *y* **by** *auto*
define *U* **where** *U* = {*x*<..}
then have *open U* **by** *simp*
moreover have *y* ∈ *U* **using** ‹*x* < *y*› *U_def* **by** *simp*
ultimately obtain *V* **where** *V* ∈ *A y* ∈ *V V* ⊆ *U*
using *topological_basisE*[*OF* ‹*topological_basis A*›] **by** *auto*
have *U* = {*y*..} **unfolding** *U_def* **using** ∗ ‹*x* < *y*› **by** *auto*
then have *V* ⊆ {*y*..} **using** ‹*V* ⊆ *U*› **by** *simp*
then have (*LEAST w*. *w* ∈ *V*) = *y* **using** ‹*y* ∈ *V*› **by** (*meson Least_equality atLeast_iff subsetCE*)
then have *y* ∈ *B1* ∪ *B2* **using** ‹*V* ∈ *A*› *B1_def* **by** *auto*
moreover have *x* < *y* ∧ *y* ≤ *y* **using** ‹*x* < *y*› **by** *simp*
ultimately show *?thesis* **by** *auto*
qed
moreover have *countable* (*B1* ∪ *B2*) **using** ‹*countable B1*› ‹*countable B2*› **by** *simp*
ultimately show *?thesis* **by** *auto*
qed

lemma *countable_separating_set_linorder2*:
shows ∃ *B*::(*'a*::{*linorder_topology*, *second_countable_topology*} *set*). *countable B* ∧ (∀ *x y*. *x* < *y* ⟶ (∃ *b* ∈ *B*. *x* ≤ *b* ∧ *b* < *y*))
proof –
obtain *A*::*'a set set* **where** *countable A topological_basis A* **using** *ex_countable_basis*

by auto
  define B1 where B1 = {(GREATEST x. x ∈ U) | U. U ∈ A}
  then have countable B1 using ‹countable A› by (simp add: Setcompr_eq_image)
  define B2 where B2 = {(SOME x. x ∈ U)| U. U ∈ A}
  then have countable B2 using ‹countable A› by (simp add: Setcompr_eq_image)
  have ∃ b ∈ B1 ∪ B2. x ≤ b ∧ b < y if x < y for x y
  proof (cases)
    assume ∃ z. x < z ∧ z < y
    then obtain z where z: x < z ∧ z < y by auto
    define U where U = {x<..<y}
    then have open U by simp
    moreover have z ∈ U using z U_def by simp
    ultimately obtain V where V ∈ A z ∈ V V ⊆ U
      using topological_basisE[OF ‹topological_basis A›] by auto
    define w where w = (SOME x. x ∈ V)
    then have w ∈ V using ‹z ∈ V› by (metis someI2)
    then have x ≤ w ∧ w < y using ‹w ∈ V› ‹V ⊆ U› U_def by fastforce
    moreover have w ∈ B1 ∪ B2 using w_def B2_def ‹V ∈ A› by auto
    ultimately show ?thesis by auto
  next
    assume ¬(∃ z. x < z ∧ z < y)
    then have ∗: ⋀z. z < y ⟹ z ≤ x using leI by blast
    define U where U = {..<y}
    then have open U by simp
    moreover have x ∈ U using ‹x < y› U_def by simp
    ultimately obtain V where V ∈ A x ∈ V V ⊆ U
      using topological_basisE[OF ‹topological_basis A›] by auto
    have U = {..x} unfolding U_def using ∗ ‹x < y› by auto
    then have V ⊆ {..x} using ‹V ⊆ U› by simp
    then have (GREATEST x. x ∈ V) = x using ‹x ∈ V› by (meson Greatest_equality atMost_iff subsetCE)
    then have x ∈ B1 ∪ B2 using ‹V ∈ A› B1_def by auto
    moreover have x ≤ x ∧ x < y using ‹x < y› by simp
    ultimately show ?thesis by auto
  qed
  moreover have countable (B1 ∪ B2) using ‹countable B1› ‹countable B2› by simp
  ultimately show ?thesis by auto
qed

lemma countable_separating_set_dense_linorder:
  shows ∃ B::('a::{linorder_topology, dense_linorder, second_countable_topology} set). countable B ∧ (∀ x y. x < y ⟶ (∃ b ∈ B. x < b ∧ b < y))
proof −
  obtain B::'a set where B: countable B ⋀x y. x < y ⟹ (∃ b ∈ B. x < b ∧ b ≤ y)
    using countable_separating_set_linorder1 by auto
  have ∃ b ∈ B. x < b ∧ b < y if x < y for x y
  proof −

obtain z **where** $x < z$ $z < y$ **using** ‹$x < y$› *dense* **by** *blast*
then obtain b **where** $b \in B$ $x < b \wedge b \leq z$ **using** $B(2)$ **by** *auto*
then have $x < b \wedge b < y$ **using** ‹$z < y$› **by** *auto*
then show *?thesis* **using** ‹$b \in B$› **by** *auto*
qed
then show *?thesis* **using** $B(1)$ **by** *auto*
qed

2.1.3 Polish spaces

Textbooks define Polish spaces as completely metrizable. We assume the topology to be complete for a given metric.

class *polish_space* = *complete_space* + *second_countable_topology*

2.1.4 Limit Points

definition (**in** *topological_space*) *islimpt*:: $'a \Rightarrow 'a$ *set* $\Rightarrow$ *bool* (**infixr** *islimpt 60*)
where x *islimpt* $S \longleftrightarrow (\forall T.\ x{\in}T \longrightarrow$ *open* $T \longrightarrow (\exists y{\in}S.\ y{\in}T \wedge y{\neq}x))$

lemma *islimptI*:
assumes $\bigwedge T.\ x \in T \Longrightarrow$ *open* $T \Longrightarrow \exists y{\in}S.\ y \in T \wedge y \neq x$
shows x *islimpt* S
using *assms* **unfolding** *islimpt_def* **by** *auto*

lemma *islimptE*:
assumes x *islimpt* S **and** $x \in T$ **and** *open* T
obtains y **where** $y \in S$ **and** $y \in T$ **and** $y \neq x$
using *assms* **unfolding** *islimpt_def* **by** *auto*

lemma *islimpt_iff_eventually*: x *islimpt* $S \longleftrightarrow \neg$ *eventually* $(\lambda y.\ y \notin S)$ (*at* x)
unfolding *islimpt_def eventually_at_topological* **by** *auto*

lemma *islimpt_subset*: x *islimpt* $S \Longrightarrow S \subseteq T \Longrightarrow x$ *islimpt* T
unfolding *islimpt_def* **by** *fast*

lemma *islimpt_UNIV_iff*: x *islimpt UNIV* $\longleftrightarrow \neg$ *open* $\{x\}$
unfolding *islimpt_def* **by** (*safe*, *fast*, *case_tac* $T = \{x\}$, *fast*, *fast*)

lemma *islimpt_punctured*: x *islimpt* $S = x$ *islimpt* $(S-\{x\})$
unfolding *islimpt_def* **by** *blast*

A perfect space has no isolated points.

lemma *islimpt_UNIV* [*simp*, *intro*]: x *islimpt UNIV*
for x :: *'a::perfect_space*
unfolding *islimpt_UNIV_iff* **by** (*rule not_open_singleton*)

lemma *closed_limpt*: *closed* $S \longleftrightarrow (\forall x.\ x$ *islimpt* $S \longrightarrow x \in S)$
unfolding *closed_def*
apply (*subst open_subopen*)

apply (*simp add*: *islimpt_def subset_eq*)
apply (*metis ComplE ComplI*)
done

lemma *islimpt_EMPTY*[*simp*]: ¬ *x islimpt* {}
by (*auto simp*: *islimpt_def*)

lemma *islimpt_Un*: *x islimpt* (*S* ∪ *T*) ⟷ *x islimpt S* ∨ *x islimpt T*
by (*simp add*: *islimpt_iff_eventually eventually_conj_iff*)

lemma *islimpt_insert*:
fixes *x* :: *'a*::*t1_space*
shows *x islimpt* (*insert a s*) ⟷ *x islimpt s*
proof
assume ∗: *x islimpt* (*insert a s*)
show *x islimpt s*
proof (*rule islimptI*)
fix *t*
assume *t*: *x* ∈ *t open t*
show ∃ *y*∈*s*. *y* ∈ *t* ∧ *y* ≠ *x*
proof (*cases x* = *a*)
case *True*
obtain *y* **where** *y* ∈ *insert a s y* ∈ *t y* ≠ *x*
using ∗ *t* **by** (*rule islimptE*)
with ‹*x* = *a*› **show** *?thesis* **by** *auto*
next
case *False*
with *t* **have** *t'*: *x* ∈ *t* − {*a*} *open* (*t* − {*a*})
by (*simp_all add*: *open_Diff*)
obtain *y* **where** *y* ∈ *insert a s y* ∈ *t* − {*a*} *y* ≠ *x*
using ∗ *t'* **by** (*rule islimptE*)
then show *?thesis* **by** *auto*
qed
qed
next
assume *x islimpt s*
then show *x islimpt* (*insert a s*)
by (*rule islimpt_subset*) *auto*
qed

lemma *islimpt_finite*:
fixes *x* :: *'a*::*t1_space*
shows *finite s* ⟹ ¬ *x islimpt s*
by (*induct set*: *finite*) (*simp_all add*: *islimpt_insert*)

lemma *islimpt_Un_finite*:
fixes *x* :: *'a*::*t1_space*
shows *finite s* ⟹ *x islimpt* (*s* ∪ *t*) ⟷ *x islimpt t*

```
  by (simp add: islimpt_Un islimpt_finite)

lemma islimpt_eq_acc_point:
  fixes l :: 'a :: t1_space
  shows l islimpt S ⟷ (∀ U. l∈U ⟶ open U ⟶ infinite (U ∩ S))
proof (safe intro!: islimptI)
  fix U
  assume l islimpt S l ∈ U open U finite (U ∩ S)
  then have l islimpt S l ∈ (U − (U ∩ S − {l})) open (U − (U ∩ S − {l}))
    by (auto intro: finite_imp_closed)
  then show False
    by (rule islimptE) auto
next
  fix T
  assume *: ∀ U. l∈U ⟶ open U ⟶ infinite (U ∩ S) l ∈ T open T
  then have infinite (T ∩ S − {l})
    by auto
  then have ∃ x. x ∈ (T ∩ S − {l})
    unfolding ex_in_conv by (intro notI) simp
  then show ∃ y∈S. y ∈ T ∧ y ≠ l
    by auto
qed

lemma acc_point_range_imp_convergent_subsequence:
  fixes l :: 'a :: first_countable_topology
  assumes l: ∀ U. l∈U ⟶ open U ⟶ infinite (U ∩ range f)
  shows ∃ r::nat⇒nat. strict_mono r ∧ (f ∘ r) ⇢ l
proof −
  from countable_basis_at_decseq[of l]
  obtain A where A:
      ⋀i. open (A i)
      ⋀i. l ∈ A i
      ⋀S. open S ⟹ l ∈ S ⟹ eventually (λi. A i ⊆ S) sequentially
    by blast
  define s where s n i = (SOME j. i < j ∧ f j ∈ A (Suc n)) for n i
  {
    fix n i
    have infinite (A (Suc n) ∩ range f − f‘{.. i})
      using l A by auto
    then have ∃ x. x ∈ A (Suc n) ∩ range f − f‘{.. i}
      unfolding ex_in_conv by (intro notI) simp
    then have ∃ j. f j ∈ A (Suc n) ∧ j ∉ {.. i}
      by auto
    then have ∃ a. i < a ∧ f a ∈ A (Suc n)
      by (auto simp: not_le)
    then have i < s n i f (s n i) ∈ A (Suc n)
      unfolding s_def by (auto intro: someI2_ex)
  }
  note s = this
```

define *r* **where** $r = rec\_nat\ (s\ 0\ 0)\ s$
have *strict_mono r*
by (*auto simp*: *r_def s strict_mono_Suc_iff*)
moreover
have $(\lambda n.\ f\ (r\ n)) \longrightarrow l$
proof (*rule topological_tendstoI*)
fix *S*
assume *open S* $l \in S$
with *A(3)* **have** *eventually* $(\lambda i.\ A\ i \subseteq S)$ *sequentially*
by *auto*
moreover
{
fix *i*
assume $Suc\ 0 \leq i$
then have $f\ (r\ i) \in A\ i$
by (*cases i*) (*simp_all add*: *r_def s*)
}
then have *eventually* $(\lambda i.\ f\ (r\ i) \in A\ i)$ *sequentially*
by (*auto simp*: *eventually_sequentially*)
ultimately show *eventually* $(\lambda i.\ f\ (r\ i) \in S)$ *sequentially*
by *eventually_elim auto*
qed
ultimately show $\exists\, r{::}nat{\Rightarrow}nat.\ strict\_mono\ r \wedge (f \circ r) \longrightarrow l$
by (*auto simp*: *convergent_def comp_def*)
qed

lemma *islimpt_range_imp_convergent_subsequence*:
fixes $l :: 'a :: \{t1\_space,\ first\_countable\_topology\}$
assumes *l*: *l islimpt* (*range f*)
shows $\exists\, r{::}nat{\Rightarrow}nat.\ strict\_mono\ r \wedge (f \circ r) \longrightarrow l$
using *l* **unfolding** *islimpt_eq_acc_point*
by (*rule acc_point_range_imp_convergent_subsequence*)

lemma *sequence_unique_limpt*:
fixes $f :: nat \Rightarrow 'a{::}t2\_space$
assumes $(f \longrightarrow l)$ *sequentially*
and l' *islimpt* (*range f*)
shows $l' = l$
proof (*rule ccontr*)
assume $l' \neq l$
obtain *s t* **where** *open s open t* $l' \in s$ $l \in t$ $s \cap t = \{\}$
using *hausdorff* [*OF* ‹$l' \neq l$›] **by** *auto*
have *eventually* $(\lambda n.\ f\ n \in t)$ *sequentially*
using *assms(1)* ‹*open t*› ‹$l \in t$› **by** (*rule topological_tendstoD*)
then obtain *N* **where** $\forall\, n{\geq}N.\ f\ n \in t$
unfolding *eventually_sequentially* **by** *auto*

have $UNIV = \{..{<}N\} \cup \{N..\}$
by *auto*

```
  then have l' islimpt (f ` ({..<N} ∪ {N..}))
    using assms(2) by simp
  then have l' islimpt (f ` {..<N} ∪ f ` {N..})
    by (simp add: image_Un)
  then have l' islimpt (f ` {N..})
    by (simp add: islimpt_Un_finite)
  then obtain y where y ∈ f ` {N..} y ∈ s y ≠ l'
    using ‹l' ∈ s› ‹open s› by (rule islimptE)
  then obtain n where N ≤ n f n ∈ s f n ≠ l'
    by auto
  with ‹∀ n≥N. f n ∈ t› have f n ∈ s ∩ t
    by simp
  with ‹s ∩ t = {}› show False
    by simp
qed
```

2.1.5 Interior of a Set

```
definition interior :: ('a::topological_space) set ⇒ 'a set where
interior S = ⋃{T. open T ∧ T ⊆ S}

lemma interiorI [intro?]:
  assumes open T and x ∈ T and T ⊆ S
  shows x ∈ interior S
  using assms unfolding interior_def by fast

lemma interiorE [elim?]:
  assumes x ∈ interior S
  obtains T where open T and x ∈ T and T ⊆ S
  using assms unfolding interior_def by fast

lemma open_interior [simp, intro]: open (interior S)
  by (simp add: interior_def open_Union)

lemma interior_subset: interior S ⊆ S
  by (auto simp: interior_def)

lemma interior_maximal: T ⊆ S ⟹ open T ⟹ T ⊆ interior S
  by (auto simp: interior_def)

lemma interior_open: open S ⟹ interior S = S
  by (intro equalityI interior_subset interior_maximal subset_refl)

lemma interior_eq: interior S = S ⟷ open S
  by (metis open_interior interior_open)

lemma open_subset_interior: open S ⟹ S ⊆ interior T ⟷ S ⊆ T
  by (metis interior_maximal interior_subset subset_trans)
```

lemma *interior_empty* [*simp*]: *interior* {} = {}
  using *open_empty* **by** (*rule interior_open*)

lemma *interior_UNIV* [*simp*]: *interior UNIV* = *UNIV*
  using *open_UNIV* **by** (*rule interior_open*)

lemma *interior_interior* [*simp*]: *interior* (*interior S*) = *interior S*
  using *open_interior* **by** (*rule interior_open*)

lemma *interior_mono*: $S \subseteq T \Longrightarrow$ *interior S* $\subseteq$ *interior T*
  by (*auto simp*: *interior_def*)

lemma *interior_unique*:
  assumes $T \subseteq S$ **and** *open T*
  assumes $\bigwedge T'.\ T' \subseteq S \Longrightarrow$ *open* $T' \Longrightarrow T' \subseteq T$
  shows *interior S* = *T*
  by (*intro equalityI assms interior_subset open_interior interior_maximal*)

lemma *interior_singleton* [*simp*]: *interior* {*a*} = {}
  for *a* :: *'a::perfect_space*
  by (*meson interior_eq interior_subset not_open_singleton subset_singletonD*)

lemma *interior_Int* [*simp*]: *interior* ($S \cap T$) = *interior S* $\cap$ *interior T*
  by (*meson Int_mono Int_subset_iff antisym_conv interior_maximal interior_subset open_Int open_interior*)

lemma *eventually_nhds_in_nhd*: $x \in$ *interior s* $\Longrightarrow$ *eventually* ($\lambda y.\ y \in s$) (*nhds x*)
  using *interior_subset*[*of s*] **by** (*subst eventually_nhds*) *blast*

lemma *interior_limit_point* [*intro*]:
  fixes *x* :: *'a::perfect_space*
  assumes *x*: $x \in$ *interior S*
  shows *x islimpt S*
  using *x islimpt_UNIV* [*of x*]
  unfolding *interior_def islimpt_def*
  apply (*clarsimp*, *rename_tac T T'*)
  apply (*drule_tac x*=$T \cap T'$ **in** *spec*)
  apply (*auto simp*: *open_Int*)
  done

lemma *interior_closed_Un_empty_interior*:
  assumes *cS*: *closed S*
    and *iT*: *interior T* = {}
  shows *interior* ($S \cup T$) = *interior S*
proof
  show *interior S* $\subseteq$ *interior* ($S \cup T$)
    by (*rule interior_mono*) (*rule Un_upper1*)
  show *interior* ($S \cup T$) $\subseteq$ *interior S*

```
  proof
    fix x
    assume x ∈ interior (S ∪ T)
    then obtain R where open R x ∈ R R ⊆ S ∪ T ..
    show x ∈ interior S
    proof (rule ccontr)
      assume x ∉ interior S
      with ‹x ∈ R› ‹open R› obtain y where y ∈ R − S
        unfolding interior_def by fast
      from ‹open R› ‹closed S› have open (R − S)
        by (rule open_Diff)
      from ‹R ⊆ S ∪ T› have R − S ⊆ T
        by fast
     from ‹y ∈ R − S› ‹open (R − S)› ‹R − S ⊆ T› ‹interior T = {}› show False
        unfolding interior_def by fast
    qed
  qed
qed

lemma interior_Times: interior (A × B) = interior A × interior B
proof (rule interior_unique)
  show interior A × interior B ⊆ A × B
    by (intro Sigma_mono interior_subset)
  show open (interior A × interior B)
    by (intro open_Times open_interior)
  fix T
  assume T ⊆ A × B and open T
  then show T ⊆ interior A × interior B
  proof safe
    fix x y
    assume (x, y) ∈ T
    then obtain C D where open C open D C × D ⊆ T x ∈ C y ∈ D
      using ‹open T› unfolding open_prod_def by fast
    then have open C open D C ⊆ A D ⊆ B x ∈ C y ∈ D
      using ‹T ⊆ A × B› by auto
    then show x ∈ interior A and y ∈ interior B
      by (auto intro: interiorI)
  qed
qed

lemma interior_Ici:
  fixes x :: 'a :: {dense_linorder,linorder_topology}
  assumes b < x
  shows interior {x ..} = {x <..}
proof (rule interior_unique)
  fix T
  assume T ⊆ {x ..} open T
  moreover have x ∉ T
  proof
```

```
    assume x ∈ T
    obtain y where y < x {y <.. x} ⊆ T
      using open_left[OF ‹open T› ‹x ∈ T› ‹b < x›] by auto
    with dense[OF ‹y < x›] obtain z where z ∈ T z < x
      by (auto simp: subset_eq Ball_def)
    with ‹T ⊆ {x ..}› show False by auto
  qed
  ultimately show T ⊆ {x <..}
    by (auto simp: subset_eq less_le)
qed auto

lemma interior_Iic:
  fixes x :: 'a ::{dense_linorder,linorder_topology}
  assumes x < b
  shows interior {.. x} = {..< x}
proof (rule interior_unique)
  fix T
  assume T ⊆ {.. x} open T
  moreover have x ∉ T
  proof
    assume x ∈ T
    obtain y where x < y {x ..< y} ⊆ T
      using open_right[OF ‹open T› ‹x ∈ T› ‹x < b›] by auto
    with dense[OF ‹x < y›] obtain z where z ∈ T x < z
      by (auto simp: subset_eq Ball_def less_le)
    with ‹T ⊆ {.. x}› show False by auto
  qed
  ultimately show T ⊆ {..< x}
    by (auto simp: subset_eq less_le)
qed auto

lemma countable_disjoint_nonempty_interior_subsets:
  fixes ℱ :: 'a::second_countable_topology set set
  assumes pw: pairwise disjnt ℱ and int: ⋀S. ⟦S ∈ ℱ; interior S = {}⟧ ⟹ S =
{}
  shows countable ℱ
proof (rule countable_image_inj_on)
  have disjoint (interior ` ℱ)
    using pw by (simp add: disjoint_image_subset interior_subset)
  then show countable (interior ` ℱ)
    by (auto intro: countable_disjoint_open_subsets)
  show inj_on interior ℱ
    using pw apply (clarsimp simp: inj_on_def pairwise_def)
    apply (metis disjnt_def disjnt_subset1 inf.orderE int interior_subset)
    done
qed
```

2.1.6 Closure of a Set

definition *closure* :: ('*a*::*topological_space*) *set* ⇒ '*a set* **where**
closure S = *S* ∪ {*x* . *x islimpt S*}

lemma *interior_closure*: *interior S* = − (*closure* (− *S*))
  by (*auto simp*: *interior_def closure_def islimpt_def*)

lemma *closure_interior*: *closure S* = − *interior* (− *S*)
  by (*simp add*: *interior_closure*)

lemma *closed_closure*[*simp*, *intro*]: *closed* (*closure S*)
  by (*simp add*: *closure_interior closed_Compl*)

lemma *closure_subset*: *S* ⊆ *closure S*
  by (*simp add*: *closure_def*)

lemma *closure_hull*: *closure S* = *closed hull S*
  by (*auto simp*: *hull_def closure_interior interior_def*)

lemma *closure_eq*: *closure S* = *S* ⟷ *closed S*
  unfolding *closure_hull* **using** *closed_Inter* **by** (*rule hull_eq*)

lemma *closure_closed* [*simp*]: *closed S* ⟹ *closure S* = *S*
  by (*simp only*: *closure_eq*)

lemma *closure_closure* [*simp*]: *closure* (*closure S*) = *closure S*
  unfolding *closure_hull* **by** (*rule hull_hull*)

lemma *closure_mono*: *S* ⊆ *T* ⟹ *closure S* ⊆ *closure T*
  unfolding *closure_hull* **by** (*rule hull_mono*)

lemma *closure_minimal*: *S* ⊆ *T* ⟹ *closed T* ⟹ *closure S* ⊆ *T*
  unfolding *closure_hull* **by** (*rule hull_minimal*)

lemma *closure_unique*:
  assumes *S* ⊆ *T*
    and *closed T*
    and ⋀*T'*. *S* ⊆ *T'* ⟹ *closed T'* ⟹ *T* ⊆ *T'*
  shows *closure S* = *T*
  using *assms* **unfolding** *closure_hull* **by** (*rule hull_unique*)

lemma *closure_empty* [*simp*]: *closure* {} = {}
  using *closed_empty* **by** (*rule closure_closed*)

lemma *closure_UNIV* [*simp*]: *closure UNIV* = *UNIV*
  using *closed_UNIV* **by** (*rule closure_closed*)

lemma *closure_Un* [*simp*]: *closure* (*S* ∪ *T*) = *closure S* ∪ *closure T*
  by (*simp add*: *closure_interior*)

lemma *closure_eq_empty* [*iff*]: *closure* $S = \{\} \longleftrightarrow S = \{\}$
  using *closure_empty closure_subset*[*of S*] **by** *blast*

lemma *closure_subset_eq*: *closure* $S \subseteq S \longleftrightarrow$ *closed* S
  using *closure_eq*[*of S*] *closure_subset*[*of S*] **by** *simp*

lemma *open_Int_closure_eq_empty*: *open* $S \Longrightarrow (S \cap$ *closure* $T) = \{\} \longleftrightarrow S \cap T = \{\}$
  using *open_subset_interior*[*of* $S - T$]
  using *interior_subset*[*of* $- T$]
  by (*auto simp*: *closure_interior*)

lemma *open_Int_closure_subset*: *open* $S \Longrightarrow S \cap$ *closure* $T \subseteq$ *closure* $(S \cap T)$
proof
  fix x
  assume $*$: *open* S $x \in S \cap$ *closure* T
  have x *islimpt* $(S \cap T)$ **if** $**$: x *islimpt* T
  proof (*rule islimptI*)
    fix A
    assume $x \in A$ *open* A
    with $*$ **have** $x \in A \cap S$ *open* $(A \cap S)$
      by (*simp_all add*: *open_Int*)
    with $**$ **obtain** y **where** $y \in T$ $y \in A \cap S$ $y \neq x$
      by (*rule islimptE*)
    then have $y \in S \cap T$ $y \in A \wedge y \neq x$
      by *simp_all*
    then show $\exists y \in (S \cap T).\ y \in A \wedge y \neq x$ **..**
  qed
  with $*$ **show** $x \in$ *closure* $(S \cap T)$
    unfolding *closure_def* **by** *blast*
qed

lemma *closure_complement*: *closure* $(- S) = -$ *interior* S
  by (*simp add*: *closure_interior*)

lemma *interior_complement*: *interior* $(- S) = -$ *closure* S
  by (*simp add*: *closure_interior*)

lemma *interior_diff*: *interior*$(S - T)$ = *interior* $S -$ *closure* T
  by (*simp add*: *Diff_eq interior_complement*)

lemma *closure_Times*: *closure* $(A \times B)$ = *closure* $A \times$ *closure* B
proof (*rule closure_unique*)
  show $A \times B \subseteq$ *closure* $A \times$ *closure* B
    by (*intro Sigma_mono closure_subset*)
  show *closed* (*closure* $A \times$ *closure* B)
    by (*intro closed_Times closed_closure*)
  fix T

```
  assume A × B ⊆ T and closed T
  then show closure A × closure B ⊆ T
    apply (simp add: closed_def open_prod_def, clarify)
    apply (rule ccontr)
    apply (drule_tac x=(a, b) in bspec, simp, clarify, rename_tac C D)
    apply (simp add: closure_interior interior_def)
    apply (drule_tac x=C in spec)
    apply (drule_tac x=D in spec, auto)
    done
qed

lemma closure_open_Int_superset:
  assumes open S S ⊆ closure T
  shows closure(S ∩ T) = closure S
proof –
  have closure S ⊆ closure(S ∩ T)
   by (metis assms closed_closure closure_minimal inf.orderE open_Int_closure_subset)
  then show ?thesis
    by (simp add: closure_mono dual_order.antisym)
qed

lemma closure_Int: closure (⋂I) ≤ ⋂{closure S |S. S ∈ I}
proof –
  {
    fix y
    assume y ∈ ⋂I
    then have y: ∀S ∈ I. y ∈ S by auto
    {
      fix S
      assume S ∈ I
      then have y ∈ closure S
        using closure_subset y by auto
    }
    then have y ∈ ⋂{closure S |S. S ∈ I}
      by auto
  }
  then have ⋂I ⊆ ⋂{closure S |S. S ∈ I}
    by auto
  moreover have closed (⋂{closure S |S. S ∈ I})
    unfolding closed_Inter closed_closure by auto
  ultimately show ?thesis using closure_hull[of ⋂I]
    hull_minimal[of ⋂I ⋂{closure S |S. S ∈ I} closed] by auto
qed

lemma islimpt_in_closure: (x islimpt S) = (x∈closure(S−{x}))
  unfolding closure_def using islimpt_punctured by blast

lemma connected_imp_connected_closure: connected S ⟹ connected (closure S)
  by (rule connectedI) (meson closure_subset open_Int open_Int_closure_eq_empty
```

subset_trans connectedD)

lemma *bdd_below_closure*:
  fixes *A* :: *real set*
  assumes *bdd_below A*
  shows *bdd_below* (*closure A*)
proof −
  from *assms* **obtain** *m* **where** $\bigwedge x.\ x \in A \Longrightarrow m \leq x$
    by (*auto simp*: *bdd_below_def*)
  then have $A \subseteq \{m..\}$ **by** *auto*
  then have *closure* $A \subseteq \{m..\}$
    using *closed_real_atLeast* **by** (*rule closure_minimal*)
  then show *?thesis*
    by (*auto simp*: *bdd_below_def*)
qed

2.1.7 Frontier (also known as boundary)

definition *frontier* :: (*'a*::*topological_space*) *set* ⇒ *'a set* **where**
frontier S = *closure S* − *interior S*

lemma *frontier_closed* [*iff*]: *closed* (*frontier S*)
  by (*simp add*: *frontier_def closed_Diff*)

lemma *frontier_closures*: *frontier S* = *closure S* ∩ *closure* (− *S*)
  by (*auto simp*: *frontier_def interior_closure*)

lemma *frontier_Int*: *frontier*(*S* ∩ *T*) = *closure*(*S* ∩ *T*) ∩ (*frontier S* ∪ *frontier T*)
proof −
  have *closure* (*S* ∩ *T*) ⊆ *closure S closure* (*S* ∩ *T*) ⊆ *closure T*
    by (*simp_all add*: *closure_mono*)
  then show *?thesis*
    by (*auto simp*: *frontier_closures*)
qed

lemma *frontier_Int_subset*: *frontier*(*S* ∩ *T*) ⊆ *frontier S* ∪ *frontier T*
  by (*auto simp*: *frontier_Int*)

lemma *frontier_Int_closed*:
  assumes *closed S closed T*
  shows *frontier*(*S* ∩ *T*) = (*frontier S* ∩ *T*) ∪ (*S* ∩ *frontier T*)
proof −
  have *closure* (*S* ∩ *T*) = *T* ∩ *S*
    using *assms* **by** (*simp add*: *Int_commute closed_Int*)
  moreover have *T* ∩ (*closure S* ∩ *closure* (− *S*)) = *frontier S* ∩ *T*
    by (*simp add*: *Int_commute frontier_closures*)
  ultimately show *?thesis*
   by (*simp add*: *Int_Un_distrib Int_assoc Int_left_commute assms frontier_closures*)

qed

lemma *frontier_subset_closed*: *closed* $S \Longrightarrow$ *frontier* $S \subseteq S$
  by (*metis frontier_def closure_closed Diff_subset*)

lemma *frontier_empty* [*simp*]: *frontier* {} = {}
  by (*simp add*: *frontier_def*)

lemma *frontier_subset_eq*: *frontier* $S \subseteq S \longleftrightarrow$ *closed* S
proof –
  {
    assume *frontier* $S \subseteq S$
    then have *closure* $S \subseteq S$
      using *interior_subset* **unfolding** *frontier_def* **by** *auto*
    then have *closed* S
      using *closure_subset_eq* **by** *auto*
  }
  then show *?thesis* **using** *frontier_subset_closed*[*of* S] **..**
qed

lemma *frontier_complement* [*simp*]: *frontier* $(-\ S)$ = *frontier* S
  by (*auto simp*: *frontier_def closure_complement interior_complement*)

lemma *frontier_Un_subset*: *frontier*$(S \cup T) \subseteq$ *frontier* $S \cup$ *frontier* T
  by (*metis compl_sup frontier_Int_subset frontier_complement*)

lemma *frontier_disjoint_eq*: *frontier* $S \cap S = \{\} \longleftrightarrow$ *open* S
  using *frontier_complement frontier_subset_eq*[*of* $-\ S$]
  unfolding *open_closed* **by** *auto*

lemma *frontier_UNIV* [*simp*]: *frontier UNIV* = {}
  using *frontier_complement frontier_empty* **by** *fastforce*

lemma *frontier_interiors*: *frontier* $s = -$ *interior*$(s) -$ *interior*$(-s)$
  by (*simp add*: *Int_commute frontier_def interior_closure*)

lemma *frontier_interior_subset*: *frontier*(*interior* S) $\subseteq$ *frontier* S
  by (*simp add*: *Diff_mono frontier_interiors interior_mono interior_subset*)

lemma *closure_Un_frontier*: *closure* $S = S \cup$ *frontier* S
proof –
  have $S \cup$ *interior* $S = S$
    using *interior_subset* **by** *auto*
  then show *?thesis*
    using *closure_subset* **by** (*auto simp*: *frontier_def*)
qed

2.1.8 Filters and the "eventually true" quantifier

Identify Trivial limits, where we can't approach arbitrarily closely.

lemma *trivial_limit_within*: *trivial_limit* (*at a within S*) ⟷ ¬ *a islimpt S*
proof
  assume *trivial_limit* (*at a within S*)
  then show ¬ *a islimpt S*
    unfolding *trivial_limit_def*
    unfolding *eventually_at_topological*
    unfolding *islimpt_def*
    apply (*clarsimp simp add*: *set_eq_iff*)
    apply (*rename_tac T*, *rule_tac x*=*T* **in** *exI*)
    apply (*clarsimp*, *drule_tac x*=*y* **in** *bspec*, *simp_all*)
    done
next
  assume ¬ *a islimpt S*
  then show *trivial_limit* (*at a within S*)
    unfolding *trivial_limit_def eventually_at_topological islimpt_def*
    by *metis*
qed

lemma *trivial_limit_at_iff*: *trivial_limit* (*at a*) ⟷ ¬ *a islimpt UNIV*
  using *trivial_limit_within* [*of a UNIV*] **by** *simp*

lemma *trivial_limit_at*: ¬ *trivial_limit* (*at a*)
  for *a* :: *'a::perfect_space*
  by (*rule at_neq_bot*)

lemma *not_trivial_limit_within*: ¬ *trivial_limit* (*at x within S*) = (*x* ∈ *closure* (*S* − {*x*}))
  using *islimpt_in_closure* **by** (*metis trivial_limit_within*)

lemma *not_in_closure_trivial_limitI*:
  x ∉ *closure s* ⟹ *trivial_limit* (*at x within s*)
  using *not_trivial_limit_within*[*of x s*]
  by *safe* (*metis Diff_empty Diff_insert0 closure_subset contra_subsetD*)

lemma *filterlim_at_within_closure_implies_filterlim*: *filterlim f l* (*at x within s*)
  if *x* ∈ *closure s* ⟹ *filterlim f l* (*at x within s*)
  by (*metis bot.extremum filterlim_filtercomap filterlim_mono not_in_closure_trivial_limitI that*)

lemma *at_within_eq_bot_iff*: *at c within A* = *bot* ⟷ *c* ∉ *closure* (*A* − {*c*})
  using *not_trivial_limit_within*[*of c A*] **by** *blast*

Some property holds "sufficiently close" to the limit point.

lemma *trivial_limit_eventually*: *trivial_limit net* ⟹ *eventually P net*
  by *simp*

lemma *trivial_limit_eq*: *trivial_limit net* ⟷ (∀ *P*. *eventually P net*)
by (*simp add*: *filter_eq_iff*)

lemma *Lim_topological*:
(*f* ⟶ *l*) *net* ⟷
trivial_limit net ∨ (∀ *S*. *open S* ⟶ *l* ∈ *S* ⟶ *eventually* (λ*x*. *f x* ∈ *S*) *net*)
unfolding *tendsto_def trivial_limit_eq* **by** *auto*

lemma *eventually_within_Un*:
eventually P (*at x within* (*s* ∪ *t*)) ⟷
eventually P (*at x within s*) ∧ *eventually P* (*at x within t*)
unfolding *eventually_at_filter*
by (*auto elim*!: *eventually_rev_mp*)

lemma *Lim_within_union*:
(*f* ⟶ *l*) (*at x within* (*s* ∪ *t*)) ⟷
(*f* ⟶ *l*) (*at x within s*) ∧ (*f* ⟶ *l*) (*at x within t*)
unfolding *tendsto_def*
by (*auto simp*: *eventually_within_Un*)

2.1.9 Limits

The expected monotonicity property.

lemma *Lim_Un*:
assumes (*f* ⟶ *l*) (*at x within S*) (*f* ⟶ *l*) (*at x within T*)
shows (*f* ⟶ *l*) (*at x within* (*S* ∪ *T*))
using *assms* **unfolding** *at_within_union* **by** (*rule filterlim_sup*)

lemma *Lim_Un_univ*:
(*f* ⟶ *l*) (*at x within S*) ⟹ (*f* ⟶ *l*) (*at x within T*) ⟹
S ∪ *T* = *UNIV* ⟹ (*f* ⟶ *l*) (*at x*)
by (*metis Lim_Un*)

Interrelations between restricted and unrestricted limits.

lemma *Lim_at_imp_Lim_at_within*: (*f* ⟶ *l*) (*at x*) ⟹ (*f* ⟶ *l*) (*at x within S*)
by (*metis order_refl filterlim_mono subset_UNIV at_le*)

lemma *eventually_within_interior*:
assumes *x* ∈ *interior S*
shows *eventually P* (*at x within S*) ⟷ *eventually P* (*at x*)
(**is** *?lhs* = *?rhs*)
proof
from *assms* **obtain** *T* **where** *T*: *open T x* ∈ *T T* ⊆ *S* **..**
{
assume *?lhs*
then obtain *A* **where** *open A* **and** *x* ∈ *A* **and** ∀ *y*∈*A*. *y* ≠ *x* ⟶ *y* ∈ *S* ⟶ *P y*
by (*auto simp*: *eventually_at_topological*)

with *T* **have** *open* (*A* ∩ *T*) **and** *x* ∈ *A* ∩ *T* **and** ∀ *y* ∈ *A* ∩ *T*. *y* ≠ *x* ⟶ *P y*
  by *auto*
 then show *?rhs*
  by (*auto simp*: *eventually_at_topological*)
next
 assume *?rhs*
 then show *?lhs*
  by (*auto elim*: *eventually_mono simp*: *eventually_at_filter*)
}
qed

lemma *at_within_interior*: *x* ∈ *interior S* ⟹ *at x within S* = *at x*
 unfolding *filter_eq_iff* **by** (*intro allI eventually_within_interior*)

lemma *Lim_within_LIMSEQ*:
 fixes *a* :: *'a*::*first_countable_topology*
 assumes ∀ *S*. (∀ *n*. *S n* ≠ *a* ∧ *S n* ∈ *T*) ∧ *S* ⟶ *a* ⟶ (λ*n*. *X* (*S n*)) ⟶ *L*
 shows (*X* ⟶ *L*) (*at a within T*)
 using *assms* **unfolding** *tendsto_def* [**where** *l*=*L*]
 by (*simp add*: *sequentially_imp_eventually_within*)

lemma *Lim_right_bound*:
 fixes *f* :: *'a* :: {*linorder_topology*, *conditionally_complete_linorder*, *no_top*} ⇒
  'b::{*linorder_topology*, *conditionally_complete_linorder*}
 assumes *mono*: ⋀*a b*. *a* ∈ *I* ⟹ *b* ∈ *I* ⟹ *x* < *a* ⟹ *a* ≤ *b* ⟹ *f a* ≤ *f b*
  and *bnd*: ⋀*a*. *a* ∈ *I* ⟹ *x* < *a* ⟹ *K* ≤ *f a*
 shows (*f* ⟶ *Inf* (*f* ‘ ({*x*<..} ∩ *I*))) (*at x within* ({*x*<..} ∩ *I*))
proof (*cases* {*x*<..} ∩ *I* = {})
 case *True*
 then show *?thesis* **by** *simp*
next
 case *False*
 show *?thesis*
 proof (*rule order_tendstoI*)
  fix *a*
  assume *a*: *a* < *Inf* (*f* ‘ ({*x*<..} ∩ *I*))
  {
   fix *y*
   assume *y* ∈ {*x*<..} ∩ *I*
   with *False bnd* **have** *Inf* (*f* ‘ ({*x*<..} ∩ *I*)) ≤ *f y*
    by (*auto intro*!: *cInf_lower bdd_belowI2*)
   with *a* **have** *a* < *f y*
    by (*blast intro*: *less_le_trans*)
  }
  then show *eventually* (λ*x*. *a* < *f x*) (*at x within* ({*x*<..} ∩ *I*))
   by (*auto simp*: *eventually_at_filter intro*: *exI*[*of* _ *1*] *zero_less_one*)
 next
  fix *a*

```
    assume Inf (f ‘ ({x<..} ∩ I)) < a
    from cInf_lessD[OF _ this] False obtain y where y: x < y y ∈ I f y < a
      by auto
    then have eventually (λx. x ∈ I ⟶ f x < a) (at_right x)
      unfolding eventually_at_right[OF ‹x < y›] by (metis less_imp_le le_less_trans mono)
    then show eventually (λx. f x < a) (at x within ({x<..} ∩ I))
      unfolding eventually_at_filter by eventually_elim simp
  qed
qed


lemma islimpt_sequential:
  fixes x :: 'a::first_countable_topology
  shows x islimpt S ⟷ (∃f. (∀n::nat. f n ∈ S − {x}) ∧ (f ⟶ x) sequentially)
    (is ?lhs = ?rhs)
proof
  assume ?lhs
  from countable_basis_at_decseq[of x] obtain A where A:
      ⋀i. open (A i)
      ⋀i. x ∈ A i
      ⋀S. open S ⟹ x ∈ S ⟹ eventually (λi. A i ⊆ S) sequentially
    by blast
  define f where f n = (SOME y. y ∈ S ∧ y ∈ A n ∧ x ≠ y) for n
  {
    fix n
    from ‹?lhs› have ∃y. y ∈ S ∧ y ∈ A n ∧ x ≠ y
      unfolding islimpt_def using A(1,2)[of n] by auto
    then have f n ∈ S ∧ f n ∈ A n ∧ x ≠ f n
      unfolding f_def by (rule someI_ex)
    then have f n ∈ S f n ∈ A n x ≠ f n by auto
  }
  then have ∀n. f n ∈ S − {x} by auto
  moreover have (λn. f n) ⟶ x
  proof (rule topological_tendstoI)
    fix S
    assume open S x ∈ S
    from A(3)[OF this] ‹⋀n. f n ∈ A n›
    show eventually (λx. f x ∈ S) sequentially
      by (auto elim!: eventually_mono)
  qed
  ultimately show ?rhs by fast
next
  assume ?rhs
  then obtain f :: nat ⇒ 'a where f: ⋀n. f n ∈ S − {x} and lim: f ⟶ x
    by auto
  show ?lhs
    unfolding islimpt_def
  proof safe
```

fix *T*
assume *open T x* ∈ *T*
from *lim*[*THEN topological_tendstoD*, *OF this*] *f*
show ∃ *y*∈*S*. *y* ∈ *T* ∧ *y* ≠ *x*
unfolding *eventually_sequentially* **by** *auto*
qed
qed

These are special for limits out of the same topological space.

lemma *Lim_within_id*: (*id* ⟶ *a*) (*at a within s*)
unfolding *id_def* **by** (*rule tendsto_ident_at*)

lemma *Lim_at_id*: (*id* ⟶ *a*) (*at a*)
unfolding *id_def* **by** (*rule tendsto_ident_at*)

It's also sometimes useful to extract the limit point from the filter.

abbreviation *netlimit* :: *'a*::*t2_space filter* ⇒ *'a*
where *netlimit F* ≡ *Lim F* (λ*x*. *x*)

lemma *netlimit_at* [*simp*]:
fixes *a* :: *'a*::{*perfect_space*,*t2_space*}
shows *netlimit* (*at a*) = *a*
using *Lim_ident_at* [*of a UNIV*] **by** *simp*

lemma *lim_within_interior*:
x ∈ *interior S* ⟹ (*f* ⟶ *l*) (*at x within S*) ⟷ (*f* ⟶ *l*) (*at x*)
by (*metis at_within_interior*)

lemma *netlimit_within_interior*:
fixes *x* :: *'a*::{*t2_space*,*perfect_space*}
assumes *x* ∈ *interior S*
shows *netlimit* (*at x within S*) = *x*
using *assms* **by** (*metis at_within_interior netlimit_at*)

Useful lemmas on closure and set of possible sequential limits.

lemma *closure_sequential*:
fixes *l* :: *'a*::*first_countable_topology*
shows *l* ∈ *closure S* ⟷ (∃ *x*. (∀ *n*. *x n* ∈ *S*) ∧ (*x* ⟶ *l*) *sequentially*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
moreover
{
assume *l* ∈ *S*
then have *?rhs* **using** *tendsto_const*[*of l sequentially*] **by** *auto*
}
moreover
{
assume *l islimpt S*

```
    then have ?rhs unfolding islimpt_sequential by auto
  }
  ultimately show ?rhs
    unfolding closure_def by auto
next
  assume ?rhs
  then show ?lhs unfolding closure_def islimpt_sequential by auto
qed

lemma closed_sequential_limits:
  fixes S :: 'a::first_countable_topology set
  shows closed S ⟷ (∀x l. (∀n. x n ∈ S) ∧ (x ⟶ l) sequentially ⟶ l ∈ S)
by (metis closure_sequential closure_subset_eq subset_iff)

lemma tendsto_If_within_closures:
  assumes f: x ∈ s ∪ (closure s ∩ closure t) ⟹
      (f ⟶ l x) (at x within s ∪ (closure s ∩ closure t))
  assumes g: x ∈ t ∪ (closure s ∩ closure t) ⟹
      (g ⟶ l x) (at x within t ∪ (closure s ∩ closure t))
  assumes x ∈ s ∪ t
  shows ((λx. if x ∈ s then f x else g x) ⟶ l x) (at x within s ∪ t)
proof -
  have *: (s ∪ t) ∩ {x. x ∈ s} = s (s ∪ t) ∩ {x. x ∉ s} = t - s
    by auto
  have (f ⟶ l x) (at x within s)
    by (rule filterlim_at_within_closure_implies_filterlim)
     (use ‹x ∈ _› in ‹auto simp: inf_commute closure_def intro: tendsto_within_subset[OF
f]›)
  moreover
  have (g ⟶ l x) (at x within t - s)
    by (rule filterlim_at_within_closure_implies_filterlim)
      (use ‹x ∈ _› in
      ‹auto intro!: tendsto_within_subset[OF g] simp: closure_def intro: islimpt_subset›)
  ultimately show ?thesis
    by (intro filterlim_at_within_If) (simp_all only: *)
qed
```

2.1.10 Compactness

```
lemma brouwer_compactness_lemma:
  fixes f :: 'a::topological_space ⇒ 'b::real_normed_vector
  assumes compact s
    and continuous_on s f
    and ¬ (∃x∈s. f x = 0)
  obtains d where 0 < d and ∀x∈s. d ≤ norm (f x)
proof (cases s = {})
  case True
  show thesis
    by (rule that [of 1]) (auto simp: True)
```

next
  case *False*
  have *continuous_on s (norm ∘ f)*
    by (*rule continuous_intros continuous_on_norm assms*(*2*))+
  with *False* **obtain** *x* **where** *x*: *x* ∈ *s* ∀ *y*∈*s*. (*norm* ∘ *f*) *x* ≤ (*norm* ∘ *f*) *y*
    using *continuous_attains_inf*[*OF assms*(*1*), *of norm* ∘ *f*]
    unfolding *o_def*
    by *auto*
  have (*norm* ∘ *f*) *x* > *0*
    using *assms*(*3*) **and** *x*(*1*)
    by *auto*
  then show *?thesis*
    by (*rule that*) (*insert x*(*2*), *auto simp*: *o_def*)
qed

Bolzano-Weierstrass property

proposition *Heine_Borel_imp_Bolzano_Weierstrass*:
  assumes *compact s*
    and *infinite t*
    and *t* ⊆ *s*
  shows ∃ *x* ∈ *s*. *x islimpt t*
proof (*rule ccontr*)
  assume ¬ (∃ *x* ∈ *s*. *x islimpt t*)
  then obtain *f* **where** *f*: ∀ *x*∈*s*. *x* ∈ *f x* ∧ *open* (*f x*) ∧ (∀ *y*∈*t*. *y* ∈ *f x* ⟶ *y* = *x*)
    unfolding *islimpt_def*
    using *bchoice*[*of s* λ *x T*. *x* ∈ *T* ∧ *open T* ∧ (∀ *y*∈*t*. *y* ∈ *T* ⟶ *y* = *x*)]
    by *auto*
  obtain *g* **where** *g*: *g* ⊆ {*t*. ∃ *x*. *x* ∈ *s* ∧ *t* = *f x*} *finite g s* ⊆ ⋃ *g*
    using *assms*(*1*)[*unfolded compact_eq_Heine_Borel*, *THEN spec*[**where** *x*={*t*. ∃ *x*. *x*∈*s* ∧ *t* = *f x*}]]
    using *f* **by** *auto*
  from *g*(*1*,*3*) **have** *g'*:∀ *x*∈*g*. ∃ *xa* ∈ *s*. *x* = *f xa*
    by *auto*
  {
    fix *x y*
    assume *x* ∈ *t y* ∈ *t f x* = *f y*
    then have *x* ∈ *f x* *y* ∈ *f x* ⟶ *y* = *x*
      using *f*[*THEN bspec*[**where** *x*=*x*]] **and** ‹*t* ⊆ *s*› **by** *auto*
    then have *x* = *y*
      using ‹*f x* = *f y*› **and** *f*[*THEN bspec*[**where** *x*=*y*]] **and** ‹*y* ∈ *t*› **and** ‹*t* ⊆ *s*›
      by *auto*
  }
  then have *inj_on f t*
    unfolding *inj_on_def* **by** *simp*
  then have *infinite* (*f* ‘ *t*)
    using *assms*(*2*) **using** *finite_imageD* **by** *auto*
  moreover

```
  {
    fix x
    assume x ∈ t f x ∉ g
    from g(3) assms(3) ‹x ∈ t› obtain h where h ∈ g and x ∈ h
      by auto
    then obtain y where y ∈ s h = f y
      using g'[THEN bspec[where x=h]] by auto
    then have y = x
      using f[THEN bspec[where x=y]] and ‹x∈t› and ‹x∈h›[unfolded ‹h = f y›]
      by auto
    then have False
      using ‹f x ∉ g› ‹h ∈ g› unfolding ‹h = f y›
      by auto
  }
  then have f ` t ⊆ g by auto
  ultimately show False
    using g(2) using finite_subset by auto
qed

lemma sequence_infinite_lemma:
  fixes f :: nat ⇒ 'a::t1_space
  assumes ∀ n. f n ≠ l
    and (f ⟶ l) sequentially
  shows infinite (range f)
proof
  assume finite (range f)
  then have l ∉ range f ∧ closed (range f)
    using ‹finite (range f)› assms(1) finite_imp_closed by blast
  then have eventually (λn. f n ∈ − range f) sequentially
    by (metis Compl_iff assms(2) open_Compl topological_tendstoD)
  then show False
    unfolding eventually_sequentially by auto
qed

lemma Bolzano_Weierstrass_imp_closed:
  fixes s :: 'a::{first_countable_topology,t2_space} set
  assumes ∀ t. infinite t ∧ t ⊆ s −−> (∃ x ∈ s. x islimpt t)
  shows closed s
proof −
  {
    fix x l
    assume as: ∀ n::nat. x n ∈ s (x ⟶ l) sequentially
    then have l ∈ s
    proof (cases ∀ n. x n ≠ l)
      case False
      then show l∈s using as(1) by auto
    next
      case True note cas = this
      with as(2) have infinite (range x)
```

```
      using sequence_infinite_lemma[of x l] by auto
    then obtain l' where l'∈s l' islimpt (range x)
      using assms[THEN spec[where x=range x]] as(1) by auto
    then show l∈s using sequence_unique_limpt[of x l l']
      using as cas by auto
  qed
 }
 then show ?thesis
   unfolding closed_sequential_limits by fast
qed

lemma closure_insert:
 fixes x :: 'a::t1_space
 shows closure (insert x s) = insert x (closure s)
 apply (rule closure_unique)
 apply (rule insert_mono [OF closure_subset])
 apply (rule closed_insert [OF closed_closure])
 apply (simp add: closure_minimal)
 done
```

In particular, some common special cases.

```
lemma compact_Un [intro]:
 assumes compact s
   and compact t
 shows  compact (s ∪ t)
proof (rule compactI)
 fix f
 assume *: Ball f open s ∪ t ⊆ ⋃f
 from * ‹compact s› obtain s' where s' ⊆ f ∧ finite s' ∧ s ⊆ ⋃s'
   unfolding compact_eq_Heine_Borel by (auto elim!: allE[of _ f])
 moreover
 from * ‹compact t› obtain t' where t' ⊆ f ∧ finite t' ∧ t ⊆ ⋃t'
   unfolding compact_eq_Heine_Borel by (auto elim!: allE[of _ f])
 ultimately show ∃f'⊆f. finite f' ∧ s ∪ t ⊆ ⋃f'
   by (auto intro!: exI[of _ s' ∪ t'])
qed

lemma compact_Union [intro]: finite S ⟹ (⋀T. T ∈ S ⟹ compact T) ⟹
compact (⋃S)
 by (induct set: finite) auto

lemma compact_UN [intro]:
 finite A ⟹ (⋀x. x ∈ A ⟹ compact (B x)) ⟹ compact (⋃x∈A. B x)
 by (rule compact_Union) auto

lemma closed_Int_compact [intro]:
 assumes closed s
   and compact t
 shows compact (s ∩ t)
```

using *compact_Int_closed* [*of t s*] *assms*
by (*simp add*: *Int_commute*)

lemma *compact_Int* [*intro*]:
fixes *s t* :: '*a* :: *t2_space set*
assumes *compact s*
and *compact t*
shows *compact* ($s \cap t$)
using *assms* **by** (*intro compact_Int_closed compact_imp_closed*)

lemma *compact_sing* [*simp*]: *compact* $\{a\}$
unfolding *compact_eq_Heine_Borel* **by** *auto*

lemma *compact_insert* [*simp*]:
assumes *compact s*
shows *compact* (*insert x s*)
proof −
have *compact* ($\{x\} \cup s$)
using *compact_sing assms* **by** (*rule compact_Un*)
then show *?thesis* **by** *simp*
qed

lemma *finite_imp_compact*: *finite s* $\Longrightarrow$ *compact s*
by (*induct set*: *finite*) *simp_all*

lemma *open_delete*:
fixes *s* :: '*a*::*t1_space set*
shows *open s* $\Longrightarrow$ *open* ($s - \{x\}$)
by (*simp add*: *open_Diff*)

Compactness expressed with filters

lemma *closure_iff_nhds_not_empty*:
$x \in$ *closure X* $\longleftrightarrow$ ($\forall A. \forall S \subseteq A.$ *open S* $\longrightarrow x \in S \longrightarrow X \cap A \neq \{\}$)
proof *safe*
assume *x*: $x \in$ *closure X*
fix *S A*
assume *open S* $x \in S$ $X \cap A = \{\}$ $S \subseteq A$
then have $x \notin$ *closure* ($-S$)
by (*auto simp*: *closure_complement subset_eq*[*symmetric*] *intro*: *interiorI*)
with *x* **have** $x \in$ *closure X* − *closure* ($-S$)
by *auto*
also have ... $\subseteq$ *closure* ($X \cap S$)
using ‹*open S*› *open_Int_closure_subset*[*of S X*] **by** (*simp add*: *closed_Compl ac_simps*)
finally have $X \cap S \neq \{\}$ **by** *auto*
then show *False* **using** ‹$X \cap A = \{\}$› ‹$S \subseteq A$› **by** *auto*
next
assume $\forall A\ S.\ S \subseteq A \longrightarrow$ *open S* $\longrightarrow x \in S \longrightarrow X \cap A \neq \{\}$
from *this*[*THEN spec*, *of* − *X*, *THEN spec*, *of* − *closure X*]

show $x \in$ *closure X*
by (*simp add*: *closure_subset open_Compl*)
qed

lemma *compact_filter*:
compact U $\longleftrightarrow$ ($\forall$ *F*. *F* $\neq$ *bot* $\longrightarrow$ *eventually* (λx. $x \in U$) *F* $\longrightarrow$ ($\exists x \in U$. *inf* (*nhds x*) *F* $\neq$ *bot*))
proof (*intro allI iffI impI compact_fip*[*THEN iffD2*] *notI*)
fix *F*
assume *compact U*
assume *F*: *F* $\neq$ *bot eventually* (λx. $x \in U$) *F*
then have $U \neq \{\}$
by (*auto simp*: *eventually_False*)

define *Z* **where** *Z* = *closure* ` {*A*. *eventually* (λx. $x \in A$) *F*}
then have $\forall z \in Z$. *closed z*
by *auto*
moreover
have *ev_Z*: $\bigwedge z$. $z \in Z \Longrightarrow$ *eventually* (λx. $x \in z$) *F*
unfolding *Z_def* **by** (*auto elim*: *eventually_mono intro*: *subsetD*[*OF closure_subset*])
have ($\forall B \subseteq Z$. *finite B* $\longrightarrow U \cap \bigcap B \neq \{\}$)
proof (*intro allI impI*)
fix *B* **assume** *finite B* $B \subseteq Z$
with ‹*finite B*› *ev_Z F*(*2*) **have** *eventually* (λx. $x \in U \cap (\bigcap B)$) *F*
by (*auto simp*: *eventually_ball_finite_distrib eventually_conj_iff*)
with *F* **show** $U \cap \bigcap B \neq \{\}$
by (*intro notI*) (*simp add*: *eventually_False*)
qed
ultimately have $U \cap \bigcap Z \neq \{\}$
using ‹*compact U*› **unfolding** *compact_fip* **by** *blast*
then obtain *x* **where** $x \in U$ **and** *x*: $\bigwedge z$. $z \in Z \Longrightarrow x \in z$
by *auto*

have $\bigwedge P$. *eventually P* (*inf* (*nhds x*) *F*) $\Longrightarrow P \neq$ *bot*
unfolding *eventually_inf eventually_nhds*
proof *safe*
fix *P Q R S*
assume *eventually R F open S* $x \in S$
with *open_Int_closure_eq_empty*[*of S* {*x*. *R x*}] *x*[*of closure* {*x*. *R x*}]
have $S \cap \{x.\ R\ x\} \neq \{\}$ **by** (*auto simp*: *Z_def*)
moreover assume *Ball S Q* $\forall x$. *Q x* $\wedge$ *R x* $\longrightarrow$ *bot x*
ultimately show *False* **by** (*auto simp*: *set_eq_iff*)
qed
with ‹$x \in U$› **show** $\exists x \in U$. *inf* (*nhds x*) *F* $\neq$ *bot*
by (*metis eventually_bot*)
next
fix *A*
assume *A*: $\forall a \in A$. *closed a* $\forall B \subseteq A$. *finite B* $\longrightarrow U \cap \bigcap B \neq \{\}$ $U \cap \bigcap A = \{\}$
define *F* **where** *F* = (*INF* $a \in$ *insert U A*. *principal a*)

```
  have F ≠ bot
    unfolding F_def
  proof (rule INF_filter_not_bot)
    fix X
    assume X: X ⊆ insert U A finite X
    with A(2)[THEN spec, of X − {U}] have U ∩ ⋂(X − {U}) ≠ {}
      by auto
    with X show (INF a∈X. principal a) ≠ bot
      by (auto simp: INF_principal_finite principal_eq_bot_iff)
  qed
  moreover
  have F ≤ principal U
    unfolding F_def by auto
  then have eventually (λx. x ∈ U) F
    by (auto simp: le_filter_def eventually_principal)
  moreover
  assume ∀F. F ≠ bot ⟶ eventually (λx. x ∈ U) F ⟶ (∃x∈U. inf (nhds x)
F ≠ bot)
  ultimately obtain x where x ∈ U and x: inf (nhds x) F ≠ bot
    by auto

  { fix V assume V ∈ A
    then have F ≤ principal V
      unfolding F_def by (intro INF_lower2[of V]) auto
    then have V: eventually (λx. x ∈ V) F
      by (auto simp: le_filter_def eventually_principal)
    have x ∈ closure V
      unfolding closure_iff_nhds_not_empty
    proof (intro impI allI)
      fix S A
      assume open S x ∈ S S ⊆ A
      then have eventually (λx. x ∈ A) (nhds x)
        by (auto simp: eventually_nhds)
      with V have eventually (λx. x ∈ V ∩ A) (inf (nhds x) F)
        by (auto simp: eventually_inf)
      with x show V ∩ A ≠ {}
        by (auto simp del: Int_iff simp add: trivial_limit_def)
    qed
    then have x ∈ V
      using ‹V ∈ A› A(1) by simp
  }
  with ‹x∈U› have x ∈ U ∩ ⋂A by auto
  with ‹U ∩ ⋂A = {}› show False by auto
qed

definition countably_compact :: ('a::topological_space) set ⇒ bool where
countably_compact U ⟷
  (∀A. countable A ⟶ (∀a∈A. open a) ⟶ U ⊆ ⋃A
    ⟶ (∃T⊆A. finite T ∧ U ⊆ ⋃T))
```

lemma *countably_compactE*:
  assumes *countably_compact s* **and** $\forall\, t{\in}C.$ *open t* **and** $s \subseteq \bigcup C$ *countable C*
  obtains C' **where** $C' \subseteq C$ **and** *finite* C' **and** $s \subseteq \bigcup C'$
  using *assms* **unfolding** *countably_compact_def* **by** *metis*

lemma *countably_compactI*:
  assumes $\bigwedge C.\ \forall\, t{\in}C.$ *open t* $\Longrightarrow s \subseteq \bigcup C \Longrightarrow$ *countable C* $\Longrightarrow (\exists\, C'{\subseteq}C.$ *finite* $C' \wedge s \subseteq \bigcup C')$
  shows *countably_compact s*
  using *assms* **unfolding** *countably_compact_def* **by** *metis*

lemma *compact_imp_countably_compact*: *compact U* $\Longrightarrow$ *countably_compact U*
  by (*auto simp*: *compact_eq_Heine_Borel countably_compact_def*)

lemma *countably_compact_imp_compact*:
  assumes *countably_compact U*
    and *ccover*: *countable B* $\forall\, b{\in}B.$ *open b*
    and *basis*: $\bigwedge T\, x.$ *open T* $\Longrightarrow x \in T \Longrightarrow x \in U \Longrightarrow \exists\, b{\in}B.\ x \in b \wedge b \cap U \subseteq T$
  shows *compact U*
  using ‹*countably_compact U*›
  unfolding *compact_eq_Heine_Borel countably_compact_def*
proof *safe*
  fix *A*
  assume *A*: $\forall\, a{\in}A.$ *open a* $U \subseteq \bigcup A$
  assume $*$: $\forall\, A.$ *countable A* $\longrightarrow (\forall\, a{\in}A.$ *open a*$) \longrightarrow U \subseteq \bigcup A \longrightarrow (\exists\, T{\subseteq}A.$ *finite T* $\wedge U \subseteq \bigcup T)$
  moreover define *C* **where** $C = \{b{\in}B.\ \exists\, a{\in}A.\ b \cap U \subseteq a\}$
  ultimately have *countable C* $\forall\, a{\in}C.$ *open a*
    unfolding *C_def* **using** *ccover* **by** *auto*
  moreover
  have $\bigcup A \cap U \subseteq \bigcup C$
  proof *safe*
    fix *x a*
    assume $x \in U\ x \in a\ a \in A$
    with *basis*[*of a x*] *A* **obtain** *b* **where** $b \in B\ x \in b\ b \cap U \subseteq a$
      by *blast*
    with ‹$a \in A$› **show** $x \in \bigcup C$
      unfolding *C_def* **by** *auto*
  qed
  then have $U \subseteq \bigcup C$ **using** ‹$U \subseteq \bigcup A$› **by** *auto*
  ultimately obtain *T* **where** *T*: $T{\subseteq}C$ *finite T* $U \subseteq \bigcup T$
    using $*$ **by** *metis*
  then have $\forall\, t{\in}T.\ \exists\, a{\in}A.\ t \cap U \subseteq a$
    by (*auto simp*: *C_def*)
  then obtain *f* **where** $\forall\, t{\in}T.\ f\ t \in A \wedge t \cap U \subseteq f\ t$
    unfolding *bchoice_iff Bex_def* **..**
  with *T* **show** $\exists\, T{\subseteq}A.$ *finite T* $\wedge U \subseteq \bigcup T$

unfolding *C_def* **by** (*intro exI*[*of* _ *f*'*T*]) *fastforce*
qed

proposition *countably_compact_imp_compact_second_countable*:
countably_compact U $\Longrightarrow$ *compact* (*U* :: '*a* :: *second_countable_topology set*)
proof (*rule countably_compact_imp_compact*)
fix *T* **and** *x* :: '*a*
assume *open T x* $\in$ *T*
from *topological_basisE*[*OF is_basis this*] **obtain** *b* **where**
b $\in$ (*SOME B*. *countable B* $\wedge$ *topological_basis B*) *x* $\in$ *b b* $\subseteq$ *T* **.**
then show $\exists$ *b*$\in$*SOME B*. *countable B* $\wedge$ *topological_basis B*. *x* $\in$ *b* $\wedge$ *b* $\cap$ *U* $\subseteq$ *T*
by *blast*
qed (*insert countable_basis topological_basis_open*[*OF is_basis*], *auto*)

lemma *countably_compact_eq_compact*:
countably_compact U $\longleftrightarrow$ *compact* (*U* :: '*a* :: *second_countable_topology set*)
using *countably_compact_imp_compact_second_countable compact_imp_countably_compact*
by *blast*

Sequential compactness

definition *seq_compact* :: '*a*::*topological_space set* $\Rightarrow$ *bool* **where**
seq_compact S $\longleftrightarrow$
($\forall$ *f*. ($\forall$ *n*. *f n* $\in$ *S*)
$\longrightarrow$ ($\exists$ *l*$\in$*S*. $\exists$ *r*::*nat*$\Rightarrow$*nat*. *strict_mono r* $\wedge$ ((*f* $\circ$ *r*) $\longrightarrow$ *l*) *sequentially*))

lemma *seq_compactI*:
assumes $\bigwedge$*f*. $\forall$ *n*. *f n* $\in$ *S* $\Longrightarrow$ $\exists$ *l*$\in$*S*. $\exists$ *r*::*nat*$\Rightarrow$*nat*. *strict_mono r* $\wedge$ ((*f* $\circ$ *r*) $\longrightarrow$ *l*) *sequentially*
shows *seq_compact S*
unfolding *seq_compact_def* **using** *assms* **by** *fast*

lemma *seq_compactE*:
assumes *seq_compact S* $\forall$ *n*. *f n* $\in$ *S*
obtains *l r* **where** *l* $\in$ *S strict_mono* (*r* :: *nat* $\Rightarrow$ *nat*) ((*f* $\circ$ *r*) $\longrightarrow$ *l*) *sequentially*
using *assms* **unfolding** *seq_compact_def* **by** *fast*

lemma *closed_sequentially*:
assumes *closed s* **and** $\forall$ *n*. *f n* $\in$ *s* **and** *f* $\longrightarrow$ *l*
shows *l* $\in$ *s*
proof (*rule ccontr*)
assume *l* $\notin$ *s*
with ‹*closed s*› **and** ‹*f* $\longrightarrow$ *l*› **have** *eventually* (λ*n*. *f n* $\in$ − *s*) *sequentially*
by (*fast intro*: *topological_tendstoD*)
with ‹$\forall$ *n*. *f n* $\in$ *s*› **show** *False*
by *simp*
qed

lemma *seq_compact_Int_closed*:
  assumes *seq_compact s* **and** *closed t*
  shows *seq_compact* (*s* ∩ *t*)
proof (*rule seq_compactI*)
  fix *f* **assume** ∀ *n*::*nat*. *f n* ∈ *s* ∩ *t*
  hence ∀ *n*. *f n* ∈ *s* **and** ∀ *n*. *f n* ∈ *t*
    by *simp_all*
  from ‹*seq_compact s*› **and** ‹∀ *n*. *f n* ∈ *s*›
  obtain *l r* **where** *l* ∈ *s* **and** *r*: *strict_mono r* **and** *l*: (*f* ∘ *r*) ⟶ *l*
    by (*rule seq_compactE*)
  from ‹∀ *n*. *f n* ∈ *t*› **have** ∀ *n*. (*f* ∘ *r*) *n* ∈ *t*
    by *simp*
  from ‹*closed t*› **and** *this* **and** *l* **have** *l* ∈ *t*
    by (*rule closed_sequentially*)
  with ‹*l* ∈ *s*› **and** *r* **and** *l* **show** ∃ *l*∈*s* ∩ *t*. ∃ *r*. *strict_mono r* ∧ (*f* ∘ *r*) ⟶ *l*
    by *fast*
qed

lemma *seq_compact_closed_subset*:
  assumes *closed s* **and** *s* ⊆ *t* **and** *seq_compact t*
  shows *seq_compact s*
  using *assms seq_compact_Int_closed* [*of t s*] **by** (*simp add*: *Int_absorb1*)

lemma *seq_compact_imp_countably_compact*:
  fixes *U* :: *'a* :: *first_countable_topology set*
  assumes *seq_compact U*
  shows *countably_compact U*
proof (*safe intro*!: *countably_compactI*)
  fix *A*
  assume *A*: ∀ *a*∈*A*. *open a U* ⊆ ⋃*A countable A*
  have *subseq*: ⋀*X*. *range X* ⊆ *U* ⟹ ∃ *r x*. *x* ∈ *U* ∧ *strict_mono* (*r* :: *nat* ⇒ *nat*) ∧ (*X* ∘ *r*) ⟶ *x*
    using ‹*seq_compact U*› **by** (*fastforce simp*: *seq_compact_def subset_eq*)
  show ∃ *T*⊆*A*. *finite T* ∧ *U* ⊆ ⋃ *T*
  proof *cases*
    assume *finite A*
    with *A* **show** *?thesis* **by** *auto*
  next
    assume *infinite A*
    then have *A* ≠ {} **by** *auto*
    show *?thesis*
    proof (*rule ccontr*)
      assume ¬ (∃ *T*⊆*A*. *finite T* ∧ *U* ⊆ ⋃ *T*)
      then have ∀ *T*. ∃ *x*. *T* ⊆ *A* ∧ *finite T* ⟶ (*x* ∈ *U* − ⋃ *T*)
        by *auto*
      then obtain *X'* **where** *T*: ⋀*T*. *T* ⊆ *A* ⟹ *finite T* ⟹ *X' T* ∈ *U* − ⋃ *T*
        by *metis*
      define *X* **where** *X n* = *X'* (*from_nat_into A* ‘ {.. *n*}) **for** *n*

```
      have X: ⋀n. X n ∈ U − (⋃i≤n. from_nat_into A i)
        using ‹A ≠ {}› unfolding X_def by (intro T) (auto intro: from_nat_into)
      then have range X ⊆ U
        by auto
      with subseq[of X] obtain r x where x ∈ U and r: strict_mono r (X ∘ r)
⟶ x
        by auto
      from ‹x∈U› ‹U ⊆ ⋃A› from_nat_into_surj[OF ‹countable A›]
      obtain n where x ∈ from_nat_into A n by auto
      with r(2) A(1) from_nat_into[OF ‹A ≠ {}›, of n]
      have eventually (λi. X (r i) ∈ from_nat_into A n) sequentially
        unfolding tendsto_def by (auto simp: comp_def)
      then obtain N where ⋀i. N ≤ i ⟹ X (r i) ∈ from_nat_into A n
        by (auto simp: eventually_sequentially)
      moreover from X have ⋀i. n ≤ r i ⟹ X (r i) ∉ from_nat_into A n
        by auto
      moreover from ‹strict_mono r›[THEN seq_suble, of max n N] have ∃i. n ≤
r i ∧ N ≤ i
        by (auto intro!: exI[of _ max n N])
      ultimately show False
        by auto
    qed
  qed
qed

lemma compact_imp_seq_compact:
  fixes U :: 'a :: first_countable_topology set
  assumes compact U
  shows seq_compact U
  unfolding seq_compact_def
proof safe
  fix X :: nat ⇒ 'a
  assume ∀n. X n ∈ U
  then have eventually (λx. x ∈ U) (filtermap X sequentially)
    by (auto simp: eventually_filtermap)
  moreover
  have filtermap X sequentially ≠ bot
    by (simp add: trivial_limit_def eventually_filtermap)
  ultimately
  obtain x where x ∈ U and x: inf (nhds x) (filtermap X sequentially) ≠ bot (is
?F ≠ _)
    using ‹compact U› by (auto simp: compact_filter)

  from countable_basis_at_decseq[of x]
  obtain A where A:
      ⋀i. open (A i)
      ⋀i. x ∈ A i
      ⋀S. open S ⟹ x ∈ S ⟹ eventually (λi. A i ⊆ S) sequentially
    by blast
```

```
  define s where s n i = (SOME j. i < j ∧ X j ∈ A (Suc n)) for n i
  {
    fix n i
    have ∃ a. i < a ∧ X a ∈ A (Suc n)
    proof (rule ccontr)
      assume ¬ (∃ a>i. X a ∈ A (Suc n))
      then have ⋀a. Suc i ≤ a ⟹ X a ∉ A (Suc n)
        by auto
      then have eventually (λx. x ∉ A (Suc n)) (filtermap X sequentially)
        by (auto simp: eventually_filtermap eventually_sequentially)
      moreover have eventually (λx. x ∈ A (Suc n)) (nhds x)
        using A(1,2)[of Suc n] by (auto simp: eventually_nhds)
      ultimately have eventually (λx. False) ?F
        by (auto simp: eventually_inf)
      with x show False
        by (simp add: eventually_False)
    qed
    then have i < s n i X (s n i) ∈ A (Suc n)
      unfolding s_def by (auto intro: someI2_ex)
  }
  note s = this
  define r where r = rec_nat (s 0 0) s
  have strict_mono r
    by (auto simp: r_def s strict_mono_Suc_iff)
  moreover
  have (λn. X (r n)) ⟶ x
  proof (rule topological_tendstoI)
    fix S
    assume open S x ∈ S
    with A(3) have eventually (λi. A i ⊆ S) sequentially
      by auto
    moreover
    {
      fix i
      assume Suc 0 ≤ i
      then have X (r i) ∈ A i
        by (cases i) (simp_all add: r_def s)
    }
    then have eventually (λi. X (r i) ∈ A i) sequentially
      by (auto simp: eventually_sequentially)
    ultimately show eventually (λi. X (r i) ∈ S) sequentially
      by eventually_elim auto
  qed
  ultimately show ∃ x ∈ U. ∃ r. strict_mono r ∧ (X ∘ r) ⟶ x
    using ‹x ∈ U› by (auto simp: convergent_def comp_def)
qed

lemma countably_compact_imp_acc_point:
  assumes countably_compact s
```

```
    and countable t
    and infinite t
    and t ⊆ s
  shows ∃ x∈s. ∀ U. x∈U ∧ open U ⟶ infinite (U ∩ t)
proof (rule ccontr)
  define C where C = (λF. interior (F ∪ (− t))) ‘ {F. finite F ∧ F ⊆ t }
  note ‹countably_compact s›
  moreover have ∀ t∈C. open t
    by (auto simp: C_def)
  moreover
  assume ¬ (∃ x∈s. ∀ U. x∈U ∧ open U ⟶ infinite (U ∩ t))
  then have s: ⋀x. x ∈ s ⟹ ∃ U. x∈U ∧ open U ∧ finite (U ∩ t) by metis
  have s ⊆ ⋃ C
    using ‹t ⊆ s›
    unfolding C_def
    apply (safe dest!: s)
    apply (rule_tac a=U ∩ t in UN_I)
    apply (auto intro!: interiorI simp add: finite_subset)
    done
  moreover
  from ‹countable t› have countable C
    unfolding C_def by (auto intro: countable_Collect_finite_subset)
  ultimately
  obtain D where D ⊆ C finite D s ⊆ ⋃ D
    by (rule countably_compactE)
  then obtain E where E: E ⊆ {F. finite F ∧ F ⊆ t } finite E
    and s: s ⊆ (⋃ F∈E. interior (F ∪ (− t)))
    by (metis (lifting) finite_subset_image C_def)
  from s ‹t ⊆ s› have t ⊆ ⋃ E
    using interior_subset by blast
  moreover have finite (⋃ E)
    using E by auto
  ultimately show False using ‹infinite t›
    by (auto simp: finite_subset)
qed

lemma countable_acc_point_imp_seq_compact:
  fixes s :: 'a::first_countable_topology set
  assumes ∀ t. infinite t ∧ countable t ∧ t ⊆ s ⟶
    (∃ x∈s. ∀ U. x∈U ∧ open U ⟶ infinite (U ∩ t))
  shows seq_compact s
proof −
  {
    fix f :: nat ⇒ 'a
    assume f: ∀ n. f n ∈ s
    have ∃ l∈s. ∃ r. strict_mono r ∧ ((f ∘ r) ⟶ l) sequentially
    proof (cases finite (range f))
      case True
      obtain l where infinite {n. f n = f l}
```

```
        using pigeonhole_infinite[OF _ True] by auto
      then obtain r :: nat ⇒ nat where strict_mono r and fr: ∀ n. f (r n) = f l
        using infinite_enumerate by blast
      then have strict_mono r ∧ (f ∘ r) ⟶ f l
        by (simp add: fr o_def)
      with f show ∃ l∈s. ∃ r. strict_mono r ∧ (f ∘ r) ⟶ l
        by auto
    next
      case False
      with f assms have ∃ x∈s. ∀ U. x∈U ∧ open U ⟶ infinite (U ∩ range f)
        by auto
      then obtain l where l ∈ s ∀ U. l∈U ∧ open U ⟶ infinite (U ∩ range f)
..
      from this(2) have ∃ r. strict_mono r ∧ ((f ∘ r) ⟶ l) sequentially
        using acc_point_range_imp_convergent_subsequence[of l f] by auto
      with ‹l ∈ s› show ∃ l∈s. ∃ r. strict_mono r ∧ ((f ∘ r) ⟶ l) sequentially ..
    qed
  }
  then show ?thesis
    unfolding seq_compact_def by auto
qed

lemma seq_compact_eq_countably_compact:
  fixes U :: 'a :: first_countable_topology set
  shows seq_compact U ⟷ countably_compact U
  using
    countable_acc_point_imp_seq_compact
    countably_compact_imp_acc_point
    seq_compact_imp_countably_compact
  by metis

lemma seq_compact_eq_acc_point:
  fixes s :: 'a :: first_countable_topology set
  shows seq_compact s ⟷
    (∀ t. infinite t ∧ countable t ∧ t ⊆ s --> (∃ x∈s. ∀ U. x∈U ∧ open U ⟶
infinite (U ∩ t)))
  using
    countable_acc_point_imp_seq_compact[of s]
    countably_compact_imp_acc_point[of s]
    seq_compact_imp_countably_compact[of s]
  by metis

lemma seq_compact_eq_compact:
  fixes U :: 'a :: second_countable_topology set
  shows seq_compact U ⟷ compact U
  using seq_compact_eq_countably_compact countably_compact_eq_compact by blast

proposition Bolzano_Weierstrass_imp_seq_compact:
  fixes s :: 'a::{t1_space, first_countable_topology} set
```

shows ∀ *t*. *infinite t* ∧ *t* ⊆ *s* ⟶ (∃ *x* ∈ *s*. *x islimpt t*) ⟹ *seq_compact s*
by (*rule countable_acc_point_imp_seq_compact*) (*metis islimpt_eq_acc_point*)

2.1.11 Cartesian products

lemma *seq_compact_Times*: *seq_compact s* ⟹ *seq_compact t* ⟹ *seq_compact* (*s* × *t*)
unfolding *seq_compact_def*
apply *clarify*
apply (*drule_tac x=fst* ∘ *f* **in** *spec*)
apply (*drule mp*, *simp add*: *mem_Times_iff*)
apply (*clarify*, *rename_tac l1 r1*)
apply (*drule_tac x=snd* ∘ *f* ∘ *r1* **in** *spec*)
apply (*drule mp*, *simp add*: *mem_Times_iff*)
apply (*clarify*, *rename_tac l2 r2*)
apply (*rule_tac x=*(*l1*, *l2*) **in** *rev_bexI*, *simp*)
apply (*rule_tac x=r1* ∘ *r2* **in** *exI*)
apply (*rule conjI*, *simp add*: *strict_mono_def*)
apply (*drule_tac f=r2* **in** *LIMSEQ_subseq_LIMSEQ*, *assumption*)
apply (*drule* (*1*) *tendsto_Pair*) **back**
apply (*simp add*: *o_def*)
done

lemma *compact_Times*:
assumes *compact s compact t*
shows *compact* (*s* × *t*)
proof (*rule compactI*)
fix *C*
assume *C*: ∀ *t*∈*C*. *open t s* × *t* ⊆ ⋃ *C*
have ∀ *x*∈*s*. ∃ *a*. *open a* ∧ *x* ∈ *a* ∧ (∃ *d*⊆*C*. *finite d* ∧ *a* × *t* ⊆ ⋃ *d*)
proof
fix *x*
assume *x* ∈ *s*
have ∀ *y*∈*t*. ∃ *a b c*. *c* ∈ *C* ∧ *open a* ∧ *open b* ∧ *x* ∈ *a* ∧ *y* ∈ *b* ∧ *a* × *b* ⊆ *c* (**is** ∀ *y*∈*t*. *?P y*)
proof
fix *y*
assume *y* ∈ *t*
with ‹*x* ∈ *s*› *C* **obtain** *c* **where** *c* ∈ *C* (*x*, *y*) ∈ *c open c* **by** *auto*
then show *?P y* **by** (*auto elim*!: *open_prod_elim*)
qed
then obtain *a b c* **where** *b*: ⋀*y*. *y* ∈ *t* ⟹ *open* (*b y*)
and *c*: ⋀*y*. *y* ∈ *t* ⟹ *c y* ∈ *C* ∧ *open* (*a y*) ∧ *open* (*b y*) ∧ *x* ∈ *a y* ∧ *y* ∈ *b y* ∧ *a y* × *b y* ⊆ *c y*
by *metis*
then have ∀ *y*∈*t*. *open* (*b y*) *t* ⊆ (⋃ *y*∈*t*. *b y*) **by** *auto*
with *compactE_image*[*OF* ‹*compact t*›] **obtain** *D* **where** *D*: *D* ⊆ *t finite D t* ⊆ (⋃ *y*∈*D*. *b y*)
by *metis*

moreover from *D c* **have** $(\bigcap y{\in}D.\ a\ y) \times t \subseteq (\bigcup y{\in}D.\ c\ y)$
by (*fastforce simp*: *subset_eq*)
ultimately show $\exists\, a.\ open\ a \wedge x \in a \wedge (\exists\, d{\subseteq}C.\ finite\ d \wedge a \times t \subseteq \bigcup d)$
using *c* **by** (*intro exI*[*of* _ *c'D*] *exI*[*of* _ $\bigcap$(*a'D*)] *conjI*) (*auto intro*!: *open_INT*)
qed
then obtain *a d* **where** *a*: $\bigwedge x.\ x{\in}s \Longrightarrow open\ (a\ x)\ s \subseteq (\bigcup x{\in}s.\ a\ x)$
and *d*: $\bigwedge x.\ x \in s \Longrightarrow d\ x \subseteq C \wedge finite\ (d\ x) \wedge a\ x \times t \subseteq \bigcup(d\ x)$
unfolding *subset_eq UN_iff* **by** *metis*
moreover
from *compactE_image*[*OF* ‹*compact s*› *a*]
obtain *e* **where** *e*: $e \subseteq s$ *finite e* **and** *s*: $s \subseteq (\bigcup x{\in}e.\ a\ x)$
by *auto*
moreover
{
from *s* **have** $s \times t \subseteq (\bigcup x{\in}e.\ a\ x \times t)$
by *auto*
also have $\ldots \subseteq (\bigcup x{\in}e.\ \bigcup(d\ x))$
using *d* ‹$e \subseteq s$› **by** (*intro UN_mono*) *auto*
finally have $s \times t \subseteq (\bigcup x{\in}e.\ \bigcup(d\ x))$ **.**
}
ultimately show $\exists\, C'{\subseteq}C.\ finite\ C' \wedge s \times t \subseteq \bigcup C'$
by (*intro exI*[*of* _ $(\bigcup x{\in}e.\ d\ x)$]) (*auto simp*: *subset_eq*)
qed

lemma *tube_lemma*:
assumes *compact K*
assumes *open W*
assumes $\{x0\} \times K \subseteq W$
shows $\exists\, X0.\ x0 \in X0 \wedge open\ X0 \wedge X0 \times K \subseteq W$
proof −
{
fix *y* **assume** $y \in K$
then have $(x0, y) \in W$ **using** *assms* **by** *auto*
with ‹*open W*›
have $\exists\, X0\ Y.\ open\ X0 \wedge open\ Y \wedge x0 \in X0 \wedge y \in Y \wedge X0 \times Y \subseteq W$
by (*rule open_prod_elim*) *blast*
}
then obtain *X0 Y* **where**
$*$: $\forall\, y \in K.\ open\ (X0\ y) \wedge open\ (Y\ y) \wedge x0 \in X0\ y \wedge y \in Y\ y \wedge X0\ y \times Y\ y \subseteq W$
by *metis*
from $*$ **have** $\forall\, t{\in}Y\ `\ K.\ open\ t\ K \subseteq \bigcup(Y\ `\ K)$ **by** *auto*
with ‹*compact K*› **obtain** *CC* **where** *CC*: $CC \subseteq Y\ `\ K$ *finite CC* $K \subseteq \bigcup CC$
by (*meson compactE*)
then obtain *c* **where** *c*: $\bigwedge C.\ C \in CC \Longrightarrow c\ C \in K \wedge C = Y\ (c\ C)$
by (*force intro*!: *choice*)
with $*$ *CC* **show** *?thesis*
by (*force intro*!: *exI*[**where** $x{=}\bigcap C{\in}CC.\ X0\ (c\ C)$])

```
qed

lemma continuous_on_prod_compactE:
  fixes fx::'a::topological_space × 'b::topological_space ⇒ 'c::metric_space
    and e::real
  assumes cont_fx: continuous_on (U × C) fx
  assumes compact C
  assumes [intro]: x0 ∈ U
  notes [continuous_intros] = continuous_on_compose2[OF cont_fx]
  assumes e > 0
  obtains X0 where x0 ∈ X0 open X0
    ∀x∈X0 ∩ U. ∀t ∈ C. dist (fx (x, t)) (fx (x0, t)) ≤ e
proof -
  define psi where psi = (λ(x, t). dist (fx (x, t)) (fx (x0, t)))
  define W0 where W0 = {(x, t) ∈ U × C. psi (x, t) < e}
  have W0_eq: W0 = psi -` {..<e} ∩ U × C
    by (auto simp: vimage_def W0_def)
  have open {..<e} by simp
  have continuous_on (U × C) psi
    by (auto intro!: continuous_intros simp: psi_def split_beta')
  from this[unfolded continuous_on_open_invariant, rule_format, OF ‹open {..<e}›]
  obtain W where W: open W W ∩ U × C = W0 ∩ U × C
    unfolding W0_eq by blast
  have {x0} × C ⊆ W ∩ U × C
    unfolding W
    by (auto simp: W0_def psi_def ‹0 < e›)
  then have {x0} × C ⊆ W by blast
  from tube_lemma[OF ‹compact C› ‹open W› this]
  obtain X0 where X0: x0 ∈ X0 open X0 X0 × C ⊆ W
    by blast

  have ∀x∈X0 ∩ U. ∀t ∈ C. dist (fx (x, t)) (fx (x0, t)) ≤ e
  proof safe
    fix x assume x: x ∈ X0 x ∈ U
    fix t assume t: t ∈ C
    have dist (fx (x, t)) (fx (x0, t)) = psi (x, t)
      by (auto simp: psi_def)
    also
    {
      have (x, t) ∈ X0 × C
        using t x
        by auto
      also note ‹. . . ⊆ W›
      finally have (x, t) ∈ W .
      with t x have (x, t) ∈ W ∩ U × C
        by blast
      also note ‹W ∩ U × C = W0 ∩ U × C›
      finally have psi (x, t) < e
        by (auto simp: W0_def)
```

  }
  finally show *dist* (*fx* (*x*, *t*)) (*fx* (*x0*, *t*)) ≤ *e* **by** *simp*
 qed
 from *X0*(*1*,*2*) *this* **show** *?thesis* **..**
qed

2.1.12 Continuity

lemma *continuous_at_imp_continuous_within*:
 continuous (*at x*) *f* ⟹ *continuous* (*at x within s*) *f*
 unfolding *continuous_within continuous_at* **using** *Lim_at_imp_Lim_at_within* **by** *auto*

lemma *Lim_trivial_limit*: *trivial_limit net* ⟹ (*f* ⟶ *l*) *net*
 by *simp*

lemmas *continuous_on* = *continuous_on_def* — legacy theorem name

lemma *continuous_within_subset*:
 continuous (*at x within s*) *f* ⟹ *t* ⊆ *s* ⟹ *continuous* (*at x within t*) *f*
 unfolding *continuous_within* **by**(*metis tendsto_within_subset*)

lemma *continuous_on_interior*:
 continuous_on s f ⟹ *x* ∈ *interior s* ⟹ *continuous* (*at x*) *f*
 by (*metis continuous_on_eq_continuous_at continuous_on_subset interiorE*)

lemma *continuous_on_eq*:
 ⟦*continuous_on s f*; ⋀*x*. *x* ∈ *s* ⟹ *f x* = *g x*⟧ ⟹ *continuous_on s g*
 unfolding *continuous_on_def tendsto_def eventually_at_topological*
 by *simp*

Characterization of various kinds of continuity in terms of sequences.

lemma *continuous_within_sequentiallyI*:
 fixes *f* :: *'a*::{*first_countable_topology*, *t2_space*} ⇒ *'b*::*topological_space*
 assumes ⋀*u*::*nat* ⇒ *'a*. *u* ⟶ *a* ⟹ (∀ *n*. *u n* ∈ *s*) ⟹ (λ*n*. *f* (*u n*)) ⟶ *f a*
 shows *continuous* (*at a within s*) *f*
 using *assms* **unfolding** *continuous_within tendsto_def*[**where** *l* = *f a*]
 by (*auto intro*!: *sequentially_imp_eventually_within*)

lemma *continuous_within_tendsto_compose*:
 fixes *f*::*'a*::*t2_space* ⇒ *'b*::*topological_space*
 assumes *continuous* (*at a within s*) *f*
   eventually (λ*n*. *x n* ∈ *s*) *F*
   (*x* ⟶ *a*) *F*
 shows ((λ*n*. *f* (*x n*)) ⟶ *f a*) *F*
proof −
 have ∗: *filterlim x* (*inf* (*nhds a*) (*principal s*)) *F*
  using *assms*(*2*) *assms*(*3*) **unfolding** *at_within_def filterlim_inf* **by** (*auto simp*:

filterlim_principal eventually_mono)
  show *?thesis*
   by (*auto simp*: *assms*(*1*) *continuous_within*[*symmetric*] *tendsto_at_within_iff_tendsto_nhds*[*symmetric*] *intro*!: *filterlim_compose*[*OF* _ ∗])
qed

lemma *continuous_within_tendsto_compose'*:
  fixes *f*::*'a*::*t2_space* ⇒ *'b*::*topological_space*
  assumes *continuous* (*at a within s*) *f*
    ⋀*n*. *x n* ∈ *s*
    (*x* ⟶ *a*) *F*
  shows ((λ*n*. *f* (*x n*)) ⟶ *f a*) *F*
 by (*auto intro*!: *continuous_within_tendsto_compose*[*OF assms*(*1*)] *simp add*: *assms*)

lemma *continuous_within_sequentially*:
  fixes *f* :: *'a*::{*first_countable_topology*, *t2_space*} ⇒ *'b*::*topological_space*
  shows *continuous* (*at a within s*) *f* ⟷
    (∀ *x*. (∀ *n*::*nat*. *x n* ∈ *s*) ∧ (*x* ⟶ *a*) *sequentially*
        ⟶ ((*f* ∘ *x*) ⟶ *f a*) *sequentially*)
  using *continuous_within_tendsto_compose'*[*of a s f* _ *sequentially*]
    continuous_within_sequentiallyI[*of a s f*]
  by (*auto simp*: *o_def*)

lemma *continuous_at_sequentiallyI*:
  fixes *f* :: *'a*::{*first_countable_topology*, *t2_space*} ⇒ *'b*::*topological_space*
  assumes ⋀*u*. *u* ⟶ *a* ⟹ (λ*n*. *f* (*u n*)) ⟶ *f a*
  shows *continuous* (*at a*) *f*
  using *continuous_within_sequentiallyI*[*of a UNIV f*] *assms* **by** *auto*

lemma *continuous_at_sequentially*:
  fixes *f* :: *'a*::*metric_space* ⇒ *'b*::*topological_space*
  shows *continuous* (*at a*) *f* ⟷
    (∀ *x*. (*x* ⟶ *a*) *sequentially* −−> ((*f* ∘ *x*) ⟶ *f a*) *sequentially*)
  using *continuous_within_sequentially*[*of a UNIV f*] **by** *simp*

lemma *continuous_on_sequentiallyI*:
  fixes *f* :: *'a*::{*first_countable_topology*, *t2_space*} ⇒ *'b*::*topological_space*
   assumes ⋀*u a*. (∀ *n*. *u n* ∈ *s*) ⟹ *a* ∈ *s* ⟹ *u* ⟶ *a* ⟹ (λ*n*. *f* (*u n*)) ⟶ *f a*
  shows *continuous_on s f*
  using *assms* **unfolding** *continuous_on_eq_continuous_within*
  using *continuous_within_sequentiallyI*[*of* _ *s f*] **by** *auto*

lemma *continuous_on_sequentially*:
  fixes *f* :: *'a*::{*first_countable_topology*, *t2_space*} ⇒ *'b*::*topological_space*
  shows *continuous_on s f* ⟷
    (∀ *x*. ∀ *a* ∈ *s*. (∀ *n*. *x*(*n*) ∈ *s*) ∧ (*x* ⟶ *a*) *sequentially*
      −−> ((*f* ∘ *x*) ⟶ *f a*) *sequentially*)
    (**is** *?lhs* = *?rhs*)

proof
  assume *?rhs*
  then show *?lhs*
    using *continuous_within_sequentially*[*of* _ *s f*]
    unfolding *continuous_on_eq_continuous_within*
    by *auto*
next
  assume *?lhs*
  then show *?rhs*
    unfolding *continuous_on_eq_continuous_within*
    using *continuous_within_sequentially*[*of* _ *s f*]
    by *auto*
qed

Continuity in terms of open preimages.

lemma *continuous_at_open*:
  continuous (*at x*) *f* $\longleftrightarrow$ ($\forall$ *t*. *open t* $\wedge$ *f x* $\in$ *t* --> ($\exists$ *s*. *open s* $\wedge$ *x* $\in$ *s* $\wedge$ ($\forall$ *x'* $\in$ *s*. (*f x'*) $\in$ *t*)))
  unfolding *continuous_within_topological* [*of x UNIV f*]
  unfolding *imp_conjL*
  by (*intro all_cong imp_cong ex_cong conj_cong refl*) *auto*

lemma *continuous_imp_tendsto*:
  assumes *continuous* (*at x0*) *f*
    and *x* $\longrightarrow$ *x0*
  shows (*f* $\circ$ *x*) $\longrightarrow$ (*f x0*)
proof (*rule topological_tendstoI*)
  fix *S*
  assume *open S f x0* $\in$ *S*
  then obtain *T* **where** *T_def*: *open T x0* $\in$ *T* $\forall$ *x*$\in$*T*. *f x* $\in$ *S*
     using *assms continuous_at_open* **by** *metis*
  then have *eventually* (λ*n*. *x n* $\in$ *T*) *sequentially*
    using *assms T_def* **by** (*auto simp*: *tendsto_def*)
  then show *eventually* (λ*n*. (*f* $\circ$ *x*) *n* $\in$ *S*) *sequentially*
    using *T_def* **by** (*auto elim*!: *eventually_mono*)
qed

2.1.13 Homeomorphisms

definition *homeomorphism s t f g* $\longleftrightarrow$
  ($\forall$ *x*$\in$*s*. (*g*(*f x*) = *x*)) $\wedge$ (*f* ` *s* = *t*) $\wedge$ *continuous_on s f* $\wedge$
  ($\forall$ *y*$\in$*t*. (*f*(*g y*) = *y*)) $\wedge$ (*g* ` *t* = *s*) $\wedge$ *continuous_on t g*

lemma *homeomorphismI* [*intro?*]:
  assumes *continuous_on S f continuous_on T g*
        f ` *S* $\subseteq$ *T g* ` *T* $\subseteq$ *S* $\bigwedge$*x*. *x* $\in$ *S* $\Longrightarrow$ *g*(*f x*) = *x* $\bigwedge$*y*. *y* $\in$ *T* $\Longrightarrow$ *f*(*g y*) = *y*
    shows *homeomorphism S T f g*
  using *assms* **by** (*force simp*: *homeomorphism_def*)

lemma *homeomorphism_translation*:
  fixes *a* :: *'a* :: *real_normed_vector*
  shows *homeomorphism* ((+) *a* ‘ *S*) *S* ((+) (− *a*)) ((+) *a*)
unfolding *homeomorphism_def* **by** (*auto simp*: *algebra_simps continuous_intros*)

lemma *homeomorphism_ident*: *homeomorphism T T* (λ*a*. *a*) (λ*a*. *a*)
  by (*rule homeomorphismI*) *auto*

lemma *homeomorphism_compose*:
  assumes *homeomorphism S T f g homeomorphism T U h k*
    shows *homeomorphism S U* (*h o f*) (*g o k*)
  using *assms*
  unfolding *homeomorphism_def*
  by (*intro conjI ballI continuous_on_compose*) (*auto simp*: *image_iff*)

lemma *homeomorphism_cong*:
  homeomorphism X' Y' f' g'
    if *homeomorphism X Y f g X'* = *X Y'* = *Y* ⋀*x*. *x* ∈ *X* ⟹ *f' x* = *f x* ⋀*y*. *y* ∈ *Y* ⟹ *g' y* = *g y*
  using *that* **by** (*auto simp add*: *homeomorphism_def*)

lemma *homeomorphism_empty* [*simp*]:
  homeomorphism {} {} *f g*
  unfolding *homeomorphism_def* **by** *auto*

lemma *homeomorphism_symD*: *homeomorphism S t f g* ⟹ *homeomorphism t S g f*
  by (*simp add*: *homeomorphism_def*)

lemma *homeomorphism_sym*: *homeomorphism S t f g* = *homeomorphism t S g f*
  by (*force simp*: *homeomorphism_def*)

definition *homeomorphic* :: *'a*::*topological_space set* ⇒ *'b*::*topological_space set* ⇒ *bool*
    (**infixr** *homeomorphic 60*)
  where *s homeomorphic t* ≡ (∃*f g*. *homeomorphism s t f g*)

lemma *homeomorphic_empty* [*iff*]:
    S homeomorphic {} ⟷ *S* = {} {} *homeomorphic S* ⟷ *S* = {}
  by (*auto simp*: *homeomorphic_def homeomorphism_def*)

lemma *homeomorphic_refl*: *s homeomorphic s*
  unfolding *homeomorphic_def homeomorphism_def*
  using *continuous_on_id*
  apply (*rule_tac x* = (λ*x*. *x*) **in** *exI*)
  apply (*rule_tac x* = (λ*x*. *x*) **in** *exI*)
  apply *blast*
  done

```
lemma homeomorphic_sym: s homeomorphic t ⟷ t homeomorphic s
  unfolding homeomorphic_def homeomorphism_def
  by blast

lemma homeomorphic_trans [trans]:
  assumes S homeomorphic T
      and T homeomorphic U
    shows S homeomorphic U
  using assms
  unfolding homeomorphic_def
by (metis homeomorphism_compose)

lemma homeomorphic_minimal:
  s homeomorphic t ⟷
    (∃f g. (∀x∈s. f(x) ∈ t ∧ (g(f(x)) = x)) ∧
           (∀y∈t. g(y) ∈ s ∧ (f(g(y)) = y)) ∧
           continuous_on s f ∧ continuous_on t g)
   (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    by (fastforce simp: homeomorphic_def homeomorphism_def)
next
  assume ?rhs
  then show ?lhs
    apply clarify
    unfolding homeomorphic_def homeomorphism_def
    by (metis equalityI image_subset_iff subsetI)
 qed

lemma homeomorphicI [intro?]:
   ⟦f ‘ S = T; g ‘ T = S;
     continuous_on S f; continuous_on T g;
     ⋀x. x ∈ S ⟹ g(f(x)) = x;
     ⋀y. y ∈ T ⟹ f(g(y)) = y⟧ ⟹ S homeomorphic T
unfolding homeomorphic_def homeomorphism_def by metis

lemma homeomorphism_of_subsets:
   ⟦homeomorphism S T f g; S' ⊆ S; T'' ⊆ T; f ‘ S' = T'⟧
    ⟹ homeomorphism S' T' f g
apply (auto simp: homeomorphism_def elim!: continuous_on_subset)
by (metis subsetD imageI)

lemma homeomorphism_apply1: ⟦homeomorphism S T f g; x ∈ S⟧ ⟹ g(f x) = x
  by (simp add: homeomorphism_def)

lemma homeomorphism_apply2: ⟦homeomorphism S T f g; x ∈ T⟧ ⟹ f(g x) =
x
  by (simp add: homeomorphism_def)
```

lemma *homeomorphism_image1*: *homeomorphism S T f g* ⟹ *f* ‘ *S* = *T*
by (*simp add*: *homeomorphism_def*)

lemma *homeomorphism_image2*: *homeomorphism S T f g* ⟹ *g* ‘ *T* = *S*
by (*simp add*: *homeomorphism_def*)

lemma *homeomorphism_cont1*: *homeomorphism S T f g* ⟹ *continuous_on S f*
by (*simp add*: *homeomorphism_def*)

lemma *homeomorphism_cont2*: *homeomorphism S T f g* ⟹ *continuous_on T g*
by (*simp add*: *homeomorphism_def*)

lemma *continuous_on_no_limpt*:
(⋀*x*. ¬ *x islimpt S*) ⟹ *continuous_on S f*
unfolding *continuous_on_def*
by (*metis UNIV_I empty_iff eventually_at_topological islimptE open_UNIV tendsto_def trivial_limit_within*)

lemma *continuous_on_finite*:
fixes *S* :: *'a*::*t1_space set*
shows *finite S* ⟹ *continuous_on S f*
by (*metis continuous_on_no_limpt islimpt_finite*)

lemma *homeomorphic_finite*:
fixes *S* :: *'a*::*t1_space set* **and** *T* :: *'b*::*t1_space set*
assumes *finite T*
shows *S homeomorphic T* ⟷ *finite S* ∧ *finite T* ∧ *card S* = *card T* (**is** *?lhs* = *?rhs*)
proof
assume *S homeomorphic T*
with *assms* **show** *?rhs*
apply (*auto simp*: *homeomorphic_def homeomorphism_def*)
apply (*metis finite_imageI*)
by (*metis card_image_le finite_imageI le_antisym*)
next
assume *R*: *?rhs*
with *finite_same_card_bij* **obtain** *h* **where** *bij_betw h S T*
by *auto*
with *R* **show** *?lhs*
apply (*auto simp*: *homeomorphic_def homeomorphism_def continuous_on_finite*)
apply (*rule_tac x*=*h* **in** *exI*)
apply (*rule_tac x*=*inv_into S h* **in** *exI*)
apply (*auto simp*: *bij_betw_inv_into_left bij_betw_inv_into_right bij_betw_imp_surj_on inv_into_into bij_betwE*)
apply (*metis bij_betw_def bij_betw_inv_into*)
done
qed

Relatively weak hypotheses if a set is compact.

```
lemma homeomorphism_compact:
  fixes f :: 'a::topological_space ⇒ 'b::t2_space
  assumes compact s continuous_on s f  f ‘ s = t  inj_on f s
  shows ∃ g. homeomorphism s t f g
proof −
  define g where g x = (SOME y. y∈s ∧ f y = x) for x
  have g: ∀ x∈s. g (f x) = x
    using assms(3) assms(4)[unfolded inj_on_def] unfolding g_def by auto
  {
    fix y
    assume y ∈ t
    then obtain x where x:f x = y x∈s
      using assms(3) by auto
    then have g (f x) = x using g by auto
    then have f (g y) = y unfolding x(1)[symmetric] by auto
  }
  then have g':∀ x∈t. f (g x) = x by auto
  moreover
  {
    fix x
    have x∈s =⇒ x ∈ g ‘ t
      using g[THEN bspec[where x=x]]
      unfolding image_iff
      using assms(3)
      by (auto intro!: bexI[where x=f x])
    moreover
    {
      assume x∈g ‘ t
      then obtain y where y:y∈t g y = x by auto
      then obtain x' where x':x'∈s f x' = y
        using assms(3) by auto
      then have x ∈ s
        unfolding g_def
        using someI2[of λb. b∈s ∧ f b = y x' λx. x∈s]
        unfolding y(2)[symmetric] and g_def
        by auto
    }
    ultimately have x∈s ←→ x ∈ g ‘ t ..
  }
  then have g ‘ t = s by auto
  ultimately show ?thesis
    unfolding homeomorphism_def homeomorphic_def
    using assms continuous_on_inv by fastforce
qed

lemma homeomorphic_compact:
  fixes f :: 'a::topological_space ⇒ 'b::t2_space
  shows compact s =⇒ continuous_on s f =⇒ (f ‘ s = t) =⇒ inj_on f s =⇒ s
homeomorphic t
```

```
  unfolding homeomorphic_def by (metis homeomorphism_compact)
```

Preservation of topological properties.

```
lemma homeomorphic_compactness: s homeomorphic t ⟹ (compact s ⟷ compact t)
  unfolding homeomorphic_def homeomorphism_def
  by (metis compact_continuous_image)
```

2.1.14 On Linorder Topologies

```
lemma islimpt_greaterThanLessThan1:
  fixes a b::'a::{linorder_topology, dense_order}
  assumes a < b
  shows  a islimpt {a<..<b}
proof (rule islimptI)
  fix T
  assume open T a ∈ T
  from open_right[OF this ‹a < b›]
  obtain c where c: a < c {a..<c} ⊆ T by auto
  with assms dense[of a min c b]
  show ∃y∈{a<..<b}. y ∈ T ∧ y ≠ a
    by (metis atLeastLessThan_iff greaterThanLessThan_iff min_less_iff_conj
      not_le order.strict_implies_order subset_eq)
qed

lemma islimpt_greaterThanLessThan2:
  fixes a b::'a::{linorder_topology, dense_order}
  assumes a < b
  shows  b islimpt {a<..<b}
proof (rule islimptI)
  fix T
  assume open T b ∈ T
  from open_left[OF this ‹a < b›]
  obtain c where c: c < b {c<..b} ⊆ T by auto
  with assms dense[of max a c b]
  show ∃y∈{a<..<b}. y ∈ T ∧ y ≠ b
    by (metis greaterThanAtMost_iff greaterThanLessThan_iff max_less_iff_conj
      not_le order.strict_implies_order subset_eq)
qed

lemma closure_greaterThanLessThan[simp]:
  fixes a b::'a::{linorder_topology, dense_order}
  shows a < b ⟹ closure {a <..< b} = {a .. b} (is _ ⟹ ?l = ?r)
proof
  have ?l ⊆ closure ?r
    by (rule closure_mono) auto
  thus closure {a<..<b} ⊆ {a..b} by simp
qed (auto simp: closure_def order.order_iff_strict islimpt_greaterThanLessThan1
  islimpt_greaterThanLessThan2)
```

lemma *closure_greaterThan*[*simp*]:
  fixes *a b*::'*a*::{*no_top*, *linorder_topology*, *dense_order*}
  shows *closure* {*a*<..} = {*a*..}
proof −
  from *gt_ex* **obtain** *b* **where** $a < b$ **by** *auto*
  hence {*a*<..} = {*a*<..<*b*} ∪ {*b*..} **by** *auto*
  also have *closure* ... = {*a*..} **using** ‹$a < b$› **unfolding** *closure_Un*
    by *auto*
  finally show *?thesis* **.**
qed

lemma *closure_lessThan*[*simp*]:
  fixes *b*::'*a*::{*no_bot*, *linorder_topology*, *dense_order*}
  shows *closure* {..<*b*} = {..*b*}
proof −
  from *lt_ex* **obtain** *a* **where** $a < b$ **by** *auto*
  hence {..<*b*} = {*a*<..<*b*} ∪ {..*a*} **by** *auto*
  also have *closure* ... = {..*b*} **using** ‹$a < b$› **unfolding** *closure_Un*
    by *auto*
  finally show *?thesis* **.**
qed

lemma *closure_atLeastLessThan*[*simp*]:
  fixes *a b*::'*a*::{*linorder_topology*, *dense_order*}
  assumes $a < b$
  shows *closure* {*a* ..< *b*} = {*a* .. *b*}
proof −
  from *assms* **have** {*a* ..< *b*} = {*a*} ∪ {*a* <..< *b*} **by** *auto*
  also have *closure* ... = {*a* .. *b*} **unfolding** *closure_Un*
    by (*auto simp*: *assms less_imp_le*)
  finally show *?thesis* **.**
qed

lemma *closure_greaterThanAtMost*[*simp*]:
  fixes *a b*::'*a*::{*linorder_topology*, *dense_order*}
  assumes $a < b$
  shows *closure* {*a* <.. *b*} = {*a* .. *b*}
proof −
  from *assms* **have** {*a* <.. *b*} = {*b*} ∪ {*a* <..< *b*} **by** *auto*
  also have *closure* ... = {*a* .. *b*} **unfolding** *closure_Un*
    by (*auto simp*: *assms less_imp_le*)
  finally show *?thesis* **.**
qed

end

2.2 Operators involving abstract topology

theory *Abstract_Topology*
  imports
    Complex_Main
    HOL−Library.Set_Idioms
    HOL−Library.FuncSet
begin

2.2.1 General notion of a topology as a value

definition *istopology* :: ('*a set* ⇒ *bool*) ⇒ *bool* **where**
  istopology $L \equiv (\forall S\ T.\ L\ S \longrightarrow L\ T \longrightarrow L\ (S \cap T)) \wedge (\forall \mathcal{K}.\ (\forall K{\in}\mathcal{K}.\ L\ K) \longrightarrow L\ (\bigcup \mathcal{K}))$

typedef '*a topology* = {*L*::('*a set*) ⇒ *bool*. *istopology L*}
  morphisms *openin topology*
  unfolding *istopology_def* **by** *blast*

lemma *istopology_openin*[*intro*]: *istopology*(*openin U*)
  using *openin*[*of U*] **by** *blast*

lemma *istopology_open*: *istopology open*
  by (*auto simp*: *istopology_def*)

lemma *topology_inverse'*: *istopology U* ⟹ *openin* (*topology U*) = *U*
  using *topology_inverse*[*unfolded mem_Collect_eq*] **.**

lemma *topology_inverse_iff*: *istopology U* ⟷ *openin* (*topology U*) = *U*
  using *topology_inverse*[*of U*] *istopology_openin*[*of topology U*] **by** *auto*

lemma *topology_eq*: *T1* = *T2* ⟷ (∀ *S*. *openin T1 S* ⟷ *openin T2 S*)
proof
  assume *T1* = *T2*
  then show ∀ *S*. *openin T1 S* ⟷ *openin T2 S* **by** *simp*
next
  assume *H*: ∀ *S*. *openin T1 S* ⟷ *openin T2 S*
  then have *openin T1* = *openin T2* **by** (*simp add*: *fun_eq_iff*)
  then have *topology* (*openin T1*) = *topology* (*openin T2*) **by** *simp*
  then show *T1* = *T2* **unfolding** *openin_inverse* **.**
qed

The "universe": the union of all sets in the topology.

definition *topspace* $T = \bigcup \{S.\ openin\ T\ S\}$

Main properties of open sets

proposition *openin_clauses*:
  fixes *U* :: '*a topology*
  shows

openin U {}
$\bigwedge S\ T$. *openin U S* $\Longrightarrow$ *openin U T* $\Longrightarrow$ *openin U* $(S \cap T)$
$\bigwedge K$. $(\forall S \in K$. *openin U S*$)$ $\Longrightarrow$ *openin U* $(\bigcup K)$
using *openin*[*of U*] **unfolding** *istopology_def* **by** *auto*

lemma *openin_subset*: *openin U S* $\Longrightarrow$ $S \subseteq$ *topspace U*
unfolding *topspace_def* **by** *blast*

lemma *openin_empty*[*simp*]: *openin U* {}
by (*rule openin_clauses*)

lemma *openin_Int*[*intro*]: *openin U S* $\Longrightarrow$ *openin U T* $\Longrightarrow$ *openin U* $(S \cap T)$
by (*rule openin_clauses*)

lemma *openin_Union*[*intro*]: $(\bigwedge S.\ S \in K \Longrightarrow$ *openin U S*$)$ $\Longrightarrow$ *openin U* $(\bigcup K)$
using *openin_clauses* **by** *blast*

lemma *openin_Un*[*intro*]: *openin U S* $\Longrightarrow$ *openin U T* $\Longrightarrow$ *openin U* $(S \cup T)$
using *openin_Union*[*of* $\{S,T\}$ *U*] **by** *auto*

lemma *openin_topspace*[*intro*, *simp*]: *openin U* (*topspace U*)
by (*force simp*: *openin_Union topspace_def*)

lemma *openin_subopen*: *openin U S* $\longleftrightarrow$ $(\forall x \in S.\ \exists T.$ *openin U T* $\wedge\ x \in T\ \wedge$ $T \subseteq S)$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?lhs*
then show *?rhs* **by** *auto*
next
assume *H*: *?rhs*
let *?t* = $\bigcup\{T.$ *openin U T* $\wedge\ T \subseteq S\}$
have *openin U ?t* **by** (*force simp*: *openin_Union*)
also have *?t* = *S* **using** *H* **by** *auto*
finally show *openin U S* **.**
qed

lemma *openin_INT* [*intro*]:
assumes *finite I*
$\bigwedge i.\ i \in I \Longrightarrow$ *openin T* $(U\ i)$
shows *openin T* $((\bigcap i \in I.\ U\ i) \cap$ *topspace T*$)$
using *assms* **by** (*induct*, *auto simp*: *inf_sup_aci*(*2*) *openin_Int*)

lemma *openin_INT2* [*intro*]:
assumes *finite I* $I \neq \{\}$
$\bigwedge i.\ i \in I \Longrightarrow$ *openin T* $(U\ i)$
shows *openin T* $(\bigcap i \in I.\ U\ i)$
proof –
have $(\bigcap i \in I.\ U\ i) \subseteq$ *topspace T*

using ‹$I \neq \{\}$› *openin_subset*[*OF assms*(*3*)] **by** *auto*
then show *?thesis*
using *openin_INT*[*of* _ _ *U*, *OF assms*(*1*) *assms*(*3*)] **by** (*simp add*: *inf.absorb2* *inf_commute*)
qed

lemma *openin_Inter* [*intro*]:
assumes *finite* $\mathcal{F}$ $\mathcal{F} \neq \{\}$ $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *openin T X* **shows** *openin T* ($\bigcap \mathcal{F}$)
by (*metis* (*full_types*) *assms openin_INT2 image_ident*)

lemma *openin_Int_Inter*:
assumes *finite* $\mathcal{F}$ *openin T U* $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *openin T X* **shows** *openin T* ($U \cap \bigcap \mathcal{F}$)
using *openin_Inter* [*of insert U* $\mathcal{F}$] *assms* **by** *auto*

Closed sets

definition *closedin* :: *'a topology* ⇒ *'a set* ⇒ *bool* **where**
closedin U S ⟷ $S \subseteq$ *topspace U* ∧ *openin U* (*topspace U* − *S*)

lemma *closedin_subset*: *closedin U S* ⟹ $S \subseteq$ *topspace U*
by (*metis closedin_def*)

lemma *closedin_empty*[*simp*]: *closedin U* {}
by (*simp add*: *closedin_def*)

lemma *closedin_topspace*[*intro*, *simp*]: *closedin U* (*topspace U*)
by (*simp add*: *closedin_def*)

lemma *closedin_Un*[*intro*]: *closedin U S* ⟹ *closedin U T* ⟹ *closedin U* ($S \cup T$)
by (*auto simp*: *Diff_Un closedin_def*)

lemma *Diff_Inter*[*intro*]: $A - \bigcap S = \bigcup \{A - s \mid s.\ s \in S\}$
by *auto*

lemma *closedin_Union*:
assumes *finite S* $\bigwedge T.\ T \in S \Longrightarrow$ *closedin U T*
shows *closedin U* ($\bigcup S$)
using *assms* **by** *induction auto*

lemma *closedin_Inter*[*intro*]:
assumes *Ke*: $K \neq \{\}$
and *Kc*: $\bigwedge S.\ S \in K \Longrightarrow$ *closedin U S*
shows *closedin U* ($\bigcap K$)
using *Ke Kc* **unfolding** *closedin_def Diff_Inter* **by** *auto*

lemma *closedin_INT*[*intro*]:
assumes $A \neq \{\}$ $\bigwedge x.\ x \in A \Longrightarrow$ *closedin U* (*B x*)

shows *closedin U* ($\bigcap x{\in}A.\ B\ x$)
using *assms* **by** *blast*

lemma *closedin_Int*[*intro*]: *closedin U S* $\Longrightarrow$ *closedin U T* $\Longrightarrow$ *closedin U* ($S \cap T$)
using *closedin_Inter*[*of* {*S*,*T*} *U*] **by** *auto*

lemma *openin_closedin_eq*: *openin U S* $\longleftrightarrow$ $S \subseteq$ *topspace U* $\wedge$ *closedin U* (*topspace U* $-$ *S*)
by (*metis Diff_subset closedin_def double_diff equalityD1 openin_subset*)

lemma *topology_finer_closedin*:
topspace X = *topspace Y* $\Longrightarrow$ ($\forall S$. *openin Y S* $\longrightarrow$ *openin X S*) $\longleftrightarrow$ ($\forall S$. *closedin Y S* $\longrightarrow$ *closedin X S*)
by (*metis closedin_def openin_closedin_eq*)

lemma *openin_closedin*: $S \subseteq$ *topspace U* $\Longrightarrow$ (*openin U S* $\longleftrightarrow$ *closedin U* (*topspace U* $-$ *S*))
by (*simp add*: *openin_closedin_eq*)

lemma *openin_diff*[*intro*]:
assumes *oS*: *openin U S*
and *cT*: *closedin U T*
shows *openin U* ($S - T$)
proof $-$
have $S - T = S \cap$ (*topspace U* $-$ *T*) **using** *openin_subset*[*of U S*] *oS cT*
by (*auto simp*: *topspace_def openin_subset*)
then show *?thesis* **using** *oS cT*
by (*auto simp*: *closedin_def*)
qed

lemma *closedin_diff*[*intro*]:
assumes *oS*: *closedin U S*
and *cT*: *openin U T*
shows *closedin U* ($S - T$)
proof $-$
have $S - T = S \cap$ (*topspace U* $-$ *T*)
using *closedin_subset*[*of U S*] *oS cT* **by** (*auto simp*: *topspace_def*)
then show *?thesis*
using *oS cT* **by** (*auto simp*: *openin_closedin_eq*)
qed

2.2.2 The discrete topology

definition *discrete_topology* **where** *discrete_topology U* $\equiv$ *topology* ($\lambda S.\ S \subseteq U$)

lemma *openin_discrete_topology* [*simp*]: *openin* (*discrete_topology U*) *S* $\longleftrightarrow$ $S \subseteq U$
proof $-$

```
  have istopology (λS. S ⊆ U)
    by (auto simp: istopology_def)
  then show ?thesis
    by (simp add: discrete_topology_def topology_inverse')
qed

lemma topspace_discrete_topology [simp]: topspace(discrete_topology U) = U
  by (meson openin_discrete_topology openin_subset openin_topspace order_refl sub-
set_antisym)

lemma closedin_discrete_topology [simp]: closedin (discrete_topology U) S ⟷ S
⊆ U
  by (simp add: closedin_def)

lemma discrete_topology_unique:
   discrete_topology U = X ⟷ topspace X = U ∧ (∀x ∈ U. openin X {x}) (is
?lhs = ?rhs)
proof
  assume R: ?rhs
  then have openin X S if S ⊆ U for S
    using openin_subopen subsetD that by fastforce
  moreover have x ∈ topspace X if openin X S and x ∈ S for x S
    using openin_subset that by blast
  ultimately
  show ?lhs
    using R by (auto simp: topology_eq)
qed auto

lemma discrete_topology_unique_alt:
  discrete_topology U = X ⟷ topspace X ⊆ U ∧ (∀x ∈ U. openin X {x})
  using openin_subset
  by (auto simp: discrete_topology_unique)

lemma subtopology_eq_discrete_topology_empty:
   X = discrete_topology {} ⟷ topspace X = {}
  using discrete_topology_unique [of {} X] by auto

lemma subtopology_eq_discrete_topology_sing:
   X = discrete_topology {a} ⟷ topspace X = {a}
  by (metis discrete_topology_unique openin_topspace singletonD)
```

2.2.3 Subspace topology

```
definition subtopology :: 'a topology ⇒ 'a set ⇒ 'a topology where
subtopology U V = topology (λT. ∃S. T = S ∩ V ∧ openin U S)

lemma istopology_subtopology: istopology (λT. ∃S. T = S ∩ V ∧ openin U S)
  (is istopology ?L)
proof –
```

have *?L* {} **by** *blast*
{
fix *A B*
assume *A*: *?L A* **and** *B*: *?L B*
from *A B* **obtain** *Sa* **and** *Sb* **where** *Sa*: *openin U Sa A* = *Sa* ∩ *V* **and** *Sb*: *openin U Sb B* = *Sb* ∩ *V*
by *blast*
have *A* ∩ *B* = (*Sa* ∩ *Sb*) ∩ *V openin U* (*Sa* ∩ *Sb*)
using *Sa Sb* **by** *blast+*
then have *?L* (*A* ∩ *B*) **by** *blast*
}
moreover
{
fix *K*
assume *K*: *K* ⊆ *Collect ?L*
have *th0*: *Collect ?L* = (λ*S*. *S* ∩ *V*) ' *Collect* (*openin U*)
by *blast*
from *K*[*unfolded th0 subset_image_iff*]
obtain *Sk* **where** *Sk*: *Sk* ⊆ *Collect* (*openin U*) *K* = (λ*S*. *S* ∩ *V*) ' *Sk*
by *blast*
have ⋃*K* = (⋃*Sk*) ∩ *V*
using *Sk* **by** *auto*
moreover have *openin U* (⋃*Sk*)
using *Sk* **by** (*auto simp*: *subset_eq*)
ultimately have *?L* (⋃*K*) **by** *blast*
}
ultimately show *?thesis*
unfolding *subset_eq mem_Collect_eq istopology_def* **by** *auto*
qed

lemma *openin_subtopology*: *openin* (*subtopology U V*) *S* ⟷ (∃ *T*. *openin U T* ∧ *S* = *T* ∩ *V*)
unfolding *subtopology_def topology_inverse'*[*OF istopology_subtopology*]
by *auto*

lemma *openin_subtopology_Int*:
openin X S ⟹ *openin* (*subtopology X T*) (*S* ∩ *T*)
using *openin_subtopology* **by** *auto*

lemma *openin_subtopology_Int2*:
openin X T ⟹ *openin* (*subtopology X S*) (*S* ∩ *T*)
using *openin_subtopology* **by** *auto*

lemma *openin_subtopology_diff_closed*:
⟦*S* ⊆ *topspace X*; *closedin X T*⟧ ⟹ *openin* (*subtopology X S*) (*S* − *T*)
unfolding *closedin_def openin_subtopology*
by (*rule_tac x*=*topspace X* − *T* **in** *exI*) *auto*

lemma *openin_relative_to*: (*openin X relative_to S*) = *openin* (*subtopology X S*)

by (*force simp*: *relative_to_def openin_subtopology*)

lemma *topspace_subtopology* [*simp*]: *topspace* (*subtopology U V*) = *topspace U* ∩ *V*
by (*auto simp*: *topspace_def openin_subtopology*)

lemma *topspace_subtopology_subset*:
$S \subseteq topspace\ X \Longrightarrow topspace(subtopology\ X\ S) = S$
by (*simp add*: *inf.absorb_iff2*)

lemma *closedin_subtopology*: *closedin* (*subtopology U V*) *S* ⟷ (∃ *T*. *closedin U T* ∧ *S* = *T* ∩ *V*)
unfolding *closedin_def topspace_subtopology*
by (*auto simp*: *openin_subtopology*)

lemma *openin_subtopology_refl*: *openin* (*subtopology U V*) *V* ⟷ *V* ⊆ *topspace U*
unfolding *openin_subtopology*
by *auto* (*metis IntD1 in_mono openin_subset*)

lemma *subtopology_subtopology*:
subtopology (*subtopology X S*) *T* = *subtopology X* (*S* ∩ *T*)
proof −
have *eq*: $\bigwedge T'.\ (\exists S'.\ T' = S' \cap T \wedge (\exists T.\ openin\ X\ T \wedge S' = T \cap S)) = (\exists Sa.\ T' = Sa \cap (S \cap T) \wedge openin\ X\ Sa)$
by (*metis inf_assoc*)
have *subtopology* (*subtopology X S*) *T* = *topology* (λ*Ta*. ∃ *Sa*. *Ta* = *Sa* ∩ *T* ∧ *openin* (*subtopology X S*) *Sa*)
by (*simp add*: *subtopology_def*)
also have ... = *subtopology X* (*S* ∩ *T*)
by (*simp add*: *openin_subtopology eq*) (*simp add*: *subtopology_def*)
finally show *?thesis* .
qed

lemma *openin_subtopology_alt*:
openin (*subtopology X U*) *S* ⟷ *S* ∈ (λ*T*. *U* ∩ *T*) ‘ *Collect* (*openin X*)
by (*simp add*: *image_iff inf_commute openin_subtopology*)

lemma *closedin_subtopology_alt*:
closedin (*subtopology X U*) *S* ⟷ *S* ∈ (λ*T*. *U* ∩ *T*) ‘ *Collect* (*closedin X*)
by (*simp add*: *image_iff inf_commute closedin_subtopology*)

lemma *subtopology_superset*:
assumes *UV*: *topspace U* ⊆ *V*
shows *subtopology U V* = *U*
proof −
{
fix *S*
{

```
      fix T
      assume T: openin U T S = T ∩ V
      from T openin_subset[OF T(1)] UV have eq: S = T
        by blast
      have openin U S
        unfolding eq using T by blast
    }
    moreover
    {
      assume S: openin U S
      then have ∃ T. openin U T ∧ S = T ∩ V
        using openin_subset[OF S] UV by auto
    }
    ultimately have (∃ T. openin U T ∧ S = T ∩ V) ⟷ openin U S
      by blast
  }
  then show ?thesis
    unfolding topology_eq openin_subtopology by blast
qed

lemma subtopology_topspace[simp]: subtopology U (topspace U) = U
  by (simp add: subtopology_superset)

lemma subtopology_UNIV[simp]: subtopology U UNIV = U
  by (simp add: subtopology_superset)

lemma subtopology_restrict:
   subtopology X (topspace X ∩ S) = subtopology X S
  by (metis subtopology_subtopology subtopology_topspace)

lemma openin_subtopology_empty:
   openin (subtopology U {}) S ⟷ S = {}
by (metis Int_empty_right openin_empty openin_subtopology)

lemma closedin_subtopology_empty:
   closedin (subtopology U {}) S ⟷ S = {}
by (metis Int_empty_right closedin_empty closedin_subtopology)

lemma closedin_subtopology_refl [simp]:
   closedin (subtopology U X) X ⟷ X ⊆ topspace U
by (metis closedin_def closedin_topspace inf.absorb_iff2 le_inf_iff topspace_subtopology)

lemma closedin_topspace_empty: topspace T = {} ⟹ (closedin T S ⟷ S = {})
  by (simp add: closedin_def)

lemma open_in_topspace_empty:
   topspace X = {} ⟹ openin X S ⟷ S = {}
  by (simp add: openin_closedin_eq)
```

lemma *openin_imp_subset*:
  openin (*subtopology U S*) *T* $\Longrightarrow$ *T* $\subseteq$ *S*
by (*metis Int_iff openin_subtopology subsetI*)

lemma *closedin_imp_subset*:
  closedin (*subtopology U S*) *T* $\Longrightarrow$ *T* $\subseteq$ *S*
by (*simp add*: *closedin_def*)

lemma *openin_open_subtopology*:
    openin X S $\Longrightarrow$ *openin* (*subtopology X S*) *T* $\longleftrightarrow$ *openin X T* $\wedge$ *T* $\subseteq$ *S*
  by (*metis inf.orderE openin_Int openin_imp_subset openin_subtopology*)

lemma *closedin_closed_subtopology*:
    closedin X S $\Longrightarrow$ (*closedin* (*subtopology X S*) *T* $\longleftrightarrow$ *closedin X T* $\wedge$ *T* $\subseteq$ *S*)
  by (*metis closedin_Int closedin_imp_subset closedin_subtopology inf.orderE*)

lemma *openin_subtopology_Un*:
  ⟦*openin* (*subtopology X T*) *S*; *openin* (*subtopology X U*) *S*⟧
  $\Longrightarrow$ *openin* (*subtopology X* (*T* $\cup$ *U*)) *S*
by (*simp add*: *openin_subtopology*) *blast*

lemma *closedin_subtopology_Un*:
  ⟦*closedin* (*subtopology X T*) *S*; *closedin* (*subtopology X U*) *S*⟧
  $\Longrightarrow$ *closedin* (*subtopology X* (*T* $\cup$ *U*)) *S*
by (*simp add*: *closedin_subtopology*) *blast*

lemma *openin_trans_full*:
  ⟦*openin* (*subtopology X U*) *S*; *openin X U*⟧ $\Longrightarrow$ *openin X S*
  by (*simp add*: *openin_open_subtopology*)

2.2.4 The canonical topology from the underlying type class

abbreviation *euclidean* :: *'a::topological_space topology*
  where *euclidean* $\equiv$ *topology open*

abbreviation *top_of_set* :: *'a::topological_space set* $\Rightarrow$ *'a topology*
  where *top_of_set* $\equiv$ *subtopology* (*topology open*)

lemma *open_openin*: *open S* $\longleftrightarrow$ *openin euclidean S*
  by (*simp add*: *istopology_open topology_inverse'*)

declare *open_openin* [*symmetric*, *simp*]

lemma *topspace_euclidean* [*simp*]: *topspace euclidean* = *UNIV*
  by (*force simp*: *topspace_def*)

lemma *topspace_euclidean_subtopology*[*simp*]: *topspace* (*top_of_set S*) = *S*
  by (*simp*)

lemma *closed_closedin*: *closed S* ⟷ *closedin euclidean S*
 by (*simp add*: *closed_def closedin_def Compl_eq_Diff_UNIV*)

declare *closed_closedin* [*symmetric*, *simp*]

lemma *openin_subtopology_self* [*simp*]: *openin* (*top_of_set S*) *S*
 by (*metis openin_topspace topspace_euclidean_subtopology*)

The most basic facts about the usual topology and metric on R

abbreviation *euclideanreal* :: *real topology*
 where *euclideanreal* ≡ *topology open*

2.2.5 Basic "localization" results are handy for connectedness.

lemma *openin_open*: *openin* (*top_of_set U*) *S* ⟷ (∃ *T*. *open T* ∧ (*S* = *U* ∩ *T*))
 by (*auto simp*: *openin_subtopology*)

lemma *openin_Int_open*:
 ⟦*openin* (*top_of_set U*) *S*; *open T*⟧
 ⟹ *openin* (*top_of_set U*) (*S* ∩ *T*)
by (*metis open_Int Int_assoc openin_open*)

lemma *openin_open_Int*[*intro*]: *open S* ⟹ *openin* (*top_of_set U*) (*U* ∩ *S*)
 by (*auto simp*: *openin_open*)

lemma *open_openin_trans*[*trans*]:
 open S ⟹ *open T* ⟹ *T* ⊆ *S* ⟹ *openin* (*top_of_set S*) *T*
 by (*metis Int_absorb1 openin_open_Int*)

lemma *open_subset*: *S* ⊆ *T* ⟹ *open S* ⟹ *openin* (*top_of_set T*) *S*
 by (*auto simp*: *openin_open*)

lemma *closedin_closed*: *closedin* (*top_of_set U*) *S* ⟷ (∃ *T*. *closed T* ∧ *S* = *U* ∩ *T*)
 by (*simp add*: *closedin_subtopology Int_ac*)

lemma *closedin_closed_Int*: *closed S* ⟹ *closedin* (*top_of_set U*) (*U* ∩ *S*)
 by (*metis closedin_closed*)

lemma *closed_subset*: *S* ⊆ *T* ⟹ *closed S* ⟹ *closedin* (*top_of_set T*) *S*
 by (*auto simp*: *closedin_closed*)

lemma *closedin_closed_subset*:
 ⟦*closedin* (*top_of_set U*) *V*; *T* ⊆ *U*; *S* = *V* ∩ *T*⟧
 ⟹ *closedin* (*top_of_set T*) *S*
 by (*metis* (*no_types*, *lifting*) *Int_assoc Int_commute closedin_closed inf.orderE*)

lemma *finite_imp_closedin*:

```
  fixes S :: 'a::t1_space set
  shows ⟦finite S; S ⊆ T⟧ ⟹ closedin (top_of_set T) S
    by (simp add: finite_imp_closed closed_subset)

lemma closedin_singleton [simp]:
  fixes a :: 'a::t1_space
  shows closedin (top_of_set U) {a} ⟷ a ∈ U
using closedin_subset  by (force intro: closed_subset)

lemma openin_euclidean_subtopology_iff:
  fixes S U :: 'a::metric_space set
  shows openin (top_of_set U) S ⟷
    S ⊆ U ∧ (∀x∈S. ∃e>0. ∀x'∈U. dist x' x < e ⟶ x'∈ S)
  (is ?lhs ⟷ ?rhs)
proof
  assume ?lhs
  then show ?rhs
    unfolding openin_open open_dist by blast
next
   define T where T = {x. ∃a∈S. ∃d>0. (∀y∈U. dist y a < d ⟶ y ∈ S) ∧
dist x a < d}
  have 1: ∀x∈T. ∃e>0. ∀y. dist y x < e ⟶ y ∈ T
    unfolding T_def
    apply clarsimp
    apply (rule_tac x=d - dist x a in exI)
    by (metis add_0_left dist_commute dist_triangle_lt less_diff_eq)
  assume ?rhs then have 2: S = U ∩ T
    unfolding T_def
    by auto (metis dist_self)
  from 1 2 show ?lhs
    unfolding openin_open open_dist by fast
qed

lemma connected_openin:
      connected S ⟷
       ¬(∃E1 E2. openin (top_of_set S) E1 ∧
                 openin (top_of_set S) E2 ∧
                 S ⊆ E1 ∪ E2 ∧ E1 ∩ E2 = {} ∧ E1 ≠ {} ∧ E2 ≠ {})
  unfolding connected_def openin_open disjoint_iff_not_equal by blast

lemma connected_openin_eq:
      connected S ⟷
       ¬(∃E1 E2. openin (top_of_set S) E1 ∧
                 openin (top_of_set S) E2 ∧
                 E1 ∪ E2 = S ∧ E1 ∩ E2 = {} ∧
                 E1 ≠ {} ∧ E2 ≠ {})
  unfolding connected_openin
  by (metis (no_types, lifting) Un_subset_iff openin_imp_subset subset_antisym)
```

lemma *connected_closedin*:
  connected S ⟷
   (∄ *E1 E2*.
    closedin (*top_of_set S*) *E1* ∧
    closedin (*top_of_set S*) *E2* ∧
    S ⊆ *E1* ∪ *E2* ∧ *E1* ∩ *E2* = {} ∧ *E1* ≠ {} ∧ *E2* ≠ {})
   (**is** *?lhs* = *?rhs*)
proof
 assume *?lhs*
 then show *?rhs*
  by (*auto simp add*: *connected_closed closedin_closed*)
next
 assume *R*: *?rhs*
 then show *?lhs*
 proof (*clarsimp simp add*: *connected_closed closedin_closed*)
  fix *A B*
  assume *s_sub*: *S* ⊆ *A* ∪ *B B* ∩ *S* ≠ {}
   and *disj*: *A* ∩ *B* ∩ *S* = {}
   and *cl*: *closed A closed B*
  have *S* ∩ (*A* ∪ *B*) = *S*
   using *s_sub*(*1*) **by** *auto*
  have *S* − *A* = *B* ∩ *S*
   using *Diff_subset_conv Un_Diff_Int disj s_sub*(*1*) **by** *auto*
  then have *S* ∩ *A* = {}
    by (*metis Diff_Diff_Int Diff_disjoint Un_Diff_Int R cl closedin_closed_Int inf_commute order_refl s_sub*(*2*))
  then show *A* ∩ *S* = {}
   by *blast*
 qed
qed

lemma *connected_closedin_eq*:
  connected S ⟷
    ¬(∃ *E1 E2*.
      closedin (*top_of_set S*) *E1* ∧
      closedin (*top_of_set S*) *E2* ∧
      E1 ∪ *E2* = *S* ∧ *E1* ∩ *E2* = {} ∧
      E1 ≠ {} ∧ *E2* ≠ {})
 unfolding *connected_closedin*
 by (*metis Un_subset_iff closedin_imp_subset subset_antisym*)

These "transitivity" results are handy too

lemma *openin_trans*[*trans*]:
 openin (*top_of_set T*) *S* ⟹ *openin* (*top_of_set U*) *T* ⟹
  openin (*top_of_set U*) *S*
 by (*metis openin_Int_open openin_open*)

lemma *openin_open_trans*: *openin* (*top_of_set T*) *S* ⟹ *open T* ⟹ *open S*
 by (*auto simp*: *openin_open intro*: *openin_trans*)

lemma *closedin_trans*[*trans*]:
closedin (*top_of_set* *T*) *S* ⟹ *closedin* (*top_of_set* *U*) *T* ⟹
closedin (*top_of_set* *U*) *S*
by (*auto simp*: *closedin_closed closed_Inter Int_assoc*)

lemma *closedin_closed_trans*: *closedin* (*top_of_set* *T*) *S* ⟹ *closed* *T* ⟹ *closed* *S*
by (*auto simp*: *closedin_closed intro*: *closedin_trans*)

lemma *openin_subtopology_Int_subset*:
⟦*openin* (*top_of_set* *u*) (*u* ∩ *S*); *v* ⊆ *u*⟧ ⟹ *openin* (*top_of_set* *v*) (*v* ∩ *S*)
by (*auto simp*: *openin_subtopology*)

lemma *openin_open_eq*: *open* *s* ⟹ (*openin* (*top_of_set* *s*) *t* ⟷ *open* *t* ∧ *t* ⊆ *s*)
using *open_subset openin_open_trans openin_subset* **by** *fastforce*

2.2.6 Derived set (set of limit points)

definition *derived_set_of* :: *'a topology* ⇒ *'a set* ⇒ *'a set* (**infixl** *derived'_set'_of* *80*)
where *X derived_set_of S* ≡
{*x* ∈ *topspace* *X*.
(∀ *T*. *x* ∈ *T* ∧ *openin* *X* *T* ⟶ (∃ *y*≠*x*. *y* ∈ *S* ∧ *y* ∈ *T*))}

lemma *derived_set_of_restrict* [*simp*]:
X derived_set_of (*topspace* *X* ∩ *S*) = *X derived_set_of S*
by (*simp add*: *derived_set_of_def*) (*metis openin_subset subset_iff*)

lemma *in_derived_set_of*:
x ∈ *X derived_set_of S* ⟷ *x* ∈ *topspace* *X* ∧ (∀ *T*. *x* ∈ *T* ∧ *openin* *X* *T* ⟶ (∃ *y*≠*x*. *y* ∈ *S* ∧ *y* ∈ *T*))
by (*simp add*: *derived_set_of_def*)

lemma *derived_set_of_subset_topspace*:
X derived_set_of S ⊆ *topspace* *X*
by (*auto simp add*: *derived_set_of_def*)

lemma *derived_set_of_subtopology*:
(*subtopology* *X* *U*) *derived_set_of S* = *U* ∩ (*X derived_set_of* (*U* ∩ *S*))
by (*simp add*: *derived_set_of_def openin_subtopology*) *blast*

lemma *derived_set_of_subset_subtopology*:
(*subtopology* *X* *S*) *derived_set_of T* ⊆ *S*
by (*simp add*: *derived_set_of_subtopology*)

lemma *derived_set_of_empty* [*simp*]: *X derived_set_of* {} = {}
by (*auto simp*: *derived_set_of_def*)

lemma *derived_set_of_mono*:

S ⊆ *T* ⟹ *X derived_set_of S* ⊆ *X derived_set_of T*
unfolding *derived_set_of_def* **by** *blast*

lemma *derived_set_of_Un*:
X derived_set_of (*S* ∪ *T*) = *X derived_set_of S* ∪ *X derived_set_of T* (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
by (*clarsimp simp*: *in_derived_set_of*) (*metis IntE IntI openin_Int*)
show *?rhs* ⊆ *?lhs*
by (*simp add*: *derived_set_of_mono*)
qed

lemma *derived_set_of_Union*:
finite $\mathcal{F}$ ⟹ *X derived_set_of* (⋃$\mathcal{F}$) = (⋃*S* ∈ $\mathcal{F}$. *X derived_set_of S*)
proof (*induction* $\mathcal{F}$ *rule*: *finite_induct*)
case (*insert S* $\mathcal{F}$)
then show *?case*
by (*simp add*: *derived_set_of_Un*)
qed *auto*

lemma *derived_set_of_topspace*:
X derived_set_of (*topspace X*) = {*x* ∈ *topspace X*. ¬ *openin X* {*x*}} (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
by (*auto simp*: *in_derived_set_of*)
show *?rhs* ⊆ *?lhs*
by (*clarsimp simp*: *in_derived_set_of*) (*metis openin_closedin_eq openin_subopen singletonD subset_eq*)
qed

lemma *discrete_topology_unique_derived_set*:
discrete_topology U = *X* ⟷ *topspace X* = *U* ∧ *X derived_set_of U* = {}
by (*auto simp*: *discrete_topology_unique derived_set_of_topspace*)

lemma *subtopology_eq_discrete_topology_eq*:
subtopology X U = *discrete_topology U* ⟷ *U* ⊆ *topspace X* ∧ *U* ∩ *X derived_set_of U* = {}
using *discrete_topology_unique_derived_set* [*of U subtopology X U*]
by (*auto simp*: *eq_commute derived_set_of_subtopology*)

lemma *subtopology_eq_discrete_topology*:
S ⊆ *topspace X* ∧ *S* ∩ *X derived_set_of S* = {}
⟹ *subtopology X S* = *discrete_topology S*
by (*simp add*: *subtopology_eq_discrete_topology_eq*)

lemma *subtopology_eq_discrete_topology_gen*:
S ∩ *X derived_set_of S* = {} ⟹ *subtopology X S* = *discrete_topology*(*topspace*

X ∩ *S*)
by (*metis Int_lower1 derived_set_of_restrict inf_assoc inf_bot_right subtopology_eq_discrete_topology_eq subtopology_subtopology subtopology_topspace*)

lemma *subtopology_discrete_topology* [*simp*]:
subtopology (*discrete_topology U*) *S* = *discrete_topology*(*U* ∩ *S*)
proof −
have (λ*T*. ∃ *Sa*. *T* = *Sa* ∩ *S* ∧ *Sa* ⊆ *U*) = (λ*Sa*. *Sa* ⊆ *U* ∧ *Sa* ⊆ *S*)
by *force*
then show *?thesis*
by (*simp add*: *subtopology_def*) (*simp add*: *discrete_topology_def*)
qed
lemma *openin_Int_derived_set_of_subset*:
openin X S ⟹ *S* ∩ *X derived_set_of T* ⊆ *X derived_set_of* (*S* ∩ *T*)
by (*auto simp*: *derived_set_of_def*)

lemma *openin_Int_derived_set_of_eq*:
assumes *openin X S*
shows *S* ∩ *X derived_set_of T* = *S* ∩ *X derived_set_of* (*S* ∩ *T*) (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
by (*simp add*: *assms openin_Int_derived_set_of_subset*)
show *?rhs* ⊆ *?lhs*
by (*metis derived_set_of_mono inf_commute inf_le1 inf_mono order_refl*)
qed

2.2.7 Closure with respect to a topological space

definition *closure_of* :: *'a topology* ⇒ *'a set* ⇒ *'a set* (**infixr** *closure'_of 80*)
where *X closure_of S* ≡ {*x* ∈ *topspace X*. ∀ *T*. *x* ∈ *T* ∧ *openin X T* ⟶ (∃ *y* ∈ *S*. *y* ∈ *T*)}

lemma *closure_of_restrict*: *X closure_of S* = *X closure_of* (*topspace X* ∩ *S*)
unfolding *closure_of_def*
using *openin_subset* **by** *blast*

lemma *in_closure_of*:
x ∈ *X closure_of S* ⟷
x ∈ *topspace X* ∧ (∀ *T*. *x* ∈ *T* ∧ *openin X T* ⟶ (∃ *y*. *y* ∈ *S* ∧ *y* ∈ *T*))
by (*auto simp*: *closure_of_def*)

lemma *closure_of*: *X closure_of S* = *topspace X* ∩ (*S* ∪ *X derived_set_of S*)
by (*fastforce simp*: *in_closure_of in_derived_set_of*)

lemma *closure_of_alt*: *X closure_of S* = *topspace X* ∩ *S* ∪ *X derived_set_of S*
using *derived_set_of_subset_topspace* [*of X S*]
unfolding *closure_of_def in_derived_set_of*
by *safe* (*auto simp*: *in_derived_set_of*)

lemma *derived_set_of_subset_closure_of*:
X derived_set_of S ⊆ *X closure_of S*
by (*fastforce simp*: *closure_of_def in_derived_set_of*)

lemma *closure_of_subtopology*:
(*subtopology X U*) *closure_of S* = *U* ∩ (*X closure_of* (*U* ∩ *S*))
unfolding *closure_of_def topspace_subtopology openin_subtopology*
by *safe* (*metis* (*full_types*) *IntI Int_iff inf.commute*)+

lemma *closure_of_empty* [*simp*]: *X closure_of* {} = {}
by (*simp add*: *closure_of_alt*)

lemma *closure_of_topspace* [*simp*]: *X closure_of topspace X* = *topspace X*
by (*simp add*: *closure_of*)

lemma *closure_of_UNIV* [*simp*]: *X closure_of UNIV* = *topspace X*
by (*simp add*: *closure_of*)

lemma *closure_of_subset_topspace*: *X closure_of S* ⊆ *topspace X*
by (*simp add*: *closure_of*)

lemma *closure_of_subset_subtopology*: (*subtopology X S*) *closure_of T* ⊆ *S*
by (*simp add*: *closure_of_subtopology*)

lemma *closure_of_mono*: *S* ⊆ *T* ⟹ *X closure_of S* ⊆ *X closure_of T*
by (*fastforce simp add*: *closure_of_def*)

lemma *closure_of_subtopology_subset*:
(*subtopology X U*) *closure_of S* ⊆ (*X closure_of S*)
unfolding *closure_of_subtopology*
by *clarsimp* (*meson closure_of_mono contra_subsetD inf.cobounded2*)

lemma *closure_of_subtopology_mono*:
T ⊆ *U* ⟹ (*subtopology X T*) *closure_of S* ⊆ (*subtopology X U*) *closure_of S*
unfolding *closure_of_subtopology*
by *auto* (*meson closure_of_mono inf_mono subset_iff*)

lemma *closure_of_Un* [*simp*]: *X closure_of* (*S* ∪ *T*) = *X closure_of S* ∪ *X closure_of*
T
by (*simp add*: *Un_assoc Un_left_commute closure_of_alt derived_set_of_Un inf_sup_distrib1*)

lemma *closure_of_Union*:
finite $\mathcal{F}$ ⟹ *X closure_of* ($\bigcup\mathcal{F}$) = ($\bigcup S \in \mathcal{F}$. *X closure_of S*)
by (*induction* $\mathcal{F}$ *rule*: *finite_induct*) *auto*

lemma *closure_of_subset*: *S* ⊆ *topspace X* ⟹ *S* ⊆ *X closure_of S*
by (*auto simp*: *closure_of_def*)

lemma *closure_of_subset_Int*: *topspace X* ∩ *S* ⊆ *X closure_of S*

by (*auto simp*: *closure_of_def*)

lemma *closure_of_subset_eq*: *S* ⊆ *topspace X* ∧ *X closure_of S* ⊆ *S* ⟷ *closedin X S*
proof −
have *openin X* (*topspace X* − *S*)
if ⋀*x*. ⟦*x* ∈ *topspace X*; ∀ *T*. *x* ∈ *T* ∧ *openin X T* ⟶ *S* ∩ *T* ≠ {}⟧ ⟹ *x* ∈ *S*
apply (*subst openin_subopen*)
by (*metis Diff_iff Diff_mono Diff_triv inf.commute openin_subset order_refl that*)
then show *?thesis*
by (*auto simp*: *closedin_def closure_of_def disjoint_iff_not_equal*)
qed

lemma *closure_of_eq*: *X closure_of S* = *S* ⟷ *closedin X S*
proof (*cases S* ⊆ *topspace X*)
case *True*
then show *?thesis*
by (*metis closure_of_subset closure_of_subset_eq set_eq_subset*)
next
case *False*
then show *?thesis*
using *closure_of closure_of_subset_eq* **by** *fastforce*
qed

lemma *closedin_contains_derived_set*:
closedin X S ⟷ *X derived_set_of S* ⊆ *S* ∧ *S* ⊆ *topspace X*
proof (*intro iffI conjI*)
show *closedin X S* ⟹ *X derived_set_of S* ⊆ *S*
using *closure_of_eq derived_set_of_subset_closure_of* **by** *fastforce*
show *closedin X S* ⟹ *S* ⊆ *topspace X*
using *closedin_subset* **by** *blast*
show *X derived_set_of S* ⊆ *S* ∧ *S* ⊆ *topspace X* ⟹ *closedin X S*
by (*metis closure_of closure_of_eq inf.absorb_iff2 sup.orderE*)
qed

lemma *derived_set_subset_gen*:
X derived_set_of S ⊆ *S* ⟷ *closedin X* (*topspace X* ∩ *S*)
by (*simp add*: *closedin_contains_derived_set derived_set_of_restrict derived_set_of_subset_topspace*)

lemma *derived_set_subset*: *S* ⊆ *topspace X* ⟹ (*X derived_set_of S* ⊆ *S* ⟷ *closedin X S*)
by (*simp add*: *closedin_contains_derived_set*)

lemma *closedin_derived_set*:
closedin (*subtopology X T*) *S* ⟷
S ⊆ *topspace X* ∧ *S* ⊆ *T* ∧ (∀ *x*. *x* ∈ *X derived_set_of S* ∧ *x* ∈ *T* ⟶ *x* ∈ *S*)
by (*auto simp*: *closedin_contains_derived_set derived_set_of_subtopology Int_absorb1*)

lemma *closedin_Int_closure_of*:
  closedin (*subtopology X S*) $T \longleftrightarrow S \cap X$ *closure_of* $T = T$
 by (*metis Int_left_absorb closure_of_eq closure_of_subtopology*)

lemma *closure_of_closedin*: *closedin X S* $\Longrightarrow$ *X closure_of S* = *S*
 by (*simp add*: *closure_of_eq*)

lemma *closure_of_eq_diff*: *X closure_of S* = *topspace X* $-$ $\bigcup${*T*. *openin X T* $\wedge$ *disjnt S T*}
 by (*auto simp*: *closure_of_def disjnt_iff*)

lemma *closedin_closure_of* [*simp*]: *closedin X* (*X closure_of S*)
 unfolding *closure_of_eq_diff* **by** *blast*

lemma *closure_of_closure_of* [*simp*]: *X closure_of* (*X closure_of S*) = *X closure_of S*
 by (*simp add*: *closure_of_eq*)

lemma *closure_of_hull*:
 assumes $S \subseteq$ *topspace X* **shows** *X closure_of S* = (*closedin X*) *hull S*
proof (*rule hull_unique* [*THEN sym*])
 show $S \subseteq X$ *closure_of S*
  by (*simp add*: *closure_of_subset assms*)
next
 show *closedin X* (*X closure_of S*)
  by *simp*
 show $\bigwedge T$. $[\![S \subseteq T$; *closedin X T*$]\!] \Longrightarrow X$ *closure_of* $S \subseteq T$
  by (*metis closure_of_eq closure_of_mono*)
qed

lemma *closure_of_minimal*:
 $[\![S \subseteq T$; *closedin X T*$]\!] \Longrightarrow$ (*X closure_of S*) $\subseteq T$
 by (*metis closure_of_eq closure_of_mono*)

lemma *closure_of_minimal_eq*:
 $[\![S \subseteq$ *topspace X*; *closedin X T*$]\!] \Longrightarrow$ (*X closure_of S*) $\subseteq T \longleftrightarrow S \subseteq T$
 by (*meson closure_of_minimal closure_of_subset subset_trans*)

lemma *closure_of_unique*:
 $[\![S \subseteq T$; *closedin X T*;
  $\bigwedge T'$. $[\![S \subseteq T'$; *closedin X T'*$]\!] \Longrightarrow T \subseteq T'$$]\!]$
  $\Longrightarrow$ *X closure_of S* = *T*
 by (*meson closedin_closure_of closedin_subset closure_of_minimal closure_of_subset eq_iff order.trans*)

lemma *closure_of_eq_empty_gen*: *X closure_of S* = {} $\longleftrightarrow$ *disjnt* (*topspace X*) *S*
 unfolding *disjnt_def closure_of_restrict* [**where** *S*=*S*]
 using *closure_of* **by** *fastforce*

lemma *closure_of_eq_empty*: $S \subseteq$ *topspace* $X \Longrightarrow X$ *closure_of* $S = \{\} \longleftrightarrow S = \{\}$
using *closure_of_subset* **by** *fastforce*

lemma *openin_Int_closure_of_subset*:
assumes *openin* X S
shows $S \cap X$ *closure_of* $T \subseteq X$ *closure_of* $(S \cap T)$
proof –
have $S \cap X$ *derived_set_of* $T = S \cap X$ *derived_set_of* $(S \cap T)$
by (*meson assms openin_Int_derived_set_of_eq*)
moreover have $S \cap (S \cap T) = S \cap T$
by *fastforce*
ultimately show *?thesis*
by (*metis closure_of_alt inf.cobounded2 inf_left_commute inf_sup_distrib1*)
qed

lemma *closure_of_openin_Int_closure_of*:
assumes *openin* X S
shows X *closure_of* $(S \cap X$ *closure_of* $T) = X$ *closure_of* $(S \cap T)$
proof
show X *closure_of* $(S \cap X$ *closure_of* $T) \subseteq X$ *closure_of* $(S \cap T)$
by (*simp add*: *assms closure_of_minimal openin_Int_closure_of_subset*)
next
show X *closure_of* $(S \cap T) \subseteq X$ *closure_of* $(S \cap X$ *closure_of* $T)$
by (*metis Int_lower1 Int_subset_iff assms closedin_closure_of closure_of_minimal_eq closure_of_mono inf_le2 le_infI1 openin_subset*)
qed

lemma *openin_Int_closure_of_eq*:
assumes *openin* X S **shows** $S \cap X$ *closure_of* $T = S \cap X$ *closure_of* $(S \cap T)$
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* $\subseteq$ *?rhs*
by (*simp add*: *assms openin_Int_closure_of_subset*)
show *?rhs* $\subseteq$ *?lhs*
by (*metis closure_of_mono inf_commute inf_le1 inf_mono order_refl*)
qed

lemma *openin_Int_closure_of_eq_empty*:
assumes *openin* X S **shows** $S \cap X$ *closure_of* $T = \{\} \longleftrightarrow S \cap T = \{\}$ (**is** *?lhs* = *?rhs*)
proof
show *?lhs* $\Longrightarrow$ *?rhs*
unfolding *disjoint_iff*
by (*meson assms in_closure_of in_mono openin_subset*)
show *?rhs* $\Longrightarrow$ *?lhs*
by (*simp add*: *assms openin_Int_closure_of_eq*)
qed

lemma *closure_of_openin_Int_superset*:
openin X S ∧ *S* ⊆ *X closure_of T*
⟹ *X closure_of* (*S* ∩ *T*) = *X closure_of S*
by (*metis closure_of_openin_Int_closure_of inf.orderE*)

lemma *closure_of_openin_subtopology_Int_closure_of*:
assumes *S*: *openin* (*subtopology X U*) *S* **and** *T* ⊆ *U*
shows *X closure_of* (*S* ∩ *X closure_of T*) = *X closure_of* (*S* ∩ *T*) (**is** *?lhs* = *?rhs*)
proof
obtain *S0* **where** *S0*: *openin X S0 S* = *S0* ∩ *U*
using *assms* **by** (*auto simp*: *openin_subtopology*)
show *?lhs* ⊆ *?rhs*
proof −
have *S0* ∩ *X closure_of T* = *S0* ∩ *X closure_of* (*S0* ∩ *T*)
by (*meson S0*(*1*) *openin_Int_closure_of_eq*)
moreover have *S0* ∩ *T* = *S0* ∩ *U* ∩ *T*
using ⟨*T* ⊆ *U*⟩ **by** *fastforce*
ultimately have *S* ∩ *X closure_of T* ⊆ *X closure_of* (*S* ∩ *T*)
using *S0*(*2*) **by** *auto*
then show *?thesis*
by (*meson closedin_closure_of closure_of_minimal*)
qed
next
show *?rhs* ⊆ *?lhs*
proof −
have *T* ∩ *S* ⊆ *T* ∪ *X derived_set_of T*
by *force*
then show *?thesis*
by (*metis Int_subset_iff S closure_of closure_of_mono inf.cobounded2 inf.coboundedI2 inf_commute openin_closedin_eq topspace_subtopology*)
qed
qed

lemma *closure_of_subtopology_open*:
openin X U ∨ *S* ⊆ *U* ⟹ (*subtopology X U*) *closure_of S* = *U* ∩ *X closure_of S*
by (*metis closure_of_subtopology inf_absorb2 openin_Int_closure_of_eq*)

lemma *discrete_topology_closure_of*:
(*discrete_topology U*) *closure_of S* = *U* ∩ *S*
by (*metis closedin_discrete_topology closure_of_restrict closure_of_unique discrete_topology_unique inf_sup_ord*(*1*) *order_refl*)

Interior with respect to a topological space.

definition *interior_of* :: *'a topology* ⇒ *'a set* ⇒ *'a set* (**infixr** *interior'_of 80*)
where *X interior_of S* ≡ {*x*. ∃ *T*. *openin X T* ∧ *x* ∈ *T* ∧ *T* ⊆ *S*}

lemma *interior_of_restrict*:

X interior_of S = X interior_of (topspace X ∩ S)
using *openin_subset* **by** (*auto simp*: *interior_of_def*)

lemma *interior_of_eq*: (*X interior_of S = S*) ⟷ *openin X S*
unfolding *interior_of_def* **using** *openin_subopen* **by** *blast*

lemma *interior_of_openin*: *openin X S* ⟹ *X interior_of S = S*
by (*simp add*: *interior_of_eq*)

lemma *interior_of_empty* [*simp*]: *X interior_of* {} = {}
by (*simp add*: *interior_of_eq*)

lemma *interior_of_topspace* [*simp*]: *X interior_of* (*topspace X*) = *topspace X*
by (*simp add*: *interior_of_eq*)

lemma *openin_interior_of* [*simp*]: *openin X* (*X interior_of S*)
unfolding *interior_of_def*
using *openin_subopen* **by** *fastforce*

lemma *interior_of_interior_of* [*simp*]:
X interior_of X interior_of S = X interior_of S
by (*simp add*: *interior_of_eq*)

lemma *interior_of_subset*: *X interior_of S* ⊆ *S*
by (*auto simp*: *interior_of_def*)

lemma *interior_of_subset_closure_of*: *X interior_of S* ⊆ *X closure_of S*
by (*metis closure_of_subset_Int dual_order.trans interior_of_restrict interior_of_subset*)

lemma *subset_interior_of_eq*: *S* ⊆ *X interior_of S* ⟷ *openin X S*
by (*metis interior_of_eq interior_of_subset subset_antisym*)

lemma *interior_of_mono*: *S* ⊆ *T* ⟹ *X interior_of S* ⊆ *X interior_of T*
by (*auto simp*: *interior_of_def*)

lemma *interior_of_maximal*: ⟦*T* ⊆ *S*; *openin X T*⟧ ⟹ *T* ⊆ *X interior_of S*
by (*auto simp*: *interior_of_def*)

lemma *interior_of_maximal_eq*: *openin X T* ⟹ *T* ⊆ *X interior_of S* ⟷ *T* ⊆ *S*
by (*meson interior_of_maximal interior_of_subset order_trans*)

lemma *interior_of_unique*:
⟦*T* ⊆ *S*; *openin X T*; ⋀*T'*. ⟦*T'* ⊆ *S*; *openin X T'*⟧ ⟹ *T'* ⊆ *T*⟧ ⟹ *X interior_of S = T*
by (*simp add*: *interior_of_maximal_eq interior_of_subset subset_antisym*)

lemma *interior_of_subset_topspace*: *X interior_of S* ⊆ *topspace X*
by (*simp add*: *openin_subset*)

lemma *interior_of_subset_subtopology*: (*subtopology X S*) *interior_of T* ⊆ *S*
by (*meson openin_imp_subset openin_interior_of*)

lemma *interior_of_Int*: *X interior_of* (*S* ∩ *T*) = *X interior_of S* ∩ *X interior_of T* (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
by (*simp add*: *interior_of_mono*)
show *?rhs* ⊆ *?lhs*
by (*meson inf_mono interior_of_maximal interior_of_subset openin_Int openin_interior_of*)
qed

lemma *interior_of_Inter_subset*: *X interior_of* (⋂ℱ) ⊆ (⋂*S* ∈ ℱ. *X interior_of S*)
by (*simp add*: *INT_greatest Inf_lower interior_of_mono*)

lemma *union_interior_of_subset*:
X interior_of S ∪ *X interior_of T* ⊆ *X interior_of* (*S* ∪ *T*)
by (*simp add*: *interior_of_mono*)

lemma *interior_of_eq_empty*:
X interior_of S = {} ⟷ (∀ *T*. *openin X T* ∧ *T* ⊆ *S* ⟶ *T* = {})
by (*metis bot.extremum_uniqueI interior_of_maximal interior_of_subset openin_interior_of*)

lemma *interior_of_eq_empty_alt*:
X interior_of S = {} ⟷ (∀ *T*. *openin X T* ∧ *T* ≠ {} ⟶ *T* − *S* ≠ {})
by (*auto simp*: *interior_of_eq_empty*)

lemma *interior_of_Union_openin_subsets*:
⋃{*T*. *openin X T* ∧ *T* ⊆ *S*} = *X interior_of S*
by (*rule interior_of_unique* [*symmetric*]) *auto*

lemma *interior_of_complement*:
X interior_of (*topspace X* − *S*) = *topspace X* − *X closure_of S*
by (*auto simp*: *interior_of_def closure_of_def*)

lemma *interior_of_closure_of*:
X interior_of S = *topspace X* − *X closure_of* (*topspace X* − *S*)
unfolding *interior_of_complement* [*symmetric*]
by (*metis Diff_Diff_Int interior_of_restrict*)

lemma *closure_of_interior_of*:
X closure_of S = *topspace X* − *X interior_of* (*topspace X* − *S*)
by (*simp add*: *interior_of_complement Diff_Diff_Int closure_of*)

lemma *closure_of_complement*: *X closure_of* (*topspace X* − *S*) = *topspace X* − *X interior_of S*
unfolding *interior_of_def closure_of_def*
by (*blast dest*: *openin_subset*)

lemma *interior_of_eq_empty_complement*:
  X *interior_of* $S = \{\} \longleftrightarrow X$ *closure_of* (*topspace* $X - S$) = *topspace* X
  using *interior_of_subset_topspace* [*of* X S] *closure_of_complement* **by** *fastforce*

lemma *closure_of_eq_topspace*:
  X *closure_of* S = *topspace* $X \longleftrightarrow X$ *interior_of* (*topspace* $X - S$) = {}
  using *closure_of_subset_topspace* [*of* X S] *interior_of_complement* **by** *fastforce*

lemma *interior_of_subtopology_subset*:
  $U \cap X$ *interior_of* $S \subseteq$ (*subtopology* X U) *interior_of* S
  by (*auto simp*: *interior_of_def openin_subtopology*)

lemma *interior_of_subtopology_subsets*:
  $T \subseteq U \Longrightarrow T \cap$ (*subtopology* X U) *interior_of* $S \subseteq$ (*subtopology* X T) *interior_of*
S
  by (*metis inf.absorb_iff2 interior_of_subtopology_subset subtopology_subtopology*)

lemma *interior_of_subtopology_mono*:
  $[\![S \subseteq T; T \subseteq U]\!] \Longrightarrow$ (*subtopology* X U) *interior_of* $S \subseteq$ (*subtopology* X T)
interior_of S
  by (*metis dual_order.trans inf.orderE inf_commute interior_of_subset interior_of_subtopology_subsets*)

lemma *interior_of_subtopology_open*:
  assumes *openin* X U
  shows (*subtopology* X U) *interior_of* $S = U \cap X$ *interior_of* S
proof −
  have $\forall A.\ U \cap X$ *closure_of* $(U \cap A) = U \cap X$ *closure_of* A
    using *assms openin_Int_closure_of_eq* **by** *blast*
  then have *topspace* $X \cap U - U \cap X$ *closure_of* (*topspace* $X \cap U - S$) = $U \cap$
(*topspace* $X - X$ *closure_of* (*topspace* $X - S$))
    by (*metis* (*no_types*) *Diff_Int_distrib Int_Diff inf_commute*)
  then show *?thesis*
   unfolding *interior_of_closure_of closure_of_subtopology_open topspace_subtopology*
    using *openin_Int_closure_of_eq* [*OF assms*]
    by (*metis assms closure_of_subtopology_open*)
qed

lemma *dense_intersects_open*:
  X *closure_of* S = *topspace* $X \longleftrightarrow (\forall T.$ *openin* X $T \wedge T \neq \{\} \longrightarrow S \cap T \neq$
$\{\})$
proof −
  have X *closure_of* S = *topspace* $X \longleftrightarrow$ (*topspace* $X - X$ *interior_of* (*topspace*
$X - S$) = *topspace* X)
    by (*simp add*: *closure_of_interior_of*)
  also have ... $\longleftrightarrow X$ *interior_of* (*topspace* $X - S$) = {}
    by (*simp add*: *closure_of_complement interior_of_eq_empty_complement*)
  also have ... $\longleftrightarrow (\forall T.$ *openin* X $T \wedge T \neq \{\} \longrightarrow S \cap T \neq \{\})$
    unfolding *interior_of_eq_empty_alt*

using *openin_subset* **by** *fastforce*
finally show *?thesis* **.**
qed

lemma *interior_of_closedin_union_empty_interior_of*:
assumes *closedin X S* **and** *disj*: *X interior_of T* = {}
shows *X interior_of* (*S* ∪ *T*) = *X interior_of S*
proof −
have *X closure_of* (*topspace X* − *T*) = *topspace X*
by (*metis Diff_Diff_Int disj closure_of_eq_topspace closure_of_restrict interior_of_closure_of*)
then show *?thesis*
unfolding *interior_of_closure_of*
by (*metis Diff_Un Diff_subset assms*(*1*) *closedin_def closure_of_openin_Int_superset*)
qed

lemma *interior_of_union_eq_empty*:
closedin X S
⟹ (*X interior_of* (*S* ∪ *T*) = {} ⟷
X interior_of S = {} ∧ *X interior_of T* = {})
by (*metis interior_of_closedin_union_empty_interior_of le_sup_iff subset_empty union_interior_of_subset*)

lemma *discrete_topology_interior_of* [*simp*]:
(*discrete_topology U*) *interior_of S* = *U* ∩ *S*
by (*simp add*: *interior_of_restrict* [*of _ S*] *interior_of_eq*)

2.2.8 Frontier with respect to topological space

definition *frontier_of* :: *'a topology* ⇒ *'a set* ⇒ *'a set* (**infixr** *frontier'_of 80*)
where *X frontier_of S* ≡ *X closure_of S* − *X interior_of S*

lemma *frontier_of_closures*:
X frontier_of S = *X closure_of S* ∩ *X closure_of* (*topspace X* − *S*)
by (*metis Diff_Diff_Int closure_of_complement closure_of_subset_topspace double_diff frontier_of_def interior_of_subset_closure_of*)

lemma *interior_of_union_frontier_of* [*simp*]:
X interior_of S ∪ *X frontier_of S* = *X closure_of S*
by (*simp add*: *frontier_of_def interior_of_subset_closure_of subset_antisym*)

lemma *frontier_of_restrict*: *X frontier_of S* = *X frontier_of* (*topspace X* ∩ *S*)
by (*metis closure_of_restrict frontier_of_def interior_of_restrict*)

lemma *closedin_frontier_of*: *closedin X* (*X frontier_of S*)
by (*simp add*: *closedin_Int frontier_of_closures*)

lemma *frontier_of_subset_topspace*: *X frontier_of S* ⊆ *topspace X*
by (*simp add*: *closedin_frontier_of closedin_subset*)

lemma *frontier_of_subset_subtopology*: (*subtopology X S*) *frontier_of T* ⊆ *S*
  by (*metis* (*no_types*) *closedin_derived_set closedin_frontier_of*)

lemma *frontier_of_subtopology_subset*:
  U ∩ (*subtopology X U*) *frontier_of S* ⊆ (*X frontier_of S*)
proof –
  have *U* ∩ *X interior_of S* − *subtopology X U interior_of S* = {}
    by (*simp add*: *interior_of_subtopology_subset*)
  moreover have *X closure_of S* ∩ *subtopology X U closure_of S* = *subtopology X U closure_of S*
    by (*meson closure_of_subtopology_subset inf.absorb_iff2*)
  ultimately show *?thesis*
    unfolding *frontier_of_def*
    by *blast*
qed

lemma *frontier_of_subtopology_mono*:
  ⟦*S* ⊆ *T*; *T* ⊆ *U*⟧ ⟹ (*subtopology X T*) *frontier_of S* ⊆ (*subtopology X U*) *frontier_of S*
  by (*simp add*: *frontier_of_def Diff_mono closure_of_subtopology_mono interior_of_subtopology_mono*)

lemma *clopenin_eq_frontier_of*:
  closedin X S ∧ *openin X S* ⟷ *S* ⊆ *topspace X* ∧ *X frontier_of S* = {}
proof (*cases S* ⊆ *topspace X*)
  case *True*
  then show *?thesis*
    by (*metis Diff_eq_empty_iff closure_of_eq closure_of_subset_eq frontier_of_def interior_of_eq interior_of_subset interior_of_union_frontier_of sup_bot_right*)
next
  case *False*
  then show *?thesis*
    by (*simp add*: *frontier_of_closures openin_closedin_eq*)
qed

lemma *frontier_of_eq_empty*:
  S ⊆ *topspace X* ⟹ (*X frontier_of S* = {} ⟷ *closedin X S* ∧ *openin X S*)
  by (*simp add*: *clopenin_eq_frontier_of*)

lemma *frontier_of_openin*:
  openin X S ⟹ *X frontier_of S* = *X closure_of S* − *S*
  by (*metis* (*no_types*) *frontier_of_def interior_of_eq*)

lemma *frontier_of_openin_straddle_Int*:
  assumes *openin X U U* ∩ *X frontier_of S* ≠ {}
  shows *U* ∩ *S* ≠ {} *U* − *S* ≠ {}
proof –
  have *U* ∩ (*X closure_of S* ∩ *X closure_of* (*topspace X* − *S*)) ≠ {}
    using *assms* **by** (*simp add*: *frontier_of_closures*)
  then show *U* ∩ *S* ≠ {}

using *assms openin_Int_closure_of_eq_empty* **by** *fastforce*
show *U* − *S* ≠ {}
proof −
have ∃ *A*. *X closure_of* (*A* − *S*) ∩ *U* ≠ {}
using ‹*U* ∩ (*X closure_of S* ∩ *X closure_of* (*topspace X* − *S*)) ≠ {}› **by** *blast*
then have ¬ *U* ⊆ *S*
by (*metis Diff_disjoint Diff_eq_empty_iff Int_Diff assms*(*1*) *inf_commute openin_Int_closure_of_eq_empty*
then show *?thesis*
by *blast*
qed
qed

lemma *frontier_of_subset_closedin*: *closedin X S* ⟹ (*X frontier_of S*) ⊆ *S*
using *closure_of_eq frontier_of_def* **by** *fastforce*

lemma *frontier_of_empty* [*simp*]: *X frontier_of* {} = {}
by (*simp add*: *frontier_of_def*)

lemma *frontier_of_topspace* [*simp*]: *X frontier_of topspace X* = {}
by (*simp add*: *frontier_of_def*)

lemma *frontier_of_subset_eq*:
assumes *S* ⊆ *topspace X*
shows (*X frontier_of S*) ⊆ *S* ⟷ *closedin X S*
proof
show *X frontier_of S* ⊆ *S* ⟹ *closedin X S*
by (*metis assms closure_of_subset_eq interior_of_subset interior_of_union_frontier_of le_sup_iff*)
show *closedin X S* ⟹ *X frontier_of S* ⊆ *S*
by (*simp add*: *frontier_of_subset_closedin*)
qed

lemma *frontier_of_complement*: *X frontier_of* (*topspace X* − *S*) = *X frontier_of S*
by (*metis Diff_Diff_Int closure_of_restrict frontier_of_closures inf_commute*)

lemma *frontier_of_disjoint_eq*:
assumes *S* ⊆ *topspace X*
shows ((*X frontier_of S*) ∩ *S* = {} ⟷ *openin X S*)
proof
assume *X frontier_of S* ∩ *S* = {}
then have *closedin X* (*topspace X* − *S*)
using *assms closure_of_subset frontier_of_def interior_of_eq interior_of_subset* **by** *fastforce*
then show *openin X S*
using *assms* **by** (*simp add*: *openin_closedin*)
next
show *openin X S* ⟹ *X frontier_of S* ∩ *S* = {}
by (*simp add*: *Diff_Diff_Int closedin_def frontier_of_openin inf.absorb_iff2 inf_commute*)
qed

lemma *frontier_of_disjoint_eq_alt*:
  $S \subseteq (topspace\ X - X\ frontier\_of\ S) \longleftrightarrow openin\ X\ S$
proof (*cases* $S \subseteq topspace\ X$)
  case *True*
  show *?thesis*
    using *True frontier_of_disjoint_eq* **by** *auto*
next
  case *False*
  then show *?thesis*
    by (*meson Diff_subset openin_subset subset_trans*)
qed

lemma *frontier_of_Int*:
  $X\ frontier\_of\ (S \cap T) =$
   $X\ closure\_of\ (S \cap T) \cap (X\ frontier\_of\ S \cup X\ frontier\_of\ T)$
proof −
  have $*$: $U \subseteq S \wedge U \subseteq T \Longrightarrow U \cap (S \cap A \cup T \cap B) = U \cap (A \cup B)$ **for** $U\ S\ T\ A\ B$ :: *'a set*
    by *blast*
  show *?thesis*
   by (*simp add*: *frontier_of_closures closure_of_mono Diff_Int* $*$ *flip*: *closure_of_Un*)
qed

lemma *frontier_of_Int_subset*: $X\ frontier\_of\ (S \cap T) \subseteq X\ frontier\_of\ S \cup X\ frontier\_of\ T$
  by (*simp add*: *frontier_of_Int*)

lemma *frontier_of_Int_closedin*:
  assumes *closedin X S closedin X T*
  shows $X\ frontier\_of(S \cap T) = X\ frontier\_of\ S \cap T \cup S \cap X\ frontier\_of\ T$ (**is** $?lhs = ?rhs$)
proof
  show $?lhs \subseteq ?rhs$
   using *assms* **by** (*force simp add*: *frontier_of_Int closedin_Int closure_of_closedin*)
  show $?rhs \subseteq ?lhs$
    using *assms frontier_of_subset_closedin*
    by (*auto simp add*: *frontier_of_Int closedin_Int closure_of_closedin*)
qed

lemma *frontier_of_Un_subset*: $X\ frontier\_of(S \cup T) \subseteq X\ frontier\_of\ S \cup X\ frontier\_of\ T$
  by (*metis Diff_Un frontier_of_Int_subset frontier_of_complement*)

lemma *frontier_of_Union_subset*:
  $finite\ \mathcal{F} \Longrightarrow X\ frontier\_of\ (\bigcup \mathcal{F}) \subseteq (\bigcup T \in \mathcal{F}.\ X\ frontier\_of\ T)$
proof (*induction* $\mathcal{F}$ *rule*: *finite_induct*)
  case (*insert* A $\mathcal{F}$)
  then show *?case*

using *frontier_of_Un_subset* **by** *fastforce*
qed *simp*

lemma *frontier_of_frontier_of_subset*:
X *frontier_of* (X *frontier_of* S) $\subseteq$ X *frontier_of* S
by (*simp add*: *closedin_frontier_of frontier_of_subset_closedin*)

lemma *frontier_of_subtopology_open*:
openin X U $\Longrightarrow$ (*subtopology* X U) *frontier_of* S = $U \cap X$ *frontier_of* S
by (*simp add*: *Diff_Int_distrib closure_of_subtopology_open frontier_of_def interior_of_subtopology_open*)

lemma *discrete_topology_frontier_of* [*simp*]:
(*discrete_topology* U) *frontier_of* S = $\{\}$
by (*simp add*: *Diff_eq discrete_topology_closure_of frontier_of_closures*)

2.2.9 Locally finite collections

definition *locally_finite_in*
where
locally_finite_in X $\mathcal{A}$ $\longleftrightarrow$
($\bigcup\mathcal{A} \subseteq$ *topspace* X) $\wedge$
($\forall x \in$ *topspace* X. $\exists V$. *openin* X V $\wedge$ $x \in V$ $\wedge$ *finite* $\{U \in \mathcal{A}.\ U \cap V \neq \{\}\}$)

lemma *finite_imp_locally_finite_in*:
$[\![$*finite* $\mathcal{A}$; $\bigcup\mathcal{A} \subseteq$ *topspace* $X]\!]$ $\Longrightarrow$ *locally_finite_in* X $\mathcal{A}$
by (*auto simp*: *locally_finite_in_def*)

lemma *locally_finite_in_subset*:
assumes *locally_finite_in* X $\mathcal{A}$ $\mathcal{B} \subseteq \mathcal{A}$
shows *locally_finite_in* X $\mathcal{B}$
proof −
have *finite* ($\mathcal{A} \cap \{U.\ U \cap V \neq \{\}\}$) $\Longrightarrow$ *finite* ($\mathcal{B} \cap \{U.\ U \cap V \neq \{\}\}$) **for** V
by (*meson* ‹$\mathcal{B} \subseteq \mathcal{A}$› *finite_subset inf_le1 inf_le2 le_inf_iff subset_trans*)
then show *?thesis*
using *assms* **unfolding** *locally_finite_in_def Int_def* **by** *fastforce*
qed

lemma *locally_finite_in_refinement*:
assumes $\mathcal{A}$: *locally_finite_in* X $\mathcal{A}$ **and** f: $\bigwedge S.\ S \in \mathcal{A} \Longrightarrow f\ S \subseteq S$
shows *locally_finite_in* X (f ‘ $\mathcal{A}$)
proof −
show *?thesis*
unfolding *locally_finite_in_def*
proof *safe*
fix x
assume $x \in$ *topspace* X
then obtain V **where** *openin* X V $x \in V$ *finite* $\{U \in \mathcal{A}.\ U \cap V \neq \{\}\}$

using $\mathcal{A}$ **unfolding** *locally_finite_in_def* **by** *blast*
moreover have $\{U \in \mathcal{A}.\ f\ U \cap V \neq \{\}\} \subseteq \{U \in \mathcal{A}.\ U \cap V \neq \{\}\}$ **for** V
using f **by** *blast*
ultimately have *finite* $\{U \in \mathcal{A}.\ f\ U \cap V \neq \{\}\}$
using *finite_subset* **by** *blast*
moreover have f ‘ $\{U \in \mathcal{A}.\ f\ U \cap V \neq \{\}\} = \{U \in f$ ‘ $\mathcal{A}.\ U \cap V \neq \{\}\}$
by *blast*
ultimately have *finite* $\{U \in f$ ‘ $\mathcal{A}.\ U \cap V \neq \{\}\}$
by (*metis* (*no_types*, *lifting*) *finite_imageI*)
then show $\exists V.$ *openin* $X\ V \wedge x \in V \wedge$ *finite* $\{U \in f$ ‘ $\mathcal{A}.\ U \cap V \neq \{\}\}$
using ‹*openin* $X\ V$› ‹$x \in V$› **by** *blast*
next
show $\bigwedge x\ xa.\ [\![xa \in \mathcal{A};\ x \in f\ xa]\!] \Longrightarrow x \in$ *topspace* X
by (*meson Sup_upper* $\mathcal{A}$ f *locally_finite_in_def subset_iff*)
qed
qed

lemma *locally_finite_in_subtopology*:
assumes $\mathcal{A}$: *locally_finite_in* $X\ \mathcal{A}\ \bigcup\mathcal{A} \subseteq S$
shows *locally_finite_in* (*subtopology* $X\ S$) $\mathcal{A}$
unfolding *locally_finite_in_def*
proof *safe*
fix x
assume x: $x \in$ *topspace* (*subtopology* $X\ S$)
then obtain V **where** *openin* $X\ V\ x \in V$ **and** *fin*: *finite* $\{U \in \mathcal{A}.\ U \cap V \neq \{\}\}$
using $\mathcal{A}$ **unfolding** *locally_finite_in_def topspace_subtopology* **by** *blast*
show $\exists V.$ *openin* (*subtopology* $X\ S$) $V \wedge x \in V \wedge$ *finite* $\{U \in \mathcal{A}.\ U \cap V \neq \{\}\}$
proof (*intro exI conjI*)
show *openin* (*subtopology* $X\ S$) $(S \cap V)$
by (*simp add*: ‹*openin* $X\ V$› *openin_subtopology_Int2*)
have $\{U \in \mathcal{A}.\ U \cap (S \cap V) \neq \{\}\} \subseteq \{U \in \mathcal{A}.\ U \cap V \neq \{\}\}$
by *auto*
with *fin* **show** *finite* $\{U \in \mathcal{A}.\ U \cap (S \cap V) \neq \{\}\}$
using *finite_subset* **by** *auto*
show $x \in S \cap V$
using x ‹$x \in V$› **by** (*simp*)
qed
next
show $\bigwedge x\ A.\ [\![x \in A;\ A \in \mathcal{A}]\!] \Longrightarrow x \in$ *topspace* (*subtopology* $X\ S$)
using *assms* **unfolding** *locally_finite_in_def topspace_subtopology* **by** *blast*
qed

lemma *closedin_locally_finite_Union*:
assumes *clo*: $\bigwedge S.\ S \in \mathcal{A} \Longrightarrow$ *closedin* $X\ S$ **and** $\mathcal{A}$: *locally_finite_in* $X\ \mathcal{A}$
shows *closedin* $X\ (\bigcup\mathcal{A})$
using $\mathcal{A}$ **unfolding** *locally_finite_in_def closedin_def*

proof *clarify*
  show *openin X* (*topspace X* − ⋃𝒜)
  proof (*subst openin_subopen*, *clarify*)
    fix *x*
    assume *x* ∈ *topspace X* **and** *x* ∉ ⋃𝒜
    then obtain *V* **where** *openin X V x* ∈ *V* **and** *fin*: *finite* {*U* ∈ 𝒜. *U* ∩ *V* ≠ {}}
      using 𝒜 **unfolding** *locally_finite_in_def* **by** *blast*
    let *?T* = *V* − ⋃{*S* ∈ 𝒜. *S* ∩ *V* ≠ {}}
    show ∃ *T*. *openin X T* ∧ *x* ∈ *T* ∧ *T* ⊆ *topspace X* − ⋃𝒜
    proof (*intro exI conjI*)
      show *openin X ?T*
      by (*metis* (*no_types*, *lifting*) *fin* ‹*openin X V*› *clo closedin_Union mem_Collect_eq openin_diff*)
      show *x* ∈ *?T*
        using ‹*x* ∉ ⋃𝒜› ‹*x* ∈ *V*› **by** *auto*
      show *?T* ⊆ *topspace X* − ⋃𝒜
        using ‹*openin X V*› *openin_subset* **by** *auto*
    qed
  qed
qed

lemma *locally_finite_in_closure*:
  assumes 𝒜: *locally_finite_in X* 𝒜
  shows *locally_finite_in X* ((λ*S*. *X closure_of S*) ‘ 𝒜)
  using 𝒜 **unfolding** *locally_finite_in_def*
proof (*intro conjI*; *clarsimp*)
  fix *x A*
  assume *x* ∈ *X closure_of A*
  then show *x* ∈ *topspace X*
    by (*meson in_closure_of*)
next
  fix *x*
  assume *x* ∈ *topspace X* **and** ⋃𝒜 ⊆ *topspace X*
  then obtain *V* **where** *V*: *openin X V x* ∈ *V* **and** *fin*: *finite* {*U* ∈ 𝒜. *U* ∩ *V* ≠ {}}
    using 𝒜 **unfolding** *locally_finite_in_def* **by** *blast*
  have *eq*: {*y* ∈ *f* ‘ 𝒜. *Q y*} = *f* ‘ {*x*. *x* ∈ 𝒜 ∧ *Q*(*f x*)} **for** *f Q*
    by *blast*
  have *eq2*: {*A* ∈ 𝒜. *X closure_of A* ∩ *V* ≠ {}} = {*A* ∈ 𝒜. *A* ∩ *V* ≠ {}}
    using *openin_Int_closure_of_eq_empty V* **by** *blast*
  have *finite* {*U* ∈ (*closure_of*) *X* ‘ 𝒜. *U* ∩ *V* ≠ {}}
    by (*simp add*: *eq eq2 fin*)
  with *V* **show** ∃ *V*. *openin X V* ∧ *x* ∈ *V* ∧ *finite* {*U* ∈ (*closure_of*) *X* ‘ 𝒜. *U* ∩ *V* ≠ {}}
    by *blast*
qed

lemma *closedin_Union_locally_finite_closure*:

locally_finite_in X $\mathcal{A}$ ⟹ closedin X (⋃((λS. X *closure_of* S) ‘ $\mathcal{A}$))
by (*metis* (*mono_tags*) *closedin_closure_of closedin_locally_finite_Union imageE locally_finite_in_closure*)

lemma *closure_of_Union_subset*: ⋃((λS. X *closure_of* S) ‘ $\mathcal{A}$) ⊆ X *closure_of* (⋃$\mathcal{A}$)
by *clarify* (*meson Union_upper closure_of_mono subsetD*)

lemma *closure_of_locally_finite_Union*:
assumes *locally_finite_in X $\mathcal{A}$*
shows X *closure_of* (⋃$\mathcal{A}$) = ⋃((λS. X *closure_of* S) ‘ $\mathcal{A}$)
proof (*rule closure_of_unique*)
show ⋃ $\mathcal{A}$ ⊆ ⋃ ((*closure_of*) X ‘ $\mathcal{A}$)
using *assms* **by** (*simp add*: *SUP_upper2 Sup_le_iff closure_of_subset locally_finite_in_def*)
show *closedin X* (⋃ ((*closure_of*) X ‘ $\mathcal{A}$))
using *assms* **by** (*simp add*: *closedin_Union_locally_finite_closure*)
show ⋀T'. ⟦⋃ $\mathcal{A}$ ⊆ T'; *closedin X T'*⟧ ⟹ ⋃ ((*closure_of*) X ‘ $\mathcal{A}$) ⊆ T'
by (*simp add*: *Sup_le_iff closure_of_minimal*)
qed

2.2.10 Continuous maps

We will need to deal with continuous maps in terms of topologies and not in terms of type classes, as defined below.

definition *continuous_map* **where**
continuous_map X Y f ≡
(∀ x ∈ *topspace* X. f x ∈ *topspace* Y) ∧
(∀ U. *openin* Y U ⟶ *openin* X {x ∈ *topspace* X. f x ∈ U})

lemma *continuous_map*:
continuous_map X Y f ⟷
f ‘ (*topspace* X) ⊆ *topspace* Y ∧ (∀ U. *openin* Y U ⟶ *openin* X {x ∈ *topspace* X. f x ∈ U})
by (*auto simp*: *continuous_map_def*)

lemma *continuous_map_image_subset_topspace*:
continuous_map X Y f ⟹ f ‘ (*topspace* X) ⊆ *topspace* Y
by (*auto simp*: *continuous_map_def*)

lemma *continuous_map_on_empty*: *topspace* X = {} ⟹ *continuous_map X Y f*
by (*auto simp*: *continuous_map_def*)

lemma *continuous_map_closedin*:
continuous_map X Y f ⟷
(∀ x ∈ *topspace* X. f x ∈ *topspace* Y) ∧
(∀ C. *closedin* Y C ⟶ *closedin* X {x ∈ *topspace* X. f x ∈ C})
proof –
have (∀ U. *openin* Y U ⟶ *openin* X {x ∈ *topspace* X. f x ∈ U}) =
(∀ C. *closedin* Y C ⟶ *closedin* X {x ∈ *topspace* X. f x ∈ C})
if ⋀x. x ∈ *topspace* X ⟹ f x ∈ *topspace* Y

proof −
have *eq*: {*x* ∈ *topspace X*. *f x* ∈ *topspace Y* ∧ *f x* ∉ *C*} = (*topspace X* − {*x* ∈ *topspace X*. *f x* ∈ *C*}) **for** *C*
using *that* **by** *blast*
show *?thesis*
proof (*intro iffI allI impI*)
fix *C*
assume ∀ *U*. *openin Y U* ⟶ *openin X* {*x* ∈ *topspace X*. *f x* ∈ *U*} **and** *closedin Y C*
then have *openin X* {*x* ∈ *topspace X*. *f x* ∈ *topspace Y* − *C*} **by** *blast*
then show *closedin X* {*x* ∈ *topspace X*. *f x* ∈ *C*}
by (*auto simp add*: *closedin_def eq*)
next
fix *U*
assume ∀ *C*. *closedin Y C* ⟶ *closedin X* {*x* ∈ *topspace X*. *f x* ∈ *C*} **and** *openin Y U*
then have *closedin X* {*x* ∈ *topspace X*. *f x* ∈ *topspace Y* − *U*} **by** *blast*
then show *openin X* {*x* ∈ *topspace X*. *f x* ∈ *U*}
by (*auto simp add*: *openin_closedin_eq eq*)
qed
qed
then show *?thesis*
by (*auto simp*: *continuous_map_def*)
qed

lemma *openin_continuous_map_preimage*:
⟦*continuous_map X Y f*; *openin Y U*⟧ ⟹ *openin X* {*x* ∈ *topspace X*. *f x* ∈ *U*}
by (*simp add*: *continuous_map_def*)

lemma *closedin_continuous_map_preimage*:
⟦*continuous_map X Y f*; *closedin Y C*⟧ ⟹ *closedin X* {*x* ∈ *topspace X*. *f x* ∈ *C*}
by (*simp add*: *continuous_map_closedin*)

lemma *openin_continuous_map_preimage_gen*:
assumes *continuous_map X Y f openin X U openin Y V*
shows *openin X* {*x* ∈ *U*. *f x* ∈ *V*}
proof −
have *eq*: {*x* ∈ *U*. *f x* ∈ *V*} = *U* ∩ {*x* ∈ *topspace X*. *f x* ∈ *V*}
using *assms*(*2*) *openin_closedin_eq* **by** *fastforce*
show *?thesis*
unfolding *eq*
using *assms openin_continuous_map_preimage* **by** *fastforce*
qed

lemma *closedin_continuous_map_preimage_gen*:
assumes *continuous_map X Y f closedin X U closedin Y V*
shows *closedin X* {*x* ∈ *U*. *f x* ∈ *V*}
proof −

```
  have eq: {x ∈ U. f x ∈ V} = U ∩ {x ∈ topspace X. f x ∈ V}
    using assms(2) closedin_def by fastforce
  show ?thesis
    unfolding eq
    using assms closedin_continuous_map_preimage by fastforce
qed

lemma continuous_map_image_closure_subset:
  assumes continuous_map X Y f
  shows f ` (X closure_of S) ⊆ Y closure_of f ` S
proof -
  have *: f ` (topspace X) ⊆ topspace Y
    by (meson assms continuous_map)
  have X closure_of T ⊆ {x ∈ X closure_of T. f x ∈ Y closure_of (f ` T)} if T ⊆
topspace X for T
  proof (rule closure_of_minimal)
    show T ⊆ {x ∈ X closure_of T. f x ∈ Y closure_of f ` T}
      using closure_of_subset * that  by (fastforce simp: in_closure_of)
  next
    show closedin X {x ∈ X closure_of T. f x ∈ Y closure_of f ` T}
      using assms closedin_continuous_map_preimage_gen by fastforce
  qed
  then have f ` (X closure_of (topspace X ∩ S)) ⊆ Y closure_of (f ` (topspace X
∩ S))
    by blast
  also have ... ⊆ Y closure_of (topspace Y ∩ f ` S)
    using * by (blast intro!: closure_of_mono)
  finally have f ` (X closure_of (topspace X ∩ S)) ⊆ Y closure_of (topspace Y ∩
f ` S) .
  then show ?thesis
    by (metis closure_of_restrict)
qed

lemma continuous_map_subset_aux1: continuous_map X Y f ⟹
      (∀ S. f ` (X closure_of S) ⊆ Y closure_of f ` S)
  using continuous_map_image_closure_subset by blast

lemma continuous_map_subset_aux2:
  assumes ∀ S. S ⊆ topspace X ⟶ f ` (X closure_of S) ⊆ Y closure_of f ` S
  shows continuous_map X Y f
  unfolding continuous_map_closedin
proof (intro conjI ballI allI impI)
  fix x
  assume x ∈ topspace X
  then show f x ∈ topspace Y
    using assms closure_of_subset_topspace by fastforce
next
  fix C
  assume closedin Y C
```

then show *closedin* X $\{x \in$ *topspace* $X.\ f\ x \in C\}$
proof (*clarsimp simp flip*: *closure_of_subset_eq*, *intro conjI*)
fix x
assume x: $x \in X$ *closure_of* $\{x \in$ *topspace* $X.\ f\ x \in C\}$
and $C \subseteq$ *topspace* Y **and** Y *closure_of* $C \subseteq C$
show $x \in$ *topspace* X
by (*meson x in_closure_of*)
have $\{a \in$ *topspace* $X.\ f\ a \in C\} \subseteq$ *topspace* X
by *simp*
moreover have Y *closure_of* f ‘ $\{a \in$ *topspace* $X.\ f\ a \in C\} \subseteq C$
by (*simp add*: ‹*closedin* Y C› *closure_of_minimal image_subset_iff*)
ultimately have f ‘ (X *closure_of* $\{a \in$ *topspace* $X.\ f\ a \in C\}) \subseteq C$
using *assms* **by** *blast*
then show $f\ x \in C$
using x **by** *auto*
qed
qed

lemma *continuous_map_eq_image_closure_subset*:
continuous_map X Y $f \longleftrightarrow (\forall S.\ f$ ‘ (X *closure_of* $S) \subseteq Y$ *closure_of* f ‘ $S)$
using *continuous_map_subset_aux1 continuous_map_subset_aux2* **by** *metis*

lemma *continuous_map_eq_image_closure_subset_alt*:
continuous_map X Y $f \longleftrightarrow (\forall S.\ S \subseteq$ *topspace* $X \longrightarrow f$ ‘ (X *closure_of* $S) \subseteq$
Y *closure_of* f ‘ $S)$
using *continuous_map_subset_aux1 continuous_map_subset_aux2* **by** *metis*

lemma *continuous_map_eq_image_closure_subset_gen*:
continuous_map X Y $f \longleftrightarrow$
f ‘ (*topspace* $X) \subseteq$ *topspace* Y $\wedge$
$(\forall S.\ f$ ‘ (X *closure_of* $S) \subseteq Y$ *closure_of* f ‘ $S)$
using *continuous_map_subset_aux1 continuous_map_subset_aux2 continuous_map_image_subset_topspace*
by *metis*

lemma *continuous_map_closure_preimage_subset*:
continuous_map X Y f
$\Longrightarrow X$ *closure_of* $\{x \in$ *topspace* $X.\ f\ x \in T\}$
$\subseteq \{x \in$ *topspace* $X.\ f\ x \in Y$ *closure_of* $T\}$
unfolding *continuous_map_closedin*
by (*rule closure_of_minimal*) (*use in_closure_of* **in** ‹*fastforce*+›)

lemma *continuous_map_frontier_frontier_preimage_subset*:
assumes *continuous_map* X Y f
shows X *frontier_of* $\{x \in$ *topspace* $X.\ f\ x \in T\} \subseteq \{x \in$ *topspace* $X.\ f\ x \in Y$
frontier_of $T\}$
proof –
have *eq*: *topspace* $X - \{x \in$ *topspace* $X.\ f\ x \in T\} = \{x \in$ *topspace* $X.\ f\ x \in$
topspace $Y - T\}$

```
    using assms unfolding continuous_map_def by blast
  have X closure_of {x ∈ topspace X. f x ∈ T} ⊆ {x ∈ topspace X. f x ∈ Y
closure_of T}
    by (simp add: assms continuous_map_closure_preimage_subset)
  moreover
  have X closure_of (topspace X − {x ∈ topspace X. f x ∈ T}) ⊆ {x ∈ topspace
X. f x ∈ Y closure_of (topspace Y − T)}
    using continuous_map_closure_preimage_subset [OF assms] eq by presburger
  ultimately show ?thesis
    by (auto simp: frontier_of_closures)
qed

lemma topology_finer_continuous_id:
  assumes topspace X = topspace Y
  shows (∀ S. openin X S ⟶ openin Y S) ⟷ continuous_map Y X id  (is ?lhs
= ?rhs)
proof
  show ?lhs ⟹ ?rhs
    unfolding continuous_map_def
    using assms openin_subopen openin_subset by fastforce
  show ?rhs ⟹ ?lhs
    unfolding continuous_map_def
    using assms openin_subopen topspace_def by fastforce
qed

lemma continuous_map_const [simp]:
   continuous_map X Y (λx. C) ⟷ topspace X = {} ∨ C ∈ topspace Y
proof (cases topspace X = {})
  case False
  show ?thesis
  proof (cases C ∈ topspace Y)
    case True
    with openin_subopen show ?thesis
      by (auto simp: continuous_map_def)
  next
    case False
    then show ?thesis
      unfolding continuous_map_def by fastforce
  qed
qed (auto simp: continuous_map_on_empty)

declare continuous_map_const [THEN iffD2, continuous_intros]

lemma continuous_map_compose [continuous_intros]:
  assumes f: continuous_map X X′ f and g: continuous_map X′ X″ g
  shows continuous_map X X″ (g ∘ f)
  unfolding continuous_map_def
proof (intro conjI ballI allI impI)
  fix x
```

assume *x* ∈ *topspace X*
then show (*g* ∘ *f*) *x* ∈ *topspace X''*
  using *assms* **unfolding** *continuous_map_def* **by** *force*
next
fix *U*
assume *openin X'' U*
have *eq*: {*x* ∈ *topspace X*. (*g* ∘ *f*) *x* ∈ *U*} = {*x* ∈ *topspace X*. *f x* ∈ {*y*. *y* ∈ *topspace X'* ∧ *g y* ∈ *U*}}
  by *auto* (*meson f continuous_map_def*)
show *openin X* {*x* ∈ *topspace X*. (*g* ∘ *f*) *x* ∈ *U*}
  unfolding *eq*
  using *assms* **unfolding** *continuous_map_def*
  using ‹*openin X'' U*› **by** *blast*
qed

lemma *continuous_map_eq*:
assumes *continuous_map X X' f* **and** ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *g x* **shows** *continuous_map X X' g*
proof −
have *eq*: {*x* ∈ *topspace X*. *f x* ∈ *U*} = {*x* ∈ *topspace X*. *g x* ∈ *U*} **for** *U*
  using *assms* **by** *auto*
show *?thesis*
  using *assms* **by** (*simp add*: *continuous_map_def eq*)
qed

lemma *restrict_continuous_map* [*simp*]:
topspace X ⊆ *S* ⟹ *continuous_map X X'* (*restrict f S*) ⟷ *continuous_map X X' f*
by (*auto simp*: *elim*!: *continuous_map_eq*)

lemma *continuous_map_in_subtopology*:
continuous_map X (*subtopology X' S*) *f* ⟷ *continuous_map X X' f* ∧ *f* ‘ (*topspace X*) ⊆ *S*
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
show *?rhs*
proof −
  have ⋀*A*. *f* ‘ (*X closure_of A*) ⊆ *subtopology X' S closure_of f* ‘ *A*
    by (*meson L continuous_map_image_closure_subset*)
  then show *?thesis*
    by (*metis* (*no_types*) *closure_of_subset_subtopology closure_of_subtopology_subset closure_of_topspace continuous_map_eq_image_closure_subset dual_order.trans*)
qed
next
assume *R*: *?rhs*
then have *eq*: {*x* ∈ *topspace X*. *f x* ∈ *U*} = {*x* ∈ *topspace X*. *f x* ∈ *U* ∧ *f x* ∈ *S*} **for** *U*
  by *auto*

show *?lhs*
using *R*
unfolding *continuous_map*
by (*auto simp*: *openin_subtopology eq*)
qed

lemma *continuous_map_from_subtopology*:
continuous_map X X′ f ⟹ *continuous_map* (*subtopology X S*) *X′ f*
by (*auto simp*: *continuous_map openin_subtopology*)

lemma *continuous_map_into_fulltopology*:
continuous_map X (*subtopology X′ T*) *f* ⟹ *continuous_map X X′ f*
by (*auto simp*: *continuous_map_in_subtopology*)

lemma *continuous_map_into_subtopology*:
⟦*continuous_map X X′ f*; *f ‘ topspace X* ⊆ *T*⟧ ⟹ *continuous_map X* (*subtopology X′ T*) *f*
by (*auto simp*: *continuous_map_in_subtopology*)

lemma *continuous_map_from_subtopology_mono*:
⟦*continuous_map* (*subtopology X T*) *X′ f*; *S* ⊆ *T*⟧
⟹ *continuous_map* (*subtopology X S*) *X′ f*
by (*metis inf.absorb_iff2 continuous_map_from_subtopology subtopology_subtopology*)

lemma *continuous_map_from_discrete_topology* [*simp*]:
continuous_map (*discrete_topology U*) *X f* ⟷ *f ‘ U* ⊆ *topspace X*
by (*auto simp*: *continuous_map_def*)

lemma *continuous_map_iff_continuous* [*simp*]: *continuous_map* (*top_of_set S*) *euclidean g* = *continuous_on S g*
by (*fastforce simp add*: *continuous_map openin_subtopology continuous_on_open_invariant*)

lemma *continuous_map_iff_continuous2* [*simp*]: *continuous_map euclidean euclidean g* = *continuous_on UNIV g*
by (*metis continuous_map_iff_continuous subtopology_UNIV*)

lemma *continuous_map_openin_preimage_eq*:
continuous_map X Y f ⟷
f ‘ (*topspace X*) ⊆ *topspace Y* ∧ (∀ *U*. *openin Y U* ⟶ *openin X* (*topspace X* ∩ *f* −‘ *U*))
by (*auto simp*: *continuous_map_def vimage_def Int_def*)

lemma *continuous_map_closedin_preimage_eq*:
continuous_map X Y f ⟷
f ‘ (*topspace X*) ⊆ *topspace Y* ∧ (∀ *U*. *closedin Y U* ⟶ *closedin X* (*topspace X* ∩ *f* −‘ *U*))
by (*auto simp*: *continuous_map_closedin vimage_def Int_def*)

lemma *continuous_map_square_root*: *continuous_map euclideanreal euclideanreal sqrt*
by (*simp add*: *continuous_at_imp_continuous_on isCont_real_sqrt*)

lemma *continuous_map_sqrt* [*continuous_intros*]:
continuous_map X euclideanreal f $\Longrightarrow$ *continuous_map X euclideanreal* ($\lambda x.$ *sqrt*(*f* *x*))
by (*meson continuous_map_compose continuous_map_eq continuous_map_square_root o_apply*)

lemma *continuous_map_id* [*simp*, *continuous_intros*]: *continuous_map X X id*
unfolding *continuous_map_def* **using** *openin_subopen topspace_def* **by** *fastforce*

declare *continuous_map_id* [*unfolded id_def*, *simp*, *continuous_intros*]

lemma *continuous_map_id_subt* [*simp*]: *continuous_map* (*subtopology X S*) *X id*
by (*simp add*: *continuous_map_from_subtopology*)

declare *continuous_map_id_subt* [*unfolded id_def*, *simp*]

lemma *continuous_map_alt*:
continuous_map T1 T2 f
= (($\forall$ *U*. *openin T2 U* $\longrightarrow$ *openin T1* (*f* −‘ *U* $\cap$ *topspace T1*)) $\wedge$ *f* ‘ *topspace T1* $\subseteq$ *topspace T2*)
by (*auto simp*: *continuous_map_def vimage_def image_def Collect_conj_eq inf_commute*)

lemma *continuous_map_open* [*intro*]:
continuous_map T1 T2 f $\Longrightarrow$ *openin T2 U* $\Longrightarrow$ *openin T1* (*f*−‘*U* $\cap$ *topspace*(*T1*))
unfolding *continuous_map_alt* **by** *auto*

lemma *continuous_map_preimage_topspace* [*intro*]:
assumes *continuous_map T1 T2 f*
shows *f*−‘(*topspace T2*) $\cap$ *topspace T1* = *topspace T1*
using *assms* **unfolding** *continuous_map_def* **by** *auto*

2.2.11 Open and closed maps (not a priori assumed continuous)

definition *open_map* :: ′*a topology* $\Rightarrow$ ′*b topology* $\Rightarrow$ (′*a* $\Rightarrow$ ′*b*) $\Rightarrow$ *bool*
where *open_map X1 X2 f* $\equiv$ $\forall$ *U*. *openin X1 U* $\longrightarrow$ *openin X2* (*f* ‘ *U*)

definition *closed_map* :: ′*a topology* $\Rightarrow$ ′*b topology* $\Rightarrow$ (′*a* $\Rightarrow$ ′*b*) $\Rightarrow$ *bool*
where *closed_map X1 X2 f* $\equiv$ $\forall$ *U*. *closedin X1 U* $\longrightarrow$ *closedin X2* (*f* ‘ *U*)

lemma *open_map_imp_subset_topspace*:
open_map X1 X2 f $\Longrightarrow$ *f* ‘ (*topspace X1*) $\subseteq$ *topspace X2*
unfolding *open_map_def* **by** (*simp add*: *openin_subset*)

lemma *open_map_on_empty*:

  topspace X = {} ⟹ *open_map X Y f*
 by (*metis empty_iff imageE in_mono open_map_def openin_subopen openin_subset*)

lemma *closed_map_on_empty*:
  topspace X = {} ⟹ *closed_map X Y f*
 by (*simp add*: *closed_map_def closedin_topspace_empty*)

lemma *closed_map_const*:
  closed_map X Y (λ*x*. *c*) ⟷ *topspace X* = {} ∨ *closedin Y* {*c*}
proof (*cases topspace X* = {})
 case *True*
 then show *?thesis*
  by (*simp add*: *closed_map_on_empty*)
next
 case *False*
 then show *?thesis*
  by (*auto simp*: *closed_map_def image_constant_conv*)
qed

lemma *open_map_imp_subset*:
  ⟦*open_map X1 X2 f*; *S* ⊆ *topspace X1*⟧ ⟹ *f* ‘ *S* ⊆ *topspace X2*
 by (*meson order_trans open_map_imp_subset_topspace subset_image_iff*)

lemma *topology_finer_open_id*:
  (∀ *S*. *openin X S* ⟶ *openin X′ S*) ⟷ *open_map X X′ id*
 unfolding *open_map_def* **by** *auto*

lemma *open_map_id*: *open_map X X id*
 unfolding *open_map_def* **by** *auto*

lemma *open_map_eq*:
  ⟦*open_map X X′ f*; ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *g x*⟧ ⟹ *open_map X X′ g*
 unfolding *open_map_def*
 by (*metis image_cong openin_subset subset_iff*)

lemma *open_map_inclusion_eq*:
 open_map (*subtopology X S*) *X id* ⟷ *openin X* (*topspace X* ∩ *S*)
proof −
 have ∗: *openin X* (*T* ∩ *S*) **if** *openin X* (*S* ∩ *topspace X*) *openin X T* **for** *T*
 proof −
  have *T* ⊆ *topspace X*
   using *that* **by** (*simp add*: *openin_subset*)
  with *that* **show** *openin X* (*T* ∩ *S*)
   by (*metis inf.absorb1 inf.left_commute inf_commute openin_Int*)
 qed
 show *?thesis*
  by (*fastforce simp add*: *open_map_def Int_commute openin_subtopology_alt intro*:
∗)
qed

lemma *open_map_inclusion*:
openin X S $\Longrightarrow$ *open_map* (*subtopology X S*) *X id*
by (*simp add*: *open_map_inclusion_eq openin_Int*)

lemma *open_map_compose*:
$[\![$*open_map X X′ f*; *open_map X′ X″ g*$]\!]$ $\Longrightarrow$ *open_map X X″* ($g \circ f$)
by (*metis* (*no_types*, *lifting*) *image_comp open_map_def*)

lemma *closed_map_imp_subset_topspace*:
closed_map X1 X2 f $\Longrightarrow$ *f* ‘ (*topspace X1*) $\subseteq$ *topspace X2*
by (*simp add*: *closed_map_def closedin_subset*)

lemma *closed_map_imp_subset*:
$[\![$*closed_map X1 X2 f*; *S* $\subseteq$ *topspace X1*$]\!]$ $\Longrightarrow$ *f* ‘ *S* $\subseteq$ *topspace X2*
using *closed_map_imp_subset_topspace* **by** *blast*

lemma *topology_finer_closed_id*:
($\forall$ *S*. *closedin X S* $\longrightarrow$ *closedin X′ S*) $\longleftrightarrow$ *closed_map X X′ id*
by (*simp add*: *closed_map_def*)

lemma *closed_map_id*: *closed_map X X id*
by (*simp add*: *closed_map_def*)

lemma *closed_map_eq*:
$[\![$*closed_map X X′ f*; $\bigwedge$*x*. *x* $\in$ *topspace X* $\Longrightarrow$ *f x* = *g x*$]\!]$ $\Longrightarrow$ *closed_map X X′ g*
unfolding *closed_map_def*
by (*metis image_cong closedin_subset subset_iff*)

lemma *closed_map_compose*:
$[\![$*closed_map X X′ f*; *closed_map X′ X″ g*$]\!]$ $\Longrightarrow$ *closed_map X X″* ($g \circ f$)
by (*metis* (*no_types*, *lifting*) *closed_map_def image_comp*)

lemma *closed_map_inclusion_eq*:
closed_map (*subtopology X S*) *X id* $\longleftrightarrow$
closedin X (*topspace X* $\cap$ *S*)
proof −
have ∗: *closedin X* (*T* $\cap$ *S*) **if** *closedin X* (*S* $\cap$ *topspace X*) *closedin X T* **for** *T*
proof −
have *T* $\subseteq$ *topspace X*
using *that* **by** (*simp add*: *closedin_subset*)
with *that* **show** *closedin X* (*T* $\cap$ *S*)
by (*metis inf.absorb1 inf.left_commute inf_commute closedin_Int*)
qed
show *?thesis*
by (*fastforce simp add*: *closed_map_def Int_commute closedin_subtopology_alt intro*: ∗)
qed

lemma *closed_map_inclusion*: *closedin X S* ⟹ *closed_map* (*subtopology X S*) *X id*
by (*simp add*: *closed_map_inclusion_eq closedin_Int*)

lemma *open_map_into_subtopology*:
⟦*open_map X X′ f*; *f ‘ topspace X* ⊆ *S*⟧ ⟹ *open_map X* (*subtopology X′ S*) *f*
unfolding *open_map_def openin_subtopology*
using *openin_subset* **by** *fastforce*

lemma *closed_map_into_subtopology*:
⟦*closed_map X X′ f*; *f ‘ topspace X* ⊆ *S*⟧ ⟹ *closed_map X* (*subtopology X′ S*) *f*
unfolding *closed_map_def closedin_subtopology*
using *closedin_subset* **by** *fastforce*

lemma *open_map_into_discrete_topology*:
open_map X (*discrete_topology U*) *f* ⟷ *f ‘* (*topspace X*) ⊆ *U*
unfolding *open_map_def openin_discrete_topology* **using** *openin_subset* **by** *blast*

lemma *closed_map_into_discrete_topology*:
closed_map X (*discrete_topology U*) *f* ⟷ *f ‘* (*topspace X*) ⊆ *U*
unfolding *closed_map_def closedin_discrete_topology* **using** *closedin_subset* **by** *blast*

lemma *bijective_open_imp_closed_map*:
⟦*open_map X X′ f*; *f ‘* (*topspace X*) = *topspace X′*; *inj_on f* (*topspace X*)⟧ ⟹ *closed_map X X′ f*
unfolding *open_map_def closed_map_def closedin_def*
by *auto* (*metis Diff_subset inj_on_image_set_diff*)

lemma *bijective_closed_imp_open_map*:
⟦*closed_map X X′ f*; *f ‘* (*topspace X*) = *topspace X′*; *inj_on f* (*topspace X*)⟧ ⟹ *open_map X X′ f*
unfolding *closed_map_def open_map_def openin_closedin_eq*
by *auto* (*metis Diff_subset inj_on_image_set_diff*)

lemma *open_map_from_subtopology*:
⟦*open_map X X′ f*; *openin X U*⟧ ⟹ *open_map* (*subtopology X U*) *X′ f*
unfolding *open_map_def openin_subtopology_alt* **by** *blast*

lemma *closed_map_from_subtopology*:
⟦*closed_map X X′ f*; *closedin X U*⟧ ⟹ *closed_map* (*subtopology X U*) *X′ f*
unfolding *closed_map_def closedin_subtopology_alt* **by** *blast*

lemma *open_map_restriction*:
assumes *f*: *open_map X X′ f* **and** *U*: {*x* ∈ *topspace X*. *f x* ∈ *V*} = *U*
shows *open_map* (*subtopology X U*) (*subtopology X′ V*) *f*
unfolding *open_map_def*
proof *clarsimp*

fix *W*
assume *openin* (*subtopology X U*) *W*
then obtain *T* **where** *openin X T W* = *T* ∩ *U*
by (*meson openin_subtopology*)
with *f U* **have** *f ‘ W* = (*f ‘ T*) ∩ *V*
unfolding *open_map_def openin_closedin_eq* **by** *auto*
then show *openin* (*subtopology X′ V*) (*f ‘ W*)
by (*metis* ‹*openin X T*› *f open_map_def openin_subtopology_Int*)
qed

lemma *closed_map_restriction*:
assumes *f*: *closed_map X X′ f* **and** *U*: {*x* ∈ *topspace X*. *f x* ∈ *V*} = *U*
shows *closed_map* (*subtopology X U*) (*subtopology X′ V*) *f*
unfolding *closed_map_def*
proof *clarsimp*
fix *W*
assume *closedin* (*subtopology X U*) *W*
then obtain *T* **where** *closedin X T W* = *T* ∩ *U*
by (*meson closedin_subtopology*)
with *f U* **have** *f ‘ W* = (*f ‘ T*) ∩ *V*
unfolding *closed_map_def closedin_def* **by** *auto*
then show *closedin* (*subtopology X′ V*) (*f ‘ W*)
by (*metis* ‹*closedin X T*› *closed_map_def closedin_subtopology f*)
qed

2.2.12 Quotient maps

definition *quotient_map* **where**
quotient_map X X′ f ⟷
f ‘ (*topspace X*) = *topspace X′* ∧
(∀ *U*. *U* ⊆ *topspace X′* ⟶ (*openin X* {*x*. *x* ∈ *topspace X* ∧ *f x* ∈ *U*} ⟷ *openin X′ U*))

lemma *quotient_map_eq*:
assumes *quotient_map X X′ f* ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *g x*
shows *quotient_map X X′ g*
proof −
have *eq*: {*x* ∈ *topspace X*. *f x* ∈ *U*} = {*x* ∈ *topspace X*. *g x* ∈ *U*} **for** *U*
using *assms* **by** *auto*
show *?thesis*
using *assms*
unfolding *quotient_map_def*
by (*metis* (*mono_tags*, *lifting*) *eq image_cong*)
qed

lemma *quotient_map_compose*:
assumes *f*: *quotient_map X X′ f* **and** *g*: *quotient_map X′ X″ g*
shows *quotient_map X X″* (*g* ∘ *f*)
unfolding *quotient_map_def*

proof (*intro conjI allI impI*)
  show $(g \circ f)$ ‘ *topspace* X = *topspace* X''
   using *assms* **by** (*simp only*: *image_comp* [*symmetric*]) (*simp add*: *quotient_map_def*)
next
  fix U''
  assume $U'' \subseteq$ *topspace* X''
  define U' **where** $U' \equiv \{y \in$ *topspace* $X'.\ g\ y \in U''\}$
  have $U' \subseteq$ *topspace* X'
    by (*auto simp add*: *U'_def*)
  then have U': *openin* X $\{x \in$ *topspace* $X.\ f\ x \in U'\}$ = *openin* X' U'
    using *assms* **unfolding** *quotient_map_def* **by** *simp*
  have *eq*: $\{x \in$ *topspace* $X.\ f\ x \in$ *topspace* $X' \wedge g\ (f\ x) \in U''\} = \{x \in$ *topspace* $X.\ (g \circ f)\ x \in U''\}$
    using *f quotient_map_def* **by** *fastforce*
  have *openin* X $\{x \in$ *topspace* $X.\ (g \circ f)\ x \in U''\}$ = *openin* X $\{x \in$ *topspace* $X.\ f\ x \in U'\}$
    using *assms* **by** (*simp add*: *quotient_map_def U'_def eq*)
  also have ... = *openin* X'' U''
    using *U'_def* ‹$U'' \subseteq$ *topspace* X''› U' *g quotient_map_def* **by** *fastforce*
  finally show *openin* X $\{x \in$ *topspace* $X.\ (g \circ f)\ x \in U''\}$ = *openin* X'' U'' **.**
qed

lemma *quotient_map_from_composition*:
  assumes *f*: *continuous_map* X X' f **and** *g*: *continuous_map* X' X'' g **and** *gf*: *quotient_map* X X'' $(g \circ f)$
  shows *quotient_map* X' X'' g
  unfolding *quotient_map_def*
proof (*intro conjI allI impI*)
  show g ‘ *topspace* X' = *topspace* X''
    using *assms* **unfolding** *continuous_map_def quotient_map_def* **by** *fastforce*
next
  fix U'' :: *'c set*
  assume U'': $U'' \subseteq$ *topspace* X''
  have *eq*: $\{x \in$ *topspace* $X.\ g\ (f\ x) \in U''\} = \{x \in$ *topspace* $X.\ f\ x \in \{y.\ y \in$ *topspace* $X' \wedge g\ y \in U''\}\}$
    using *continuous_map_def f* **by** *fastforce*
  show *openin* X' $\{x \in$ *topspace* $X'.\ g\ x \in U''\}$ = *openin* X'' U''
    using *assms* **unfolding** *continuous_map_def quotient_map_def*
    by (*metis* (*mono_tags*, *lifting*) *Collect_cong* U'' *comp_apply eq*)
qed

lemma *quotient_imp_continuous_map*:
    quotient_map X X' $f \Longrightarrow$ *continuous_map* X X' f
  by (*simp add*: *continuous_map openin_subset quotient_map_def*)

lemma *quotient_imp_surjective_map*:
    quotient_map X X' $f \Longrightarrow f$ ‘ (*topspace* X) = *topspace* X'
  by (*simp add*: *quotient_map_def*)

lemma *quotient_map_closedin*:
quotient_map X X′ f ⟷
f ‘ (*topspace X*) = *topspace X′* ∧
(∀ *U*. *U* ⊆ *topspace X′* ⟶ (*closedin X* {*x*. *x* ∈ *topspace X* ∧ *f x* ∈ *U*}
⟷ *closedin X′ U*))
proof −
have *eq*: (*topspace X* − {*x* ∈ *topspace X*. *f x* ∈ *U′*}) = {*x* ∈ *topspace X*. *f x* ∈
topspace X′ ∧ *f x* ∉ *U′*}
if *f* ‘ *topspace X* = *topspace X′ U′* ⊆ *topspace X′* **for** *U′*
using *that* **by** *auto*
have (∀ *U*⊆*topspace X′*. *openin X* {*x* ∈ *topspace X*. *f x* ∈ *U*} = *openin X′ U*)
=
(∀ *U*⊆*topspace X′*. *closedin X* {*x* ∈ *topspace X*. *f x* ∈ *U*} = *closedin X′*
U)
if *f* ‘ *topspace X* = *topspace X′*
proof (*rule iffI*; *intro allI impI subsetI*)
fix *U′*
assume ∗[*rule_format*]: ∀ *U*⊆*topspace X′*. *openin X* {*x* ∈ *topspace X*. *f x* ∈
U} = *openin X′ U*
and *U′*: *U′* ⊆ *topspace X′*
show *closedin X* {*x* ∈ *topspace X*. *f x* ∈ *U′*} = *closedin X′ U′*
using *U′* **by** (*auto simp add*: *closedin_def simp flip*: ∗ [*of topspace X′* − *U′*]
eq [*OF that*])
next
fix *U′* :: ′*b set*
assume ∗[*rule_format*]: ∀ *U*⊆*topspace X′*. *closedin X* {*x* ∈ *topspace X*. *f x* ∈
U} = *closedin X′ U*
and *U′*: *U′* ⊆ *topspace X′*
show *openin X* {*x* ∈ *topspace X*. *f x* ∈ *U′*} = *openin X′ U′*
using *U′* **by** (*auto simp add*: *openin_closedin_eq simp flip*: ∗ [*of topspace X′*
− *U′*] *eq* [*OF that*])
qed
then show *?thesis*
unfolding *quotient_map_def* **by** *force*
qed

lemma *continuous_open_imp_quotient_map*:
assumes *continuous_map X X′ f* **and** *om*: *open_map X X′ f* **and** *feq*: *f* ‘ (*topspace*
X) = *topspace X′*
shows *quotient_map X X′ f*
proof −
{ fix *U*
assume *U*: *U* ⊆ *topspace X′* **and** *openin X* {*x* ∈ *topspace X*. *f x* ∈ *U*}
then have *ope*: *openin X′* (*f* ‘ {*x* ∈ *topspace X*. *f x* ∈ *U*})
using *om* **unfolding** *open_map_def* **by** *blast*
then have *openin X′ U*
using *U feq* **by** (*subst openin_subopen*) *force*
}
moreover have *openin X* {*x* ∈ *topspace X*. *f x* ∈ *U*} **if** *U* ⊆ *topspace X′* **and**

openin X′ U **for** *U*
  using *that assms* **unfolding** *continuous_map_def* **by** *blast*
 ultimately show *?thesis*
  unfolding *quotient_map_def* **using** *assms* **by** *blast*
qed

lemma *continuous_closed_imp_quotient_map*:
 assumes *continuous_map X X′ f* **and** *om*: *closed_map X X′ f* **and** *feq*: *f ‘ (topspace X) = topspace X′*
 shows *quotient_map X X′ f*
proof −
 have *f ‘ {x ∈ topspace X. f x ∈ U} = U* **if** *U ⊆ topspace X′* **for** *U*
  using *that feq* **by** *auto*
 with *assms* **show** *?thesis*
  unfolding *quotient_map_closedin closed_map_def continuous_map_closedin* **by** *auto*
qed

lemma *continuous_open_quotient_map*:
 ⟦*continuous_map X X′ f*; *open_map X X′ f*⟧ ⟹ *quotient_map X X′ f* ⟷ *f ‘ (topspace X) = topspace X′*
 by (*meson continuous_open_imp_quotient_map quotient_map_def*)

lemma *continuous_closed_quotient_map*:
 ⟦*continuous_map X X′ f*; *closed_map X X′ f*⟧ ⟹ *quotient_map X X′ f* ⟷ *f ‘ (topspace X) = topspace X′*
 by (*meson continuous_closed_imp_quotient_map quotient_map_def*)

lemma *injective_quotient_map*:
 assumes *inj_on f (topspace X)*
 shows *quotient_map X X′ f* ⟷
   continuous_map X X′ f ∧ *open_map X X′ f* ∧ *closed_map X X′ f* ∧ *f ‘ (topspace X) = topspace X′*
   (**is** *?lhs = ?rhs*)
proof
 assume *L*: *?lhs*
 have *open_map X X′ f*
 proof (*clarsimp simp add*: *open_map_def*)
  fix *U*
  assume *openin X U*
  then have *U ⊆ topspace X*
   by (*simp add*: *openin_subset*)
  moreover have *{x ∈ topspace X. f x ∈ f ‘ U} = U*
   using ‹*U ⊆ topspace X*› *assms inj_onD* **by** *fastforce*
  ultimately show *openin X′ (f ‘ U)*
   using *L* **unfolding** *quotient_map_def*
   by (*metis (no_types, lifting) Collect_cong* ‹*openin X U*› *image_mono*)
 qed
 moreover have *closed_map X X′ f*

proof (*clarsimp simp add*: *closed_map_def*)
fix *U*
assume *closedin X U*
then have $U \subseteq$ *topspace X*
by (*simp add*: *closedin_subset*)
moreover have $\{x \in$ *topspace X*. $f\,x \in f$ ‘ $U\} = U$
using ‹$U \subseteq$ *topspace X*› *assms inj_onD* **by** *fastforce*
ultimately show *closedin X′* (*f* ‘ *U*)
using *L* **unfolding** *quotient_map_closedin*
by (*metis* (*no_types*, *lifting*) *Collect_cong* ‹*closedin X U*› *image_mono*)
qed
ultimately show *?rhs*
using *L* **by** (*simp add*: *quotient_imp_continuous_map quotient_imp_surjective_map*)
next
assume *?rhs*
then show *?lhs*
by (*simp add*: *continuous_closed_imp_quotient_map*)
qed

lemma *continuous_compose_quotient_map*:
assumes *f*: *quotient_map X X′ f* **and** *g*: *continuous_map X X″* ($g \circ f$)
shows *continuous_map X′ X″ g*
unfolding *quotient_map_def continuous_map_def*
proof (*intro conjI ballI allI impI*)
show $\bigwedge x'$. $x' \in$ *topspace X′* $\Longrightarrow g\,x' \in$ *topspace X″*
using *assms* **unfolding** *quotient_map_def*
by (*metis* (*no_types*, *hide_lams*) *continuous_map_image_subset_topspace image_comp image_subset_iff*)
next
fix *U″* :: *′c set*
assume *U″*: *openin X″ U″*
have *f* ‘ *topspace X* = *topspace X′*
by (*simp add*: *f quotient_imp_surjective_map*)
then have *eq*: $\{x \in$ *topspace X*. $f\,x \in$ *topspace X′* $\wedge g\,(f\,x) \in U\} = \{x \in$ *topspace X*. $g\,(f\,x) \in U\}$ **for** *U*
by *auto*
have *openin X* $\{x \in$ *topspace X*. $f\,x \in$ *topspace X′* $\wedge g\,(f\,x) \in U''\}$
unfolding *eq* **using** *U″ g openin_continuous_map_preimage* **by** *fastforce*
then have $*$: *openin X* $\{x \in$ *topspace X*. $f\,x \in \{x \in$ *topspace X′*. $g\,x \in U''\}\}$
by *auto*
show *openin X′* $\{x \in$ *topspace X′*. $g\,x \in U''\}$
using *f* **unfolding** *quotient_map_def*
by (*metis* (*no_types*) *Collect_subset* $*$)
qed

lemma *continuous_compose_quotient_map_eq*:
quotient_map X X′ f $\Longrightarrow$ *continuous_map X X″* ($g \circ f$) $\longleftrightarrow$ *continuous_map X′ X″ g*
using *continuous_compose_quotient_map continuous_map_compose quotient_imp_continuous_map*

by *blast*

lemma *quotient_map_compose_eq*:
quotient_map X X′ f ⟹ *quotient_map X X″* (*g* ∘ *f*) ⟷ *quotient_map X′ X″ g*
by (*meson continuous_compose_quotient_map_eq quotient_imp_continuous_map quotient_map_compose quotient_map_from_composition*)

lemma *quotient_map_restriction*:
assumes *quo*: *quotient_map X Y f* **and** *U*: {*x* ∈ *topspace X*. *f x* ∈ *V*} = *U* **and** *disj*: *openin Y V* ∨ *closedin Y V*
shows *quotient_map* (*subtopology X U*) (*subtopology Y V*) *f*
using *disj*
proof
assume *V*: *openin Y V*
with *U* **have** *sub*: *U* ⊆ *topspace X V* ⊆ *topspace Y*
by (*auto simp*: *openin_subset*)
have *fim*: *f* ‘ *topspace X* = *topspace Y*
and *Y*: ⋀*U*. *U* ⊆ *topspace Y* ⟹ *openin X* {*x* ∈ *topspace X*. *f x* ∈ *U*} = *openin Y U*
using *quo* **unfolding** *quotient_map_def* **by** *auto*
have *openin X U*
using *U V Y sub*(*2*) **by** *blast*
show *?thesis*
unfolding *quotient_map_def*
proof (*intro conjI allI impI*)
show *f* ‘ *topspace* (*subtopology X U*) = *topspace* (*subtopology Y V*)
using *sub U fim* **by** (*auto*)
next
fix *Y′* :: *′b set*
assume *Y′* ⊆ *topspace* (*subtopology Y V*)
then have *Y′* ⊆ *topspace Y Y′* ⊆ *V*
by (*simp_all*)
then have *eq*: {*x* ∈ *topspace X*. *x* ∈ *U* ∧ *f x* ∈ *Y′*} = {*x* ∈ *topspace X*. *f x* ∈ *Y′*}
using *U* **by** *blast*
then show *openin* (*subtopology X U*) {*x* ∈ *topspace* (*subtopology X U*). *f x* ∈ *Y′*} = *openin* (*subtopology Y V*) *Y′*
using *U V Y* ‹*openin X U*› ‹*Y′* ⊆ *topspace Y*› ‹*Y′* ⊆ *V*›
by (*simp add*: *openin_open_subtopology eq*) (*auto simp*: *openin_closedin_eq*)
qed
next
assume *V*: *closedin Y V*
with *U* **have** *sub*: *U* ⊆ *topspace X V* ⊆ *topspace Y*
by (*auto simp*: *closedin_subset*)
have *fim*: *f* ‘ *topspace X* = *topspace Y*
and *Y*: ⋀*U*. *U* ⊆ *topspace Y* ⟹ *closedin X* {*x* ∈ *topspace X*. *f x* ∈ *U*} = *closedin Y U*
using *quo* **unfolding** *quotient_map_closedin* **by** *auto*

have *closedin X U*
using *U V Y sub*(*2*) **by** *blast*
show *?thesis*
unfolding *quotient_map_closedin*
proof (*intro conjI allI impI*)
show *f ' topspace* (*subtopology X U*) = *topspace* (*subtopology Y V*)
using *sub U fim* **by** (*auto*)
next
fix *Y' :: 'b set*
assume *Y'* ⊆ *topspace* (*subtopology Y V*)
then have *Y'* ⊆ *topspace Y Y'* ⊆ *V*
by (*simp_all*)
then have *eq*: {*x* ∈ *topspace X*. *x* ∈ *U* ∧ *f x* ∈ *Y'*} = {*x* ∈ *topspace X*. *f x* ∈ *Y'*}
using *U* **by** *blast*
then show *closedin* (*subtopology X U*) {*x* ∈ *topspace* (*subtopology X U*). *f x* ∈ *Y'*} = *closedin* (*subtopology Y V*) *Y'*
using *U V Y* ‹*closedin X U*› ‹*Y'* ⊆ *topspace Y*› ‹*Y'* ⊆ *V*›
by (*simp add*: *closedin_closed_subtopology eq*) (*auto simp*: *closedin_def*)
qed
qed

lemma *quotient_map_saturated_open*:
quotient_map X Y f ⟷
continuous_map X Y f ∧ *f '* (*topspace X*) = *topspace Y* ∧
(∀ *U*. *openin X U* ∧ {*x* ∈ *topspace X*. *f x* ∈ *f ' U*} ⊆ *U* ⟶ *openin Y* (*f ' U*))
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
then have *fim*: *f ' topspace X* = *topspace Y*
and *Y*: ⋀*U*. *U* ⊆ *topspace Y* ⟹ *openin Y U* = *openin X* {*x* ∈ *topspace X*. *f x* ∈ *U*}
unfolding *quotient_map_def* **by** *auto*
show *?rhs*
proof (*intro conjI allI impI*)
show *continuous_map X Y f*
by (*simp add*: *L quotient_imp_continuous_map*)
show *f ' topspace X* = *topspace Y*
by (*simp add*: *fim*)
next
fix *U :: 'a set*
assume *U*: *openin X U* ∧ {*x* ∈ *topspace X*. *f x* ∈ *f ' U*} ⊆ *U*
then have *sub*: *f ' U* ⊆ *topspace Y* **and** *eq*: {*x* ∈ *topspace X*. *f x* ∈ *f ' U*} = *U*
using *fim openin_subset* **by** *fastforce+*
show *openin Y* (*f ' U*)
by (*simp add*: *sub Y eq U*)
qed

```
next
  assume ?rhs
  then have YX: ⋀U. openin Y U ⟹ openin X {x ∈ topspace X. f x ∈ U}
       and fim: f ` topspace X = topspace Y
      and XY: ⋀U. ⟦openin X U; {x ∈ topspace X. f x ∈ f ` U} ⊆ U⟧ ⟹ openin
Y (f ` U)
    by (auto simp: quotient_map_def continuous_map_def)
  show ?lhs
  proof (simp add: quotient_map_def fim, intro allI impI iffI)
    fix U :: 'b set
    assume U ⊆ topspace Y and X: openin X {x ∈ topspace X. f x ∈ U}
    have feq: f ` {x ∈ topspace X. f x ∈ U} = U
      using ‹U ⊆ topspace Y› fim by auto
    show openin Y U
      using XY [OF X] by (simp add: feq)
  next
    fix U :: 'b set
    assume U ⊆ topspace Y and Y: openin Y U
    show openin X {x ∈ topspace X. f x ∈ U}
      by (metis YX [OF Y])
  qed
qed
```

2.2.13 Separated Sets

```
definition separatedin :: 'a topology ⇒ 'a set ⇒ 'a set ⇒ bool
  where separatedin X S T ≡
           S ⊆ topspace X ∧ T ⊆ topspace X ∧
           S ∩ X closure_of T = {} ∧ T ∩ X closure_of S = {}

lemma separatedin_empty [simp]:
     separatedin X S {} ⟷ S ⊆ topspace X
     separatedin X {} S ⟷ S ⊆ topspace X
  by (simp_all add: separatedin_def)

lemma separatedin_refl [simp]:
     separatedin X S S ⟷ S = {}
proof −
  have ⋀x. ⟦separatedin X S S; x ∈ S⟧ ⟹ False
    by (metis all_not_in_conv closure_of_subset inf.orderE separatedin_def)
  then show ?thesis
    by auto
qed

lemma separatedin_sym:
     separatedin X S T ⟷ separatedin X T S
  by (auto simp: separatedin_def)

lemma separatedin_imp_disjoint:
```

separatedin X S T ⟹ *disjnt S T*
by (*meson closure_of_subset disjnt_def disjnt_subset2 separatedin_def*)

lemma *separatedin_mono*:
⟦*separatedin X S T*; *S'* ⊆ *S*; *T'* ⊆ *T*⟧ ⟹ *separatedin X S' T'*
unfolding *separatedin_def*
using *closure_of_mono* **by** *blast*

lemma *separatedin_open_sets*:
⟦*openin X S*; *openin X T*⟧ ⟹ *separatedin X S T* ⟷ *disjnt S T*
unfolding *disjnt_def separatedin_def*
by (*auto simp*: *openin_Int_closure_of_eq_empty openin_subset*)

lemma *separatedin_closed_sets*:
⟦*closedin X S*; *closedin X T*⟧ ⟹ *separatedin X S T* ⟷ *disjnt S T*
unfolding *closure_of_eq disjnt_def separatedin_def*
by (*metis closedin_def closure_of_eq inf_commute*)

lemma *separatedin_subtopology*:
separatedin (*subtopology X U*) *S T* ⟷ *S* ⊆ *U* ∧ *T* ⊆ *U* ∧ *separatedin X S T* (**is** *?lhs* = *?rhs*)
by (*auto simp*: *separatedin_def closure_of_subtopology Int_ac disjoint_iff elim*!: *inf.orderE*)

lemma *separatedin_discrete_topology*:
separatedin (*discrete_topology U*) *S T* ⟷ *S* ⊆ *U* ∧ *T* ⊆ *U* ∧ *disjnt S T*
by (*metis openin_discrete_topology separatedin_def separatedin_open_sets topspace_discrete_topology*)

lemma *separated_eq_distinguishable*:
separatedin X {*x*} {*y*} ⟷
x ∈ *topspace X* ∧ *y* ∈ *topspace X* ∧
(∃ *U*. *openin X U* ∧ *x* ∈ *U* ∧ (*y* ∉ *U*)) ∧
(∃ *v*. *openin X v* ∧ *y* ∈ *v* ∧ (*x* ∉ *v*))
by (*force simp*: *separatedin_def closure_of_def*)

lemma *separatedin_Un* [*simp*]:
separatedin X S (*T* ∪ *U*) ⟷ *separatedin X S T* ∧ *separatedin X S U*
separatedin X (*S* ∪ *T*) *U* ⟷ *separatedin X S U* ∧ *separatedin X T U*
by (*auto simp*: *separatedin_def*)

lemma *separatedin_Union*:
finite $\mathcal{F}$ ⟹ *separatedin X S* (⋃$\mathcal{F}$) ⟷ *S* ⊆ *topspace X* ∧ (∀ *T* ∈ $\mathcal{F}$. *separatedin X S T*)
finite $\mathcal{F}$ ⟹ *separatedin X* (⋃$\mathcal{F}$) *S* ⟷ (∀ *T* ∈ $\mathcal{F}$. *separatedin X S T*) ∧ *S* ⊆ *topspace X*
by (*auto simp*: *separatedin_def closure_of_Union*)

lemma *separatedin_openin_diff*:
⟦*openin X S*; *openin X T*⟧ ⟹ *separatedin X* (*S* − *T*) (*T* − *S*)

unfolding *separatedin_def*
by (*metis Diff_Int_distrib2 Diff_disjoint Diff_empty Diff_mono empty_Diff empty_subsetI openin_Int_closure_of_eq_empty openin_subset*)

lemma *separatedin_closedin_diff*:
assumes *closedin X S closedin X T*
shows *separatedin X* $(S - T)$ $(T - S)$
proof −
have $S - T \subseteq$ *topspace X* $T - S \subseteq$ *topspace X*
using *assms closedin_subset* **by** *auto*
with *assms* **show** *?thesis*
by (*simp add*: *separatedin_def Diff_Int_distrib2 closure_of_minimal inf_absorb2*)
qed

lemma *separation_closedin_Un_gen*:
separatedin X S T $\longleftrightarrow$
$S \subseteq$ *topspace X* $\wedge$ $T \subseteq$ *topspace X* $\wedge$ *disjnt S T* $\wedge$
closedin (*subtopology X* $(S \cup T)$) S $\wedge$
closedin (*subtopology X* $(S \cup T)$) T
by (*auto simp add*: *separatedin_def closedin_Int_closure_of disjnt_iff dest*: *closure_of_subset*)

lemma *separation_openin_Un_gen*:
separatedin X S T $\longleftrightarrow$
$S \subseteq$ *topspace X* $\wedge$ $T \subseteq$ *topspace X* $\wedge$ *disjnt S T* $\wedge$
openin (*subtopology X* $(S \cup T)$) S $\wedge$
openin (*subtopology X* $(S \cup T)$) T
unfolding *openin_closedin_eq topspace_subtopology separation_closedin_Un_gen disjnt_def*
by (*auto simp*: *Diff_triv Int_commute Un_Diff inf_absorb1 topspace_def*)

2.2.14 Homeomorphisms

(1-way and 2-way versions may be useful in places)

definition *homeomorphic_map* :: $'a$ *topology* $\Rightarrow$ $'b$ *topology* $\Rightarrow$ $('a \Rightarrow 'b) \Rightarrow$ *bool*
where
homeomorphic_map X Y f $\equiv$ *quotient_map X Y f* $\wedge$ *inj_on f* (*topspace X*)

definition *homeomorphic_maps* :: $'a$ *topology* $\Rightarrow$ $'b$ *topology* $\Rightarrow$ $('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'a) \Rightarrow$ *bool*
where
homeomorphic_maps X Y f g $\equiv$
continuous_map X Y f $\wedge$ *continuous_map Y X g* $\wedge$
$(\forall x \in$ *topspace X*. $g(f\,x) = x) \wedge (\forall y \in$ *topspace Y*. $f(g\,y) = y)$

lemma *homeomorphic_map_eq*:
$[\![$*homeomorphic_map X Y f*; $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow f\,x = g\,x]\!] \Longrightarrow$ *homeomorphic_map X Y g*

by (*meson homeomorphic_map_def inj_on_cong quotient_map_eq*)

lemma *homeomorphic_maps_eq*:
⟦*homeomorphic_maps X Y f g*;
$\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow f\ x = f'\ x$; $\bigwedge y.\ y \in$ *topspace Y* $\Longrightarrow g\ y = g'\ y$⟧
$\Longrightarrow$ *homeomorphic_maps X Y f' g'*
unfolding *homeomorphic_maps_def*
by (*metis continuous_map_eq continuous_map_eq_image_closure_subset_gen image_subset_iff*)

lemma *homeomorphic_maps_sym*:
homeomorphic_maps X Y f g $\longleftrightarrow$ *homeomorphic_maps Y X g f*
by (*auto simp*: *homeomorphic_maps_def*)

lemma *homeomorphic_maps_id*:
homeomorphic_maps X Y id id $\longleftrightarrow Y = X$ (**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
then have *topspace X* = *topspace Y*
by (*auto simp*: *homeomorphic_maps_def continuous_map_def*)
with *L* **show** *?rhs*
unfolding *homeomorphic_maps_def*
by (*metis topology_finer_continuous_id topology_eq*)
next
assume *?rhs*
then show *?lhs*
unfolding *homeomorphic_maps_def* **by** *auto*
qed

lemma *homeomorphic_map_id* [*simp*]: *homeomorphic_map X Y id* $\longleftrightarrow Y = X$
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
then have *eq*: *topspace X* = *topspace Y*
by (*auto simp*: *homeomorphic_map_def continuous_map_def quotient_map_def*)
then have $\bigwedge S$. *openin X S* $\longrightarrow$ *openin Y S*
by (*meson L homeomorphic_map_def injective_quotient_map topology_finer_open_id*)
then show *?rhs*
using *L* **unfolding** *homeomorphic_map_def*
by (*metis eq quotient_imp_continuous_map topology_eq topology_finer_continuous_id*)
next
assume *?rhs*
then show *?lhs*
unfolding *homeomorphic_map_def*
by (*simp add*: *closed_map_id continuous_closed_imp_quotient_map*)
qed

lemma *homeomorphic_map_compose*:
assumes *homeomorphic_map X Y f homeomorphic_map Y X'' g*

shows *homeomorphic_map X X'' (g ∘ f)*
proof –
have *inj_on g (f ‘ topspace X)*
by (*metis* (*no_types*) *assms homeomorphic_map_def quotient_imp_surjective_map*)
then show *?thesis*
using *assms* **by** (*meson comp_inj_on homeomorphic_map_def quotient_map_compose_eq*)
qed

lemma *homeomorphic_maps_compose*:
homeomorphic_maps X Y f h ∧
homeomorphic_maps Y X'' g k
⟹ *homeomorphic_maps X X'' (g ∘ f) (h ∘ k)*
unfolding *homeomorphic_maps_def*
by (*auto simp*: *continuous_map_compose*; *simp add*: *continuous_map_def*)

lemma *homeomorphic_eq_everything_map*:
homeomorphic_map X Y f ⟷
continuous_map X Y f ∧ *open_map X Y f* ∧ *closed_map X Y f* ∧
f ‘ (topspace X) = topspace Y ∧ *inj_on f (topspace X)*
unfolding *homeomorphic_map_def*
by (*force simp*: *injective_quotient_map intro*: *injective_quotient_map*)

lemma *homeomorphic_imp_continuous_map*:
homeomorphic_map X Y f ⟹ *continuous_map X Y f*
by (*simp add*: *homeomorphic_eq_everything_map*)

lemma *homeomorphic_imp_open_map*:
homeomorphic_map X Y f ⟹ *open_map X Y f*
by (*simp add*: *homeomorphic_eq_everything_map*)

lemma *homeomorphic_imp_closed_map*:
homeomorphic_map X Y f ⟹ *closed_map X Y f*
by (*simp add*: *homeomorphic_eq_everything_map*)

lemma *homeomorphic_imp_surjective_map*:
homeomorphic_map X Y f ⟹ *f ‘ (topspace X) = topspace Y*
by (*simp add*: *homeomorphic_eq_everything_map*)

lemma *homeomorphic_imp_injective_map*:
homeomorphic_map X Y f ⟹ *inj_on f (topspace X)*
by (*simp add*: *homeomorphic_eq_everything_map*)

lemma *bijective_open_imp_homeomorphic_map*:
⟦*continuous_map X Y f*; *open_map X Y f*; *f ‘ (topspace X) = topspace Y*; *inj_on f (topspace X)*⟧
⟹ *homeomorphic_map X Y f*
by (*simp add*: *homeomorphic_map_def continuous_open_imp_quotient_map*)

lemma *bijective_closed_imp_homeomorphic_map*:

⟦*continuous_map X Y f*; *closed_map X Y f*; *f ‘ (topspace X) = topspace Y*; *inj_on f (topspace X)*⟧
⟹ *homeomorphic_map X Y f*
by (*simp add*: *continuous_closed_quotient_map homeomorphic_map_def*)

lemma *open_eq_continuous_inverse_map*:
assumes *X*: ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* ∈ *topspace Y* ∧ *g*(*f x*) = *x*
and *Y*: ⋀*y*. *y* ∈ *topspace Y* ⟹ *g y* ∈ *topspace X* ∧ *f*(*g y*) = *y*
shows *open_map X Y f* ⟷ *continuous_map Y X g*
proof −
have *eq*: {*x* ∈ *topspace Y*. *g x* ∈ *U*} = *f ‘ U* **if** *openin X U* **for** *U*
using *openin_subset* [*OF that*] **by** (*force simp*: *X Y image_iff*)
show *?thesis*
by (*auto simp*: *Y open_map_def continuous_map_def eq*)
qed

lemma *closed_eq_continuous_inverse_map*:
assumes *X*: ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* ∈ *topspace Y* ∧ *g*(*f x*) = *x*
and *Y*: ⋀*y*. *y* ∈ *topspace Y* ⟹ *g y* ∈ *topspace X* ∧ *f*(*g y*) = *y*
shows *closed_map X Y f* ⟷ *continuous_map Y X g*
proof −
have *eq*: {*x* ∈ *topspace Y*. *g x* ∈ *U*} = *f ‘ U* **if** *closedin X U* **for** *U*
using *closedin_subset* [*OF that*] **by** (*force simp*: *X Y image_iff*)
show *?thesis*
by (*auto simp*: *Y closed_map_def continuous_map_closedin eq*)
qed

lemma *homeomorphic_maps_map*:
homeomorphic_maps X Y f g ⟷
homeomorphic_map X Y f ∧ *homeomorphic_map Y X g* ∧
(∀ *x* ∈ *topspace X*. *g*(*f x*) = *x*) ∧ (∀ *y* ∈ *topspace Y*. *f*(*g y*) = *y*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then have *L*: *continuous_map X Y f continuous_map Y X g* ∀ *x*∈*topspace X*. *g* (*f x*) = *x* ∀ *x*′∈*topspace Y*. *f* (*g x*′) = *x*′
by (*auto simp*: *homeomorphic_maps_def*)
show *?rhs*
proof (*intro conjI bijective_open_imp_homeomorphic_map L*)
show *open_map X Y f*
using *L* **using** *open_eq_continuous_inverse_map* [*of concl*: *X Y f g*] **by** (*simp add*: *continuous_map_def*)
show *open_map Y X g*
using *L* **using** *open_eq_continuous_inverse_map* [*of concl*: *Y X g f*] **by** (*simp add*: *continuous_map_def*)
show *f ‘ topspace X* = *topspace Y g ‘ topspace Y* = *topspace X*
using *L* **by** (*force simp*: *continuous_map_closedin*)+
show *inj_on f* (*topspace X*) *inj_on g* (*topspace Y*)
using *L* **unfolding** *inj_on_def* **by** *metis*+

```
  qed
next
  assume ?rhs
  then show ?lhs
    by (auto simp: homeomorphic_maps_def homeomorphic_imp_continuous_map)
qed

lemma homeomorphic_maps_imp_map:
    homeomorphic_maps X Y f g ⟹ homeomorphic_map X Y f
  using homeomorphic_maps_map by blast

lemma homeomorphic_map_maps:
     homeomorphic_map X Y f ⟷ (∃ g. homeomorphic_maps X Y f g)
  (is ?lhs = ?rhs)
proof
  assume ?lhs
  then have L: continuous_map X Y f open_map X Y f closed_map X Y f
    f ` (topspace X) = topspace Y inj_on f (topspace X)
    by (auto simp: homeomorphic_eq_everything_map)
  have X: ⋀x. x ∈ topspace X ⟹ f x ∈ topspace Y ∧ inv_into (topspace X) f (f
x) = x
    using L by auto
  have Y: ⋀y. y ∈ topspace Y ⟹ inv_into (topspace X) f y ∈ topspace X ∧ f
(inv_into (topspace X) f y) = y
    by (simp add: L f_inv_into_f inv_into_into)
  have homeomorphic_maps X Y f (inv_into (topspace X) f)
    unfolding homeomorphic_maps_def
  proof (intro conjI L)
    show continuous_map Y X (inv_into (topspace X) f)
      by (simp add: L X Y flip: open_eq_continuous_inverse_map [where f=f])
  next
    show ∀ x∈topspace X. inv_into (topspace X) f (f x) = x
         ∀ y∈topspace Y. f (inv_into (topspace X) f y) = y
      using X Y by auto
  qed
  then show ?rhs
    by metis
next
  assume ?rhs
  then show ?lhs
    using homeomorphic_maps_map by blast
qed

lemma homeomorphic_maps_involution:
   ⟦continuous_map X X f; ⋀x. x ∈ topspace X ⟹ f(f x) = x⟧ ⟹ homeomor-
phic_maps X X f f
  by (auto simp: homeomorphic_maps_def)

lemma homeomorphic_map_involution:
```

⟦*continuous_map X X f*; ⋀*x*. *x* ∈ *topspace X* ⟹ *f*(*f x*) = *x*⟧ ⟹ *homeomorphic_map X X f*
using *homeomorphic_maps_involution homeomorphic_maps_map* **by** *blast*

lemma *homeomorphic_map_openness*:
assumes *hom*: *homeomorphic_map X Y f* **and** *U*: *U* ⊆ *topspace X*
shows *openin Y* (*f* ‘ *U*) ⟷ *openin X U*
proof –
obtain *g* **where** *homeomorphic_maps X Y f g*
using *assms* **by** (*auto simp*: *homeomorphic_map_maps*)
then have *g*: *homeomorphic_map Y X g* **and** *gf*: ⋀*x*. *x* ∈ *topspace X* ⟹ *g*(*f x*) = *x*
by (*auto simp*: *homeomorphic_maps_map*)
then have *openin X U* ⟹ *openin Y* (*f* ‘ *U*)
using *hom homeomorphic_imp_open_map open_map_def* **by** *blast*
show *openin Y* (*f* ‘ *U*) = *openin X U*
proof
assume *L*: *openin Y* (*f* ‘ *U*)
have *U* = *g* ‘ (*f* ‘ *U*)
using *U gf* **by** *force*
then show *openin X U*
by (*metis L homeomorphic_imp_open_map open_map_def g*)
next
assume *openin X U*
then show *openin Y* (*f* ‘ *U*)
using *hom homeomorphic_imp_open_map open_map_def* **by** *blast*
qed
qed

lemma *homeomorphic_map_closedness*:
assumes *hom*: *homeomorphic_map X Y f* **and** *U*: *U* ⊆ *topspace X*
shows *closedin Y* (*f* ‘ *U*) ⟷ *closedin X U*
proof –
obtain *g* **where** *homeomorphic_maps X Y f g*
using *assms* **by** (*auto simp*: *homeomorphic_map_maps*)
then have *g*: *homeomorphic_map Y X g* **and** *gf*: ⋀*x*. *x* ∈ *topspace X* ⟹ *g*(*f x*) = *x*
by (*auto simp*: *homeomorphic_maps_map*)
then have *closedin X U* ⟹ *closedin Y* (*f* ‘ *U*)
using *hom homeomorphic_imp_closed_map closed_map_def* **by** *blast*
show *closedin Y* (*f* ‘ *U*) = *closedin X U*
proof
assume *L*: *closedin Y* (*f* ‘ *U*)
have *U* = *g* ‘ (*f* ‘ *U*)
using *U gf* **by** *force*
then show *closedin X U*
by (*metis L homeomorphic_imp_closed_map closed_map_def g*)
next

assume *closedin X U*
then show *closedin Y (f ‘ U)*
using *hom homeomorphic_imp_closed_map closed_map_def* **by** *blast*
qed
qed

lemma *homeomorphic_map_openness_eq*:
homeomorphic_map X Y f ⟹ *openin X U* ⟷ *U* ⊆ *topspace X* ∧ *openin Y (f ‘ U)*
by (*meson homeomorphic_map_openness openin_closedin_eq*)

lemma *homeomorphic_map_closedness_eq*:
homeomorphic_map X Y f ⟹ *closedin X U* ⟷ *U* ⊆ *topspace X* ∧ *closedin Y (f ‘ U)*
by (*meson closedin_subset homeomorphic_map_closedness*)

lemma *all_openin_homeomorphic_image*:
assumes *homeomorphic_map X Y f*
shows (∀ *V*. *openin Y V* ⟶ *P V*) ⟷ (∀ *U*. *openin X U* ⟶ *P*(*f ‘ U*)) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*meson assms homeomorphic_map_openness_eq*)
next
assume *?rhs*
then show *?lhs*
by (*metis* (*no_types*, *lifting*) *assms homeomorphic_imp_surjective_map homeomorphic_map_openness openin_subset subset_image_iff*)
qed

lemma *all_closedin_homeomorphic_image*:
assumes *homeomorphic_map X Y f*
shows (∀ *V*. *closedin Y V* ⟶ *P V*) ⟷ (∀ *U*. *closedin X U* ⟶ *P*(*f ‘ U*)) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*meson assms homeomorphic_map_closedness_eq*)
next
assume *?rhs*
then show *?lhs*
by (*metis* (*no_types*, *lifting*) *assms homeomorphic_imp_surjective_map homeomorphic_map_closedness closedin_subset subset_image_iff*)
qed

lemma *homeomorphic_map_derived_set_of*:
assumes *hom*: *homeomorphic_map X Y f* **and** *S*: *S* ⊆ *topspace X*

shows *Y derived_set_of* $(f\ `\ S) = f\ `\ (X\ derived\_set\_of\ S)$
proof −
have *fim*: $f\ `\ (topspace\ X) = topspace\ Y$ **and** *inj*: *inj_on f* (*topspace X*)
using *hom* **by** (*auto simp*: *homeomorphic_eq_everything_map*)
have *iff*: $(\forall T.\ x \in T \wedge openin\ X\ T \longrightarrow (\exists y.\ y \neq x \wedge y \in S \wedge y \in T)) = (\forall T.\ T \subseteq topspace\ Y \longrightarrow f\,x \in T \longrightarrow openin\ Y\ T \longrightarrow (\exists y.\ y \neq f\,x \wedge y \in f\ `\ S \wedge y \in T))$
if $x \in topspace\ X$ **for** *x*
proof −
have §: $(x \in T \wedge openin\ X\ T) = (T \subseteq topspace\ X \wedge f\,x \in f\ `\ T \wedge openin\ Y\ (f\ `\ T))$ **for** *T*
by (*meson hom homeomorphic_map_openness_eq inj inj_on_image_mem_iff that*)
moreover have $(\exists y.\ y \neq x \wedge y \in S \wedge y \in T) = (\exists y.\ y \neq f\,x \wedge y \in f\ `\ S \wedge y \in f\ `\ T)$ (**is** *?lhs* = *?rhs*)
if $T \subseteq topspace\ X \wedge f\,x \in f\ `\ T \wedge openin\ Y\ (f\ `\ T)$ **for** *T*
proof
show *?lhs* $\Longrightarrow$ *?rhs*
by (*meson* § *imageI inj inj_on_eq_iff inj_on_subset that*)
show *?rhs* $\Longrightarrow$ *?lhs*
using *S inj inj_onD that* **by** *fastforce*
qed
ultimately show *?thesis*
by (*auto simp flip*: *fim simp*: *all_subset_image*)
qed
have ∗: $[\![T = f\ `\ S;\ \bigwedge x.\ x \in S \Longrightarrow P\ x \longleftrightarrow Q(f\ x)]\!] \Longrightarrow \{y.\ y \in T \wedge Q\ y\} = f\ `\ \{x \in S.\ P\ x\}$ **for** *T S P Q*
by *auto*
show *?thesis*
unfolding *derived_set_of_def*
by (*rule* ∗) (*use fim iff openin_subset* **in** *force*)+
qed

lemma *homeomorphic_map_closure_of*:
assumes *hom*: *homeomorphic_map X Y f* **and** *S*: $S \subseteq topspace\ X$
shows *Y closure_of* $(f\ `\ S) = f\ `\ (X\ closure\_of\ S)$
unfolding *closure_of*
using *homeomorphic_imp_surjective_map* [*OF hom*] *S*
by (*auto simp*: *in_derived_set_of homeomorphic_map_derived_set_of* [*OF assms*])

lemma *homeomorphic_map_interior_of*:
assumes *hom*: *homeomorphic_map X Y f* **and** *S*: $S \subseteq topspace\ X$
shows *Y interior_of* $(f\ `\ S) = f\ `\ (X\ interior\_of\ S)$
proof −
{ fix *y*
assume $y \in topspace\ Y$ **and** $y \notin Y\ closure\_of\ (topspace\ Y - f\ `\ S)$
then have $y \in f\ `\ (topspace\ X - X\ closure\_of\ (topspace\ X - S))$
using *homeomorphic_eq_everything_map* [*THEN iffD1*, *OF hom*] *homeomor-*

```
phic_map_closure_of [OF hom]
    by (metis DiffI Diff_subset S closure_of_subset_topspace inj_on_image_set_diff)
}
  moreover
  { fix x
    assume x ∈ topspace X
    then have f x ∈ topspace Y
      using hom homeomorphic_imp_surjective_map by blast }
  moreover
  { fix x
    assume x ∈ topspace X and x ∉ X closure_of (topspace X − S) and f x ∈ Y
closure_of (topspace Y − f ‘ S)
    then have False
      using homeomorphic_map_closure_of [OF hom] hom
      unfolding homeomorphic_eq_everything_map
    by (metis Diff_subset S closure_of_subset_topspace inj_on_image_mem_iff inj_on_image_set_diff)
  }
  ultimately show ?thesis
    by (auto simp: interior_of_closure_of)
qed

lemma homeomorphic_map_frontier_of:
  assumes hom: homeomorphic_map X Y f and S: S ⊆ topspace X
  shows Y frontier_of (f ‘ S) = f ‘ (X frontier_of S)
  unfolding frontier_of_def
proof (intro equalityI subsetI DiffI)
  fix y
  assume y ∈ Y closure_of f ‘ S − Y interior_of f ‘ S
  then show y ∈ f ‘ (X closure_of S − X interior_of S)
    using S hom homeomorphic_map_closure_of homeomorphic_map_interior_of by
fastforce
next
  fix y
  assume y ∈ f ‘ (X closure_of S − X interior_of S)
  then show y ∈ Y closure_of f ‘ S
    using S hom homeomorphic_map_closure_of by fastforce
next
  fix x
  assume x ∈ f ‘ (X closure_of S − X interior_of S)
  then obtain y where y: x = f y y ∈ X closure_of S y ∉ X interior_of S
    by blast
  then have y ∈ topspace X
    by (simp add: in_closure_of)
  then have f y ∉ f ‘ (X interior_of S)
   by (meson hom homeomorphic_map_def inj_on_image_mem_iff interior_of_subset_topspace
y(3))
  then show x ∉ Y interior_of f ‘ S
    using S hom homeomorphic_map_interior_of y(1) by blast
qed
```

lemma *homeomorphic_maps_subtopologies*:
⟦*homeomorphic_maps X Y f g*; *f* ‘ (*topspace X* ∩ *S*) = *topspace Y* ∩ *T*⟧
⟹ *homeomorphic_maps* (*subtopology X S*) (*subtopology Y T*) *f g*
unfolding *homeomorphic_maps_def*
by (*force simp*: *continuous_map_from_subtopology continuous_map_in_subtopology*)

lemma *homeomorphic_maps_subtopologies_alt*:
⟦*homeomorphic_maps X Y f g*; *f* ‘ (*topspace X* ∩ *S*) ⊆ *T*; *g* ‘ (*topspace Y* ∩ *T*) ⊆ *S*⟧
⟹ *homeomorphic_maps* (*subtopology X S*) (*subtopology Y T*) *f g*
unfolding *homeomorphic_maps_def*
by (*force simp*: *continuous_map_from_subtopology continuous_map_in_subtopology*)

lemma *homeomorphic_map_subtopologies*:
⟦*homeomorphic_map X Y f*; *f* ‘ (*topspace X* ∩ *S*) = *topspace Y* ∩ *T*⟧
⟹ *homeomorphic_map* (*subtopology X S*) (*subtopology Y T*) *f*
by (*meson homeomorphic_map_maps homeomorphic_maps_subtopologies*)

lemma *homeomorphic_map_subtopologies_alt*:
assumes *hom*: *homeomorphic_map X Y f*
and *S*: ⋀*x*. ⟦*x* ∈ *topspace X*; *f x* ∈ *topspace Y*⟧ ⟹ *f x* ∈ *T* ⟷ *x* ∈ *S*
shows *homeomorphic_map* (*subtopology X S*) (*subtopology Y T*) *f*
proof −
have *homeomorphic_maps* (*subtopology X S*) (*subtopology Y T*) *f g*
if *homeomorphic_maps X Y f g* **for** *g*
proof (*rule homeomorphic_maps_subtopologies* [*OF that*])
show *f* ‘ (*topspace X* ∩ *S*) = *topspace Y* ∩ *T*
using *that S*
apply (*auto simp*: *homeomorphic_maps_def continuous_map_def*)
by (*metis IntI image_iff*)
qed
then show *?thesis*
using *hom* **by** (*meson homeomorphic_map_maps*)
qed

2.2.15 Relation of homeomorphism between topological spaces

definition *homeomorphic_space* (**infixr** *homeomorphic′_space 50*)
where *X homeomorphic_space Y* ≡ ∃*f g*. *homeomorphic_maps X Y f g*

lemma *homeomorphic_space_refl*: *X homeomorphic_space X*
by (*meson homeomorphic_maps_id homeomorphic_space_def*)

lemma *homeomorphic_space_sym*:
X homeomorphic_space Y ⟷ *Y homeomorphic_space X*
unfolding *homeomorphic_space_def* **by** (*metis homeomorphic_maps_sym*)

lemma *homeomorphic_space_trans* [*trans*]:

⟦*X1 homeomorphic_space X2*; *X2 homeomorphic_space X3*⟧ ⟹ *X1 homeomorphic_space X3*
unfolding *homeomorphic_space_def* **by** (*metis homeomorphic_maps_compose*)

lemma *homeomorphic_space*:
X homeomorphic_space Y ⟷ (∃*f*. *homeomorphic_map X Y f*)
by (*simp add*: *homeomorphic_map_maps homeomorphic_space_def*)

lemma *homeomorphic_maps_imp_homeomorphic_space*:
homeomorphic_maps X Y f g ⟹ *X homeomorphic_space Y*
unfolding *homeomorphic_space_def* **by** *metis*

lemma *homeomorphic_map_imp_homeomorphic_space*:
homeomorphic_map X Y f ⟹ *X homeomorphic_space Y*
unfolding *homeomorphic_map_maps*
using *homeomorphic_space_def* **by** *blast*

lemma *homeomorphic_empty_space*:
X homeomorphic_space Y ⟹ *topspace X* = {} ⟷ *topspace Y* = {}
by (*metis homeomorphic_imp_surjective_map homeomorphic_space image_is_empty*)

lemma *homeomorphic_empty_space_eq*:
assumes *topspace X* = {}
shows *X homeomorphic_space Y* ⟷ *topspace Y* = {}
proof −
have ∀*f t*. *continuous_map X* (*t*::′*b topology*) *f*
using *assms continuous_map_on_empty* **by** *blast*
then show *?thesis*
by (*metis* (*no_types*) *assms continuous_map_on_empty empty_iff homeomorphic_empty_space homeomorphic_maps_def homeomorphic_space_def*)
qed

2.2.16 Connected topological spaces

definition *connected_space* :: ′*a topology* ⇒ *bool* **where**
connected_space X ≡
¬(∃ *E1 E2*. *openin X E1* ∧ *openin X E2* ∧
topspace X ⊆ *E1* ∪ *E2* ∧ *E1* ∩ *E2* = {} ∧ *E1* ≠ {} ∧ *E2* ≠ {})

definition *connectedin* :: ′*a topology* ⇒ ′*a set* ⇒ *bool* **where**
connectedin X S ≡ *S* ⊆ *topspace X* ∧ *connected_space* (*subtopology X S*)

lemma *connected_spaceD*:
⟦*connected_space X*;
openin X U; *openin X V*; *topspace X* ⊆ *U* ∪ *V*; *U* ∩ *V* = {}; *U* ≠ {}; *V* ≠ {}⟧ ⟹ *False*
by (*auto simp*: *connected_space_def*)

lemma *connectedin_subset_topspace*: *connectedin X S* ⟹ *S* ⊆ *topspace X*

by (*simp add*: *connectedin_def*)

lemma *connectedin_topspace*:
connectedin X (*topspace X*) ⟷ *connected_space X*
by (*simp add*: *connectedin_def*)

lemma *connected_space_subtopology*:
connectedin X S ⟹ *connected_space* (*subtopology X S*)
by (*simp add*: *connectedin_def*)

lemma *connectedin_subtopology*:
connectedin (*subtopology X S*) *T* ⟷ *connectedin X T* ∧ *T* ⊆ *S*
by (*force simp*: *connectedin_def subtopology_subtopology inf_absorb2*)

lemma *connected_space_eq*:
connected_space X ⟷
(∄ *E1 E2*. *openin X E1* ∧ *openin X E2* ∧ *E1* ∪ *E2* = *topspace X* ∧ *E1* ∩ *E2* = {} ∧ *E1* ≠ {} ∧ *E2* ≠ {})
unfolding *connected_space_def*
by (*metis openin_Un openin_subset subset_antisym*)

lemma *connected_space_closedin*:
connected_space X ⟷
(∄ *E1 E2*. *closedin X E1* ∧ *closedin X E2* ∧ *topspace X* ⊆ *E1* ∪ *E2* ∧
E1 ∩ *E2* = {} ∧ *E1* ≠ {} ∧ *E2* ≠ {}) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then have *L*: ⋀*E1 E2*. ⟦*openin X E1*; *E1* ∩ *E2* = {}; *topspace X* ⊆ *E1* ∪ *E2*; *openin X E2*⟧ ⟹ *E1* = {} ∨ *E2* = {}
by (*simp add*: *connected_space_def*)
show *?rhs*
unfolding *connected_space_def*
proof *clarify*
fix *E1 E2*
assume *closedin X E1* **and** *closedin X E2* **and** *topspace X* ⊆ *E1* ∪ *E2* **and** *E1* ∩ *E2* = {}
and *E1* ≠ {} **and** *E2* ≠ {}
have *E1* ∪ *E2* = *topspace X*
by (*meson Un_subset_iff* ‹*closedin X E1*› ‹*closedin X E2*› ‹*topspace X* ⊆ *E1* ∪ *E2*› *closedin_def subset_antisym*)
then have *topspace X* − *E2* = *E1*
using ‹*E1* ∩ *E2* = {}› **by** *fastforce*
then have *topspace X* = *E1*
using ‹*E1* ≠ {}› *L* ‹*closedin X E1*› ‹*closedin X E2*› **by** *blast*
then show *False*
using ‹*E1* ∩ *E2* = {}› ‹*E1* ∪ *E2* = *topspace X*› ‹*E2* ≠ {}› **by** *blast*
qed
next
assume *R*: *?rhs*

```
  show ?lhs
    unfolding connected_space_def
  proof clarify
    fix E1 E2
    assume openin X E1 and openin X E2 and topspace X ⊆ E1 ∪ E2 and E1
∩ E2 = {}
      and E1 ≠ {} and E2 ≠ {}
    have E1 ∪ E2 = topspace X
      by (meson Un_subset_iff ‹openin X E1› ‹openin X E2› ‹topspace X ⊆ E1 ∪
E2› openin_closedin_eq subset_antisym)
    then have topspace X − E2 = E1
      using ‹E1 ∩ E2 = {}› by fastforce
    then have topspace X = E1
      using ‹E1 ≠ {}› R ‹openin X E1› ‹openin X E2› by blast
    then show False
      using ‹E1 ∩ E2 = {}› ‹E1 ∪ E2 = topspace X› ‹E2 ≠ {}› by blast
  qed
qed

lemma connected_space_closedin_eq:
     connected_space X ⟷
       (∄E1 E2. closedin X E1 ∧ closedin X E2 ∧
                E1 ∪ E2 = topspace X ∧ E1 ∩ E2 = {} ∧ E1 ≠ {} ∧ E2 ≠ {})
  by (metis closedin_Un closedin_def connected_space_closedin subset_antisym)

lemma connected_space_clopen_in:
     connected_space X ⟷
        (∀T. openin X T ∧ closedin X T ⟶ T = {} ∨ T = topspace X)
proof −
  have eq: openin X E1 ∧ openin X E2 ∧ E1 ∪ E2 = topspace X ∧ E1 ∩ E2 =
{} ∧ P
       ⟷ E2 = topspace X − E1 ∧ openin X E1 ∧ openin X E2 ∧ P for E1 E2
P
    using openin_subset by blast
  show ?thesis
    unfolding connected_space_eq eq closedin_def
    by (auto simp: openin_closedin_eq)
qed

lemma connectedin:
     connectedin X S ⟷
        S ⊆ topspace X ∧
         (∄E1 E2.
             openin X E1 ∧ openin X E2 ∧
             S ⊆ E1 ∪ E2 ∧ E1 ∩ E2 ∩ S = {} ∧ E1 ∩ S ≠ {} ∧ E2 ∩ S ≠ {})
(is ?lhs = ?rhs)
proof −
  have *: (∃E1:: 'a set. ∃E2:: 'a set. (∃T1:: 'a set. P1 T1 ∧ E1 = f1 T1) ∧
(∃T2:: 'a set. P2 T2 ∧ E2 = f2 T2) ∧
```

R E1 E2) ⟷ (∃ *T1 T2*. *P1 T1* ∧ *P2 T2* ∧ *R*(*f1 T1*) (*f2 T2*)) **for** *P1 f1 P2 f2 R*
  by *auto*
 show *?thesis*
  unfolding *connectedin_def connected_space_def openin_subtopology topspace_subtopology* *
  by (*intro conj_cong arg_cong* [**where** *f*=*Not*] *ex_cong1*; *blast dest*!: *openin_subset*)
qed

lemma *connectedin_iff_connected* [*simp*]: *connectedin euclidean S* ⟷ *connected S*
 by (*simp add*: *connected_def connectedin*)

lemma *connectedin_closedin*:
 connectedin X S ⟷
  S ⊆ *topspace X* ∧
  ¬(∃ *E1 E2*. *closedin X E1* ∧ *closedin X E2* ∧
   S ⊆ (*E1* ∪ *E2*) ∧
   (*E1* ∩ *E2* ∩ *S* = {}) ∧
   ¬(*E1* ∩ *S* = {}) ∧ ¬(*E2* ∩ *S* = {}))
proof −
 have ∗: (∃ *E1*:: *'a set*. ∃ *E2*:: *'a set*. (∃ *T1*:: *'a set*. *P1 T1* ∧ *E1* = *f1 T1*) ∧ (∃ *T2*:: *'a set*. *P2 T2* ∧ *E2* = *f2 T2*) ∧
  R E1 E2) ⟷ (∃ *T1 T2*. *P1 T1* ∧ *P2 T2* ∧ *R*(*f1 T1*) (*f2 T2*)) **for** *P1 f1 P2 f2 R*
  by *auto*
 show *?thesis*
  unfolding *connectedin_def connected_space_closedin closedin_subtopology topspace_subtopology* *
  by (*intro conj_cong arg_cong* [**where** *f*=*Not*] *ex_cong1*; *blast dest*!: *openin_subset*)
qed

lemma *connectedin_empty* [*simp*]: *connectedin X* {}
 by (*simp add*: *connectedin*)

lemma *connected_space_topspace_empty*:
  topspace X = {} ⟹ *connected_space X*
 using *connectedin_topspace* **by** *fastforce*

lemma *connectedin_sing* [*simp*]: *connectedin X* {*a*} ⟷ *a* ∈ *topspace X*
 by (*simp add*: *connectedin*)

lemma *connectedin_absolute* [*simp*]:
 connectedin (*subtopology X S*) *S* ⟷ *connectedin X S*
 by (*simp add*: *connectedin_subtopology*)

lemma *connectedin_Union*:
 assumes $\mathcal{U}$: ⋀*S*. *S* ∈ $\mathcal{U}$ ⟹ *connectedin X S* **and** *ne*: ⋂$\mathcal{U}$ ≠ {}
 shows *connectedin X* (⋃$\mathcal{U}$)
proof −

```
  have ⋃𝒰 ⊆ topspace X
    using 𝒰 by (simp add: Union_least connectedin_def)
  moreover have False
    if openin X E1 openin X E2 and cover: ⋃𝒰 ⊆ E1 ∪ E2 and disj: E1 ∩ E2
∩ ⋃𝒰 = {}
       and overlap1: E1 ∩ ⋃𝒰 ≠ {} and overlap2: E2 ∩ ⋃𝒰 ≠ {}
      for E1 E2
  proof -
    have disjS: E1 ∩ E2 ∩ S = {} if S ∈ 𝒰 for S
      using Diff_triv that disj by auto
    have coverS: S ⊆ E1 ∪ E2 if S ∈ 𝒰 for S
      using that cover by blast
    have 𝒰 ≠ {}
      using overlap1 by blast
    obtain a where a: ⋀U. U ∈ 𝒰 ⟹ a ∈ U
      using ne by force
    with ‹𝒰 ≠ {}› have a ∈ ⋃𝒰
      by blast
    then consider a ∈ E1 | a ∈ E2
      using ‹⋃𝒰 ⊆ E1 ∪ E2› by auto
    then show False
    proof cases
      case 1
      then obtain b S where b ∈ E2 b ∈ S S ∈ 𝒰
        using overlap2 by blast
      then show ?thesis
         using 1 ‹openin X E1› ‹openin X E2› disjS coverS a [OF ‹S ∈ 𝒰›] 𝒰[OF
‹S ∈ 𝒰›]
        unfolding connectedin
        by (meson disjoint_iff_not_equal)
    next
      case 2
      then obtain b S where b ∈ E1 b ∈ S S ∈ 𝒰
        using overlap1 by blast
      then show ?thesis
         using 2 ‹openin X E1› ‹openin X E2› disjS coverS a [OF ‹S ∈ 𝒰›] 𝒰[OF
‹S ∈ 𝒰›]
        unfolding connectedin
        by (meson disjoint_iff_not_equal)
    qed
  qed
  ultimately show ?thesis
    unfolding connectedin by blast
qed

lemma connectedin_Un:
    ⟦connectedin X S; connectedin X T; S ∩ T ≠ {}⟧ ⟹ connectedin X (S ∪ T)
  using connectedin_Union [of {S,T}] by auto
```

lemma *connected_space_subconnected*:
connected_space X ⟷ (∀ *x* ∈ *topspace X*. ∀ *y* ∈ *topspace X*. ∃ *S*. *connectedin X S* ∧ *x* ∈ *S* ∧ *y* ∈ *S*) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
using *connectedin_topspace* **by** *blast*
next
assume *R* [*rule_format*]: *?rhs*
have *False* **if** *openin X U openin X V* **and** *disj*: *U* ∩ *V* = {} **and** *cover*: *topspace X* ⊆ *U* ∪ *V*
and *U* ≠ {} *V* ≠ {} **for** *U V*
proof −
obtain *u v* **where** *u* ∈ *U v* ∈ *V*
using ‹*U* ≠ {}› ‹*V* ≠ {}› **by** *auto*
then obtain *T* **where** *u* ∈ *T v* ∈ *T* **and** *T*: *connectedin X T*
using *R* [*of u v*] *that*
by (*meson* ‹*openin X U*› ‹*openin X V*› *subsetD openin_subset*)
then show *False*
using *that* **unfolding** *connectedin*
by (*metis IntI* ‹*u* ∈ *U*› ‹*v* ∈ *V*› *empty_iff inf_bot_left subset_trans*)
qed
then show *?lhs*
by (*auto simp*: *connected_space_def*)
qed

lemma *connectedin_intermediate_closure_of*:
assumes *connectedin X S S* ⊆ *T T* ⊆ *X closure_of S*
shows *connectedin X T*
proof −
have *S*: *S* ⊆ *topspace X* **and** *T*: *T* ⊆ *topspace X*
using *assms* **by** (*meson closure_of_subset_topspace dual_order.trans*)+
have §: ⋀*E1 E2*. ⟦*openin X E1*; *openin X E2*; *E1* ∩ *S* = {} ∨ *E2* ∩ *S* = {}⟧ ⟹ *E1* ∩ *T* = {} ∨ *E2* ∩ *T* = {}
using *assms* **unfolding** *disjoint_iff* **by** (*meson in_closure_of subsetD*)
then show *?thesis*
using *assms*
unfolding *connectedin closure_of_subset_topspace S T*
by (*metis Int_empty_right T dual_order.trans inf.orderE inf_left_commute*)
qed

lemma *connectedin_closure_of*:
connectedin X S ⟹ *connectedin X* (*X closure_of S*)
by (*meson closure_of_subset connectedin_def connectedin_intermediate_closure_of subset_refl*)

lemma *connectedin_separation*:
connectedin X S ⟷
S ⊆ *topspace X* ∧

```
        (∄ C1 C2. C1 ∪ C2 = S ∧ C1 ≠ {} ∧ C2 ≠ {} ∧ C1 ∩ X closure_of C2
= {} ∧ C2 ∩ X closure_of C1 = {}) (is ?lhs = ?rhs)
 unfolding connectedin_def connected_space_closedin_eq closedin_Int_closure_of topspace_subtopology
  apply (intro conj_cong refl arg_cong [where f=Not])
  apply (intro ex_cong1 iffI, blast)
  using closure_of_subset_Int by force

lemma connectedin_eq_not_separated:
   connectedin X S ⟷
        S ⊆ topspace X ∧
        (∄ C1 C2. C1 ∪ C2 = S ∧ C1 ≠ {} ∧ C2 ≠ {} ∧ separatedin X C1 C2)
  unfolding separatedin_def by (metis connectedin_separation sup.boundedE)

lemma connectedin_eq_not_separated_subset:
  connectedin X S ⟷
      S ⊆ topspace X ∧ (∄ C1 C2. S ⊆ C1 ∪ C2 ∧ S ∩ C1 ≠ {} ∧ S ∩ C2 ≠ {}
∧ separatedin X C1 C2)
proof –
  have ∀ C1 C2. S ⊆ C1 ∪ C2 ⟶ S ∩ C1 = {} ∨ S ∩ C2 = {} ∨ ¬ separatedin
X C1 C2
    if ⋀C1 C2. C1 ∪ C2 = S ⟶ C1 = {} ∨ C2 = {} ∨ ¬ separatedin X C1 C2
  proof (intro allI)
    fix C1 C2
    show S ⊆ C1 ∪ C2 ⟶ S ∩ C1 = {} ∨ S ∩ C2 = {} ∨ ¬ separatedin X C1
C2
      using that [of S ∩ C1 S ∩ C2]
      by (auto simp: separatedin_mono)
  qed
  then show ?thesis
    by (metis Un_Int_eq(1) Un_Int_eq(2) connectedin_eq_not_separated order_refl)
qed

lemma connected_space_eq_not_separated:
     connected_space X ⟷
      (∄ C1 C2. C1 ∪ C2 = topspace X ∧ C1 ≠ {} ∧ C2 ≠ {} ∧ separatedin X
C1 C2)
  by (simp add: connectedin_eq_not_separated flip: connectedin_topspace)

lemma connected_space_eq_not_separated_subset:
  connected_space X ⟷
    (∄ C1 C2. topspace X ⊆ C1 ∪ C2 ∧ C1 ≠ {} ∧ C2 ≠ {} ∧ separatedin X C1
C2)
 by (metis connected_space_eq_not_separated le_sup_iff separatedin_def subset_antisym)

lemma connectedin_subset_separated_union:
     ⟦connectedin X C; separatedin X S T; C ⊆ S ∪ T⟧ ⟹ C ⊆ S ∨ C ⊆ T
  unfolding connectedin_eq_not_separated_subset by blast

lemma connectedin_nonseparated_union:
```

assumes *connectedin X S connectedin X T* ¬*separatedin X S T*
shows *connectedin X* (*S* ∪ *T*)
proof −
have ⋀*C1 C2*. ⟦*T* ⊆ *C1* ∪ *C2*; *S* ⊆ *C1* ∪ *C2*⟧ ⟹
S ∩ *C1* = {} ∧ *T* ∩ *C1* = {} ∨ *S* ∩ *C2* = {} ∧ *T* ∩ *C2* = {} ∨ ¬
separatedin X C1 C2
using *assms*
unfolding *connectedin_eq_not_separated_subset*
by (*metis* (*no_types*, *lifting*) *assms connectedin_subset_separated_union inf.orderE separatedin_empty*(*1*) *separatedin_mono separatedin_sym*)
then show *?thesis*
unfolding *connectedin_eq_not_separated_subset*
by (*simp add*: *assms*(*1*) *assms*(*2*) *connectedin_subset_topspace Int_Un_distrib2*)
qed

lemma *connected_space_closures*:
connected_space X ⟷
(∄ *e1 e2*. *e1* ∪ *e2* = *topspace X* ∧ *X closure_of e1* ∩ *X closure_of e2* = {}
∧ *e1* ≠ {} ∧ *e2* ≠ {})
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
unfolding *connected_space_closedin_eq*
by (*metis Un_upper1 Un_upper2 closedin_closure_of closure_of_Un closure_of_eq_empty closure_of_topspace*)
next
assume *?rhs*
then show *?lhs*
unfolding *connected_space_closedin_eq*
by (*metis closure_of_eq*)
qed

lemma *connectedin_inter_frontier_of*:
assumes *connectedin X S S* ∩ *T* ≠ {} *S* − *T* ≠ {}
shows *S* ∩ *X frontier_of T* ≠ {}
proof −
have *S* ⊆ *topspace X* **and** ∗:
⋀*E1 E2*. *openin X E1* ⟶ *openin X E2* ⟶ *E1* ∩ *E2* ∩ *S* = {} ⟶ *S* ⊆ *E1*
∪ *E2* ⟶ *E1* ∩ *S* = {} ∨ *E2* ∩ *S* = {}
using ‹*connectedin X S*› **by** (*auto simp*: *connectedin*)
moreover
have *S* − (*topspace X* ∩ *T*) ≠ {}
using *assms*(*3*) **by** *blast*
moreover
have *S* ∩ *topspace X* ∩ *T* ≠ {}
using *assms*(*1*) *assms*(*2*) *connectedin* **by** *fastforce*
moreover
have *False* **if** *S* ∩ *T* ≠ {} *S* − *T* ≠ {} *T* ⊆ *topspace X S* ∩ *X frontier_of T* =

{} **for** *T*
  proof –
    have *null*: *S* ∩ (*X closure_of T* − *X interior_of T*) = {}
      using *that* **unfolding** *frontier_of_def* **by** *blast*
    have *X interior_of T* ∩ (*topspace X* − *X closure_of T*) ∩ *S* = {}
      by (*metis Diff_disjoint inf_bot_left interior_of_Int interior_of_complement interior_of_empty*)
    moreover have *S* ⊆ *X interior_of T* ∪ (*topspace X* − *X closure_of T*)
      using *that* ‹*S* ⊆ *topspace X*› *null* **by** *auto*
    moreover have *S* ∩ *X interior_of T* ≠ {}
      using *closure_of_subset that*(*1*) *that*(*3*) *null* **by** *fastforce*
    ultimately have *S* ∩ *X interior_of* (*topspace X* − *T*) = {}
      by (*metis* ∗ *inf_commute interior_of_complement openin_interior_of*)
    then have *topspace* (*subtopology X S*) ∩ *X interior_of T* = *S*
      using ‹*S* ⊆ *topspace X*› *interior_of_complement null* **by** *fastforce*
    then show *?thesis*
      using *that* **by** (*metis Diff_eq_empty_iff inf_le2 interior_of_subset subset_trans*)
  qed
  ultimately show *?thesis*
    by (*metis Int_lower1 frontier_of_restrict inf_assoc*)
qed

lemma *connectedin_continuous_map_image*:
  assumes *f*: *continuous_map X Y f* **and** *connectedin X S*
  shows *connectedin Y* (*f* ‘ *S*)
proof –
  have *S* ⊆ *topspace X* **and** ∗:
    ⋀*E1 E2*. *openin X E1* ⟶ *openin X E2* ⟶ *E1* ∩ *E2* ∩ *S* = {} ⟶ *S* ⊆ *E1* ∪ *E2* ⟶ *E1* ∩ *S* = {} ∨ *E2* ∩ *S* = {}
    using ‹*connectedin X S*› **by** (*auto simp*: *connectedin*)
  show *?thesis*
    unfolding *connectedin connected_space_def*
  proof (*intro conjI notI*; *clarify*)
    show *f x* ∈ *topspace Y* **if** *x* ∈ *S* **for** *x*
     using ‹*S* ⊆ *topspace X*› *continuous_map_image_subset_topspace f that* **by** *blast*
  next
    fix *U V*
    let *?U* = {*x* ∈ *topspace X*. *f x* ∈ *U*}
    let *?V* = {*x* ∈ *topspace X*. *f x* ∈ *V*}
    assume *UV*: *openin Y U openin Y V f* ‘ *S* ⊆ *U* ∪ *V U* ∩ *V* ∩ *f* ‘ *S* = {} *U* ∩ *f* ‘ *S* ≠ {} *V* ∩ *f* ‘ *S* ≠ {}
    then have *1*: *?U* ∩ *?V* ∩ *S* = {}
      by *auto*
    have *2*: *openin X ?U openin X ?V*
      using ‹*openin Y U*› ‹*openin Y V*› *continuous_map f* **by** *fastforce*+
    show *False*
      using ∗ [*of ?U ?V*] *UV* ‹*S* ⊆ *topspace X*›
      by (*auto simp*: *1 2*)
  qed

qed

lemma *homeomorphic_connected_space*:
  X homeomorphic_space Y $\Longrightarrow$ *connected_space X* $\longleftrightarrow$ *connected_space Y*
 unfolding *homeomorphic_space_def homeomorphic_maps_def*
 by (*metis connected_space_subconnected connectedin_continuous_map_image connectedin_topspace continuous_map_image_subset_topspace image_eqI image_subset_iff*)

lemma *homeomorphic_map_connectedness*:
 assumes *f*: *homeomorphic_map X Y f* **and** *U*: $U \subseteq$ *topspace X*
 shows *connectedin Y* (*f* ' *U*) $\longleftrightarrow$ *connectedin X U*
proof –
 have *1*: *f* ' $U \subseteq$ *topspace Y* $\longleftrightarrow$ $U \subseteq$ *topspace X*
  using *U f homeomorphic_imp_surjective_map* **by** *blast*
 moreover have *connected_space* (*subtopology Y* (*f* ' *U*)) $\longleftrightarrow$ *connected_space* (*subtopology X U*)
 proof (*rule homeomorphic_connected_space*)
  have *f* ' $U \subseteq$ *topspace Y*
   by (*simp add*: *U 1*)
  then have *topspace Y* $\cap$ *f* ' *U* = *f* ' *U*
   by (*simp add*: *subset_antisym*)
  then show *subtopology Y* (*f* ' *U*) *homeomorphic_space subtopology X U*
   by (*metis* (*no_types*) *Int_subset_iff U f homeomorphic_map_imp_homeomorphic_space homeomorphic_map_subtopologies homeomorphic_space_sym subset_antisym subset_refl*)
 qed
 ultimately show *?thesis*
  by (*auto simp*: *connectedin_def*)
qed

lemma *homeomorphic_map_connectedness_eq*:
 homeomorphic_map X Y f
  $\Longrightarrow$ *connectedin X U* $\longleftrightarrow$
   $U \subseteq$ *topspace X* $\wedge$ *connectedin Y* (*f* ' *U*)
 using *homeomorphic_map_connectedness connectedin_subset_topspace* **by** *metis*

lemma *connectedin_discrete_topology*:
 connectedin (*discrete_topology U*) *S* $\longleftrightarrow$ $S \subseteq U \wedge (\exists a.\ S \subseteq \{a\})$
proof (*cases* $S \subseteq U$)
 case *True*
 show *?thesis*
 proof (*cases* $S = \{\}$)
  case *False*
  moreover have *connectedin* (*discrete_topology U*) *S* $\longleftrightarrow$ $(\exists a.\ S = \{a\})$
  proof
   show *connectedin* (*discrete_topology U*) *S* $\Longrightarrow \exists a.\ S = \{a\}$
    using *False connectedin_inter_frontier_of insert_Diff* **by** *fastforce*
  qed (*use True* **in** *auto*)
  ultimately show *?thesis*
   by *auto*

qed *simp*
next
case *False*
then show *?thesis*
by (*simp add*: *connectedin_def*)
qed

lemma *connected_space_discrete_topology*:
connected_space (*discrete_topology* U) $\longleftrightarrow$ ($\exists a.\ U \subseteq \{a\}$)
by (*metis connectedin_discrete_topology connectedin_topspace order_refl topspace_discrete_topology*)

2.2.17 Compact sets

definition *compactin* **where**
compactin X S $\longleftrightarrow$
$S \subseteq$ *topspace* X $\wedge$
$(\forall \mathcal{U}.\ (\forall U \in \mathcal{U}.\ openin\ X\ U) \wedge S \subseteq \bigcup \mathcal{U}$
$\longrightarrow (\exists \mathcal{F}.\ finite\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge S \subseteq \bigcup \mathcal{F}))$

definition *compact_space* **where**
compact_space X $\equiv$ *compactin* X (*topspace* X)

lemma *compact_space_alt*:
compact_space X $\longleftrightarrow$
$(\forall \mathcal{U}.\ (\forall U \in \mathcal{U}.\ openin\ X\ U) \wedge topspace\ X \subseteq \bigcup \mathcal{U}$
$\longrightarrow (\exists \mathcal{F}.\ finite\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge topspace\ X \subseteq \bigcup \mathcal{F}))$
by (*simp add*: *compact_space_def compactin_def*)

lemma *compact_space*:
compact_space X $\longleftrightarrow$
$(\forall \mathcal{U}.\ (\forall U \in \mathcal{U}.\ openin\ X\ U) \wedge \bigcup \mathcal{U} = topspace\ X$
$\longrightarrow (\exists \mathcal{F}.\ finite\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge \bigcup \mathcal{F} = topspace\ X))$
unfolding *compact_space_alt*
using *openin_subset* **by** *fastforce*

lemma *compactinD*:
$[\![compactin\ X\ S;\ \bigwedge U.\ U \in \mathcal{U} \Longrightarrow openin\ X\ U;\ S \subseteq \bigcup \mathcal{U}]\!] \Longrightarrow \exists \mathcal{F}.\ finite\ \mathcal{F} \wedge \mathcal{F}$
$\subseteq \mathcal{U} \wedge S \subseteq \bigcup \mathcal{F}$
by (*auto simp*: *compactin_def*)

lemma *compactin_euclidean_iff* [*simp*]: *compactin euclidean* S $\longleftrightarrow$ *compact* S
by (*simp add*: *compact_eq_Heine_Borel compactin_def*) *meson*

lemma *compactin_absolute* [*simp*]:
compactin (*subtopology* X S) S $\longleftrightarrow$ *compactin* X S
proof –
have *eq*: $(\forall U \in \mathcal{U}.\ \exists Y.\ openin\ X\ Y \wedge U = Y \cap S) \longleftrightarrow \mathcal{U} \subseteq (\lambda Y.\ Y \cap S)$ ‘
$\{y.\ openin\ X\ y\}$ **for** $\mathcal{U}$
by *auto*

show *?thesis*
by (*auto simp*: *compactin_def openin_subtopology eq imp_conjL all_subset_image ex_finite_subset_image*)
qed

lemma *compactin_subspace*: *compactin X S* $\longleftrightarrow$ *S* $\subseteq$ *topspace X* $\wedge$ *compact_space* (*subtopology X S*)
unfolding *compact_space_def topspace_subtopology*
by (*metis compactin_absolute compactin_def inf.absorb2*)

lemma *compact_space_subtopology*: *compactin X S* $\Longrightarrow$ *compact_space* (*subtopology X S*)
by (*simp add*: *compactin_subspace*)

lemma *compactin_subtopology*: *compactin* (*subtopology X S*) *T* $\longleftrightarrow$ *compactin X T* $\wedge$ *T* $\subseteq$ *S*
by (*metis compactin_subspace inf.absorb_iff2 le_inf_iff subtopology_subtopology topspace_subtopology*)

lemma *compactin_subset_topspace*: *compactin X S* $\Longrightarrow$ *S* $\subseteq$ *topspace X*
by (*simp add*: *compactin_subspace*)

lemma *compactin_contractive*:
$[\![$*compactin X' S*; *topspace X' = topspace X*;
$\bigwedge$*U*. *openin X U* $\Longrightarrow$ *openin X' U*$]\!]$ $\Longrightarrow$ *compactin X S*
by (*simp add*: *compactin_def*)

lemma *finite_imp_compactin*:
$[\![$*S* $\subseteq$ *topspace X*; *finite S*$]\!]$ $\Longrightarrow$ *compactin X S*
by (*metis compactin_subspace compact_space finite_UnionD inf.absorb_iff2 order_refl topspace_subtopology*)

lemma *compactin_empty* [*iff*]: *compactin X* {}
by (*simp add*: *finite_imp_compactin*)

lemma *compact_space_topspace_empty*:
topspace X = {} $\Longrightarrow$ *compact_space X*
by (*simp add*: *compact_space_def*)

lemma *finite_imp_compactin_eq*:
finite S $\Longrightarrow$ (*compactin X S* $\longleftrightarrow$ *S* $\subseteq$ *topspace X*)
using *compactin_subset_topspace finite_imp_compactin* **by** *blast*

lemma *compactin_sing* [*simp*]: *compactin X* {*a*} $\longleftrightarrow$ *a* $\in$ *topspace X*
by (*simp add*: *finite_imp_compactin_eq*)

lemma *closed_compactin*:
assumes *XK*: *compactin X K* **and** *C* $\subseteq$ *K* **and** *XC*: *closedin X C*
shows *compactin X C*

unfolding *compactin_def*
proof (*intro conjI allI impI*)
show $C \subseteq$ *topspace* X
by (*simp add*: *XC closedin_subset*)
next
fix $\mathcal{U}$ :: *'a set set*
assume $\mathcal{U}$: *Ball* $\mathcal{U}$ (*openin* X) $\wedge$ $C \subseteq \bigcup\mathcal{U}$
have ($\forall$ $U\in$*insert* (*topspace* X − C) $\mathcal{U}$. *openin* X U)
using *XC* $\mathcal{U}$ **by** *blast*
moreover have $K \subseteq \bigcup$(*insert* (*topspace* X − C) $\mathcal{U}$)
using $\mathcal{U}$ *XK compactin_subset_topspace* **by** *fastforce*
ultimately obtain $\mathcal{F}$ **where** *finite* $\mathcal{F}$ $\mathcal{F} \subseteq$ *insert* (*topspace* X − C) $\mathcal{U}$ $K \subseteq \bigcup\mathcal{F}$
using *assms* **unfolding** *compactin_def* **by** *metis*
moreover have *openin* X (*topspace* X − C)
using *XC* **by** *auto*
ultimately show $\exists\mathcal{F}$. *finite* $\mathcal{F}$ $\wedge$ $\mathcal{F} \subseteq \mathcal{U}$ $\wedge$ $C \subseteq \bigcup\mathcal{F}$
using ‹$C \subseteq K$›
by (*rule_tac x*=$\mathcal{F}$ − {*topspace* X − C} **in** *exI*) *auto*
qed

lemma *closedin_compact_space*:
⟦*compact_space* X; *closedin* X S⟧ $\Longrightarrow$ *compactin* X S
by (*simp add*: *closed_compactin closedin_subset compact_space_def*)

lemma *compact_Int_closedin*:
assumes *compactin* X S *closedin* X T **shows** *compactin* X ($S \cap T$)
proof −
have *compactin* (*subtopology* X S) ($S \cap T$)
by (*metis assms closedin_compact_space closedin_subtopology compactin_subspace inf_commute*)
then show *?thesis*
by (*simp add*: *compactin_subtopology*)
qed

lemma *closed_Int_compactin*: ⟦*closedin* X S; *compactin* X T⟧ $\Longrightarrow$ *compactin* X ($S \cap T$)
by (*metis compact_Int_closedin inf_commute*)

lemma *compactin_Un*:
assumes S: *compactin* X S **and** T: *compactin* X T **shows** *compactin* X ($S \cup T$)
unfolding *compactin_def*
proof (*intro conjI allI impI*)
show $S \cup T \subseteq$ *topspace* X
using *assms* **by** (*auto simp*: *compactin_def*)
next
fix $\mathcal{U}$ :: *'a set set*
assume $\mathcal{U}$: *Ball* $\mathcal{U}$ (*openin* X) $\wedge$ $S \cup T \subseteq \bigcup\mathcal{U}$

with S **obtain** $\mathcal{F}$ **where** $\mathcal{V}$: *finite* $\mathcal{F}$ $\mathcal{F} \subseteq \mathcal{U}$ $S \subseteq \bigcup\mathcal{F}$
  unfolding *compactin_def* **by** (*meson sup.bounded_iff*)
obtain $\mathcal{W}$ **where** *finite* $\mathcal{W}$ $\mathcal{W} \subseteq \mathcal{U}$ $T \subseteq \bigcup\mathcal{W}$
  using $\mathcal{U}$ T
  unfolding *compactin_def* **by** (*meson sup.bounded_iff*)
with $\mathcal{V}$ **show** $\exists\mathcal{V}$. *finite* $\mathcal{V} \wedge \mathcal{V} \subseteq \mathcal{U} \wedge S \cup T \subseteq \bigcup\mathcal{V}$
  by (*rule_tac* $x=\mathcal{F} \cup \mathcal{W}$ **in** *exI*) *auto*
qed

lemma *compactin_Union*:
  $[\![$*finite* $\mathcal{F}$; $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *compactin* X $S]\!] \Longrightarrow$ *compactin* X $(\bigcup\mathcal{F})$
by (*induction rule*: *finite_induct*) (*simp_all add*: *compactin_Un*)

lemma *compactin_subtopology_imp_compact*:
  assumes *compactin* (*subtopology* X S) K **shows** *compactin* X K
  using *assms*
proof (*clarsimp simp add*: *compactin_def*)
  fix $\mathcal{U}$
  define $\mathcal{V}$ **where** $\mathcal{V} \equiv (\lambda U.\ U \cap S)$ ` $\mathcal{U}$
  assume $K \subseteq$ *topspace* X **and** $K \subseteq S$ **and** $\forall x{\in}\mathcal{U}$. *openin* X x **and** $K \subseteq \bigcup\mathcal{U}$
  then have $\forall V \in \mathcal{V}$. *openin* (*subtopology* X S) V $K \subseteq \bigcup\mathcal{V}$
    unfolding $\mathcal{V}$*_def* **by** (*auto simp*: *openin_subtopology*)
  moreover
  assume $\forall\mathcal{U}$. ($\forall x{\in}\mathcal{U}$. *openin* (*subtopology* X S) x) $\wedge$ $K \subseteq \bigcup\mathcal{U} \longrightarrow (\exists\mathcal{F}$. *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge K \subseteq \bigcup\mathcal{F})$
  ultimately obtain $\mathcal{F}$ **where** *finite* $\mathcal{F}$ $\mathcal{F} \subseteq \mathcal{V}$ $K \subseteq \bigcup\mathcal{F}$
    by *meson*
  then have $\mathcal{F}$: $\exists U.\ U \in \mathcal{U} \wedge V = U \cap S$ **if** $V \in \mathcal{F}$ **for** V
    unfolding $\mathcal{V}$*_def* **using** *that* **by** *blast*
  let *?*$\mathcal{F}$ = (λF. @U. $U \in \mathcal{U} \wedge F = U \cap S$) ` $\mathcal{F}$
  show $\exists\mathcal{F}$. *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge K \subseteq \bigcup\mathcal{F}$
  proof (*intro exI conjI*)
    show *finite ?*$\mathcal{F}$
      using ‹*finite* $\mathcal{F}$› **by** *blast*
    show *?*$\mathcal{F} \subseteq \mathcal{U}$
      using *someI_ex* [*OF* $\mathcal{F}$] **by** *blast*
    show $K \subseteq \bigcup$ *?*$\mathcal{F}$
    proof *clarsimp*
      fix x
      assume $x \in K$
      then show $\exists V \in \mathcal{F}.\ x \in$ (*SOME* U. $U \in \mathcal{U} \wedge V = U \cap S$)
        using ‹$K \subseteq \bigcup\mathcal{F}$› *someI_ex* [*OF* $\mathcal{F}$]
        by (*metis* (*no_types*, *lifting*) *IntD1 Union_iff subsetCE*)
    qed
  qed
qed

lemma *compact_imp_compactin_subtopology*:
  assumes *compactin* X K $K \subseteq S$ **shows** *compactin* (*subtopology* X S) K

```
  using assms
proof (clarsimp simp add: compactin_def)
  fix 𝒰 :: 'a set set
  define 𝒱 where 𝒱 ≡ {V. openin X V ∧ (∃ U ∈ 𝒰. U = V ∩ S)}
  assume K ⊆ S and K ⊆ topspace X and ∀ U∈𝒰. openin (subtopology X S) U
and K ⊆ ⋃𝒰
  then have ∀ V ∈ 𝒱. openin X V K ⊆ ⋃𝒱
    unfolding 𝒱_def by (fastforce simp: subset_eq openin_subtopology)+
  moreover
  assume ∀𝒰. (∀ U∈𝒰. openin X U) ∧ K ⊆ ⋃𝒰 ⟶ (∃ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ∧
K ⊆ ⋃ℱ)
  ultimately obtain ℱ where finite ℱ ℱ ⊆ 𝒱 K ⊆ ⋃ℱ
    by meson
  let ?ℱ = (λF. F ∩ S) ` ℱ
  show ∃ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ∧ K ⊆ ⋃ℱ
  proof (intro exI conjI)
    show finite ?ℱ
      using ‹finite ℱ› by blast
    show ?ℱ ⊆ 𝒰
      using 𝒱_def ‹ℱ ⊆ 𝒱› by blast
    show K ⊆ ⋃ ?ℱ
      using ‹K ⊆ ⋃ℱ› assms(2) by auto
  qed
qed


proposition compact_space_fip:
   compact_space X ⟷
     (∀𝒰. (∀ C∈𝒰. closedin X C) ∧ (∀ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ⟶ ⋂ℱ ≠ {}) ⟶
⋂𝒰 ≠ {})
   (is _ = ?rhs)
proof (cases topspace X = {})
  case True
  then show ?thesis
unfolding compact_space_def
 by (metis Sup_bot_conv(1) closedin_topspace_empty compactin_empty finite.emptyI
finite_UnionD order_refl)
next
  case False
  show ?thesis
  proof safe
    fix 𝒰 :: 'a set set
    assume * [rule_format]: ∀ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ⟶ ⋂ℱ ≠ {}
    define 𝒱 where 𝒱 ≡ (λS. topspace X − S) ` 𝒰
    assume clo: ∀ C∈𝒰. closedin X C and [simp]: ⋂𝒰 = {}
    then have ∀ V ∈ 𝒱. openin X V topspace X ⊆ ⋃𝒱
      by (auto simp: 𝒱_def)
   moreover assume [unfolded compact_space_alt, rule_format, of 𝒱]: compact_space
X
```

ultimately obtain $\mathcal{F}$ **where** $\mathcal{F}$: *finite* $\mathcal{F}$ $\mathcal{F} \subseteq \mathcal{U}$ *topspace* $X \subseteq$ *topspace* $X - \bigcap\mathcal{F}$
  by (*auto simp*: *ex_finite_subset_image* $\mathcal{V}$*_def*)
 moreover have $\mathcal{F} \neq \{\}$
  using $\mathcal{F}$ ‹*topspace* $X \neq \{\}$› **by** *blast*
 ultimately show *False*
  using ∗ [*of* $\mathcal{F}$]
  by *auto* (*metis Diff_iff Inter_iff clo closedin_def subsetD*)
next
 assume *R* [*rule_format*]: *?rhs*
 show *compact_space X*
  unfolding *compact_space_alt*
 proof *clarify*
  fix $\mathcal{U}$ :: *'a set set*
  define $\mathcal{V}$ **where** $\mathcal{V} \equiv (\lambda S.\ topspace\ X - S)$ ‘ $\mathcal{U}$
  assume $\forall C \in \mathcal{U}$. *openin X C* **and** *topspace* $X \subseteq \bigcup\mathcal{U}$
  with ‹*topspace* $X \neq \{\}$› **have** ∗: $\forall V \in \mathcal{V}$. *closedin X V* $\mathcal{U} \neq \{\}$
   by (*auto simp*: $\mathcal{V}$*_def*)
  show $\exists\mathcal{F}$. *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge$ *topspace* $X \subseteq \bigcup\mathcal{F}$
  proof (*rule ccontr*; *simp*)
   assume $\forall\mathcal{F}\subseteq\mathcal{U}$. *finite* $\mathcal{F} \longrightarrow \neg$ *topspace* $X \subseteq \bigcup\mathcal{F}$
   then have $\forall\mathcal{F}$. *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{V} \longrightarrow \bigcap\mathcal{F} \neq \{\}$
    by (*simp add*: $\mathcal{V}$*_def all_finite_subset_image*)
   with ‹*topspace* $X \subseteq \bigcup\mathcal{U}$› **show** *False*
    using *R* [*of* $\mathcal{V}$] ∗ **by** (*simp add*: $\mathcal{V}$*_def*)
  qed
 qed
qed
qed

corollary *compactin_fip*:
 compactin X S ⟷
  $S \subseteq$ *topspace* $X \wedge$
  $(\forall\mathcal{U}.\ (\forall C\in\mathcal{U}.\ closedin\ X\ C) \wedge (\forall\mathcal{F}.\ finite\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \longrightarrow S \cap \bigcap\mathcal{F} \neq \{\})$
$\longrightarrow S \cap \bigcap\mathcal{U} \neq \{\})$
proof (*cases* $S = \{\}$)
 case *False*
 show *?thesis*
 proof (*cases* $S \subseteq$ *topspace X*)
  case *True*
  then have *compactin X S* ⟷
    $(\forall\mathcal{U}.\ \mathcal{U} \subseteq (\lambda T.\ S \cap T)$ ‘ $\{T.\ closedin\ X\ T\} \longrightarrow$
     $(\forall\mathcal{F}.\ finite\ \mathcal{F} \longrightarrow \mathcal{F} \subseteq \mathcal{U} \longrightarrow \bigcap\mathcal{F} \neq \{\}) \longrightarrow \bigcap\mathcal{U} \neq \{\})$
   by (*simp add*: *compact_space_fip compactin_subspace closedin_subtopology image_def subset_eq Int_commute imp_conjL*)
  also have ... = $(\forall\mathcal{U}\subseteq Collect\ (closedin\ X).\ (\forall\mathcal{F}.\ finite\ \mathcal{F} \longrightarrow \mathcal{F} \subseteq (\cap)\ S$ ‘ $\mathcal{U}$
$\longrightarrow \bigcap\mathcal{F} \neq \{\}) \longrightarrow \bigcap\ ((\cap)\ S$ ‘ $\mathcal{U}) \neq \{\})$
   by (*simp add*: *all_subset_image*)
  also have ... = $(\forall\mathcal{U}.\ (\forall C\in\mathcal{U}.\ closedin\ X\ C) \wedge (\forall\mathcal{F}.\ finite\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \longrightarrow$

$S \cap \bigcap\mathcal{F} \neq \{\}) \longrightarrow S \cap \bigcap\mathcal{U} \neq \{\})$
  proof –
    have *eq*: $((\forall\mathcal{F}.$ *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \longrightarrow \bigcap\ ((\cap)\ S\ `\ \mathcal{F}) \neq \{\}) \longrightarrow \bigcap\ ((\cap)\ S\ `\ \mathcal{U}) \neq \{\}) \longleftrightarrow$
        $((\forall\mathcal{F}.$ *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \longrightarrow S \cap \bigcap\mathcal{F} \neq \{\}) \longrightarrow S \cap \bigcap\mathcal{U} \neq \{\})$ **for** $\mathcal{U}$
      by *simp* (*use* ‹$S \neq \{\}$› **in** *blast*)
    show *?thesis*
      unfolding *imp_conjL* [*symmetric*] *all_finite_subset_image eq* **by** *blast*
  qed
  finally show *?thesis*
    using *True* **by** *simp*
 qed (*simp add*: *compactin_subspace*)
qed *force*

corollary *compact_space_imp_nest*:
 fixes *C* :: *nat* ⇒ *'a set*
 assumes *compact_space X* **and** *clo*: $\bigwedge n.$ *closedin X* (*C n*)
  and *ne*: $\bigwedge n.\ C\ n \neq \{\}$ **and** *inc*: $\bigwedge m\ n.\ m \leq n \Longrightarrow C\ n \subseteq C\ m$
 shows $(\bigcap n.\ C\ n) \neq \{\}$
proof –
 let $?\mathcal{U} =$ *range* $(\lambda n. \bigcap m \leq n.\ C\ m)$
 have *closedin X A* **if** $A \in ?\mathcal{U}$ **for** *A*
  using *that clo* **by** *auto*
 moreover have $(\bigcap n{\in}K. \bigcap m \leq n.\ C\ m) \neq \{\}$ **if** *finite K* **for** *K*
 proof –
  obtain *n* **where** $\bigwedge k.\ k \in K \Longrightarrow k \leq n$
   using *Max.coboundedI* ‹*finite K*› **by** *blast*
  with *inc* **have** $C\ n \subseteq (\bigcap n{\in}K. \bigcap m \leq n.\ C\ m)$
  by *blast*
 with *ne* [*of n*] **show** *?thesis*
  by *blast*
 qed
 ultimately show *?thesis*
  using ‹*compact_space X*› [*unfolded compact_space_fip*, *rule_format*, *of* $?\mathcal{U}$]
  by (*simp add*: *all_finite_subset_image INT_extend_simps UN_atMost_UNIV del*: *INT_simps*)
qed

lemma *compactin_discrete_topology*:
  compactin (*discrete_topology X*) *S* ⟷ $S \subseteq X \wedge$ *finite S* (**is** *?lhs* = *?rhs*)
proof (*intro iffI conjI*)
 assume *L*: *?lhs*
 then show $S \subseteq X$
  by (*auto simp*: *compactin_def*)
 have ∗: $\bigwedge\mathcal{U}.$ *Ball* $\mathcal{U}$ (*openin* (*discrete_topology X*)) $\wedge S \subseteq \bigcup\mathcal{U} \Longrightarrow$
    $(\exists\mathcal{F}.$ *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge S \subseteq \bigcup\mathcal{F})$
  using *L* **by** (*auto simp*: *compactin_def*)
 show *finite S*

using ∗ [*of* (λ*x*. {*x*}) ' *X*] ‹*S* ⊆ *X*›
by *clarsimp* (*metis UN_singleton finite_subset_image infinite_super*)
next
assume *?rhs*
then show *?lhs*
by (*simp add*: *finite_imp_compactin*)
qed

lemma *compact_space_discrete_topology*: *compact_space*(*discrete_topology X*) ⟷ *finite X*
by (*simp add*: *compactin_discrete_topology compact_space_def*)

lemma *compact_space_imp_Bolzano_Weierstrass*:
assumes *compact_space X infinite S S* ⊆ *topspace X*
shows *X derived_set_of S* ≠ {}
proof
assume *X*: *X derived_set_of S* = {}
then have *closedin X S*
by (*simp add*: *closedin_contains_derived_set assms*)
then have *compactin X S*
by (*rule closedin_compact_space* [*OF* ‹*compact_space X*›])
with *X* **show** *False*
by (*metis* ‹*infinite S*› *compactin_subspace compact_space_discrete_topology inf_bot_right subtopology_eq_discrete_topology_eq*)
qed

lemma *compactin_imp_Bolzano_Weierstrass*:
⟦*compactin X S*; *infinite T* ∧ *T* ⊆ *S*⟧ ⟹ *S* ∩ *X derived_set_of T* ≠ {}
using *compact_space_imp_Bolzano_Weierstrass* [*of subtopology X S*]
by (*simp add*: *compactin_subspace derived_set_of_subtopology inf_absorb2*)

lemma *compact_closure_of_imp_Bolzano_Weierstrass*:
⟦*compactin X* (*X closure_of S*); *infinite T*; *T* ⊆ *S*; *T* ⊆ *topspace X*⟧ ⟹ *X derived_set_of T* ≠ {}
using *closure_of_mono closure_of_subset compactin_imp_Bolzano_Weierstrass* **by** *fastforce*

lemma *discrete_compactin_eq_finite*:
S ∩ *X derived_set_of S* = {} ⟹ *compactin X S* ⟷ *S* ⊆ *topspace X* ∧ *finite S*
by (*meson compactin_imp_Bolzano_Weierstrass finite_imp_compactin_eq order_refl*)

lemma *discrete_compact_space_eq_finite*:
X derived_set_of (*topspace X*) = {} ⟹ (*compact_space X* ⟷ *finite*(*topspace X*))
by (*metis compact_space_discrete_topology discrete_topology_unique_derived_set*)

lemma *image_compactin*:
assumes *cpt*: *compactin X S* **and** *cont*: *continuous_map X Y f*
shows *compactin Y* (*f* ' *S*)

```
  unfolding compactin_def
proof (intro conjI allI impI)
  show f ‘ S ⊆ topspace Y
   using compactin_subset_topspace cont continuous_map_image_subset_topspace cpt
by blast
next
  fix 𝒰 :: 'b set set
  assume 𝒰: Ball 𝒰 (openin Y) ∧ f ‘ S ⊆ ⋃𝒰
  define 𝒱 where 𝒱 ≡ (λU. {x ∈ topspace X. f x ∈ U}) ‘ 𝒰
  have S ⊆ topspace X
    and ∗: ⋀𝒰. ⟦∀ U∈𝒰. openin X U; S ⊆ ⋃𝒰⟧ ⟹ ∃ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ∧ S
⊆ ⋃ℱ
    using cpt by (auto simp: compactin_def)
  obtain ℱ where ℱ: finite ℱ ℱ ⊆ 𝒱 S ⊆ ⋃ℱ
  proof −
    have 1: ∀ U∈𝒱. openin X U
      unfolding 𝒱_def using 𝒰 cont[unfolded continuous_map] by blast
    have 2: S ⊆ ⋃𝒱
      unfolding 𝒱_def using compactin_subset_topspace cpt 𝒰 by fastforce
    show thesis
      using ∗ [OF 1 2] that by metis
  qed
  have ∀ v ∈ 𝒱. ∃ U. U ∈ 𝒰 ∧ v = {x ∈ topspace X. f x ∈ U}
    using 𝒱_def by blast
  then obtain U where U: ∀ v ∈ 𝒱. U v ∈ 𝒰 ∧ v = {x ∈ topspace X. f x ∈ U
v}
    by metis
  show ∃ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ∧ f ‘ S ⊆ ⋃ℱ
  proof (intro conjI exI)
    show finite (U ‘ ℱ)
      by (simp add: ‹finite ℱ›)
  next
    show U ‘ ℱ ⊆ 𝒰
      using ‹ℱ ⊆ 𝒱› U by auto
  next
    show f ‘ S ⊆ ⋃ (U ‘ ℱ)
      using ℱ(2−3) U UnionE subset_eq U by fastforce
  qed
qed


lemma homeomorphic_compact_space:
  assumes X homeomorphic_space Y
  shows compact_space X ⟷ compact_space Y
    using homeomorphic_space_sym
     by (metis assms compact_space_def homeomorphic_eq_everything_map homeo-
morphic_space image_compactin)

lemma homeomorphic_map_compactness:
```

assumes *hom*: *homeomorphic_map X Y f* **and** *U*: $U \subseteq$ *topspace X*
shows *compactin Y* $(f$ ‘ $U) \longleftrightarrow$ *compactin X U*
proof −
have *f* ‘ $U \subseteq$ *topspace Y*
using *hom U homeomorphic_imp_surjective_map* **by** *blast*
moreover have *homeomorphic_map* (*subtopology X U*) (*subtopology Y* (*f* ‘ *U*)) *f*
using *U hom homeomorphic_imp_surjective_map* **by** (*blast intro*: *homeomorphic_map_subtopologies*)
then have *compact_space* (*subtopology Y* (*f* ‘ *U*)) = *compact_space* (*subtopology X U*)
using *homeomorphic_compact_space homeomorphic_map_imp_homeomorphic_space* **by** *blast*
ultimately show *?thesis*
by (*simp add*: *compactin_subspace U*)
qed

lemma *homeomorphic_map_compactness_eq*:
homeomorphic_map X Y f
$\Longrightarrow$ *compactin X U* $\longleftrightarrow$ $U \subseteq$ *topspace X* $\wedge$ *compactin Y* (*f* ‘ *U*)
by (*meson compactin_subset_topspace homeomorphic_map_compactness*)

2.2.18 Embedding maps

definition *embedding_map*
where *embedding_map X Y f* $\equiv$ *homeomorphic_map X* (*subtopology Y* (*f* ‘ (*topspace X*))) *f*

lemma *embedding_map_eq*:
⟦*embedding_map X Y f*; $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow f\,x = g\,x$⟧ $\Longrightarrow$ *embedding_map X Y g*
unfolding *embedding_map_def*
by (*metis homeomorphic_map_eq image_cong*)

lemma *embedding_map_compose*:
assumes *embedding_map X X′ f embedding_map X′ X″ g*
shows *embedding_map X X″* $(g \circ f)$
proof −
have *hm*: *homeomorphic_map X* (*subtopology X′* (*f* ‘ *topspace X*)) *f homeomorphic_map X′* (*subtopology X″* (*g* ‘ *topspace X′*)) *g*
using *assms* **by** (*auto simp*: *embedding_map_def*)
then obtain *C* **where** *g* ‘ *topspace X′* $\cap$ *C* = $(g \circ f)$ ‘ *topspace X*
by (*metis* (*no_types*) *Int_absorb1 continuous_map_image_subset_topspace continuous_map_in_subtopology homeomorphic_eq_everything_map image_comp image_mono*)
then have *homeomorphic_map* (*subtopology X′* (*f* ‘ *topspace X*)) (*subtopology X″* ($(g \circ f)$ ‘ *topspace X*)) *g*
by (*metis hm homeomorphic_imp_surjective_map homeomorphic_map_subtopologies image_comp subtopology_subtopology topspace_subtopology*)
then show *?thesis*

unfolding *embedding_map_def*
using *hm(1) homeomorphic_map_compose* **by** *blast*
qed

lemma *surjective_embedding_map*:
embedding_map X Y f ∧ *f* ' (*topspace X*) = *topspace Y* ⟷ *homeomorphic_map X Y f*
by (*force simp*: *embedding_map_def homeomorphic_eq_everything_map*)

lemma *embedding_map_in_subtopology*:
embedding_map X (*subtopology Y S*) *f* ⟷ *embedding_map X Y f* ∧ *f* ' (*topspace X*) ⊆ *S* (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
unfolding *embedding_map_def*
by (*metis continuous_map_in_subtopology homeomorphic_imp_continuous_map inf_absorb2 subtopology_subtopology*)
qed (*simp add*: *embedding_map_def inf.absorb_iff2 subtopology_subtopology*)

lemma *injective_open_imp_embedding_map*:
⟦*continuous_map X Y f*; *open_map X Y f*; *inj_on f* (*topspace X*)⟧ ⟹ *embedding_map X Y f*
unfolding *embedding_map_def*
by (*simp add*: *continuous_map_in_subtopology continuous_open_quotient_map eq_iff homeomorphic_map_def open_map_imp_subset open_map_into_subtopology*)

lemma *injective_closed_imp_embedding_map*:
⟦*continuous_map X Y f*; *closed_map X Y f*; *inj_on f* (*topspace X*)⟧ ⟹ *embedding_map X Y f*
unfolding *embedding_map_def*
by (*simp add*: *closed_map_imp_subset closed_map_into_subtopology continuous_closed_quotient_map*
continuous_map_in_subtopology dual_order.eq_iff homeomorphic_map_def)

lemma *embedding_map_imp_homeomorphic_space*:
embedding_map X Y f ⟹ *X homeomorphic_space* (*subtopology Y* (*f* ' (*topspace X*)))
unfolding *embedding_map_def*
using *homeomorphic_space* **by** *blast*

lemma *embedding_imp_closed_map*:
⟦*embedding_map X Y f*; *closedin Y* (*f* ' *topspace X*)⟧ ⟹ *closed_map X Y f*
unfolding *closed_map_def*
by (*auto simp*: *closedin_closed_subtopology embedding_map_def homeomorphic_map_closedness_eq*)

2.2.19 Retraction and section maps

definition *retraction_maps* :: *'a topology* ⇒ *'b topology* ⇒ (*'a* ⇒ *'b*) ⇒ (*'b* ⇒ *'a*) ⇒ *bool*

where *retraction_maps X Y f g* ≡
continuous_map X Y f ∧ *continuous_map Y X g* ∧ (∀ *x* ∈ *topspace Y*. *f*(*g x*) = *x*)

definition *section_map* :: *'a topology* ⇒ *'b topology* ⇒ (*'a* ⇒ *'b*) ⇒ *bool*
where *section_map X Y f* ≡ ∃ *g*. *retraction_maps Y X g f*

definition *retraction_map* :: *'a topology* ⇒ *'b topology* ⇒ (*'a* ⇒ *'b*) ⇒ *bool*
where *retraction_map X Y f* ≡ ∃ *g*. *retraction_maps X Y f g*

lemma *retraction_maps_eq*:
⟦*retraction_maps X Y f g*; ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *f' x*; ⋀*x*. *x* ∈ *topspace Y* ⟹ *g x* = *g' x*⟧
⟹ *retraction_maps X Y f' g'*
unfolding *retraction_maps_def* **by** (*metis* (*no_types*, *lifting*) *continuous_map_def continuous_map_eq*)

lemma *section_map_eq*:
⟦*section_map X Y f*; ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *g x*⟧ ⟹ *section_map X Y g*
unfolding *section_map_def* **using** *retraction_maps_eq* **by** *blast*

lemma *retraction_map_eq*:
⟦*retraction_map X Y f*; ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *g x*⟧ ⟹ *retraction_map X Y g*
unfolding *retraction_map_def* **using** *retraction_maps_eq* **by** *blast*

lemma *homeomorphic_imp_retraction_maps*:
homeomorphic_maps X Y f g ⟹ *retraction_maps X Y f g*
by (*simp add*: *homeomorphic_maps_def retraction_maps_def*)

lemma *section_and_retraction_eq_homeomorphic_map*:
section_map X Y f ∧ *retraction_map X Y f* ⟷ *homeomorphic_map X Y f* (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then obtain *g g'* **where** *f*: *continuous_map X Y f*
and *g*: *continuous_map Y X g* ∀ *x*∈*topspace X*. *g* (*f x*) = *x*
and *g'*: *continuous_map Y X g'* ∀ *x*∈*topspace Y*. *f* (*g' x*) = *x*
by (*auto simp*: *retraction_map_def retraction_maps_def section_map_def*)
then have *homeomorphic_maps X Y f g*
by (*force simp add*: *homeomorphic_maps_def continuous_map_def*)
then show *?rhs*
using *homeomorphic_map_maps* **by** *blast*
next
assume *?rhs*
then show *?lhs*
unfolding *retraction_map_def section_map_def*
by (*meson homeomorphic_imp_retraction_maps homeomorphic_map_maps homeomorphic_maps_sym*)

qed

lemma *section_imp_embedding_map*:
section_map X Y f ⟹ *embedding_map X Y f*
unfolding *section_map_def embedding_map_def homeomorphic_map_maps retraction_maps_def homeomorphic_maps_def*
by (*force simp*: *continuous_map_in_subtopology continuous_map_from_subtopology*)

lemma *retraction_imp_quotient_map*:
assumes *retraction_map X Y f*
shows *quotient_map X Y f*
unfolding *quotient_map_def*
proof (*intro conjI subsetI allI impI*)
show *f ' topspace X* = *topspace Y*
using *assms* **by** (*force simp*: *retraction_map_def retraction_maps_def continuous_map_def*)
next
fix *U*
assume *U*: *U* ⊆ *topspace Y*
have *openin Y U*
if ∀ *x*∈*topspace Y*. *g x* ∈ *topspace X* ∀ *x*∈*topspace Y*. *f* (*g x*) = *x*
openin Y {*x* ∈ *topspace Y*. *g x* ∈ {*x* ∈ *topspace X*. *f x* ∈ *U*}} **for** *g*
using *openin_subopen U that* **by** *fastforce*
then show *openin X* {*x* ∈ *topspace X*. *f x* ∈ *U*} = *openin Y U*
using *assms* **by** (*auto simp*: *retraction_map_def retraction_maps_def continuous_map_def*)
qed

lemma *retraction_maps_compose*:
⟦*retraction_maps X Y f f'*; *retraction_maps Y Z g g'*⟧ ⟹ *retraction_maps X Z* (*g* ∘ *f*) (*f'* ∘ *g'*)
by (*clarsimp simp*: *retraction_maps_def continuous_map_compose*) (*simp add*: *continuous_map_def*)

lemma *retraction_map_compose*:
⟦*retraction_map X Y f*; *retraction_map Y Z g*⟧ ⟹ *retraction_map X Z* (*g* ∘ *f*)
by (*meson retraction_map_def retraction_maps_compose*)

lemma *section_map_compose*:
⟦*section_map X Y f*; *section_map Y Z g*⟧ ⟹ *section_map X Z* (*g* ∘ *f*)
by (*meson retraction_maps_compose section_map_def*)

lemma *surjective_section_eq_homeomorphic_map*:
section_map X Y f ∧ *f '* (*topspace X*) = *topspace Y* ⟷ *homeomorphic_map X Y f*
by (*meson section_and_retraction_eq_homeomorphic_map section_imp_embedding_map surjective_embedding_map*)

lemma *surjective_retraction_or_section_map*:

f ‘ (*topspace X*) = *topspace Y* ⟹ *retraction_map X Y f* ∨ *section_map X Y f* ⟷ *retraction_map X Y f*
using *section_and_retraction_eq_homeomorphic_map surjective_section_eq_homeomorphic_map*
by *fastforce*

lemma *retraction_imp_surjective_map*:
retraction_map X Y f ⟹ *f* ‘ (*topspace X*) = *topspace Y*
by (*simp add*: *retraction_imp_quotient_map quotient_imp_surjective_map*)

lemma *section_imp_injective_map*:
⟦*section_map X Y f*; *x* ∈ *topspace X*; *y* ∈ *topspace X*⟧ ⟹ *f x* = *f y* ⟷ *x* = *y*
by (*metis* (*mono_tags*, *hide_lams*) *retraction_maps_def section_map_def*)

lemma *retraction_maps_to_retract_maps*:
retraction_maps X Y r s
⟹ *retraction_maps X* (*subtopology X* (*s* ‘ (*topspace Y*))) (*s* ∘ *r*) *id*
unfolding *retraction_maps_def*
by (*auto simp*: *continuous_map_compose continuous_map_into_subtopology continuous_map_from_subtopology*)

2.2.20 Continuity

lemma *continuous_on_open*:
continuous_on S f ⟷
(∀ *T*. *openin* (*top_of_set* (*f* ‘ *S*)) *T* ⟶
openin (*top_of_set S*) (*S* ∩ *f* −‘ *T*))
unfolding *continuous_on_open_invariant openin_open Int_def vimage_def Int_commute*
by (*simp add*: *imp_ex imageI conj_commute eq_commute cong*: *conj_cong*)

lemma *continuous_on_closed*:
continuous_on S f ⟷
(∀ *T*. *closedin* (*top_of_set* (*f* ‘ *S*)) *T* ⟶
closedin (*top_of_set S*) (*S* ∩ *f* −‘ *T*))
unfolding *continuous_on_closed_invariant closedin_closed Int_def vimage_def Int_commute*
by (*simp add*: *imp_ex imageI conj_commute eq_commute cong*: *conj_cong*)

lemma *continuous_on_imp_closedin*:
assumes *continuous_on S f closedin* (*top_of_set* (*f* ‘ *S*)) *T*
shows *closedin* (*top_of_set S*) (*S* ∩ *f* −‘ *T*)
using *assms continuous_on_closed* **by** *blast*

lemma *continuous_map_subtopology_eu* [*simp*]:
continuous_map (*top_of_set S*) (*subtopology euclidean T*) *h* ⟷ *continuous_on S h* ∧ *h* ‘ *S* ⊆ *T*
by (*simp add*: *continuous_map_in_subtopology*)

lemma *continuous_map_euclidean_top_of_set*:
assumes *eq*: *f* −‘ *S* = *UNIV* **and** *cont*: *continuous_on UNIV f*
shows *continuous_map euclidean* (*top_of_set S*) *f*

by (*simp add*: *cont continuous_map_into_subtopology eq image_subset_iff_subset_vimage*)

2.2.21 Half-global and completely global cases

lemma *continuous_openin_preimage_gen*:
assumes *continuous_on S f open T*
shows *openin* (*top_of_set S*) ($S \cap f -` T$)
proof –
have ∗: ($S \cap f -` T$) = ($S \cap f -` (T \cap f ` S)$)
by *auto*
have *openin* (*top_of_set* ($f ` S$)) ($T \cap f ` S$)
using *openin_open_Int*[*of T f ` S*, *OF assms*(*2*)] **unfolding** *openin_open* **by** *auto*
then show *?thesis*
using *assms*(*1*)[*unfolded continuous_on_open*, *THEN spec*[**where** $x=T \cap f `$ *S*]]
using ∗ **by** *auto*
qed

lemma *continuous_closedin_preimage*:
assumes *continuous_on S f* **and** *closed T*
shows *closedin* (*top_of_set S*) ($S \cap f -` T$)
proof –
have ∗: ($S \cap f -` T$) = ($S \cap f -` (T \cap f ` S)$)
by *auto*
have *closedin* (*top_of_set* ($f ` S$)) ($T \cap f ` S$)
using *closedin_closed_Int*[*of T f ` S*, *OF assms*(*2*)]
by (*simp add*: *Int_commute*)
then show *?thesis*
using *assms*(*1*)[*unfolded continuous_on_closed*, *THEN spec*[**where** $x=T \cap f `$ *S*]]
using ∗ **by** *auto*
qed

lemma *continuous_openin_preimage_eq*:
continuous_on S f ⟷ (∀ *T*. *open T* ⟶ *openin* (*top_of_set S*) ($S \cap f -` T$))
by (*metis Int_commute continuous_on_open_invariant open_openin openin_subtopology*)

lemma *continuous_closedin_preimage_eq*:
continuous_on S f ⟷
(∀ *T*. *closed T* ⟶ *closedin* (*top_of_set S*) ($S \cap f -` T$))
by (*metis Int_commute closedin_closed continuous_on_closed_invariant*)

lemma *continuous_open_preimage*:
assumes *contf*: *continuous_on S f* **and** *open S open T*
shows *open* ($S \cap f -` T$)
proof–
obtain *U* **where** *open U* ($S \cap f -` T$) = $S \cap U$
using *continuous_openin_preimage_gen*[*OF contf* ‹*open T*›]

unfolding *openin_open* **by** *auto*
then show *?thesis*
using *open_Int*[*of S U*, *OF* ‹*open S*›] **by** *auto*
qed

lemma *continuous_closed_preimage*:
assumes *contf*: *continuous_on S f* **and** *closed S closed T*
shows *closed* (*S* ∩ *f* −‘ *T*)
proof−
obtain *U* **where** *closed U* (*S* ∩ *f* −‘ *T*) = *S* ∩ *U*
using *continuous_closedin_preimage*[*OF contf* ‹*closed T*›]
unfolding *closedin_closed* **by** *auto*
then show *?thesis* **using** *closed_Int*[*of S U*, *OF* ‹*closed S*›] **by** *auto*
qed

lemma *continuous_open_vimage*: *open S* ⟹ (⋀*x*. *continuous* (*at x*) *f*) ⟹ *open* (*f* −‘ *S*)
by (*metis continuous_on_eq_continuous_within open_vimage*)

lemma *continuous_closed_vimage*: *closed S* ⟹ (⋀*x*. *continuous* (*at x*) *f*) ⟹ *closed* (*f* −‘ *S*)
by (*simp add*: *closed_vimage continuous_on_eq_continuous_within*)

lemma *Times_in_interior_subtopology*:
assumes (*x*, *y*) ∈ *U openin* (*top_of_set* (*S* × *T*)) *U*
obtains *V W* **where** *openin* (*top_of_set S*) *V x* ∈ *V*
openin (*top_of_set T*) *W y* ∈ *W* (*V* × *W*) ⊆ *U*
proof −
from *assms* **obtain** *E* **where** *open E U* = *S* × *T* ∩ *E* (*x*, *y*) ∈ *E x* ∈ *S y* ∈ *T*
by (*auto simp*: *openin_open*)
from *open_prod_elim*[*OF* ‹*open E*› ‹(*x*, *y*) ∈ *E*›]
obtain *E1 E2* **where** *open E1 open E2* (*x*, *y*) ∈ *E1* × *E2 E1* × *E2* ⊆ *E*
by *blast*
show *?thesis*
proof
show *openin* (*top_of_set S*) (*E1* ∩ *S*)
openin (*top_of_set T*) (*E2* ∩ *T*)
using ‹*open E1*› ‹*open E2*›
by (*auto simp*: *openin_open*)
show *x* ∈ *E1* ∩ *S y* ∈ *E2* ∩ *T*
using ‹(*x*, *y*) ∈ *E1* × *E2*› ‹*x* ∈ *S*› ‹*y* ∈ *T*› **by** *auto*
show (*E1* ∩ *S*) × (*E2* ∩ *T*) ⊆ *U*
using ‹*E1* × *E2* ⊆ *E*› ‹*U* = _›
by (*auto simp*:)
qed
qed

lemma *closedin_Times*:
closedin (*top_of_set S*) *S'* ⟹ *closedin* (*top_of_set T*) *T'* ⟹

```
    closedin (top_of_set (S × T)) (S′ × T′)
  unfolding closedin_closed using closed_Times by blast

lemma openin_Times:
  openin (top_of_set S) S′ ⟹ openin (top_of_set T) T′ ⟹
    openin (top_of_set (S × T)) (S′ × T′)
  unfolding openin_open using open_Times by blast

lemma openin_Times_eq:
  fixes S :: 'a::topological_space set and T :: 'b::topological_space set
  shows
    openin (top_of_set (S × T)) (S′ × T′) ⟷
      S′ = {} ∨ T′ = {} ∨ openin (top_of_set S) S′ ∧ openin (top_of_set T) T′
    (is ?lhs = ?rhs)
proof (cases S′ = {} ∨ T′ = {})
  case True
  then show ?thesis by auto
next
  case False
  then obtain x y where x ∈ S′ y ∈ T′
    by blast
  show ?thesis
  proof
    assume ?lhs
    have openin (top_of_set S) S′
    proof (subst openin_subopen, clarify)
      show ∃ U. openin (top_of_set S) U ∧ x ∈ U ∧ U ⊆ S′ if x ∈ S′ for x
        using that ‹y ∈ T′› Times_in_interior_subtopology [OF _ ‹?lhs›, of x y]
        by simp (metis mem_Sigma_iff subsetD subsetI)
    qed
    moreover have openin (top_of_set T) T′
    proof (subst openin_subopen, clarify)
      show ∃ U. openin (top_of_set T) U ∧ y ∈ U ∧ U ⊆ T′ if y ∈ T′ for y
        using that ‹x ∈ S′› Times_in_interior_subtopology [OF _ ‹?lhs›, of x y]
        by simp (metis mem_Sigma_iff subsetD subsetI)
    qed
    ultimately show ?rhs
      by simp
  next
    assume ?rhs
    with False show ?lhs
      by (simp add: openin_Times)
  qed
qed

lemma Lim_transform_within_openin:
  assumes f: (f ⟶ l) (at a within T)
    and openin (top_of_set T) S a ∈ S
    and eq: ⋀x. ⟦x ∈ S; x ≠ a⟧ ⟹ f x = g x
```

shows $(g \longrightarrow l)$ (*at a within T*)
proof −
have $\forall_F$ *x in at a within T*. $x \in T \wedge x \neq a$
by (*simp add*: *eventually_at_filter*)
moreover
from ‹*openin* _ _› **obtain** *U* **where** *open U S* = *T* ∩ *U*
by (*auto simp*: *openin_open*)
then have $a \in U$ **using** ‹$a \in S$› **by** *auto*
from *topological_tendstoD*[*OF tendsto_ident_at* ‹*open U*› ‹$a \in U$›]
have $\forall_F$ *x in at a within T*. $x \in U$ **by** *auto*
ultimately
have $\forall_F$ *x in at a within T*. *f x* = *g x*
by *eventually_elim* (*auto simp*: ‹*S* = _› *eq*)
with *f* **show** *?thesis*
by (*rule Lim_transform_eventually*)
qed

lemma *continuous_on_open_gen*:
assumes *f* ‘ *S* ⊆ *T*
shows *continuous_on S f* ⟷
(∀ *U*. *openin* (*top_of_set T*) *U*
⟶ *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *U*))
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*clarsimp simp add*: *continuous_openin_preimage_eq openin_open*)
(*metis Int_assoc assms image_subset_iff_subset_vimage inf.absorb_iff1*)
next
assume *R* [*rule_format*]: *?rhs*
show *?lhs*
proof (*clarsimp simp add*: *continuous_openin_preimage_eq*)
fix *U*::′*a set*
assume *open U*
then have *openin* (*top_of_set S*) (*S* ∩ *f* −‘ (*U* ∩ *T*))
by (*metis R inf_commute openin_open*)
then show *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *U*)
by (*metis Int_assoc Int_commute assms image_subset_iff_subset_vimage inf.absorb_iff2 vimage_Int*)
qed
qed

lemma *continuous_openin_preimage*:
⟦*continuous_on S f*; *f* ‘ *S* ⊆ *T*; *openin* (*top_of_set T*) *U*⟧
⟹ *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *U*)
by (*simp add*: *continuous_on_open_gen*)

lemma *continuous_on_closed_gen*:
assumes *f* ‘ *S* ⊆ *T*

```
  shows continuous_on S f ⟷
           (∀ U. closedin (top_of_set T) U
                ⟶ closedin (top_of_set S) (S ∩ f −` U))
    (is ?lhs = ?rhs)
proof −
  have ∗: U ⊆ T ⟹ S ∩ f −` (T − U) = S − (S ∩ f −` U) for U
    using assms by blast
  show ?thesis
  proof
    assume L: ?lhs
    show ?rhs
    proof clarify
      fix U
      assume closedin (top_of_set T) U
      then show closedin (top_of_set S) (S ∩ f −` U)
        using L unfolding continuous_on_open_gen [OF assms]
        by (metis ∗ closedin_def inf_le1 topspace_euclidean_subtopology)
    qed
  next
    assume R [rule_format]: ?rhs
    show ?lhs
      unfolding continuous_on_open_gen [OF assms]
      by (metis ∗ R inf_le1 openin_closedin_eq topspace_euclidean_subtopology)
  qed
qed

lemma continuous_closedin_preimage_gen:
  assumes continuous_on S f f ` S ⊆ T closedin (top_of_set T) U
    shows closedin (top_of_set S) (S ∩ f −` U)
using assms continuous_on_closed_gen by blast

lemma continuous_transform_within_openin:
  assumes continuous (at a within T) f
    and openin (top_of_set T) S a ∈ S
    and eq: ⋀x. x ∈ S ⟹ f x = g x
  shows continuous (at a within T) g
  using assms by (simp add: Lim_transform_within_openin continuous_within)
```

2.2.22 The topology generated by some (open) subsets

In the definition below of a generated topology, the *Empty* case is not necessary, as it follows from *UN* taking for *K* the empty set. However, it is convenient to have, and is never a problem in proofs, so I prefer to write it down explicitly.

We do not require *UNIV* to be an open set, as this will not be the case in applications. (We are thinking of a topology on a subset of *UNIV*, the remaining part of *UNIV* being irrelevant.)

```
inductive generate_topology_on for S where
```

```
  Empty: generate_topology_on S {}
| Int: generate_topology_on S a ⟹ generate_topology_on S b ⟹ generate_topology_on
S (a ∩ b)
| UN: (⋀k. k ∈ K ⟹ generate_topology_on S k) ⟹ generate_topology_on S (⋃K)
| Basis: s ∈ S ⟹ generate_topology_on S s

lemma istopology_generate_topology_on:
  istopology (generate_topology_on S)
unfolding istopology_def by (auto intro: generate_topology_on.intros)
```

The basic property of the topology generated by a set S is that it is the smallest topology containing all the elements of S:

```
lemma generate_topology_on_coarsest:
  assumes T: istopology T ⋀s. s ∈ S ⟹ T s
        and gen: generate_topology_on S s0
  shows T s0
  using gen
by (induct rule: generate_topology_on.induct) (use T in ‹auto simp: istopology_def›)

abbreviation topology_generated_by::('a set set) ⇒ ('a topology)
  where topology_generated_by S ≡ topology (generate_topology_on S)

lemma openin_topology_generated_by_iff:
  openin (topology_generated_by S) s ⟷ generate_topology_on S s
  using topology_inverse'[OF istopology_generate_topology_on[of S]] by simp

lemma openin_topology_generated_by:
  openin (topology_generated_by S) s ⟹ generate_topology_on S s
using openin_topology_generated_by_iff by auto

lemma topology_generated_by_topspace [simp]:
  topspace (topology_generated_by S) = (⋃S)
proof
  {
    fix s assume openin (topology_generated_by S) s
    then have generate_topology_on S s by (rule openin_topology_generated_by)
    then have s ⊆ (⋃S) by (induct, auto)
  }
  then show topspace (topology_generated_by S) ⊆ (⋃S)
    unfolding topspace_def by auto
next
  have generate_topology_on S (⋃S)
   using generate_topology_on.UN[OF generate_topology_on.Basis, of S S] by simp
  then show (⋃S) ⊆ topspace (topology_generated_by S)
    unfolding topspace_def using openin_topology_generated_by_iff by auto
qed

lemma topology_generated_by_Basis:
  s ∈ S ⟹ openin (topology_generated_by S) s
```

by (*simp only*: *openin_topology_generated_by_iff*, *auto simp*: *generate_topology_on.Basis*)

lemma *generate_topology_on_Inter*:
$[\![finite\ \mathcal{F};\ \bigwedge K.\ K \in \mathcal{F} \Longrightarrow generate\_topology\_on\ \mathcal{S}\ K;\ \mathcal{F} \neq \{\}]\!] \Longrightarrow generate\_topology\_on\ \mathcal{S}\ (\bigcap \mathcal{F})$
by (*induction* $\mathcal{F}$ *rule*: *finite_induct*; *force intro*: *generate_topology_on.intros*)

2.2.23 Topology bases and sub-bases

lemma *istopology_base_alt*:
$istopology\ (arbitrary\ union\_of\ P) \longleftrightarrow$
$(\forall S\ T.\ (arbitrary\ union\_of\ P)\ S \wedge (arbitrary\ union\_of\ P)\ T$
$\longrightarrow (arbitrary\ union\_of\ P)\ (S \cap T))$
by (*simp add*: *istopology_def*) (*blast intro*: *arbitrary_union_of_Union*)

lemma *istopology_base_eq*:
$istopology\ (arbitrary\ union\_of\ P) \longleftrightarrow$
$(\forall S\ T.\ P\ S \wedge P\ T \longrightarrow (arbitrary\ union\_of\ P)\ (S \cap T))$
by (*simp add*: *istopology_base_alt arbitrary_union_of_Int_eq*)

lemma *istopology_base*:
$(\bigwedge S\ T.\ [\![P\ S;\ P\ T]\!] \Longrightarrow P(S \cap T)) \Longrightarrow istopology\ (arbitrary\ union\_of\ P)$
by (*simp add*: *arbitrary_def istopology_base_eq union_of_inc*)

lemma *openin_topology_base_unique*:
$openin\ X = arbitrary\ union\_of\ P \longleftrightarrow$
$(\forall V.\ P\ V \longrightarrow openin\ X\ V) \wedge (\forall U\ x.\ openin\ X\ U \wedge x \in U \longrightarrow (\exists V.\ P\ V \wedge x \in V \wedge V \subseteq U))$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*auto simp*: *union_of_def arbitrary_def*)
next
assume *R*: *?rhs*
then have $*$: $\exists \mathcal{U} \subseteq Collect\ P.\ \bigcup \mathcal{U} = S$ **if** *openin X S* **for** *S*
using *that* **by** (*rule_tac* $x = \{V.\ P\ V \wedge V \subseteq S\}$ **in** *exI*) *fastforce*
from *R* **show** *?lhs*
by (*fastforce simp add*: *union_of_def arbitrary_def intro*: $*$)
qed

lemma *topology_base_unique*:
assumes $\bigwedge S.\ P\ S \Longrightarrow openin\ X\ S$
$\bigwedge U\ x.\ [\![openin\ X\ U;\ x \in U]\!] \Longrightarrow \exists B.\ P\ B \wedge x \in B \wedge B \subseteq U$
shows $topology\ (arbitrary\ union\_of\ P) = X$
proof −
have $X = topology\ (openin\ X)$
by (*simp add*: *openin_inverse*)
also from *assms* **have** $openin\ X = arbitrary\ union\_of\ P$

by (*subst openin_topology_base_unique*) *auto*
finally show *?thesis* **..**
qed

lemma *topology_bases_eq_aux*:
$[\![$(*arbitrary union_of P*) *S*;
$\bigwedge U\, x.\ [\![P\ U;\ x \in U]\!] \Longrightarrow \exists\, V.\ Q\ V \wedge x \in V \wedge V \subseteq U]\!]$
$\Longrightarrow$ (*arbitrary union_of Q*) *S*
by (*metis arbitrary_union_of_alt arbitrary_union_of_idempot*)

lemma *topology_bases_eq*:
$[\![\bigwedge U\, x.\ [\![P\ U;\ x \in U]\!] \Longrightarrow \exists\, V.\ Q\ V \wedge x \in V \wedge V \subseteq U;$
$\bigwedge V\, x.\ [\![Q\ V;\ x \in V]\!] \Longrightarrow \exists\, U.\ P\ U \wedge x \in U \wedge U \subseteq V]\!]$
$\Longrightarrow$ *topology* (*arbitrary union_of P*) =
topology (*arbitrary union_of Q*)
by (*fastforce intro*: *arg_cong* [**where** *f*=*topology*] *elim*: *topology_bases_eq_aux*)

lemma *istopology_subbase*:
istopology (*arbitrary union_of* (*finite intersection_of P relative_to S*))
by (*simp add*: *finite_intersection_of_Int istopology_base relative_to_Int*)

lemma *openin_subbase*:
openin (*topology* (*arbitrary union_of* (*finite intersection_of B relative_to U*))) *S*
$\longleftrightarrow$ (*arbitrary union_of* (*finite intersection_of B relative_to U*)) *S*
by (*simp add*: *istopology_subbase topology_inverse'*)

lemma *topspace_subbase* [*simp*]:
topspace(*topology* (*arbitrary union_of* (*finite intersection_of B relative_to U*))) =
U (**is** *?lhs* = _)
proof
show *?lhs* $\subseteq$ *U*
by (*metis arbitrary_union_of_relative_to openin_subbase openin_topspace relative_to_imp_subset*)
show *U* $\subseteq$ *?lhs*
by (*metis arbitrary_union_of_inc finite_intersection_of_empty inf.orderE istopology_subbase*
openin_subset relative_to_inc subset_UNIV topology_inverse')
qed

lemma *minimal_topology_subbase*:
assumes *X*: $\bigwedge S.\ P\ S \Longrightarrow$ *openin X S* **and** *openin X U*
and *S*: *openin*(*topology*(*arbitrary union_of* (*finite intersection_of P relative_to U*))) *S*
shows *openin X S*
proof −
have (*arbitrary union_of* (*finite intersection_of P relative_to U*)) *S*
using *S openin_subbase* **by** *blast*
with *X* ‹*openin X U*› **show** *?thesis*
by (*force simp add*: *union_of_def intersection_of_def relative_to_def intro*: *openin_Int_Inter*)

qed

lemma *istopology_subbase_UNIV*:
  istopology (*arbitrary union_of* (*finite intersection_of P*))
  by (*simp add*: *istopology_base finite_intersection_of_Int*)

lemma *generate_topology_on_eq*:
  generate_topology_on S = *arbitrary union_of finite′ intersection_of* ($\lambda x.\ x \in S$)
(**is** *?lhs* = *?rhs*)
proof (*intro ext iffI*)
  fix A
  assume *?lhs* A
  then show *?rhs* A
  proof *induction*
    case (*Int a b*)
    then show *?case*
      by (*metis* (*mono_tags*, *lifting*) *istopology_base_alt finite′_intersection_of_Int istopology_base*)
  next
    case (*UN K*)
    then show *?case*
      by (*simp add*: *arbitrary_union_of_Union*)
  next
    case (*Basis s*)
    then show *?case*
      by (*simp add*: *Sup_upper arbitrary_union_of_inc finite′_intersection_of_inc relative_to_subset*)
  qed *auto*
next
  fix A
  assume *?rhs* A
  then obtain $\mathcal{U}$ **where** $\mathcal{U}$: $\bigwedge T.\ T \in \mathcal{U} \Longrightarrow \exists \mathcal{F}.\ finite'\ \mathcal{F} \wedge \mathcal{F} \subseteq S \wedge \bigcap \mathcal{F} = T$ **and** *eq*: $A = \bigcup \mathcal{U}$
    unfolding *union_of_def intersection_of_def* **by** *auto*
  show *?lhs* A
    unfolding *eq*
  proof (*rule generate_topology_on.UN*)
    fix T
    assume $T \in \mathcal{U}$
    with $\mathcal{U}$ **obtain** $\mathcal{F}$ **where** *finite′* $\mathcal{F}$ $\mathcal{F} \subseteq S$ $\bigcap \mathcal{F} = T$
      by *blast*
    have *generate_topology_on* S ($\bigcap \mathcal{F}$)
    proof (*rule generate_topology_on_Inter*)
      show *finite* $\mathcal{F}$ $\mathcal{F} \neq \{\}$
        by (*auto simp*: ‹*finite′* $\mathcal{F}$›)
      show $\bigwedge K.\ K \in \mathcal{F} \Longrightarrow$ *generate_topology_on* S K
        by (*metis* ‹$\mathcal{F} \subseteq S$› *generate_topology_on.simps subset_iff*)
    qed

then show *generate_topology_on S T*
using ‹$\bigcap\mathcal{F}$ = *T*› **by** *blast*
qed
qed

lemma *continuous_on_generated_topo_iff*:
continuous_map T1 (*topology_generated_by S*) *f* ⟷
((∀ *U*. *U* ∈ *S* ⟶ *openin T1* (*f*−‘*U* ∩ *topspace*(*T1*))) ∧ (*f*‘(*topspace T1*) ⊆ (⋃ *S*)))
unfolding *continuous_map_alt topology_generated_by_topspace*
proof (*auto simp add*: *topology_generated_by_Basis*)
assume *H*: ∀ *U*. *U* ∈ *S* ⟶ *openin T1* (*f* −‘ *U* ∩ *topspace T1*)
fix *U* **assume** *openin* (*topology_generated_by S*) *U*
then have *generate_topology_on S U* **by** (*rule openin_topology_generated_by*)
then show *openin T1* (*f* −‘ *U* ∩ *topspace T1*)
proof (*induct*)
fix *a b*
assume *H*: *openin T1* (*f* −‘ *a* ∩ *topspace T1*) *openin T1* (*f* −‘ *b* ∩ *topspace T1*)
have *f* −‘ (*a* ∩ *b*) ∩ *topspace T1* = (*f*−‘*a* ∩ *topspace T1*) ∩ (*f*−‘*b* ∩ *topspace T1*)
by *auto*
then show *openin T1* (*f* −‘ (*a* ∩ *b*) ∩ *topspace T1*) **using** *H* **by** *auto*
next
fix *K*
assume *H*: *openin T1* (*f* −‘ *k* ∩ *topspace T1*) **if** *k*∈ *K* **for** *k*
define *L* **where** *L* = {*f* −‘ *k* ∩ *topspace T1*|*k*. *k* ∈ *K*}
have ∗: *openin T1 l* **if** *l* ∈*L* **for** *l* **using** *that H* **unfolding** *L_def* **by** *auto*
have *openin T1* (⋃*L*) **using** *openin_Union*[*OF* ∗] **by** *simp*
moreover have (⋃*L*) = (*f* −‘ ⋃*K* ∩ *topspace T1*) **unfolding** *L_def* **by** *auto*
ultimately show *openin T1* (*f* −‘ ⋃*K* ∩ *topspace T1*) **by** *simp*
qed (*auto simp add*: *H*)
qed

lemma *continuous_on_generated_topo*:
assumes ⋀*U*. *U* ∈*S* ⟹ *openin T1* (*f*−‘*U* ∩ *topspace*(*T1*))
f‘(*topspace T1*) ⊆ (⋃ *S*)
shows *continuous_map T1* (*topology_generated_by S*) *f*
using *assms continuous_on_generated_topo_iff* **by** *blast*

2.2.24 Pullback topology

Pulling back a topology by map gives again a topology. *subtopology* is a special case of this notion, pulling back by the identity. We introduce the general notion as we will need it to define the strong operator topology on the space of continuous linear operators, by pulling back the product topology on the space of all functions.

pullback_topology A f T is the pullback of the topology *T* by the map *f* on the set A.

```
definition pullback_topology::('a set) ⇒ ('a ⇒ 'b) ⇒ ('b topology) ⇒ ('a topology)
  where pullback_topology A f T = topology (λS. ∃ U. openin T U ∧ S = f−‘U
∩ A)

lemma istopology_pullback_topology:
  istopology (λS. ∃ U. openin T U ∧ S = f−‘U ∩ A)
  unfolding istopology_def proof (auto)
  fix K assume ∀ S∈K. ∃ U. openin T U ∧ S = f −‘ U ∩ A
  then have ∃ U. ∀ S∈K. openin T (U S) ∧ S = f−‘(U S) ∩ A
    by (rule bchoice)
  then obtain U where U: ∀ S∈K. openin T (U S) ∧ S = f−‘(U S) ∩ A
    by blast
  define V where V = (⋃ S∈K. U S)
  have openin T V ⋃ K = f −‘ V ∩ A unfolding V_def using U by auto
  then show ∃ V. openin T V ∧ ⋃ K = f −‘ V ∩ A by auto
qed

lemma openin_pullback_topology:
  openin (pullback_topology A f T) S ⟷ (∃ U. openin T U ∧ S = f−‘U ∩ A)
unfolding pullback_topology_def topology_inverse'[OF istopology_pullback_topology]
by auto

lemma topspace_pullback_topology:
  topspace (pullback_topology A f T) = f−‘(topspace T) ∩ A
by (auto simp add: topspace_def openin_pullback_topology)

proposition continuous_map_pullback [intro]:
  assumes continuous_map T1 T2 g
  shows continuous_map (pullback_topology A f T1) T2 (g o f)
unfolding continuous_map_alt
proof (auto)
  fix U::'b set assume openin T2 U
  then have openin T1 (g−‘U ∩ topspace T1)
    using assms unfolding continuous_map_alt by auto
  have (g o f)−‘U ∩ topspace (pullback_topology A f T1) = (g o f)−‘U ∩ A ∩
f−‘(topspace T1)
    unfolding topspace_pullback_topology by auto
  also have ... = f−‘(g−‘U ∩ topspace T1) ∩ A
    by auto
  also have openin (pullback_topology A f T1) (...)
    unfolding openin_pullback_topology using ‹openin T1 (g−‘U ∩ topspace T1)›
by auto
  finally show openin (pullback_topology A f T1) ((g ∘ f) −‘ U ∩ topspace
(pullback_topology A f T1))
    by auto
next
  fix x assume x ∈ topspace (pullback_topology A f T1)
  then have f x ∈ topspace T1
```

unfolding *topspace_pullback_topology* **by** *auto*
then show *g (f x) ∈ topspace T2*
using *assms* **unfolding** *continuous_map_def* **by** *auto*
qed

proposition *continuous_map_pullback'* [*intro*]:
assumes *continuous_map T1 T2 (f o g) topspace T1 ⊆ g−'A*
shows *continuous_map T1 (pullback_topology A f T2) g*
unfolding *continuous_map_alt*
proof (*auto*)
fix *U* **assume** *openin (pullback_topology A f T2) U*
then have *∃ V. openin T2 V ∧ U = f−'V ∩ A*
unfolding *openin_pullback_topology* **by** *auto*
then obtain *V* **where** *openin T2 V U = f−'V ∩ A*
by *blast*
then have *g −' U ∩ topspace T1 = g−'(f−'V ∩ A) ∩ topspace T1*
by *blast*
also have ... = *(f o g)−'V ∩ (g−'A ∩ topspace T1)*
by *auto*
also have ... = *(f o g)−'V ∩ topspace T1*
using *assms(2)* **by** *auto*
also have *openin T1 (...)*
using *assms(1)* ‹*openin T2 V*› **by** *auto*
finally show *openin T1 (g −' U ∩ topspace T1)* **by** *simp*
next
fix *x* **assume** *x ∈ topspace T1*
have *(f o g) x ∈ topspace T2*
using *assms(1)* ‹*x ∈ topspace T1*› **unfolding** *continuous_map_def* **by** *auto*
then have *g x ∈ f−'(topspace T2)*
unfolding *comp_def* **by** *blast*
moreover have *g x ∈ A* **using** *assms(2)* ‹*x ∈ topspace T1*› **by** *blast*
ultimately show *g x ∈ topspace (pullback_topology A f T2)*
unfolding *topspace_pullback_topology* **by** *blast*
qed

2.2.25 Proper maps (not a priori assumed continuous)

definition *proper_map*
where
proper_map X Y f ≡
closed_map X Y f ∧ (∀ y ∈ topspace Y. compactin X {x ∈ topspace X. f x = y})

lemma *proper_imp_closed_map*:
proper_map X Y f ⟹ closed_map X Y f
by (*simp add*: *proper_map_def*)

lemma *proper_map_imp_subset_topspace*:
proper_map X Y f ⟹ f ' (topspace X) ⊆ topspace Y

```
  by (simp add: closed_map_imp_subset_topspace proper_map_def)

lemma closed_injective_imp_proper_map:
  assumes f: closed_map X Y f and inj: inj_on f (topspace X)
  shows proper_map X Y f
  unfolding proper_map_def
proof (clarsimp simp: f)
  show compactin X {x ∈ topspace X. f x = y}
    if y ∈ topspace Y for y
  proof −
    have {x ∈ topspace X. f x = y} = {} ∨ (∃ a ∈ topspace X. {x ∈ topspace X.
f x = y} = {a})
      using inj_on_eq_iff [OF inj] by auto
    then show ?thesis
      using that by (metis (no_types, lifting) compactin_empty compactin_sing)
  qed
qed

lemma injective_imp_proper_eq_closed_map:
  inj_on f (topspace X) ⟹ (proper_map X Y f ⟷ closed_map X Y f)
  using closed_injective_imp_proper_map proper_imp_closed_map by blast

lemma homeomorphic_imp_proper_map:
  homeomorphic_map X Y f ⟹ proper_map X Y f
 by (simp add: closed_injective_imp_proper_map homeomorphic_eq_everything_map)

lemma compactin_proper_map_preimage:
  assumes f: proper_map X Y f and compactin Y K
  shows compactin X {x. x ∈ topspace X ∧ f x ∈ K}
proof −
  have f ` (topspace X) ⊆ topspace Y
    by (simp add: f proper_map_imp_subset_topspace)
  have ∗: ⋀y. y ∈ topspace Y ⟹ compactin X {x ∈ topspace X. f x = y}
    using f by (auto simp: proper_map_def)
  show ?thesis
    unfolding compactin_def
  proof clarsimp
    show ∃ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ∧ {x ∈ topspace X. f x ∈ K} ⊆ ⋃ℱ
      if 𝒰: ∀ U∈𝒰. openin X U and sub: {x ∈ topspace X. f x ∈ K} ⊆ ⋃𝒰
      for 𝒰
    proof −
      have ∀ y ∈ K. ∃𝒱. finite 𝒱 ∧ 𝒱 ⊆ 𝒰  ∧ {x ∈ topspace X. f x = y} ⊆ ⋃𝒱
      proof
        fix y
        assume y ∈ K
        then have compactin X {x ∈ topspace X. f x = y}
          by (metis ∗ ‹compactin Y K› compactin_subspace subsetD)
        with ‹y ∈ K› show ∃𝒱. finite 𝒱 ∧ 𝒱 ⊆ 𝒰  ∧ {x ∈ topspace X. f x = y} ⊆
⋃𝒱
```

unfolding *compactin_def* **using** $\mathcal{U}$ *sub* **by** *fastforce*
qed
then obtain $\mathcal{V}$ **where** $\mathcal{V}$: $\bigwedge y.\ y \in K \Longrightarrow$ *finite* $(\mathcal{V}\ y) \wedge \mathcal{V}\ y \subseteq \mathcal{U} \ \wedge \{x \in$ *topspace* $X.\ f\,x = y\} \subseteq \bigcup(\mathcal{V}\ y)$
by (*metis* (*full_types*))
define F **where** $F \equiv \lambda y.$ *topspace* $Y - f$ ' (*topspace* $X - \bigcup(\mathcal{V}\ y))$
have $\exists\mathcal{F}.$ *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq F$ ' $K \wedge K \subseteq \bigcup\mathcal{F}$
proof (*rule compactinD* [*OF* ‹*compactin* Y K›])
have $\bigwedge x.\ x \in K \Longrightarrow$ *closedin* Y (f ' (*topspace* $X - \bigcup(\mathcal{V}\ x)))$
using f **unfolding** *proper_map_def closed_map_def*
by (*meson* $\mathcal{U}$ $\mathcal{V}$ *openin_Union openin_closedin_eq subsetD*)
then show *openin* Y U **if** $U \in F$ ' K **for** U
using *that* **by** (*auto simp*: *F_def*)
show $K \subseteq \bigcup(F$ ' $K)$
using $\mathcal{V}$ ‹*compactin* Y K› **unfolding** *F_def compactin_def* **by** *fastforce*
qed
then obtain J **where** *finite* J $J \subseteq K$ **and** J: $K \subseteq \bigcup(F$ ' $J)$
by (*auto simp*: *ex_finite_subset_image*)
show *?thesis*
unfolding *F_def*
proof (*intro exI conjI*)
show *finite* $(\bigcup(\mathcal{V}$ ' $J))$
using $\mathcal{V}$ ‹$J \subseteq K$› ‹*finite* J› **by** *blast*
show $\bigcup(\mathcal{V}$ ' $J) \subseteq \mathcal{U}$
using $\mathcal{V}$ ‹$J \subseteq K$› **by** *blast*
show $\{x \in$ *topspace* $X.\ f\,x \in K\} \subseteq \bigcup(\bigcup(\mathcal{V}$ ' $J))$
using J ‹$J \subseteq K$› **unfolding** *F_def* **by** *auto*
qed
qed
qed
qed

lemma *compact_space_proper_map_preimage*:
assumes f: *proper_map* X Y f **and** *fim*: f ' (*topspace* X) = *topspace* Y **and** *compact_space* Y
shows *compact_space* X
proof –
have *eq*: *topspace* $X = \{x \in$ *topspace* $X.\ f\,x \in$ *topspace* $Y\}$
using *fim* **by** *blast*
moreover have *compactin* Y (*topspace* Y)
using ‹*compact_space* Y› *compact_space_def* **by** *auto*
ultimately show *?thesis*
unfolding *compact_space_def*
using *eq* f *compactin_proper_map_preimage* **by** *fastforce*
qed

lemma *proper_map_alt*:
proper_map X Y f $\longleftrightarrow$

```
    closed_map X Y f ∧ (∀K. compactin Y K ⟶ compactin X {x. x ∈ topspace
X ∧ f x ∈ K})
  proof (intro iffI conjI allI impI)
  show compactin X {x ∈ topspace X. f x ∈ K}
    if proper_map X Y f and compactin Y K for K
    using that by (simp add: compactin_proper_map_preimage)
  show proper_map X Y f
    if f: closed_map X Y f ∧ (∀K. compactin Y K ⟶ compactin X {x ∈ topspace
X. f x ∈ K})
  proof −
    have compactin X {x ∈ topspace X. f x = y} if y ∈ topspace Y for y
    proof −
      have compactin X {x ∈ topspace X. f x ∈ {y}}
        using f compactin_sing that by fastforce
      then show ?thesis
        by auto
    qed
    with f show ?thesis
      by (auto simp: proper_map_def)
  qed
qed (simp add: proper_imp_closed_map)

lemma proper_map_on_empty:
   topspace X = {} ⟹ proper_map X Y f
  by (auto simp: proper_map_def closed_map_on_empty)

lemma proper_map_id [simp]:
   proper_map X X id
proof (clarsimp simp: proper_map_alt closed_map_id)
  fix K
  assume K: compactin X K
  then have {a ∈ topspace X. a ∈ K} = K
    by (simp add: compactin_subspace subset_antisym subset_iff)
  then show compactin X {a ∈ topspace X. a ∈ K}
    using K by auto
qed

lemma proper_map_compose:
  assumes proper_map X Y f proper_map Y Z g
  shows proper_map X Z (g ∘ f)
proof −
  have closed_map X Y f and f: ⋀K. compactin Y K ⟹ compactin X {x ∈
topspace X. f x ∈ K}
    and closed_map Y Z g and g: ⋀K. compactin Z K ⟹ compactin Y {x ∈
topspace Y. g x ∈ K}
    using assms by (auto simp: proper_map_alt)
  show ?thesis
    unfolding proper_map_alt
  proof (intro conjI allI impI)
```

show *closed_map X Z* (*g* ∘ *f*)
using ‹*closed_map X Y f*› ‹*closed_map Y Z g*› *closed_map_compose* **by** *blast*
have {*x* ∈ *topspace X*. *g* (*f x*) ∈ *K*} = {*x* ∈ *topspace X*. *f x* ∈ {*b* ∈ *topspace Y*. *g b* ∈ *K*}} **for** *K*
using ‹*closed_map X Y f*› *closed_map_imp_subset_topspace* **by** *blast*
then show *compactin X* {*x* ∈ *topspace X*. (*g* ∘ *f*) *x* ∈ *K*}
if *compactin Z K* **for** *K*
using *f* [*OF g* [*OF that*]] **by** *auto*
qed
qed

lemma *proper_map_const*:
proper_map X Y (λ*x*. *c*) ⟷ *compact_space X* ∧ (*topspace X* = {} ∨ *closedin Y* {*c*})
proof (*cases topspace X* = {})
case *True*
then show *?thesis*
by (*simp add*: *compact_space_topspace_empty proper_map_on_empty*)
next
case *False*
have ∗: *compactin X* {*x* ∈ *topspace X*. *c* = *y*} **if** *compact_space X* **for** *y*
proof (*cases c* = *y*)
case *True*
then show *?thesis*
using *compact_space_def* ‹*compact_space X*› **by** *auto*
qed *auto*
then show *?thesis*
using *closed_compactin closedin_subset*
by (*force simp*: *False proper_map_def closed_map_const compact_space_def*)
qed

lemma *proper_map_inclusion*:
s ⊆ *topspace X*
⟹ *proper_map* (*subtopology X s*) *X id* ⟷ *closedin X s* ∧ (∀ *k*. *compactin X k* ⟶ *compactin X* (*s* ∩ *k*))
by (*auto simp*: *proper_map_alt closed_map_inclusion_eq inf.absorb_iff2 Collect_conj_eq compactin_subtopology intro*: *closed_Int_compactin*)

2.2.26 Perfect maps (proper, continuous and surjective)

definition *perfect_map*
where *perfect_map X Y f* ≡ *continuous_map X Y f* ∧ *proper_map X Y f* ∧ *f* ‘ (*topspace X*) = *topspace Y*

lemma *homeomorphic_imp_perfect_map*:
homeomorphic_map X Y f ⟹ *perfect_map X Y f*
by (*simp add*: *homeomorphic_eq_everything_map homeomorphic_imp_proper_map perfect_map_def*)

lemma *perfect_imp_quotient_map*:
perfect_map X Y f ⟹ *quotient_map X Y f*
by (*simp add*: *continuous_closed_imp_quotient_map perfect_map_def proper_map_def*)

lemma *homeomorphic_eq_injective_perfect_map*:
homeomorphic_map X Y f ⟷ *perfect_map X Y f* ∧ *inj_on f* (*topspace X*)
using *homeomorphic_imp_perfect_map homeomorphic_map_def perfect_imp_quotient_map*
by *blast*

lemma *perfect_injective_eq_homeomorphic_map*:
perfect_map X Y f ∧ *inj_on f* (*topspace X*) ⟷ *homeomorphic_map X Y f*
by (*simp add*: *homeomorphic_eq_injective_perfect_map*)

lemma *perfect_map_id* [*simp*]: *perfect_map X X id*
by (*simp add*: *homeomorphic_imp_perfect_map*)

lemma *perfect_map_compose*:
⟦*perfect_map X Y f*; *perfect_map Y Z g*⟧ ⟹ *perfect_map X Z* (*g* ∘ *f*)
by (*meson continuous_map_compose perfect_imp_quotient_map perfect_map_def proper_map_compose quotient_map_compose_eq quotient_map_def*)

lemma *perfect_imp_continuous_map*:
perfect_map X Y f ⟹ *continuous_map X Y f*
using *perfect_map_def* **by** *blast*

lemma *perfect_imp_closed_map*:
perfect_map X Y f ⟹ *closed_map X Y f*
by (*simp add*: *perfect_map_def proper_map_def*)

lemma *perfect_imp_proper_map*:
perfect_map X Y f ⟹ *proper_map X Y f*
by (*simp add*: *perfect_map_def*)

lemma *perfect_imp_surjective_map*:
perfect_map X Y f ⟹ *f* ‘ (*topspace X*) = *topspace Y*
by (*simp add*: *perfect_map_def*)

end

2.3 Abstract Topology 2

theory *Abstract_Topology_2*
imports
Elementary_Topology
Abstract_Topology
HOL−Library.Indicator_Function
begin

Combination of Elementary and Abstract Topology

```
lemma approachable_lt_le2:
  (∃(d::real) > 0. ∀x. Q x ⟶ f x < d ⟶ P x) ⟷ (∃d>0. ∀x. f x ≤ d ⟶
Q x ⟶ P x)
 apply auto
 apply (rule_tac x=d/2 in exI, auto)
 done

lemma triangle_lemma:
 fixes x y z :: real
 assumes x: 0 ≤ x
  and y: 0 ≤ y
  and z: 0 ≤ z
  and xy: x^2 ≤ y^2 + z^2
 shows x ≤ y + z
proof -
 have y^2 + z^2 ≤ y^2 + 2 * y * z + z^2
  using z y by simp
 with xy have th: x^2 ≤ (y + z)^2
  by (simp add: power2_eq_square field_simps)
 from y z have yz: y + z ≥ 0
  by arith
 from power2_le_imp_le[OF th yz] show ?thesis .
qed

lemma isCont_indicator:
 fixes x :: 'a::t2_space
 shows isCont (indicator A :: 'a ⇒ real) x = (x ∉ frontier A)
proof auto
 fix x
 assume cts_at: isCont (indicator A :: 'a ⇒ real) x and fr: x ∈ frontier A
 with continuous_at_open have 1: ∀V::real set. open V ∧ indicator A x ∈ V ⟶
  (∃U::'a set. open U ∧ x ∈ U ∧ (∀y∈U. indicator A y ∈ V)) by auto
 show False
 proof (cases x ∈ A)
  assume x: x ∈ A
  hence indicator A x ∈ ({0<..<2} :: real set) by simp
  hence ∃U. open U ∧ x ∈ U ∧ (∀y∈U. indicator A y ∈ ({0<..<2} :: real set))
   using 1 open_greaterThanLessThan by blast
  then guess U .. note U = this
  hence ∀y∈U. indicator A y > (0::real)
   unfolding greaterThanLessThan_def by auto
  hence U ⊆ A using indicator_eq_0_iff by force
  hence x ∈ interior A using U interiorI by auto
  thus ?thesis using fr unfolding frontier_def by simp
 next
  assume x: x ∉ A
  hence indicator A x ∈ ({−1<..<1} :: real set) by simp
  hence ∃U. open U ∧ x ∈ U ∧ (∀y∈U. indicator A y ∈ ({−1<..<1} :: real
set))
```

```
      using 1 open_greaterThanLessThan by blast
    then guess U .. note U = this
    hence ∀y∈U. indicator A y < (1::real)
      unfolding greaterThanLessThan_def by auto
    hence U ⊆ −A by auto
    hence x ∈ interior (−A) using U interiorI by auto
    thus ?thesis using fr interior_complement unfolding frontier_def by auto
  qed
next
  assume nfr: x ∉ frontier A
  hence x ∈ interior A ∨ x ∈ interior (−A)
    by (auto simp: frontier_def closure_interior)
  thus isCont ((indicator A)::'a ⇒ real) x
  proof
    assume int: x ∈ interior A
    then obtain U where U: open U x ∈ U U ⊆ A unfolding interior_def by auto
    hence ∀y∈U. indicator A y = (1::real) unfolding indicator_def by auto
    hence continuous_on U (indicator A) by (simp add: indicator_eq_1_iff)
    thus ?thesis using U continuous_on_eq_continuous_at by auto
  next
    assume ext: x ∈ interior (−A)
    then obtain U where U: open U x ∈ U U ⊆ −A unfolding interior_def by auto
    then have continuous_on U (indicator A)
      using continuous_on_topological by (auto simp: subset_iff)
    thus ?thesis using U continuous_on_eq_continuous_at by auto
  qed
qed

lemma closedin_limpt:
  closedin (top_of_set T) S ⟷ S ⊆ T ∧ (∀x. x islimpt S ∧ x ∈ T ⟶ x ∈ S)
  apply (simp add: closedin_closed, safe)
   apply (simp add: closed_limpt islimpt_subset)
  apply (rule_tac x=closure S in exI, simp)
  apply (force simp: closure_def)
  done

lemma closedin_closed_eq: closed S ⟹ closedin (top_of_set S) T ⟷ closed T ∧ T ⊆ S
  by (meson closedin_limpt closed_subset closedin_closed_trans)

lemma connected_closed_set:
   closed S
   ⟹ connected S ⟷ (∄A B. closed A ∧ closed B ∧ A ≠ {} ∧ B ≠ {} ∧ A ∪ B = S ∧ A ∩ B = {})
  unfolding connected_closedin_eq closedin_closed_eq connected_closedin_eq by blast
```

If a connnected set is written as the union of two nonempty closed sets, then these sets have to intersect.

lemma *connected_as_closed_union*:
 assumes *connected* C $C = A \cup B$ *closed* A *closed* B $A \neq \{\}$ $B \neq \{\}$
 shows $A \cap B \neq \{\}$
by (*metis assms closed_Un connected_closed_set*)

lemma *closedin_subset_trans*:
 closedin (*top_of_set* U) $S \Longrightarrow S \subseteq T \Longrightarrow T \subseteq U \Longrightarrow$
  closedin (*top_of_set* T) S
 by (*meson closedin_limpt subset_iff*)

lemma *openin_subset_trans*:
 openin (*top_of_set* U) $S \Longrightarrow S \subseteq T \Longrightarrow T \subseteq U \Longrightarrow$
  openin (*top_of_set* T) S
 by (*auto simp*: *openin_open*)

lemma *closedin_compact*:
 ⟦*compact* S; *closedin* (*top_of_set* S) T⟧ $\Longrightarrow$ *compact* T
by (*metis closedin_closed compact_Int_closed*)

lemma *closedin_compact_eq*:
 fixes S :: *'a*::*t2_space set*
 shows
  compact S
   $\Longrightarrow$ (*closedin* (*top_of_set* S) $T \longleftrightarrow$
    compact $T \wedge T \subseteq S$)
by (*metis closedin_imp_subset closedin_compact closed_subset compact_imp_closed*)

2.3.1 Closure

lemma *euclidean_closure_of* [*simp*]: *euclidean closure_of* S = *closure* S
 by (*auto simp*: *closure_of_def closure_def islimpt_def*)

lemma *closure_openin_Int_closure*:
 assumes *ope*: *openin* (*top_of_set* U) S **and** $T \subseteq U$
 shows *closure*($S \cap$ *closure* T) = *closure*($S \cap T$)
proof
 obtain V **where** *open* V **and** S: $S = U \cap V$
  using *ope* **using** *openin_open* **by** *metis*
 show *closure* ($S \cap$ *closure* T) $\subseteq$ *closure* ($S \cap T$)
  proof (*clarsimp simp*: S)
   fix x
   assume $x \in$ *closure* ($U \cap V \cap$ *closure* T)
   then have $V \cap$ *closure* $T \subseteq A \Longrightarrow x \in$ *closure* A **for** A
     by (*metis closure_mono subsetD inf.coboundedI2 inf_assoc*)
   then have $x \in$ *closure* ($T \cap V$)
    by (*metis* ‹*open* V› *closure_closure inf_commute open_Int_closure_subset*)
   then show $x \in$ *closure* ($U \cap V \cap T$)
    by (*metis* ‹$T \subseteq U$› *inf.absorb_iff2 inf_assoc inf_commute*)

qed
next
show *closure* $(S \cap T) \subseteq$ *closure* $(S \cap$ *closure* $T)$
by (*meson Int_mono closure_mono closure_subset order_refl*)
qed

corollary *infinite_openin*:
fixes S :: *'a* :: *t1_space set*
shows ⟦*openin* (*top_of_set* U) S; $x \in S$; x *islimpt* U⟧ $\Longrightarrow$ *infinite* S
by (*clarsimp simp add*: *openin_open islimpt_eq_acc_point inf_commute*)

lemma *closure_Int_ballI*:
assumes $\bigwedge U$. ⟦*openin* (*top_of_set* S) U; $U \neq \{\}$⟧ $\Longrightarrow T \cap U \neq \{\}$
shows $S \subseteq$ *closure* T
proof (*clarsimp simp*: *closure_iff_nhds_not_empty*)
fix x **and** A **and** V
assume $x \in S$ $V \subseteq A$ *open* V $x \in V$ $T \cap A = \{\}$
then have *openin* (*top_of_set* S) $(A \cap V \cap S)$
by (*auto simp*: *openin_open intro*!: *exI*[**where** x=V])
moreover have $A \cap V \cap S \neq \{\}$ **using** ‹$x \in V$› ‹$V \subseteq A$› ‹$x \in S$›
by *auto*
ultimately have $T \cap (A \cap V \cap S) \neq \{\}$
by (*rule assms*)
with ‹$T \cap A = \{\}$› **show** *False* **by** *auto*
qed

2.3.2 Frontier

lemma *euclidean_interior_of* [*simp*]: *euclidean interior_of* S = *interior* S
by (*auto simp*: *interior_of_def interior_def*)

lemma *euclidean_frontier_of* [*simp*]: *euclidean frontier_of* S = *frontier* S
by (*auto simp*: *frontier_of_def frontier_def*)

lemma *connected_Int_frontier*:
⟦*connected* s; $s \cap t \neq \{\}$; $s - t \neq \{\}$⟧ $\Longrightarrow$ ($s \cap$ *frontier* $t \neq \{\}$)
apply (*simp add*: *frontier_interiors connected_openin*, *safe*)
apply (*drule_tac* x=$s \cap$ *interior* t **in** *spec*, *safe*)
apply (*drule_tac* [2] x=$s \cap$ *interior* $(-t)$ **in** *spec*)
apply (*auto simp*: *disjoint_eq_subset_Compl dest*: *interior_subset* [*THEN subsetD*])
done

2.3.3 Compactness

lemma *openin_delete*:
fixes a :: *'a* :: *t1_space*
shows *openin* (*top_of_set* u) s
$\Longrightarrow$ *openin* (*top_of_set* u) $(s - \{a\})$
by (*metis Int_Diff open_delete openin_open*)

lemma *compact_eq_openin_cover*:
  compact S ⟷
    (∀ *C*. (∀ *c*∈*C*. *openin* (*top_of_set S*) *c*) ∧ *S* ⊆ ⋃ *C* ⟶
      (∃ *D*⊆*C*. *finite D* ∧ *S* ⊆ ⋃ *D*))
proof *safe*
  fix *C*
  assume *compact S* **and** ∀ *c*∈*C*. *openin* (*top_of_set S*) *c* **and** *S* ⊆ ⋃ *C*
  then have ∀ *c*∈{*T*. *open T* ∧ *S* ∩ *T* ∈ *C*}. *open c* **and** *S* ⊆ ⋃{*T*. *open T* ∧ *S* ∩ *T* ∈ *C*}
    unfolding *openin_open* **by** *force+*
  with ‹*compact S*› **obtain** *D* **where** *D* ⊆ {*T*. *open T* ∧ *S* ∩ *T* ∈ *C*} **and** *finite D* **and** *S* ⊆ ⋃ *D*
    by (*meson compactE*)
  then have *image* (λ*T*. *S* ∩ *T*) *D* ⊆ *C* ∧ *finite* (*image* (λ*T*. *S* ∩ *T*) *D*) ∧ *S* ⊆ ⋃(*image* (λ*T*. *S* ∩ *T*) *D*)
    by *auto*
  then show ∃ *D*⊆*C*. *finite D* ∧ *S* ⊆ ⋃ *D* **..**
next
  assume *1*: ∀ *C*. (∀ *c*∈*C*. *openin* (*top_of_set S*) *c*) ∧ *S* ⊆ ⋃ *C* ⟶
        (∃ *D*⊆*C*. *finite D* ∧ *S* ⊆ ⋃ *D*)
  show *compact S*
  proof (*rule compactI*)
    fix *C*
    let *?C* = *image* (λ*T*. *S* ∩ *T*) *C*
    assume ∀ *t*∈*C*. *open t* **and** *S* ⊆ ⋃ *C*
    then have (∀ *c*∈*?C*. *openin* (*top_of_set S*) *c*) ∧ *S* ⊆ ⋃ *?C*
      unfolding *openin_open* **by** *auto*
    with *1* **obtain** *D* **where** *D* ⊆ *?C* **and** *finite D* **and** *S* ⊆ ⋃ *D*
      by *metis*
    let *?D* = *inv_into C* (λ*T*. *S* ∩ *T*) ‘ *D*
    have *?D* ⊆ *C* ∧ *finite ?D* ∧ *S* ⊆ ⋃ *?D*
    proof (*intro conjI*)
      from ‹*D* ⊆ *?C*› **show** *?D* ⊆ *C*
        by (*fast intro*: *inv_into_into*)
      from ‹*finite D*› **show** *finite ?D*
        by (*rule finite_imageI*)
      from ‹*S* ⊆ ⋃ *D*› **show** *S* ⊆ ⋃ *?D*
        apply (*rule subset_trans*)
        by (*metis Int_Union Int_lower2* ‹*D* ⊆ (∩) *S* ‘ *C*› *image_inv_into_cancel*)
    qed
    then show ∃ *D*⊆*C*. *finite D* ∧ *S* ⊆ ⋃ *D* **..**
  qed
qed

2.3.4 Continuity

lemma *interior_image_subset*:
  assumes *inj f* ⋀*x*. *continuous* (*at x*) *f*

shows *interior* $(f$ ‘ $S) \subseteq f$ ‘ $(interior\ S)$
proof
fix x **assume** $x \in interior\ (f$ ‘ $S)$
then obtain T **where** *as*: *open* T $x \in T$ $T \subseteq f$ ‘ S **..**
then have $x \in f$ ‘ S **by** *auto*
then obtain y **where** y: $y \in S$ $x = f\ y$ **by** *auto*
have *open* $(f$ −‘ $T)$
using *assms* ‹*open T*› **by** (*simp add*: *continuous_at_imp_continuous_on open_vimage*)
moreover have $y \in vimage\ f\ T$
using ‹$x = f\ y$› ‹$x \in T$› **by** *simp*
moreover have $vimage\ f\ T \subseteq S$
using ‹$T \subseteq image\ f\ S$› ‹*inj f*› **unfolding** *inj_on_def subset_eq* **by** *auto*
ultimately have $y \in interior\ S$ **..**
with ‹$x = f\ y$› **show** $x \in f$ ‘ *interior* S **..**
qed

2.3.5 Equality of continuous functions on closure and related results

lemma *continuous_closedin_preimage_constant*:
fixes f :: _ ⇒ *'b::t1_space*
shows *continuous_on* $S\ f \Longrightarrow closedin\ (top\_of\_set\ S)\ \{x \in S.\ f\ x = a\}$
using *continuous_closedin_preimage*[*of S f* $\{a\}$] **by** (*simp add*: *vimage_def Collect_conj_eq*)

lemma *continuous_closed_preimage_constant*:
fixes f :: _ ⇒ *'b::t1_space*
shows *continuous_on* $S\ f \Longrightarrow closed\ S \Longrightarrow closed\ \{x \in S.\ f\ x = a\}$
using *continuous_closed_preimage*[*of S f* $\{a\}$] **by** (*simp add*: *vimage_def Collect_conj_eq*)

lemma *continuous_constant_on_closure*:
fixes f :: _ ⇒ *'b::t1_space*
assumes *continuous_on* (*closure S*) f
and $\bigwedge x.\ x \in S \Longrightarrow f\ x = a$
and $x \in closure\ S$
shows $f\ x = a$
using *continuous_closed_preimage_constant*[*of closure S f a*]
assms closure_minimal[*of S* $\{x \in closure\ S.\ f\ x = a\}$] *closure_subset*
unfolding *subset_eq*
by *auto*

lemma *image_closure_subset*:
assumes *contf*: *continuous_on* (*closure S*) f
and *closed* T
and $(f$ ‘ $S) \subseteq T$
shows f ‘ $(closure\ S) \subseteq T$
proof −
have $S \subseteq \{x \in closure\ S.\ f\ x \in T\}$

using *assms(3) closure_subset* **by** *auto*
moreover have *closed* (*closure S* ∩ *f* −‘ *T*)
using *continuous_closed_preimage*[*OF contf*] ‹*closed T*› **by** *auto*
ultimately have *closure S* = (*closure S* ∩ *f* −‘ *T*)
using *closure_minimal*[*of S* (*closure S* ∩ *f* −‘ *T*)] **by** *auto*
then show *?thesis* **by** *auto*
qed

2.3.6 A function constant on a set

definition *constant_on* (**infixl** (*constant′_on*) *50*)
where *f constant_on A* ≡ ∃ *y*. ∀ *x*∈*A*. *f x* = *y*

lemma *constant_on_subset*: ⟦*f constant_on A*; *B* ⊆ *A*⟧ ⟹ *f constant_on B*
unfolding *constant_on_def* **by** *blast*

lemma *injective_not_constant*:
fixes *S* :: ′*a*::{*perfect_space*} *set*
shows ⟦*open S*; *inj_on f S*; *f constant_on S*⟧ ⟹ *S* = {}
unfolding *constant_on_def*
by (*metis equals0I inj_on_contraD islimpt_UNIV islimpt_def*)

lemma *constant_on_closureI*:
fixes *f* :: _ ⇒ ′*b*::*t1_space*
assumes *cof*: *f constant_on S* **and** *contf*: *continuous_on* (*closure S*) *f*
shows *f constant_on* (*closure S*)
using *continuous_constant_on_closure* [*OF contf*] *cof* **unfolding** *constant_on_def*
by *metis*

2.3.7 Continuity relative to a union.

lemma *continuous_on_Un_local*:
⟦*closedin* (*top_of_set* (*s* ∪ *t*)) *s*; *closedin* (*top_of_set* (*s* ∪ *t*)) *t*;
continuous_on s f; *continuous_on t f*⟧
⟹ *continuous_on* (*s* ∪ *t*) *f*
unfolding *continuous_on closedin_limpt*
by (*metis Lim_trivial_limit Lim_within_union Un_iff trivial_limit_within*)

lemma *continuous_on_cases_local*:
⟦*closedin* (*top_of_set* (*s* ∪ *t*)) *s*; *closedin* (*top_of_set* (*s* ∪ *t*)) *t*;
continuous_on s f; *continuous_on t g*;
⋀*x*. ⟦*x* ∈ *s* ∧ ¬*P x* ∨ *x* ∈ *t* ∧ *P x*⟧ ⟹ *f x* = *g x*⟧
⟹ *continuous_on* (*s* ∪ *t*) (λ*x*. *if P x then f x else g x*)
by (*rule continuous_on_Un_local*) (*auto intro*: *continuous_on_eq*)

lemma *continuous_on_cases_le*:
fixes *h* :: ′*a* :: *topological_space* ⇒ *real*
assumes *continuous_on* {*t* ∈ *s*. *h t* ≤ *a*} *f*
and *continuous_on* {*t* ∈ *s*. *a* ≤ *h t*} *g*
and *h*: *continuous_on s h*

and $\bigwedge t.\ [\![t \in s;\ h\ t = a]\!] \Longrightarrow f\ t = g\ t$
shows *continuous_on* $s\ (\lambda t.\ \textit{if}\ h\ t \leq a\ \textit{then}\ f(t)\ \textit{else}\ g(t))$
proof −
have s: $s = (s \cap h -\text{`}\ \textit{atMost}\ a) \cup (s \cap h -\text{`}\ \textit{atLeast}\ a)$
by *force*
have *1*: *closedin* (*top_of_set* s) ($s \cap h -$‘ *atMost* a)
by (*rule continuous_closedin_preimage* [*OF h closed_atMost*])
have *2*: *closedin* (*top_of_set* s) ($s \cap h -$‘ *atLeast* a)
by (*rule continuous_closedin_preimage* [*OF h closed_atLeast*])
have *eq*: $s \cap h -\text{`}\ \{..a\} = \{t \in s.\ h\ t \leq a\}\ s \cap h -\text{`}\ \{a..\} = \{t \in s.\ a \leq h\ t\}$
by *auto*
show *?thesis*
apply (*rule continuous_on_subset* [*of s*, *OF _ order_refl*])
apply (*subst s*)
apply (*rule continuous_on_cases_local*)
using *1 2 s assms* **apply** (*auto simp*: *eq*)
done
qed

lemma *continuous_on_cases_1*:
fixes *s* :: *real set*
assumes *continuous_on* $\{t \in s.\ t \leq a\}\ f$
and *continuous_on* $\{t \in s.\ a \leq t\}\ g$
and $a \in s \Longrightarrow f\ a = g\ a$
shows *continuous_on* $s\ (\lambda t.\ \textit{if}\ t \leq a\ \textit{then}\ f(t)\ \textit{else}\ g(t))$
using *assms*
by (*auto intro*: *continuous_on_cases_le* [**where** $h = id$, *simplified*])

2.3.8 Inverse function property for open/closed maps

lemma *continuous_on_inverse_open_map*:
assumes *contf*: *continuous_on S f*
and *imf*: $f\ \text{`}\ S = T$
and *injf*: $\bigwedge x.\ x \in S \Longrightarrow g\ (f\ x) = x$
and *oo*: $\bigwedge U$. *openin* (*top_of_set S*) $U \Longrightarrow$ *openin* (*top_of_set T*) (f ‘ U)
shows *continuous_on T g*
proof −
from *imf injf* **have** *gTS*: $g\ \text{`}\ T = S$
by *force*
from *imf injf* **have** *fU*: $U \subseteq S \Longrightarrow (f\ \text{`}\ U) = T \cap g -\text{`}\ U$ **for** U
by *force*
show *?thesis*
by (*simp add*: *continuous_on_open* [*of T g*] *gTS*) (*metis openin_imp_subset fU oo*)
qed

lemma *continuous_on_inverse_closed_map*:
assumes *contf*: *continuous_on S f*
and *imf*: $f\ \text{`}\ S = T$

and *injf*: $\bigwedge x.\ x \in S \Longrightarrow g(f\,x) = x$
and *oo*: $\bigwedge U.$ *closedin* (*top_of_set S*) $U \Longrightarrow$ *closedin* (*top_of_set T*) (f ‘ U)
shows *continuous_on T g*
proof −
from *imf injf* **have** *gTS*: g ‘ $T = S$
by *force*
from *imf injf* **have** *fU*: $U \subseteq S \Longrightarrow (f$ ‘ $U) = T \cap g$ −‘ U **for** U
by *force*
show *?thesis*
by (*simp add*: *continuous_on_closed* [*of T g*] *gTS*) (*metis closedin_imp_subset fU oo*)
qed

lemma *homeomorphism_injective_open_map*:
assumes *contf*: *continuous_on S f*
and *imf*: f ‘ $S = T$
and *injf*: *inj_on f S*
and *oo*: $\bigwedge U.$ *openin* (*top_of_set S*) $U \Longrightarrow$ *openin* (*top_of_set T*) (f ‘ U)
obtains *g* **where** *homeomorphism S T f g*
proof
have *continuous_on T* (*inv_into S f*)
by (*metis contf continuous_on_inverse_open_map imf injf inv_into_f_f oo*)
with *imf injf contf* **show** *homeomorphism S T f* (*inv_into S f*)
by (*auto simp*: *homeomorphism_def*)
qed

lemma *homeomorphism_injective_closed_map*:
assumes *contf*: *continuous_on S f*
and *imf*: f ‘ $S = T$
and *injf*: *inj_on f S*
and *oo*: $\bigwedge U.$ *closedin* (*top_of_set S*) $U \Longrightarrow$ *closedin* (*top_of_set T*) (f ‘ U)
obtains *g* **where** *homeomorphism S T f g*
proof
have *continuous_on T* (*inv_into S f*)
by (*metis contf continuous_on_inverse_closed_map imf injf inv_into_f_f oo*)
with *imf injf contf* **show** *homeomorphism S T f* (*inv_into S f*)
by (*auto simp*: *homeomorphism_def*)
qed

lemma *homeomorphism_imp_open_map*:
assumes *hom*: *homeomorphism S T f g*
and *oo*: *openin* (*top_of_set S*) U
shows *openin* (*top_of_set T*) (f ‘ U)
proof −
from *hom oo* **have** [*simp*]: f ‘ $U = T \cap g$ −‘ U
using *openin_subset* **by** (*fastforce simp*: *homeomorphism_def rev_image_eqI*)
from *hom* **have** *continuous_on T g*
unfolding *homeomorphism_def* **by** *blast*
moreover have g ‘ $T = S$

```
    by (metis hom homeomorphism_def)
  ultimately show ?thesis
    by (simp add: continuous_on_open oo)
qed

lemma homeomorphism_imp_closed_map:
  assumes hom: homeomorphism S T f g
    and oo: closedin (top_of_set S) U
  shows closedin (top_of_set T) (f ` U)
proof –
  from hom oo have [simp]: f ` U = T ∩ g –` U
    using closedin_subset by (fastforce simp: homeomorphism_def rev_image_eqI)
  from hom have continuous_on T g
    unfolding homeomorphism_def by blast
  moreover have g ` T = S
    by (metis hom homeomorphism_def)
  ultimately show ?thesis
    by (simp add: continuous_on_closed oo)
qed
```

2.3.9 Seperability

```
lemma subset_second_countable:
  obtains 𝓑 :: 'a:: second_countable_topology set set
    where countable 𝓑
          {} ∉ 𝓑
          ⋀C. C ∈ 𝓑 ⟹ openin(top_of_set S) C
          ⋀T. openin(top_of_set S) T ⟹ ∃𝒰. 𝒰 ⊆ 𝓑 ∧ T = ⋃𝒰
proof –
  obtain 𝓑 :: 'a set set
    where countable 𝓑
      and opeB: ⋀C. C ∈ 𝓑 ⟹ openin(top_of_set S) C
      and 𝓑:    ⋀T. openin(top_of_set S) T ⟹ ∃𝒰. 𝒰 ⊆ 𝓑 ∧ T = ⋃𝒰
  proof –
    obtain 𝒞 :: 'a set set
      where countable 𝒞 and ope: ⋀C. C ∈ 𝒞 ⟹ open C
        and 𝒞: ⋀S. open S ⟹ ∃ U. U ⊆ 𝒞 ∧ S = ⋃ U
      by (metis univ_second_countable that)
    show ?thesis
    proof
      show countable ((λC. S ∩ C) ` 𝒞)
        by (simp add: ‹countable 𝒞›)
      show ⋀C. C ∈ (∩) S ` 𝒞 ⟹ openin (top_of_set S) C
        using ope by auto
      show ⋀T. openin (top_of_set S) T ⟹ ∃𝒰⊆(∩) S ` 𝒞. T = ⋃𝒰
        by (metis 𝒞 image_mono inf_Sup openin_open)
    qed
  qed
  show ?thesis
```

proof
show *countable* ($\mathcal{B}$ − {{}})
using ‹*countable* $\mathcal{B}$› **by** *blast*
show $\bigwedge C$. ⟦$C \in \mathcal{B}$ − {{}}⟧ ⟹ *openin* (*top_of_set S*) *C*
by (*simp add*: ‹$\bigwedge C$. $C \in \mathcal{B}$ ⟹ *openin* (*top_of_set S*) *C*›)
show $\exists \mathcal{U} \subseteq \mathcal{B}$ − {{}}. $T = \bigcup \mathcal{U}$ **if** *openin* (*top_of_set S*) *T* **for** *T*
using $\mathcal{B}$ [*OF that*]
apply *clarify*
apply (*rule_tac x*=$\mathcal{U}$ − {{}} **in** *exI*, *auto*)
done
qed *auto*
qed

lemma *Lindelof_openin*:
fixes $\mathcal{F}$ :: *'a::second_countable_topology set set*
assumes $\bigwedge S$. $S \in \mathcal{F}$ ⟹ *openin* (*top_of_set U*) *S*
obtains $\mathcal{F}'$ **where** $\mathcal{F}' \subseteq \mathcal{F}$ *countable* $\mathcal{F}'$ $\bigcup \mathcal{F}' = \bigcup \mathcal{F}$
proof −
have $\bigwedge S$. $S \in \mathcal{F}$ ⟹ ∃ *T*. *open T* ∧ $S = U \cap T$
using *assms* **by** (*simp add*: *openin_open*)
then obtain *tf* **where** *tf*: $\bigwedge S$. $S \in \mathcal{F}$ ⟹ *open* (*tf S*) ∧ ($S = U \cap$ *tf S*)
by *metis*
have [*simp*]: $\bigwedge \mathcal{F}'$. $\mathcal{F}' \subseteq \mathcal{F}$ ⟹ $\bigcup \mathcal{F}' = U \cap \bigcup$(*tf* ‘ $\mathcal{F}'$)
using *tf* **by** *fastforce*
obtain $\mathcal{G}$ **where** *countable* $\mathcal{G} \wedge \mathcal{G} \subseteq$ *tf* ‘ $\mathcal{F}$ $\bigcup \mathcal{G} = \bigcup$(*tf* ‘ $\mathcal{F}$)
using *tf* **by** (*force intro*: *Lindelof* [*of tf* ‘ $\mathcal{F}$])
then obtain $\mathcal{F}'$ **where** $\mathcal{F}'$: $\mathcal{F}' \subseteq \mathcal{F}$ *countable* $\mathcal{F}'$ $\bigcup \mathcal{F}' = \bigcup \mathcal{F}$
by (*clarsimp simp add*: *countable_subset_image*)
then show *?thesis* ..
qed

2.3.10 Closed Maps

lemma *continuous_imp_closed_map*:
fixes *f* :: *'a::t2_space* ⇒ *'b::t2_space*
assumes *closedin* (*top_of_set S*) *U*
continuous_on S f f ‘ *S* = *T compact S*
shows *closedin* (*top_of_set T*) (*f* ‘ *U*)
by (*metis assms closedin_compact_eq compact_continuous_image continuous_on_subset subset_image_iff*)

lemma *closed_map_restrict*:
assumes *cloU*: *closedin* (*top_of_set* ($S \cap f$ −‘ T')) *U*
and *cc*: $\bigwedge U$. *closedin* (*top_of_set S*) *U* ⟹ *closedin* (*top_of_set T*) (*f* ‘ *U*)
and $T' \subseteq T$
shows *closedin* (*top_of_set T'*) (*f* ‘ *U*)
proof −
obtain *V* **where** *closed V* $U = S \cap f$ −‘ $T' \cap V$
using *cloU* **by** (*auto simp*: *closedin_closed*)

```
  with cc [of S ∩ V] ‹T' ⊆ T› show ?thesis
    by (fastforce simp add: closedin_closed)
qed
```

2.3.11 Open Maps

```
lemma open_map_restrict:
  assumes opeU: openin (top_of_set (S ∩ f -` T')) U
    and oo: ⋀U. openin (top_of_set S) U ⟹ openin (top_of_set T) (f ` U)
    and T' ⊆ T
  shows openin (top_of_set T') (f ` U)
proof -
  obtain V where open V U = S ∩ f -` T' ∩ V
    using opeU by (auto simp: openin_open)
  with oo [of S ∩ V] ‹T' ⊆ T› show ?thesis
    by (fastforce simp add: openin_open)
qed
```

2.3.12 Quotient maps

```
lemma quotient_map_imp_continuous_open:
  assumes T: f ` S ⊆ T
      and ope: ⋀U. U ⊆ T
              ⟹ (openin (top_of_set S) (S ∩ f -` U) ⟷
                   openin (top_of_set T) U)
    shows continuous_on S f
proof -
  have [simp]: S ∩ f -` f ` S = S by auto
  show ?thesis
    by (meson T continuous_on_open_gen ope openin_imp_subset)
qed

lemma quotient_map_imp_continuous_closed:
  assumes T: f ` S ⊆ T
      and ope: ⋀U. U ⊆ T
                  ⟹ (closedin (top_of_set S) (S ∩ f -` U) ⟷
                       closedin (top_of_set T) U)
    shows continuous_on S f
proof -
  have [simp]: S ∩ f -` f ` S = S by auto
  show ?thesis
    by (meson T closedin_imp_subset continuous_on_closed_gen ope)
qed

lemma open_map_imp_quotient_map:
  assumes contf: continuous_on S f
      and T: T ⊆ f ` S
      and ope: ⋀T. openin (top_of_set S) T
                   ⟹ openin (top_of_set (f ` S)) (f ` T)
    shows openin (top_of_set S) (S ∩ f -` T) =
```

openin (*top_of_set* (*f* ‘ *S*)) *T*
proof −
have *T* = *f* ‘ (*S* ∩ *f* −‘ *T*)
using *T* **by** *blast*
then show *?thesis*
using *ope contf continuous_on_open* **by** *metis*
qed

lemma *closed_map_imp_quotient_map*:
assumes *contf*: *continuous_on S f*
and *T*: *T* ⊆ *f* ‘ *S*
and *ope*: ⋀*T*. *closedin* (*top_of_set S*) *T*
⟹ *closedin* (*top_of_set* (*f* ‘ *S*)) (*f* ‘ *T*)
shows *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *T*) ⟷
openin (*top_of_set* (*f* ‘ *S*)) *T*
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then have ∗: *closedin* (*top_of_set S*) (*S* − (*S* ∩ *f* −‘ *T*))
using *closedin_diff* **by** *fastforce*
have [*simp*]: (*f* ‘ *S* − *f* ‘ (*S* − (*S* ∩ *f* −‘ *T*))) = *T*
using *T* **by** *blast*
show *?rhs*
using *ope* [*OF* ∗, *unfolded closedin_def*] **by** *auto*
next
assume *?rhs*
with *contf* **show** *?lhs*
by (*auto simp*: *continuous_on_open*)
qed

lemma *continuous_right_inverse_imp_quotient_map*:
assumes *contf*: *continuous_on S f* **and** *imf*: *f* ‘ *S* ⊆ *T*
and *contg*: *continuous_on T g* **and** *img*: *g* ‘ *T* ⊆ *S*
and *fg* [*simp*]: ⋀*y*. *y* ∈ *T* ⟹ *f*(*g y*) = *y*
and *U*: *U* ⊆ *T*
shows *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *U*) ⟷
openin (*top_of_set T*) *U*
(**is** *?lhs* = *?rhs*)
proof −
have *f*: ⋀*Z*. *openin* (*top_of_set* (*f* ‘ *S*)) *Z* ⟹
openin (*top_of_set S*) (*S* ∩ *f* −‘ *Z*)
and *g*: ⋀*Z*. *openin* (*top_of_set* (*g* ‘ *T*)) *Z* ⟹
openin (*top_of_set T*) (*T* ∩ *g* −‘ *Z*)
using *contf contg* **by** (*auto simp*: *continuous_on_open*)
show *?thesis*
proof
have *T* ∩ *g* −‘ (*g* ‘ *T* ∩ (*S* ∩ *f* −‘ *U*)) = {*x* ∈ *T*. *f* (*g x*) ∈ *U*}
using *imf img* **by** *blast*
also have ... = *U*

```
      using U by auto
    finally have eq: T ∩ g −‘ (g ‘ T ∩ (S ∩ f −‘ U)) = U .
    assume ?lhs
    then have ∗: openin (top_of_set (g ‘ T)) (g ‘ T ∩ (S ∩ f −‘ U))
     by (meson img openin_Int openin_subtopology_Int_subset openin_subtopology_self)
    show ?rhs
      using g [OF ∗] eq by auto
  next
    assume rhs: ?rhs
    show ?lhs
     by (metis f fg image_eqI image_subset_iff imf img openin_subopen openin_subtopology_self
openin_trans rhs)
  qed
qed

lemma continuous_left_inverse_imp_quotient_map:
  assumes continuous_on S f
      and continuous_on (f ‘ S) g
      and ⋀x. x ∈ S =⇒ g(f x) = x
      and U ⊆ f ‘ S
    shows openin (top_of_set S) (S ∩ f −‘ U) ←→
           openin (top_of_set (f ‘ S)) U
apply (rule continuous_right_inverse_imp_quotient_map)
using assms apply force+
done

lemma continuous_imp_quotient_map:
  fixes f :: 'a::t2_space ⇒ 'b::t2_space
  assumes continuous_on S f f ‘ S = T compact S U ⊆ T
    shows openin (top_of_set S) (S ∩ f −‘ U) ←→
           openin (top_of_set T) U
 by (metis (no_types, lifting) assms closed_map_imp_quotient_map continuous_imp_closed_map)
```

2.3.13 Pasting lemmas for functions, for of casewise definitions

on open sets

```
lemma pasting_lemma:
  assumes ope: ⋀i. i ∈ I =⇒ openin X (T i)
      and cont: ⋀i. i ∈ I =⇒ continuous_map(subtopology X (T i)) Y (f i)
      and f: ⋀i j x. [[i ∈ I; j ∈ I; x ∈ topspace X ∩ T i ∩ T j]] =⇒ f i x = f j x
      and g: ⋀x. x ∈ topspace X =⇒ ∃j. j ∈ I ∧ x ∈ T j ∧ g x = f j x
    shows continuous_map X Y g
  unfolding continuous_map_openin_preimage_eq
proof (intro conjI allI impI)
  show g ‘ topspace X ⊆ topspace Y
    using g cont continuous_map_image_subset_topspace by fastforce
next
  fix U
```

assume *Y*: *openin Y U*
have *T*: *T i* ⊆ *topspace X* **if** *i* ∈ *I* **for** *i*
using *ope* **by** (*simp add*: *openin_subset that*)
have ∗: *topspace X* ∩ *g* −' *U* = (⋃*i* ∈ *I*. *T i* ∩ *f i* −' *U*)
using *f g T* **by** *fastforce*
have ⋀*i*. *i* ∈ *I* ⟹ *openin X* (*T i* ∩ *f i* −' *U*)
using *cont* **unfolding** *continuous_map_openin_preimage_eq*
by (*metis Y T inf.commute inf_absorb1 ope topspace_subtopology openin_trans_full*)
then show *openin X* (*topspace X* ∩ *g* −' *U*)
by (*auto simp*: ∗)
qed

lemma *pasting_lemma_exists*:
assumes *X*: *topspace X* ⊆ (⋃*i* ∈ *I*. *T i*)
and *ope*: ⋀*i*. *i* ∈ *I* ⟹ *openin X* (*T i*)
and *cont*: ⋀*i*. *i* ∈ *I* ⟹ *continuous_map* (*subtopology X* (*T i*)) *Y* (*f i*)
and *f*: ⋀*i j x*. ⟦*i* ∈ *I*; *j* ∈ *I*; *x* ∈ *topspace X* ∩ *T i* ∩ *T j*⟧ ⟹ *f i x* = *f j x*
obtains *g* **where** *continuous_map X Y g* ⋀*x i*. ⟦*i* ∈ *I*; *x* ∈ *topspace X* ∩ *T i*⟧ ⟹ *g x* = *f i x*
proof
let *?h* = λ*x*. *f* (*SOME i*. *i* ∈ *I* ∧ *x* ∈ *T i*) *x*
show *continuous_map X Y ?h*
apply (*rule pasting_lemma* [*OF ope cont*])
apply (*blast intro*: *f*)+
by (*metis* (*no_types*, *lifting*) *UN_E X subsetD someI_ex*)
show *f* (*SOME i*. *i* ∈ *I* ∧ *x* ∈ *T i*) *x* = *f i x* **if** *i* ∈ *I x* ∈ *topspace X* ∩ *T i* **for** *i x*
by (*metis* (*no_types*, *lifting*) *IntD2 IntI f someI_ex that*)
qed

lemma *pasting_lemma_locally_finite*:
assumes *fin*: ⋀*x*. *x* ∈ *topspace X* ⟹ ∃ *V*. *openin X V* ∧ *x* ∈ *V* ∧ *finite* {*i* ∈ *I*. *T i* ∩ *V* ≠ {}}
and *clo*: ⋀*i*. *i* ∈ *I* ⟹ *closedin X* (*T i*)
and *cont*: ⋀*i*. *i* ∈ *I* ⟹ *continuous_map*(*subtopology X* (*T i*)) *Y* (*f i*)
and *f*: ⋀*i j x*. ⟦*i* ∈ *I*; *j* ∈ *I*; *x* ∈ *topspace X* ∩ *T i* ∩ *T j*⟧ ⟹ *f i x* = *f j x*
and *g*: ⋀*x*. *x* ∈ *topspace X* ⟹ ∃*j*. *j* ∈ *I* ∧ *x* ∈ *T j* ∧ *g x* = *f j x*
shows *continuous_map X Y g*
unfolding *continuous_map_closedin_preimage_eq*
proof (*intro conjI allI impI*)
show *g* ' *topspace X* ⊆ *topspace Y*
using *g cont continuous_map_image_subset_topspace* **by** *fastforce*
next
fix *U*
assume *Y*: *closedin Y U*
have *T*: *T i* ⊆ *topspace X* **if** *i* ∈ *I* **for** *i*
using *clo* **by** (*simp add*: *closedin_subset that*)
have ∗: *topspace X* ∩ *g* −' *U* = (⋃*i* ∈ *I*. *T i* ∩ *f i* −' *U*)
using *f g T* **by** *fastforce*

have *cTf*: $\bigwedge i.\ i \in I \Longrightarrow$ *closedin X* $(T\ i \cap f\ i$ −‘ $U)$
using *cont* **unfolding** *continuous_map_closedin_preimage_eq topspace_subtopology*
by (*simp add*: *Int_absorb1 T Y clo closedin_closed_subtopology*)
have *sub*: $\{Z \in (\lambda i.\ T\ i \cap f\ i$ −‘ $U)$ ‘ $I.\ Z \cap V \neq \{\}\}$
$\subseteq (\lambda i.\ T\ i \cap f\ i$ −‘ $U)$ ‘ $\{i \in I.\ T\ i \cap V \neq \{\}\}$ **for** V
by *auto*
have *1*: $(\bigcup i{\in}I.\ T\ i \cap f\ i$ −‘ $U) \subseteq$ *topspace X*
using T **by** *blast*
then have *lf*: *locally_finite_in X* $((\lambda i.\ T\ i \cap f\ i$ −‘ $U)$ ‘ $I)$
unfolding *locally_finite_in_def*
using *finite_subset* [*OF sub*] *fin* **by** *force*
show *closedin X* (*topspace X* $\cap\ g$ −‘ U)
apply (*subst* ∗)
apply (*rule closedin_locally_finite_Union*)
apply (*auto intro*: *cTf lf*)
done
qed

Likewise on closed sets, with a finiteness assumption

lemma *pasting_lemma_closed*:
assumes *fin*: *finite I*
and *clo*: $\bigwedge i.\ i \in I \Longrightarrow$ *closedin X* $(T\ i)$
and *cont*: $\bigwedge i.\ i \in I \Longrightarrow$ *continuous_map*(*subtopology X* $(T\ i)$) Y $(f\ i)$
and f: $\bigwedge i\ j\ x.\ [\![i \in I;\ j \in I;\ x \in$ *topspace X* $\cap\ T\ i \cap T\ j]\!] \Longrightarrow f\ i\ x = f\ j\ x$
and g: $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow \exists j.\ j \in I \wedge x \in T\ j \wedge g\ x = f\ j\ x$
shows *continuous_map X Y g*
using *pasting_lemma_locally_finite* [*OF _ clo cont f g*] *fin* **by** *auto*

lemma *pasting_lemma_exists_locally_finite*:
assumes *fin*: $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow \exists V.$ *openin X V* $\wedge\ x \in V\ \wedge$ *finite* $\{i \in I.\ T\ i \cap V \neq \{\}\}$
and X: *topspace X* $\subseteq \bigcup (T$ ‘ $I)$
and *clo*: $\bigwedge i.\ i \in I \Longrightarrow$ *closedin X* $(T\ i)$
and *cont*: $\bigwedge i.\ i \in I \Longrightarrow$ *continuous_map*(*subtopology X* $(T\ i)$) Y $(f\ i)$
and f: $\bigwedge i\ j\ x.\ [\![i \in I;\ j \in I;\ x \in$ *topspace X* $\cap\ T\ i \cap T\ j]\!] \Longrightarrow f\ i\ x = f\ j\ x$
and g: $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow \exists j.\ j \in I \wedge x \in T\ j \wedge g\ x = f\ j\ x$
obtains g **where** *continuous_map X Y g* $\bigwedge x\ i.\ [\![i \in I;\ x \in$ *topspace X* $\cap\ T\ i]\!] \Longrightarrow g\ x = f\ i\ x$
proof
show *continuous_map X Y* $(\lambda x.\ f(@i.\ i \in I \wedge x \in T\ i)\ x)$
apply (*rule pasting_lemma_locally_finite* [*OF fin*])
apply (*blast intro*: *assms*)+
by (*metis* (*no_types*, *lifting*) *UN_E X set_rev_mp someI_ex*)
next
fix $x\ i$
assume $i \in I$ **and** $x \in$ *topspace X* $\cap\ T\ i$
show f (*SOME* $i.\ i \in I \wedge x \in T\ i)\ x = f\ i\ x$
apply (*rule someI2_ex*)

using ‹$i \in I$› ‹$x \in$ *topspace* $X \cap T\ i$› **apply** *blast*
by (*meson Int_iff* ‹$i \in I$› ‹$x \in$ *topspace* $X \cap T\ i$› f)
qed

lemma *pasting_lemma_exists_closed*:
assumes *fin*: *finite* I
and X: *topspace* $X \subseteq \bigcup (T\ `\ I)$
and *clo*: $\bigwedge i.\ i \in I \Longrightarrow$ *closedin* $X\ (T\ i)$
and *cont*: $\bigwedge i.\ i \in I \Longrightarrow$ *continuous_map*(*subtopology* $X\ (T\ i)$) $Y\ (f\ i)$
and f: $\bigwedge i\ j\ x.\ [\![i \in I;\ j \in I;\ x \in$ *topspace* $X \cap T\ i \cap T\ j]\!] \Longrightarrow f\ i\ x = f\ j\ x$
obtains g **where** *continuous_map* $X\ Y\ g$ $\bigwedge x\ i.\ [\![i \in I;\ x \in$ *topspace* $X \cap T\ i]\!]$
$\Longrightarrow g\ x = f\ i\ x$
proof
show *continuous_map* $X\ Y$ ($\lambda x.\ f$ (*SOME* $i.\ i \in I \wedge x \in T\ i$) x)
apply (*rule pasting_lemma_closed* [*OF* ‹*finite* I› *clo cont*])
apply (*blast intro*: f)+
by (*metis* (*mono_tags*, *lifting*) *UN_iff* X *someI_ex subset_iff*)
next
fix $x\ i$
assume $i \in I$ $x \in$ *topspace* $X \cap T\ i$
then show f (*SOME* $i.\ i \in I \wedge x \in T\ i$) $x = f\ i\ x$
by (*metis* (*no_types*, *lifting*) *IntD2 IntI f someI_ex*)
qed

lemma *continuous_map_cases*:
assumes f: *continuous_map* (*subtopology* X (X *closure_of* $\{x.\ P\ x\}$)) $Y\ f$
and g: *continuous_map* (*subtopology* X (X *closure_of* $\{x.\ \neg\ P\ x\}$)) $Y\ g$
and *fg*: $\bigwedge x.\ x \in X$ *frontier_of* $\{x.\ P\ x\} \Longrightarrow f\ x = g\ x$
shows *continuous_map* $X\ Y$ ($\lambda x.$ *if* $P\ x$ *then* $f\ x$ *else* $g\ x$)
proof (*rule pasting_lemma_closed*)
let *?f* $= \lambda b.$ *if* b *then* f *else* g
let *?g* $= \lambda x.$ *if* $P\ x$ *then* $f\ x$ *else* $g\ x$
let *?T* $= \lambda b.$ *if* b *then* X *closure_of* $\{x.\ P\ x\}$ *else* X *closure_of* $\{x.\ {\sim}P\ x\}$
show *finite* {*True*,*False*} **by** *auto*
have *eq*: *topspace* $X -$ *Collect* $P =$ *topspace* $X \cap \{x.\ \neg\ P\ x\}$
by *blast*
show *?f* $i\ x =$ *?f* $j\ x$
if $i \in$ {*True*,*False*} $j \in$ {*True*,*False*} **and** x: $x \in$ *topspace* $X \cap$ *?T* $i \cap$ *?T* j
for $i\ j\ x$
proof −
have $f\ x = g\ x$
if $i\ \neg\ j$
apply (*rule fg*)
unfolding *frontier_of_closures eq*
using x *that closure_of_restrict* **by** *fastforce*
moreover
have $g\ x = f\ x$
if $x \in X$ *closure_of* $\{x.\ \neg\ P\ x\}$ $x \in X$ *closure_of Collect* P $\neg\ i\ j$ **for** x
apply (*rule fg* [*symmetric*])

```
        unfolding frontier_of_closures eq
        using x that closure_of_restrict by fastforce
    ultimately show ?thesis
      using that by (auto simp flip: closure_of_restrict)
  qed
  show ∃j. j ∈ {True,False} ∧ x ∈ ?T j ∧ (if P x then f x else g x) = ?f j x
    if x ∈ topspace X for x
    apply simp
    apply safe
    apply (metis Int_iff closure_of inf_sup_absorb mem_Collect_eq that)
    by (metis DiffI eq closure_of_subset_Int contra_subsetD mem_Collect_eq that)
qed (auto simp: f g)

lemma continuous_map_cases_alt:
  assumes f: continuous_map (subtopology X (X closure_of {x ∈ topspace X. P
x})) Y f
      and g: continuous_map (subtopology X (X closure_of {x ∈ topspace X. ~P
x})) Y g
      and fg: ⋀x. x ∈ X frontier_of {x ∈ topspace X. P x} ⟹ f x = g x
    shows continuous_map X Y (λx. if P x then f x else g x)
  apply (rule continuous_map_cases)
  using assms
   apply (simp_all add: Collect_conj_eq closure_of_restrict [symmetric] frontier_of_restrict
[symmetric])
  done

lemma continuous_map_cases_function:
  assumes contp: continuous_map X Z p
    and contf: continuous_map (subtopology X {x ∈ topspace X. p x ∈ Z closure_of
U}) Y f
    and contg: continuous_map (subtopology X {x ∈ topspace X. p x ∈ Z closure_of
(topspace Z − U)}) Y g
    and fg: ⋀x. ⟦x ∈ topspace X; p x ∈ Z frontier_of U⟧ ⟹ f x = g x
  shows continuous_map X Y (λx. if p x ∈ U then f x else g x)
proof (rule continuous_map_cases_alt)
  show continuous_map (subtopology X (X closure_of {x ∈ topspace X. p x ∈ U}))
Y f
  proof (rule continuous_map_from_subtopology_mono)
    let ?T = {x ∈ topspace X. p x ∈ Z closure_of U}
    show continuous_map (subtopology X ?T) Y f
      by (simp add: contf)
    show X closure_of {x ∈ topspace X. p x ∈ U} ⊆ ?T
      by (rule continuous_map_closure_preimage_subset [OF contp])
  qed
  show continuous_map (subtopology X (X closure_of {x ∈ topspace X. p x ∉ U}))
Y g
  proof (rule continuous_map_from_subtopology_mono)
    let ?T = {x ∈ topspace X. p x ∈ Z closure_of (topspace Z − U)}
    show continuous_map (subtopology X ?T) Y g
```

by (*simp add*: *contg*)
have *X closure_of* {*x* ∈ *topspace X*. *p x* ∉ *U*} ⊆ *X closure_of* {*x* ∈ *topspace X*. *p x* ∈ *topspace Z* − *U*}
apply (*rule closure_of_mono*)
using *continuous_map_closedin contp* **by** *fastforce*
then show *X closure_of* {*x* ∈ *topspace X*. *p x* ∉ *U*} ⊆ *?T*
by (*rule order_trans* [*OF _ continuous_map_closure_preimage_subset* [*OF contp*]])
qed
next
show *f x* = *g x* **if** *x* ∈ *X frontier_of* {*x* ∈ *topspace X*. *p x* ∈ *U*} **for** *x*
using *that continuous_map_frontier_frontier_preimage_subset* [*OF contp*, *of U*] *fg* **by** *blast*
qed

2.3.14 Retractions

definition *retraction* :: ('*a*::*topological_space*) *set* ⇒ '*a set* ⇒ ('*a* ⇒ '*a*) ⇒ *bool*
where *retraction S T r* ⟷
T ⊆ *S* ∧ *continuous_on S r* ∧ *r* ‘ *S* ⊆ *T* ∧ (∀ *x*∈*T*. *r x* = *x*)

definition *retract_of* (**infixl** *retract'_of 50*) **where**
T retract_of S ⟷ (∃ *r*. *retraction S T r*)

lemma *retraction_idempotent*: *retraction S T r* ⟹ *x* ∈ *S* ⟹ *r* (*r x*) = *r x*
unfolding *retraction_def* **by** *auto*

Preservation of fixpoints under (more general notion of) retraction

lemma *invertible_fixpoint_property*:
fixes *S* :: '*a*::*topological_space set*
and *T* :: '*b*::*topological_space set*
assumes *contt*: *continuous_on T i*
and *i* ‘ *T* ⊆ *S*
and *contr*: *continuous_on S r*
and *r* ‘ *S* ⊆ *T*
and *ri*: ⋀*y*. *y* ∈ *T* ⟹ *r* (*i y*) = *y*
and *FP*: ⋀*f*. ⟦*continuous_on S f*; *f* ‘ *S* ⊆ *S*⟧ ⟹ ∃ *x*∈*S*. *f x* = *x*
and *contg*: *continuous_on T g*
and *g* ‘ *T* ⊆ *T*
obtains *y* **where** *y* ∈ *T* **and** *g y* = *y*
proof −
have ∃ *x*∈*S*. (*i* ∘ *g* ∘ *r*) *x* = *x*
proof (*rule FP*)
show *continuous_on S* (*i* ∘ *g* ∘ *r*)
by (*meson contt contr assms*(*4*) *contg assms*(*8*) *continuous_on_compose continuous_on_subset*)
show (*i* ∘ *g* ∘ *r*) ‘ *S* ⊆ *S*
using *assms*(*2*,*4*,*8*) **by** *force*
qed

```
  then obtain x where x: x ∈ S (i ∘ g ∘ r) x = x ..
  then have *: g (r x) ∈ T
    using assms(4,8) by auto
  have r ((i ∘ g ∘ r) x) = r x
    using x by auto
  then show ?thesis
    using * ri that by auto
qed

lemma homeomorphic_fixpoint_property:
  fixes S :: 'a::topological_space set
    and T :: 'b::topological_space set
  assumes S homeomorphic T
  shows (∀f. continuous_on S f ∧ f ‘ S ⊆ S ⟶ (∃x∈S. f x = x)) ⟷
         (∀g. continuous_on T g ∧ g ‘ T ⊆ T ⟶ (∃y∈T. g y = y))
         (is ?lhs = ?rhs)
proof −
  obtain r i where r:
      ∀x∈S. i (r x) = x r ‘ S = T continuous_on S r
      ∀y∈T. r (i y) = y i ‘ T = S continuous_on T i
    using assms unfolding homeomorphic_def homeomorphism_def by blast
  show ?thesis
  proof
    assume ?lhs
    with r show ?rhs
      by (metis invertible_fixpoint_property[of T i S r] order_refl)
  next
    assume ?rhs
    with r show ?lhs
      by (metis invertible_fixpoint_property[of S r T i] order_refl)
  qed
qed

lemma retract_fixpoint_property:
  fixes f :: 'a::topological_space ⇒ 'b::topological_space
    and S :: 'a set
  assumes T retract_of S
    and FP: ⋀f. ⟦continuous_on S f; f ‘ S ⊆ S⟧ ⟹ ∃x∈S. f x = x
    and contg: continuous_on T g
    and g ‘ T ⊆ T
  obtains y where y ∈ T and g y = y
proof −
  obtain h where retraction S T h
    using assms(1) unfolding retract_of_def ..
  then show ?thesis
    unfolding retraction_def
    using invertible_fixpoint_property[OF continuous_on_id _ _ _ _ FP]
    by (metis assms(4) contg image_ident that)
qed
```

lemma *retraction*:
retraction S T r $\longleftrightarrow$
$T \subseteq S \wedge$ *continuous_on* $S\ r \wedge r\ `\ S = T \wedge (\forall x \in T.\ r\ x = x)$
by (*force simp*: *retraction_def*)

lemma *retractionE*: — yields properties normalized wrt. simp – less likely to loop
assumes *retraction S T r*
obtains $T = r\ `\ S$ $r\ `\ S \subseteq S$ *continuous_on* $S\ r$ $\bigwedge x.\ x \in S \Longrightarrow r\ (r\ x) = r\ x$
proof (*rule that*)
from *retraction* [*of S T r*] *assms*
have $T \subseteq S$ *continuous_on* $S\ r$ $r\ `\ S = T$ **and** $\forall x \in T.\ r\ x = x$
by *simp_all*
then show $T = r\ `\ S$ $r\ `\ S \subseteq S$ *continuous_on* $S\ r$
by *simp_all*
from ‹$\forall x \in T.\ r\ x = x$› **have** $r\ x = x$ **if** $x \in T$ **for** x
using *that* **by** *simp*
with ‹$r\ `\ S = T$› **show** $r\ (r\ x) = r\ x$ **if** $x \in S$ **for** x
using *that* **by** *auto*
qed

lemma *retract_ofE*: — yields properties normalized wrt. simp – less likely to loop
assumes *T retract_of S*
obtains r **where** $T = r\ `\ S$ $r\ `\ S \subseteq S$ *continuous_on* $S\ r$ $\bigwedge x.\ x \in S \Longrightarrow r\ (r\ x) = r\ x$
proof –
from *assms* **obtain** r **where** *retraction S T r*
by (*auto simp add*: *retract_of_def*)
with *that* **show** *thesis*
by (*auto elim*: *retractionE*)
qed

lemma *retract_of_imp_extensible*:
assumes *S retract_of T* **and** *continuous_on S f* **and** $f\ `\ S \subseteq U$
obtains g **where** *continuous_on* $T\ g$ $g\ `\ T \subseteq U$ $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
proof –
from ‹*S retract_of T*› **obtain** r **where** *retraction T S r*
by (*auto simp add*: *retract_of_def*)
show *thesis*
by (*rule that* [*of* $f \circ r$])
(*use* ‹*continuous_on S f*› ‹$f\ `\ S \subseteq U$› ‹*retraction T S r*› **in** ‹*auto simp*: *continuous_on_compose2 retraction*›)
qed

lemma *idempotent_imp_retraction*:
assumes *continuous_on S f* **and** $f\ `\ S \subseteq S$ **and** $\bigwedge x.\ x \in S \Longrightarrow f(f\ x) = f\ x$
shows *retraction S* $(f\ `\ S)$ f
by (*simp add*: *assms retraction*)

lemma *retraction_subset*:
  assumes *retraction S T r* **and** $T \subseteq s'$ **and** $s' \subseteq S$
  shows *retraction s' T r*
  unfolding *retraction_def*
  by (*metis assms continuous_on_subset image_mono retraction*)

lemma *retract_of_subset*:
  assumes *T retract_of S* **and** $T \subseteq s'$ **and** $s' \subseteq S$
    shows *T retract_of s'*
by (*meson assms retract_of_def retraction_subset*)

lemma *retraction_refl* [*simp*]: *retraction S S* ($\lambda x.\ x$)
by (*simp add*: *retraction*)

lemma *retract_of_refl* [*iff*]: *S retract_of S*
  unfolding *retract_of_def retraction_def*
  using *continuous_on_id* **by** *blast*

lemma *retract_of_imp_subset*:
   S retract_of T $\Longrightarrow S \subseteq T$
by (*simp add*: *retract_of_def retraction_def*)

lemma *retract_of_empty* [*simp*]:
     ({} *retract_of S*) $\longleftrightarrow S = \{\}$ (*S retract_of* {}) $\longleftrightarrow S = \{\}$
by (*auto simp*: *retract_of_def retraction_def*)

lemma *retract_of_singleton* [*iff*]: ({*x*} *retract_of S*) $\longleftrightarrow x \in S$
  unfolding *retract_of_def retraction_def* **by** *force*

lemma *retraction_comp*:
   ⟦*retraction S T f*; *retraction T U g*⟧
       $\Longrightarrow$ *retraction S U* ($g \circ f$)
apply (*auto simp*: *retraction_def intro*: *continuous_on_compose2*)
by *blast*

lemma *retract_of_trans* [*trans*]:
  assumes *S retract_of T* **and** *T retract_of U*
    shows *S retract_of U*
using *assms* **by** (*auto simp*: *retract_of_def intro*: *retraction_comp*)

lemma *closedin_retract*:
  fixes *S* :: *'a* :: *t2_space set*
  assumes *S retract_of T*
    shows *closedin* (*top_of_set T*) *S*
proof –
  obtain *r* **where** *r*: $S \subseteq T$ *continuous_on T r r* ‘ $T \subseteq S$ $\bigwedge x.\ x \in S \Longrightarrow r\ x = x$
    using *assms* **by** (*auto simp*: *retract_of_def retraction_def*)
  have $S = \{x \in T.\ x = r\ x\}$
    using *r* **by** *auto*

also have ... = $T \cap ((\lambda x.\ (x,\ r\ x))\ -\text{`}\ (\{y.\ \exists\, x.\ y = (x,\ x)\}))$
unfolding *vimage_def mem_Times_iff fst_conv snd_conv*
using *r*
by *auto*
also have *closedin* (*top_of_set* T) ...
by (*rule continuous_closedin_preimage*) (*auto intro*!: *closed_diagonal continuous_on_Pair r*)
finally show *?thesis* **.**
qed

lemma *closedin_self* [*simp*]: *closedin* (*top_of_set S*) *S*
by *simp*

lemma *retract_of_closed*:
fixes *S* :: *'a* :: *t2_space set*
shows ⟦*closed T*; *S retract_of T*⟧ ⟹ *closed S*
by (*metis closedin_retract closedin_closed_eq*)

lemma *retract_of_compact*:
⟦*compact T*; *S retract_of T*⟧ ⟹ *compact S*
by (*metis compact_continuous_image retract_of_def retraction*)

lemma *retract_of_connected*:
⟦*connected T*; *S retract_of T*⟧ ⟹ *connected S*
by (*metis Topological_Spaces.connected_continuous_image retract_of_def retraction*)

lemma *retraction_openin_vimage_iff*:
openin (*top_of_set S*) ($S \cap r$ −‘ *U*) ⟷ *openin* (*top_of_set T*) *U*
if *retraction*: *retraction S T r* **and** $U \subseteq T$
using *retraction* **apply** (*rule retractionE*)
apply (*rule continuous_right_inverse_imp_quotient_map* [**where** *g*=*r*])
using ‹$U \subseteq T$› **apply** (*auto elim*: *continuous_on_subset*)
done

lemma *retract_of_Times*:
⟦*S retract_of s'*; *T retract_of t'*⟧ ⟹ ($S \times T$) *retract_of* ($s' \times t'$)
apply (*simp add*: *retract_of_def retraction_def Sigma_mono*, *clarify*)
apply (*rename_tac f g*)
apply (*rule_tac x*=$\lambda z.$ ((*f* ∘ *fst*) *z*, (*g* ∘ *snd*) *z*) **in** *exI*)
apply (*rule conjI continuous_intros* | *erule continuous_on_subset* | *force*)+
done

2.3.15 Retractions on a topological space

definition *retract_of_space* :: *'a set* ⇒ *'a topology* ⇒ *bool* (**infix** *retract'_of'_space* *50*)
where *S retract_of_space X*
≡ $S \subseteq$ *topspace X* ∧ (∃ *r*. *continuous_map X* (*subtopology X S*) *r* ∧ (∀ *x* ∈

S. *r x* = *x*))

lemma *retract_of_space_retraction_maps*:
S retract_of_space X ⟷ *S* ⊆ *topspace X* ∧ (∃ *r*. *retraction_maps X* (*subtopology X S*) *r id*)
by (*auto simp*: *retract_of_space_def retraction_maps_def*)

lemma *retract_of_space_section_map*:
S retract_of_space X ⟷ *S* ⊆ *topspace X* ∧ *section_map* (*subtopology X S*) *X id*
unfolding *retract_of_space_def retraction_maps_def section_map_def*
by (*auto simp*: *continuous_map_from_subtopology*)

lemma *retract_of_space_imp_subset*:
S retract_of_space X ⟹ *S* ⊆ *topspace X*
by (*simp add*: *retract_of_space_def*)

lemma *retract_of_space_topspace*:
topspace X retract_of_space X
using *retract_of_space_def* **by** *force*

lemma *retract_of_space_empty* [*simp*]:
{} *retract_of_space X* ⟷ *topspace X* = {}
by (*auto simp*: *continuous_map_def retract_of_space_def*)

lemma *retract_of_space_singleton* [*simp*]:
{*a*} *retract_of_space X* ⟷ *a* ∈ *topspace X*
proof −
have *continuous_map X* (*subtopology X* {*a*}) (λ*x*. *a*) ∧ (λ*x*. *a*) *a* = *a* **if** *a* ∈ *topspace X*
using *that* **by** *simp*
then show *?thesis*
by (*force simp*: *retract_of_space_def*)
qed

lemma *retract_of_space_clopen*:
assumes *openin X S closedin X S S* = {} ⟹ *topspace X* = {}
shows *S retract_of_space X*
proof (*cases S* = {})
case *False*
then obtain *a* **where** *a* ∈ *S*
by *blast*
show *?thesis*
unfolding *retract_of_space_def*
proof (*intro exI conjI*)
show *S* ⊆ *topspace X*
by (*simp add*: *assms closedin_subset*)
have *continuous_map X X* (λ*x*. *if x* ∈ *S then x else a*)
proof (*rule continuous_map_cases*)
show *continuous_map* (*subtopology X* (*X closure_of* {*x*. *x* ∈ *S*})) *X* (λ*x*. *x*)

```
      by (simp add: continuous_map_from_subtopology)
    show continuous_map (subtopology X (X closure_of {x. x ∉ S})) X (λx. a)
      using ‹S ⊆ topspace X› ‹a ∈ S› by force
    show x = a if x ∈ X frontier_of {x. x ∈ S} for x
      using assms that clopenin_eq_frontier_of by fastforce
  qed
  then show continuous_map X (subtopology X S) (λx. if x ∈ S then x else a)
   using ‹S ⊆ topspace X› ‹a ∈ S› by (auto simp: continuous_map_in_subtopology)
 qed auto
qed (use assms in auto)

lemma retract_of_space_disjoint_union:
 assumes openin X S openin X T and ST: disjnt S T S ∪ T = topspace X and
S = {} ⟹ topspace X = {}
 shows S retract_of_space X
proof (rule retract_of_space_clopen)
 have S ∩ T = {}
  by (meson ST disjnt_def)
 then have S = topspace X − T
  using ST by auto
 then show closedin X S
  using ‹openin X T› by blast
qed (auto simp: assms)

lemma retraction_maps_section_image1:
 assumes retraction_maps X Y r s
 shows s ` (topspace Y) retract_of_space X
 unfolding retract_of_space_section_map
proof
 show s ` topspace Y ⊆ topspace X
    using assms continuous_map_image_subset_topspace retraction_maps_def by
blast
 show section_map (subtopology X (s ` topspace Y)) X id
  unfolding section_map_def
  using assms retraction_maps_to_retract_maps by blast
qed

lemma retraction_maps_section_image2:
  retraction_maps X Y r s
    ⟹ subtopology X (s ` (topspace Y)) homeomorphic_space Y
 using embedding_map_imp_homeomorphic_space homeomorphic_space_sym section_imp_embedding_map
    section_map_def by blast
```

2.3.16 Paths and path-connectedness

```
definition pathin :: 'a topology ⇒ (real ⇒ 'a) ⇒ bool where
  pathin X g ≡ continuous_map (subtopology euclideanreal {0..1}) X g

lemma pathin_compose:
```

⟦pathin X g; continuous_map X Y f⟧ ⟹ pathin Y (f ∘ g)
by (simp add: continuous_map_compose pathin_def)

lemma pathin_subtopology:
pathin (subtopology X S) g ⟷ pathin X g ∧ (∀x ∈ {0..1}. g x ∈ S)
by (auto simp: pathin_def continuous_map_in_subtopology)

lemma pathin_const:
pathin X (λx. a) ⟷ a ∈ topspace X
by (simp add: pathin_def)

lemma path_start_in_topspace: pathin X g ⟹ g 0 ∈ topspace X
by (force simp: pathin_def continuous_map)

lemma path_finish_in_topspace: pathin X g ⟹ g 1 ∈ topspace X
by (force simp: pathin_def continuous_map)

lemma path_image_subset_topspace: pathin X g ⟹ g ‘ ({0..1}) ⊆ topspace X
by (force simp: pathin_def continuous_map)

definition path_connected_space :: 'a topology ⇒ bool
where path_connected_space X ≡ ∀x ∈ topspace X. ∀ y ∈ topspace X. ∃g. pathin X g ∧ g 0 = x ∧ g 1 = y

definition path_connectedin :: 'a topology ⇒ 'a set ⇒ bool
where path_connectedin X S ≡ S ⊆ topspace X ∧ path_connected_space(subtopology X S)

lemma path_connectedin_absolute [simp]:
path_connectedin (subtopology X S) S ⟷ path_connectedin X S
by (simp add: path_connectedin_def subtopology_subtopology)

lemma path_connectedin_subset_topspace:
path_connectedin X S ⟹ S ⊆ topspace X
by (simp add: path_connectedin_def)

lemma path_connectedin_subtopology:
path_connectedin (subtopology X S) T ⟷ path_connectedin X T ∧ T ⊆ S
by (auto simp: path_connectedin_def subtopology_subtopology inf.absorb2)

lemma path_connectedin:
path_connectedin X S ⟷
S ⊆ topspace X ∧
(∀x ∈ S. ∀y ∈ S. ∃g. pathin X g ∧ g ‘ {0..1} ⊆ S ∧ g 0 = x ∧ g 1 = y)
unfolding path_connectedin_def path_connected_space_def pathin_def continuous_map_in_subtopology
by (intro conj_cong refl ball_cong) (simp_all add: inf.absorb_iff2)

lemma path_connectedin_topspace:
path_connectedin X (topspace X) ⟷ path_connected_space X

by (*simp add*: *path_connectedin_def*)

lemma *path_connected_imp_connected_space*:
assumes *path_connected_space X*
shows *connected_space X*
proof −
have ∗: ∃ *S*. *connectedin X S* ∧ *g 0* ∈ *S* ∧ *g 1* ∈ *S* **if** *pathin X g* **for** *g*
proof (*intro exI conjI*)
have *continuous_map* (*subtopology euclideanreal* {*0..1*}) *X g*
using *connectedin_absolute that* **by** (*simp add*: *pathin_def*)
then show *connectedin X* (*g* ‘ {*0..1*})
by (*rule connectedin_continuous_map_image*) *auto*
qed *auto*
show *?thesis*
using *assms*
by (*auto intro*: ∗ *simp add*: *path_connected_space_def connected_space_subconnected Ball_def*)
qed

lemma *path_connectedin_imp_connectedin*:
path_connectedin X S ⟹ *connectedin X S*
by (*simp add*: *connectedin_def path_connected_imp_connected_space path_connectedin_def*)

lemma *path_connected_space_topspace_empty*:
topspace X = {} ⟹ *path_connected_space X*
by (*simp add*: *path_connected_space_def*)

lemma *path_connectedin_empty* [*simp*]: *path_connectedin X* {}
by (*simp add*: *path_connectedin*)

lemma *path_connectedin_singleton* [*simp*]: *path_connectedin X* {*a*} ⟷ *a* ∈ *topspace X*
proof
show *path_connectedin X* {*a*} ⟹ *a* ∈ *topspace X*
by (*simp add*: *path_connectedin*)
show *a* ∈ *topspace X* ⟹ *path_connectedin X* {*a*}
unfolding *path_connectedin*
using *pathin_const* **by** *fastforce*
qed

lemma *path_connectedin_continuous_map_image*:
assumes *f*: *continuous_map X Y f* **and** *S*: *path_connectedin X S*
shows *path_connectedin Y* (*f* ‘ *S*)
proof −
have *fX*: *f* ‘ (*topspace X*) ⊆ *topspace Y*
by (*metis f continuous_map_image_subset_topspace*)
show *?thesis*
unfolding *path_connectedin*
proof (*intro conjI ballI*; *clarify?*)

fix *x*
assume $x \in S$
show $f\ x \in$ *topspace Y*
by (*meson S fX* ‹$x \in S$› *image_subset_iff path_connectedin_subset_topspace set_mp*)
next
fix *x y*
assume $x \in S$ **and** $y \in S$
then obtain *g* **where** *g*: *pathin X g* g ‘ $\{0..1\} \subseteq S$ $g\ 0 = x$ $g\ 1 = y$
using *S* **by** (*force simp*: *path_connectedin*)
show $\exists g.$ *pathin Y g* $\wedge$ g ‘ $\{0..1\} \subseteq f$ ‘ $S \wedge g\ 0 = f\ x \wedge g\ 1 = f\ y$
proof (*intro exI conjI*)
show *pathin Y* $(f \circ g)$
using ‹*pathin X g*› *f pathin_compose* **by** *auto*
qed (*use g* **in** *auto*)
qed
qed

lemma *path_connectedin_discrete_topology*:
path_connectedin (*discrete_topology U*) $S \longleftrightarrow S \subseteq U \wedge (\exists a.\ S \subseteq \{a\})$
apply *safe*
using *path_connectedin_subset_topspace* **apply** *fastforce*
apply (*meson connectedin_discrete_topology path_connectedin_imp_connectedin*)
using *subset_singletonD* **by** *fastforce*

lemma *path_connected_space_discrete_topology*:
path_connected_space (*discrete_topology U*) $\longleftrightarrow (\exists a.\ U \subseteq \{a\})$
by (*metis path_connectedin_discrete_topology path_connectedin_topspace path_connected_space_topspace_empty subset_singletonD topspace_discrete_topology*)

lemma *homeomorphic_path_connected_space_imp*:
⟦*path_connected_space X*; *X homeomorphic_space Y*⟧ $\Longrightarrow$ *path_connected_space Y*
unfolding *homeomorphic_space_def homeomorphic_maps_def*
by (*metis* (*no_types*, *hide_lams*) *continuous_map_closedin continuous_map_image_subset_topspace imageI order_class.order.antisym path_connectedin_continuous_map_image path_connectedin_topspace subsetI*)

lemma *homeomorphic_path_connected_space*:
X homeomorphic_space Y $\Longrightarrow$ *path_connected_space X* $\longleftrightarrow$ *path_connected_space Y*
by (*meson homeomorphic_path_connected_space_imp homeomorphic_space_sym*)

lemma *homeomorphic_map_path_connectedness*:
assumes *homeomorphic_map X Y f* $U \subseteq$ *topspace X*
shows *path_connectedin Y* (f ‘ U) $\longleftrightarrow$ *path_connectedin X U*
unfolding *path_connectedin_def*
proof (*intro conj_cong homeomorphic_path_connected_space*)

show (*f* ‘ *U* ⊆ *topspace Y*) = (*U* ⊆ *topspace X*)
using *assms homeomorphic_imp_surjective_map* **by** *blast*
next
assume *U* ⊆ *topspace X*
show *subtopology Y* (*f* ‘ *U*) *homeomorphic_space subtopology X U*
using *assms* **unfolding** *homeomorphic_eq_everything_map*
by (*metis* (*no_types*, *hide_lams*) *assms homeomorphic_map_subtopologies homeomorphic_space homeomorphic_space_sym image_mono inf.absorb_iff2*)
qed

lemma *homeomorphic_map_path_connectedness_eq*:
homeomorphic_map X Y f ⟹ *path_connectedin X U* ⟷ *U* ⊆ *topspace X* ∧ *path_connectedin Y* (*f* ‘ *U*)
by (*meson homeomorphic_map_path_connectedness path_connectedin_def*)

2.3.17 Connected components

definition *connected_component_of* :: ′*a topology* ⇒ ′*a* ⇒ ′*a* ⇒ *bool*
where *connected_component_of X x y* ≡
∃ *T*. *connectedin X T* ∧ *x* ∈ *T* ∧ *y* ∈ *T*

abbreviation *connected_component_of_set*
where *connected_component_of_set X x* ≡ *Collect* (*connected_component_of X x*)

definition *connected_components_of* :: ′*a topology* ⇒ (′*a set*) *set*
where *connected_components_of X* ≡ *connected_component_of_set X* ‘ *topspace X*

lemma *connected_component_in_topspace*:
connected_component_of X x y ⟹ *x* ∈ *topspace X* ∧ *y* ∈ *topspace X*
by (*meson connected_component_of_def connectedin_subset_topspace in_mono*)

lemma *connected_component_of_refl*:
connected_component_of X x x ⟷ *x* ∈ *topspace X*
by (*meson connected_component_in_topspace connected_component_of_def connectedin_sing insertI1*)

lemma *connected_component_of_sym*:
connected_component_of X x y ⟷ *connected_component_of X y x*
by (*meson connected_component_of_def*)

lemma *connected_component_of_trans*:
⟦*connected_component_of X x y*; *connected_component_of X y z*⟧
⟹ *connected_component_of X x z*
unfolding *connected_component_of_def*
using *connectedin_Un* **by** *blast*

lemma *connected_component_of_mono*:
⟦*connected_component_of* (*subtopology X S*) *x y*; *S* ⊆ *T*⟧
⟹ *connected_component_of* (*subtopology X T*) *x y*

by (*metis connected_component_of_def connectedin_subtopology inf.absorb_iff2 subtopology_subtopology*)

lemma *connected_component_of_set*:
connected_component_of_set X x = {*y*. ∃ *T*. *connectedin X T* ∧ *x* ∈ *T* ∧ *y* ∈ *T*}
by (*meson connected_component_of_def*)

lemma *connected_component_of_subset_topspace*:
connected_component_of_set X x ⊆ *topspace X*
using *connected_component_in_topspace* **by** *force*

lemma *connected_component_of_eq_empty*:
connected_component_of_set X x = {} ⟷ (*x* ∉ *topspace X*)
using *connected_component_in_topspace connected_component_of_refl* **by** *fastforce*

lemma *connected_space_iff_connected_component*:
connected_space X ⟷ (∀ *x* ∈ *topspace X*. ∀ *y* ∈ *topspace X*. *connected_component_of X x y*)
by (*simp add*: *connected_component_of_def connected_space_subconnected*)

lemma *connected_space_imp_connected_component_of*:
⟦*connected_space X*; *a* ∈ *topspace X*; *b* ∈ *topspace X*⟧
⟹ *connected_component_of X a b*
by (*simp add*: *connected_space_iff_connected_component*)

lemma *connected_space_connected_component_set*:
connected_space X ⟷ (∀ *x* ∈ *topspace X*. *connected_component_of_set X x* = *topspace X*)
using *connected_component_of_subset_topspace connected_space_iff_connected_component*
by *fastforce*

lemma *connected_component_of_maximal*:
⟦*connectedin X S*; *x* ∈ *S*⟧ ⟹ *S* ⊆ *connected_component_of_set X x*
by (*meson Ball_Collect connected_component_of_def*)

lemma *connected_component_of_equiv*:
connected_component_of X x y ⟷
x ∈ *topspace X* ∧ *y* ∈ *topspace X* ∧ *connected_component_of X x* = *connected_component_of X y*
apply (*simp add*: *connected_component_in_topspace fun_eq_iff*)
by (*meson connected_component_of_refl connected_component_of_sym connected_component_of_trans*)

lemma *connected_component_of_disjoint*:
disjnt (*connected_component_of_set X x*) (*connected_component_of_set X y*)
⟷ ~(*connected_component_of X x y*)
using *connected_component_of_equiv* **unfolding** *disjnt_iff* **by** *force*

lemma *connected_component_of_eq*:
connected_component_of X x = *connected_component_of X y* ⟷

$(x \notin topspace\ X) \wedge (y \notin topspace\ X) \vee$
$x \in topspace\ X \wedge y \in topspace\ X \wedge$
connected_component_of X x y
by (*metis Collect_empty_eq_bot connected_component_of_eq_empty connected_component_of_equiv*)

lemma *connectedin_connected_component_of*:
connectedin X (*connected_component_of_set X x*)
proof −
have *connected_component_of_set X x* $= \bigcup \{T.\ connectedin\ X\ T \wedge x \in T\}$
by (*auto simp*: *connected_component_of_def*)
then show *?thesis*
apply (*rule ssubst*)
by (*blast intro*: *connectedin_Union*)
qed

lemma *Union_connected_components_of*:
$\bigcup$(*connected_components_of X*) = *topspace X*
unfolding *connected_components_of_def*
apply (*rule equalityI*)
apply (*simp add*: *SUP_least connected_component_of_subset_topspace*)
using *connected_component_of_refl* **by** *fastforce*

lemma *connected_components_of_maximal*:
⟦$C \in$ *connected_components_of X*; *connectedin X S*; ~*disjnt C S*⟧ $\Longrightarrow S \subseteq C$
unfolding *connected_components_of_def disjnt_def*
apply *clarify*
by (*metis Int_emptyI connected_component_of_def connected_component_of_trans mem_Collect_eq*)

lemma *pairwise_disjoint_connected_components_of*:
pairwise disjnt (*connected_components_of X*)
unfolding *connected_components_of_def pairwise_def*
apply *clarify*
by (*metis connected_component_of_disjoint connected_component_of_equiv*)

lemma *complement_connected_components_of_Union*:
$C \in$ *connected_components_of X*
$\Longrightarrow$ *topspace X* − $C = \bigcup$ (*connected_components_of X* − $\{C\}$)
apply (*rule equalityI*)
using *Union_connected_components_of* **apply** *fastforce*
by (*metis Diff_cancel Diff_subset Union_connected_components_of cSup_singleton diff_Union_pairwise_disjoint equalityE insert_subsetI pairwise_disjoint_connected_components_of*)

lemma *nonempty_connected_components_of*:
$C \in$ *connected_components_of X* $\Longrightarrow C \neq \{\}$
unfolding *connected_components_of_def*
by (*metis* (*no_types*, *lifting*) *connected_component_of_eq_empty imageE*)

lemma *connected_components_of_subset*:
  $C \in$ *connected_components_of* $X \Longrightarrow C \subseteq$ *topspace* X
  using *Union_connected_components_of* **by** *fastforce*

lemma *connectedin_connected_components_of*:
  assumes $C \in$ *connected_components_of* X
  shows *connectedin* X C
proof −
  have $C \in$ *connected_component_of_set* X ‘ *topspace* X
    using *assms connected_components_of_def* **by** *blast*
then show *?thesis*
  using *connectedin_connected_component_of* **by** *fastforce*
qed

lemma *connected_component_in_connected_components_of*:
  connected_component_of_set X $a \in$ *connected_components_of* $X \longleftrightarrow a \in$ *topspace* X
  apply (*rule iffI*)
  using *connected_component_of_eq_empty nonempty_connected_components_of* **apply** *fastforce*
  by (*simp add*: *connected_components_of_def*)

lemma *connected_space_iff_components_eq*:
  connected_space $X \longleftrightarrow (\forall C \in$ *connected_components_of* $X. \forall C' \in$ *connected_components_of* $X. C = C')$
  apply (*rule iffI*)
  apply (*force simp*: *connected_components_of_def connected_space_connected_component_set image_iff*)
  by (*metis connected_component_in_connected_components_of connected_component_of_refl connected_space_iff_connected_component mem_Collect_eq*)

lemma *connected_components_of_eq_empty*:
  connected_components_of $X = \{\} \longleftrightarrow$ *topspace* $X = \{\}$
  by (*simp add*: *connected_components_of_def*)

lemma *connected_components_of_empty_space*:
  topspace $X = \{\} \Longrightarrow$ *connected_components_of* $X = \{\}$
  by (*simp add*: *connected_components_of_eq_empty*)

lemma *connected_components_of_subset_sing*:
  connected_components_of $X \subseteq \{S\} \longleftrightarrow$ *connected_space* $X \wedge$ (*topspace* $X = \{\}$ $\vee$ *topspace* $X = S$)
proof (*cases topspace* $X = \{\}$)
  case *True*
  then show *?thesis*
   by (*simp add*: *connected_components_of_empty_space connected_space_topspace_empty*)
next
  case *False*
  then show *?thesis*

by (*metis* (*no_types*, *hide_lams*) *Union_connected_components_of ccpo_Sup_singleton connected_components_of_eq_empty connected_space_iff_components_eq insertI1 singletonD subsetI subset_singleton_iff*)
qed

lemma *connected_space_iff_components_subset_singleton*:
connected_space X ⟷ (∃ *a*. *connected_components_of X* ⊆ {*a*})
by (*simp add*: *connected_components_of_subset_sing*)

lemma *connected_components_of_eq_singleton*:
connected_components_of X = {*S*}
⟷ *connected_space X* ∧ *topspace X* ≠ {} ∧ *S* = *topspace X*
by (*metis ccpo_Sup_singleton connected_components_of_subset_sing insert_not_empty subset_singleton_iff*)

lemma *connected_components_of_connected_space*:
connected_space X ⟹ *connected_components_of X* = (*if topspace X* = {} *then* {} *else* {*topspace X*})
by (*simp add*: *connected_components_of_eq_empty connected_components_of_eq_singleton*)

lemma *exists_connected_component_of_superset*:
assumes *connectedin X S* **and** *ne*: *topspace X* ≠ {}
shows ∃ *C*. *C* ∈ *connected_components_of X* ∧ *S* ⊆ *C*
proof (*cases S* = {})
case *True*
then show *?thesis*
using *ne connected_components_of_def* **by** *blast*
next
case *False*
then show *?thesis*
by (*meson all_not_in_conv assms*(*1*) *connected_component_in_connected_components_of connected_component_of_maximal connectedin_subset_topspace in_mono*)
qed

lemma *closedin_connected_components_of*:
assumes *C* ∈ *connected_components_of X*
shows *closedin X C*
proof –
obtain *x* **where** *x* ∈ *topspace X* **and** *x*: *C* = *connected_component_of_set X x*
using *assms* **by** (*auto simp*: *connected_components_of_def*)
have *connected_component_of_set X x* ⊆ *topspace X*
by (*simp add*: *connected_component_of_subset_topspace*)
moreover have *X closure_of connected_component_of_set X x* ⊆ *connected_component_of_set X x*
proof (*rule connected_component_of_maximal*)
show *connectedin X* (*X closure_of connected_component_of_set X x*)
by (*simp add*: *connectedin_closure_of connectedin_connected_component_of*)
show *x* ∈ *X closure_of connected_component_of_set X x*

```
    by (simp add: ‹x ∈ topspace X› closure_of connected_component_of_refl)
  qed
  ultimately
  show ?thesis
    using closure_of_subset_eq x by auto
qed

lemma closedin_connected_component_of:
   closedin X (connected_component_of_set X x)
 by (metis closedin_connected_components_of closedin_empty connected_component_in_connected_components_of
connected_component_of_eq_empty)

lemma connected_component_of_eq_overlap:
   connected_component_of_set X x = connected_component_of_set X y ⟷
      (x ∉ topspace X) ∧ (y ∉ topspace X) ∨
      ~(connected_component_of_set X x ∩ connected_component_of_set X y = {})
  using connected_component_of_equiv by fastforce

lemma connected_component_of_nonoverlap:
   connected_component_of_set X x ∩ connected_component_of_set X y = {} ⟷
     (x ∉ topspace X) ∨ (y ∉ topspace X) ∨
     ~(connected_component_of_set X x = connected_component_of_set X y)
   by (metis connected_component_of_eq_empty connected_component_of_eq_overlap
inf.idem)

lemma connected_component_of_overlap:
  ~(connected_component_of_set X x ∩ connected_component_of_set X y = {}) ⟷
    x ∈ topspace X ∧ y ∈ topspace X ∧
    connected_component_of_set X x = connected_component_of_set X y
  by (meson connected_component_of_nonoverlap)

end
```

2.4 Connected Components

```
theory Connected
  imports
    Abstract_Topology_2
begin
```

2.4.1 Connectedness

```
lemma connected_local:
 connected S ⟷
  ¬ (∃ e1 e2.
      openin (top_of_set S) e1 ∧
      openin (top_of_set S) e2 ∧
      S ⊆ e1 ∪ e2 ∧
      e1 ∩ e2 = {} ∧
```

$e1 \neq \{\} \wedge$
$e2 \neq \{\})$
unfolding *connected_def openin_open*
by *safe blast+*

lemma *exists_diff*:
fixes P :: *'a set* $\Rightarrow$ *bool*
shows $(\exists S.\ P\ (-\ S)) \longleftrightarrow (\exists S.\ P\ S)$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof −
have *?rhs* **if** *?lhs*
using *that* **by** *blast*
moreover have $P\ (-\ (-\ S))$ **if** $P\ S$ **for** S
proof −
have $S = -\ (-\ S)$ **by** *simp*
with *that* **show** *?thesis* **by** *metis*
qed
ultimately show *?thesis* **by** *metis*
qed

lemma *connected_clopen*: *connected* $S \longleftrightarrow$
$(\forall T.$ *openin* (*top_of_set* S) $T \wedge$
closedin (*top_of_set* S) $T \longrightarrow T = \{\} \vee T = S)$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof −
have $\neg$ *connected* $S \longleftrightarrow$
$(\exists e1\ e2.\ open\ e1 \wedge open\ (-\ e2) \wedge S \subseteq e1 \cup (-\ e2) \wedge e1 \cap (-\ e2) \cap S = \{\}$
$\wedge\ e1 \cap S \neq \{\} \wedge (-\ e2) \cap S \neq \{\})$
unfolding *connected_def openin_open closedin_closed*
by (*metis double_complement*)
then have *th0*: *connected* $S \longleftrightarrow$
$\neg\ (\exists e2\ e1.\ closed\ e2 \wedge open\ e1 \wedge S \subseteq e1 \cup (-\ e2) \wedge e1 \cap (-\ e2) \cap S = \{\}$
$\wedge\ e1 \cap S \neq \{\} \wedge (-\ e2) \cap S \neq \{\})$
(**is** _ $\longleftrightarrow \neg\ (\exists e2\ e1.\ ?P\ e2\ e1)$)
by (*simp add*: *closed_def*) *metis*
have *th1*: *?rhs* $\longleftrightarrow \neg\ (\exists t'\ t.\ closed\ t' \wedge t = S \cap t' \wedge t \neq \{\} \wedge t \neq S \wedge (\exists t'.\ open\ t'$
$\wedge\ t = S \cap t'))$
(**is** _ $\longleftrightarrow \neg\ (\exists t'\ t.\ ?Q\ t'\ t)$)
unfolding *connected_def openin_open closedin_closed* **by** *auto*
have $(\exists e1.\ ?P\ e2\ e1) \longleftrightarrow (\exists t.\ ?Q\ e2\ t)$ **for** $e2$
proof −
have $?P\ e2\ e1 \longleftrightarrow (\exists t.\ closed\ e2 \wedge t = S \cap e2 \wedge open\ e1 \wedge t = S \cap e1 \wedge t \neq \{\}$
$\wedge\ t \neq S)$ **for** $e1$
by *auto*
then show *?thesis*
by *metis*
qed
then have $\forall e2.\ (\exists e1.\ ?P\ e2\ e1) \longleftrightarrow (\exists t.\ ?Q\ e2\ t)$
by *blast*
then show *?thesis*

by (*simp add*: *th0 th1*)
qed

2.4.2 Connected components, considered as a connectedness relation or a set

definition *connected_component S x y* ≡ ∃ *T*. *connected T* ∧ *T* ⊆ *S* ∧ *x* ∈ *T* ∧ *y* ∈ *T*

abbreviation *connected_component_set S x* ≡ *Collect* (*connected_component S x*)

lemma *connected_componentI*:
connected T ⟹ *T* ⊆ *S* ⟹ *x* ∈ *T* ⟹ *y* ∈ *T* ⟹ *connected_component S x y*
by (*auto simp*: *connected_component_def*)

lemma *connected_component_in*: *connected_component S x y* ⟹ *x* ∈ *S* ∧ *y* ∈ *S*
by (*auto simp*: *connected_component_def*)

lemma *connected_component_refl*: *x* ∈ *S* ⟹ *connected_component S x x*
by (*auto simp*: *connected_component_def*) (*use connected_sing* **in** *blast*)

lemma *connected_component_refl_eq* [*simp*]: *connected_component S x x* ⟷ *x* ∈ *S*
by (*auto simp*: *connected_component_refl*) (*auto simp*: *connected_component_def*)

lemma *connected_component_sym*: *connected_component S x y* ⟹ *connected_component S y x*
by (*auto simp*: *connected_component_def*)

lemma *connected_component_trans*:
connected_component S x y ⟹ *connected_component S y z* ⟹ *connected_component S x z*
unfolding *connected_component_def*
by (*metis Int_iff Un_iff Un_subset_iff equals0D connected_Un*)

lemma *connected_component_of_subset*:
connected_component S x y ⟹ *S* ⊆ *T* ⟹ *connected_component T x y*
by (*auto simp*: *connected_component_def*)

lemma *connected_component_Union*: *connected_component_set S x* = ⋃{*T*. *connected T* ∧ *x* ∈ *T* ∧ *T* ⊆ *S*}
by (*auto simp*: *connected_component_def*)

lemma *connected_connected_component* [*iff*]: *connected* (*connected_component_set S x*)
by (*auto simp*: *connected_component_Union intro*: *connected_Union*)

lemma *connected_iff_eq_connected_component_set*:
connected S ⟷ (∀ *x* ∈ *S*. *connected_component_set S x* = *S*)

proof (*cases* $S = \{\}$)
  case *True*
  then show *?thesis* **by** *simp*
next
  case *False*
  then obtain x **where** $x \in S$ **by** *auto*
  show *?thesis*
  proof
    assume *connected S*
    then show $\forall x \in S.$ *connected_component_set* $S\ x = S$
      by (*force simp*: *connected_component_def*)
  next
    assume $\forall x \in S.$ *connected_component_set* $S\ x = S$
    then show *connected S*
      by (*metis* ‹$x \in S$› *connected_connected_component*)
  qed
qed

lemma *connected_component_subset*: *connected_component_set* $S\ x \subseteq S$
  using *connected_component_in* **by** *blast*

lemma *connected_component_eq_self*: *connected* $S \Longrightarrow x \in S \Longrightarrow$ *connected_component_set* $S\ x = S$
  by (*simp add*: *connected_iff_eq_connected_component_set*)

lemma *connected_iff_connected_component*:
  connected $S \longleftrightarrow (\forall x \in S. \forall y \in S.$ *connected_component* $S\ x\ y)$
  using *connected_component_in* **by** (*auto simp*: *connected_iff_eq_connected_component_set*)

lemma *connected_component_maximal*:
  $x \in T \Longrightarrow$ *connected* $T \Longrightarrow T \subseteq S \Longrightarrow T \subseteq$ (*connected_component_set* $S\ x$)
  using *connected_component_eq_self connected_component_of_subset* **by** *blast*

lemma *connected_component_mono*:
  $S \subseteq T \Longrightarrow$ *connected_component_set* $S\ x \subseteq$ *connected_component_set* $T\ x$
  by (*simp add*: *Collect_mono connected_component_of_subset*)

lemma *connected_component_eq_empty* [*simp*]: *connected_component_set* $S\ x = \{\} \longleftrightarrow x \notin S$
  using *connected_component_refl* **by** (*fastforce simp*: *connected_component_in*)

lemma *connected_component_set_empty* [*simp*]: *connected_component_set* $\{\}\ x = \{\}$
  using *connected_component_eq_empty* **by** *blast*

lemma *connected_component_eq*:
  $y \in$ *connected_component_set* $S\ x \Longrightarrow$ (*connected_component_set* $S\ y =$ *connected_component_set* $S\ x$)
  by (*metis* (*no_types*, *lifting*)

Collect_cong connected_component_sym connected_component_trans mem_Collect_eq)

lemma *closed_connected_component*:
  assumes *S*: *closed S*
  shows *closed* (*connected_component_set S x*)
proof (*cases x* ∈ *S*)
  case *False*
  then show *?thesis*
    by (*metis connected_component_eq_empty closed_empty*)
next
  case *True*
  show *?thesis*
    unfolding *closure_eq* [*symmetric*]
  proof
    show *closure* (*connected_component_set S x*) ⊆ *connected_component_set S x*
      apply (*rule connected_component_maximal*)
        apply (*simp add*: *closure_def True*)
       apply (*simp add*: *connected_imp_connected_closure*)
      apply (*simp add*: *S closure_minimal connected_component_subset*)
      done
  next
    show *connected_component_set S x* ⊆ *closure* (*connected_component_set S x*)
      by (*simp add*: *closure_subset*)
  qed
qed

lemma *connected_component_disjoint*:
  connected_component_set S a ∩ *connected_component_set S b* = {} ⟷
    a ∉ *connected_component_set S b*
  apply (*auto simp*: *connected_component_eq*)
  using *connected_component_eq connected_component_sym*
  apply *blast*
  done

lemma *connected_component_nonoverlap*:
  connected_component_set S a ∩ *connected_component_set S b* = {} ⟷
    a ∉ *S* ∨ *b* ∉ *S* ∨ *connected_component_set S a* ≠ *connected_component_set S b*
  apply (*auto simp*: *connected_component_in*)
  using *connected_component_refl_eq*
    apply *blast*
   apply (*metis connected_component_eq mem_Collect_eq*)
  apply (*metis connected_component_eq mem_Collect_eq*)
  done

lemma *connected_component_overlap*:
  connected_component_set S a ∩ *connected_component_set S b* ≠ {} ⟷
    a ∈ *S* ∧ *b* ∈ *S* ∧ *connected_component_set S a* = *connected_component_set S b*
  by (*auto simp*: *connected_component_nonoverlap*)

lemma *connected_component_sym_eq*: *connected_component S x y* ⟷ *connected_component S y x*
  using *connected_component_sym* **by** *blast*

lemma *connected_component_eq_eq*:
  connected_component_set S x = *connected_component_set S y* ⟷
    $x \notin S \wedge y \notin S \vee x \in S \wedge y \in S \wedge$ *connected_component S x y*
  apply (*cases* $y \in S$, *simp*)
  apply (*metis connected_component_eq connected_component_eq_empty connected_component_refl_eq mem_Collect_eq*)
  apply (*cases* $x \in S$, *simp*)
   apply (*metis connected_component_eq_empty*)
  using *connected_component_eq_empty*
  apply *blast*
  done

lemma *connected_iff_connected_component_eq*:
 connected S ⟷ ($\forall x \in S. \forall y \in S.$ *connected_component_set S x* = *connected_component_set S y*)
  by (*simp add*: *connected_component_eq_eq connected_iff_connected_component*)

lemma *connected_component_idemp*:
 connected_component_set (*connected_component_set S x*) *x* = *connected_component_set S x*
  apply (*rule subset_antisym*)
   apply (*simp add*: *connected_component_subset*)
  apply (*metis connected_component_eq_empty connected_component_maximal*
      connected_component_refl_eq connected_connected_component mem_Collect_eq set_eq_subset)
  done

lemma *connected_component_unique*:
  ⟦$x \in c$; $c \subseteq S$; *connected c*;
    $\bigwedge c'$. ⟦$x \in c'$; $c' \subseteq S$; *connected c'*⟧ $\Longrightarrow c' \subseteq c$⟧
      $\Longrightarrow$ *connected_component_set S x* = *c*
  apply (*rule subset_antisym*)
   apply (*meson connected_component_maximal connected_component_subset connected_connected_component contra_subsetD*)
  by (*simp add*: *connected_component_maximal*)

lemma *joinable_connected_component_eq*:
  ⟦*connected T*; $T \subseteq S$;
    connected_component_set S x $\cap T \neq \{\}$;
    connected_component_set S y $\cap T \neq \{\}$⟧
    $\Longrightarrow$ *connected_component_set S x* = *connected_component_set S y*
apply (*simp add*: *ex_in_conv* [*symmetric*])
apply (*rule connected_component_eq*)
by (*metis* (*no_types*, *hide_lams*) *connected_component_eq_eq connected_component_in connected_component_maximal subsetD mem_Collect_eq*)

lemma *Union_connected_component*: $\bigcup$(*connected_component_set S* ‘ *S*) = *S*
  apply (*rule subset_antisym*)
  apply (*simp add*: *SUP_least connected_component_subset*)
  using *connected_component_refl_eq*
  by *force*

lemma *complement_connected_component_unions*:
    S − *connected_component_set S x* =
    $\bigcup$(*connected_component_set S* ‘ *S* − {*connected_component_set S x*})
  apply (*subst Union_connected_component* [*symmetric*], *auto*)
  apply (*metis connected_component_eq_eq connected_component_in*)
  by (*metis connected_component_eq mem_Collect_eq*)

lemma *connected_component_intermediate_subset*:
    ⟦*connected_component_set U a* ⊆ *T*; *T* ⊆ *U*⟧
    ⟹ *connected_component_set T a* = *connected_component_set U a*
  apply (*case_tac a* ∈ *U*)
  apply (*simp add*: *connected_component_maximal connected_component_mono subset_antisym*)
  using *connected_component_eq_empty* **by** *blast*

2.4.3 The set of connected components of a set

definition *components*:: ′*a*::*topological_space set* ⇒ ′*a set set*
  where *components S* ≡ *connected_component_set S* ‘ *S*

lemma *components_iff*: *S* ∈ *components U* ⟷ (∃ *x*. *x* ∈ *U* ∧ *S* = *connected_component_set U x*)
  by (*auto simp*: *components_def*)

lemma *componentsI*: *x* ∈ *U* ⟹ *connected_component_set U x* ∈ *components U*
  by (*auto simp*: *components_def*)

lemma *componentsE*:
  assumes *S* ∈ *components U*
  obtains *x* **where** *x* ∈ *U S* = *connected_component_set U x*
  using *assms* **by** (*auto simp*: *components_def*)

lemma *Union_components* [*simp*]: $\bigcup$(*components u*) = *u*
  apply (*rule subset_antisym*)
  using *Union_connected_component components_def* **apply** *fastforce*
  apply (*metis Union_connected_component components_def set_eq_subset*)
  done

lemma *pairwise_disjoint_components*: *pairwise* (λ*X Y*. *X* ∩ *Y* = {}) (*components u*)

apply (*simp add*: *pairwise_def*)
apply (*auto simp*: *components_iff*)
apply (*metis connected_component_eq_eq connected_component_in*)+
done

lemma *in_components_nonempty*: $c \in components\ s \Longrightarrow c \neq \{\}$
by (*metis components_iff connected_component_eq_empty*)

lemma *in_components_subset*: $c \in components\ s \Longrightarrow c \subseteq s$
using *Union_components* **by** *blast*

lemma *in_components_connected*: $c \in components\ s \Longrightarrow connected\ c$
by (*metis components_iff connected_connected_component*)

lemma *in_components_maximal*:
$c \in components\ s \longleftrightarrow$
$c \neq \{\} \wedge c \subseteq s \wedge connected\ c \wedge (\forall d.\ d \neq \{\} \wedge c \subseteq d \wedge d \subseteq s \wedge connected\ d \longrightarrow d = c)$
apply (*rule iffI*)
apply (*simp add*: *in_components_nonempty in_components_connected*)
apply (*metis* (*full_types*) *components_iff connected_component_eq_self connected_component_intermediat connected_component_refl in_components_subset mem_Collect_eq rev_subsetD*)
apply (*metis bot.extremum_uniqueI components_iff connected_component_eq_empty connected_component_maximal connected_component_subset connected_connected_component subset_emptyI*)
done

lemma *joinable_components_eq*:
$connected\ t \wedge t \subseteq s \wedge c1 \in components\ s \wedge c2 \in components\ s \wedge c1 \cap t \neq \{\} \wedge c2 \cap t \neq \{\} \Longrightarrow c1 = c2$
by (*metis* (*full_types*) *components_iff joinable_connected_component_eq*)

lemma *closed_components*: $[\![closed\ s;\ c \in components\ s]\!] \Longrightarrow closed\ c$
by (*metis closed_connected_component components_iff*)

lemma *components_nonoverlap*:
$[\![c \in components\ s;\ c' \in components\ s]\!] \Longrightarrow (c \cap c' = \{\}) \longleftrightarrow (c \neq c')$
apply (*auto simp*: *in_components_nonempty components_iff*)
using *connected_component_refl* **apply** *blast*
apply (*metis connected_component_eq_eq connected_component_in*)
by (*metis connected_component_eq mem_Collect_eq*)

lemma *components_eq*: $[\![c \in components\ s;\ c' \in components\ s]\!] \Longrightarrow (c = c' \longleftrightarrow c \cap c' \neq \{\})$
by (*metis components_nonoverlap*)

lemma *components_eq_empty* [*simp*]: $components\ s = \{\} \longleftrightarrow s = \{\}$
by (*simp add*: *components_def*)

lemma *components_empty* [*simp*]: *components* {} = {}
by *simp*

lemma *connected_eq_connected_components_eq*: *connected s* ⟷ (∀ *c* ∈ *components s*. ∀ *c'* ∈ *components s*. *c* = *c'*)
by (*metis* (*no_types*, *hide_lams*) *components_iff connected_component_eq_eq connected_iff_connected_component*)

lemma *components_eq_sing_iff*: *components s* = {*s*} ⟷ *connected s* ∧ *s* ≠ {}
apply (*rule iffI*)
using *in_components_connected* **apply** *fastforce*
apply *safe*
using *Union_components* **apply** *fastforce*
apply (*metis components_iff connected_component_eq_self*)
using *in_components_maximal*
apply *auto*
done

lemma *components_eq_sing_exists*: (∃ *a*. *components s* = {*a*}) ⟷ *connected s* ∧ *s* ≠ {}
apply (*rule iffI*)
using *connected_eq_connected_components_eq* **apply** *fastforce*
apply (*metis components_eq_sing_iff*)
done

lemma *connected_eq_components_subset_sing*: *connected s* ⟷ *components s* ⊆ {*s*}
by (*metis Union_components components_empty components_eq_sing_iff connected_empty insert_subset order_refl subset_singletonD*)

lemma *connected_eq_components_subset_sing_exists*: *connected s* ⟷ (∃ *a*. *components s* ⊆ {*a*})
by (*metis components_eq_sing_exists connected_eq_components_subset_sing empty_iff subset_iff subset_singletonD*)

lemma *in_components_self*: *s* ∈ *components s* ⟷ *connected s* ∧ *s* ≠ {}
by (*metis components_empty components_eq_sing_iff empty_iff in_components_connected insertI1*)

lemma *components_maximal*: ⟦*c* ∈ *components s*; *connected t*; *t* ⊆ *s*; *c* ∩ *t* ≠ {}⟧ ⟹ *t* ⊆ *c*
apply (*simp add*: *components_def ex_in_conv* [*symmetric*], *clarify*)
by (*meson connected_component_def connected_component_trans*)

lemma *exists_component_superset*: ⟦*t* ⊆ *s*; *s* ≠ {}; *connected t*⟧ ⟹ ∃ *c*. *c* ∈ *components s* ∧ *t* ⊆ *c*
apply (*cases t* = {}, *force*)
apply (*metis components_def ex_in_conv connected_component_maximal contra_subsetD image_eqI*)
done

lemma *components_intermediate_subset*: ⟦$s \in$ *components u*; $s \subseteq t$; $t \subseteq u$⟧ $\Longrightarrow s \in$ *components t*
apply (*auto simp*: *components_iff*)
apply (*metis connected_component_eq_empty connected_component_intermediate_subset*)
done

lemma *in_components_unions_complement*: $c \in$ *components s* $\Longrightarrow s - c = \bigcup$(*components* $s - \{c\}$)
by (*metis complement_connected_component_unions components_def components_iff*)

lemma *connected_intermediate_closure*:
assumes *cs*: *connected s* **and** *st*: $s \subseteq t$ **and** *ts*: $t \subseteq$ *closure s*
shows *connected t*
proof (*rule connectedI*)
fix *A B*
assume *A*: *open A* **and** *B*: *open B* **and** *Alap*: $A \cap t \neq \{\}$ **and** *Blap*: $B \cap t \neq \{\}$
and *disj*: $A \cap B \cap t = \{\}$ **and** *cover*: $t \subseteq A \cup B$
have *disjs*: $A \cap B \cap s = \{\}$
using *disj st* **by** *auto*
have $A \cap$ *closure s* $\neq \{\}$
using *Alap Int_absorb1 ts* **by** *blast*
then have *Alaps*: $A \cap s \neq \{\}$
by (*simp add*: *A open_Int_closure_eq_empty*)
have $B \cap$ *closure s* $\neq \{\}$
using *Blap Int_absorb1 ts* **by** *blast*
then have *Blaps*: $B \cap s \neq \{\}$
by (*simp add*: *B open_Int_closure_eq_empty*)
then show *False*
using *cs* [*unfolded connected_def*] *A B disjs Alaps Blaps cover st*
by *blast*
qed

lemma *closedin_connected_component*: *closedin* (*top_of_set s*) (*connected_component_set s x*)
proof (*cases connected_component_set s x* = {})
case *True*
then show *?thesis*
by (*metis closedin_empty*)
next
case *False*
then obtain *y* **where** *y*: *connected_component s x y*
by *blast*
have ∗: *connected_component_set s x* $\subseteq s \cap$ *closure* (*connected_component_set s x*)
by (*auto simp*: *closure_def connected_component_in*)
have *connected_component s x y* $\Longrightarrow s \cap$ *closure* (*connected_component_set s x*) $\subseteq$ *connected_component_set s x*

apply (*rule connected_component_maximal*, *simp*)
using *closure_subset connected_component_in* **apply** *fastforce*
using ∗ *connected_intermediate_closure* **apply** *blast+*
done
with *y* ∗ **show** *?thesis*
by (*auto simp*: *closedin_closed*)
qed

lemma *closedin_component*:
$C \in components\ s \Longrightarrow closedin\ (top\_of\_set\ s)\ C$
using *closedin_connected_component componentsE* **by** *blast*

2.4.4 Proving a function is constant on a connected set by proving that a level set is open

lemma *continuous_levelset_openin_cases*:
fixes $f :: \_ \Rightarrow 'b::t1\_space$
shows $connected\ s \Longrightarrow continuous\_on\ s\ f \Longrightarrow$
$openin\ (top\_of\_set\ s)\ \{x \in s.\ f\ x = a\}$
$\Longrightarrow (\forall x \in s.\ f\ x \neq a) \lor (\forall x \in s.\ f\ x = a)$
unfolding *connected_clopen*
using *continuous_closedin_preimage_constant* **by** *auto*

lemma *continuous_levelset_openin*:
fixes $f :: \_ \Rightarrow 'b::t1\_space$
shows $connected\ s \Longrightarrow continuous\_on\ s\ f \Longrightarrow$
$openin\ (top\_of\_set\ s)\ \{x \in s.\ f\ x = a\} \Longrightarrow$
$(\exists x \in s.\ f\ x = a) \Longrightarrow (\forall x \in s.\ f\ x = a)$
using *continuous_levelset_openin_cases*[*of s f*]
by *meson*

lemma *continuous_levelset_open*:
fixes $f :: \_ \Rightarrow 'b::t1\_space$
assumes *connected s*
and *continuous_on s f*
and $open\ \{x \in s.\ f\ x = a\}$
and $\exists x \in s.\ f\ x = a$
shows $\forall x \in s.\ f\ x = a$
using *continuous_levelset_openin*[*OF assms*(*1*,*2*), *of a*, *unfolded openin_open*]
using *assms* (*3*,*4*)
by *fast*

2.4.5 Preservation of Connectedness

lemma *homeomorphic_connectedness*:
assumes *s homeomorphic t*
shows $connected\ s \longleftrightarrow connected\ t$
using *assms* **unfolding** *homeomorphic_def homeomorphism_def* **by** (*metis connected_continuous_image*)

lemma *connected_monotone_quotient_preimage*:
  assumes *connected T*
      and *contf*: *continuous_on S f* **and** *fim*: f ‘ $S = T$
      and *opT*: $\bigwedge U.\ U \subseteq T$
              $\Longrightarrow$ *openin* (*top_of_set S*) ($S \cap f$ −‘ U) $\longleftrightarrow$
                openin (*top_of_set T*) U
      and *connT*: $\bigwedge y.\ y \in T \Longrightarrow$ *connected* ($S \cap f$ −‘ $\{y\}$)
    shows *connected S*
proof (*rule connectedI*)
  fix *U V*
  assume *open U* **and** *open V* **and** $U \cap S \neq \{\}$ **and** $V \cap S \neq \{\}$
    and $U \cap V \cap S = \{\}$ **and** $S \subseteq U \cup V$
  moreover
  have *disjoint*: f ‘ ($S \cap U$) $\cap$ f ‘ ($S \cap V$) = {}
  proof −
    have *False* **if** $y \in f$ ‘ ($S \cap U$) $\cap$ f ‘ ($S \cap V$) **for** *y*
    proof −
      have $y \in T$
        using *fim that* **by** *blast*
      show *?thesis*
        using *connectedD* [*OF connT* [*OF* ‹$y \in T$›] ‹*open U*› ‹*open V*›]
            ‹$S \subseteq U \cup V$› ‹$U \cap V \cap S = \{\}$› *that* **by** *fastforce*
    qed
    then show *?thesis* **by** *blast*
  qed
  ultimately have *UU*: ($S \cap f$ −‘ f ‘ ($S \cap U$)) = $S \cap U$ **and** *VV*: ($S \cap f$ −‘ f
‘ ($S \cap V$)) = $S \cap V$
    by *auto*
  have *opeU*: *openin* (*top_of_set T*) (f ‘ ($S \cap U$))
    by (*metis UU* ‹*open U*› *fim image_Int_subset le_inf_iff opT openin_open_Int*)
  have *opeV*: *openin* (*top_of_set T*) (f ‘ ($S \cap V$))
   by (*metis opT fim VV* ‹*open V*› *openin_open_Int image_Int_subset inf.bounded_iff*)
  have $T \subseteq f$ ‘ ($S \cap U$) $\cup$ f ‘ ($S \cap V$)
    using ‹$S \subseteq U \cup V$› *fim* **by** *auto*
  then show *False*
    using ‹*connected T*› *disjoint opeU opeV* ‹$U \cap S \neq \{\}$› ‹$V \cap S \neq \{\}$›
    by (*auto simp*: *connected_openin*)
qed

lemma *connected_open_monotone_preimage*:
  assumes *contf*: *continuous_on S f* **and** *fim*: f ‘ $S = T$
    and *ST*: $\bigwedge C.$ *openin* (*top_of_set S*) $C \Longrightarrow$ *openin* (*top_of_set T*) (f ‘ C)
    and *connT*: $\bigwedge y.\ y \in T \Longrightarrow$ *connected* ($S \cap f$ −‘ $\{y\}$)
    and *connected C* $C \subseteq T$
  shows *connected* ($S \cap f$ −‘ C)
proof −
  have *contf'*: *continuous_on* ($S \cap f$ −‘ C) *f*
    by (*meson contf continuous_on_subset inf_le1*)

```
  have eqC: f ` (S ∩ f -` C) = C
    using ‹C ⊆ T› fim by blast
  show ?thesis
   proof (rule connected_monotone_quotient_preimage [OF ‹connected C› contf'
eqC])
    show connected (S ∩ f -` C ∩ f -` {y}) if y ∈ C for y
    proof -
      have S ∩ f -` C ∩ f -` {y} = S ∩ f -` {y}
        using that by blast
      moreover have connected (S ∩ f -` {y})
        using ‹C ⊆ T› connT that by blast
      ultimately show ?thesis
        by metis
    qed
    have ⋀U. openin (top_of_set (S ∩ f -` C)) U
               ⟹ openin (top_of_set C) (f ` U)
      using open_map_restrict [OF _ ST ‹C ⊆ T›] by metis
    then show ⋀D. D ⊆ C
          ⟹ openin (top_of_set (S ∩ f -` C)) (S ∩ f -` C ∩ f -` D) =
              openin (top_of_set C) D
      using open_map_imp_quotient_map [of (S ∩ f -` C) f] contf' by (simp add:
eqC)
  qed
qed


lemma connected_closed_monotone_preimage:
  assumes contf: continuous_on S f and fim: f ` S = T
    and ST: ⋀C. closedin (top_of_set S) C ⟹ closedin (top_of_set T) (f ` C)
    and connT: ⋀y. y ∈ T ⟹ connected (S ∩ f -` {y})
    and connected C C ⊆ T
  shows connected (S ∩ f -` C)
proof -
  have contf': continuous_on (S ∩ f -` C) f
    by (meson contf continuous_on_subset inf_le1)
  have eqC: f ` (S ∩ f -` C) = C
    using ‹C ⊆ T› fim by blast
  show ?thesis
   proof (rule connected_monotone_quotient_preimage [OF ‹connected C› contf'
eqC])
    show connected (S ∩ f -` C ∩ f -` {y}) if y ∈ C for y
    proof -
      have S ∩ f -` C ∩ f -` {y} = S ∩ f -` {y}
        using that by blast
      moreover have connected (S ∩ f -` {y})
        using ‹C ⊆ T› connT that by blast
      ultimately show ?thesis
        by metis
    qed
```

```
    have ⋀U. closedin (top_of_set (S ∩ f −‘ C)) U
            ⟹ closedin (top_of_set C) (f ‘ U)
      using closed_map_restrict [OF _ ST ‹C ⊆ T›] by metis
    then show ⋀D. D ⊆ C
          ⟹ openin (top_of_set (S ∩ f −‘ C)) (S ∩ f −‘ C ∩ f −‘ D) =
              openin (top_of_set C) D
     using closed_map_imp_quotient_map [of (S ∩ f −‘ C) f] contf' by (simp add:
eqC)
  qed
qed
```

2.4.6 Lemmas about components

See Newman IV, 3.3 and 3.4.

```
lemma connected_Un_clopen_in_complement:
  fixes S U :: 'a::metric_space set
  assumes connected S connected U S ⊆ U
      and opeT: openin (top_of_set (U − S)) T
      and cloT: closedin (top_of_set (U − S)) T
    shows connected (S ∪ T)
proof −
  have ∗: ⟦⋀x y. P x y ⟷ P y x; ⋀x y. P x y ⟹ S ⊆ x ∨ S ⊆ y;
          ⋀x y. ⟦P x y; S ⊆ x⟧ ⟹ False⟧ ⟹ ¬(∃x y. (P x y)) for P
    by metis
  show ?thesis
    unfolding connected_closedin_eq
  proof (rule ∗)
    fix H1 H2
    assume H: closedin (top_of_set (S ∪ T)) H1 ∧
              closedin (top_of_set (S ∪ T)) H2 ∧
              H1 ∪ H2 = S ∪ T ∧ H1 ∩ H2 = {} ∧ H1 ≠ {} ∧ H2 ≠ {}
    then have clo: closedin (top_of_set S) (S ∩ H1)
                   closedin (top_of_set S) (S ∩ H2)
      by (metis Un_upper1 closedin_closed_subset inf_commute)+
    have Seq: S ∩ (H1 ∪ H2) = S
      by (simp add: H)
    have S ∩ ((S ∪ T) ∩ H1) ∪ S ∩ ((S ∪ T) ∩ H2) = S
      using Seq by auto
    moreover have H1 ∩ (S ∩ ((S ∪ T) ∩ H2)) = {}
      using H by blast
    ultimately have S ∩ H1 = {} ∨ S ∩ H2 = {}
      by (metis (no_types) H Int_assoc ‹S ∩ (H1 ∪ H2) = S› ‹connected S›
          clo Seq connected_closedin inf_bot_right inf_le1)
    then show S ⊆ H1 ∨ S ⊆ H2
      using H ‹connected S› unfolding connected_closedin by blast
  next
    fix H1 H2
    assume H: closedin (top_of_set (S ∪ T)) H1 ∧
              closedin (top_of_set (S ∪ T)) H2 ∧
```

```
          H1 ∪ H2 = S ∪ T ∧ H1 ∩ H2 = {} ∧ H1 ≠ {} ∧ H2 ≠ {}
      and S ⊆ H1
    then have H2T: H2 ⊆ T
      by auto
    have T ⊆ U
      using Diff_iff opeT openin_imp_subset by auto
    with ‹S ⊆ U› have Ueq: U = (U − S) ∪ (S ∪ T)
      by auto
    have openin (top_of_set ((U − S) ∪ (S ∪ T))) H2
    proof (rule openin_subtopology_Un)
      show openin (top_of_set (S ∪ T)) H2
        using ‹H2 ⊆ T› apply (auto simp: openin_closedin_eq)
        by (metis Diff_Diff_Int Diff_disjoint Diff_partition Diff_subset H Int_absorb1
Un_Diff)
      then show openin (top_of_set (U − S)) H2
        by (meson H2T Un_upper2 opeT openin_subset_trans openin_trans)
    qed
    moreover have closedin (top_of_set ((U − S) ∪ (S ∪ T))) H2
    proof (rule closedin_subtopology_Un)
      show closedin (top_of_set (U − S)) H2
        using H H2T cloT closedin_subset_trans
        by (blast intro: closedin_subtopology_Un closedin_trans)
    qed (simp add: H)
    ultimately
    have H2: H2 = {} ∨ H2 = U
      using Ueq ‹connected U› unfolding connected_clopen by metis
    then have H2 ⊆ S
      by (metis Diff_partition H Un_Diff_cancel Un_subset_iff ‹H2 ⊆ T› assms(3)
inf.orderE opeT openin_imp_subset)
    moreover have T ⊆ H2 − S
      by (metis (no_types) H2 H opeT openin_closedin_eq topspace_euclidean_subtopology)
    ultimately show False
      using H ‹S ⊆ H1› by blast
  qed blast
qed


proposition component_diff_connected:
  fixes S :: 'a::metric_space set
  assumes connected S connected U S ⊆ U and C: C ∈ components (U − S)
  shows connected(U − C)
  using ‹connected S› unfolding connected_closedin_eq not_ex de_Morgan_conj
proof clarify
  fix H3 H4
  assume clo3: closedin (top_of_set (U − C)) H3
    and clo4: closedin (top_of_set (U − C)) H4
    and H3 ∪ H4 = U − C and H3 ∩ H4 = {} and H3 ≠ {} and H4 ≠ {}
    and * [rule_format]:
    ∀ H1 H2. ¬ closedin (top_of_set S) H1 ∨
```

¬ *closedin* (*top_of_set S*) *H2* ∨
H1 ∪ *H2* ≠ *S* ∨ *H1* ∩ *H2* ≠ {} ∨ ¬ *H1* ≠ {} ∨ ¬ *H2* ≠ {}
then have *H3* ⊆ *U*−*C* **and** *ope3*: *openin* (*top_of_set* (*U* − *C*)) (*U* − *C* − *H3*)
and *H4* ⊆ *U*−*C* **and** *ope4*: *openin* (*top_of_set* (*U* − *C*)) (*U* − *C* − *H4*)
by (*auto simp*: *closedin_def*)
have *C* ≠ {} *C* ⊆ *U*−*S connected C*
using *C in_components_nonempty in_components_subset in_components_maximal*
by *blast*+
have *cCH3*: *connected* (*C* ∪ *H3*)
proof (*rule connected_Un_clopen_in_complement* [*OF* ‹*connected C*› ‹*connected U*› _ _ *clo3*])
show *openin* (*top_of_set* (*U* − *C*)) *H3*
apply (*simp add*: *openin_closedin_eq* ‹*H3* ⊆ *U* − *C*›)
apply (*simp add*: *closedin_subtopology*)
by (*metis Diff_cancel Diff_triv Un_Diff clo4* ‹*H3* ∩ *H4* = {}› ‹*H3* ∪ *H4* = *U* − *C*› *closedin_closed inf_commute sup_bot.left_neutral*)
qed (*use clo3* ‹*C* ⊆ *U* − *S*› **in** *auto*)
have *cCH4*: *connected* (*C* ∪ *H4*)
proof (*rule connected_Un_clopen_in_complement* [*OF* ‹*connected C*› ‹*connected U*› _ _ *clo4*])
show *openin* (*top_of_set* (*U* − *C*)) *H4*
apply (*simp add*: *openin_closedin_eq* ‹*H4* ⊆ *U* − *C*›)
apply (*simp add*: *closedin_subtopology*)
by (*metis Diff_cancel Int_commute Un_Diff Un_Diff_Int* ‹*H3* ∩ *H4* = {}› ‹*H3* ∪ *H4* = *U* − *C*› *clo3 closedin_closed*)
qed (*use clo4* ‹*C* ⊆ *U* − *S*› **in** *auto*)
have *closedin* (*top_of_set S*) (*S* ∩ *H3*) *closedin* (*top_of_set S*) (*S* ∩ *H4*)
using *clo3 clo4* ‹*S* ⊆ *U*› ‹*C* ⊆ *U* − *S*› **by** (*auto simp*: *closedin_closed*)
moreover have *S* ∩ *H3* ≠ {}
using *components_maximal* [*OF C cCH3*] ‹*C* ≠ {}› ‹*C* ⊆ *U* − *S*› ‹*H3* ≠ {}› ‹*H3* ⊆ *U* − *C*› **by** *auto*
moreover have *S* ∩ *H4* ≠ {}
using *components_maximal* [*OF C cCH4*] ‹*C* ≠ {}› ‹*C* ⊆ *U* − *S*› ‹*H4* ≠ {}› ‹*H4* ⊆ *U* − *C*› **by** *auto*
ultimately show *False*
using ∗ [*of S* ∩ *H3 S* ∩ *H4*] ‹*H3* ∩ *H4* = {}› ‹*C* ⊆ *U* − *S*› ‹*H3* ∪ *H4* = *U* − *C*› ‹*S* ⊆ *U*›
by *auto*
qed

2.4.7 Constancy of a function from a connected set into a finite, disconnected or discrete set

Still missing: versions for a set that is smaller than R, or countable.

lemma *continuous_disconnected_range_constant*:
assumes *S*: *connected S*
and *conf*: *continuous_on S f*
and *fim*: *f* ‘ *S* ⊆ *t*
and *cct*: ⋀*y*. *y* ∈ *t* ⟹ *connected_component_set t y* = {*y*}

```
    shows f constant_on S
proof (cases S = {})
  case True then show ?thesis
    by (simp add: constant_on_def)
next
  case False
  { fix x assume x ∈ S
    then have f ` S ⊆ {f x}
     by (metis connected_continuous_image conf connected_component_maximal fim
image_subset_iff rev_image_eqI S cct)
  }
  with False show ?thesis
    unfolding constant_on_def by blast
qed
```

This proof requires the existence of two separate values of the range type.

```
lemma finite_range_constant_imp_connected:
  assumes ⋀f::'a::topological_space ⇒ 'b::real_normed_algebra_1.
             ⟦continuous_on S f; finite(f ` S)⟧ ⟹ f constant_on S
    shows connected S
proof –
  { fix t u
    assume clt: closedin (top_of_set S) t
       and clu: closedin (top_of_set S) u
       and tue: t ∩ u = {} and tus: t ∪ u = S
    have conif: continuous_on S (λx. if x ∈ t then 0 else 1)
      apply (subst tus [symmetric])
      apply (rule continuous_on_cases_local)
      using clt clu tue
      apply (auto simp: tus)
      done
    have fi: finite ((λx. if x ∈ t then 0 else 1) ` S)
      by (rule finite_subset [of _ {0,1}]) auto
    have t = {} ∨ u = {}
      using assms [OF conif fi] tus [symmetric]
       by (auto simp: Ball_def constant_on_def) (metis IntI empty_iff one_neq_zero
tue)
  }
  then show ?thesis
    by (simp add: connected_closedin_eq)
qed

end
theory Abstract_Limits
  imports
    Abstract_Topology
begin
```

2.4.8 nhdsin and atin

definition *nhdsin* :: $'a$ *topology* ⇒ $'a$ ⇒ $'a$ *filter*
where *nhdsin X a* =
(*if a* ∈ *topspace X then* (*INF S*∈{*S*. *openin X S* ∧ *a* ∈ *S*}. *principal S*) *else bot*)

definition *atin* :: $'a$ *topology* ⇒ $'a$ ⇒ $'a$ *filter*
where *atin X a* ≡ *inf* (*nhdsin X a*) (*principal* (*topspace X* − {*a*}))

lemma *nhdsin_degenerate* [*simp*]: *a* ∉ *topspace X* ⟹ *nhdsin X a* = *bot*
and *atin_degenerate* [*simp*]: *a* ∉ *topspace X* ⟹ *atin X a* = *bot*
by (*simp_all add*: *nhdsin_def atin_def*)

lemma *eventually_nhdsin*:
eventually P (*nhdsin X a*) ⟷ *a* ∉ *topspace X* ∨ (∃ *S*. *openin X S* ∧ *a* ∈ *S* ∧ (∀ *x*∈*S*. *P x*))
proof (*cases a* ∈ *topspace X*)
case *True*
hence *nhdsin X a* = (*INF S*∈{*S*. *openin X S* ∧ *a* ∈ *S*}. *principal S*)
by (*simp add*: *nhdsin_def*)
also have *eventually P* ... ⟷ (∃ *S*. *openin X S* ∧ *a* ∈ *S* ∧ (∀ *x*∈*S*. *P x*))
using *True* **by** (*subst eventually_INF_base*) (*auto simp*: *eventually_principal*)
finally show *?thesis* **using** *True* **by** *simp*
qed *auto*

lemma *eventually_atin*:
eventually P (*atin X a*) ⟷ *a* ∉ *topspace X* ∨
(∃ *U*. *openin X U* ∧ *a* ∈ *U* ∧ (∀ *x* ∈ *U* − {*a*}. *P x*))
proof (*cases a* ∈ *topspace X*)
case *True*
hence *eventually P* (*atin X a*) ⟷ (∃ *S*. *openin X S* ∧
a ∈ *S* ∧ (∀ *x*∈*S*. *x* ∈ *topspace X* ∧ *x* ≠ *a* ⟶ *P x*))
by (*simp add*: *atin_def eventually_inf_principal eventually_nhdsin*)
also have ... ⟷ (∃ *U*. *openin X U* ∧ *a* ∈ *U* ∧ (∀ *x* ∈ *U* − {*a*}. *P x*))
using *openin_subset* **by** (*intro ex_cong*) *auto*
finally show *?thesis* **by** (*simp add*: *True*)
qed *auto*

2.4.9 Limits in a topological space

definition *limitin* :: $'a$ *topology* ⇒ ($'b$ ⇒ $'a$) ⇒ $'a$ ⇒ $'b$ *filter* ⇒ *bool* **where**
limitin X f l F ≡ *l* ∈ *topspace X* ∧ (∀ *U*. *openin X U* ∧ *l* ∈ *U* ⟶ *eventually* (λ*x*. *f x* ∈ *U*) *F*)

lemma *limitin_canonical_iff* [*simp*]: *limitin euclidean f l F* ⟷ (*f* ⟶ *l*) *F*
by (*auto simp*: *limitin_def tendsto_def*)

lemma *limitin_topspace*: *limitin X f l F* ⟹ *l* ∈ *topspace X*

by (*simp add*: *limitin_def*)

lemma *limitin_const_iff* [*simp*]: *limitin X* (λ*a*. *l*) *l F* ⟷ *l* ∈ *topspace X*
by (*simp add*: *limitin_def*)

lemma *limitin_const*: *limitin euclidean* (λ*a*. *l*) *l F*
by *simp*

lemma *limitin_eventually*:
⟦*l* ∈ *topspace X*; *eventually* (λ*x*. *f x* = *l*) *F*⟧ ⟹ *limitin X f l F*
by (*auto simp*: *limitin_def eventually_mono*)

lemma *limitin_subsequence*:
⟦*strict_mono r*; *limitin X f l sequentially*⟧ ⟹ *limitin X* (*f* ∘ *r*) *l sequentially*
unfolding *limitin_def* **using** *eventually_subseq* **by** *fastforce*

lemma *limitin_subtopology*:
limitin (*subtopology X S*) *f l F*
⟷ *l* ∈ *S* ∧ *eventually* (λ*a*. *f a* ∈ *S*) *F* ∧ *limitin X f l F* (**is** *?lhs* = *?rhs*)
proof (*cases l* ∈ *S* ∩ *topspace X*)
case *True*
show *?thesis*
proof
assume *L*: *?lhs*
with *True*
have $\forall_F$ *b in F*. *f b* ∈ *topspace X* ∩ *S*
by (*metis* (*no_types*) *limitin_def openin_topspace topspace_subtopology*)
with *L* **show** *?rhs*
apply (*clarsimp simp add*: *limitin_def eventually_mono openin_subtopology_alt*)
apply (*drule_tac x*=*S* ∩ *U* **in** *spec*, *force simp*: *elim*: *eventually_mono*)
done
next
assume *?rhs*
then show *?lhs*
using *eventually_elim2*
by (*fastforce simp add*: *limitin_def openin_subtopology_alt*)
qed
qed (*auto simp*: *limitin_def*)

lemma *limitin_canonical_iff_gen* [*simp*]:
assumes *open S*
shows *limitin* (*top_of_set S*) *f l F* ⟷ (*f* ⟶ *l*) *F* ∧ *l* ∈ *S*
using *assms* **by** (*auto simp*: *limitin_subtopology tendsto_def*)

lemma *limitin_sequentially*:
limitin X S l sequentially ⟷
l ∈ *topspace X* ∧ (∀ *U*. *openin X U* ∧ *l* ∈ *U* ⟶ (∃ *N*. ∀ *n*. *N* ≤ *n* ⟶ *S n*
∈ *U*))

by (*simp add*: *limitin_def eventually_sequentially*)

lemma *limitin_sequentially_offset*:
limitin X f l sequentially $\Longrightarrow$ *limitin X* ($\lambda i.$ *f* (*i* + *k*)) *l sequentially*
unfolding *limitin_sequentially*
by (*metis add.commute le_add2 order_trans*)

lemma *limitin_sequentially_offset_rev*:
assumes *limitin X* ($\lambda i.$ *f* (*i* + *k*)) *l sequentially*
shows *limitin X f l sequentially*
proof −
have $\exists N. \forall n \geq N.$ *f n* $\in$ *U* **if** *U*: *openin X U l* $\in$ *U* **for** *U*
proof −
obtain *N* **where** $\bigwedge n.\ n \geq N \Longrightarrow f\ (n + k) \in U$
using *assms U* **unfolding** *limitin_sequentially* **by** *blast*
then have $\forall n \geq N+k.\ f\ n \in U$
by (*metis add_leD2 le_add_diff_inverse ordered_cancel_comm_monoid_diff_class.le_diff_conv2 add.commute*)
then show *?thesis* **..**
qed
with *assms* **show** *?thesis*
unfolding *limitin_sequentially*
by *simp*
qed

lemma *limitin_atin*:
limitin Y f y (*atin X x*) $\longleftrightarrow$
y $\in$ *topspace Y* $\wedge$
(*x* $\in$ *topspace X*
$\longrightarrow$ ($\forall V.$ *openin Y V* $\wedge$ *y* $\in$ *V*
$\longrightarrow$ ($\exists U.$ *openin X U* $\wedge$ *x* $\in$ *U* $\wedge$ *f* ‘ (*U* − {*x*}) $\subseteq$ *V*)))
by (*auto simp*: *limitin_def eventually_atin image_subset_iff*)

lemma *limitin_atin_self*:
limitin Y f (*f a*) (*atin X a*) $\longleftrightarrow$
f a $\in$ *topspace Y* $\wedge$
(*a* $\in$ *topspace X*
$\longrightarrow$ ($\forall V.$ *openin Y V* $\wedge$ *f a* $\in$ *V*
$\longrightarrow$ ($\exists U.$ *openin X U* $\wedge$ *a* $\in$ *U* $\wedge$ *f* ‘ *U* $\subseteq$ *V*)))
unfolding *limitin_atin* **by** *fastforce*

lemma *limitin_trivial*:
⟦*trivial_limit F*; *y* $\in$ *topspace X*⟧ $\Longrightarrow$ *limitin X f y F*
by (*simp add*: *limitin_def*)

lemma *limitin_transform_eventually*:
⟦*eventually* ($\lambda x.$ *f x* = *g x*) *F*; *limitin X f l F*⟧ $\Longrightarrow$ *limitin X g l F*
unfolding *limitin_def* **using** *eventually_elim2* **by** *fastforce*

```
lemma continuous_map_limit:
  assumes continuous_map X Y g and f: limitin X f l F
  shows limitin Y (g ∘ f) (g l) F
proof –
  have g l ∈ topspace Y
    by (meson assms continuous_map_def limitin_topspace)
  moreover
  have ⋀U. ⟦∀ V. openin X V ∧ l ∈ V ⟶ (∀F x in F. f x ∈ V); openin Y U;
g l ∈ U⟧
          ⟹ ∀F x in F. g (f x) ∈ U
    using assms eventually_mono
    by (fastforce simp: limitin_def dest!: openin_continuous_map_preimage)
  ultimately show ?thesis
    using f by (fastforce simp add: limitin_def)
qed
```

2.4.10 Pointwise continuity in topological spaces

```
definition topcontinuous_at where
  topcontinuous_at X Y f x ⟷
     x ∈ topspace X ∧
     (∀ x ∈ topspace X. f x ∈ topspace Y) ∧
     (∀ V. openin Y V ∧ f x ∈ V
         ⟶ (∃ U. openin X U ∧ x ∈ U ∧ (∀ y ∈ U. f y ∈ V)))

lemma topcontinuous_at_atin:
   topcontinuous_at X Y f x ⟷
       x ∈ topspace X ∧
       (∀ x ∈ topspace X. f x ∈ topspace Y) ∧
       limitin Y f (f x) (atin X x)
  unfolding topcontinuous_at_def
  by (fastforce simp add: limitin_atin)+

lemma continuous_map_eq_topcontinuous_at:
   continuous_map X Y f ⟷ (∀ x ∈ topspace X. topcontinuous_at X Y f x)
   (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    by (auto simp: continuous_map_def topcontinuous_at_def)
next
  assume R: ?rhs
  then show ?lhs
    apply (auto simp: continuous_map_def topcontinuous_at_def)
    apply (subst openin_subopen, safe)
    apply (drule bspec, assumption)
    using openin_subset[of X] apply (auto simp: subset_iff dest!: spec)
    done
qed
```

lemma *continuous_map_atin*:
continuous_map X Y f ⟷ (∀ *x* ∈ *topspace X*. *limitin Y f* (*f x*) (*atin X x*))
by (*auto simp*: *limitin_def topcontinuous_at_atin continuous_map_eq_topcontinuous_at*)

lemma *limitin_continuous_map*:
⟦*continuous_map X Y f*; *a* ∈ *topspace X*; *f a* = *b*⟧ ⟹ *limitin Y f b* (*atin X a*)
by (*auto simp*: *continuous_map_atin*)

2.4.11 Combining theorems for continuous functions into the reals

lemma *continuous_map_canonical_const* [*continuous_intros*]:
continuous_map X euclidean (λ*x*. *c*)
by *simp*

lemma *continuous_map_real_mult* [*continuous_intros*]:
⟦*continuous_map X euclideanreal f*; *continuous_map X euclideanreal g*⟧
⟹ *continuous_map X euclideanreal* (λ*x*. *f x* ∗ *g x*)
by (*simp add*: *continuous_map_atin tendsto_mult*)

lemma *continuous_map_real_pow* [*continuous_intros*]:
continuous_map X euclideanreal f ⟹ *continuous_map X euclideanreal* (λ*x*. *f x* ˆ *n*)
by (*induction n*) (*auto simp*: *continuous_map_real_mult*)

lemma *continuous_map_real_mult_left*:
continuous_map X euclideanreal f ⟹ *continuous_map X euclideanreal* (λ*x*. *c* ∗ *f x*)
by (*simp add*: *continuous_map_atin tendsto_mult*)

lemma *continuous_map_real_mult_left_eq*:
continuous_map X euclideanreal (λ*x*. *c* ∗ *f x*) ⟷ *c* = *0* ∨ *continuous_map X euclideanreal f*
proof (*cases c* = *0*)
case *False*
have *continuous_map X euclideanreal* (λ*x*. *c* ∗ *f x*) ⟹ *continuous_map X euclideanreal f*
apply (*frule continuous_map_real_mult_left* [**where** *c*=*inverse c*])
apply (*simp add*: *field_simps False*)
done
with *False* **show** *?thesis*
using *continuous_map_real_mult_left* **by** *blast*
qed *simp*

lemma *continuous_map_real_mult_right*:
continuous_map X euclideanreal f ⟹ *continuous_map X euclideanreal* (λ*x*. *f x* ∗ *c*)
by (*simp add*: *continuous_map_atin tendsto_mult*)

lemma *continuous_map_real_mult_right_eq*:
continuous_map X euclideanreal (λ*x*. *f x* ∗ *c*) ⟷ *c* = *0* ∨ *continuous_map X euclideanreal f*
by (*simp add*: *mult.commute flip*: *continuous_map_real_mult_left_eq*)

lemma *continuous_map_minus* [*continuous_intros*]:
fixes *f* :: *'a*⇒*'b*::*real_normed_vector*
shows *continuous_map X euclidean f* ⟹ *continuous_map X euclidean* (λ*x*. − *f x*)
by (*simp add*: *continuous_map_atin tendsto_minus*)

lemma *continuous_map_minus_eq* [*simp*]:
fixes *f* :: *'a*⇒*'b*::*real_normed_vector*
shows *continuous_map X euclidean* (λ*x*. − *f x*) ⟷ *continuous_map X euclidean f*
using *continuous_map_minus add.inverse_inverse continuous_map_eq* **by** *fastforce*

lemma *continuous_map_add* [*continuous_intros*]:
fixes *f* :: *'a*⇒*'b*::*real_normed_vector*
shows ⟦*continuous_map X euclidean f*; *continuous_map X euclidean g*⟧ ⟹ *continuous_map X euclidean* (λ*x*. *f x* + *g x*)
by (*simp add*: *continuous_map_atin tendsto_add*)

lemma *continuous_map_diff* [*continuous_intros*]:
fixes *f* :: *'a*⇒*'b*::*real_normed_vector*
shows ⟦*continuous_map X euclidean f*; *continuous_map X euclidean g*⟧ ⟹ *continuous_map X euclidean* (λ*x*. *f x* − *g x*)
by (*simp add*: *continuous_map_atin tendsto_diff*)

lemma *continuous_map_norm* [*continuous_intros*]:
fixes *f* :: *'a*⇒*'b*::*real_normed_vector*
shows *continuous_map X euclidean f* ⟹ *continuous_map X euclidean* (λ*x*. *norm*(*f x*))
by (*simp add*: *continuous_map_atin tendsto_norm*)

lemma *continuous_map_real_abs* [*continuous_intros*]:
continuous_map X euclideanreal f ⟹ *continuous_map X euclideanreal* (λ*x*. *abs*(*f x*))
by (*simp add*: *continuous_map_atin tendsto_rabs*)

lemma *continuous_map_real_max* [*continuous_intros*]:
⟦*continuous_map X euclideanreal f*; *continuous_map X euclideanreal g*⟧
⟹ *continuous_map X euclideanreal* (λ*x*. *max* (*f x*) (*g x*))
by (*simp add*: *continuous_map_atin tendsto_max*)

lemma *continuous_map_real_min* [*continuous_intros*]:
⟦*continuous_map X euclideanreal f*; *continuous_map X euclideanreal g*⟧
⟹ *continuous_map X euclideanreal* (λ*x*. *min* (*f x*) (*g x*))

by (*simp add*: *continuous_map_atin tendsto_min*)

lemma *continuous_map_sum* [*continuous_intros*]:
fixes *f* :: *'a⇒'b⇒'c::real_normed_vector*
shows ⟦*finite I*; ⋀*i*. *i* ∈ *I* ⟹ *continuous_map X euclidean* (λ*x*. *f x i*)⟧
⟹ *continuous_map X euclidean* (λ*x*. *sum* (*f x*) *I*)
by (*simp add*: *continuous_map_atin tendsto_sum*)

lemma *continuous_map_prod* [*continuous_intros*]:
⟦*finite I*;
⋀*i*. *i* ∈ *I* ⟹ *continuous_map X euclideanreal* (λ*x*. *f x i*)⟧
⟹ *continuous_map X euclideanreal* (λ*x*. *prod* (*f x*) *I*)
by (*simp add*: *continuous_map_atin tendsto_prod*)

lemma *continuous_map_real_inverse* [*continuous_intros*]:
⟦*continuous_map X euclideanreal f*; ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* ≠ *0*⟧
⟹ *continuous_map X euclideanreal* (λ*x*. *inverse*(*f x*))
by (*simp add*: *continuous_map_atin tendsto_inverse*)

lemma *continuous_map_real_divide* [*continuous_intros*]:
⟦*continuous_map X euclideanreal f*; *continuous_map X euclideanreal g*; ⋀*x*. *x* ∈ *topspace X* ⟹ *g x* ≠ *0*⟧
⟹ *continuous_map X euclideanreal* (λ*x*. *f x* / *g x*)
by (*simp add*: *continuous_map_atin tendsto_divide*)

end

Chapter 3

Functional Analysis

3.1 A decision procedure for metric spaces

theory *Metric_Arith*
  imports *HOL.Real_Vector_Spaces*
begin

A port of the decision procedure "Formalization of metric spaces in HOL Light" [3] for the type class *metric_space*, with the *Argo* solver as backend.

named_theorems *metric_prenex*
named_theorems *metric_nnf*
named_theorems *metric_unfold*
named_theorems *metric_pre_arith*

lemmas *pre_arith_simps* =
  max.bounded_iff max_less_iff_conj
  le_max_iff_disj less_max_iff_disj
  simp_thms HOL.eq_commute
declare *pre_arith_simps* [*metric_pre_arith*]

lemmas *unfold_simps* =
  Un_iff subset_iff disjoint_iff_not_equal
  Ball_def Bex_def
declare *unfold_simps* [*metric_unfold*]

declare *HOL.nnf_simps*(*4*) [*metric_prenex*]

lemma *imp_prenex* [*metric_prenex*]:
  $\bigwedge P\ Q.\ (\exists x.\ P\ x) \longrightarrow Q \equiv \forall x.\ (P\ x \longrightarrow Q)$
  $\bigwedge P\ Q.\ P \longrightarrow (\exists x.\ Q\ x) \equiv \exists x.\ (P \longrightarrow Q\ x)$
  $\bigwedge P\ Q.\ (\forall x.\ P\ x) \longrightarrow Q \equiv \exists x.\ (P\ x \longrightarrow Q)$
  $\bigwedge P\ Q.\ P \longrightarrow (\forall x.\ Q\ x) \equiv \forall x.\ (P \longrightarrow Q\ x)$
  by *simp_all*

lemma *ex_prenex* [*metric_prenex*]:

$\bigwedge P\ Q.\ (\exists x.\ P\ x) \wedge Q \equiv \exists x.\ (P\ x \wedge Q)$
$\bigwedge P\ Q.\ P \wedge (\exists x.\ Q\ x) \equiv \exists x.\ (P \wedge Q\ x)$
$\bigwedge P\ Q.\ (\exists x.\ P\ x) \vee Q \equiv \exists x.\ (P\ x \vee Q)$
$\bigwedge P\ Q.\ P \vee (\exists x.\ Q\ x) \equiv \exists x.\ (P \vee Q\ x)$
$\bigwedge P.\ \neg(\exists x.\ P\ x) \equiv \forall x.\ \neg P\ x$
by *simp_all*

lemma *all_prenex* [*metric_prenex*]:
$\bigwedge P\ Q.\ (\forall x.\ P\ x) \wedge Q \equiv \forall x.\ (P\ x \wedge Q)$
$\bigwedge P\ Q.\ P \wedge (\forall x.\ Q\ x) \equiv \forall x.\ (P \wedge Q\ x)$
$\bigwedge P\ Q.\ (\forall x.\ P\ x) \vee Q \equiv \forall x.\ (P\ x \vee Q)$
$\bigwedge P\ Q.\ P \vee (\forall x.\ Q\ x) \equiv \forall x.\ (P \vee Q\ x)$
$\bigwedge P.\ \neg(\forall x.\ P\ x) \equiv \exists x.\ \neg P\ x$
by *simp_all*

lemma *nnf_thms* [*metric_nnf*]:
$(\neg\ (P \wedge Q)) = (\neg\ P \vee \neg\ Q)$
$(\neg\ (P \vee Q)) = (\neg\ P \wedge \neg\ Q)$
$(P \longrightarrow Q) = (\neg\ P \vee Q)$
$(P = Q) \longleftrightarrow (P \vee \neg\ Q) \wedge (\neg\ P \vee Q)$
$(\neg\ (P = Q)) \longleftrightarrow (\neg\ P \vee \neg\ Q) \wedge (P \vee Q)$
$(\neg\ \neg\ P) = P$
by *blast+*

lemmas *nnf_simps* = *nnf_thms linorder_not_less linorder_not_le*
declare *nnf_simps*[*metric_nnf*]

lemma *ball_insert*: $(\forall x{\in}insert\ a\ B.\ P\ x) = (P\ a \wedge (\forall x{\in}B.\ P\ x))$
by *blast*

lemma *Sup_insert_insert*:
fixes *a*::*real*
shows *Sup* (*insert a* (*insert b s*)) = *Sup* (*insert* (*max a b*) *s*)
by (*simp add*: *Sup_real_def*)

lemma *real_abs_dist*: $|dist\ x\ y| = dist\ x\ y$
by *simp*

lemma *maxdist_thm* [*THEN HOL.eq_reflection*]:
assumes *finite s* $x \in s$ $y \in s$
defines $\bigwedge a.\ f\ a \equiv |dist\ x\ a - dist\ a\ y|$
shows *dist x y* = *Sup* (*f* ‘ *s*)
proof −
have $dist\ x\ y \leq Sup\ (f\ `\ s)$
proof −
have *finite* (*f* ‘ *s*)
by (*simp add*: ‹*finite s*›)
moreover have $|dist\ x\ y - dist\ y\ y| \in f\ `\ s$
by (*metis* ‹$y \in s$› *f_def imageI*)

```
    ultimately show ?thesis
      using le_cSup_finite by simp
  qed
  also have Sup (f ‘ s) ≤ dist x y
    using ‹x ∈ s› cSUP_least[of s f] abs_dist_diff_le
    unfolding f_def
    by blast
  finally show ?thesis .
qed

theorem metric_eq_thm [THEN HOL.eq_reflection]:
  x ∈ s ⟹ y ∈ s ⟹ x = y ⟷ (∀ a∈s. dist x a = dist y a)
  by auto

ML_file metric_arith.ML

method_setup metric = ‹
  Scan.succeed (SIMPLE_METHOD' o MetricArith.metric_arith_tac)
› prove simple linear statements in metric spaces (∀∃p fragment)

end
```

3.2 Elementary Metric Spaces

```
theory Elementary_Metric_Spaces
  imports
    Abstract_Topology_2
    Metric_Arith
begin
```

3.2.1 Open and closed balls

```
definition ball :: 'a::metric_space ⇒ real ⇒ 'a set
  where ball x e = {y. dist x y < e}

definition cball :: 'a::metric_space ⇒ real ⇒ 'a set
  where cball x e = {y. dist x y ≤ e}

definition sphere :: 'a::metric_space ⇒ real ⇒ 'a set
  where sphere x e = {y. dist x y = e}

lemma mem_ball [simp, metric_unfold]: y ∈ ball x e ⟷ dist x y < e
  by (simp add: ball_def)

lemma mem_cball [simp, metric_unfold]: y ∈ cball x e ⟷ dist x y ≤ e
  by (simp add: cball_def)

lemma mem_sphere [simp]: y ∈ sphere x e ⟷ dist x y = e
  by (simp add: sphere_def)
```

lemma *ball_trivial* [*simp*]: *ball x 0* = {}
  by (*simp add*: *ball_def*)

lemma *cball_trivial* [*simp*]: *cball x 0* = {*x*}
  by (*simp add*: *cball_def*)

lemma *sphere_trivial* [*simp*]: *sphere x 0* = {*x*}
  by (*simp add*: *sphere_def*)

lemma *disjoint_ballI*: *dist x y* ≥ *r*+*s* ⟹ *ball x r* ∩ *ball y s* = {}
  using *dist_triangle_less_add not_le* **by** *fastforce*

lemma *disjoint_cballI*: *dist x y* > *r* + *s* ⟹ *cball x r* ∩ *cball y s* = {}
 by (*metis add_mono disjoint_iff_not_equal dist_triangle2 dual_order.trans leD mem_cball*)

lemma *sphere_empty* [*simp*]: *r* < *0* ⟹ *sphere a r* = {}
  for *a* :: ′*a*::*metric_space*
  by *auto*

lemma *centre_in_ball* [*simp*]: *x* ∈ *ball x e* ⟷ *0* < *e*
  by *simp*

lemma *centre_in_cball* [*simp*]: *x* ∈ *cball x e* ⟷ *0* ≤ *e*
  by *simp*

lemma *ball_subset_cball* [*simp*, *intro*]: *ball x e* ⊆ *cball x e*
  by (*simp add*: *subset_eq*)

lemma *mem_ball_imp_mem_cball*: *x* ∈ *ball y e* ⟹ *x* ∈ *cball y e*
  by *auto*

lemma *sphere_cball* [*simp*,*intro*]: *sphere z r* ⊆ *cball z r*
  by *force*

lemma *cball_diff_sphere*: *cball a r* − *sphere a r* = *ball a r*
  by *auto*

lemma *subset_ball*[*intro*]: *d* ≤ *e* ⟹ *ball x d* ⊆ *ball x e*
  by *auto*

lemma *subset_cball*[*intro*]: *d* ≤ *e* ⟹ *cball x d* ⊆ *cball x e*
  by *auto*

lemma *mem_ball_leI*: *x* ∈ *ball y e* ⟹ *e* ≤ *f* ⟹ *x* ∈ *ball y f*
  by *auto*

lemma *mem_cball_leI*: *x* ∈ *cball y e* ⟹ *e* ≤ *f* ⟹ *x* ∈ *cball y f*
  by *auto*

```
lemma cball_trans: y ∈ cball z b ⟹ x ∈ cball y a ⟹ x ∈ cball z (b + a)
  by metric

lemma ball_max_Un: ball a (max r s) = ball a r ∪ ball a s
  by auto

lemma ball_min_Int: ball a (min r s) = ball a r ∩ ball a s
  by auto

lemma cball_max_Un: cball a (max r s) = cball a r ∪ cball a s
  by auto

lemma cball_min_Int: cball a (min r s) = cball a r ∩ cball a s
  by auto

lemma cball_diff_eq_sphere: cball a r − ball a r = sphere a r
  by auto

lemma open_ball [intro, simp]: open (ball x e)
proof −
  have open (dist x −‘ {..<e})
    by (intro open_vimage open_lessThan continuous_intros)
  also have dist x −‘ {..<e} = ball x e
    by auto
  finally show ?thesis .
qed

lemma open_contains_ball: open S ⟷ (∀x∈S. ∃e>0. ball x e ⊆ S)
  by (simp add: open_dist subset_eq Ball_def dist_commute)

lemma openI [intro?]: (⋀x. x∈S ⟹ ∃e>0. ball x e ⊆ S) ⟹ open S
  by (auto simp: open_contains_ball)

lemma openE[elim?]:
  assumes open S x∈S
  obtains e where e>0 ball x e ⊆ S
  using assms unfolding open_contains_ball by auto

lemma open_contains_ball_eq: open S ⟹ x∈S ⟷ (∃e>0. ball x e ⊆ S)
  by (metis open_contains_ball subset_eq centre_in_ball)

lemma ball_eq_empty[simp]: ball x e = {} ⟷ e ≤ 0
  unfolding mem_ball set_eq_iff
  by (simp add: not_less) metric

lemma ball_empty: e ≤ 0 ⟹ ball x e = {}
  by simp
```

lemma *closed_cball* [*iff*]: *closed* (*cball x e*)
proof −
  have *closed* (*dist x* −‘ {..*e*})
    by (*intro closed_vimage closed_atMost continuous_intros*)
  also have *dist x* −‘ {..*e*} = *cball x e*
    by *auto*
  finally show *?thesis* **.**
qed

lemma *open_contains_cball*: *open S* ⟷ (∀ *x*∈*S*. ∃ *e*>*0*. *cball x e* ⊆ *S*)
proof −
  {
    fix *x* **and** *e*::*real*
    assume *x*∈*S e*>*0 ball x e* ⊆ *S*
    then have ∃ *d*>*0*. *cball x d* ⊆ *S*
      unfolding *subset_eq* **by** (*rule_tac x*=*e*/*2* **in** *exI*, *auto*)
  }
  moreover
  {
    fix *x* **and** *e*::*real*
    assume *x*∈*S e*>*0 cball x e* ⊆ *S*
    then have ∃ *d*>*0*. *ball x d* ⊆ *S*
      using *mem_ball_imp_mem_cball* **by** *blast*
  }
  ultimately show *?thesis*
    unfolding *open_contains_ball* **by** *auto*
qed

lemma *open_contains_cball_eq*: *open S* ⟹ (∀ *x*. *x* ∈ *S* ⟷ (∃ *e*>*0*. *cball x e* ⊆ *S*))
  by (*metis open_contains_cball subset_eq order_less_imp_le centre_in_cball*)

lemma *eventually_nhds_ball*: *d* > *0* ⟹ *eventually* (λ*x*. *x* ∈ *ball z d*) (*nhds z*)
  by (*rule eventually_nhds_in_open*) *simp_all*

lemma *eventually_at_ball*: *d* > *0* ⟹ *eventually* (λ*t*. *t* ∈ *ball z d* ∧ *t* ∈ *A*) (*at z within A*)
  unfolding *eventually_at* **by** (*intro exI*[*of* _ *d*]) (*simp_all add*: *dist_commute*)

lemma *eventually_at_ball'*: *d* > *0* ⟹ *eventually* (λ*t*. *t* ∈ *ball z d* ∧ *t* ≠ *z* ∧ *t* ∈ *A*) (*at z within A*)
  unfolding *eventually_at* **by** (*intro exI*[*of* _ *d*]) (*simp_all add*: *dist_commute*)

lemma *at_within_ball*: *e* > *0* ⟹ *dist x y* < *e* ⟹ *at y within ball x e* = *at y*
  by (*subst at_within_open*) *auto*

lemma *atLeastAtMost_eq_cball*:
  fixes *a b*::*real*
  shows {*a* .. *b*} = *cball* ((*a* + *b*)/*2*) ((*b* − *a*)/*2*)

```
  by (auto simp: dist_real_def field_simps)

lemma cball_eq_atLeastAtMost:
  fixes a b::real
  shows cball a b = {a − b .. a + b}
  by (auto simp: dist_real_def)

lemma greaterThanLessThan_eq_ball:
  fixes a b::real
  shows {a <..< b} = ball ((a + b)/2) ((b − a)/2)
  by (auto simp: dist_real_def field_simps)

lemma ball_eq_greaterThanLessThan:
  fixes a b::real
  shows ball a b = {a − b <..< a + b}
  by (auto simp: dist_real_def)

lemma interior_ball [simp]: interior (ball x e) = ball x e
  by (simp add: interior_open)

lemma cball_eq_empty [simp]: cball x e = {} ⟷ e < 0
  apply (simp add: set_eq_iff not_le)
  apply (metis zero_le_dist dist_self order_less_le_trans)
  done

lemma cball_empty [simp]: e < 0 ⟹ cball x e = {}
  by simp

lemma cball_sing:
  fixes x :: 'a::metric_space
  shows e = 0 ⟹ cball x e = {x}
  by simp

lemma ball_divide_subset: d ≥ 1 ⟹ ball x (e/d) ⊆ ball x e
  by (metis ball_eq_empty div_by_1 frac_le linear subset_ball zero_less_one)

lemma ball_divide_subset_numeral: ball x (e / numeral w) ⊆ ball x e
  using ball_divide_subset one_le_numeral by blast

lemma cball_divide_subset: d ≥ 1 ⟹ cball x (e/d) ⊆ cball x e
  apply (cases e < 0, simp add: field_split_simps)
  by (metis div_by_1 frac_le less_numeral_extra(1) not_le order_refl subset_cball)

lemma cball_divide_subset_numeral: cball x (e / numeral w) ⊆ cball x e
  using cball_divide_subset one_le_numeral by blast

lemma cball_scale:
  assumes a ≠ 0
  shows  (λx. a *R x) ` cball c r = cball (a *R c :: 'a :: real_normed_vector) (|a|
```

∗ *r*)
proof −
have *1*: (λ*x*. *a* $*_R$ *x*) ‘ *cball c r* ⊆ *cball* (*a* $*_R$ *c*) (|*a*| ∗ *r*) **if** *a* ≠ *0* **for** *a r* **and** *c* :: ′*a*
proof *safe*
fix *x*
assume *x*: *x* ∈ *cball c r*
have *dist* (*a* $*_R$ *c*) (*a* $*_R$ *x*) = *norm* (*a* $*_R$ *c* − *a* $*_R$ *x*)
by (*auto simp*: *dist_norm*)
also have *a* $*_R$ *c* − *a* $*_R$ *x* = *a* $*_R$ (*c* − *x*)
by (*simp add*: *algebra_simps*)
finally show *a* $*_R$ *x* ∈ *cball* (*a* $*_R$ *c*) (|*a*| ∗ *r*)
using *that x* **by** (*auto simp*: *dist_norm*)
qed

have *cball* (*a* $*_R$ *c*) (|*a*| ∗ *r*) = (λ*x*. *a* $*_R$ *x*) ‘ (λ*x*. *inverse a* $*_R$ *x*) ‘ *cball* (*a* $*_R$ *c*) (|*a*| ∗ *r*)
unfolding *image_image* **using** *assms* **by** *simp*
also have . . . ⊆ (λ*x*. *a* $*_R$ *x*) ‘ *cball* (*inverse a* $*_R$ (*a* $*_R$ *c*)) (|*inverse a*| ∗ (|*a*| ∗ *r*))
using *assms* **by** (*intro image_mono 1*) *auto*
also have . . . = (λ*x*. *a* $*_R$ *x*) ‘ *cball c r*
using *assms* **by** (*simp add*: *algebra_simps*)
finally have *cball* (*a* $*_R$ *c*) (|*a*| ∗ *r*) ⊆ (λ*x*. *a* $*_R$ *x*) ‘ *cball c r* **.**
moreover from *assms* **have** (λ*x*. *a* $*_R$ *x*) ‘ *cball c r* ⊆ *cball* (*a* $*_R$ *c*) (|*a*| ∗ *r*)
by (*intro 1*) *auto*
ultimately show *?thesis* **by** *blast*
qed

lemma *ball_scale*:
assumes *a* ≠ *0*
shows (λ*x*. *a* $*_R$ *x*) ‘ *ball c r* = *ball* (*a* $*_R$ *c* :: ′*a* :: *real_normed_vector*) (|*a*| ∗ *r*)
proof −
have *1*: (λ*x*. *a* $*_R$ *x*) ‘ *ball c r* ⊆ *ball* (*a* $*_R$ *c*) (|*a*| ∗ *r*) **if** *a* ≠ *0* **for** *a r* **and** *c* :: ′*a*
proof *safe*
fix *x*
assume *x*: *x* ∈ *ball c r*
have *dist* (*a* $*_R$ *c*) (*a* $*_R$ *x*) = *norm* (*a* $*_R$ *c* − *a* $*_R$ *x*)
by (*auto simp*: *dist_norm*)
also have *a* $*_R$ *c* − *a* $*_R$ *x* = *a* $*_R$ (*c* − *x*)
by (*simp add*: *algebra_simps*)
finally show *a* $*_R$ *x* ∈ *ball* (*a* $*_R$ *c*) (|*a*| ∗ *r*)
using *that x* **by** (*auto simp*: *dist_norm*)
qed

have *ball* (*a* $*_R$ *c*) (|*a*| ∗ *r*) = (λ*x*. *a* $*_R$ *x*) ‘ (λ*x*. *inverse a* $*_R$ *x*) ‘ *ball* (*a* $*_R$ *c*) (|*a*| ∗ *r*)

unfolding *image_image* **using** *assms* **by** *simp*
also have ... ⊆ (λ*x*. *a* $*_R$ *x*) ' *ball* (*inverse a* $*_R$ (*a* $*_R$ *c*)) (|*inverse a*| ∗ (|*a*| ∗ *r*))
using *assms* **by** (*intro image_mono 1*) *auto*
also have ... = (λ*x*. *a* $*_R$ *x*) ' *ball c r*
using *assms* **by** (*simp add*: *algebra_simps*)
finally have *ball* (*a* $*_R$ *c*) (|*a*| ∗ *r*) ⊆ (λ*x*. *a* $*_R$ *x*) ' *ball c r* .
moreover from *assms* **have** (λ*x*. *a* $*_R$ *x*) ' *ball c r* ⊆ *ball* (*a* $*_R$ *c*) (|*a*| ∗ *r*)
by (*intro 1*) *auto*
ultimately show *?thesis* **by** *blast*
qed

3.2.2 Limit Points

lemma *islimpt_approachable*:
fixes *x* :: *'a*::*metric_space*
shows *x islimpt S* ⟷ (∀ *e*>*0*. ∃ *x'*∈*S*. *x'* ≠ *x* ∧ *dist x' x* < *e*)
unfolding *islimpt_iff_eventually eventually_at* **by** *fast*

lemma *islimpt_approachable_le*: *x islimpt S* ⟷ (∀ *e*>*0*. ∃ *x'*∈ *S*. *x'* ≠ *x* ∧ *dist x' x* ≤ *e*)
for *x* :: *'a*::*metric_space*
unfolding *islimpt_approachable*
using *approachable_lt_le2* [**where** *f*=λ*y*. *dist y x* **and** *P*=λ*y*. *y* ∉ *S* ∨ *y* = *x* **and** *Q*=λ*x*. *True*]
by *auto*

lemma *limpt_of_limpts*: *x islimpt* {*y*. *y islimpt S*} ⟹ *x islimpt S*
for *x* :: *'a*::*metric_space*
apply (*clarsimp simp add*: *islimpt_approachable*)
apply (*drule_tac x*=*e*/*2* **in** *spec*)
apply (*auto simp*: *simp del*: *less_divide_eq_numeral1*)
apply (*drule_tac x*=*dist x' x* **in** *spec*)
apply (*auto simp del*: *less_divide_eq_numeral1*)
apply *metric*
done

lemma *closed_limpts*: *closed* {*x*::*'a*::*metric_space*. *x islimpt S*}
using *closed_limpt limpt_of_limpts* **by** *blast*

lemma *limpt_of_closure*: *x islimpt closure S* ⟷ *x islimpt S*
for *x* :: *'a*::*metric_space*
by (*auto simp*: *closure_def islimpt_Un dest*: *limpt_of_limpts*)

lemma *islimpt_eq_infinite_ball*: *x islimpt S* ⟷ (∀ *e*>*0*. *infinite*(*S* ∩ *ball x e*))
apply (*simp add*: *islimpt_eq_acc_point*, *safe*)
apply (*metis Int_commute open_ball centre_in_ball*)
by (*metis open_contains_ball Int_mono finite_subset inf_commute subset_refl*)

lemma *islimpt_eq_infinite_cball*: *x islimpt S* ⟷ (∀ *e*>*0*. *infinite*(*S* ∩ *cball x e*))
 apply (*simp add*: *islimpt_eq_infinite_ball*, *safe*)
  apply (*meson Int_mono ball_subset_cball finite_subset order_refl*)
 by (*metis open_ball centre_in_ball finite_Int inf.absorb_iff2 inf_assoc open_contains_cball_eq*)

3.2.3 Perfect Metric Spaces

lemma *perfect_choose_dist*: $0 < r \Longrightarrow \exists a.\ a \neq x \wedge dist\ a\ x < r$
 for *x* :: *'a*::{*perfect_space*,*metric_space*}
 using *islimpt_UNIV* [*of x*] **by** (*simp add*: *islimpt_approachable*)

lemma *cball_eq_sing*:
 fixes *x* :: *'a*::{*metric_space*,*perfect_space*}
 shows $cball\ x\ e = \{x\} \longleftrightarrow e = 0$
proof (*rule linorder_cases*)
 assume *e*: $0 < e$
 obtain *a* **where** $a \neq x$ $dist\ a\ x < e$
  using *perfect_choose_dist* [*OF e*] **by** *auto*
 then have $a \neq x$ $dist\ x\ a \leq e$
  by (*auto simp*: *dist_commute*)
 with *e* **show** *?thesis* **by** (*auto simp*: *set_eq_iff*)
qed *auto*

3.2.4 ?

lemma *finite_ball_include*:
 fixes *a* :: *'a*::*metric_space*
 assumes *finite S*
 shows $\exists e>0.\ S \subseteq ball\ a\ e$
 using *assms*
proof *induction*
 case (*insert x S*)
 then obtain *e0* **where** *e0*>*0* **and** *e0*:$S \subseteq ball\ a\ e0$ **by** *auto*
 define *e* **where** $e = max\ e0\ (2 * dist\ a\ x)$
 have *e*>*0* **unfolding** *e_def* **using** ‹*e0*>*0*› **by** *auto*
 moreover have $insert\ x\ S \subseteq ball\ a\ e$
  using *e0* ‹*e*>*0*› **unfolding** *e_def* **by** *auto*
 ultimately show *?case* **by** *auto*
qed (*auto intro*: *zero_less_one*)

lemma *finite_set_avoid*:
 fixes *a* :: *'a*::*metric_space*
 assumes *finite S*
 shows $\exists d>0.\ \forall x\in S.\ x \neq a \longrightarrow d \leq dist\ a\ x$
 using *assms*
proof *induction*
 case (*insert x S*)
 then obtain *d* **where** $d > 0$ **and** *d*: $\forall x\in S.\ x \neq a \longrightarrow d \leq dist\ a\ x$
  by *blast*
 show *?case*

```
  proof (cases x = a)
    case True
    with ‹d > 0 ›d show ?thesis by auto
  next
    case False
    let ?d = min d (dist a x)
    from False ‹d > 0› have dp: ?d > 0
      by auto
    from d have d': ∀x∈S. x ≠ a ⟶ ?d ≤ dist a x
      by auto
    with dp False show ?thesis
      by (metis insert_iff le_less min_less_iff_conj not_less)
  qed
qed (auto intro: zero_less_one)

lemma discrete_imp_closed:
  fixes S :: 'a::metric_space set
  assumes e: 0 < e
    and d: ∀x ∈ S. ∀y ∈ S. dist y x < e ⟶ y = x
  shows closed S
proof –
  have False if C: ⋀e. e>0 ⟹ ∃x'∈S. x' ≠ x ∧ dist x' x < e for x
  proof –
    from e have e2: e/2 > 0 by arith
    from C[rule_format, OF e2] obtain y where y: y ∈ S y ≠ x dist y x < e/2
      by blast
    from e2 y(2) have mp: min (e/2) (dist x y) > 0
      by simp
    from d y C[OF mp] show ?thesis
      by metric
  qed
  then show ?thesis
    by (metis islimpt_approachable closed_limpt [where 'a='a])
qed
```

3.2.5 Interior

```
lemma mem_interior: x ∈ interior S ⟷ (∃e>0. ball x e ⊆ S)
  using open_contains_ball_eq [where S=interior S]
  by (simp add: open_subset_interior)

lemma mem_interior_cball: x ∈ interior S ⟷ (∃e>0. cball x e ⊆ S)
  by (meson ball_subset_cball interior_subset mem_interior open_contains_cball open_interior
    subset_trans)
```

3.2.6 Frontier

```
lemma frontier_straddle:
  fixes a :: 'a::metric_space
```

shows *a* ∈ *frontier S* ⟷ (∀ *e*>*0*. (∃ *x*∈*S*. *dist a x* < *e*) ∧ (∃ *x*. *x* ∉ *S* ∧ *dist a x* < *e*))
unfolding *frontier_def closure_interior*
by (*auto simp*: *mem_interior subset_eq ball_def*)

3.2.7 Limits

proposition *Lim*: (*f* ⟶ *l*) *net* ⟷ *trivial_limit net* ∨ (∀ *e*>*0*. *eventually* (λ*x*. *dist* (*f x*) *l* < *e*) *net*)
by (*auto simp*: *tendsto_iff trivial_limit_eq*)

Show that they yield usual definitions in the various cases.

proposition *Lim_within_le*: (*f* ⟶ *l*)(*at a within S*) ⟷
(∀ *e*>*0*. ∃ *d*>*0*. ∀ *x*∈*S*. *0* < *dist x a* ∧ *dist x a* ≤ *d* ⟶ *dist* (*f x*) *l* < *e*)
by (*auto simp*: *tendsto_iff eventually_at_le*)

proposition *Lim_within*: (*f* ⟶ *l*) (*at a within S*) ⟷
(∀ *e* >*0*. ∃ *d*>*0*. ∀ *x* ∈ *S*. *0* < *dist x a* ∧ *dist x a* < *d* ⟶ *dist* (*f x*) *l* < *e*)
by (*auto simp*: *tendsto_iff eventually_at*)

corollary *Lim_withinI* [*intro?*]:
assumes ⋀*e*. *e* > *0* ⟹ ∃ *d*>*0*. ∀ *x* ∈ *S*. *0* < *dist x a* ∧ *dist x a* < *d* ⟶ *dist* (*f x*) *l* ≤ *e*
shows (*f* ⟶ *l*) (*at a within S*)
apply (*simp add*: *Lim_within*, *clarify*)
apply (*rule ex_forward* [*OF assms* [*OF half_gt_zero*]], *auto*)
done

proposition *Lim_at*: (*f* ⟶ *l*) (*at a*) ⟷
(∀ *e* >*0*. ∃ *d*>*0*. ∀ *x*. *0* < *dist x a* ∧ *dist x a* < *d* ⟶ *dist* (*f x*) *l* < *e*)
by (*auto simp*: *tendsto_iff eventually_at*)

lemma *Lim_transform_within_set*:
fixes *a* :: *'a*::*metric_space* **and** *l* :: *'b*::*metric_space*
shows ⟦(*f* ⟶ *l*) (*at a within S*); *eventually* (λ*x*. *x* ∈ *S* ⟷ *x* ∈ *T*) (*at a*)⟧
⟹ (*f* ⟶ *l*) (*at a within T*)
apply (*clarsimp simp*: *eventually_at Lim_within*)
apply (*drule_tac x*=*e* **in** *spec*, *clarify*)
apply (*rename_tac k*)
apply (*rule_tac x*=*min d k* **in** *exI*, *simp*)
done

Another limit point characterization.

lemma *limpt_sequential_inj*:
fixes *x* :: *'a*::*metric_space*
shows *x islimpt S* ⟷
(∃ *f*. (∀ *n*::*nat*. *f n* ∈ *S* − {*x*}) ∧ *inj f* ∧ (*f* ⟶ *x*) *sequentially*)
(**is** *?lhs* = *?rhs*)
proof

```
  assume ?lhs
  then have ∀ e>0. ∃ x'∈S. x' ≠ x ∧ dist x' x < e
    by (force simp: islimpt_approachable)
  then obtain y where y: ⋀e. e>0 ⟹ y e ∈ S ∧ y e ≠ x ∧ dist (y e) x < e
    by metis
  define f where f ≡ rec_nat (y 1) (λn fn. y (min (inverse(2 ^ (Suc n))) (dist
fn x)))
  have [simp]: f 0 = y 1
             f(Suc n) = y (min (inverse(2 ^ (Suc n))) (dist (f n) x)) for n
    by (simp_all add: f_def)
  have f: f n ∈ S ∧ (f n ≠ x) ∧ dist (f n) x < inverse(2 ^ n) for n
  proof (induction n)
    case 0 show ?case
      by (simp add: y)
  next
    case (Suc n) then show ?case
      apply (auto simp: y)
     by (metis half_gt_zero_iff inverse_positive_iff_positive less_divide_eq_numeral1(1)
min_less_iff_conj y zero_less_dist_iff zero_less_numeral zero_less_power)
  qed
  show ?rhs
  proof (rule_tac x=f in exI, intro conjI allI)
    show ⋀n. f n ∈ S − {x}
      using f by blast
    have dist (f n) x < dist (f m) x if m < n for m n
    using that
    proof (induction n)
      case 0 then show ?case by simp
    next
      case (Suc n)
      then consider m < n | m = n using less_Suc_eq by blast
      then show ?case
      proof cases
        assume m < n
         have dist (f(Suc n)) x = dist (y (min (inverse(2 ^ (Suc n))) (dist (f n)
x))) x
          by simp
        also have ... < dist (f n) x
          by (metis dist_pos_lt f min.strict_order_iff min_less_iff_conj y)
        also have ... < dist (f m) x
          using Suc.IH ‹m < n› by blast
        finally show ?thesis .
      next
        assume m = n then show ?case
             by simp (metis dist_pos_lt f half_gt_zero_iff inverse_positive_iff_positive
min_less_iff_conj y zero_less_numeral zero_less_power)
      qed
    qed
    then show inj f
```

```
      by (metis less_irrefl linorder_injI)
    show f ⟶ x
      apply (rule tendstoI)
      apply (rule_tac c=nat (ceiling(1/e)) in eventually_sequentiallyI)
      apply (rule less_trans [OF f [THEN conjunct2, THEN conjunct2]])
      apply (simp add: field_simps)
      by (meson le_less_trans mult_less_cancel_left not_le of_nat_less_two_power)
  qed
next
  assume ?rhs
  then show ?lhs
    by (fastforce simp add: islimpt_approachable lim_sequentially)
qed

lemma Lim_dist_ubound:
  assumes ¬(trivial_limit net)
    and (f ⟶ l) net
    and eventually (λx. dist a (f x) ≤ e) net
  shows dist a l ≤ e
  using assms by (fast intro: tendsto_le tendsto_intros)
```

3.2.8 Continuity

Derive the epsilon-delta forms, which we often use as "definitions"

```
proposition continuous_within_eps_delta:
  continuous (at x within s) f ⟷ (∀ e>0. ∃ d>0. ∀ x'∈ s.  dist x' x < d -->
dist (f x') (f x) < e)
  unfolding continuous_within and Lim_within  by fastforce

corollary continuous_at_eps_delta:
  continuous (at x) f ⟷ (∀ e > 0. ∃ d > 0. ∀ x'. dist x' x < d ⟶ dist (f x') (f
x) < e)
  using continuous_within_eps_delta [of x UNIV f] by simp

lemma continuous_at_right_real_increasing:
  fixes f :: real ⇒ real
  assumes nondecF: ⋀x y. x ≤ y ⟹ f x ≤ f y
  shows continuous (at_right a) f ⟷ (∀ e>0. ∃ d>0. f (a + d) − f a < e)
  apply (simp add: greaterThan_def dist_real_def continuous_within Lim_within_le)
  apply (intro all_cong ex_cong, safe)
  apply (erule_tac x=a + d in allE, simp)
  apply (simp add: nondecF field_simps)
  apply (drule nondecF, simp)
  done

lemma continuous_at_left_real_increasing:
  assumes nondecF: ⋀ x y. x ≤ y ⟹ f x ≤ ((f y) :: real)
  shows (continuous (at_left (a :: real)) f) = (∀ e > 0. ∃ delta > 0. f a − f (a −
delta) < e)
```

```
  apply (simp add: lessThan_def dist_real_def continuous_within Lim_within_le)
  apply (intro all_cong ex_cong, safe)
  apply (erule_tac x=a − d in allE, simp)
  apply (simp add: nondecF field_simps)
  apply (cut_tac x=a − d and y=x in nondecF, simp_all)
  done
```

Versions in terms of open balls.

```
lemma continuous_within_ball:
  continuous (at x within s) f ⟷
    (∀ e > 0. ∃ d > 0. f ` (ball x d ∩ s) ⊆ ball (f x) e)
  (is ?lhs = ?rhs)
proof
  assume ?lhs
  {
    fix e :: real
    assume e > 0
    then obtain d where d: d>0 ∀ xa∈s. 0 < dist xa x ∧ dist xa x < d ⟶ dist
(f xa) (f x) < e
      using ‹?lhs›[unfolded continuous_within Lim_within] by auto
    {
      fix y
      assume y ∈ f ` (ball x d ∩ s)
      then have y ∈ ball (f x) e
        using d(2)
        using ‹e > 0›
        by (auto simp: dist_commute)
    }
    then have ∃ d>0. f ` (ball x d ∩ s) ⊆ ball (f x) e
      using ‹d > 0›
      unfolding subset_eq ball_def by (auto simp: dist_commute)
  }
  then show ?rhs by auto
next
  assume ?rhs
  then show ?lhs
    unfolding continuous_within Lim_within ball_def subset_eq
    apply (auto simp: dist_commute)
    apply (erule_tac x=e in allE, auto)
    done
qed

lemma continuous_at_ball:
  continuous (at x) f ⟷ (∀ e>0. ∃ d>0. f ` (ball x d) ⊆ ball (f x) e) (is ?lhs =
?rhs)
proof
  assume ?lhs
  then show ?rhs
    unfolding continuous_at Lim_at subset_eq Ball_def Bex_def image_iff mem_ball
```

```
    by (metis dist_commute dist_pos_lt dist_self)
next
  assume ?rhs
  then show ?lhs
    unfolding continuous_at Lim_at subset_eq Ball_def Bex_def image_iff mem_ball
    by (metis dist_commute)
qed
```

Define setwise continuity in terms of limits within the set.

```
lemma continuous_on_iff:
  continuous_on s f ⟷
    (∀x∈s. ∀e>0. ∃d>0. ∀x'∈s. dist x' x < d ⟶ dist (f x') (f x) < e)
  unfolding continuous_on_def Lim_within
  by (metis dist_pos_lt dist_self)

lemma continuous_within_E:
  assumes continuous (at x within s) f e>0
  obtains d where d>0 ⋀x'. ⟦x'∈ s; dist x' x ≤ d⟧ ⟹ dist (f x') (f x) < e
  using assms apply (simp add: continuous_within_eps_delta)
  apply (drule spec [of _ e], clarify)
  apply (rule_tac d=d/2 in that, auto)
  done

lemma continuous_onI [intro?]:
  assumes ⋀x e. ⟦e > 0; x ∈ s⟧ ⟹ ∃d>0. ∀x'∈s. dist x' x < d ⟶ dist (f x')
(f x) ≤ e
  shows continuous_on s f
apply (simp add: continuous_on_iff, clarify)
apply (rule ex_forward [OF assms [OF half_gt_zero]], auto)
done
```

Some simple consequential lemmas.

```
lemma continuous_onE:
    assumes continuous_on s f x∈s e>0
    obtains d where d>0 ⋀x'. ⟦x' ∈ s; dist x' x ≤ d⟧ ⟹ dist (f x') (f x) < e
  using assms
  apply (simp add: continuous_on_iff)
  apply (elim ballE allE)
  apply (auto intro: that [where d=d/2 for d])
  done
```

The usual transformation theorems.

```
lemma continuous_transform_within:
  fixes f g :: 'a::metric_space ⇒ 'b::topological_space
  assumes continuous (at x within s) f
    and 0 < d
    and x ∈ s
    and ⋀x'. ⟦x' ∈ s; dist x' x < d⟧ ⟹ f x' = g x'
  shows continuous (at x within s) g
```

```
  using assms
  unfolding continuous_within
  by (force intro: Lim_transform_within)
```

3.2.9 Closure and Limit Characterization

```
lemma closure_approachable:
  fixes S :: 'a::metric_space set
  shows x ∈ closure S ⟷ (∀ e>0. ∃ y∈S. dist y x < e)
  apply (auto simp: closure_def islimpt_approachable)
  apply (metis dist_self)
  done

lemma closure_approachable_le:
  fixes S :: 'a::metric_space set
  shows x ∈ closure S ⟷ (∀ e>0. ∃ y∈S. dist y x ≤ e)
  unfolding closure_approachable
  using dense by force

lemma closure_approachableD:
  assumes x ∈ closure S e>0
  shows ∃ y∈S. dist x y < e
  using assms unfolding closure_approachable by (auto simp: dist_commute)

lemma closed_approachable:
  fixes S :: 'a::metric_space set
  shows closed S ⟹ (∀ e>0. ∃ y∈S. dist y x < e) ⟷ x ∈ S
  by (metis closure_closed closure_approachable)

lemma closure_contains_Inf:
  fixes S :: real set
  assumes S ≠ {} bdd_below S
  shows Inf S ∈ closure S
proof -
  have *: ∀ x∈S. Inf S ≤ x
    using cInf_lower[of _ S] assms by metis
  {
    fix e :: real
    assume e > 0
    then have Inf S < Inf S + e by simp
    with assms obtain x where x ∈ S x < Inf S + e
      by (subst (asm) cInf_less_iff) auto
    with * have ∃ x∈S. dist x (Inf S) < e
      by (intro bexI[of _ x]) (auto simp: dist_real_def)
  }
  then show ?thesis unfolding closure_approachable by auto
qed

lemma closure_contains_Sup:
```

fixes *S* :: *real set*
assumes *S* ≠ {} *bdd_above S*
shows *Sup S* ∈ *closure S*
proof −
have ∗: ∀ *x*∈*S*. *x* ≤ *Sup S*
using *cSup_upper*[*of* _ *S*] *assms* **by** *metis*
{
fix *e* :: *real*
assume *e* > *0*
then have *Sup S* − *e* < *Sup S* **by** *simp*
with *assms* **obtain** *x* **where** *x* ∈ *S Sup S* − *e* < *x*
by (*subst* (*asm*) *less_cSup_iff*) *auto*
with ∗ **have** ∃ *x*∈*S*. *dist x* (*Sup S*) < *e*
by (*intro bexI*[*of* _ *x*]) (*auto simp*: *dist_real_def*)
}
then show *?thesis* **unfolding** *closure_approachable* **by** *auto*
qed

lemma *not_trivial_limit_within_ball*:
¬ *trivial_limit* (*at x within S*) ⟷ (∀ *e*>*0*. *S* ∩ *ball x e* − {*x*} ≠ {})
(**is** *?lhs* ⟷ *?rhs*)
proof
show *?rhs* **if** *?lhs*
proof −
{
fix *e* :: *real*
assume *e* > *0*
then obtain *y* **where** *y* ∈ *S* − {*x*} **and** *dist y x* < *e*
using ‹*?lhs*› *not_trivial_limit_within*[*of x S*] *closure_approachable*[*of x S* − {*x*}]
by *auto*
then have *y* ∈ *S* ∩ *ball x e* − {*x*}
unfolding *ball_def* **by** (*simp add*: *dist_commute*)
then have *S* ∩ *ball x e* − {*x*} ≠ {} **by** *blast*
}
then show *?thesis* **by** *auto*
qed
show *?lhs* **if** *?rhs*
proof −
{
fix *e* :: *real*
assume *e* > *0*
then obtain *y* **where** *y* ∈ *S* ∩ *ball x e* − {*x*}
using ‹*?rhs*› **by** *blast*
then have *y* ∈ *S* − {*x*} **and** *dist y x* < *e*
unfolding *ball_def* **by** (*simp_all add*: *dist_commute*)
then have ∃ *y* ∈ *S* − {*x*}. *dist y x* < *e*
by *auto*
}

```
    then show ?thesis
      using not_trivial_limit_within[of x S] closure_approachable[of x S − {x}]
      by auto
  qed
qed
```

3.2.10 Boundedness

```
definition (in metric_space) bounded :: 'a set ⇒ bool
  where bounded S ⟷ (∃x e. ∀y∈S. dist x y ≤ e)

lemma bounded_subset_cball: bounded S ⟷ (∃e x. S ⊆ cball x e ∧ 0 ≤ e)
  unfolding bounded_def subset_eq by auto (meson order_trans zero_le_dist)

lemma bounded_any_center: bounded S ⟷ (∃e. ∀y∈S. dist a y ≤ e)
  unfolding bounded_def
  by auto (metis add.commute add_le_cancel_right dist_commute dist_triangle_le)

lemma bounded_iff: bounded S ⟷ (∃a. ∀x∈S. norm x ≤ a)
  unfolding bounded_any_center [where a=0]
  by (simp add: dist_norm)

lemma bdd_above_norm: bdd_above (norm ` X) ⟷ bounded X
  by (simp add: bounded_iff bdd_above_def)

lemma bounded_norm_comp: bounded ((λx. norm (f x)) ` S) = bounded (f ` S)
  by (simp add: bounded_iff)

lemma boundedI:
  assumes ⋀x. x ∈ S ⟹ norm x ≤ B
  shows bounded S
  using assms bounded_iff by blast

lemma bounded_empty [simp]: bounded {}
  by (simp add: bounded_def)

lemma bounded_subset: bounded T ⟹ S ⊆ T ⟹ bounded S
  by (metis bounded_def subset_eq)

lemma bounded_interior[intro]: bounded S ⟹ bounded(interior S)
  by (metis bounded_subset interior_subset)

lemma bounded_closure[intro]:
  assumes bounded S
  shows bounded (closure S)
proof −
  from assms obtain x and a where a: ∀y∈S. dist x y ≤ a
    unfolding bounded_def by auto
  {
```

fix *y*
assume *y* ∈ *closure S*
then obtain *f* **where** *f*: ∀ *n*. *f n* ∈ *S* (*f* ⟶ *y*) *sequentially*
unfolding *closure_sequential* **by** *auto*
have ∀ *n*. *f n* ∈ *S* ⟶ *dist x* (*f n*) ≤ *a* **using** *a* **by** *simp*
then have *eventually* (λ*n*. *dist x* (*f n*) ≤ *a*) *sequentially*
by (*simp add*: *f*(*1*))
then have *dist x y* ≤ *a*
using *Lim_dist_ubound f*(*2*) *trivial_limit_sequentially* **by** *blast*
}
then show *?thesis*
unfolding *bounded_def* **by** *auto*
qed

lemma *bounded_closure_image*: *bounded* (*f* ' *closure S*) ⟹ *bounded* (*f* ' *S*)
by (*simp add*: *bounded_subset closure_subset image_mono*)

lemma *bounded_cball*[*simp*,*intro*]: *bounded* (*cball x e*)
unfolding *bounded_def* **using** *mem_cball* **by** *blast*

lemma *bounded_ball*[*simp*,*intro*]: *bounded* (*ball x e*)
by (*metis ball_subset_cball bounded_cball bounded_subset*)

lemma *bounded_Un*[*simp*]: *bounded* (*S* ∪ *T*) ⟷ *bounded S* ∧ *bounded T*
by (*auto simp*: *bounded_def*) (*metis Un_iff bounded_any_center le_max_iff_disj*)

lemma *bounded_Union*[*intro*]: *finite F* ⟹ ∀ *S*∈*F*. *bounded S* ⟹ *bounded* (⋃ *F*)
by (*induct rule*: *finite_induct*[*of F*]) *auto*

lemma *bounded_UN* [*intro*]: *finite A* ⟹ ∀ *x*∈*A*. *bounded* (*B x*) ⟹ *bounded* (⋃ *x*∈*A*. *B x*)
by *auto*

lemma *bounded_insert* [*simp*]: *bounded* (*insert x S*) ⟷ *bounded S*
proof −
have ∀ *y*∈{*x*}. *dist x y* ≤ *0*
by *simp*
then have *bounded* {*x*}
unfolding *bounded_def* **by** *fast*
then show *?thesis*
by (*metis insert_is_Un bounded_Un*)
qed

lemma *bounded_subset_ballI*: *S* ⊆ *ball x r* ⟹ *bounded S*
by (*meson bounded_ball bounded_subset*)

lemma *bounded_subset_ballD*:
assumes *bounded S* **shows** ∃ *r*. *0* < *r* ∧ *S* ⊆ *ball x r*
proof −

```
  obtain e::real and y where S ⊆ cball y e 0 ≤ e
    using assms by (auto simp: bounded_subset_cball)
  then show ?thesis
    by (intro exI[where x=dist x y + e + 1]) metric
qed

lemma finite_imp_bounded [intro]: finite S ⟹ bounded S
  by (induct set: finite) simp_all

lemma bounded_Int[intro]: bounded S ∨ bounded T ⟹ bounded (S ∩ T)
  by (metis Int_lower1 Int_lower2 bounded_subset)

lemma bounded_diff[intro]: bounded S ⟹ bounded (S − T)
  by (metis Diff_subset bounded_subset)

lemma bounded_dist_comp:
  assumes bounded (f ` S) bounded (g ` S)
  shows bounded ((λx. dist (f x) (g x)) ` S)
proof −
  from assms obtain M1 M2 where ∗: dist (f x) undefined ≤ M1 dist undefined
(g x) ≤ M2 if x ∈ S for x
    by (auto simp: bounded_any_center[of _ undefined] dist_commute)
  have dist (f x) (g x) ≤ M1 + M2 if x ∈ S for x
    using ∗[OF that]
    by metric
  then show ?thesis
    by (auto intro!: boundedI)
qed

lemma bounded_Times:
  assumes bounded s bounded t
  shows bounded (s × t)
proof −
  obtain x y a b where ∀z∈s. dist x z ≤ a ∀z∈t. dist y z ≤ b
    using assms [unfolded bounded_def] by auto
  then have ∀z∈s × t. dist (x, y) z ≤ sqrt (a^2 + b^2)
    by (auto simp: dist_Pair_Pair real_sqrt_le_mono add_mono power_mono)
  then show ?thesis unfolding bounded_any_center [where a=(x, y)] by auto
qed
```

3.2.11 Compactness

```
lemma compact_imp_bounded:
  assumes compact U
  shows bounded U
proof −
  have compact U ∀x∈U. open (ball x 1) U ⊆ (⋃x∈U. ball x 1)
    using assms by auto
  then obtain D where D: D ⊆ U finite D U ⊆ (⋃x∈D. ball x 1)
```

by (*metis compactE_image*)
from ‹*finite D*› **have** *bounded* ($\bigcup$*x*∈*D*. *ball x 1*)
by (*simp add*: *bounded_UN*)
then show *bounded U* **using** ‹*U* ⊆ ($\bigcup$*x*∈*D*. *ball x 1*)›
by (*rule bounded_subset*)
qed

lemma *closure_Int_ball_not_empty*:
assumes *S* ⊆ *closure T x* ∈ *S r* > *0*
shows *T* ∩ *ball x r* ≠ {}
using *assms centre_in_ball closure_iff_nhds_not_empty* **by** *blast*

lemma *compact_sup_maxdistance*:
fixes *S* :: *'a*::*metric_space set*
assumes *compact S*
and *S* ≠ {}
shows ∃ *x*∈*S*. ∃ *y*∈*S*. ∀ *u*∈*S*. ∀ *v*∈*S*. *dist u v* ≤ *dist x y*
proof −
have *compact* (*S* × *S*)
using ‹*compact S*› **by** (*intro compact_Times*)
moreover have *S* × *S* ≠ {}
using ‹*S* ≠ {}› **by** *auto*
moreover have *continuous_on* (*S* × *S*) (λ*x*. *dist* (*fst x*) (*snd x*))
by (*intro continuous_at_imp_continuous_on ballI continuous_intros*)
ultimately show *?thesis*
using *continuous_attains_sup*[*of S* × *S* λ*x*. *dist* (*fst x*) (*snd x*)] **by** *auto*
qed

Totally bounded

lemma *cauchy_def*: *Cauchy S* ⟷ (∀ *e*>*0*. ∃ *N*. ∀ *m n*. *m* ≥ *N* ∧ *n* ≥ *N* ⟶ *dist* (*S m*) (*S n*) < *e*)
unfolding *Cauchy_def* **by** *metis*

proposition *seq_compact_imp_totally_bounded*:
assumes *seq_compact S*
shows ∀ *e*>*0*. ∃ *k*. *finite k* ∧ *k* ⊆ *S* ∧ *S* ⊆ ($\bigcup$*x*∈*k*. *ball x e*)
proof −
{ **fix** *e*::*real* **assume** *e* > *0* **assume** ∗: $\bigwedge$*k*. *finite k* ⟹ *k* ⊆ *S* ⟹ ¬ *S* ⊆ ($\bigcup$*x*∈*k*. *ball x e*)
let *?Q* = λ*x n r*. *r* ∈ *S* ∧ (∀ *m* < (*n*::*nat*). ¬ (*dist* (*x m*) *r* < *e*))
have ∃ *x*. ∀ *n*::*nat*. *?Q x n* (*x n*)
proof (*rule dependent_wellorder_choice*)
fix *n x* **assume** $\bigwedge$*y*. *y* < *n* ⟹ *?Q x y* (*x y*)
then have ¬ *S* ⊆ ($\bigcup$*x*∈*x* ‘ {*0*..<*n*}. *ball x e*)
using ∗[*of x* ‘ {*0* ..< *n*}] **by** (*auto simp*: *subset_eq*)
then obtain *z* **where** *z*:*z*∈*S z* ∉ ($\bigcup$*x*∈*x* ‘ {*0*..<*n*}. *ball x e*)
unfolding *subset_eq* **by** *auto*
show ∃ *r*. *?Q x n r*

```
      using z by auto
    qed simp
    then obtain x where ∀ n::nat. x n ∈ S and x:⋀n m. m < n ⟹ ¬ (dist (x
m) (x n) < e)
      by blast
    then obtain l r where l ∈ S and r:strict_mono r and ((x ∘ r) ⟶ l)
sequentially
      using assms by (metis seq_compact_def)
    then have Cauchy (x ∘ r)
      using LIMSEQ_imp_Cauchy by auto
    then obtain N::nat where ⋀m n. N ≤ m ⟹ N ≤ n ⟹ dist ((x ∘ r) m)
((x ∘ r) n) < e
      unfolding cauchy_def using ‹e > 0› by blast
    then have False
      using x[of r N r (N+1)] r by (auto simp: strict_mono_def) }
  then show ?thesis
    by metis
qed
```

Heine-Borel theorem

```
proposition seq_compact_imp_Heine_Borel:
  fixes S :: 'a :: metric_space set
  assumes seq_compact S
  shows compact S
proof -
  from seq_compact_imp_totally_bounded[OF ‹seq_compact S›]
  obtain f where f: ∀ e>0. finite (f e) ∧ f e ⊆ S ∧ S ⊆ (⋃x∈f e. ball x e)
    unfolding choice_iff' ..
  define K where K = (λ(x, r). ball x r) ` ((⋃e ∈ ℚ ∩ {0 <..}. f e) × ℚ)
  have countably_compact S
    using ‹seq_compact S› by (rule seq_compact_imp_countably_compact)
  then show compact S
  proof (rule countably_compact_imp_compact)
    show countable K
      unfolding K_def using f
      by (auto intro: countable_finite countable_subset countable_rat
              intro!: countable_image countable_SIGMA countable_UN)
    show ∀ b∈K. open b by (auto simp: K_def)
  next
    fix T x
    assume T: open T x ∈ T and x: x ∈ S
    from openE[OF T] obtain e where 0 < e ball x e ⊆ T
      by auto
    then have 0 < e/2 ball x (e/2) ⊆ T
      by auto
    from Rats_dense_in_real[OF ‹0 < e/2›] obtain r where r ∈ ℚ 0 < r r < e/2
      by auto
    from f[rule_format, of r] ‹0 < r› ‹x ∈ S› obtain k where k ∈ f r x ∈ ball k r
```

by *auto*
from ‹$r \in \mathbb{Q}$› ‹$0 < r$› ‹$k \in f\ r$› **have** *ball k r* $\in K$
by (*auto simp*: *K_def*)
then show $\exists\, b{\in}K.\ x \in b \wedge b \cap S \subseteq T$
proof (*rule bexI*[*rotated*], *safe*)
fix *y*
assume $y \in$ *ball k r*
with ‹$r < e/2$› ‹$x \in$ *ball k r*› **have** *dist x y* $< e$
by (*intro dist_triangle_half_r* [*of k _ e*]) (*auto simp*: *dist_commute*)
with ‹*ball x e* $\subseteq T$› **show** $y \in T$
by *auto*
next
show $x \in$ *ball k r* **by** *fact*
qed
qed
qed

proposition *compact_eq_seq_compact_metric*:
compact (*S* :: *'a*::*metric_space set*) $\longleftrightarrow$ *seq_compact S*
using *compact_imp_seq_compact seq_compact_imp_Heine_Borel* **by** *blast*

proposition *compact_def*: — this is the definition of compactness in HOL Light
compact (*S* :: *'a*::*metric_space set*) $\longleftrightarrow$
($\forall f.\ (\forall n.\ f\ n \in S) \longrightarrow (\exists\, l{\in}S.\ \exists\, r{::}nat{\Rightarrow}nat.$ *strict_mono r* $\wedge\ (f \circ r) \longrightarrow$
l))
unfolding *compact_eq_seq_compact_metric seq_compact_def* **by** *auto*

Complete the chain of compactness variants

proposition *compact_eq_Bolzano_Weierstrass*:
fixes *S* :: *'a*::*metric_space set*
shows *compact S* $\longleftrightarrow$ ($\forall\, T.$ *infinite T* $\wedge\ T \subseteq S \longrightarrow (\exists\, x \in S.\ x$ *islimpt T*))
using *Bolzano_Weierstrass_imp_seq_compact Heine_Borel_imp_Bolzano_Weierstrass compact_eq_seq_compact_metric*
by *blast*

proposition *Bolzano_Weierstrass_imp_bounded*:
($\bigwedge T.$ ⟦*infinite T*; $T \subseteq S$⟧ $\Longrightarrow (\exists\, x \in S.\ x$ *islimpt T*)) $\Longrightarrow$ *bounded S*
using *compact_imp_bounded* **unfolding** *compact_eq_Bolzano_Weierstrass* **by** *metis*

3.2.12 Banach fixed point theorem

theorem *banach_fix*:— TODO: rename to *Banach_fix*
assumes *s*: *complete s s* $\neq$ {}
and *c*: $0 \leq c\ c < 1$
and *f*: *f* ‘ *s* $\subseteq$ *s*
and *lipschitz*: $\forall\, x{\in}s.\ \forall\, y{\in}s.$ *dist* (*f x*) (*f y*) $\leq c *$ *dist x y*
shows $\exists!x{\in}s.\ f\ x = x$
proof −
from *c* **have** $1 - c > 0$ **by** *simp*

```
  from s(2) obtain z0 where z0: z0 ∈ s by blast
  define z where z n = (f ^^ n) z0 for n
  with f z0 have z_in_s: z n ∈ s for n :: nat
    by (induct n) auto
  define d where d = dist (z 0) (z 1)

  have fzn: f (z n) = z (Suc n) for n
    by (simp add: z_def)
  have cf_z: dist (z n) (z (Suc n)) ≤ (c ^ n) * d for n :: nat
  proof (induct n)
    case 0
    then show ?case
      by (simp add: d_def)
  next
    case (Suc m)
    with ‹0 ≤ c› have c * dist (z m) (z (Suc m)) ≤ c ^ Suc m * d
      using mult_left_mono[of dist (z m) (z (Suc m)) c ^ m * d c] by simp
    then show ?case
      using lipschitz[THEN bspec[where x=z m], OF z_in_s, THEN bspec[where
x=z (Suc m)], OF z_in_s]
      by (simp add: fzn mult_le_cancel_left)
  qed

  have cf_z2: (1 − c) * dist (z m) (z (m + n)) ≤ (c ^ m) * d * (1 − c ^ n) for
n m :: nat
  proof (induct n)
    case 0
    show ?case by simp
  next
    case (Suc k)
    from c have (1 − c) * dist (z m) (z (m + Suc k)) ≤
        (1 − c) * (dist (z m) (z (m + k)) + dist (z (m + k)) (z (Suc (m + k))))
      by (simp add: dist_triangle)
    also from c cf_z[of m + k] have ... ≤ (1 − c) * (dist (z m) (z (m + k)) +
c ^ (m + k) * d)
      by simp
    also from Suc have ... ≤ c ^ m * d * (1 − c ^ k) + (1 − c) * c ^ (m + k)
* d
      by (simp add: field_simps)
    also have ... = (c ^ m) * (d * (1 − c ^ k) + (1 − c) * c ^ k * d)
      by (simp add: power_add field_simps)
    also from c have ... ≤ (c ^ m) * d * (1 − c ^ Suc k)
      by (simp add: field_simps)
    finally show ?case by simp
  qed

  have ∃N. ∀m n. N ≤ m ∧ N ≤ n ⟶ dist (z m) (z n) < e if e > 0 for e
  proof (cases d = 0)
```

```
  case True
  from ‹1 − c > 0› have (1 − c) ∗ x ≤ 0 ⟷ x ≤ 0 for x
    by (simp add: mult_le_0_iff)
  with c cf_z2[of 0] True have z n = z0 for n
    by (simp add: z_def)
  with ‹e > 0› show ?thesis by simp
next
  case False
  with zero_le_dist[of z 0 z 1] have d > 0
    by (metis d_def less_le)
  with ‹1 − c > 0› ‹e > 0› have 0 < e ∗ (1 − c) / d
    by simp
  with c obtain N where N: c ^ N < e ∗ (1 − c) / d
    using real_arch_pow_inv[of e ∗ (1 − c) / d c] by auto
  have ∗: dist (z m) (z n) < e if m > n and as: m ≥ N n ≥ N for m n :: nat
  proof −
    from c ‹n ≥ N› have ∗: c ^ n ≤ c ^ N
      using power_decreasing[OF ‹n≥N›, of c] by simp
    from c ‹m > n› have 1 − c ^ (m − n) > 0
      using power_strict_mono[of c 1 m − n] by simp
    with ‹d > 0› ‹0 < 1 − c› have ∗∗: d ∗ (1 − c ^ (m − n)) / (1 − c) > 0
      by simp
    from cf_z2[of n m − n] ‹m > n›
    have dist (z m) (z n) ≤ c ^ n ∗ d ∗ (1 − c ^ (m − n)) / (1 − c)
     by (simp add: pos_le_divide_eq[OF ‹1 − c > 0›] mult.commute dist_commute)
    also have ... ≤ c ^ N ∗ d ∗ (1 − c ^ (m − n)) / (1 − c)
      using mult_right_mono[OF ∗ order_less_imp_le[OF ∗∗]]
      by (simp add: mult.assoc)
    also have ... < (e ∗ (1 − c) / d) ∗ d ∗ (1 − c ^ (m − n)) / (1 − c)
      using mult_strict_right_mono[OF N ∗∗] by (auto simp: mult.assoc)
    also from c ‹d > 0› ‹1 − c > 0› have ... = e ∗ (1 − c ^ (m − n))
      by simp
    also from c ‹1 − c ^ (m − n) > 0› ‹e > 0› have ... ≤ e
      using mult_right_le_one_le[of e 1 − c ^ (m − n)] by auto
    finally show ?thesis by simp
  qed
  have dist (z n) (z m) < e if N ≤ m N ≤ n for m n :: nat
  proof (cases n = m)
    case True
    with ‹e > 0› show ?thesis by simp
  next
    case False
    with ∗[of n m] ∗[of m n] and that show ?thesis
      by (auto simp: dist_commute nat_neq_iff)
  qed
  then show ?thesis by auto
qed
then have Cauchy z
  by (simp add: cauchy_def)
```

```
  then obtain x where x∈s and x:(z ⟶ x) sequentially
    using s(1)[unfolded compact_def complete_def, THEN spec[where x=z]] and
z_in_s by auto

  define e where e = dist (f x) x
  have e = 0
  proof (rule ccontr)
    assume e ≠ 0
    then have e > 0
      unfolding e_def using zero_le_dist[of f x x]
      by (metis dist_eq_0_iff dist_nz e_def)
    then obtain N where N:∀n≥N. dist (z n) x < e/2
      using x[unfolded lim_sequentially, THEN spec[where x=e/2]] by auto
    then have N':dist (z N) x < e/2 by auto
    have *: c * dist (z N) x ≤ dist (z N) x
      unfolding mult_le_cancel_right2
      using zero_le_dist[of z N x] and c
      by (metis dist_eq_0_iff dist_nz order_less_asym less_le)
    have dist (f (z N)) (f x) ≤ c * dist (z N) x
      using lipschitz[THEN bspec[where x=z N], THEN bspec[where x=x]]
      using z_in_s[of N] ‹x∈s›
      using c
      by auto
    also have ... < e/2
      using N' and c using * by auto
    finally show False
      unfolding fzn
      using N[THEN spec[where x=Suc N]] and dist_triangle_half_r[of z (Suc N)
f x e x]
      unfolding e_def
      by auto
  qed
  then have f x = x by (auto simp: e_def)
  moreover have y = x if f y = y y ∈ s for y
  proof -
    from ‹x ∈ s› ‹f x = x› that have dist x y ≤ c * dist x y
      using lipschitz[THEN bspec[where x=x], THEN bspec[where x=y]] by simp
    with c and zero_le_dist[of x y] have dist x y = 0
      by (simp add: mult_le_cancel_right1)
    then show ?thesis by simp
  qed
  ultimately show ?thesis
    using ‹x∈s› by blast
qed
```

3.2.13 Edelstein fixed point theorem

```
theorem Edelstein_fix:
  fixes S :: 'a::metric_space set
```

assumes *S*: *compact S S* ≠ {}
and *gs*: (*g* ‘ *S*) ⊆ *S*
and *dist*: ∀ *x*∈*S*. ∀ *y*∈*S*. *x* ≠ *y* ⟶ *dist* (*g x*) (*g y*) < *dist x y*
shows ∃!*x*∈*S*. *g x* = *x*
proof −
let *?D* = (λ*x*. (*x*, *x*)) ‘ *S*
have *D*: *compact ?D ?D* ≠ {}
by (*rule compact_continuous_image*)
(*auto intro*!: *S continuous_Pair continuous_ident simp*: *continuous_on_eq_continuous_within*)

have ⋀*x y e*. *x* ∈ *S* ⟹ *y* ∈ *S* ⟹ *0* < *e* ⟹ *dist y x* < *e* ⟹ *dist* (*g y*) (*g x*) < *e*
using *dist* **by** *fastforce*
then have *continuous_on S g*
by (*auto simp*: *continuous_on_iff*)
then have *cont*: *continuous_on ?D* (λ*x*. *dist* ((*g* ∘ *fst*) *x*) (*snd x*))
unfolding *continuous_on_eq_continuous_within*
by (*intro continuous_dist ballI continuous_within_compose*)
(*auto intro*!: *continuous_fst continuous_snd continuous_ident simp*: *image_image*)

obtain *a* **where** *a* ∈ *S* **and** *le*: ⋀*x*. *x* ∈ *S* ⟹ *dist* (*g a*) *a* ≤ *dist* (*g x*) *x*
using *continuous_attains_inf*[*OF D cont*] **by** *auto*

have *g a* = *a*
proof (*rule ccontr*)
assume *g a* ≠ *a*
with ‹*a* ∈ *S*› *gs* **have** *dist* (*g* (*g a*)) (*g a*) < *dist* (*g a*) *a*
by (*intro dist*[*rule_format*]) *auto*
moreover have *dist* (*g a*) *a* ≤ *dist* (*g* (*g a*)) (*g a*)
using ‹*a* ∈ *S*› *gs* **by** (*intro le*) *auto*
ultimately show *False* **by** *auto*
qed
moreover have ⋀*x*. *x* ∈ *S* ⟹ *g x* = *x* ⟹ *x* = *a*
using *dist*[*THEN bspec*[**where** *x*=*a*]] ‹*g a* = *a*› **and** ‹*a*∈*S*› **by** *auto*
ultimately show ∃!*x*∈*S*. *g x* = *x*
using ‹*a* ∈ *S*› **by** *blast*
qed

3.2.14 The diameter of a set

definition *diameter* :: ′*a*::*metric_space set* ⇒ *real* **where**
diameter S = (*if S* = {} *then 0 else SUP* (*x*,*y*)∈*S*×*S*. *dist x y*)

lemma *diameter_empty* [*simp*]: *diameter*{} = *0*
by (*auto simp*: *diameter_def*)

lemma *diameter_singleton* [*simp*]: *diameter*{*x*} = *0*
by (*auto simp*: *diameter_def*)

```
lemma diameter_le:
  assumes S ≠ {} ∨ 0 ≤ d
    and no: ⋀x y. ⟦x ∈ S; y ∈ S⟧ ⟹ norm(x − y) ≤ d
  shows diameter S ≤ d
  using assms
  by (auto simp: dist_norm diameter_def intro: cSUP_least)

lemma diameter_bounded_bound:
  fixes S :: 'a :: metric_space set
  assumes S: bounded S x ∈ S y ∈ S
  shows dist x y ≤ diameter S
proof −
  from S obtain z d where z: ⋀x. x ∈ S ⟹ dist z x ≤ d
    unfolding bounded_def by auto
  have bdd_above (case_prod dist ` (S×S))
  proof (intro bdd_aboveI, safe)
    fix a b
    assume a ∈ S b ∈ S
    with z[of a] z[of b] dist_triangle[of a b z]
    show dist a b ≤ 2 ∗ d
      by (simp add: dist_commute)
  qed
  moreover have (x,y) ∈ S×S using S by auto
  ultimately have dist x y ≤ (SUP (x,y)∈S×S. dist x y)
    by (rule cSUP_upper2) simp
  with ‹x ∈ S› show ?thesis
    by (auto simp: diameter_def)
qed

lemma diameter_lower_bounded:
  fixes S :: 'a :: metric_space set
  assumes S: bounded S
    and d: 0 < d d < diameter S
  shows ∃x∈S. ∃y∈S. d < dist x y
proof (rule ccontr)
  assume contr: ¬ ?thesis
  moreover have S ≠ {}
    using d by (auto simp: diameter_def)
  ultimately have diameter S ≤ d
    by (auto simp: not_less diameter_def intro!: cSUP_least)
  with ‹d < diameter S› show False by auto
qed

lemma diameter_bounded:
  assumes bounded S
  shows ∀x∈S. ∀y∈S. dist x y ≤ diameter S
    and ∀d>0. d < diameter S ⟶ (∃x∈S. ∃y∈S. dist x y > d)
  using diameter_bounded_bound[of S] diameter_lower_bounded[of S] assms
  by auto
```

lemma *bounded_two_points*: *bounded S* ⟷ (∃ *e*. ∀ *x*∈*S*. ∀ *y*∈*S*. *dist x y* ≤ *e*)
  by (*meson bounded_def diameter_bounded*(*1*))

lemma *diameter_compact_attained*:
  assumes *compact S*
    and *S* ≠ {}
  shows ∃ *x*∈*S*. ∃ *y*∈*S*. *dist x y* = *diameter S*
proof −
  have *b*: *bounded S* **using** *assms*(*1*)
    by (*rule compact_imp_bounded*)
  then obtain *x y* **where** *xys*: *x*∈*S y*∈*S*
    and *xy*: ∀ *u*∈*S*. ∀ *v*∈*S*. *dist u v* ≤ *dist x y*
    using *compact_sup_maxdistance*[*OF assms*] **by** *auto*
  then have *diameter S* ≤ *dist x y*
    unfolding *diameter_def*
    apply *clarsimp*
    apply (*rule cSUP_least*, *fast*+)
    done
  then show *?thesis*
    by (*metis b diameter_bounded_bound order_antisym xys*)
qed

lemma *diameter_ge_0*:
  assumes *bounded S* **shows** *0* ≤ *diameter S*
  by (*metis all_not_in_conv assms diameter_bounded_bound diameter_empty dist_self order_refl*)

lemma *diameter_subset*:
  assumes *S* ⊆ *T bounded T*
  shows *diameter S* ≤ *diameter T*
proof (*cases S* = {} ∨ *T* = {})
  case *True*
  with *assms* **show** *?thesis*
    by (*force simp*: *diameter_ge_0*)
next
  case *False*
  then have *bdd_above* ((λ*x*. *case x of* (*x*, *xa*) ⇒ *dist x xa*) ‘ (*T* × *T*))
    using ‹*bounded T*› *diameter_bounded_bound* **by** (*force simp*: *bdd_above_def*)
  with *False* ‹*S* ⊆ *T*› **show** *?thesis*
    apply (*simp add*: *diameter_def*)
    apply (*rule cSUP_subset_mono*, *auto*)
    done
qed

lemma *diameter_closure*:
  assumes *bounded S*
  shows *diameter*(*closure S*) = *diameter S*
proof (*rule order_antisym*)

```
  have False if diameter S < diameter (closure S)
  proof -
    define d where d = diameter(closure S) − diameter(S)
    have d > 0
      using that by (simp add: d_def)
    then have diameter(closure(S)) − d / 2 < diameter(closure(S))
      by simp
   have dd: diameter (closure S) − d / 2 = (diameter(closure(S)) + diameter(S))
/ 2
      by (simp add: d_def field_split_simps)
     have bocl: bounded (closure S)
      using assms by blast
    moreover have 0 ≤ diameter S
      using assms diameter_ge_0 by blast
     ultimately obtain x y where x ∈ closure S y ∈ closure S and xy: diame-
ter(closure(S)) − d / 2 < dist x y
     using diameter_bounded(2) [OF bocl, rule_format, of diameter(closure(S)) −
d / 2] ‹d > 0› d_def by auto
    then obtain x' y' where x'y': x' ∈ S dist x' x < d/4 y' ∈ S dist y' y < d/4
      using closure_approachable
      by (metis ‹0 < d› zero_less_divide_iff zero_less_numeral)
    then have dist x' y' ≤ diameter S
      using assms diameter_bounded_bound by blast
    with x'y' have dist x y ≤ d / 4 + diameter S + d / 4
      by (meson add_mono_thms_linordered_semiring(1) dist_triangle dist_triangle3
less_eq_real_def order_trans)
    then show ?thesis
      using xy d_def by linarith
  qed
  then show diameter (closure S) ≤ diameter S
    by fastforce
  next
    show diameter S ≤ diameter (closure S)
      by (simp add: assms bounded_closure closure_subset diameter_subset)
qed

proposition Lebesgue_number_lemma:
  assumes compact S 𝒞 ≠ {} S ⊆ ⋃𝒞 and ope: ⋀B. B ∈ 𝒞 ⟹ open B
  obtains δ where 0 < δ ⋀T. ⟦T ⊆ S; diameter T < δ⟧ ⟹ ∃B ∈ 𝒞. T ⊆ B
proof (cases S = {})
  case True
  then show ?thesis
    by (metis ‹𝒞 ≠ {}› zero_less_one empty_subsetI equals0I subset_trans that)
next
  case False
  { fix x assume x ∈ S
    then obtain C where C: x ∈ C C ∈ 𝒞
      using ‹S ⊆ ⋃𝒞› by blast
    then obtain r where r: r>0 ball x (2∗r) ⊆ C
```

by (*metis mult.commute mult_2_right not_le ope openE field_sum_of_halves zero_le_numeral zero_less_mult_iff*)
then have $\exists\, r\ C.\ r > 0 \wedge ball\ x\ (2{*}r) \subseteq C \wedge C \in \mathcal{C}$
using *C* **by** *blast*
}
then obtain *r* **where** *r*: $\bigwedge x.\ x \in S \Longrightarrow r\ x > 0 \wedge (\exists\, C \in \mathcal{C}.\ ball\ x\ (2{*}r\ x) \subseteq C)$
by *metis*
then have $S \subseteq (\bigcup x \in S.\ ball\ x\ (r\ x))$
by *auto*
then obtain $\mathcal{T}$ **where** *finite* $\mathcal{T}$ $S \subseteq \bigcup\mathcal{T}$ **and** $\mathcal{T}$: $\mathcal{T} \subseteq (\lambda x.\ ball\ x\ (r\ x))$ ‘ S
by (*rule compactE* [*OF* ‹*compact S*›]) *auto*
then obtain *S0* **where** $S0 \subseteq S$ *finite S0* **and** *S0*: $\mathcal{T} = (\lambda x.\ ball\ x\ (r\ x))$ ‘ *S0*
by (*meson finite_subset_image*)
then have $S0 \neq \{\}$
using *False* ‹$S \subseteq \bigcup\mathcal{T}$› **by** *auto*
define δ **where** $\delta = Inf\ (r$ ‘ $S0)$
have $\delta > 0$
using ‹*finite S0*› ‹$S0 \subseteq S$› ‹$S0 \neq \{\}$› *r* **by** (*auto simp*: *δ_def finite_less_Inf_iff*)
show *?thesis*
proof
show $0 < \delta$
by (*simp add*: ‹$0 < \delta$›)
show $\exists\, B \in \mathcal{C}.\ T \subseteq B$ **if** $T \subseteq S$ **and** *dia*: *diameter* $T < \delta$ **for** *T*
proof (*cases* $T = \{\}$)
case *True*
then show *?thesis*
using ‹$\mathcal{C} \neq \{\}$› **by** *blast*
next
case *False*
then obtain *y* **where** $y \in T$ **by** *blast*
then have $y \in S$
using ‹$T \subseteq S$› **by** *auto*
then obtain *x* **where** $x \in S0$ **and** *x*: $y \in ball\ x\ (r\ x)$
using ‹$S \subseteq \bigcup\mathcal{T}$› *S0 that* **by** *blast*
have $ball\ y\ \delta \subseteq ball\ y\ (r\ x)$
by (*metis δ_def* ‹$S0 \neq \{\}$› ‹*finite S0*› ‹$x \in S0$› *empty_is_image finite_imageI finite_less_Inf_iff imageI less_irrefl not_le subset_ball*)
also have ... $\subseteq ball\ x\ (2{*}r\ x)$
using *x* **by** *metric*
finally obtain *C* **where** $C \in \mathcal{C}$ $ball\ y\ \delta \subseteq C$
by (*meson r* ‹$S0 \subseteq S$› ‹$x \in S0$› *dual_order.trans subsetCE*)
have *bounded T*
using ‹*compact S*› *bounded_subset compact_imp_bounded* ‹$T \subseteq S$› **by** *blast*
then have $T \subseteq ball\ y\ \delta$
using ‹$y \in T$› *dia diameter_bounded_bound* **by** *fastforce*
then show *?thesis*
apply (*rule_tac x=C* **in** *bexI*)
using ‹$ball\ y\ \delta \subseteq C$› ‹$C \in \mathcal{C}$› **by** *auto*

```
    qed
  qed
qed
```

3.2.15 Metric spaces with the Heine-Borel property

A metric space (or topological vector space) is said to have the Heine-Borel property if every closed and bounded subset is compact.

```
class heine_borel = metric_space +
  assumes bounded_imp_convergent_subsequence:
    bounded (range f) ⟹ ∃ l r. strict_mono (r::nat⇒nat) ∧ ((f ∘ r) ⟶ l) sequentially

proposition bounded_closed_imp_seq_compact:
  fixes S::'a::heine_borel set
  assumes bounded S
    and closed S
  shows seq_compact S
proof (unfold seq_compact_def, clarify)
  fix f :: nat ⇒ 'a
  assume f: ∀ n. f n ∈ S
  with ‹bounded S› have bounded (range f)
    by (auto intro: bounded_subset)
  obtain l r where r: strict_mono (r :: nat ⇒ nat) and l: ((f ∘ r) ⟶ l) sequentially
    using bounded_imp_convergent_subsequence [OF ‹bounded (range f)›] by auto
  from f have fr: ∀ n. (f ∘ r) n ∈ S
    by simp
  have l ∈ S using ‹closed S› fr l
    by (rule closed_sequentially)
  show ∃ l∈S. ∃ r. strict_mono r ∧ ((f ∘ r) ⟶ l) sequentially
    using ‹l ∈ S› r l by blast
qed

lemma compact_eq_bounded_closed:
  fixes S :: 'a::heine_borel set
  shows compact S ⟷ bounded S ∧ closed S
  using bounded_closed_imp_seq_compact compact_eq_seq_compact_metric compact_imp_bounded compact_imp_closed
  by auto

lemma compact_Inter:
  fixes ℱ :: 'a :: heine_borel set set
  assumes com: ⋀S. S ∈ ℱ ⟹ compact S and ℱ ≠ {}
  shows compact(⋂ ℱ)
  using assms
  by (meson Inf_lower all_not_in_conv bounded_subset closed_Inter compact_eq_bounded_closed)

lemma compact_closure [simp]:
```

fixes *S* :: *'a::heine_borel set*
shows *compact(closure S)* ⟷ *bounded S*
by (*meson bounded_closure bounded_subset closed_closure closure_subset compact_eq_bounded_closed*)

instance *real* :: *heine_borel*
proof
fix *f* :: *nat* ⇒ *real*
assume *f*: *bounded* (*range f*)
obtain *r* :: *nat* ⇒ *nat* **where** *r*: *strict_mono r monoseq* (*f* ∘ *r*)
unfolding *comp_def* **by** (*metis seq_monosub*)
then have *Bseq* (*f* ∘ *r*)
unfolding *Bseq_eq_bounded* **using** *f*
by (*metis BseqI' bounded_iff comp_apply rangeI*)
with *r* **show** ∃ *l r*. *strict_mono r* ∧ (*f* ∘ *r*) ⟶ *l*
using *Bseq_monoseq_convergent*[*of f* ∘ *r*] **by** (*auto simp*: *convergent_def*)
qed

lemma *compact_lemma_general*:
fixes *f* :: *nat* ⇒ *'a*
fixes *proj*::*'a* ⇒ *'b* ⇒ *'c*::*heine_borel* (**infixl** *proj 60*)
fixes *unproj*:: (*'b* ⇒ *'c*) ⇒ *'a*
assumes *finite_basis*: *finite basis*
assumes *bounded_proj*: ⋀*k*. *k* ∈ *basis* ⟹ *bounded* ((λ*x*. *x proj k*) ' *range f*)
assumes *proj_unproj*: ⋀*e k*. *k* ∈ *basis* ⟹ (*unproj e*) *proj k* = *e k*
assumes *unproj_proj*: ⋀*x*. *unproj* (λ*k*. *x proj k*) = *x*
shows ∀ *d*⊆*basis*. ∃ *l*::*'a*. ∃ *r*::*nat*⇒*nat*.
strict_mono r ∧ (∀ *e*>*0*. *eventually* (λ*n*. ∀ *i*∈*d*. *dist* (*f* (*r n*) *proj i*) (*l proj i*) < *e*) *sequentially*)
proof *safe*
fix *d* :: *'b set*
assume *d*: *d* ⊆ *basis*
with *finite_basis* **have** *finite d*
by (*blast intro*: *finite_subset*)
from *this d* **show** ∃ *l*::*'a*. ∃ *r*::*nat*⇒*nat*. *strict_mono r* ∧
(∀ *e*>*0*. *eventually* (λ*n*. ∀ *i*∈*d*. *dist* (*f* (*r n*) *proj i*) (*l proj i*) < *e*) *sequentially*)
proof (*induct d*)
case *empty*
then show *?case*
unfolding *strict_mono_def* **by** *auto*
next
case (*insert k d*)
have *k*[*intro*]: *k* ∈ *basis*
using *insert* **by** *auto*
have *s'*: *bounded* ((λ*x*. *x proj k*) ' *range f*)
using *k*
by (*rule bounded_proj*)
obtain *l1*::*'a* **and** *r1* **where** *r1*: *strict_mono r1*
and *lr1*: ∀ *e* > *0*. *eventually* (λ*n*. ∀ *i*∈*d*. *dist* (*f* (*r1 n*) *proj i*) (*l1 proj i*) < *e*) *sequentially*

```
      using insert(3) using insert(4) by auto
    have f': ∀ n. f (r1 n) proj k ∈ (λx. x proj k) ` range f
      by simp
    have bounded (range (λi. f (r1 i) proj k))
      by (metis (lifting) bounded_subset f' image_subsetI s')
    then obtain l2 r2 where r2:strict_mono r2 and lr2:((λi. f (r1 (r2 i)) proj
k) ⟶ l2) sequentially
      using bounded_imp_convergent_subsequence[of λi. f (r1 i) proj k]
      by (auto simp: o_def)
    define r where r = r1 ∘ r2
    have r:strict_mono r
      using r1 and r2 unfolding r_def o_def strict_mono_def by auto
    moreover
    define l where l = unproj (λi. if i = k then l2 else l1 proj i)
    {
      fix e::real
      assume e > 0
      from lr1 ‹e > 0› have N1: eventually (λn. ∀ i∈d. dist (f (r1 n) proj i) (l1
proj i) < e) sequentially
        by blast
      from lr2 ‹e > 0› have N2:eventually (λn. dist (f (r1 (r2 n)) proj k) l2 <
e) sequentially
        by (rule tendstoD)
      from r2 N1 have N1': eventually (λn. ∀ i∈d. dist (f (r1 (r2 n)) proj i) (l1
proj i) < e) sequentially
        by (rule eventually_subseq)
      have eventually (λn. ∀ i∈(insert k d). dist (f (r n) proj i) (l proj i) < e)
sequentially
        using N1' N2
      by eventually_elim (insert insert.prems, auto simp: l_def r_def o_def proj_unproj)
    }
    ultimately show ?case by auto
  qed
qed

lemma bounded_fst: bounded s ⟹ bounded (fst ` s)
  unfolding bounded_def
  by (metis (erased, hide_lams) dist_fst_le image_iff order_trans)

lemma bounded_snd: bounded s ⟹ bounded (snd ` s)
  unfolding bounded_def
  by (metis (no_types, hide_lams) dist_snd_le image_iff order.trans)

instance prod :: (heine_borel, heine_borel) heine_borel
proof
  fix f :: nat ⇒ 'a × 'b
  assume f: bounded (range f)
  then have bounded (fst ` range f)
    by (rule bounded_fst)
```

```
  then have s1: bounded (range (fst ∘ f))
    by (simp add: image_comp)
  obtain l1 r1 where r1: strict_mono r1 and l1: (λn. fst (f (r1 n))) ⟶ l1
    using bounded_imp_convergent_subsequence [OF s1] unfolding o_def by fast
  from f have s2: bounded (range (snd ∘ f ∘ r1))
    by (auto simp: image_comp intro: bounded_snd bounded_subset)
  obtain l2 r2 where r2: strict_mono r2 and l2: ((λn. snd (f (r1 (r2 n)))) ⟶
l2) sequentially
    using bounded_imp_convergent_subsequence [OF s2]
    unfolding o_def by fast
  have l1': ((λn. fst (f (r1 (r2 n)))) ⟶ l1) sequentially
    using LIMSEQ_subseq_LIMSEQ [OF l1 r2] unfolding o_def .
  have l: ((f ∘ (r1 ∘ r2)) ⟶ (l1, l2)) sequentially
    using tendsto_Pair [OF l1' l2] unfolding o_def by simp
  have r: strict_mono (r1 ∘ r2)
    using r1 r2 unfolding strict_mono_def by simp
  show ∃ l r. strict_mono r ∧ ((f ∘ r) ⟶ l) sequentially
    using l r by fast
qed
```

3.2.16 Completeness

```
proposition (in metric_space) completeI:
  assumes ⋀f. ∀ n. f n ∈ s ⟹ Cauchy f ⟹ ∃ l∈s. f ⟶ l
  shows complete s
  using assms unfolding complete_def by fast

proposition (in metric_space) completeE:
  assumes complete s and ∀ n. f n ∈ s and Cauchy f
  obtains l where l ∈ s and f ⟶ l
  using assms unfolding complete_def by fast


lemma compact_imp_complete:
  fixes s :: 'a::metric_space set
  assumes compact s
  shows complete s
proof −
  {
    fix f
    assume as: (∀ n::nat. f n ∈ s) Cauchy f
    from as(1) obtain l r where lr: l∈s strict_mono r (f ∘ r) ⟶ l
      using assms unfolding compact_def by blast

    note lr' = seq_suble [OF lr(2)]
    {
      fix e :: real
      assume e > 0
      from as(2) obtain N where N:∀ m n. N ≤ m ∧ N ≤ n ⟶ dist (f m) (f
```

```
n) < e/2
        unfolding cauchy_def
        using ‹e > 0›
        apply (erule_tac x=e/2 in allE, auto)
        done
      from lr(3)[unfolded lim_sequentially, THEN spec[where x=e/2]]
      obtain M where M:∀n≥M. dist ((f ∘ r) n) l < e/2
        using ‹e > 0› by auto
      {
        fix n :: nat
        assume n: n ≥ max N M
        have dist ((f ∘ r) n) l < e/2
          using n M by auto
        moreover have r n ≥ N
          using lr'[of n] n by auto
        then have dist (f n) ((f ∘ r) n) < e/2
          using N and n by auto
        ultimately have dist (f n) l < e using n M
          by metric
      }
      then have ∃N. ∀n≥N. dist (f n) l < e by blast
    }
    then have ∃l∈s. (f ⟶ l) sequentially using ‹l∈s›
      unfolding lim_sequentially by auto
  }
  then show ?thesis unfolding complete_def by auto
qed

proposition compact_eq_totally_bounded:
  compact s ⟷ complete s ∧ (∀e>0. ∃k. finite k ∧ s ⊆ (⋃x∈k. ball x e))
    (is _ ⟷ ?rhs)
proof
  assume assms: ?rhs
  then obtain k where k: ⋀e. 0 < e ⟹ finite (k e) ⋀e. 0 < e ⟹ s ⊆ (⋃x∈k
e. ball x e)
    by (auto simp: choice_iff')

  show compact s
  proof cases
    assume s = {}
    then show compact s by (simp add: compact_def)
  next
    assume s ≠ {}
    show ?thesis
      unfolding compact_def
    proof safe
      fix f :: nat ⇒ 'a
      assume f: ∀n. f n ∈ s
```

define *e* **where** *e n* = *1* / (*2* ∗ *Suc n*) **for** *n*
then have [*simp*]: ⋀*n*. *0* < *e n* **by** *auto*
define *B* **where** *B n U* = (*SOME b*. *infinite* {*n*. *f n* ∈ *b*} ∧ (∃ *x*. *b* ⊆ *ball x* (*e n*) ∩ *U*)) **for** *n U*
{
  fix *n U*
  assume *infinite* {*n*. *f n* ∈ *U*}
  then have ∃ *b*∈*k* (*e n*). *infinite* {*i*∈{*n*. *f n* ∈ *U*}. *f i* ∈ *ball b* (*e n*)}
    using *k f* **by** (*intro pigeonhole_infinite_rel*) (*auto simp*: *subset_eq*)
  then obtain *a* **where**
    a ∈ *k* (*e n*)
    infinite {*i* ∈ {*n*. *f n* ∈ *U*}. *f i* ∈ *ball a* (*e n*)} **..**
  then have ∃ *b*. *infinite* {*i*. *f i* ∈ *b*} ∧ (∃ *x*. *b* ⊆ *ball x* (*e n*) ∩ *U*)
    by (*intro exI*[*of* _ *ball a* (*e n*) ∩ *U*] *exI*[*of* _ *a*]) (*auto simp*: *ac_simps*)
  from *someI_ex*[*OF this*]
  have *infinite* {*i*. *f i* ∈ *B n U*} ∃ *x*. *B n U* ⊆ *ball x* (*e n*) ∩ *U*
    unfolding *B_def* **by** *auto*
}
note *B* = *this*

define *F* **where** *F* = *rec_nat* (*B 0 UNIV*) *B*
{
  fix *n*
  have *infinite* {*i*. *f i* ∈ *F n*}
    by (*induct n*) (*auto simp*: *F_def B*)
}
then have *F*: ⋀*n*. ∃ *x*. *F* (*Suc n*) ⊆ *ball x* (*e n*) ∩ *F n*
  using *B* **by** (*simp add*: *F_def*)
then have *F_dec*: ⋀*m n*. *m* ≤ *n* ⟹ *F n* ⊆ *F m*
  using *decseq_SucI*[*of F*] **by** (*auto simp*: *decseq_def*)

obtain *sel* **where** *sel*: ⋀*k i*. *i* < *sel k i* ⋀*k i*. *f* (*sel k i*) ∈ *F k*
proof (*atomize_elim*, *unfold all_conj_distrib*[*symmetric*], *intro choice allI*)
  fix *k i*
  have *infinite* ({*n*. *f n* ∈ *F k*} − {.. *i*})
    using ‹*infinite* {*n*. *f n* ∈ *F k*}› **by** *auto*
  from *infinite_imp_nonempty*[*OF this*]
  show ∃ *x*>*i*. *f x* ∈ *F k*
    by (*simp add*: *set_eq_iff not_le conj_commute*)
qed

define *t* **where** *t* = *rec_nat* (*sel 0 0*) (λ*n i*. *sel* (*Suc n*) *i*)
have *strict_mono t*
  unfolding *strict_mono_Suc_iff* **by** (*simp add*: *t_def sel*)
moreover have ∀ *i*. (*f* ∘ *t*) *i* ∈ *s*
  using *f* **by** *auto*
moreover
have *t*: (*f* ∘ *t*) *n* ∈ *F n* **for** *n*
  by (*cases n*) (*simp_all add*: *t_def sel*)

```
    have Cauchy (f ∘ t)
    proof (safe intro!: metric_CauchyI exI elim!: nat_approx_posE)
      fix r :: real and N n m
      assume 1 / Suc N < r Suc N ≤ n Suc N ≤ m
      then have (f ∘ t) n ∈ F (Suc N) (f ∘ t) m ∈ F (Suc N) 2 * e N < r
        using F_dec t by (auto simp: e_def field_simps)
      with F[of N] obtain x where dist x ((f ∘ t) n) < e N dist x ((f ∘ t) m)
< e N
        by (auto simp: subset_eq)
      with ‹2 * e N < r› show dist ((f ∘ t) m) ((f ∘ t) n) < r
        by metric
    qed

    ultimately show ∃ l∈s. ∃ r. strict_mono r ∧ (f ∘ r) ⟶ l
      using assms unfolding complete_def by blast
  qed
 qed
qed (metis compact_imp_complete compact_imp_seq_compact seq_compact_imp_totally_bounded)

lemma cauchy_imp_bounded:
  assumes Cauchy s
  shows bounded (range s)
proof –
  from assms obtain N :: nat where ∀ m n. N ≤ m ∧ N ≤ n ⟶ dist (s m) (s
n) < 1
    unfolding cauchy_def by force
  then have N:∀ n. N ≤ n ⟶ dist (s N) (s n) < 1 by auto
  moreover
  have bounded (s ‘ {0..N})
    using finite_imp_bounded[of s ‘ {1..N}] by auto
  then obtain a where a:∀ x∈s ‘ {0..N}. dist (s N) x ≤ a
    unfolding bounded_any_center [where a=s N] by auto
  ultimately show ?thesis
    unfolding bounded_any_center [where a=s N]
    apply (rule_tac x=max a 1 in exI, auto)
    apply (erule_tac x=y in allE)
    apply (erule_tac x=y in ballE, auto)
    done
qed

instance heine_borel < complete_space
proof
  fix f :: nat ⇒ 'a assume Cauchy f
  then have bounded (range f)
    by (rule cauchy_imp_bounded)
  then have compact (closure (range f))
    unfolding compact_eq_bounded_closed by auto
  then have complete (closure (range f))
```

```
    by (rule compact_imp_complete)
  moreover have ∀ n. f n ∈ closure (range f)
    using closure_subset [of range f] by auto
  ultimately have ∃ l∈closure (range f). (f ⟶ l) sequentially
    using ‹Cauchy f› unfolding complete_def by auto
  then show convergent f
    unfolding convergent_def by auto
qed

lemma complete_UNIV: complete (UNIV :: ('a::complete_space) set)
proof (rule completeI)
  fix f :: nat ⇒ 'a assume Cauchy f
  then have convergent f by (rule Cauchy_convergent)
  then show ∃ l∈UNIV. f ⟶ l unfolding convergent_def by simp
qed

lemma complete_imp_closed:
  fixes S :: 'a::metric_space set
  assumes complete S
  shows closed S
proof (unfold closed_sequential_limits, clarify)
  fix f x assume ∀ n. f n ∈ S and f ⟶ x
  from ‹f ⟶ x› have Cauchy f
    by (rule LIMSEQ_imp_Cauchy)
  with ‹complete S› and ‹∀ n. f n ∈ S› obtain l where l ∈ S and f ⟶ l
    by (rule completeE)
  from ‹f ⟶ x› and ‹f ⟶ l› have x = l
    by (rule LIMSEQ_unique)
  with ‹l ∈ S› show x ∈ S
    by simp
qed

lemma complete_Int_closed:
  fixes S :: 'a::metric_space set
  assumes complete S and closed t
  shows complete (S ∩ t)
proof (rule completeI)
  fix f assume ∀ n. f n ∈ S ∩ t and Cauchy f
  then have ∀ n. f n ∈ S and ∀ n. f n ∈ t
    by simp_all
  from ‹complete S› obtain l where l ∈ S and f ⟶ l
    using ‹∀ n. f n ∈ S› and ‹Cauchy f› by (rule completeE)
  from ‹closed t› and ‹∀ n. f n ∈ t› and ‹f ⟶ l› have l ∈ t
    by (rule closed_sequentially)
  with ‹l ∈ S› and ‹f ⟶ l› show ∃ l∈S ∩ t. f ⟶ l
    by fast
qed

lemma complete_closed_subset:
```

fixes *S* :: *'a::metric_space set*
assumes *closed S* **and** *S* ⊆ *t* **and** *complete t*
shows *complete S*
using *assms complete_Int_closed* [*of t S*] **by** (*simp add*: *Int_absorb1*)

lemma *complete_eq_closed*:
fixes *S* :: (*'a::complete_space*) *set*
shows *complete S* ⟷ *closed S*
proof
assume *closed S* **then show** *complete S*
using *subset_UNIV complete_UNIV* **by** (*rule complete_closed_subset*)
next
assume *complete S* **then show** *closed S*
by (*rule complete_imp_closed*)
qed

lemma *convergent_eq_Cauchy*:
fixes *S* :: *nat* ⇒ *'a::complete_space*
shows (∃ *l*. (*S* ⟶ *l*) *sequentially*) ⟷ *Cauchy S*
unfolding *Cauchy_convergent_iff convergent_def* **..**

lemma *convergent_imp_bounded*:
fixes *S* :: *nat* ⇒ *'a::metric_space*
shows (*S* ⟶ *l*) *sequentially* ⟹ *bounded* (*range S*)
by (*intro cauchy_imp_bounded LIMSEQ_imp_Cauchy*)

lemma *frontier_subset_compact*:
fixes *S* :: *'a::heine_borel set*
shows *compact S* ⟹ *frontier S* ⊆ *S*
using *frontier_subset_closed compact_eq_bounded_closed*
by *blast*

lemma *continuous_closed_imp_Cauchy_continuous*:
fixes *S* :: (*'a::complete_space*) *set*
shows ⟦*continuous_on S f*; *closed S*; *Cauchy σ*; ⋀*n*. (*σ n*) ∈ *S*⟧ ⟹ *Cauchy*(*f* ∘ *σ*)
apply (*simp add*: *complete_eq_closed* [*symmetric*] *continuous_on_sequentially*)
by (*meson LIMSEQ_imp_Cauchy complete_def*)

lemma *banach_fix_type*:
fixes *f*::*'a::complete_space*⇒*'a*
assumes *c*:*0* ≤ *c c* < *1*
and *lipschitz*:∀ *x*. ∀ *y*. *dist* (*f x*) (*f y*) ≤ *c* ∗ *dist x y*
shows ∃!*x*. (*f x* = *x*)
using *assms banach_fix*[*OF complete_UNIV UNIV_not_empty assms*(*1*,*2*) *subset_UNIV*, *of f*]
by *auto*

3.2.17 Finite intersection property

Also developed in HOL's toplogical spaces theory, but the Heine-Borel type class isn't available there.

lemma *closed_imp_fip*:
  fixes S :: *'a::heine_borel set*
  assumes *closed* S
    and T: $T \in \mathcal{F}$ *bounded* T
    and *clof*: $\bigwedge T.\ T \in \mathcal{F} \Longrightarrow$ *closed* T
    and *none*: $\bigwedge \mathcal{F}'.\ [\![$*finite* $\mathcal{F}'$; $\mathcal{F}' \subseteq \mathcal{F}]\!] \Longrightarrow S \cap \bigcap \mathcal{F}' \neq \{\}$
   shows $S \cap \bigcap \mathcal{F} \neq \{\}$
proof –
  have *compact* $(S \cap T)$
   using ‹*closed* S› *clof compact_eq_bounded_closed* T **by** *blast*
  then have $(S \cap T) \cap \bigcap \mathcal{F} \neq \{\}$
   apply (*rule compact_imp_fip*)
    apply (*simp add*: *clof*)
    by (*metis Int_assoc complete_lattice_class.Inf_insert finite_insert insert_subset none* ‹$T \in \mathcal{F}$›)
  then show *?thesis* **by** *blast*
qed

lemma *closed_imp_fip_compact*:
  fixes S :: *'a::heine_borel set*
  shows
   $[\![$*closed* S; $\bigwedge T.\ T \in \mathcal{F} \Longrightarrow$ *compact* T;
    $\bigwedge \mathcal{F}'.\ [\![$*finite* $\mathcal{F}'$; $\mathcal{F}' \subseteq \mathcal{F}]\!] \Longrightarrow S \cap \bigcap \mathcal{F}' \neq \{\}]\!]$
     $\Longrightarrow S \cap \bigcap \mathcal{F} \neq \{\}$
by (*metis Inf_greatest closed_imp_fip compact_eq_bounded_closed empty_subsetI finite.emptyI inf.orderE*)

lemma *closed_fip_Heine_Borel*:
  fixes $\mathcal{F}$ :: *'a::heine_borel set set*
  assumes *closed* S $T \in \mathcal{F}$ *bounded* T
    and $\bigwedge T.\ T \in \mathcal{F} \Longrightarrow$ *closed* T
    and $\bigwedge \mathcal{F}'.\ [\![$*finite* $\mathcal{F}'$; $\mathcal{F}' \subseteq \mathcal{F}]\!] \Longrightarrow \bigcap \mathcal{F}' \neq \{\}$
   shows $\bigcap \mathcal{F} \neq \{\}$
proof –
  have *UNIV* $\cap \bigcap \mathcal{F} \neq \{\}$
   using *assms closed_imp_fip* [*OF closed_UNIV*] **by** *auto*
  then show *?thesis* **by** *simp*
qed

lemma *compact_fip_Heine_Borel*:
  fixes $\mathcal{F}$ :: *'a::heine_borel set set*
  assumes *clof*: $\bigwedge T.\ T \in \mathcal{F} \Longrightarrow$ *compact* T
    and *none*: $\bigwedge \mathcal{F}'.\ [\![$*finite* $\mathcal{F}'$; $\mathcal{F}' \subseteq \mathcal{F}]\!] \Longrightarrow \bigcap \mathcal{F}' \neq \{\}$
   shows $\bigcap \mathcal{F} \neq \{\}$
by (*metis InterI all_not_in_conv clof closed_fip_Heine_Borel compact_eq_bounded_closed*

```
none)

lemma compact_sequence_with_limit:
  fixes f :: nat ⇒ 'a::heine_borel
  shows (f ⟶ l) sequentially ⟹ compact (insert l (range f))
apply (simp add: compact_eq_bounded_closed, auto)
apply (simp add: convergent_imp_bounded)
by (simp add: closed_limpt islimpt_insert sequence_unique_limpt)
```

3.2.18 Properties of Balls and Spheres

```
lemma compact_cball[simp]:
  fixes x :: 'a::heine_borel
  shows compact (cball x e)
  using compact_eq_bounded_closed bounded_cball closed_cball
  by blast

lemma compact_frontier_bounded[intro]:
  fixes S :: 'a::heine_borel set
  shows bounded S ⟹ compact (frontier S)
  unfolding frontier_def
  using compact_eq_bounded_closed
  by blast

lemma compact_frontier[intro]:
  fixes S :: 'a::heine_borel set
  shows compact S ⟹ compact (frontier S)
  using compact_eq_bounded_closed compact_frontier_bounded
  by blast
```

3.2.19 Distance from a Set

```
lemma distance_attains_sup:
  assumes compact s s ≠ {}
  shows ∃x∈s. ∀y∈s. dist a y ≤ dist a x
proof (rule continuous_attains_sup [OF assms])
  {
    fix x
    assume x∈s
    have (dist a ⟶ dist a x) (at x within s)
      by (intro tendsto_dist tendsto_const tendsto_ident_at)
  }
  then show continuous_on s (dist a)
    unfolding continuous_on ..
qed
```

For *minimal* distance, we only need closure, not compactness.

```
lemma distance_attains_inf:
  fixes a :: 'a::heine_borel
```

assumes *closed s* **and** $s \neq \{\}$
obtains x **where** $x \in s$ $\bigwedge y.\ y \in s \Longrightarrow dist\ a\ x \leq dist\ a\ y$
proof −
from *assms* **obtain** b **where** $b \in s$ **by** *auto*
let $?B = s \cap cball\ a\ (dist\ b\ a)$
have $?B \neq \{\}$ **using** ‹$b \in s$›
by (*auto simp*: *dist_commute*)
moreover have *continuous_on ?B* (*dist a*)
by (*auto intro*!: *continuous_at_imp_continuous_on continuous_dist continuous_ident continuous_const*)
moreover have *compact ?B*
by (*intro closed_Int_compact* ‹*closed s*› *compact_cball*)
ultimately obtain x **where** $x \in ?B\ \forall y \in ?B.\ dist\ a\ x \leq dist\ a\ y$
by (*metis continuous_attains_inf*)
with *that* **show** *?thesis* **by** *fastforce*
qed

3.2.20 Infimum Distance

definition *infdist x A* = (*if* $A = \{\}$ *then 0 else INF* $a \in A.$ *dist x a*)

lemma *bdd_below_image_dist*[*intro*, *simp*]: *bdd_below* (*dist x* ‘ *A*)
by (*auto intro*!: *zero_le_dist*)

lemma *infdist_notempty*: $A \neq \{\} \Longrightarrow infdist\ x\ A = (INF\ a \in A.\ dist\ x\ a)$
by (*simp add*: *infdist_def*)

lemma *infdist_nonneg*: $0 \leq infdist\ x\ A$
by (*auto simp*: *infdist_def intro*: *cINF_greatest*)

lemma *infdist_le*: $a \in A \Longrightarrow infdist\ x\ A \leq dist\ x\ a$
by (*auto intro*: *cINF_lower simp add*: *infdist_def*)

lemma *infdist_le2*: $a \in A \Longrightarrow dist\ x\ a \leq d \Longrightarrow infdist\ x\ A \leq d$
by (*auto intro*!: *cINF_lower2 simp add*: *infdist_def*)

lemma *infdist_zero*[*simp*]: $a \in A \Longrightarrow infdist\ a\ A = 0$
by (*auto intro*!: *antisym infdist_nonneg infdist_le2*)

lemma *infdist_Un_min*:
assumes $A \neq \{\}$ $B \neq \{\}$
shows $infdist\ x\ (A \cup B) = min\ (infdist\ x\ A)\ (infdist\ x\ B)$
using *assms* **by** (*simp add*: *infdist_def cINF_union inf_real_def*)

lemma *infdist_triangle*: $infdist\ x\ A \leq infdist\ y\ A + dist\ x\ y$
proof (*cases* $A = \{\}$)
case *True*
then show *?thesis* **by** (*simp add*: *infdist_def*)
next

```
  case False
  then obtain a where a ∈ A by auto
  have infdist x A ≤ Inf {dist x y + dist y a |a. a ∈ A}
  proof (rule cInf_greatest)
    from ‹A ≠ {}› show {dist x y + dist y a |a. a ∈ A} ≠ {}
      by simp
    fix d
    assume d ∈ {dist x y + dist y a |a. a ∈ A}
    then obtain a where d: d = dist x y + dist y a a ∈ A
      by auto
    show infdist x A ≤ d
      unfolding infdist_notempty[OF ‹A ≠ {}›]
    proof (rule cINF_lower2)
      show a ∈ A by fact
      show dist x a ≤ d
        unfolding d by (rule dist_triangle)
    qed simp
  qed
  also have ... = dist x y + infdist y A
  proof (rule cInf_eq, safe)
    fix a
    assume a ∈ A
    then show dist x y + infdist y A ≤ dist x y + dist y a
      by (auto intro: infdist_le)
  next
    fix i
    assume inf: ⋀d. d ∈ {dist x y + dist y a |a. a ∈ A} ⟹ i ≤ d
    then have i − dist x y ≤ infdist y A
      unfolding infdist_notempty[OF ‹A ≠ {}›] using ‹a ∈ A›
      by (intro cINF_greatest) (auto simp: field_simps)
    then show i ≤ dist x y + infdist y A
      by simp
  qed
  finally show ?thesis by simp
qed

lemma infdist_triangle_abs: |infdist x A − infdist y A| ≤ dist x y
  by (metis (full_types) abs_diff_le_iff diff_le_eq dist_commute infdist_triangle)

lemma in_closure_iff_infdist_zero:
  assumes A ≠ {}
  shows x ∈ closure A ⟷ infdist x A = 0
proof
  assume x ∈ closure A
  show infdist x A = 0
  proof (rule ccontr)
    assume infdist x A ≠ 0
    with infdist_nonneg[of x A] have infdist x A > 0
      by auto
```

then have *ball x* (*infdist x A*) ∩ *closure A* = {}
apply *auto*
apply (*metis* ‹*x* ∈ *closure A*› *closure_approachable dist_commute infdist_le not_less*)
done
then have *x* ∉ *closure A*
by (*metis* ‹*0* < *infdist x A*› *centre_in_ball disjoint_iff_not_equal*)
then show *False* **using** ‹*x* ∈ *closure A*› **by** *simp*
qed
next
assume *x*: *infdist x A* = *0*
then obtain *a* **where** *a* ∈ *A*
by *atomize_elim* (*metis all_not_in_conv assms*)
show *x* ∈ *closure A*
unfolding *closure_approachable*
apply *safe*
proof (*rule ccontr*)
fix *e* :: *real*
assume *e* > *0*
assume ¬ (∃ *y*∈*A*. *dist y x* < *e*)
then have *infdist x A* ≥ *e* **using** ‹*a* ∈ *A*›
unfolding *infdist_def*
by (*force simp*: *dist_commute intro*: *cINF_greatest*)
with *x* ‹*e* > *0*› **show** *False* **by** *auto*
qed
qed

lemma *in_closed_iff_infdist_zero*:
assumes *closed A A* ≠ {}
shows *x* ∈ *A* ⟷ *infdist x A* = *0*
proof −
have *x* ∈ *closure A* ⟷ *infdist x A* = *0*
by (*rule in_closure_iff_infdist_zero*) *fact*
with *assms* **show** *?thesis* **by** *simp*
qed

lemma *infdist_pos_not_in_closed*:
assumes *closed S S* ≠ {} *x* ∉ *S*
shows *infdist x S* > *0*
using *in_closed_iff_infdist_zero*[*OF assms*(*1*) *assms*(*2*), *of x*] *assms*(*3*) *infdist_nonneg le_less* **by** *fastforce*

lemma
infdist_attains_inf:
fixes *X*::′*a*::*heine_borel set*
assumes *closed X*
assumes *X* ≠ {}
obtains *x* **where** *x* ∈ *X infdist y X* = *dist y x*
proof −

```
  have bdd_below (dist y ‘ X)
    by auto
  from distance_attains_inf[OF assms, of y]
  obtain x where INF: x ∈ X ⋀z. z ∈ X ⟹ dist y x ≤ dist y z by auto
  have infdist y X = dist y x
    by (auto simp: infdist_def assms
     intro!: antisym cINF_lower[OF _ ‹x ∈ X›] cINF_greatest[OF assms(2) INF(2)])
  with ‹x ∈ X› show ?thesis ..
qed
```

Every metric space is a T4 space:

```
instance metric_space ⊆ t4_space
proof
  fix S T::'a set assume H: closed S closed T S ∩ T = {}
  consider S = {} | T = {} | S ≠ {} ∧ T ≠ {} by auto
  then show ∃ U V. open U ∧ open V ∧ S ⊆ U ∧ T ⊆ V ∧ U ∩ V = {}
  proof (cases)
    case 1
    show ?thesis
      apply (rule exI[of _ {}], rule exI[of _ UNIV]) using 1 by auto
  next
    case 2
    show ?thesis
      apply (rule exI[of _ UNIV], rule exI[of _ {}]) using 2 by auto
  next
    case 3
    define U where U = (⋃x∈S. ball x ((infdist x T)/2))
    have A: open U unfolding U_def by auto
    have infdist x T > 0 if x ∈ S for x
      using H that 3 by (auto intro!: infdist_pos_not_in_closed)
    then have B: S ⊆ U unfolding U_def by auto
    define V where V = (⋃x∈T. ball x ((infdist x S)/2))
    have C: open V unfolding V_def by auto
    have infdist x S > 0 if x ∈ T for x
      using H that 3 by (auto intro!: infdist_pos_not_in_closed)
    then have D: T ⊆ V unfolding V_def by auto

    have (ball x ((infdist x T)/2)) ∩ (ball y ((infdist y S)/2)) = {} if x ∈ S y ∈
T for x y
    proof auto
      fix z assume H: dist x z * 2 < infdist x T dist y z * 2 < infdist y S
      have 2 * dist x y ≤ 2 * dist x z + 2 * dist y z
        by metric
      also have ... < infdist x T + infdist y S
        using H by auto
      finally have dist x y < infdist x T ∨ dist x y < infdist y S
        by auto
      then show False
       using infdist_le[OF ‹x ∈ S›, of y] infdist_le[OF ‹y ∈ T›, of x] by (auto simp
```

```
add: dist_commute)
    qed
    then have E: U ∩ V = {}
      unfolding U_def V_def by auto
    show ?thesis
      apply (rule exI[of _ U], rule exI[of _ V]) using A B C D E by auto
  qed
qed

lemma tendsto_infdist [tendsto_intros]:
  assumes f: (f ⟶ l) F
  shows ((λx. infdist (f x) A) ⟶ infdist l A) F
proof (rule tendstoI)
  fix e ::real
  assume e > 0
  from tendstoD[OF f this]
  show eventually (λx. dist (infdist (f x) A) (infdist l A) < e) F
  proof (eventually_elim)
    fix x
    from infdist_triangle[of l A f x] infdist_triangle[of f x A l]
    have dist (infdist (f x) A) (infdist l A) ≤ dist (f x) l
      by (simp add: dist_commute dist_real_def)
    also assume dist (f x) l < e
    finally show dist (infdist (f x) A) (infdist l A) < e .
  qed
qed

lemma continuous_infdist[continuous_intros]:
  assumes continuous F f
  shows continuous F (λx. infdist (f x) A)
  using assms unfolding continuous_def by (rule tendsto_infdist)

lemma continuous_on_infdist [continuous_intros]:
  assumes continuous_on S f
  shows continuous_on S (λx. infdist (f x) A)
using assms unfolding continuous_on by (auto intro: tendsto_infdist)

lemma compact_infdist_le:
  fixes A::'a::heine_borel set
  assumes A ≠ {}
  assumes compact A
  assumes e > 0
  shows compact {x. infdist x A ≤ e}
proof -
  from continuous_closed_vimage[of {0..e} λx. infdist x A]
    continuous_infdist[OF continuous_ident, of _ UNIV A]
  have closed {x. infdist x A ≤ e} by (auto simp: vimage_def infdist_nonneg)
  moreover
  from assms obtain x0 b where b: ⋀x. x ∈ A ⟹ dist x0 x ≤ b closed A
```

```
    by (auto simp: compact_eq_bounded_closed bounded_def)
  {
    fix y
    assume infdist y A ≤ e
    moreover
    from infdist_attains_inf[OF ‹closed A› ‹A ≠ {}›, of y]
    obtain z where z ∈ A infdist y A = dist y z by blast
    ultimately
    have dist x0 y ≤ b + e using b by metric
  } then
  have bounded {x. infdist x A ≤ e}
   by (auto simp: bounded_any_center[where a=x0] intro!: exI[where x=b + e])
  ultimately show compact {x. infdist x A ≤ e}
    by (simp add: compact_eq_bounded_closed)
qed
```

3.2.21 Separation between Points and Sets

```
proposition separate_point_closed:
  fixes s :: 'a::heine_borel set
  assumes closed s and a ∉ s
  shows ∃ d>0. ∀ x∈s. d ≤ dist a x
proof (cases s = {})
  case True
  then show ?thesis by(auto intro!: exI[where x=1])
next
  case False
  from assms obtain x where x∈s ∀ y∈s. dist a x ≤ dist a y
    using ‹s ≠ {}› by (blast intro: distance_attains_inf [of s a])
  with ‹x∈s› show ?thesis using dist_pos_lt[of a x] and‹a ∉ s›
    by blast
qed

proposition separate_compact_closed:
  fixes s t :: 'a::heine_borel set
  assumes compact s
    and t: closed t s ∩ t = {}
  shows ∃ d>0. ∀ x∈s. ∀ y∈t. d ≤ dist x y
proof cases
  assume s ≠ {} ∧ t ≠ {}
  then have s ≠ {} t ≠ {} by auto
  let ?inf = λx. infdist x t
  have continuous_on s ?inf
     by (auto intro!: continuous_at_imp_continuous_on continuous_infdist continu-
ous_ident)
  then obtain x where x: x ∈ s ∀ y∈s. ?inf x ≤ ?inf y
    using continuous_attains_inf[OF ‹compact s› ‹s ≠ {}›] by auto
  then have 0 < ?inf x
   using t ‹t ≠ {}› in_closed_iff_infdist_zero by (auto simp: less_le infdist_nonneg)
```

```
  moreover have ∀x'∈s. ∀y∈t. ?inf x ≤ dist x' y
    using x by (auto intro: order_trans infdist_le)
  ultimately show ?thesis by auto
qed (auto intro!: exI[of _ 1])

proposition separate_closed_compact:
  fixes s t :: 'a::heine_borel set
  assumes closed s
    and compact t
    and s ∩ t = {}
  shows ∃d>0. ∀x∈s. ∀y∈t. d ≤ dist x y
proof −
  have *: t ∩ s = {}
    using assms(3) by auto
  show ?thesis
   using separate_compact_closed[OF assms(2,1) *] by (force simp: dist_commute)
qed

proposition compact_in_open_separated:
  fixes A::'a::heine_borel set
  assumes A ≠ {}
  assumes compact A
  assumes open B
  assumes A ⊆ B
  obtains e where e > 0 {x. infdist x A ≤ e} ⊆ B
proof atomize_elim
  have closed (− B) compact A − B ∩ A = {}
    using assms by (auto simp: open_Diff compact_eq_bounded_closed)
  from separate_closed_compact[OF this]
  obtain d'::real where d': d'>0 ⋀x y. x ∉ B ⟹ y ∈ A ⟹ d' ≤ dist x y
    by auto
  define d where d = d' / 2
  hence d>0 d < d' using d' by auto
  with d' have d: ⋀x y. x ∉ B ⟹ y ∈ A ⟹ d < dist x y
    by force
  show ∃e>0. {x. infdist x A ≤ e} ⊆ B
  proof (rule ccontr)
    assume ∄e. 0 < e ∧ {x. infdist x A ≤ e} ⊆ B
    with ‹d > 0› obtain x where x: infdist x A ≤ d x ∉ B
      by auto
    from assms have closed A A ≠ {} by (auto simp: compact_eq_bounded_closed)
    from infdist_attains_inf[OF this]
    obtain y where y: y ∈ A infdist x A = dist x y
      by auto
    have dist x y ≤ d using x y by simp
    also have ... < dist x y using y d x by auto
    finally show False by simp
  qed
qed
```

3.2.22 Uniform Continuity

lemma *uniformly_continuous_onE*:
  assumes *uniformly_continuous_on s f 0 < e*
  obtains *d* **where** *d>0* ⋀*x x'*. ⟦*x*∈*s*; *x'*∈*s*; *dist x' x < d*⟧ ⟹ *dist* (*f x'*) (*f x*) < *e*
using *assms*
by (*auto simp*: *uniformly_continuous_on_def*)

lemma *uniformly_continuous_on_sequentially*:
  uniformly_continuous_on s f ⟷ (∀ *x y*. (∀ *n*. *x n* ∈ *s*) ∧ (∀ *n*. *y n* ∈ *s*) ∧
    (λ*n*. *dist* (*x n*) (*y n*)) ⟶ *0* ⟶ (λ*n*. *dist* (*f*(*x n*)) (*f*(*y n*))) ⟶ *0*) (**is** *?lhs* = *?rhs*)
proof
  assume *?lhs*
  {
    fix *x y*
    assume *x*: ∀ *n*. *x n* ∈ *s*
      and *y*: ∀ *n*. *y n* ∈ *s*
      and *xy*: ((λ*n*. *dist* (*x n*) (*y n*)) ⟶ *0*) *sequentially*
    {
      fix *e* :: *real*
      assume *e* > *0*
      then obtain *d* **where** *d* > *0* **and** *d*: ∀ *x*∈*s*. ∀ *x'*∈*s*. *dist x' x* < *d* ⟶ *dist* (*f x'*) (*f x*) < *e*
        using ‹*?lhs*›[*unfolded uniformly_continuous_on_def*, *THEN spec*[**where** *x*=*e*]] **by** *auto*
      obtain *N* **where** *N*: ∀ *n*≥*N*. *dist* (*x n*) (*y n*) < *d*
        using *xy*[*unfolded lim_sequentially dist_norm*] **and** ‹*d*>*0*› **by** *auto*
      {
        fix *n*
        assume *n*≥*N*
        then have *dist* (*f* (*x n*)) (*f* (*y n*)) < *e*
          using *N*[*THEN spec*[**where** *x*=*n*]]
          using *d*[*THEN bspec*[**where** *x*=*x n*], *THEN bspec*[**where** *x*=*y n*]]
          using *x* **and** *y*
          by (*simp add*: *dist_commute*)
      }
      then have ∃ *N*. ∀ *n*≥*N*. *dist* (*f* (*x n*)) (*f* (*y n*)) < *e*
        by *auto*
    }
    then have ((λ*n*. *dist* (*f*(*x n*)) (*f*(*y n*))) ⟶ *0*) *sequentially*
      unfolding *lim_sequentially* **and** *dist_real_def* **by** *auto*
  }
  then show *?rhs* **by** *auto*
next
  assume *?rhs*
  {
    assume ¬ *?lhs*
    then obtain *e* **where** *e* > *0* ∀ *d*>*0*. ∃ *x*∈*s*. ∃ *x'*∈*s*. *dist x' x* < *d* ∧ ¬ *dist* (*f*

x′) (f x) < e
      unfolding *uniformly_continuous_on_def* **by** *auto*
    then obtain *fa* **where** *fa*:
      ∀ x. 0 < x ⟶ fst (fa x) ∈ s ∧ snd (fa x) ∈ s ∧ dist (fst (fa x)) (snd (fa x)) < x ∧ ¬ dist (f (fst (fa x))) (f (snd (fa x))) < e
      using *choice*[*of* λd x. d>0 ⟶ fst x ∈ s ∧ snd x ∈ s ∧ dist (snd x) (fst x) < d ∧ ¬ dist (f (snd x)) (f (fst x)) < e]
      unfolding *Bex_def*
      by (*auto simp*: *dist_commute*)
    define *x* **where** x n = fst (fa (inverse (real n + 1))) **for** *n*
    define *y* **where** y n = snd (fa (inverse (real n + 1))) **for** *n*
    have *xyn*: ∀ n. x n ∈ s ∧ y n ∈ s
      and *xy0*: ∀ n. dist (x n) (y n) < inverse (real n + 1)
      and *fxy*:∀ n. ¬ dist (f (x n)) (f (y n)) < e
      unfolding *x_def* **and** *y_def* **using** *fa*
      by *auto*
    {
      fix e :: *real*
      assume e > 0
      then obtain N :: *nat* **where** N ≠ 0 **and** N: 0 < inverse (real N) ∧ inverse (real N) < e
        unfolding *real_arch_inverse*[*of* e] **by** *auto*
      {
        fix n :: *nat*
        assume n ≥ N
        then have inverse (real n + 1) < inverse (real N)
          using *of_nat_0_le_iff* **and** ‹N≠0› **by** *auto*
        also have ... < e **using** N **by** *auto*
        finally have inverse (real n + 1) < e **by** *auto*
        then have dist (x n) (y n) < e
          using *xy0*[*THEN spec*[**where** x=n]] **by** *auto*
      }
      then have ∃ N. ∀ n≥N. dist (x n) (y n) < e **by** *auto*
    }
    then have ∀ e>0. ∃ N. ∀ n≥N. dist (f (x n)) (f (y n)) < e
      using ‹?rhs›[*THEN spec*[**where** x=x], *THEN spec*[**where** x=y]] **and** *xyn*
      unfolding *lim_sequentially dist_real_def* **by** *auto*
    then have *False* **using** *fxy* **and** ‹e>0› **by** *auto*
  }
  then show *?lhs*
    unfolding *uniformly_continuous_on_def* **by** *blast*
qed

3.2.23 Continuity on a Compact Domain Implies Uniform Continuity

From the proof of the Heine-Borel theorem: Lemma 2 in section 3.7, page 69 of J. C. Burkill and H. Burkill. A Second Course in Mathematical Analysis (CUP, 2002)

```
lemma Heine_Borel_lemma:
  assumes compact S and Ssub: S ⊆ ⋃𝒢 and opn: ⋀G. G ∈ 𝒢 ⟹ open G
  obtains e where 0 < e ⋀x. x ∈ S ⟹ ∃ G ∈ 𝒢. ball x e ⊆ G
proof –
  have False if neg: ⋀e. 0 < e ⟹ ∃ x ∈ S. ∀ G ∈ 𝒢. ¬ ball x e ⊆ G
  proof –
    have ∃ x ∈ S. ∀ G ∈ 𝒢. ¬ ball x (1 / Suc n) ⊆ G for n
      using neg by simp
    then obtain f where ⋀n. f n ∈ S and fG: ⋀G n. G ∈ 𝒢 ⟹ ¬ ball (f n) (1
/ Suc n) ⊆ G
      by metis
    then obtain l r where l ∈ S strict_mono r and to_l: (f ∘ r) ⟶ l
      using ‹compact S› compact_def that by metis
    then obtain G where l ∈ G G ∈ 𝒢
      using Ssub by auto
    then obtain e where 0 < e and e: ⋀z. dist z l < e ⟹ z ∈ G
      using opn open_dist by blast
    obtain N1 where N1: ⋀n. n ≥ N1 ⟹ dist (f (r n)) l < e/2
      using to_l apply (simp add: lim_sequentially)
      using ‹0 < e› half_gt_zero that by blast
    obtain N2 where N2: of_nat N2 > 2/e
      using reals_Archimedean2 by blast
    obtain x where x ∈ ball (f (r (max N1 N2))) (1 / real (Suc (r (max N1
N2)))) and x ∉ G
      using fG [OF ‹G ∈ 𝒢›, of r (max N1 N2)] by blast
    then have dist (f (r (max N1 N2))) x < 1 / real (Suc (r (max N1 N2)))
      by simp
    also have ... ≤ 1 / real (Suc (max N1 N2))
      apply (simp add: field_split_simps del: max.bounded_iff)
      using ‹strict_mono r› seq_suble by blast
    also have ... ≤ 1 / real (Suc N2)
      by (simp add: field_simps)
    also have ... < e/2
      using N2 ‹0 < e› by (simp add: field_simps)
    finally have dist (f (r (max N1 N2))) x < e/2 .
    moreover have dist (f (r (max N1 N2))) l < e/2
      using N1 max.cobounded1 by blast
    ultimately have dist x l < e
      by metric
    then show ?thesis
      using e ‹x ∉ G› by blast
  qed
  then show ?thesis
    by (meson that)
qed

lemma compact_uniformly_equicontinuous:
  assumes compact S
    and cont: ⋀x e. ⟦x ∈ S; 0 < e⟧
```

$\Longrightarrow \exists\, d.\ 0 < d \wedge$
$(\forall f \in \mathcal{F}.\ \forall x' \in S.\ dist\ x'\ x < d \longrightarrow dist\ (f\ x')\ (f\ x) < e)$
and $0 < e$
obtains d **where** $0 < d$
$\bigwedge f\ x\ x'.\ [\![f \in \mathcal{F};\ x \in S;\ x' \in S;\ dist\ x'\ x < d]\!] \Longrightarrow dist\ (f\ x')\ (f\ x)$
$< e$
proof −
obtain d **where** *d_pos*: $\bigwedge x\ e.\ [\![x \in S;\ 0 < e]\!] \Longrightarrow 0 < d\ x\ e$
and *d_dist* : $\bigwedge x\ x'\ e\ f.\ [\![dist\ x'\ x < d\ x\ e;\ x \in S;\ x' \in S;\ 0 < e;\ f \in \mathcal{F}]\!] \Longrightarrow$
$dist\ (f\ x')\ (f\ x) < e$
using *cont* **by** *metis*
let $?\mathcal{G} = ((\lambda x.\ ball\ x\ (d\ x\ (e/2)))\ `\ S)$
have *Ssub*: $S \subseteq \bigcup\ ?\mathcal{G}$
by *clarsimp* (*metis d_pos* ‹$0 < e$› *dist_self half_gt_zero_iff*)
then obtain k **where** $0 < k$ **and** k: $\bigwedge x.\ x \in S \Longrightarrow \exists\, G \in\ ?\mathcal{G}.\ ball\ x\ k \subseteq G$
by (*rule Heine_Borel_lemma* [*OF* ‹*compact S*›]) *auto*
moreover have $dist\ (f\ v)\ (f\ u) < e$ **if** $f \in \mathcal{F}\ \ u \in S\ \ v \in S\ \ dist\ v\ u < k$ **for** $f\ u\ v$
proof −
obtain G **where** $G \in\ ?\mathcal{G}\ \ u \in G\ \ v \in G$
using k *that*
by (*metis* ‹$dist\ v\ u < k$› ‹$u \in S$› ‹$0 < k$› *centre_in_ball subsetD dist_commute*
mem_ball)
then obtain w **where** w: $dist\ w\ u < d\ w\ (e/2)\ \ dist\ w\ v < d\ w\ (e/2)\ \ w \in S$
by *auto*
with *that d_dist* **have** $dist\ (f\ w)\ (f\ v) < e/2$
by (*metis* ‹$0 < e$› *dist_commute half_gt_zero*)
moreover
have $dist\ (f\ w)\ (f\ u) < e/2$
using *that d_dist w* **by** (*metis* ‹$0 < e$› *dist_commute divide_pos_pos zero_less_numeral*)
ultimately show *?thesis*
using *dist_triangle_half_r* **by** *blast*
qed
ultimately show *?thesis* **using** *that* **by** *blast*
qed

corollary *compact_uniformly_continuous*:
fixes f :: $'a$ :: *metric_space* ⇒ $'b$ :: *metric_space*
assumes f: *continuous_on* $S\ f$ **and** S: *compact* S
shows *uniformly_continuous_on* $S\ f$
using f
unfolding *continuous_on_iff uniformly_continuous_on_def*
by (*force intro*: *compact_uniformly_equicontinuous* [*OF* S, *of* $\{f\}$])

3.2.24 Theorems relating continuity and uniform continuity to closures

lemma *continuous_on_closure*:
continuous_on (*closure* S) $f \longleftrightarrow$
$(\forall\, x\ e.\ x \in closure\ S \wedge 0 < e$

```
        ⟶ (∃ d. 0 < d ∧ (∀ y. y ∈ S ∧ dist y x < d ⟶ dist (f y) (f x) < e)))
  (is ?lhs = ?rhs)
proof
  assume ?lhs then show ?rhs
    unfolding continuous_on_iff by (metis Un_iff closure_def)
next
  assume R [rule_format]: ?rhs
  show ?lhs
  proof
    fix x and e::real
    assume 0 < e and x: x ∈ closure S
    obtain δ::real where δ > 0
               and δ: ⋀y. ⟦y ∈ S; dist y x < δ⟧ ⟹ dist (f y) (f x) < e/2
      using R [of x e/2] ‹0 < e› x by auto
    have dist (f y) (f x) ≤ e if y: y ∈ closure S and dyx: dist y x < δ/2 for y
    proof −
      obtain δ'::real where δ' > 0
                 and δ': ⋀z. ⟦z ∈ S; dist z y < δ'⟧ ⟹ dist (f z) (f y) < e/2
        using R [of y e/2] ‹0 < e› y by auto
      obtain z where z ∈ S and z: dist z y < min δ' δ / 2
        using closure_approachable y
       by (metis ‹0 < δ'› ‹0 < δ› divide_pos_pos min_less_iff_conj zero_less_numeral)
      have dist (f z) (f y) < e/2
        using δ' [OF ‹z ∈ S›] z ‹0 < δ'› by metric
      moreover have dist (f z) (f x) < e/2
        using δ[OF ‹z ∈ S›] z dyx by metric
      ultimately show ?thesis
        by metric
    qed
    then show ∃ d>0. ∀ x'∈closure S. dist x' x < d ⟶ dist (f x') (f x) ≤ e
      by (rule_tac x=δ/2 in exI) (simp add: ‹δ > 0›)
  qed
qed

lemma continuous_on_closure_sequentially:
  fixes f :: 'a::metric_space ⇒ 'b :: metric_space
  shows
   continuous_on (closure S) f ⟷
    (∀ x a. a ∈ closure S ∧ (∀ n. x n ∈ S) ∧ x ⟶ a ⟶ (f ∘ x) ⟶ f a)
   (is ?lhs = ?rhs)
proof −
  have continuous_on (closure S) f ⟷
          (∀ x ∈ closure S. continuous (at x within S) f)
    by (force simp: continuous_on_closure continuous_within_eps_delta)
  also have ... = ?rhs
    by (force simp: continuous_within_sequentially)
  finally show ?thesis .
qed
```

lemma *uniformly_continuous_on_closure*:
fixes *f* :: *'a*::*metric_space* ⇒ *'b*::*metric_space*
assumes *ucont*: *uniformly_continuous_on S f*
and *cont*: *continuous_on* (*closure S*) *f*
shows *uniformly_continuous_on* (*closure S*) *f*
unfolding *uniformly_continuous_on_def*
proof (*intro allI impI*)
fix *e*::*real*
assume *0 < e*
then obtain *d*::*real*
where *d>0*
and *d*: ⋀*x x'*. ⟦*x*∈*S*; *x'*∈*S*; *dist x' x < d*⟧ ⟹ *dist* (*f x'*) (*f x*) < *e/3*
using *ucont* [*unfolded uniformly_continuous_on_def*, *rule_format*, *of e/3*] **by** *auto*
show ∃ *d>0*. ∀ *x*∈*closure S*. ∀ *x'*∈*closure S*. *dist x' x < d* ⟶ *dist* (*f x'*) (*f x*) < *e*
proof (*rule exI* [**where** *x=d/3*], *clarsimp simp*: ‹*d > 0*›)
fix *x y*
assume *x*: *x* ∈ *closure S* **and** *y*: *y* ∈ *closure S* **and** *dyx*: *dist y x ∗ 3 < d*
obtain *d1*::*real* **where** *d1 > 0*
and *d1*: ⋀*w*. ⟦*w* ∈ *closure S*; *dist w x < d1*⟧ ⟹ *dist* (*f w*) (*f x*) < *e/3*
using *cont* [*unfolded continuous_on_iff*, *rule_format*, *of x e/3*] ‹*0 < e*› *x* **by** *auto*
obtain *x'* **where** *x'* ∈ *S* **and** *x'*: *dist x' x < min d1* (*d* / *3*)
using *closure_approachable* [*of x S*]
by (*metis* ‹*0 < d1*› ‹*0 < d*› *divide_pos_pos min_less_iff_conj x zero_less_numeral*)
obtain *d2*::*real* **where** *d2 > 0*
and *d2*: ∀ *w* ∈ *closure S*. *dist w y < d2* ⟶ *dist* (*f w*) (*f y*) < *e/3*
using *cont* [*unfolded continuous_on_iff*, *rule_format*, *of y e/3*] ‹*0 < e*› *y* **by** *auto*
obtain *y'* **where** *y'* ∈ *S* **and** *y'*: *dist y' y < min d2* (*d* / *3*)
using *closure_approachable* [*of y S*]
by (*metis* ‹*0 < d2*› ‹*0 < d*› *divide_pos_pos min_less_iff_conj y zero_less_numeral*)
have *dist x' x < d/3* **using** *x'* **by** *auto*
then have *dist x' y' < d*
using *dyx y'* **by** *metric*
then have *dist* (*f x'*) (*f y'*) < *e/3*
by (*rule d* [*OF* ‹*y'* ∈ *S*› ‹*x'* ∈ *S*›])
moreover have *dist* (*f x'*) (*f x*) < *e/3* **using** ‹*x'* ∈ *S*› *closure_subset x' d1*
by (*simp add*: *closure_def*)
moreover have *dist* (*f y'*) (*f y*) < *e/3* **using** ‹*y'* ∈ *S*› *closure_subset y' d2*
by (*simp add*: *closure_def*)
ultimately show *dist* (*f y*) (*f x*) < *e* **by** *metric*
qed
qed

lemma *uniformly_continuous_on_extension_at_closure*:
fixes *f*::*'a*::*metric_space* ⇒ *'b*::*complete_space*
assumes *uc*: *uniformly_continuous_on X f*

assumes $x \in$ *closure* X
obtains l **where** $(f \longrightarrow l)$ (*at* x *within* X)
proof −
from *assms* **obtain** *xs* **where** *xs*: $xs \longrightarrow x$ $\bigwedge n.$ $xs\ n \in X$
by (*auto simp*: *closure_sequential*)

from *uniformly_continuous_on_Cauchy*[*OF uc LIMSEQ_imp_Cauchy*, *OF xs*]
obtain l **where** l: $(\lambda n.\ f\ (xs\ n)) \longrightarrow l$
by *atomize_elim* (*simp only*: *convergent_eq_Cauchy*)

have $(f \longrightarrow l)$ (*at* x *within* X)
proof (*safe intro*!: *Lim_within_LIMSEQ*)
fix xs'
assume $\forall n.\ xs'\ n \neq x \wedge xs'\ n \in X$
and xs': $xs' \longrightarrow x$
then have $xs'\ n \neq x$ $xs'\ n \in X$ **for** n **by** *auto*

from *uniformly_continuous_on_Cauchy*[*OF uc LIMSEQ_imp_Cauchy*, *OF* ‹xs' $\longrightarrow x$› ‹xs' _ $\in X$›]
obtain l' **where** l': $(\lambda n.\ f\ (xs'\ n)) \longrightarrow l'$
by *atomize_elim* (*simp only*: *convergent_eq_Cauchy*)

show $(\lambda n.\ f\ (xs'\ n)) \longrightarrow l$
proof (*rule tendstoI*)
fix e::*real* **assume** $e > 0$
define e' **where** $e' \equiv e/2$
have $e' > 0$ **using** ‹$e > 0$› **by** (*simp add*: *e'_def*)

have $\forall_F\ n$ *in sequentially*. *dist* $(f\ (xs\ n))\ l < e'$
by (*simp add*: ‹$0 < e'$› *l tendstoD*)
moreover
from *uc*[*unfolded uniformly_continuous_on_def*, *rule_format*, *OF* ‹$e' > 0$›]
obtain d **where** d: $d > 0$ $\bigwedge x\ x'.\ x \in X \Longrightarrow x' \in X \Longrightarrow$ *dist* $x\ x' < d \Longrightarrow$ *dist* $(f\ x)\ (f\ x') < e'$
by *auto*
have $\forall_F\ n$ *in sequentially*. *dist* $(xs\ n)\ (xs'\ n) < d$
by (*auto intro*!: ‹$0 < d$› *order_tendstoD tendsto_eq_intros xs xs'*)
ultimately
show $\forall_F\ n$ *in sequentially*. *dist* $(f\ (xs'\ n))\ l < e$
proof *eventually_elim*
case (*elim n*)
have *dist* $(f\ (xs'\ n))\ l \leq$ *dist* $(f\ (xs\ n))\ (f\ (xs'\ n)) +$ *dist* $(f\ (xs\ n))\ l$
by *metric*
also have *dist* $(f\ (xs\ n))\ (f\ (xs'\ n)) < e'$
by (*auto intro*!: *d xs* ‹xs' _ $\in$ _› *elim*)
also note ‹*dist* $(f\ (xs\ n))\ l < e'$›
also have $e' + e' = e$ **by** (*simp add*: *e'_def*)
finally show *?case* **by** *simp*
qed

qed
qed
thus *?thesis* **..**
qed

lemma *uniformly_continuous_on_extension_on_closure*:
fixes f::'*a*::*metric_space* ⇒ '*b*::*complete_space*
assumes *uc*: *uniformly_continuous_on X f*
obtains *g* **where** *uniformly_continuous_on* (*closure X*) *g* ⋀*x*. *x* ∈ *X* ⟹ *f x* = *g x*
⋀*Y h x*. *X* ⊆ *Y* ⟹ *Y* ⊆ *closure X* ⟹ *continuous_on Y h* ⟹ (⋀*x*. *x* ∈ *X* ⟹ *f x* = *h x*) ⟹ *x* ∈ *Y* ⟹ *h x* = *g x*
proof –
from *uc* **have** *cont_f*: *continuous_on X f*
by (*simp add*: *uniformly_continuous_imp_continuous*)
obtain *y* **where** *y*: (*f* ⟶ *y x*) (*at x within X*) **if** *x* ∈ *closure X* **for** *x*
apply *atomize_elim*
apply (*rule choice*)
using *uniformly_continuous_on_extension_at_closure*[*OF assms*]
by *metis*
let *?g* = λ*x*. *if x* ∈ *X then f x else y x*

have *uniformly_continuous_on* (*closure X*) *?g*
unfolding *uniformly_continuous_on_def*
proof *safe*
fix *e*::*real* **assume** *e* > *0*
define *e'* **where** *e'* ≡ *e* / *3*
have *e'* > *0* **using** ‹*e* > *0*› **by** (*simp add*: *e'_def*)
from *uc*[*unfolded uniformly_continuous_on_def*, *rule_format*, *OF* ‹*0* < *e'*›]
obtain *d* **where** *d* > *0* **and** *d*: ⋀*x x'*. *x* ∈ *X* ⟹ *x'* ∈ *X* ⟹ *dist x' x* < *d* ⟹ *dist* (*f x'*) (*f x*) < *e'*
by *auto*
define *d'* **where** *d'* = *d* / *3*
have *d'* > *0* **using** ‹*d* > *0*› **by** (*simp add*: *d'_def*)
show ∃ *d*>*0*. ∀ *x*∈*closure X*. ∀ *x'*∈*closure X*. *dist x' x* < *d* ⟶ *dist* (*?g x'*) (*?g x*) < *e*
proof (*safe intro*!: *exI*[**where** *x*=*d'*] ‹*d'* > *0*›)
fix *x x'* **assume** *x*: *x* ∈ *closure X* **and** *x'*: *x'* ∈ *closure X* **and** *dist*: *dist x' x* < *d'*
then obtain *xs xs'* **where** *xs*: *xs* ⟶ *x* ⋀*n*. *xs n* ∈ *X*
and *xs'*: *xs'* ⟶ *x'* ⋀*n*. *xs' n* ∈ *X*
by (*auto simp*: *closure_sequential*)
have ∀$_F$ *n in sequentially*. *dist* (*xs' n*) *x'* < *d'*
and ∀$_F$ *n in sequentially*. *dist* (*xs n*) *x* < *d'*
by (*auto intro*!: ‹*0* < *d'*› *order_tendstoD tendsto_eq_intros xs xs'*)
moreover
have (λ*x*. *f* (*xs x*)) ⟶ *y x* **if** *x* ∈ *closure X x* ∉ *X xs* ⟶ *x* ⋀*n*. *xs n* ∈ *X* **for** *xs x*
using *that not_eventuallyD*

by (*force intro*!: *filterlim_compose*[*OF y*[*OF* ‹*x* ∈ *closure X*›]] *simp*: *filterlim_at*)
then have (λ*x*. *f* (*xs*′ *x*)) ⟶ *?g x*′ (λ*x*. *f* (*xs x*)) ⟶ *?g x*
using *x x*′
by (*auto intro*!: *continuous_on_tendsto_compose*[*OF cont_f*] *simp*: *xs*′ *xs*)
then have ∀$_F$ *n in sequentially*. *dist* (*f* (*xs*′ *n*)) (*?g x*′) < *e*′
∀$_F$ *n in sequentially*. *dist* (*f* (*xs n*)) (*?g x*) < *e*′
by (*auto intro*!: ‹*0* < *e*′› *order_tendstoD tendsto_eq_intros*)
ultimately
have ∀$_F$ *n in sequentially*. *dist* (*?g x*′) (*?g x*) < *e*
proof *eventually_elim*
case (*elim n*)
have *dist* (*?g x*′) (*?g x*) ≤
dist (*f* (*xs*′ *n*)) (*?g x*′) + *dist* (*f* (*xs*′ *n*)) (*f* (*xs n*)) + *dist* (*f* (*xs n*)) (*?g x*)
by (*metis add.commute add_le_cancel_left dist_commute dist_triangle dist_triangle_le*)
also
from ‹*dist* (*xs*′ *n*) *x*′ < *d*′› ‹*dist x*′ *x* < *d*′› ‹*dist* (*xs n*) *x* < *d*′›
have *dist* (*xs*′ *n*) (*xs n*) < *d* **unfolding** *d*′*_def* **by** *metric*
with ‹*xs* _ ∈ *X*› ‹*xs*′ _ ∈ *X*› **have** *dist* (*f* (*xs*′ *n*)) (*f* (*xs n*)) < *e*′
by (*rule d*)
also note ‹*dist* (*f* (*xs*′ *n*)) (*?g x*′) < *e*′›
also note ‹*dist* (*f* (*xs n*)) (*?g x*) < *e*′›
finally show *?case* **by** (*simp add*: *e*′*_def*)
qed
then show *dist* (*?g x*′) (*?g x*) < *e* **by** *simp*
qed
qed
moreover have *f x* = *?g x* **if** *x* ∈ *X* **for** *x* **using** *that* **by** *simp*
moreover
{
fix *Y h x*
assume *Y*: *x* ∈ *Y X* ⊆ *Y Y* ⊆ *closure X* **and** *cont_h*: *continuous_on Y h*
and *extension*: (⋀*x*. *x* ∈ *X* ⟹ *f x* = *h x*)
{
assume *x* ∉ *X*
have *x* ∈ *closure X* **using** *Y* **by** *auto*
then obtain *xs* **where** *xs*: *xs* ⟶ *x* ⋀*n*. *xs n* ∈ *X*
by (*auto simp*: *closure_sequential*)
from *continuous_on_tendsto_compose*[*OF cont_h xs*(*1*)] *xs*(*2*) *Y*
have *hx*: (λ*x*. *f* (*xs x*)) ⟶ *h x*
by (*auto simp*: *subsetD extension*)
then have (λ*x*. *f* (*xs x*)) ⟶ *y x*
using ‹*x* ∉ *X*› *not_eventuallyD xs*(*2*)
by (*force intro*!: *filterlim_compose*[*OF y*[*OF* ‹*x* ∈ *closure X*›]] *simp*: *filterlim_at xs*)
with *hx* **have** *h x* = *y x* **by** (*rule LIMSEQ_unique*)
} **then**

have *h x* = *?g x*
using *extension* **by** *auto*
}
ultimately show *?thesis* **..**
qed

lemma *bounded_uniformly_continuous_image*:
fixes *f* :: *'a* :: *heine_borel* ⇒ *'b* :: *heine_borel*
assumes *uniformly_continuous_on S f bounded S*
shows *bounded*(*f* ' *S*)
by (*metis* (*no_types*, *lifting*) *assms bounded_closure_image compact_closure compact_continuous_image compact_eq_bounded_closed image_cong uniformly_continuous_imp_continuous uniformly_continuous_on_extension_on_closure*)

3.2.25 With Abstract Topology (TODO: move and remove dependency?)

lemma *openin_contains_ball*:
openin (*top_of_set T*) *S* ⟷
S ⊆ *T* ∧ (∀ *x* ∈ *S*. ∃ *e*. *0* < *e* ∧ *ball x e* ∩ *T* ⊆ *S*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
apply (*simp add*: *openin_open*)
apply (*metis Int_commute Int_mono inf.cobounded2 open_contains_ball order_refl subsetCE*)
done
next
assume *?rhs*
then show *?lhs*
apply (*simp add*: *openin_euclidean_subtopology_iff*)
by (*metis* (*no_types*) *Int_iff dist_commute inf.absorb_iff2 mem_ball*)
qed

lemma *openin_contains_cball*:
openin (*top_of_set T*) *S* ⟷
S ⊆ *T* ∧ (∀ *x* ∈ *S*. ∃ *e*. *0* < *e* ∧ *cball x e* ∩ *T* ⊆ *S*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*force simp add*: *openin_contains_ball intro*: *exI* [**where** *x*=_/*2*])
next
assume *?rhs*
then show *?lhs*
by (*force simp add*: *openin_contains_ball*)
qed

3.2.26 Closed Nest

Bounded closed nest property (proof does not use Heine-Borel)

```
lemma bounded_closed_nest:
  fixes S :: nat ⇒ ('a::heine_borel) set
  assumes ⋀n. closed (S n)
      and ⋀n. S n ≠ {}
      and ⋀m n. m ≤ n ⟹ S n ⊆ S m
      and bounded (S 0)
  obtains a where ⋀n. a ∈ S n
proof –
  from assms(2) obtain x where x: ∀ n. x n ∈ S n
    using choice[of λn x. x ∈ S n] by auto
  from assms(4,1) have seq_compact (S 0)
    by (simp add: bounded_closed_imp_seq_compact)
  then obtain l r where lr: l ∈ S 0 strict_mono r (x ∘ r) ⟶ l
    using x and assms(3) unfolding seq_compact_def by blast
  have ∀ n. l ∈ S n
  proof
    fix n :: nat
    have closed (S n)
      using assms(1) by simp
    moreover have ∀ i. (x ∘ r) i ∈ S i
      using x and assms(3) and lr(2) [THEN seq_suble] by auto
    then have ∀ i. (x ∘ r) (i + n) ∈ S n
      using assms(3) by (fast intro!: le_add2)
    moreover have (λi. (x ∘ r) (i + n)) ⟶ l
      using lr(3) by (rule LIMSEQ_ignore_initial_segment)
    ultimately show l ∈ S n
      by (rule closed_sequentially)
  qed
  then show ?thesis
    using that by blast
qed
```

Decreasing case does not even need compactness, just completeness.

```
lemma decreasing_closed_nest:
  fixes S :: nat ⇒ ('a::complete_space) set
  assumes ⋀n. closed (S n)
          ⋀n. S n ≠ {}
          ⋀m n. m ≤ n ⟹ S n ⊆ S m
          ⋀e. e>0 ⟹ ∃ n. ∀ x∈S n. ∀ y∈S n. dist x y < e
  obtains a where ⋀n. a ∈ S n
proof –
  have ∀ n. ∃ x. x ∈ S n
    using assms(2) by auto
  then have ∃ t. ∀ n. t n ∈ S n
    using choice[of λn x. x ∈ S n] by auto
  then obtain t where t: ∀ n. t n ∈ S n by auto
```

```
 {
  fix e :: real
  assume e > 0
  then obtain N where N: ∀x∈S N. ∀y∈S N. dist x y < e
    using assms(4) by blast
  {
    fix m n :: nat
    assume N ≤ m ∧ N ≤ n
    then have t m ∈ S N t n ∈ S N
      using assms(3) t unfolding  subset_eq t by blast+
    then have dist (t m) (t n) < e
      using N by auto
  }
  then have ∃N. ∀m n. N ≤ m ∧ N ≤ n ⟶ dist (t m) (t n) < e
    by auto
 }
 then have Cauchy t
   unfolding cauchy_def by auto
 then obtain l where l:(t ⟶ l) sequentially
   using complete_UNIV unfolding complete_def by auto
 { fix n :: nat
   { fix e :: real
     assume e > 0
     then obtain N :: nat where N: ∀n≥N. dist (t n) l < e
       using l[unfolded lim_sequentially] by auto
     have t (max n N) ∈ S n
       by (meson assms(3) contra_subsetD max.cobounded1 t)
     then have ∃y∈S n. dist y l < e
       using N max.cobounded2 by blast
   }
   then have l ∈ S n
     using closed_approachable[of S n l] assms(1) by auto
 }
 then show ?thesis
   using that by blast
qed
```

Strengthen it to the intersection actually being a singleton.

```
lemma decreasing_closed_nest_sing:
 fixes S :: nat ⇒ 'a::complete_space set
 assumes ⋀n. closed(S n)
         ⋀n. S n ≠ {}
         ⋀m n. m ≤ n ⟹ S n ⊆ S m
         ⋀e. e>0 ⟹ ∃n. ∀x ∈ (S n). ∀ y∈(S n). dist x y < e
 shows ∃a. ⋂(range S) = {a}
proof –
 obtain a where a: ∀n. a ∈ S n
   using decreasing_closed_nest[of S] using assms by auto
 { fix b
```

```
    assume b: b ∈ ⋂(range S)
    { fix e :: real
      assume e > 0
      then have dist a b < e
        using assms(4) and b and a by blast
    }
    then have dist a b = 0
      by (metis dist_eq_0_iff dist_nz less_le)
  }
  with a have ⋂(range S) = {a}
    unfolding image_def by auto
  then show ?thesis ..
qed
```

3.2.27 Making a continuous function avoid some value in a neighbourhood

```
lemma continuous_within_avoid:
  fixes f :: 'a::metric_space ⇒ 'b::t1_space
  assumes continuous (at x within s) f
    and f x ≠ a
  shows ∃ e>0. ∀ y ∈ s. dist x y < e --> f y ≠ a
proof -
  obtain U where open U and f x ∈ U and a ∉ U
    using t1_space [OF ‹f x ≠ a›] by fast
  have (f ⟶ f x) (at x within s)
    using assms(1) by (simp add: continuous_within)
  then have eventually (λy. f y ∈ U) (at x within s)
    using ‹open U› and ‹f x ∈ U›
    unfolding tendsto_def by fast
  then have eventually (λy. f y ≠ a) (at x within s)
    using ‹a ∉ U› by (fast elim: eventually_mono)
  then show ?thesis
    using ‹f x ≠ a› by (auto simp: dist_commute eventually_at)
qed

lemma continuous_at_avoid:
  fixes f :: 'a::metric_space ⇒ 'b::t1_space
  assumes continuous (at x) f
    and f x ≠ a
  shows ∃ e>0. ∀ y. dist x y < e ⟶ f y ≠ a
  using assms continuous_within_avoid[of x UNIV f a] by simp

lemma continuous_on_avoid:
  fixes f :: 'a::metric_space ⇒ 'b::t1_space
  assumes continuous_on s f
    and x ∈ s
    and f x ≠ a
  shows ∃ e>0. ∀ y ∈ s. dist x y < e ⟶ f y ≠ a
```

using *assms(1)[unfolded continuous_on_eq_continuous_within, THEN bspec*[**where** *x=x*],
OF assms(2)] continuous_within_avoid[of x s f a]
using *assms(3)*
by *auto*

lemma *continuous_on_open_avoid*:
fixes *f* :: *'a::metric_space ⇒ 'b::t1_space*
assumes *continuous_on s f*
and *open s*
and $x \in s$
and $f\ x \neq a$
shows $\exists\, e>0.\ \forall\, y.\ dist\ x\ y < e \longrightarrow f\ y \neq a$
using *assms(1)[unfolded continuous_on_eq_continuous_at[OF assms(2)], THEN bspec*[**where** *x=x*], *OF assms(3)]*
using *continuous_at_avoid[of x f a] assms(4)*
by *auto*

3.2.28 Consequences for Real Numbers

lemma *closed_contains_Inf*:
fixes *S* :: *real set*
shows $S \neq \{\} \Longrightarrow$ *bdd_below S* $\Longrightarrow$ *closed S* $\Longrightarrow$ *Inf S* $\in S$
by (*metis closure_contains_Inf closure_closed*)

lemma *closed_subset_contains_Inf*:
fixes *A C* :: *real set*
shows *closed C* $\Longrightarrow A \subseteq C \Longrightarrow A \neq \{\} \Longrightarrow$ *bdd_below A* $\Longrightarrow$ *Inf A* $\in C$
by (*metis closure_contains_Inf closure_minimal subset_eq*)

lemma *closed_contains_Sup*:
fixes *S* :: *real set*
shows $S \neq \{\} \Longrightarrow$ *bdd_above S* $\Longrightarrow$ *closed S* $\Longrightarrow$ *Sup S* $\in S$
by (*subst closure_closed[symmetric], assumption, rule closure_contains_Sup*)

lemma *closed_subset_contains_Sup*:
fixes *A C* :: *real set*
shows *closed C* $\Longrightarrow A \subseteq C \Longrightarrow A \neq \{\} \Longrightarrow$ *bdd_above A* $\Longrightarrow$ *Sup A* $\in C$
by (*metis closure_contains_Sup closure_minimal subset_eq*)

lemma *atLeastAtMost_subset_contains_Inf*:
fixes *A* :: *real set* **and** *a b* :: *real*
shows $A \neq \{\} \Longrightarrow a \leq b \Longrightarrow A \subseteq \{a..b\} \Longrightarrow$ *Inf A* $\in \{a..b\}$
by (*rule closed_subset_contains_Inf*)
(*auto intro*: *closed_real_atLeastAtMost intro*!: *bdd_belowI[of A a]*)

lemma *bounded_real*: *bounded* (*S::real set*) $\longleftrightarrow (\exists\, a.\ \forall\, x{\in}S.\ |x| \leq a)$
by (*simp add*: *bounded_iff*)

lemma *bounded_imp_bdd_above*: *bounded S* $\Longrightarrow$ *bdd_above* (*S* :: *real set*)
  by (*auto simp*: *bounded_def bdd_above_def dist_real_def*)
    (*metis abs_le_D1 abs_minus_commute diff_le_eq*)

lemma *bounded_imp_bdd_below*: *bounded S* $\Longrightarrow$ *bdd_below* (*S* :: *real set*)
  by (*auto simp*: *bounded_def bdd_below_def dist_real_def*)
    (*metis abs_le_D1 add.commute diff_le_eq*)

lemma *bounded_has_Sup*:
  fixes *S* :: *real set*
  assumes *bounded S*
    and $S \neq \{\}$
  shows $\forall x \in S.\ x \leq Sup\ S$
    and $\forall b.\ (\forall x \in S.\ x \leq b) \longrightarrow Sup\ S \leq b$
proof
  show $\forall b.\ (\forall x \in S.\ x \leq b) \longrightarrow Sup\ S \leq b$
    using *assms* **by** (*metis cSup_least*)
qed (*metis cSup_upper assms*(*1*) *bounded_imp_bdd_above*)

lemma *Sup_insert*:
  fixes *S* :: *real set*
  shows *bounded S* $\Longrightarrow$ *Sup* (*insert x S*) = (*if S* = {} *then x else max x* (*Sup S*))
  by (*auto simp*: *bounded_imp_bdd_above sup_max cSup_insert_If*)

lemma *bounded_has_Inf*:
  fixes *S* :: *real set*
  assumes *bounded S*
    and $S \neq \{\}$
  shows $\forall x \in S.\ x \geq Inf\ S$
    and $\forall b.\ (\forall x \in S.\ x \geq b) \longrightarrow Inf\ S \geq b$
proof
  show $\forall b.\ (\forall x \in S.\ x \geq b) \longrightarrow Inf\ S \geq b$
    using *assms* **by** (*metis cInf_greatest*)
qed (*metis cInf_lower assms*(*1*) *bounded_imp_bdd_below*)

lemma *Inf_insert*:
  fixes *S* :: *real set*
  shows *bounded S* $\Longrightarrow$ *Inf* (*insert x S*) = (*if S* = {} *then x else min x* (*Inf S*))
  by (*auto simp*: *bounded_imp_bdd_below inf_min cInf_insert_If*)

lemma *open_real*:
  fixes *s* :: *real set*
  shows *open s* $\longleftrightarrow$ $(\forall x \in s.\ \exists e{>}0.\ \forall x'.\ |x' - x| < e$ --> $x' \in s)$
  unfolding *open_dist dist_norm* **by** *simp*

lemma *islimpt_approachable_real*:
  fixes *s* :: *real set*
  shows *x islimpt s* $\longleftrightarrow$ $(\forall e{>}0.\ \exists x' \in s.\ x' \neq x \wedge |x' - x| < e)$
  unfolding *islimpt_approachable dist_norm* **by** *simp*

lemma *closed_real*:
fixes s :: *real set*
shows *closed* $s \longleftrightarrow (\forall x. (\forall e>0. \ \exists x' \in s. x' \neq x \wedge |x' - x| < e) \longrightarrow x \in s)$
unfolding *closed_limpt islimpt_approachable dist_norm* **by** *simp*

lemma *continuous_at_real_range*:
fixes f :: *'a::real_normed_vector* ⇒ *real*
shows *continuous* (*at* x) $f \longleftrightarrow (\forall e>0. \exists d>0. \forall x'. norm(x' - x) < d \dashrightarrow |f x' - f x| < e)$
unfolding *continuous_at*
unfolding *Lim_at*
unfolding *dist_norm*
apply *auto*
apply (*erule_tac x=e* **in** *allE*, *auto*)
apply (*rule_tac x=d* **in** *exI*, *auto*)
apply (*erule_tac x=x′* **in** *allE*, *auto*)
apply (*erule_tac x=e* **in** *allE*, *auto*)
done

lemma *continuous_on_real_range*:
fixes f :: *'a::real_normed_vector* ⇒ *real*
shows *continuous_on* s f ⟷
$(\forall x \in s. \forall e>0. \exists d>0. (\forall x' \in s. norm(x' - x) < d \longrightarrow |f x' - f x| < e))$
unfolding *continuous_on_iff dist_norm* **by** *simp*

lemma *continuous_on_closed_Collect_le*:
fixes f g :: *'a::topological_space* ⇒ *real*
assumes f: *continuous_on* s f **and** g: *continuous_on* s g **and** s: *closed* s
shows *closed* $\{x \in s. f x \leq g x\}$
proof −
have *closed* $((\lambda x. g x - f x) -` \{0..\} \cap s)$
using *closed_real_atLeast continuous_on_diff* [*OF g f*]
by (*simp add*: *continuous_on_closed_vimage* [*OF s*])
also have $((\lambda x. g x - f x) -` \{0..\} \cap s) = \{x\in s. f x \leq g x\}$
by *auto*
finally show *?thesis* **.**
qed

lemma *continuous_le_on_closure*:
fixes *a::real*
assumes f: *continuous_on* (*closure* s) f
and x: $x \in closure(s)$
and *xlo*: $\bigwedge x. x \in s \Longrightarrow f(x) \leq a$
shows $f(x) \leq a$
using *image_closure_subset* [*OF f*, **where** $T = \{x. x \leq a\}$] *assms*
continuous_on_closed_Collect_le[*of UNIV* $\lambda x. x$ $\lambda x. a$]
by *auto*

lemma *continuous_ge_on_closure*:
  fixes *a*::*real*
  assumes *f*: *continuous_on* (*closure s*) *f*
    and *x*: *x* ∈ *closure*(*s*)
    and *xlo*: ⋀*x*. *x* ∈ *s* ==> *f*(*x*) ≥ *a*
   shows *f*(*x*) ≥ *a*
  using *image_closure_subset* [*OF f*, **where** *T*= {*x*. *a* ≤ *x*}] *assms*
   continuous_on_closed_Collect_le[*of UNIV* λ*x*. *a* λ*x*. *x*]
  by *auto*

3.2.29 The infimum of the distance between two sets

definition *setdist* :: ′*a*::*metric_space set* ⇒ ′*a set* ⇒ *real* **where**
  setdist s t ≡
    (*if s* = {} ∨ *t* = {} *then 0*
     else Inf {*dist x y*| *x y*. *x* ∈ *s* ∧ *y* ∈ *t*})

lemma *setdist_empty1* [*simp*]: *setdist* {} *t* = *0*
  by (*simp add*: *setdist_def*)

lemma *setdist_empty2* [*simp*]: *setdist t* {} = *0*
  by (*simp add*: *setdist_def*)

lemma *setdist_pos_le* [*simp*]: *0* ≤ *setdist s t*
  by (*auto simp*: *setdist_def ex_in_conv* [*symmetric*] *intro*: *cInf_greatest*)

lemma *le_setdistI*:
  assumes *s* ≠ {} *t* ≠ {} ⋀*x y*. ⟦*x* ∈ *s*; *y* ∈ *t*⟧ ⟹ *d* ≤ *dist x y*
   shows *d* ≤ *setdist s t*
  using *assms*
  by (*auto simp*: *setdist_def Set*.*ex_in_conv* [*symmetric*] *intro*: *cInf_greatest*)

lemma *setdist_le_dist*: ⟦*x* ∈ *s*; *y* ∈ *t*⟧ ⟹ *setdist s t* ≤ *dist x y*
  unfolding *setdist_def*
  by (*auto intro*!: *bdd_belowI* [**where** *m*=*0*] *cInf_lower*)

lemma *le_setdist_iff*:
    d ≤ *setdist S T* ⟷
    (∀ *x* ∈ *S*. ∀ *y* ∈ *T*. *d* ≤ *dist x y*) ∧ (*S* = {} ∨ *T* = {} ⟶ *d* ≤ *0*)
  apply (*cases S* = {} ∨ *T* = {})
  apply (*force simp add*: *setdist_def*)
  apply (*intro iffI conjI*)
  using *setdist_le_dist* **apply** *fastforce*
  apply (*auto simp*: *intro*: *le_setdistI*)
  done

lemma *setdist_ltE*:
  assumes *setdist S T* < *b S* ≠ {} *T* ≠ {}
   obtains *x y* **where** *x* ∈ *S y* ∈ *T dist x y* < *b*

using *assms*
by (*auto simp*: *not_le* [*symmetric*] *le_setdist_iff*)

lemma *setdist_refl*: *setdist S S* = *0*
  apply (*cases S* = {})
  apply (*force simp add*: *setdist_def*)
  apply (*rule antisym* [*OF _ setdist_pos_le*])
  apply (*metis all_not_in_conv dist_self setdist_le_dist*)
  done

lemma *setdist_sym*: *setdist S T* = *setdist T S*
  by (*force simp*: *setdist_def dist_commute intro*!: *arg_cong* [**where** *f*=*Inf*])

lemma *setdist_triangle*: *setdist S T* ≤ *setdist S* {*a*} + *setdist* {*a*} *T*
proof (*cases S* = {} ∨ *T* = {})
  case *True* **then show** *?thesis*
    using *setdist_pos_le* **by** *fastforce*
next
  case *False*
  then have ⋀*x*. *x* ∈ *S* ⟹ *setdist S T* − *dist x a* ≤ *setdist* {*a*} *T*
    apply (*intro le_setdistI*)
    apply (*simp_all add*: *algebra_simps*)
    apply (*metis dist_commute dist_triangle3 order_trans* [*OF setdist_le_dist*])
    done
  then have *setdist S T* − *setdist* {*a*} *T* ≤ *setdist S* {*a*}
    using *False* **by** (*fastforce intro*: *le_setdistI*)
  then show *?thesis*
    by (*simp add*: *algebra_simps*)
qed

lemma *setdist_singletons* [*simp*]: *setdist* {*x*} {*y*} = *dist x y*
  by (*simp add*: *setdist_def*)

lemma *setdist_Lipschitz*: |*setdist* {*x*} *S* − *setdist* {*y*} *S*| ≤ *dist x y*
  apply (*subst setdist_singletons* [*symmetric*])
  by (*metis abs_diff_le_iff diff_le_eq setdist_triangle setdist_sym*)

lemma *continuous_at_setdist* [*continuous_intros*]: *continuous* (*at x*) (λ*y*. (*setdist* {*y*} *S*))
  by (*force simp*: *continuous_at_eps_delta dist_real_def intro*: *le_less_trans* [*OF setdist_Lipschitz*])

lemma *continuous_on_setdist* [*continuous_intros*]: *continuous_on T* (λ*y*. (*setdist* {*y*} *S*))
  by (*metis continuous_at_setdist continuous_at_imp_continuous_on*)

lemma *uniformly_continuous_on_setdist*: *uniformly_continuous_on T* (λ*y*. (*setdist* {*y*} *S*))
  by (*force simp*: *uniformly_continuous_on_def dist_real_def intro*: *le_less_trans* [*OF*

setdist_Lipschitz])

lemma *setdist_subset_right*: ⟦*T* ≠ {}; *T* ⊆ *u*⟧ ⟹ *setdist S u* ≤ *setdist S T*
  apply (*cases S* = {} ∨ *u* = {}, *force*)
 apply (*auto simp*: *setdist_def intro*!: *bdd_belowI* [**where** *m*=*0*] *cInf_superset_mono*)
  done

lemma *setdist_subset_left*: ⟦*S* ≠ {}; *S* ⊆ *T*⟧ ⟹ *setdist T u* ≤ *setdist S u*
  by (*metis setdist_subset_right setdist_sym*)

lemma *setdist_closure_1* [*simp*]: *setdist* (*closure S*) *T* = *setdist S T*
proof (*cases S* = {} ∨ *T* = {})
  case *True* **then show** *?thesis* **by** *force*
next
  case *False*
  { **fix** *y*
    assume *y* ∈ *T*
    have *continuous_on* (*closure S*) (λ*a*. *dist a y*)
      by (*auto simp*: *continuous_intros dist_norm*)
    then have ∗: ⋀*x*. *x* ∈ *closure S* ⟹ *setdist S T* ≤ *dist x y*
      by (*fast intro*: *setdist_le_dist* ‹*y* ∈ *T*› *continuous_ge_on_closure*)
  } **note** ∗ = *this*
  show *?thesis*
    apply (*rule antisym*)
     using *False closure_subset* **apply** (*blast intro*: *setdist_subset_left*)
    using *False* ∗ **apply** (*force intro*!: *le_setdistI*)
    done
qed

lemma *setdist_closure_2* [*simp*]: *setdist T* (*closure S*) = *setdist T S*
  by (*metis setdist_closure_1 setdist_sym*)

lemma *setdist_eq_0I*: ⟦*x* ∈ *S*; *x* ∈ *T*⟧ ⟹ *setdist S T* = *0*
  by (*metis antisym dist_self setdist_le_dist setdist_pos_le*)

lemma *setdist_unique*:
  ⟦*a* ∈ *S*; *b* ∈ *T*; ⋀*x y*. *x* ∈ *S* ∧ *y* ∈ *T* ==> *dist a b* ≤ *dist x y*⟧
   ⟹ *setdist S T* = *dist a b*
  by (*force simp add*: *setdist_le_dist le_setdist_iff intro*: *antisym*)

lemma *setdist_le_sing*: *x* ∈ *S* ==> *setdist S T* ≤ *setdist* {*x*} *T*
  using *setdist_subset_left* **by** *auto*

lemma *infdist_eq_setdist*: *infdist x A* = *setdist* {*x*} *A*
  by (*simp add*: *infdist_def setdist_def Setcompr_eq_image*)

lemma *setdist_eq_infdist*: *setdist A B* = (*if A* = {} *then 0 else INF a*∈*A*. *infdist a B*)
proof −

have *Inf* $\{dist\ x\ y\ |x\ y.\ x \in A \wedge y \in B\} = (INF\ x{\in}A.\ Inf\ (dist\ x\ `\ B))$
if $b \in B$ $a \in A$ **for** a b
proof (*rule order_antisym*)
have *Inf* $\{dist\ x\ y\ |x\ y.\ x \in A \wedge y \in B\} \le Inf\ (dist\ x\ `\ B)$
if $b \in B$ $a \in A$ $x \in A$ **for** x
proof −
have ∗: $\bigwedge b'.\ b' \in B \Longrightarrow Inf\ \{dist\ x\ y\ |x\ y.\ x \in A \wedge y \in B\} \le dist\ x\ b'$
by (*metis* (*mono_tags*, *lifting*) *ex_in_conv setdist_def setdist_le_dist that*(*3*))
show *?thesis*
using *that* **by** (*subst conditionally_complete_lattice_class.le_cInf_iff*) (*auto simp*: ∗)+
qed
then show *Inf* $\{dist\ x\ y\ |x\ y.\ x \in A \wedge y \in B\} \le (INF\ x{\in}A.\ Inf\ (dist\ x\ `\ B))$
using *that*
by (*subst conditionally_complete_lattice_class.le_cInf_iff*) (*auto simp*: *bdd_below_def*)
next
have ∗: $\bigwedge x\ y.\ [\![b \in B;\ a \in A;\ x \in A;\ y \in B]\!] \Longrightarrow \exists\, a{\in}A.\ Inf\ (dist\ a\ `\ B) \le dist\ x\ y$
by (*meson bdd_below_image_dist cINF_lower*)
show $(INF\ x{\in}A.\ Inf\ (dist\ x\ `\ B)) \le Inf\ \{dist\ x\ y\ |x\ y.\ x \in A \wedge y \in B\}$
proof (*rule conditionally_complete_lattice_class.cInf_mono*)
show *bdd_below* $((\lambda x.\ Inf\ (dist\ x\ `\ B))\ `\ A)$
by (*metis* (*no_types*, *lifting*) *bdd_belowI2 ex_in_conv infdist_def infdist_nonneg that*(*1*))
qed (*use that* **in** ‹*auto simp*: ∗›)
qed
then show *?thesis*
by (*auto simp*: *setdist_def infdist_def*)
qed

lemma *infdist_mono*:
assumes $A \subseteq B$ $A \ne \{\}$
shows *infdist* x $B \le$ *infdist* x A
by (*simp add*: *assms infdist_eq_setdist setdist_subset_right*)

lemma *infdist_singleton* [*simp*]:
infdist x $\{y\}$ = *dist* x y
by (*simp add*: *infdist_eq_setdist*)

proposition *setdist_attains_inf*:
assumes *compact* B $B \ne \{\}$
obtains y **where** $y \in B$ *setdist* A B = *infdist* y A
proof (*cases* $A = \{\}$)
case *True*
then show *thesis*
by (*metis assms diameter_compact_attained infdist_def setdist_def that*)
next
case *False*
obtain y **where** $y \in B$ **and** *min*: $\bigwedge y'.\ y' \in B \Longrightarrow infdist\ y\ A \le infdist\ y'\ A$

```
    by (metis continuous_attains_inf [OF assms continuous_on_infdist] continu-
ous_on_id)
  show thesis
  proof
    have setdist A B = (INF y∈B. infdist y A)
      by (metis ‹B ≠ {}› setdist_eq_infdist setdist_sym)
    also have ... = infdist y A
    proof (rule order_antisym)
      show (INF y∈B. infdist y A) ≤ infdist y A
      proof (rule cInf_lower)
        show infdist y A ∈ (λy. infdist y A) ‘ B
          using ‹y ∈ B› by blast
        show bdd_below ((λy. infdist y A) ‘ B)
          by (meson bdd_belowI2 infdist_nonneg)
      qed
    next
      show infdist y A ≤ (INF y∈B. infdist y A)
        by (simp add: ‹B ≠ {}› cINF_greatest min)
    qed
    finally show setdist A B = infdist y A .
  qed (fact ‹y ∈ B›)
qed

end
```

3.3 Elementary Normed Vector Spaces

```
theory Elementary_Normed_Spaces
  imports
  HOL−Library.FuncSet
  Elementary_Metric_Spaces Cartesian_Space
  Connected
begin
```

3.3.1 Orthogonal Transformation of Balls

3.3.2 Various Lemmas Combining Imports

```
lemma open_sums:
  fixes T :: ('b::real_normed_vector) set
  assumes open S ∨ open T
  shows open (⋃x∈ S. ⋃y ∈ T. {x + y})
  using assms
proof
  assume S: open S
  show ?thesis
  proof (clarsimp simp: open_dist)
    fix x y
    assume x ∈ S y ∈ T
```

with *S* **obtain** *e* **where** *e* > *0* **and** *e*: ⋀*x'*. *dist x' x* < *e* ⟹ *x'* ∈ *S*
by (*auto simp*: *open_dist*)
then have ⋀*z*. *dist z* (*x* + *y*) < *e* ⟹ ∃ *x*∈*S*. ∃ *y*∈*T*. *z* = *x* + *y*
by (*metis* ‹*y* ∈ *T*› *diff_add_cancel dist_add_cancel2*)
then show ∃ *e*>*0*. ∀ *z*. *dist z* (*x* + *y*) < *e* ⟶ (∃ *x*∈*S*. ∃ *y*∈*T*. *z* = *x* + *y*)
using ‹*0* < *e*› ‹*x* ∈ *S*› **by** *blast*
qed
next
assume *T*: *open T*
show *?thesis*
proof (*clarsimp simp*: *open_dist*)
fix *x y*
assume *x* ∈ *S y* ∈ *T*
with *T* **obtain** *e* **where** *e* > *0* **and** *e*: ⋀*x'*. *dist x' y* < *e* ⟹ *x'* ∈ *T*
by (*auto simp*: *open_dist*)
then have ⋀*z*. *dist z* (*x* + *y*) < *e* ⟹ ∃ *x*∈*S*. ∃ *y*∈*T*. *z* = *x* + *y*
by (*metis* ‹*x* ∈ *S*› *add_diff_cancel_left' add_diff_eq diff_diff_add dist_norm*)
then show ∃ *e*>*0*. ∀ *z*. *dist z* (*x* + *y*) < *e* ⟶ (∃ *x*∈*S*. ∃ *y*∈*T*. *z* = *x* + *y*)
using ‹*0* < *e*› ‹*y* ∈ *T*› **by** *blast*
qed
qed

lemma *image_orthogonal_transformation_ball*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'a*
assumes *orthogonal_transformation f*
shows *f* ‘ *ball x r* = *ball* (*f x*) *r*
proof (*intro equalityI subsetI*)
fix *y* **assume** *y* ∈ *f* ‘ *ball x r*
with *assms* **show** *y* ∈ *ball* (*f x*) *r*
by (*auto simp*: *orthogonal_transformation_isometry*)
next
fix *y* **assume** *y*: *y* ∈ *ball* (*f x*) *r*
then obtain *z* **where** *z*: *y* = *f z*
using *assms orthogonal_transformation_surj* **by** *blast*
with *y assms* **show** *y* ∈ *f* ‘ *ball x r*
by (*auto simp*: *orthogonal_transformation_isometry*)
qed

lemma *image_orthogonal_transformation_cball*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'a*
assumes *orthogonal_transformation f*
shows *f* ‘ *cball x r* = *cball* (*f x*) *r*
proof (*intro equalityI subsetI*)
fix *y* **assume** *y* ∈ *f* ‘ *cball x r*
with *assms* **show** *y* ∈ *cball* (*f x*) *r*
by (*auto simp*: *orthogonal_transformation_isometry*)
next
fix *y* **assume** *y*: *y* ∈ *cball* (*f x*) *r*
then obtain *z* **where** *z*: *y* = *f z*

```
    using assms orthogonal_transformation_surj by blast
  with y assms show y ∈ f ` cball x r
    by (auto simp: orthogonal_transformation_isometry)
qed
```

3.3.3 Support

```
definition (in monoid_add) support_on :: 'b set ⇒ ('b ⇒ 'a) ⇒ 'b set
  where support_on S f = {x∈S. f x ≠ 0}

lemma in_support_on: x ∈ support_on S f ⟷ x ∈ S ∧ f x ≠ 0
  by (simp add: support_on_def)

lemma support_on_simps[simp]:
  support_on {} f = {}
  support_on (insert x S) f =
    (if f x = 0 then support_on S f else insert x (support_on S f))
  support_on (S ∪ T) f = support_on S f ∪ support_on T f
  support_on (S ∩ T) f = support_on S f ∩ support_on T f
  support_on (S − T) f = support_on S f − support_on T f
  support_on (f ` S) g = f ` (support_on S (g ∘ f))
  unfolding support_on_def by auto

lemma support_on_cong:
  (⋀x. x ∈ S ⟹ f x = 0 ⟷ g x = 0) ⟹ support_on S f = support_on S g
  by (auto simp: support_on_def)

lemma support_on_if: a ≠ 0 ⟹ support_on A (λx. if P x then a else 0) = {x∈A.
P x}
  by (auto simp: support_on_def)

lemma support_on_if_subset: support_on A (λx. if P x then a else 0) ⊆ {x ∈ A. P
x}
  by (auto simp: support_on_def)

lemma finite_support[intro]: finite S ⟹ finite (support_on S f)
  unfolding support_on_def by auto


definition (in comm_monoid_add) supp_sum :: ('b ⇒ 'a) ⇒ 'b set ⇒ 'a
  where supp_sum f S = (∑x∈support_on S f. f x)

lemma supp_sum_empty[simp]: supp_sum f {} = 0
  unfolding supp_sum_def by auto

lemma supp_sum_insert[simp]:
  finite (support_on S f) ⟹
    supp_sum f (insert x S) = (if x ∈ S then supp_sum f S else f x + supp_sum f S)
  by (simp add: supp_sum_def in_support_on insert_absorb)
```

lemma *supp_sum_divide_distrib*: *supp_sum f A* / (*r*::′*a*::*field*) = *supp_sum* (λ*n*. *f n* / *r*) *A*
  by (*cases r* = *0*)
    (*auto simp*: *supp_sum_def sum_divide_distrib intro*!: *sum.cong support_on_cong*)

3.3.4 Intervals

lemma *image_affinity_interval*:
  fixes *c* :: ′*a*::*ordered_real_vector*
  shows ((λ*x*. *m* $*_R$ *x* + *c*) ‘ {*a*..*b*}) =
      (*if* {*a*..*b*}={} *then* {}
       else if 0 ≤ *m then* {*m* $*_R$ *a* + *c* .. *m* $*_R$ *b* + *c*}
       else {*m* $*_R$ *b* + *c* .. *m* $*_R$ *a* + *c*})
    (**is** *?lhs* = *?rhs*)
proof (*cases m*=*0*)
  case *True*
  then show *?thesis*
    by *force*
next
  case *False*
  show *?thesis*
  proof
    show *?lhs* ⊆ *?rhs*
      by (*auto simp*: *scaleR_left_mono scaleR_left_mono_neg*)
    show *?rhs* ⊆ *?lhs*
    proof (*clarsimp*, *intro conjI impI subsetI*)
      show ⟦*0* ≤ *m*; *a* ≤ *b*; *x* ∈ {*m* $*_R$ *a* + *c*..*m* $*_R$ *b* + *c*}⟧
          ⟹ *x* ∈ (λ*x*. *m* $*_R$ *x* + *c*) ‘ {*a*..*b*} **for** *x*
        using *False*
        by (*rule_tac x*=*inverse m* $*_R$ (*x*−*c*) **in** *image_eqI*)
          (*auto simp*: *pos_le_divideR_eq pos_divideR_le_eq le_diff_eq diff_le_eq*)
      show ⟦¬ *0* ≤ *m*; *a* ≤ *b*; *x* ∈ {*m* $*_R$ *b* + *c*..*m* $*_R$ *a* + *c*}⟧
          ⟹ *x* ∈ (λ*x*. *m* $*_R$ *x* + *c*) ‘ {*a*..*b*} **for** *x*
        by (*rule_tac x*=*inverse m* $*_R$ (*x*−*c*) **in** *image_eqI*)
          (*auto simp add*: *neg_le_divideR_eq neg_divideR_le_eq le_diff_eq diff_le_eq*)
    qed
  qed
qed

3.3.5 Limit Points

lemma *islimpt_ball*:
  fixes *x y* :: ′*a*::{*real_normed_vector*,*perfect_space*}
  shows *y islimpt ball x e* ⟷ *0* < *e* ∧ *y* ∈ *cball x e*
  (**is** *?lhs* ⟷ *?rhs*)
proof
  show *?rhs* **if** *?lhs*
  proof
    {

```
      assume e ≤ 0
      then have ∗: ball x e = {}
        using ball_eq_empty[of x e] by auto
      have False using ‹?lhs›
        unfolding ∗ using islimpt_EMPTY[of y] by auto
    }
    then show e > 0 by (metis not_less)
    show y ∈ cball x e
      using closed_cball[of x e] islimpt_subset[of y ball x e cball x e]
        ball_subset_cball[of x e] ‹?lhs›
      unfolding closed_limpt by auto
  qed
  show ?lhs if ?rhs
  proof −
    from that have e > 0 by auto
    {
      fix d :: real
      assume d > 0
      have ∃x'∈ball x e. x' ≠ y ∧ dist x' y < d
      proof (cases d ≤ dist x y)
        case True
        then show ?thesis
        proof (cases x = y)
          case True
          then have False
            using ‹d ≤ dist x y› ‹d>0› by auto
          then show ?thesis
            by auto
        next
          case False
          have dist x (y − (d / (2 ∗ dist y x)) ∗R (y − x)) =
            norm (x − y + (d / (2 ∗ norm (y − x))) ∗R (y − x))
           unfolding mem_cball mem_ball dist_norm diff_diff_eq2 diff_add_eq[symmetric]
            by auto
          also have ... = |− 1 + d / (2 ∗ norm (x − y))| ∗ norm (x − y)
            using scaleR_left_distrib[of − 1 d / (2 ∗ norm (y − x)), symmetric, of
y − x]
            unfolding scaleR_minus_left scaleR_one
            by (auto simp: norm_minus_commute)
          also have ... = |− norm (x − y) + d / 2|
            unfolding abs_mult_pos[of norm (x − y), OF norm_ge_zero[of x − y]]
            unfolding distrib_right using ‹x≠y›  by auto
          also have ... ≤ e − d/2 using ‹d ≤ dist x y› and ‹d>0› and ‹?rhs›
            by (auto simp: dist_norm)
          finally have y − (d / (2 ∗ dist y x)) ∗R (y − x) ∈ ball x e using ‹d>0›
            by auto
          moreover
          have (d / (2∗dist y x)) ∗R (y − x) ≠ 0
            using ‹x≠y›[unfolded dist_nz] ‹d>0› unfolding scaleR_eq_0_iff
```

```
        by (auto simp: dist_commute)
      moreover
      have dist (y − (d / (2 ∗ dist y x)) ∗R (y − x)) y < d
        using ‹0 < d› by (fastforce simp: dist_norm)
      ultimately show ?thesis
        by (rule_tac x = y − (d / (2∗dist y x)) ∗R (y − x) in bexI) auto
    qed
  next
    case False
    then have d > dist x y by auto
    show ∃ x′ ∈ ball x e. x′ ≠ y ∧ dist x′ y < d
    proof (cases x = y)
      case True
      obtain z where z: z ≠ y dist z y < min e d
        using perfect_choose_dist[of min e d y]
        using ‹d > 0› ‹e>0› by auto
      show ?thesis
        by (metis True z dist_commute mem_ball min_less_iff_conj)
    next
      case False
      then show ?thesis
        using ‹d>0› ‹d > dist x y› ‹?rhs› by force
    qed
  qed
 }
 then show ?thesis
   unfolding mem_cball islimpt_approachable mem_ball by auto
qed
qed

lemma closure_ball_lemma:
  fixes x y :: ′a::real_normed_vector
  assumes x ≠ y
  shows y islimpt ball x (dist x y)
proof (rule islimptI)
  fix T
  assume y ∈ T open T
  then obtain r where 0 < r ∀ z. dist z y < r ⟶ z ∈ T
    unfolding open_dist by fast
  — choose point between x and y, within distance r of y.
  define k where k = min 1 (r / (2 ∗ dist x y))
  define z where z = y + scaleR k (x − y)
  have z_def2: z = x + scaleR (1 − k) (y − x)
    unfolding z_def by (simp add: algebra_simps)
  have dist z y < r
    unfolding z_def k_def using ‹0 < r›
    by (simp add: dist_norm min_def)
  then have z ∈ T
    using ‹∀ z. dist z y < r ⟶ z ∈ T› by simp
```

have *dist x z < dist x y*
using ‹*0 < r*› *assms* **by** (*simp add*: *z_def2 k_def dist_norm norm_minus_commute*)

then have *z ∈ ball x* (*dist x y*)
by *simp*
have *z ≠ y*
unfolding *z_def k_def* **using** ‹*x ≠ y*› ‹*0 < r*›
by (*simp add*: *min_def*)
show ∃ *z*∈*ball x* (*dist x y*). *z ∈ T ∧ z ≠ y*
using ‹*z ∈ ball x* (*dist x y*)› ‹*z ∈ T*› ‹*z ≠ y*›
by *fast*
qed

3.3.6 Balls and Spheres in Normed Spaces

lemma *mem_ball_0* [*simp*]: *x ∈ ball 0 e ⟷ norm x < e*
for *x* :: *'a*::*real_normed_vector*
by *simp*

lemma *mem_cball_0* [*simp*]: *x ∈ cball 0 e ⟷ norm x ≤ e*
for *x* :: *'a*::*real_normed_vector*
by *simp*

lemma *closure_ball* [*simp*]:
fixes *x* :: *'a*::*real_normed_vector*
assumes *0 < e*
shows *closure* (*ball x e*) = *cball x e*
proof
show *closure* (*ball x e*) ⊆ *cball x e*
using *closed_cball closure_minimal* **by** *blast*
have ⋀*y*. *dist x y < e ∨ dist x y = e ⟹ y ∈ closure* (*ball x e*)
by (*metis Un_iff assms closure_ball_lemma closure_def dist_eq_0_iff mem_Collect_eq mem_ball*)
then show *cball x e ⊆ closure* (*ball x e*)
by *force*
qed

lemma *mem_sphere_0* [*simp*]: *x ∈ sphere 0 e ⟷ norm x = e*
for *x* :: *'a*::*real_normed_vector*
by *simp*

lemma *interior_cball* [*simp*]:
fixes *x* :: *'a*::{*real_normed_vector*, *perfect_space*}
shows *interior* (*cball x e*) = *ball x e*
proof (*cases e ≥ 0*)
case *False* **note** *cs* = *this*
from *cs* **have** *null*: *ball x e* = {}
using *ball_empty*[*of e x*] **by** *auto*

```
  moreover
  have cball x e = {}
  proof (rule equals0I)
    fix y
    assume y ∈ cball x e
    then show False
         by (metis ball_eq_empty null cs dist_eq_0_iff dist_le_zero_iff empty_subsetI
mem_cball
          subset_antisym subset_ball)
  qed
  then have interior (cball x e) = {}
    using interior_empty by auto
  ultimately show ?thesis by blast
next
  case True note cs = this
  have ball x e ⊆ cball x e
    using ball_subset_cball by auto
  moreover
  {
    fix S y
    assume as: S ⊆ cball x e open S y∈S
    then obtain d where d>0 and d: ∀ x'. dist x' y < d ⟶ x' ∈ S
      unfolding open_dist by blast
    then obtain xa where xa_y: xa ≠ y and xa: dist xa y < d
      using perfect_choose_dist [of d] by auto
    have xa ∈ S
      using d[THEN spec[where x = xa]]
      using xa by (auto simp: dist_commute)
    then have xa_cball: xa ∈ cball x e
      using as(1) by auto
    then have y ∈ ball x e
    proof (cases x = y)
      case True
      then have e > 0 using cs order.order_iff_strict xa_cball xa_y by fastforce
      then show y ∈ ball x e
        using ‹x = y › by simp
    next
      case False
      have dist (y + (d / 2 / dist y x) *R (y − x)) y < d
        unfolding dist_norm
        using ‹d>0› norm_ge_zero[of y − x] ‹x ≠ y› by auto
      then have *: y + (d / 2 / dist y x) *R (y − x) ∈ cball x e
        using d as(1)[unfolded subset_eq] by blast
      have y − x ≠ 0 using ‹x ≠ y› by auto
      hence **:d / (2 ∗ norm (y − x)) > 0
        unfolding zero_less_norm_iff [symmetric] using ‹d>0› by auto
      have dist (y + (d / 2 / dist y x) *R (y − x)) x =
        norm (y + (d / (2 ∗ norm (y − x))) *R y − (d / (2 ∗ norm (y − x))) *R
x − x)
```

```
        by (auto simp: dist_norm algebra_simps)
      also have ... = norm ((1 + d / (2 * norm (y − x))) *R (y − x))
        by (auto simp: algebra_simps)
      also have ... = |1 + d / (2 * norm (y − x))| * norm (y − x)
        using ** by auto
      also have ... = (dist y x) + d/2
        using ** by (auto simp: distrib_right dist_norm)
      finally have e ≥ dist x y +d/2
        using *[unfolded mem_cball] by (auto simp: dist_commute)
      then show y ∈ ball x e
        unfolding mem_ball using ‹d>0› by auto
    qed
  }
  then have ∀ S ⊆ cball x e. open S ⟶ S ⊆ ball x e
    by auto
  ultimately show ?thesis
    using interior_unique[of ball x e cball x e]
    using open_ball[of x e]
    by auto
qed

lemma frontier_ball [simp]:
  fixes a :: 'a::real_normed_vector
  shows 0 < e ⟹ frontier (ball a e) = sphere a e
  by (force simp: frontier_def)

lemma frontier_cball [simp]:
  fixes a :: 'a::{real_normed_vector, perfect_space}
  shows frontier (cball a e) = sphere a e
  by (force simp: frontier_def)

corollary compact_sphere [simp]:
  fixes a :: 'a::{real_normed_vector,perfect_space,heine_borel}
  shows compact (sphere a r)
using compact_frontier [of cball a r] by simp

corollary bounded_sphere [simp]:
  fixes a :: 'a::{real_normed_vector,perfect_space,heine_borel}
  shows bounded (sphere a r)
by (simp add: compact_imp_bounded)

corollary closed_sphere  [simp]:
  fixes a :: 'a::{real_normed_vector,perfect_space,heine_borel}
  shows closed (sphere a r)
by (simp add: compact_imp_closed)

lemma image_add_ball [simp]:
  fixes a :: 'a::real_normed_vector
  shows (+) b ` ball a r = ball (a+b) r
```

proof –
 { **fix** *x* :: *'a*
  assume *dist* (*a* + *b*) *x* < *r*
  moreover
  have *b* + (*x* − *b*) = *x*
   by *simp*
  ultimately have *x* ∈ (+) *b* ' *ball a r*
   by (*metis add.commute dist_add_cancel image_eqI mem_ball*) }
 then show *?thesis*
  by (*auto simp*: *add.commute*)
qed

lemma *image_add_cball* [*simp*]:
 fixes *a* :: *'a::real_normed_vector*
 shows (+) *b* ' *cball a r* = *cball* (*a*+*b*) *r*
proof –
 have ⋀*x*. *dist* (*a* + *b*) *x* ≤ *r* ⟹ ∃ *y*∈*cball a r*. *x* = *b* + *y*
  by (*metis* (*no_types*) *add.commute diff_add_cancel dist_add_cancel2 mem_cball*)
 then show *?thesis*
  by (*force simp*: *add.commute*)
qed

3.3.7 Various Lemmas on Normed Algebras

lemma *closed_of_nat_image*: *closed* (*of_nat* ' *A* :: *'a::real_normed_algebra_1 set*)
 by (*rule discrete_imp_closed*[*of 1*]) (*auto simp*: *dist_of_nat*)

lemma *closed_of_int_image*: *closed* (*of_int* ' *A* :: *'a::real_normed_algebra_1 set*)
 by (*rule discrete_imp_closed*[*of 1*]) (*auto simp*: *dist_of_int*)

lemma *closed_Nats* [*simp*]: *closed* (ℕ :: *'a* :: *real_normed_algebra_1 set*)
 unfolding *Nats_def* **by** (*rule closed_of_nat_image*)

lemma *closed_Ints* [*simp*]: *closed* (ℤ :: *'a* :: *real_normed_algebra_1 set*)
 unfolding *Ints_def* **by** (*rule closed_of_int_image*)

lemma *closed_subset_Ints*:
 fixes *A* :: *'a* :: *real_normed_algebra_1 set*
 assumes *A* ⊆ ℤ
 shows *closed A*
proof (*intro discrete_imp_closed*[*OF zero_less_one*] *ballI impI*, *goal_cases*)
 case (*1 x y*)
 with *assms* **have** *x* ∈ ℤ **and** *y* ∈ ℤ **by** *auto*
 with ‹*dist y x* < *1*› **show** *y* = *x*
  by (*auto elim*!: *Ints_cases simp*: *dist_of_int*)
qed

3.3.8 Filters

definition *indirection* :: *'a::real_normed_vector* ⇒ *'a* ⇒ *'a filter* (**infixr** *indirection* *70*)
  where *a indirection v* = *at a within* {*b*. ∃ *c*≥*0*. *b* − *a* = *scaleR c v*}

3.3.9 Trivial Limits

lemma *trivial_limit_at_infinity*:
  ¬ *trivial_limit* (*at_infinity* :: (*'a*::{*real_normed_vector*,*perfect_space*}) *filter*)
proof −
  obtain *x*::*'a* **where** *x* ≠ *0*
    by (*meson perfect_choose_dist zero_less_one*)
  then have *b* ≤ *norm* ((*b* / *norm x*) $*_R$ *x*) **for** *b*
    by *simp*
  then show *?thesis*
    unfolding *trivial_limit_def eventually_at_infinity*
    by *blast*
qed

lemma *at_within_ball_bot_iff*:
  fixes *x y* :: *'a*::{*real_normed_vector*,*perfect_space*}
  shows *at x within ball y r* = *bot* ⟷ (*r*=*0* ∨ *x* ∉ *cball y r*)
  unfolding *trivial_limit_within*
  by (*metis* (*no_types*) *cball_empty equals0D islimpt_ball less_linear*)

3.3.10 Limits

proposition *Lim_at_infinity*: (*f* ⟶ *l*) *at_infinity* ⟷ (∀ *e*>*0*. ∃ *b*. ∀ *x*. *norm x* ≥ *b* ⟶ *dist* (*f x*) *l* < *e*)
  by (*auto simp*: *tendsto_iff eventually_at_infinity*)

corollary *Lim_at_infinityI* [*intro?*]:
  assumes ⋀*e*. *e* > *0* ⟹ ∃ *B*. ∀ *x*. *norm x* ≥ *B* ⟶ *dist* (*f x*) *l* ≤ *e*
  shows (*f* ⟶ *l*) *at_infinity*
proof −
  have ⋀*e*. *e* > *0* ⟹ ∃ *B*. ∀ *x*. *norm x* ≥ *B* ⟶ *dist* (*f x*) *l* < *e*
    by (*meson assms dense le_less_trans*)
  then show *?thesis*
    using *Lim_at_infinity* **by** *blast*
qed

lemma *Lim_transform_within_set_eq*:
  fixes *a* :: *'a*::*metric_space* **and** *l* :: *'b*::*metric_space*
  shows *eventually* (λ*x*. *x* ∈ *S* ⟷ *x* ∈ *T*) (*at a*)
      ⟹ ((*f* ⟶ *l*) (*at a within S*) ⟷ (*f* ⟶ *l*) (*at a within T*))
  by (*force intro*: *Lim_transform_within_set elim*: *eventually_mono*)

lemma *Lim_null*:
  fixes *f* :: *'a* ⇒ *'b*::*real_normed_vector*

shows $(f \longrightarrow l)$ *net* $\longleftrightarrow ((\lambda x.\ f(x) - l) \longrightarrow 0)$ *net*
by (*simp add*: *Lim dist_norm*)

lemma *Lim_null_comparison*:
fixes f :: $'a \Rightarrow 'b$::*real_normed_vector*
assumes *eventually* $(\lambda x.\ norm\ (f\ x) \leq g\ x)$ *net* $(g \longrightarrow 0)$ *net*
shows $(f \longrightarrow 0)$ *net*
using *assms*(*2*)
proof (*rule metric_tendsto_imp_tendsto*)
show *eventually* $(\lambda x.\ dist\ (f\ x)\ 0 \leq dist\ (g\ x)\ 0)$ *net*
using *assms*(*1*) **by** (*rule eventually_mono*) (*simp add*: *dist_norm*)
qed

lemma *Lim_transform_bound*:
fixes f :: $'a \Rightarrow 'b$::*real_normed_vector*
and g :: $'a \Rightarrow 'c$::*real_normed_vector*
assumes *eventually* $(\lambda n.\ norm\ (f\ n) \leq norm\ (g\ n))$ *net*
and $(g \longrightarrow 0)$ *net*
shows $(f \longrightarrow 0)$ *net*
using *assms*(*1*) *tendsto_norm_zero* [*OF assms*(*2*)]
by (*rule Lim_null_comparison*)

lemma *lim_null_mult_right_bounded*:
fixes f :: $'a \Rightarrow 'b$::*real_normed_div_algebra*
assumes f: $(f \longrightarrow 0)$ F **and** g: *eventually* $(\lambda x.\ norm(g\ x) \leq B)$ F
shows $((\lambda z.\ f\ z * g\ z) \longrightarrow 0)$ F
proof −
have $((\lambda x.\ norm\ (f\ x) * norm\ (g\ x)) \longrightarrow 0)$ F
proof (*rule Lim_null_comparison*)
show $\forall_F\ x$ *in* F. $norm\ (norm\ (f\ x) * norm\ (g\ x)) \leq norm\ (f\ x) * B$
by (*simp add*: *eventually_mono* [*OF g*] *mult_left_mono*)
show $((\lambda x.\ norm\ (f\ x) * B) \longrightarrow 0)$ F
by (*simp add*: *f tendsto_mult_left_zero tendsto_norm_zero*)
qed
then show *?thesis*
by (*subst tendsto_norm_zero_iff* [*symmetric*]) (*simp add*: *norm_mult*)
qed

lemma *lim_null_mult_left_bounded*:
fixes f :: $'a \Rightarrow 'b$::*real_normed_div_algebra*
assumes g: *eventually* $(\lambda x.\ norm(g\ x) \leq B)$ F **and** f: $(f \longrightarrow 0)$ F
shows $((\lambda z.\ g\ z * f\ z) \longrightarrow 0)$ F
proof −
have $((\lambda x.\ norm\ (g\ x) * norm\ (f\ x)) \longrightarrow 0)$ F
proof (*rule Lim_null_comparison*)
show $\forall_F\ x$ *in* F. $norm\ (norm\ (g\ x) * norm\ (f\ x)) \leq B * norm\ (f\ x)$
by (*simp add*: *eventually_mono* [*OF g*] *mult_right_mono*)
show $((\lambda x.\ B * norm\ (f\ x)) \longrightarrow 0)$ F
by (*simp add*: *f tendsto_mult_right_zero tendsto_norm_zero*)

```
  qed
  then show ?thesis
    by (subst tendsto_norm_zero_iff [symmetric]) (simp add: norm_mult)
qed

lemma lim_null_scaleR_bounded:
  assumes f: (f ⟶ 0) net and gB: eventually (λa. f a = 0 ∨ norm(g a) ≤
B) net
    shows ((λn. f n *R g n) ⟶ 0) net
proof
  fix ε::real
  assume 0 < ε
  then have B: 0 < ε / (abs B + 1) by simp
  have *: |f x| * norm (g x) < ε if f: |f x| * (|B| + 1) < ε and g: norm (g x) ≤
B for x
  proof -
    have |f x| * norm (g x) ≤ |f x| * B
      by (simp add: mult_left_mono g)
    also have ... ≤ |f x| * (|B| + 1)
      by (simp add: mult_left_mono)
    also have ... < ε
      by (rule f)
    finally show ?thesis .
  qed
  have ⋀x. ⟦|f x| < ε / (|B| + 1); norm (g x) ≤ B⟧ ⟹ |f x| * norm (g x) < ε
    by (simp add: * pos_less_divide_eq)
  then show ∀F x in net. dist (f x *R g x) 0 < ε
    using ‹0 < ε› by (auto intro: eventually_mono [OF eventually_conj [OF tend-
stoD [OF f B] gB]])
qed

lemma Lim_norm_ubound:
  fixes f :: 'a ⇒ 'b::real_normed_vector
  assumes ¬(trivial_limit net) (f ⟶ l) net eventually (λx. norm(f x) ≤ e) net
  shows norm(l) ≤ e
  using assms by (fast intro: tendsto_le tendsto_intros)

lemma Lim_norm_lbound:
  fixes f :: 'a ⇒ 'b::real_normed_vector
  assumes ¬ trivial_limit net
    and (f ⟶ l) net
    and eventually (λx. e ≤ norm (f x)) net
  shows e ≤ norm l
  using assms by (fast intro: tendsto_le tendsto_intros)
```

Limit under bilinear function

```
lemma Lim_bilinear:
  assumes (f ⟶ l) net
    and (g ⟶ m) net
```

and *bounded_bilinear h*
shows $((\lambda x.\ h\ (f\ x)\ (g\ x)) \longrightarrow (h\ l\ m))$ *net*
using ‹*bounded_bilinear h*› ‹$(f \longrightarrow l)$ *net*› ‹$(g \longrightarrow m)$ *net*›
by (*rule bounded_bilinear.tendsto*)

lemma *Lim_at_zero*:
fixes *a* :: *'a::real_normed_vector*
and *l* :: *'b::topological_space*
shows $(f \longrightarrow l)\ (at\ a) \longleftrightarrow ((\lambda x.\ f(a + x)) \longrightarrow l)\ (at\ 0)$
using *LIM_offset_zero LIM_offset_zero_cancel* **..**

3.3.11 Limit Point of Filter

lemma *netlimit_at_vector*:
fixes *a* :: *'a::real_normed_vector*
shows *netlimit* (*at a*) = *a*
proof (*cases* $\exists x.\ x \neq a$)
case *True* **then obtain** *x* **where** *x*: $x \neq a$ **..**
have $\bigwedge d.\ 0 < d \Longrightarrow \exists x.\ x \neq a \wedge norm\ (x - a) < d$
by (*rule_tac x=a + scaleR (d / 2) (sgn (x − a))* **in** *exI*) (*simp add*: *norm_sgn sgn_zero_iff x*)
then have ¬ *trivial_limit* (*at a*)
by (*auto simp*: *trivial_limit_def eventually_at dist_norm*)
then show *?thesis*
by (*rule Lim_ident_at* [*of a UNIV*])
qed *simp*

3.3.12 Boundedness

lemma *continuous_on_closure_norm_le*:
fixes *f* :: *'a::metric_space ⇒ 'b::real_normed_vector*
assumes *continuous_on* (*closure s*) *f*
and $\forall y \in s.\ norm(f\ y) \leq b$
and $x \in (closure\ s)$
shows *norm* (*f x*) $\leq b$
proof −
have ∗: *f ‘ s* ⊆ *cball 0 b*
using *assms*(*2*)[*unfolded mem_cball_0*[*symmetric*]] **by** *auto*
show *?thesis*
by (*meson* ∗ *assms*(*1*) *assms*(*3*) *closed_cball image_closure_subset image_subset_iff mem_cball_0*)
qed

lemma *bounded_pos*: *bounded S* $\longleftrightarrow (\exists b>0.\ \forall x\in S.\ norm\ x \leq b)$
unfolding *bounded_iff*
by (*meson less_imp_le not_le order_trans zero_less_one*)

lemma *bounded_pos_less*: *bounded S* $\longleftrightarrow (\exists b>0.\ \forall x\in S.\ norm\ x < b)$
by (*metis bounded_pos le_less_trans less_imp_le linordered_field_no_ub*)

```
lemma Bseq_eq_bounded:
  fixes f :: nat ⇒ 'a::real_normed_vector
  shows Bseq f ⟷ bounded (range f)
  unfolding Bseq_def bounded_pos by auto

lemma bounded_linear_image:
  assumes bounded S
    and bounded_linear f
  shows bounded (f ` S)
proof −
  from assms(1) obtain b where b > 0 and b: ∀x∈S. norm x ≤ b
    unfolding bounded_pos by auto
  from assms(2) obtain B where B: B > 0 ∀x. norm (f x) ≤ B ∗ norm x
    using bounded_linear.pos_bounded by (auto simp: ac_simps)
  show ?thesis
    unfolding bounded_pos
  proof (intro exI, safe)
    show norm (f x) ≤ B ∗ b if x ∈ S for x
      by (meson B b less_imp_le mult_left_mono order_trans that)
  qed (use ‹b > 0› ‹B > 0› in auto)
qed

lemma bounded_scaling:
  fixes S :: 'a::real_normed_vector set
  shows bounded S ⟹ bounded ((λx. c ∗R x) ` S)
  by (simp add: bounded_linear_image bounded_linear_scaleR_right)

lemma bounded_scaleR_comp:
  fixes f :: 'a ⇒ 'b::real_normed_vector
  assumes bounded (f ` S)
  shows bounded ((λx. r ∗R f x) ` S)
  using bounded_scaling[of f ` S r] assms
  by (auto simp: image_image)

lemma bounded_translation:
  fixes S :: 'a::real_normed_vector set
  assumes bounded S
  shows bounded ((λx. a + x) ` S)
proof −
  from assms obtain b where b: b > 0 ∀x∈S. norm x ≤ b
    unfolding bounded_pos by auto
  {
    fix x
    assume x ∈ S
    then have norm (a + x) ≤ b + norm a
      using norm_triangle_ineq[of a x] b by auto
  }
  then show ?thesis
    unfolding bounded_pos
```

using *norm_ge_zero*[*of a*] *b*(*1*) **and** *add_strict_increasing*[*of b 0 norm a*]
by (*auto intro*!: *exI*[*of _ b + norm a*])
qed

lemma *bounded_translation_minus*:
fixes *S* :: *'a*::*real_normed_vector set*
shows *bounded S* $\Longrightarrow$ *bounded* ((λx. *x* − *a*) ‘ *S*)
using *bounded_translation* [*of S* −*a*] **by** *simp*

lemma *bounded_uminus* [*simp*]:
fixes *X* :: *'a*::*real_normed_vector set*
shows *bounded* (*uminus* ‘ *X*) $\longleftrightarrow$ *bounded X*
by (*auto simp*: *bounded_def dist_norm*; *rule_tac x*=−*x* **in** *exI*; *force simp*: *add.commute norm_minus_commute*)

lemma *uminus_bounded_comp* [*simp*]:
fixes *f* :: *'a* ⇒ *'b*::*real_normed_vector*
shows *bounded* ((λx. − *f x*) ‘ *S*) $\longleftrightarrow$ *bounded* (*f* ‘ *S*)
using *bounded_uminus*[*of f* ‘ *S*]
by (*auto simp*: *image_image*)

lemma *bounded_plus_comp*:
fixes *f g*::*'a* ⇒ *'b*::*real_normed_vector*
assumes *bounded* (*f* ‘ *S*)
assumes *bounded* (*g* ‘ *S*)
shows *bounded* ((λx. *f x* + *g x*) ‘ *S*)
proof −
{
fix *B C*
assume $\bigwedge x$. *x*∈*S* $\Longrightarrow$ *norm* (*f x*) ≤ *B* $\bigwedge x$. *x*∈*S* $\Longrightarrow$ *norm* (*g x*) ≤ *C*
then have $\bigwedge x$. *x* ∈ *S* $\Longrightarrow$ *norm* (*f x* + *g x*) ≤ *B* + *C*
by (*auto intro*!: *norm_triangle_le add_mono*)
} **then show** *?thesis*
using *assms* **by** (*fastforce simp*: *bounded_iff*)
qed

lemma *bounded_plus*:
fixes *S* ::*'a*::*real_normed_vector set*
assumes *bounded S bounded T*
shows *bounded* ((λ(*x*,*y*). *x* + *y*) ‘ (*S* × *T*))
using *bounded_plus_comp* [*of fst S* × *T snd*] *assms*
by (*auto simp*: *split_def split*: *if_split_asm*)

lemma *bounded_minus_comp*:
bounded (*f* ‘ *S*) $\Longrightarrow$ *bounded* (*g* ‘ *S*) $\Longrightarrow$ *bounded* ((λx. *f x* − *g x*) ‘ *S*)
for *f g*::*'a* ⇒ *'b*::*real_normed_vector*
using *bounded_plus_comp*[*of f S* λx. − *g x*]
by *auto*

```
lemma bounded_minus:
  fixes S ::'a::real_normed_vector set
  assumes bounded S bounded T
  shows bounded ((λ(x,y). x − y) ‘ (S × T))
  using bounded_minus_comp [of fst S × T snd] assms
  by (auto simp: split_def split: if_split_asm)

lemma not_bounded_UNIV[simp]:
  ¬ bounded (UNIV :: 'a::{real_normed_vector, perfect_space} set)
proof (auto simp: bounded_pos not_le)
  obtain x :: 'a where x ≠ 0
    using perfect_choose_dist [OF zero_less_one] by fast
  fix b :: real
  assume b: b >0
  have b1: b +1 ≥ 0
    using b by simp
  with ‹x ≠ 0› have b < norm (scaleR (b + 1) (sgn x))
    by (simp add: norm_sgn)
  then show ∃x::'a. b < norm x ..
qed

corollary cobounded_imp_unbounded:
    fixes S :: 'a::{real_normed_vector, perfect_space} set
    shows bounded (− S) =⇒ ¬ bounded S
  using bounded_Un [of S −S]  by (simp)
```

3.3.13 Relations among convergence and absolute convergence for power series

```
lemma summable_imp_bounded:
  fixes f :: nat ⇒ 'a::real_normed_vector
  shows summable f =⇒ bounded (range f)
by (frule summable_LIMSEQ_zero) (simp add: convergent_imp_bounded)

lemma summable_imp_sums_bounded:
   summable f =⇒ bounded (range (λn. sum f {..<n}))
by (auto simp: summable_def sums_def dest: convergent_imp_bounded)

lemma power_series_conv_imp_absconv_weak:
  fixes a:: nat ⇒ 'a::{real_normed_div_algebra,banach} and w :: 'a
  assumes sum: summable (λn. a n ∗ z ˆ n) and no: norm w < norm z
    shows summable (λn. of_real(norm(a n)) ∗ w ˆ n)
proof −
  obtain M where M: ⋀x. norm (a x ∗ z ˆ x) ≤ M
    using summable_imp_bounded [OF sum] by (force simp: bounded_iff)
  show ?thesis
  proof (rule series_comparison_complex)
    have ⋀n. norm (a n) ∗ norm z ˆ n ≤ M
      by (metis (no_types) M norm_mult norm_power)
```

then show *summable* ($\lambda n.$ *complex_of_real* (*norm* (*a n*) $*$ *norm w* ^ *n*))
using *Abel_lemma no norm_ge_zero summable_of_real* **by** *blast*
qed (*auto simp*: *norm_mult norm_power*)
qed

3.3.14 Normed spaces with the Heine-Borel property

lemma *not_compact_UNIV*[*simp*]:
fixes *s* :: *'a*::{*real_normed_vector*,*perfect_space*,*heine_borel*} *set*
shows $\neg$ *compact* (*UNIV*::*'a set*)
by (*simp add*: *compact_eq_bounded_closed*)

lemma *not_compact_space_euclideanreal* [*simp*]: $\neg$ *compact_space euclideanreal*
by (*simp add*: *compact_space_def*)

Representing sets as the union of a chain of compact sets.

lemma *closed_Union_compact_subsets*:
fixes *S* :: *'a*::{*heine_borel*,*real_normed_vector*} *set*
assumes *closed S*
obtains *F* **where** $\bigwedge n.$ *compact*(*F n*) $\bigwedge n.$ *F n* $\subseteq$ *S* $\bigwedge n.$ *F n* $\subseteq$ *F*(*Suc n*)
($\bigcup n.$ *F n*) = *S* $\bigwedge K.$ $[\![$*compact K*; *K* $\subseteq$ *S*$]\!] \Longrightarrow \exists N. \forall n \geq N.$ *K* $\subseteq$
F n
proof
show *compact* (*S* $\cap$ *cball 0* (*of_nat n*)) **for** *n*
using *assms compact_eq_bounded_closed* **by** *auto*
next
show ($\bigcup n.$ *S* $\cap$ *cball 0* (*real n*)) = *S*
by (*auto simp*: *real_arch_simple*)
next
fix *K* :: *'a set*
assume *compact K K* $\subseteq$ *S*
then obtain *N* **where** *K* $\subseteq$ *cball 0 N*
by (*meson bounded_pos mem_cball_0 compact_imp_bounded subsetI*)
then show $\exists N. \forall n \geq N.$ *K* $\subseteq$ *S* $\cap$ *cball 0* (*real n*)
by (*metis of_nat_le_iff Int_subset_iff* ‹*K* $\subseteq$ *S*› *real_arch_simple subset_cball subset_trans*)
qed *auto*

3.3.15 Intersecting chains of compact sets and the Baire property

proposition *bounded_closed_chain*:
fixes $\mathcal{F}$ :: *'a*::*heine_borel set set*
assumes *B* $\in \mathcal{F}$ *bounded B* **and** $\mathcal{F}$: $\bigwedge S.$ *S* $\in \mathcal{F} \Longrightarrow$ *closed S* **and** {} $\notin \mathcal{F}$
and *chain*: $\bigwedge S\ T.$ *S* $\in \mathcal{F} \wedge$ *T* $\in \mathcal{F} \Longrightarrow$ *S* $\subseteq$ *T* $\vee$ *T* $\subseteq$ *S*
shows $\bigcap \mathcal{F} \neq$ {}
proof −
have *B* $\cap \bigcap \mathcal{F} \neq$ {}
proof (*rule compact_imp_fip*)

```
    show compact B ⋀T. T ∈ ℱ ⟹ closed T
      by (simp_all add: assms compact_eq_bounded_closed)
    show ⟦finite 𝒢; 𝒢 ⊆ ℱ⟧ ⟹ B ∩ ⋂𝒢 ≠ {} for 𝒢
    proof (induction 𝒢 rule: finite_induct)
      case empty
      with assms show ?case by force
    next
      case (insert U 𝒢)
      then have U ∈ ℱ and ne: B ∩ ⋂𝒢 ≠ {} by auto
      then consider B ⊆ U | U ⊆ B
          using ‹B ∈ ℱ› chain by blast
        then show ?case
        proof cases
          case 1
          then show ?thesis
            using Int_left_commute ne by auto
        next
          case 2
          have U ≠ {}
            using ‹U ∈ ℱ› ‹{} ∉ ℱ› by blast
          moreover
          have False if ⋀x. x ∈ U ⟹ ∃Y∈𝒢. x ∉ Y
          proof -
            have ⋀x. x ∈ U ⟹ ∃Y∈𝒢. Y ⊆ U
              by (metis chain contra_subsetD insert.prems insert_subset that)
            then obtain Y where Y ∈ 𝒢 Y ⊆ U
              by (metis all_not_in_conv ‹U ≠ {}›)
            moreover obtain x where x ∈ ⋂𝒢
              by (metis Int_emptyI ne)
            ultimately show ?thesis
              by (metis Inf_lower subset_eq that)
          qed
          with 2 show ?thesis
            by blast
        qed
      qed
  qed
  then show ?thesis by blast
qed

corollary compact_chain:
  fixes ℱ :: 'a::heine_borel set set
  assumes ⋀S. S ∈ ℱ ⟹ compact S {} ∉ ℱ
          ⋀S T. S ∈ ℱ ∧ T ∈ ℱ ⟹ S ⊆ T ∨ T ⊆ S
    shows ⋂ ℱ ≠ {}
proof (cases ℱ = {})
  case True
  then show ?thesis by auto
next
```

case *False*
show *?thesis*
by (*metis False all_not_in_conv assms compact_imp_bounded compact_imp_closed bounded_closed_chain*)
qed

lemma *compact_nest*:
fixes F :: $'a$::*linorder* ⇒ $'b$::*heine_borel set*
assumes F: $\bigwedge n.$ *compact*($F\ n$) $\bigwedge n.$ $F\ n \neq \{\}$ **and** *mono*: $\bigwedge m\ n.\ m \leq n \Longrightarrow F\ n \subseteq F\ m$
shows $\bigcap$(*range* F) $\neq \{\}$
proof −
have ∗: $\bigwedge S\ T.\ S \in$ *range* $F \wedge T \in$ *range* $F \Longrightarrow S \subseteq T \vee T \subseteq S$
by (*metis mono image_iff le_cases*)
show *?thesis*
using F **by** (*intro compact_chain* [*OF* _ _ ∗]; *blast dest*: ∗)
qed

The Baire property of dense sets

theorem *Baire*:
fixes S::$'a$::{*real_normed_vector*,*heine_borel*} *set*
assumes *closed* S *countable* $\mathcal{G}$
and *ope*: $\bigwedge T.\ T \in \mathcal{G} \Longrightarrow$ *openin* (*top_of_set* S) $T \wedge S \subseteq$ *closure* T
shows $S \subseteq$ *closure*($\bigcap \mathcal{G}$)
proof (*cases* $\mathcal{G} = \{\}$)
case *True*
then show *?thesis*
using *closure_subset* **by** *auto*
next
let *?g* = *from_nat_into* $\mathcal{G}$
case *False*
then have *gin*: *?g* $n \in \mathcal{G}$ **for** n
by (*simp add*: *from_nat_into*)
show *?thesis*
proof (*clarsimp simp*: *closure_approachable*)
fix x **and** e::*real*
assume $x \in S$ $0 < e$
obtain *TF* **where** *opeF*: $\bigwedge n.$ *openin* (*top_of_set* S) (*TF* n)
and *ne*: $\bigwedge n.$ *TF* $n \neq \{\}$
and *subg*: $\bigwedge n.\ S \cap$ *closure*(*TF* n) $\subseteq$ *?g* n
and *subball*: $\bigwedge n.$ *closure*(*TF* n) $\subseteq$ *ball* x e
and *decr*: $\bigwedge n.$ *TF*(*Suc* n) $\subseteq$ *TF* n
proof −
have ∗: $\exists Y.$ (*openin* (*top_of_set* S) $Y \wedge Y \neq \{\} \wedge$
$S \cap$ *closure* $Y \subseteq$ *?g* $n \wedge$ *closure* $Y \subseteq$ *ball* x e) $\wedge Y \subseteq U$
if *opeU*: *openin* (*top_of_set* S) U **and** $U \neq \{\}$ **and** *cloU*: *closure* $U \subseteq$ *ball* x e **for** U n
proof −
obtain T **where** T: *open* T $U = T \cap S$

```
      using ‹openin (top_of_set S) U› by (auto simp: openin_subtopology)
    with ‹U ≠ {}› have T ∩ closure (?g n) ≠ {}
      using gin ope by fastforce
    then have T ∩ ?g n ≠ {}
      using ‹open T› open_Int_closure_eq_empty by blast
    then obtain y where y ∈ U y ∈ ?g n
      using T ope [of ?g n, OF gin] by (blast dest: openin_imp_subset)
    moreover have openin (top_of_set S) (U ∩ ?g n)
      using gin ope opeU by blast
    ultimately obtain d where U: U ∩ ?g n ⊆ S and d > 0 and d: ball y
d ∩ S ⊆ U ∩ ?g n
      by (force simp: openin_contains_ball)
    show ?thesis
    proof (intro exI conjI)
      show openin (top_of_set S) (S ∩ ball y (d/2))
        by (simp add: openin_open_Int)
      show S ∩ ball y (d/2) ≠ {}
        using ‹0 < d› ‹y ∈ U› opeU openin_imp_subset by fastforce
      have S ∩ closure (S ∩ ball y (d/2)) ⊆ S ∩ closure (ball y (d/2))
        using closure_mono by blast
      also have ... ⊆ ?g n
        using ‹d > 0› d by force
      finally show S ∩ closure (S ∩ ball y (d/2)) ⊆ ?g n .
      have closure (S ∩ ball y (d/2)) ⊆ S ∩ ball y d
      proof −
        have closure (ball y (d/2)) ⊆ ball y d
          using ‹d > 0› by auto
        then have closure (S ∩ ball y (d/2)) ⊆ ball y d
          by (meson closure_mono inf.cobounded2 subset_trans)
        then show ?thesis
          by (simp add: ‹closed S› closure_minimal)
      qed
      also have ... ⊆ ball x e
        using cloU closure_subset d by blast
      finally show closure (S ∩ ball y (d/2)) ⊆ ball x e .
      show S ∩ ball y (d/2) ⊆ U
        using ball_divide_subset_numeral d by blast
    qed
  qed
  let ?Φ = λn X. openin (top_of_set S) X ∧ X ≠ {} ∧
                S ∩ closure X ⊆ ?g n ∧ closure X ⊆ ball x e
  have closure (S ∩ ball x (e/2)) ⊆ closure(ball x (e/2))
    by (simp add: closure_mono)
  also have ... ⊆ ball x e
    using ‹e > 0› by auto
  finally have closure (S ∩ ball x (e/2)) ⊆ ball x e .
  moreover haveopenin (top_of_set S) (S ∩ ball x (e/2)) S ∩ ball x (e/2) ≠
{}
    using ‹0 < e› ‹x ∈ S› by auto
```

```
      ultimately obtain Y where Y: ?Φ 0 Y ∧ Y ⊆ S ∩ ball x (e/2)
          using ∗ [of S ∩ ball x (e/2) 0] by metis
      show thesis
      proof (rule exE [OF dependent_nat_choice])
        show ∃ x. ?Φ 0 x
          using Y by auto
        show ∃ Y. ?Φ (Suc n) Y ∧ Y ⊆ X if ?Φ n X for X n
          using that by (blast intro: ∗)
      qed (use that in metis)
    qed
    have (⋂ n. S ∩ closure (TF n)) ≠ {}
    proof (rule compact_nest)
      show ⋀n. compact (S ∩ closure (TF n))
      by (metis closed_closure subball bounded_subset_ballI compact_eq_bounded_closed
closed_Int_compact [OF ‹closed S›])
      show ⋀n. S ∩ closure (TF n) ≠ {}
        by (metis Int_absorb1 opeF ‹closed S› closure_eq_empty closure_minimal ne
openin_imp_subset)
      show ⋀m n. m ≤ n ⟹ S ∩ closure (TF n) ⊆ S ∩ closure (TF m)
        by (meson closure_mono decr dual_order.refl inf_mono lift_Suc_antimono_le)
    qed
    moreover have (⋂ n. S ∩ closure (TF n)) ⊆ {y ∈ ⋂𝒢. dist y x < e}
    proof (clarsimp, intro conjI)
      fix y
      assume y ∈ S and y: ∀ n. y ∈ closure (TF n)
      then show ∀ T∈𝒢. y ∈ T
        by (metis Int_iff from_nat_into_surj [OF ‹countable 𝒢›] subsetD subg)
      show dist y x < e
        by (metis y dist_commute mem_ball subball subsetCE)
    qed
    ultimately show ∃ y ∈ ⋂𝒢. dist y x < e
      by auto
  qed
qed
```

3.3.16 Continuity

Structural rules for uniform continuity

```
lemma (in bounded_linear) uniformly_continuous_on[continuous_intros]:
  fixes g :: _::metric_space ⇒ _
  assumes uniformly_continuous_on s g
  shows uniformly_continuous_on s (λx. f (g x))
  using assms unfolding uniformly_continuous_on_sequentially
  unfolding dist_norm tendsto_norm_zero_iff diff[symmetric]
  by (auto intro: tendsto_zero)

lemma uniformly_continuous_on_dist[continuous_intros]:
  fixes f g :: 'a::metric_space ⇒ 'b::metric_space
  assumes uniformly_continuous_on s f
```

```
    and uniformly_continuous_on s g
  shows uniformly_continuous_on s (λx. dist (f x) (g x))
proof -
  {
    fix a b c d :: 'b
    have |dist a b - dist c d| ≤ dist a c + dist b d
      using dist_triangle2 [of a b c] dist_triangle2 [of b c d]
      using dist_triangle3 [of c d a] dist_triangle [of a d b]
      by arith
  } note le = this
  {
    fix x y
    assume f: (λn. dist (f (x n)) (f (y n))) ⟶ 0
    assume g: (λn. dist (g (x n)) (g (y n))) ⟶ 0
    have (λn. |dist (f (x n)) (g (x n)) - dist (f (y n)) (g (y n))|) ⟶ 0
      by (rule Lim_transform_bound [OF _ tendsto_add_zero [OF f g]],
        simp add: le)
  }
  then show ?thesis
    using assms unfolding uniformly_continuous_on_sequentially
    unfolding dist_real_def by simp
qed

lemma uniformly_continuous_on_cmul_right [continuous_intros]:
  fixes f :: 'a::real_normed_vector ⇒ 'b::real_normed_algebra
  shows uniformly_continuous_on s f ⟹ uniformly_continuous_on s (λx. f x * c)
  using bounded_linear.uniformly_continuous_on[OF bounded_linear_mult_left] .

lemma uniformly_continuous_on_cmul_left[continuous_intros]:
  fixes f :: 'a::real_normed_vector ⇒ 'b::real_normed_algebra
  assumes uniformly_continuous_on s f
    shows uniformly_continuous_on s (λx. c * f x)
by (metis assms bounded_linear.uniformly_continuous_on bounded_linear_mult_right)

lemma uniformly_continuous_on_norm[continuous_intros]:
  fixes f :: 'a :: metric_space ⇒ 'b :: real_normed_vector
  assumes uniformly_continuous_on s f
  shows uniformly_continuous_on s (λx. norm (f x))
  unfolding norm_conv_dist using assms
  by (intro uniformly_continuous_on_dist uniformly_continuous_on_const)

lemma uniformly_continuous_on_cmul[continuous_intros]:
  fixes f :: 'a::metric_space ⇒ 'b::real_normed_vector
  assumes uniformly_continuous_on s f
  shows uniformly_continuous_on s (λx. c *R f(x))
  using bounded_linear_scaleR_right assms
  by (rule bounded_linear.uniformly_continuous_on)

lemma dist_minus:
```

```
  fixes x y :: 'a::real_normed_vector
  shows dist (− x) (− y) = dist x y
  unfolding dist_norm minus_diff_minus norm_minus_cancel ..

lemma uniformly_continuous_on_minus[continuous_intros]:
  fixes f :: 'a::metric_space ⇒ 'b::real_normed_vector
  shows uniformly_continuous_on s f ⟹ uniformly_continuous_on s (λx. − f x)
  unfolding uniformly_continuous_on_def dist_minus .

lemma uniformly_continuous_on_add[continuous_intros]:
  fixes f g :: 'a::metric_space ⇒ 'b::real_normed_vector
  assumes uniformly_continuous_on s f
    and uniformly_continuous_on s g
  shows uniformly_continuous_on s (λx. f x + g x)
  using assms
  unfolding uniformly_continuous_on_sequentially
  unfolding dist_norm tendsto_norm_zero_iff add_diff_add
  by (auto intro: tendsto_add_zero)

lemma uniformly_continuous_on_diff[continuous_intros]:
  fixes f :: 'a::metric_space ⇒ 'b::real_normed_vector
  assumes uniformly_continuous_on s f
    and uniformly_continuous_on s g
  shows uniformly_continuous_on s (λx. f x − g x)
  using assms uniformly_continuous_on_add [of s f − g]
    by (simp add: fun_Compl_def uniformly_continuous_on_minus)
```

3.3.17 Arithmetic Preserves Topological Properties

```
lemma open_scaling[intro]:
  fixes s :: 'a::real_normed_vector set
  assumes c ≠ 0
    and open s
  shows open((λx. c *R x) ` s)
proof −
  {
    fix x
    assume x ∈ s
    then obtain e where e>0
        and e:∀x'. dist x' x < e ⟶ x' ∈ s using assms(2)[unfolded open_dist,
THEN bspec[where x=x]]
      by auto
    have e * |c| > 0
      using assms(1)[unfolded zero_less_abs_iff[symmetric]] ‹e>0› by auto
    moreover
    {
      fix y
      assume dist y (c *R x) < e * |c|
      then have norm (c *R ((1 / c) *R y − x)) < e * norm c
```

```
        by (simp add: ‹c ≠ 0› dist_norm scale_right_diff_distrib)
      then have norm ((1 / c) *R y − x) < e
        by (simp add: ‹c ≠ 0›)
      then have y ∈ (*R) c ‘ s
        using rev_image_eqI[of (1 / c) *R y s y (*R) c]
        by (simp add: ‹c ≠ 0› dist_norm e)
    }
    ultimately have ∃e>0. ∀x'. dist x' (c *R x) < e ⟶ x' ∈ (*R) c ‘ s
      by (rule_tac x=e * |c| in exI, auto)
  }
  then show ?thesis unfolding open_dist by auto
qed

lemma minus_image_eq_vimage:
  fixes A :: 'a::ab_group_add set
  shows (λx. − x) ‘ A = (λx. − x) −‘ A
  by (auto intro!: image_eqI [where f=λx. − x])

lemma open_negations:
  fixes S :: 'a::real_normed_vector set
  shows open S ⟹ open ((λx. − x) ‘ S)
  using open_scaling [of − 1 S] by simp

lemma open_translation:
  fixes S :: 'a::real_normed_vector set
  assumes open S
  shows open((λx. a + x) ‘ S)
proof −
  {
    fix x
    have continuous (at x) (λx. x − a)
      by (intro continuous_diff continuous_ident continuous_const)
  }
  moreover have {x. x − a ∈ S} = (+) a ‘ S
    by force
  ultimately show ?thesis
    by (metis assms continuous_open_vimage vimage_def)
qed

lemma open_translation_subtract:
  fixes S :: 'a::real_normed_vector set
  assumes open S
  shows open ((λx. x − a) ‘ S)
  using assms open_translation [of S − a] by (simp cong: image_cong_simp)

lemma open_neg_translation:
  fixes S :: 'a::real_normed_vector set
  assumes open S
  shows open((λx. a − x) ‘ S)
```

```
  using open_translation[OF open_negations[OF assms], of a]
  by (auto simp: image_image)

lemma open_affinity:
  fixes S :: 'a::real_normed_vector set
  assumes open S  c ≠ 0
  shows open ((λx. a + c *R x) ` S)
proof −
  have *: (λx. a + c *R x) = (λx. a + x) ∘ (λx. c *R x)
    unfolding o_def ..
  have (+) a ` (*R) c ` S = ((+) a ∘ (*R) c) ` S
    by auto
  then show ?thesis
    using assms open_translation[of (*R) c ` S a]
    unfolding *
    by auto
qed

lemma interior_translation:
  interior ((+) a ` S) = (+) a ` (interior S) for S :: 'a::real_normed_vector set
proof (rule set_eqI, rule)
  fix x
  assume x ∈ interior ((+) a ` S)
  then obtain e where e > 0 and e: ball x e ⊆ (+) a ` S
    unfolding mem_interior by auto
  then have ball (x − a) e ⊆ S
    unfolding subset_eq Ball_def mem_ball dist_norm
    by (auto simp: diff_diff_eq)
  then show x ∈ (+) a ` interior S
    unfolding image_iff
    by (metis ‹0 < e› add.commute diff_add_cancel mem_interior)
next
  fix x
  assume x ∈ (+) a ` interior S
  then obtain y e where e > 0 and e: ball y e ⊆ S and y: x = a + y
    unfolding image_iff Bex_def mem_interior by auto
  {
    fix z
    have *: a + y − z = y + a − z by auto
    assume z ∈ ball x e
    then have z − a ∈ S
      using e[unfolded subset_eq, THEN bspec[where x=z − a]]
      unfolding mem_ball dist_norm y group_add_class.diff_diff_eq2 *
      by auto
    then have z ∈ (+) a ` S
      unfolding image_iff by (auto intro!: bexI[where x=z − a])
  }
  then have ball x e ⊆ (+) a ` S
    unfolding subset_eq by auto
```

then show *x* ∈ *interior* ((+) *a* ‘ *S*)
  unfolding *mem_interior* **using** ‹*e* > *0*› **by** *auto*
qed

lemma *interior_translation_subtract*:
 interior ((λ*x*. *x* − *a*) ‘ *S*) = (λ*x*. *x* − *a*) ‘ *interior S* **for** *S* :: ′*a*::*real_normed_vector set*
 using *interior_translation* [*of* − *a*] **by** (*simp cong*: *image_cong_simp*)

lemma *compact_scaling*:
 fixes *s* :: ′*a*::*real_normed_vector set*
 assumes *compact s*
 shows *compact* ((λ*x*. *c* $*_R$ *x*) ‘ *s*)
proof −
 let *?f* = λ*x*. *scaleR c x*
 have ∗: *bounded_linear ?f* **by** (*rule bounded_linear_scaleR_right*)
 show *?thesis*
  using *compact_continuous_image*[*of s ?f*] *continuous_at_imp_continuous_on*[*of s ?f*]
  using *linear_continuous_at*[*OF* ∗] *assms*
  by *auto*
qed

lemma *compact_negations*:
 fixes *s* :: ′*a*::*real_normed_vector set*
 assumes *compact s*
 shows *compact* ((λ*x*. − *x*) ‘ *s*)
 using *compact_scaling* [*OF assms*, *of* − *1*] **by** *auto*

lemma *compact_sums*:
 fixes *s t* :: ′*a*::*real_normed_vector set*
 assumes *compact s*
  and *compact t*
 shows *compact* {*x* + *y* | *x y*. *x* ∈ *s* ∧ *y* ∈ *t*}
proof −
 have ∗: {*x* + *y* | *x y*. *x* ∈ *s* ∧ *y* ∈ *t*} = (λ*z*. *fst z* + *snd z*) ‘ (*s* × *t*)
  by (*fastforce simp*: *image_iff*)
 have *continuous_on* (*s* × *t*) (λ*z*. *fst z* + *snd z*)
  unfolding *continuous_on* **by** (*rule ballI*) (*intro tendsto_intros*)
 then show *?thesis*
  unfolding ∗ **using** *compact_continuous_image compact_Times* [*OF assms*] **by** *auto*
qed

lemma *compact_differences*:
 fixes *s t* :: ′*a*::*real_normed_vector set*
 assumes *compact s*
  and *compact t*

shows *compact* $\{x - y \mid x\ y.\ x \in s \wedge y \in t\}$
proof−
have $\{x - y \mid x\ y.\ x{\in}s \wedge y \in t\} = \{x + y \mid x\ y.\ x \in s \wedge y \in (uminus\ `\ t)\}$
using *diff_conv_add_uminus* **by** *force*
then show *?thesis*
using *compact_sums*[*OF assms*(*1*) *compact_negations*[*OF assms*(*2*)]] **by** *auto*
qed

lemma *compact_translation*:
compact ((+) *a* ‘ *s*) **if** *compact s* **for** *s* :: ′*a*::*real_normed_vector set*
proof −
have $\{x + y \mid x\ y.\ x \in s \wedge y \in \{a\}\} = (\lambda x.\ a + x)\ `\ s$
by *auto*
then show *?thesis*
using *compact_sums* [*OF that compact_sing* [*of a*]] **by** *auto*
qed

lemma *compact_translation_subtract*:
compact ((λ*x*. *x* − *a*) ‘ *s*) **if** *compact s* **for** *s* :: ′*a*::*real_normed_vector set*
using *that compact_translation* [*of s* − *a*] **by** (*simp cong*: *image_cong_simp*)

lemma *compact_affinity*:
fixes *s* :: ′*a*::*real_normed_vector set*
assumes *compact s*
shows *compact* $((\lambda x.\ a + c *_R x)\ `\ s)$
proof −
have $(+)\ a\ `\ (*_R)\ c\ `\ s = (\lambda x.\ a + c *_R x)\ `\ s$
by *auto*
then show *?thesis*
using *compact_translation*[*OF compact_scaling*[*OF assms*], *of a c*] **by** *auto*
qed

lemma *closed_scaling*:
fixes *S* :: ′*a*::*real_normed_vector set*
assumes *closed S*
shows *closed* $((\lambda x.\ c *_R x)\ `\ S)$
proof (*cases c* = *0*)
case *True* **then show** *?thesis*
by (*auto simp*: *image_constant_conv*)
next
case *False*
from *assms* **have** *closed* $((\lambda x.\ inverse\ c *_R x)\ -`\ S)$
by (*simp add*: *continuous_closed_vimage*)
also have $(\lambda x.\ inverse\ c *_R x)\ -`\ S = (\lambda x.\ c *_R x)\ `\ S$
using ‹*c* ≠ *0*› **by** (*auto elim*: *image_eqI* [*rotated*])
finally show *?thesis* **.**
qed

lemma *closed_negations*:

```
  fixes S :: 'a::real_normed_vector set
  assumes closed S
  shows closed ((λx. −x) ‘ S)
  using closed_scaling[OF assms, of − 1] by simp

lemma compact_closed_sums:
  fixes S :: 'a::real_normed_vector set
  assumes compact S and closed T
  shows closed (⋃x∈ S. ⋃y ∈ T. {x + y})
proof −
  let ?S = {x + y |x y. x ∈ S ∧ y ∈ T}
  {
    fix x l
    assume as: ∀ n. x n ∈ ?S (x ⟶ l) sequentially
    from as(1) obtain f where f: ∀ n. x n = fst (f n) + snd (f n) ∀ n. fst (f n)
∈ S ∀ n. snd (f n) ∈ T
      using choice[of λn y. x n = (fst y) + (snd y) ∧ fst y ∈ S ∧ snd y ∈ T] by
auto
    obtain l' r where l'∈S and r: strict_mono r and lr: (((λn. fst (f n)) ∘ r)
⟶ l') sequentially
      using assms(1)[unfolded compact_def, THEN spec[where x=λ n. fst (f n)]]
using f(2) by auto
    have ((λn. snd (f (r n))) ⟶ l − l') sequentially
      using tendsto_diff[OF LIMSEQ_subseq_LIMSEQ[OF as(2) r] lr] and f(1)
      unfolding o_def
      by auto
    then have l − l' ∈ T
      using assms(2)[unfolded closed_sequential_limits,
        THEN spec[where x=λ n. snd (f (r n))],
        THEN spec[where x=l − l']]
      using f(3)
      by auto
    then have l ∈ ?S
      using ‹l' ∈ S› by force
  }
  moreover have ?S = (⋃x∈ S. ⋃y ∈ T. {x + y})
    by force
  ultimately show ?thesis
    unfolding closed_sequential_limits
    by (metis (no_types, lifting))
qed

lemma closed_compact_sums:
  fixes S T :: 'a::real_normed_vector set
  assumes closed S compact T
  shows closed (⋃x∈ S. ⋃y ∈ T. {x + y})
proof −
  have (⋃x∈ T. ⋃y ∈ S. {x + y}) = (⋃x∈ S. ⋃y ∈ T. {x + y})
    by auto
```

then show *?thesis*
  using *compact_closed_sums*[*OF assms*(*2*,*1*)] **by** *simp*
qed

lemma *compact_closed_differences*:
  fixes *S T* :: *'a*::*real_normed_vector set*
  assumes *compact S closed T*
  shows *closed* ($\bigcup x \in S. \bigcup y \in T. \{x - y\}$)
proof –
  have ($\bigcup x \in S. \bigcup y \in$ *uminus* ‘ $T. \{x + y\}$) = ($\bigcup x \in S. \bigcup y \in T. \{x - y\}$)
    by *force*
  then show *?thesis*
    by (*metis assms closed_negations compact_closed_sums*)
qed

lemma *closed_compact_differences*:
  fixes *S T* :: *'a*::*real_normed_vector set*
  assumes *closed S compact T*
  shows *closed* ($\bigcup x \in S. \bigcup y \in T. \{x - y\}$)
proof –
  have ($\bigcup x \in S. \bigcup y \in$ *uminus* ‘ $T. \{x + y\}$) = $\{x - y \mid x\ y.\ x \in S \wedge y \in T\}$
    by *auto*
 then show *?thesis*
  using *closed_compact_sums*[*OF assms*(*1*) *compact_negations*[*OF assms*(*2*)]] **by** *simp*
qed

lemma *closed_translation*:
  closed ((+) *a* ‘ *S*) **if** *closed S* **for** *a* :: *'a*::*real_normed_vector*
proof –
  have ($\bigcup x \in \{a\}. \bigcup y \in S. \{x + y\}$) = ((+) *a* ‘ *S*) **by** *auto*
  then show *?thesis*
    using *compact_closed_sums* [*OF compact_sing* [*of a*] *that*] **by** *auto*
qed

lemma *closed_translation_subtract*:
  closed (($\lambda x.\ x - a$) ‘ *S*) **if** *closed S* **for** *a* :: *'a*::*real_normed_vector*
  using *that closed_translation* [*of S* $- a$] **by** (*simp cong*: *image_cong_simp*)

lemma *closure_translation*:
  closure ((+) *a* ‘ *s*) = (+) *a* ‘ *closure s* **for** *a* :: *'a*::*real_normed_vector*
proof –
  have ∗: (+) *a* ‘ (− *s*) = − (+) *a* ‘ *s*
    by (*auto intro*!: *image_eqI* [**where** $x = x - a$ **for** *x*])
  show *?thesis*
    using *interior_translation* [*of a* − *s*, *symmetric*]
    by (*simp add*: *closure_interior translation_Compl* ∗)
qed

lemma *closure_translation_subtract*:
closure ((λ*x*. *x* − *a*) ‘ *s*) = (λ*x*. *x* − *a*) ‘ *closure s* **for** *a* :: ′*a*::*real_normed_vector*
using *closure_translation* [*of* − *a s*] **by** (*simp cong*: *image_cong_simp*)

lemma *frontier_translation*:
frontier ((+) *a* ‘ *s*) = (+) *a* ‘ *frontier s* **for** *a* :: ′*a*::*real_normed_vector*
by (*auto simp add*: *frontier_def translation_diff interior_translation closure_translation*)

lemma *frontier_translation_subtract*:
frontier ((+) *a* ‘ *s*) = (+) *a* ‘ *frontier s* **for** *a* :: ′*a*::*real_normed_vector*
by (*auto simp add*: *frontier_def translation_diff interior_translation closure_translation*)

lemma *sphere_translation*:
sphere (*a* + *c*) *r* = (+) *a* ‘ *sphere c r* **for** *a* :: ′*n*::*real_normed_vector*
by (*auto simp*: *dist_norm algebra_simps intro*!: *image_eqI* [**where** *x* = *x* − *a* **for** *x*])

lemma *sphere_translation_subtract*:
sphere (*c* − *a*) *r* = (λ*x*. *x* − *a*) ‘ *sphere c r* **for** *a* :: ′*n*::*real_normed_vector*
using *sphere_translation* [*of* − *a c*] **by** (*simp cong*: *image_cong_simp*)

lemma *cball_translation*:
cball (*a* + *c*) *r* = (+) *a* ‘ *cball c r* **for** *a* :: ′*n*::*real_normed_vector*
by (*auto simp*: *dist_norm algebra_simps intro*!: *image_eqI* [**where** *x* = *x* − *a* **for** *x*])

lemma *cball_translation_subtract*:
cball (*c* − *a*) *r* = (λ*x*. *x* − *a*) ‘ *cball c r* **for** *a* :: ′*n*::*real_normed_vector*
using *cball_translation* [*of* − *a c*] **by** (*simp cong*: *image_cong_simp*)

lemma *ball_translation*:
ball (*a* + *c*) *r* = (+) *a* ‘ *ball c r* **for** *a* :: ′*n*::*real_normed_vector*
by (*auto simp*: *dist_norm algebra_simps intro*!: *image_eqI* [**where** *x* = *x* − *a* **for** *x*])

lemma *ball_translation_subtract*:
ball (*c* − *a*) *r* = (λ*x*. *x* − *a*) ‘ *ball c r* **for** *a* :: ′*n*::*real_normed_vector*
using *ball_translation* [*of* − *a c*] **by** (*simp cong*: *image_cong_simp*)

3.3.18 Homeomorphisms

lemma *homeomorphic_scaling*:
fixes *S* :: ′*a*::*real_normed_vector set*
assumes *c* ≠ *0*
shows *S homeomorphic* ((λ*x*. *c* $*_R$ *x*) ‘ *S*)
unfolding *homeomorphic_minimal*
apply (*rule_tac x*=λ*x*. *c* $*_R$ *x* **in** *exI*)
apply (*rule_tac x*=λ*x*. (*1* / *c*) $*_R$ *x* **in** *exI*)
using *assms* **by** (*auto simp*: *continuous_intros*)

lemma *homeomorphic_translation*:
  fixes *S* :: *'a::real_normed_vector set*
  shows *S homeomorphic* ((*λx. a + x*) ‘ *S*)
  unfolding *homeomorphic_minimal*
  apply (*rule_tac x=λx. a + x* **in** *exI*)
  apply (*rule_tac x=λx. −a + x* **in** *exI*)
  by (*auto simp*: *continuous_intros*)

lemma *homeomorphic_affinity*:
  fixes *S* :: *'a::real_normed_vector set*
  assumes *c ≠ 0*
  shows *S homeomorphic* ((*λx. a + c* $*_R$ *x*) ‘ *S*)
proof −
  have ∗: (+) *a* ‘ (∗$_R$) *c* ‘ *S* = (*λx. a + c* $*_R$ *x*) ‘ *S* **by** *auto*
  show *?thesis*
   by (*metis* ∗ *assms homeomorphic_scaling homeomorphic_trans homeomorphic_translation*)
qed

lemma *homeomorphic_balls*:
  fixes *a b* ::*'a::real_normed_vector*
  assumes *0 < d 0 < e*
  shows (*ball a d*) *homeomorphic* (*ball b e*) (**is** *?th*)
    and (*cball a d*) *homeomorphic* (*cball b e*) (**is** *?cth*)
proof −
  show *?th* **unfolding** *homeomorphic_minimal*
    apply(*rule_tac x=λx. b* + (*e/d*) $*_R$ (*x* − *a*) **in** *exI*)
    apply(*rule_tac x=λx. a* + (*d/e*) $*_R$ (*x* − *b*) **in** *exI*)
    using *assms*
   by (*auto intro*!: *continuous_intros simp*: *dist_commute dist_norm pos_divide_less_eq*)
  show *?cth* **unfolding** *homeomorphic_minimal*
    apply(*rule_tac x=λx. b* + (*e/d*) $*_R$ (*x* − *a*) **in** *exI*)
    apply(*rule_tac x=λx. a* + (*d/e*) $*_R$ (*x* − *b*) **in** *exI*)
    using *assms*
   by (*auto intro*!: *continuous_intros simp*: *dist_commute dist_norm pos_divide_le_eq*)
qed

lemma *homeomorphic_spheres*:
  fixes *a b* ::*'a::real_normed_vector*
  assumes *0 < d 0 < e*
  shows (*sphere a d*) *homeomorphic* (*sphere b e*)
unfolding *homeomorphic_minimal*
    apply(*rule_tac x=λx. b* + (*e/d*) $*_R$ (*x* − *a*) **in** *exI*)
    apply(*rule_tac x=λx. a* + (*d/e*) $*_R$ (*x* − *b*) **in** *exI*)
    using *assms*
   by (*auto intro*!: *continuous_intros simp*: *dist_commute dist_norm pos_divide_less_eq*)

lemma *homeomorphic_ball01_UNIV*:
  ball (*0::'a::real_normed_vector*) *1 homeomorphic* (*UNIV*:: *'a set*)

```
  (is ?B homeomorphic ?U)
proof
  have x ∈ (λz. z /R (1 − norm z)) ‘ ball 0 1 for x::'a
    apply (rule_tac x=x /R (1 + norm x) in image_eqI)
     apply (auto simp: field_split_simps)
    using norm_ge_zero [of x] apply linarith+
    done
  then show (λz::'a. z /R (1 − norm z)) ‘ ?B = ?U
    by blast
  have x ∈ range (λz. (1 / (1 + norm z)) *R z) if norm x < 1 for x::'a
    using that
   by (rule_tac x=x /R (1 − norm x) in image_eqI) (auto simp: field_split_simps)
  then show (λz::'a. z /R (1 + norm z)) ‘ ?U = ?B
    by (force simp: field_split_simps dest: add_less_zeroD)
  show continuous_on (ball 0 1) (λz. z /R (1 − norm z))
    by (rule continuous_intros | force)+
  have 0: ⋀z. 1 + norm z ≠ 0
    by (metis (no_types) le_add_same_cancel1 norm_ge_zero not_one_le_zero)
  then show continuous_on UNIV (λz. z /R (1 + norm z))
    by (auto intro!: continuous_intros)
  show ⋀x. x ∈ ball 0 1 ⟹
         x /R (1 − norm x) /R (1 + norm (x /R (1 − norm x))) = x
    by (auto simp: field_split_simps)
  show ⋀y. y /R (1 + norm y) /R (1 − norm (y /R (1 + norm y))) = y
    using 0 by (auto simp: field_split_simps)
qed

proposition homeomorphic_ball_UNIV:
  fixes a ::'a::real_normed_vector
  assumes 0 < r shows ball a r homeomorphic (UNIV:: 'a set)
 using assms homeomorphic_ball01_UNIV homeomorphic_balls(1) homeomorphic_trans
zero_less_one by blast
```

3.3.19 Discrete

```
lemma finite_implies_discrete:
  fixes S :: 'a::topological_space set
  assumes finite (f ‘ S)
  shows (∀x ∈ S. ∃e>0. ∀y. y ∈ S ∧ f y ≠ f x ⟶ e ≤ norm (f y − f x))
proof −
  have ∃e>0. ∀y. y ∈ S ∧ f y ≠ f x ⟶ e ≤ norm (f y − f x) if x ∈ S for x
  proof (cases f ‘ S − {f x} = {})
    case True
    with zero_less_numeral show ?thesis
      by (fastforce simp add: Set.image_subset_iff cong: conj_cong)
  next
    case False
    then obtain z where z ∈ S f z ≠ f x
      by blast
```

moreover have *finn*: *finite* {*norm* (*z* − *f x*) |*z*. *z* ∈ *f* ' *S* − {*f x*}}
  using *assms* **by** *simp*
ultimately have ∗: *0* < *Inf*{*norm*(*z* − *f x*) | *z*. *z* ∈ *f* ' *S* − {*f x*}}
  by (*force intro*: *finite_imp_less_Inf*)
show *?thesis*
  by (*force intro*!: ∗ *cInf_le_finite* [*OF finn*])
qed
with *assms* **show** *?thesis*
  by *blast*
qed

3.3.20 Completeness of "Isometry" (up to constant bounds)

lemma *cauchy_isometric*:— TODO: rename lemma to *Cauchy_isometric*
  assumes *e*: *e* > *0*
    and *s*: *subspace s*
    and *f*: *bounded_linear f*
    and *normf*: ∀ *x*∈*s*. *norm* (*f x*) ≥ *e* ∗ *norm x*
    and *xs*: ∀ *n*. *x n* ∈ *s*
    and *cf*: *Cauchy* (*f* ∘ *x*)
  shows *Cauchy x*
proof −
  interpret *f*: *bounded_linear f* **by** *fact*
  have ∃ *N*. ∀ *n*≥*N*. *norm* (*x n* − *x N*) < *d* **if** *d* > *0* **for** *d* :: *real*
  proof −
    from *that* **obtain** *N* **where** *N*: ∀ *n*≥*N*. *norm* (*f* (*x n*) − *f* (*x N*)) < *e* ∗ *d*
      using *cf*[*unfolded Cauchy_def o_def dist_norm*, *THEN spec*[**where** *x*=*e*∗*d*]] *e*
      by *auto*
    have *norm* (*x n* − *x N*) < *d* **if** *n* ≥ *N* **for** *n*
    proof −
      have *e* ∗ *norm* (*x n* − *x N*) ≤ *norm* (*f* (*x n* − *x N*))
        using *subspace_diff*[*OF s*, *of x n x N*]
        using *xs*[*THEN spec*[**where** *x*=*N*]] **and** *xs*[*THEN spec*[**where** *x*=*n*]]
        using *normf*[*THEN bspec*[**where** *x*=*x n* − *x N*]]
        by *auto*
      also have *norm* (*f* (*x n* − *x N*)) < *e* ∗ *d*
        using ‹*N* ≤ *n*› *N* **unfolding** *f*.*diff*[*symmetric*] **by** *auto*
      finally show *?thesis*
        using ‹*e*>*0*› **by** *simp*
    qed
    then show *?thesis* **by** *auto*
  qed
  then show *?thesis*
    by (*simp add*: *Cauchy_altdef2 dist_norm*)
qed

lemma *complete_isometric_image*:
  assumes *0* < *e*
    and *s*: *subspace s*

```
    and f: bounded_linear f
    and normf: ∀x∈s. norm(f x) ≥ e * norm(x)
    and cs: complete s
  shows complete (f ` s)
proof -
  have ∃l∈f ` s. (g ⟶ l) sequentially
    if as:∀n::nat. g n ∈ f ` s and cfg:Cauchy g for g
  proof -
    from that obtain x where ∀n. x n ∈ s ∧ g n = f (x n)
      using choice[of λ n xa. xa ∈ s ∧ g n = f xa] by auto
    then have x: ∀n. x n ∈ s ∀n. g n = f (x n) by auto
    then have f ∘ x = g by (simp add: fun_eq_iff)
    then obtain l where l∈s and l:(x ⟶ l) sequentially
      using cs[unfolded complete_def, THEN spec[where x=x]]
      using cauchy_isometric[OF ‹0 < e› s f normf] and cfg and x(1)
      by auto
    then show ?thesis
      using linear_continuous_at[OF f, unfolded continuous_at_sequentially, THEN
spec[where x=x], of l]
      by (auto simp: ‹f ∘ x = g›)
  qed
  then show ?thesis
    unfolding complete_def by auto
qed
```

3.3.21 Connected Normed Spaces

```
lemma compact_components:
  fixes s :: 'a::heine_borel set
  shows ⟦compact s; c ∈ components s⟧ ⟹ compact c
by (meson bounded_subset closed_components in_components_subset compact_eq_bounded_closed)

lemma discrete_subset_disconnected:
  fixes S :: 'a::topological_space set
  fixes t :: 'b::real_normed_vector set
  assumes conf: continuous_on S f
      and no: ⋀x. x ∈ S ⟹ ∃e>0. ∀y. y ∈ S ∧ f y ≠ f x ⟶ e ≤ norm (f y −
f x)
   shows f ` S ⊆ {y. connected_component_set (f ` S) y = {y}}
proof -
  { fix x assume x: x ∈ S
    then obtain e where e>0 and ele: ⋀y. ⟦y ∈ S; f y ≠ f x⟧ ⟹ e ≤ norm (f
y − f x)
      using conf no [OF x] by auto
    then have e2: 0 ≤ e / 2
      by simp
    define F where F ≡ connected_component_set (f ` S) (f x)
    have False if y ∈ S and ccs: f y ∈ F and not: f y ≠ f x for y
    proof -
```

define *C* **where** $C \equiv$ *cball* (*f x*) (*e*/*2*)
define *D* **where** $D \equiv -$ *ball* (*f x*) *e*
have *disj*: $C \cap D = \{\}$
  unfolding *C_def D_def* **using** ‹*0* < *e*› **by** *fastforce*
moreover have *FCD*: $F \subseteq C \cup D$
proof −
  have $t \in C \vee t \in D$ **if** $t \in F$ **for** *t*
  proof −
    obtain *y* **where** $y \in S$ $t = f\, y$
      using *F_def* ‹$t \in F$› *connected_component_in* **by** *blast*
    then show *?thesis*
      by (*metis C_def ComplI D_def centre_in_cball dist_norm e2 ele mem_ball norm_minus_commute not_le*)
  qed
  then show *?thesis*
    by *auto*
qed
ultimately have $C \cap F = \{\} \vee D \cap F = \{\}$
  using *connected_closed* [*of F*] ‹*e*>*0*› *not*
  unfolding *C_def D_def*
  by (*metis Elementary_Metric_Spaces.open_ball F_def closed_cball connected_connected_component inf_bot_left open_closed*)
moreover have $C \cap F \neq \{\}$
  unfolding *disjoint_iff*
  by (*metis FCD ComplD image_eqI mem_Collect_eq subsetD x D_def F_def Un_iff* ‹*0* < *e*› *centre_in_ball connected_component_refl_eq*)
moreover have $D \cap F \neq \{\}$
  unfolding *disjoint_iff*
  by (*metis ComplI D_def ccs dist_norm ele mem_ball norm_minus_commute not not_le that*(*1*))
ultimately show *?thesis* **by** *metis*
qed
moreover have *connected_component_set* (*f* ‘ *S*) (*f x*) ⊆ *f* ‘ *S*
  by (*auto simp*: *connected_component_in*)
ultimately have *connected_component_set* (*f* ‘ *S*) (*f x*) = {*f x*}
  by (*auto simp*: *x F_def*)
}
with *assms* **show** *?thesis*
  by *blast*
qed

lemma *continuous_disconnected_range_constant_eq*:
  (*connected S* ⟷
    (∀ *f*::′*a*::*topological_space* ⇒ ′*b*::*real_normed_algebra_1*.
     ∀ *t*. *continuous_on S f* ∧ *f* ‘ *S* ⊆ *t* ∧ (∀ *y* ∈ *t*. *connected_component_set t y* = {*y*})
     ⟶ *f constant_on S*)) (**is** *?thesis1*)
and *continuous_discrete_range_constant_eq*:
  (*connected S* ⟷

```
        (∀f::'a::topological_space ⇒ 'b::real_normed_algebra_1.
         continuous_on S f ∧
          (∀x ∈ S. ∃e. 0 < e ∧ (∀y. y ∈ S ∧ (f y ≠ f x) ⟶ e ≤ norm(f y − f
x)))
         ⟶ f constant_on S)) (is ?thesis2)
  and continuous_finite_range_constant_eq:
      (connected S ⟷
        (∀f::'a::topological_space ⇒ 'b::real_normed_algebra_1.
         continuous_on S f ∧ finite (f ` S)
         ⟶ f constant_on S)) (is ?thesis3)
proof −
  have ∗: ⋀s t u v. ⟦s ⟹ t; t ⟹ u; u ⟹ v; v ⟹ s⟧
    ⟹ (s ⟷ t) ∧ (s ⟷ u) ∧ (s ⟷ v)
    by blast
  have ?thesis1 ∧ ?thesis2 ∧ ?thesis3
    apply (rule ∗)
    using continuous_disconnected_range_constant apply metis
    apply clarify
    apply (frule discrete_subset_disconnected; blast)
    apply (blast dest: finite_implies_discrete)
    apply (blast intro!: finite_range_constant_imp_connected)
    done
  then show ?thesis1 ?thesis2 ?thesis3
    by blast+
qed

lemma continuous_discrete_range_constant:
  fixes f :: 'a::topological_space ⇒ 'b::real_normed_algebra_1
  assumes S: connected S
      and continuous_on S f
      and ⋀x. x ∈ S ⟹ ∃e>0. ∀y. y ∈ S ∧ f y ≠ f x ⟶ e ≤ norm (f y − f x)
    shows f constant_on S
  using continuous_discrete_range_constant_eq [THEN iffD1, OF S] assms by blast

lemma continuous_finite_range_constant:
  fixes f :: 'a::topological_space ⇒ 'b::real_normed_algebra_1
  assumes connected S
      and continuous_on S f
      and finite (f ` S)
    shows f constant_on S
  using assms continuous_finite_range_constant_eq  by blast

end
```

3.4 Linear Decision Procedure for Normed Spaces

```
theory Norm_Arith
imports HOL−Library.Sum_of_Squares
begin
```

lemma *sum_sqs_eq*:
fixes x::'a::*idom* **shows** $x * x + y * y = x * (y * 2) \Longrightarrow y = x$
by *algebra*

lemma *norm_cmul_rule_thm*:
fixes x :: 'a::*real_normed_vector*
shows $b \geq norm\ x \Longrightarrow |c| * b \geq norm\ (scaleR\ c\ x)$
unfolding *norm_scaleR*
apply (*erule mult_left_mono*)
apply *simp*
done

lemma *norm_add_rule_thm*:
fixes *x1 x2* :: 'a::*real_normed_vector*
shows $norm\ x1 \leq b1 \Longrightarrow norm\ x2 \leq b2 \Longrightarrow norm\ (x1 + x2) \leq b1 + b2$
by (*rule order_trans* [*OF norm_triangle_ineq add_mono*])

lemma *ge_iff_diff_ge_0*:
fixes a :: 'a::*linordered_ring*
shows $a \geq b \equiv a - b \geq 0$
by (*simp add*: *field_simps*)

lemma *pth_1*:
fixes x :: 'a::*real_normed_vector*
shows $x \equiv scaleR\ 1\ x$ **by** *simp*

lemma *pth_2*:
fixes x :: 'a::*real_normed_vector*
shows $x - y \equiv x + -y$
by (*atomize* (*full*)) *simp*

lemma *pth_3*:
fixes x :: 'a::*real_normed_vector*
shows $- x \equiv scaleR\ (-1)\ x$
by *simp*

lemma *pth_4*:
fixes x :: 'a::*real_normed_vector*
shows $scaleR\ 0\ x \equiv 0$
and $scaleR\ c\ 0 = (0::'a)$
by *simp_all*

lemma *pth_5*:
fixes x :: 'a::*real_normed_vector*
shows $scaleR\ c\ (scaleR\ d\ x) \equiv scaleR\ (c * d)\ x$
by *simp*

lemma *pth_6*:
  fixes *x* :: *'a::real_normed_vector*
  shows *scaleR c (x + y) ≡ scaleR c x + scaleR c y*
  by (*simp add*: *scaleR_right_distrib*)

lemma *pth_7*:
  fixes *x* :: *'a::real_normed_vector*
  shows *0 + x ≡ x*
    and *x + 0 ≡ x*
  by *simp_all*

lemma *pth_8*:
  fixes *x* :: *'a::real_normed_vector*
  shows *scaleR c x + scaleR d x ≡ scaleR (c + d) x*
  by (*simp add*: *scaleR_left_distrib*)

lemma *pth_9*:
  fixes *x* :: *'a::real_normed_vector*
  shows (*scaleR c x + z*) + *scaleR d x ≡ scaleR (c + d) x + z*
    and *scaleR c x +* (*scaleR d x + z*) *≡ scaleR (c + d) x + z*
    and (*scaleR c x + w*) + (*scaleR d x + z*) *≡ scaleR (c + d) x + (w + z)*
  by (*simp_all add*: *algebra_simps*)

lemma *pth_a*:
  fixes *x* :: *'a::real_normed_vector*
  shows *scaleR 0 x + y ≡ y*
  by *simp*

lemma *pth_b*:
  fixes *x* :: *'a::real_normed_vector*
  shows *scaleR c x + scaleR d y ≡ scaleR c x + scaleR d y*
    and (*scaleR c x + z*) + *scaleR d y ≡ scaleR c x + (z + scaleR d y)*
    and *scaleR c x +* (*scaleR d y + z*) *≡ scaleR c x + (scaleR d y + z)*
    and (*scaleR c x + w*) + (*scaleR d y + z*) *≡ scaleR c x + (w + (scaleR d y + z))*
  by (*simp_all add*: *algebra_simps*)

lemma *pth_c*:
  fixes *x* :: *'a::real_normed_vector*
  shows *scaleR c x + scaleR d y ≡ scaleR d y + scaleR c x*
    and (*scaleR c x + z*) + *scaleR d y ≡ scaleR d y + (scaleR c x + z)*
    and *scaleR c x +* (*scaleR d y + z*) *≡ scaleR d y + (scaleR c x + z)*
    and (*scaleR c x + w*) + (*scaleR d y + z*) *≡ scaleR d y + ((scaleR c x + w) + z)*
  by (*simp_all add*: *algebra_simps*)

lemma *pth_d*:
  fixes *x* :: *'a::real_normed_vector*

shows $x + 0 \equiv x$
by *simp*

lemma *norm_imp_pos_and_ge*:
fixes x :: *'a::real_normed_vector*
shows *norm* $x \equiv n \Longrightarrow$ *norm* $x \geq 0 \wedge n \geq$ *norm* x
by *atomize auto*

lemma *real_eq_0_iff_le_ge_0*:
fixes x :: *real*
shows $x = 0 \equiv x \geq 0 \wedge - x \geq 0$
by *arith*

lemma *norm_pths*:
fixes x :: *'a::real_normed_vector*
shows $x = y \longleftrightarrow$ *norm* $(x - y) \leq 0$
and $x \neq y \longleftrightarrow \neg$ (*norm* $(x - y) \leq 0$)
using *norm_ge_zero*[*of* $x - y$] **by** *auto*

lemmas *arithmetic_simps* =
arith_simps
add_numeral_special
add_neg_numeral_special
mult_1_left
mult_1_right

ML_file ‹*normarith.ML*›

method_setup *norm* = ‹
Scan.succeed (*SIMPLE_METHOD' o NormArith.norm_arith_tac*)
› *prove simple linear statements about vector norms*

Hence more metric properties.

proposition *dist_triangle_add*:
fixes $x\ y\ x'\ y'$ :: *'a::real_normed_vector*
shows *dist* $(x + y)\ (x' + y') \leq$ *dist* $x\ x'$ + *dist* $y\ y'$
by *norm*

lemma *dist_triangle_add_half*:
fixes $x\ x'\ y\ y'$ :: *'a::real_normed_vector*
shows *dist* $x\ x' < e / 2 \Longrightarrow$ *dist* $y\ y' < e / 2 \Longrightarrow$ *dist*$(x + y)\ (x' + y') < e$
by *norm*

end

Chapter 4

Vector Analysis

theory *Topology_Euclidean_Space*
  imports
    Elementary_Normed_Spaces
    Linear_Algebra
    Norm_Arith
begin

4.1 Elementary Topology in Euclidean Space

lemma *euclidean_dist_l2*:
  fixes *x y* :: *'a* :: *euclidean_space*
  shows *dist x y* = *L2_set* (λ*i*. *dist* (*x* • *i*) (*y* • *i*)) *Basis*
  unfolding *dist_norm norm_eq_sqrt_inner L2_set_def*
  by (*subst euclidean_inner*) (*simp add*: *power2_eq_square inner_diff_left*)

lemma *norm_nth_le*: *norm* (*x* • *i*) ≤ *norm x* **if** *i* ∈ *Basis*
proof −
  have $(x \cdot i)^2 = (\sum i{\in}\{i\}.\ (x \cdot i)^2)$
    by *simp*
  also have ... $\leq (\sum i{\in}Basis.\ (x \cdot i)^2)$
    by (*intro sum_mono2*) (*auto simp*: *that*)
  finally show *?thesis*
    unfolding *norm_conv_dist euclidean_dist_l2*[*of x*] *L2_set_def*
    by (*auto intro*!: *real_le_rsqrt*)
qed

4.1.1 Continuity of the representation WRT an orthogonal basis

lemma *orthogonal_Basis*: *pairwise orthogonal Basis*
  by (*simp add*: *inner_not_same_Basis orthogonal_def pairwise_def*)

lemma *representation_bound*:
  fixes *B* :: *'N*::*real_inner set*

assumes *finite B independent B b* ∈ *B* **and** *orth*: *pairwise orthogonal B*
obtains *m* **where** $m > 0$ $\bigwedge x.\ x \in span\ B \Longrightarrow |representation\ B\ x\ b| \leq m * norm\ x$
proof
fix *x*
assume *x*: *x* ∈ *span B*
have $b \neq 0$
using ‹*independent B*› ‹*b* ∈ *B*› *dependent_zero* **by** *blast*
have [*simp*]: $b \cdot b' = (if\ b' = b\ then\ (norm\ b)^2\ else\ 0)$
if *b* ∈ *B b*′ ∈ *B* **for** *b b*′
using *orth* **by** (*simp add*: *orthogonal_def pairwise_def norm_eq_sqrt_inner that*)
have $norm\ x = norm\ (\sum b{\in}B.\ representation\ B\ x\ b *_R b)$
using *real_vector.sum_representation_eq* [*OF* ‹*independent B*› *x* ‹*finite B*› *order_refl*]
by *simp*
also have $\ldots = sqrt\ ((\sum b{\in}B.\ representation\ B\ x\ b *_R b) \cdot (\sum b{\in}B.\ representation\ B\ x\ b *_R b))$
by (*simp add*: *norm_eq_sqrt_inner*)
also have $\ldots = sqrt\ (\sum b{\in}B.\ (representation\ B\ x\ b *_R b) \cdot (representation\ B\ x\ b *_R b))$
using ‹*finite B*›
by (*simp add*: *inner_sum_left inner_sum_right if_distrib* [*of* λ*x*. _ ∗ *x*] *cong*: *if_cong sum.cong_simp*)
also have $\ldots = sqrt\ (\sum b{\in}B.\ (norm\ (representation\ B\ x\ b *_R b))^2)$
by (*simp add*: *mult.commute mult.left_commute power2_eq_square*)
also have $\ldots = sqrt\ (\sum b{\in}B.\ (representation\ B\ x\ b)^2 * (norm\ b)^2)$
by (*simp add*: *norm_mult power_mult_distrib*)
finally have $norm\ x = sqrt\ (\sum b{\in}B.\ (representation\ B\ x\ b)^2 * (norm\ b)^2)$ **.**
moreover
have $sqrt\ ((representation\ B\ x\ b)^2 * (norm\ b)^2) \leq sqrt\ (\sum b{\in}B.\ (representation\ B\ x\ b)^2 * (norm\ b)^2)$
using ‹*b* ∈ *B*› ‹*finite B*› **by** (*auto intro*: *member_le_sum*)
then have $|representation\ B\ x\ b| \leq (1\ /\ norm\ b) * sqrt\ (\sum b{\in}B.\ (representation\ B\ x\ b)^2 * (norm\ b)^2)$
using ‹$b \neq 0$› **by** (*simp add*: *field_split_simps real_sqrt_mult del*: *real_sqrt_le_iff*)
ultimately show $|representation\ B\ x\ b| \leq (1\ /\ norm\ b) * norm\ x$
by *simp*
next
show $0 < 1\ /\ norm\ b$
using ‹*independent B*› ‹*b* ∈ *B*› *dependent_zero* **by** *auto*
qed

lemma *continuous_on_representation*:
fixes *B* :: ′*N*::*euclidean_space set*
assumes *finite B independent B b* ∈ *B pairwise orthogonal B*
shows *continuous_on* (*span B*) (λ*x*. *representation B x b*)
proof
show $\exists d{>}0.\ \forall x'{\in}span\ B.\ dist\ x'\ x < d \longrightarrow dist\ (representation\ B\ x'\ b)\ (representation\ B\ x\ b) \leq e$

```
    if e > 0 x ∈ span B for x e
  proof –
    obtain m where m > 0 and m: ⋀x. x ∈ span B ⟹ |representation B x b|
≤ m ∗ norm x
      using assms representation_bound by blast
    show ?thesis
      unfolding dist_norm
    proof (intro exI conjI ballI impI)
      show e/m > 0
        by (simp add: ‹e > 0› ‹m > 0›)
      show norm (representation B x' b − representation B x b) ≤ e
        if x': x' ∈ span B and less: norm (x'−x) < e/m for x'
      proof –
        have |representation B (x'−x) b| ≤ m ∗ norm (x'−x)
          using m [of x'−x] ‹x ∈ span B› span_diff x' by blast
        also have . . . < e
          by (metis ‹m > 0› less mult.commute pos_less_divide_eq)
        finally have |representation B (x'−x) b| ≤ e by simp
        then show ?thesis
          by (simp add: ‹x ∈ span B› ‹independent B› representation_diff x')
      qed
    qed
  qed
qed
```

4.1.2 Balls in Euclidean Space

```
lemma cball_subset_cball_iff:
  fixes a :: 'a :: euclidean_space
  shows cball a r ⊆ cball a' r' ⟷ dist a a' + r ≤ r' ∨ r < 0
    (is ?lhs ⟷ ?rhs)
proof
  assume ?lhs
  then show ?rhs
  proof (cases r < 0)
    case True
    then show ?rhs by simp
  next
    case False
    then have [simp]: r ≥ 0 by simp
    have norm (a − a') + r ≤ r'
    proof (cases a = a')
      case True
      then show ?thesis
          using subsetD [where c = a + r ∗R (SOME i. i ∈ Basis), OF ‹?lhs›]
subsetD [where c = a, OF ‹?lhs›]
        by (force simp: SOME_Basis dist_norm)
    next
      case False
```

have *norm* $(a' - (a + (r / norm (a - a')) *_R (a - a'))) = norm (a' - a - (r / norm (a - a')) *_R (a - a'))$
by (*simp add*: *algebra_simps*)
also have ... $= norm ((-1 - (r / norm (a - a'))) *_R (a - a'))$
by (*simp add*: *algebra_simps*)
also from ‹$a \neq a'$› **have** ... $= |- norm (a - a') - r|$
by *simp* (*simp add*: *field_simps*)
finally have [*simp*]: $norm (a' - (a + (r / norm (a - a')) *_R (a - a'))) = |norm (a - a') + r|$
by *linarith*
from ‹$a \neq a'$› **show** *?thesis*
using *subsetD* [**where** $c = a' + (1 + r / norm(a - a')) *_R (a - a')$, *OF* ‹*?lhs*›]
by (*simp add*: *dist_norm scaleR_add_left*)
qed
then show *?rhs*
by (*simp add*: *dist_norm*)
qed
qed *metric*

lemma *cball_subset_ball_iff*: $cball\ a\ r \subseteq ball\ a'\ r' \longleftrightarrow dist\ a\ a' + r < r' \vee r < 0$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
for a :: *'a::euclidean_space*
proof
assume *?lhs*
then show *?rhs*
proof (*cases* $r < 0$)
case *True* **then**
show *?rhs* **by** *simp*
next
case *False*
then have [*simp*]: $r \geq 0$ **by** *simp*
have $norm (a - a') + r < r'$
proof (*cases* $a = a'$)
case *True*
then show *?thesis*
using *subsetD* [**where** $c = a + r *_R (SOME\ i.\ i \in Basis)$, *OF* ‹*?lhs*›] *subsetD* [**where** $c = a$, *OF* ‹*?lhs*›]
by (*force simp*: *SOME_Basis dist_norm*)
next
case *False*
have *False* **if** $norm (a - a') + r \geq r'$
proof −
from *that* **have** $|r' - norm (a - a')| \leq r$
by (*simp split*: *abs_split*)
(*metis* ‹$0 \leq r$› ‹*?lhs*› *centre_in_cball dist_commute dist_norm less_asym mem_ball subset_eq*)
then show *?thesis*
using *subsetD* [**where** $c = a + (r' / norm(a - a') - 1) *_R (a - a')$,

```
OF ‹?lhs›] ‹a ≠ a'›
        apply (simp add: dist_norm)
        apply (simp add: scaleR_left_diff_distrib)
        apply (simp add: field_simps)
        done
     qed
     then show ?thesis by force
   qed
   then show ?rhs by (simp add: dist_norm)
 qed
next
 assume ?rhs
 then show ?lhs
   by metric
qed

lemma ball_subset_cball_iff: ball a r ⊆ cball a' r' ⟷ dist a a' + r ≤ r' ∨ r ≤ 0
 (is ?lhs = ?rhs)
 for a :: 'a::euclidean_space
proof (cases r ≤ 0)
 case True
 then show ?thesis
   by metric
next
 case False
 show ?thesis
 proof
   assume ?lhs
   then have (cball a r ⊆ cball a' r')
     by (metis False closed_cball closure_ball closure_closed closure_mono not_less)
   with False show ?rhs
     by (fastforce iff: cball_subset_cball_iff)
 next
   assume ?rhs
   with False show ?lhs
     by metric
 qed
qed

lemma ball_subset_ball_iff:
 fixes a :: 'a :: euclidean_space
 shows ball a r ⊆ ball a' r' ⟷ dist a a' + r ≤ r' ∨ r ≤ 0
      (is ?lhs = ?rhs)
proof (cases r ≤ 0)
 case True then show ?thesis
   by metric
next
 case False show ?thesis
 proof
```

```
    assume ?lhs
    then have 0 < r'
      using False by metric
    then have (cball a r ⊆ cball a' r')
      by (metis False‹?lhs› closure_ball closure_mono not_less)
    then show ?rhs
      using False cball_subset_cball_iff by fastforce
  qed metric
qed


lemma ball_eq_ball_iff:
  fixes x :: 'a :: euclidean_space
  shows ball x d = ball y e ⟷ d ≤ 0 ∧ e ≤ 0 ∨ x=y ∧ d=e
        (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
  proof (cases d ≤ 0 ∨ e ≤ 0)
    case True
      with ‹?lhs› show ?rhs
        by safe (simp_all only: ball_eq_empty [of y e, symmetric] ball_eq_empty [of x
d, symmetric])
  next
    case False
    with ‹?lhs› show ?rhs
     apply (auto simp: set_eq_subset ball_subset_ball_iff dist_norm norm_minus_commute
algebra_simps)
     apply (metis add_le_same_cancel1 le_add_same_cancel1 norm_ge_zero norm_pths(2)
order_trans)
      apply (metis add_increasing2 add_le_imp_le_right eq_iff norm_ge_zero)
      done
  qed
next
  assume ?rhs then show ?lhs
    by (auto simp: set_eq_subset ball_subset_ball_iff)
qed

lemma cball_eq_cball_iff:
  fixes x :: 'a :: euclidean_space
  shows cball x d = cball y e ⟷ d < 0 ∧ e < 0 ∨ x=y ∧ d=e
        (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
  proof (cases d < 0 ∨ e < 0)
    case True
      with ‹?lhs› show ?rhs
        by safe (simp_all only: cball_eq_empty [of y e, symmetric] cball_eq_empty [of
```

```
x d, symmetric])
  next
    case False
    with ‹?lhs› show ?rhs
     apply (auto simp: set_eq_subset cball_subset_cball_iff dist_norm norm_minus_commute
algebra_simps)
     apply (metis add_le_same_cancel1 le_add_same_cancel1 norm_ge_zero norm_pths(2)
order_trans)
      apply (metis add_increasing2 add_le_imp_le_right eq_iff norm_ge_zero)
      done
  qed
next
  assume ?rhs then show ?lhs
    by (auto simp: set_eq_subset cball_subset_cball_iff)
qed

lemma ball_eq_cball_iff:
  fixes x :: 'a :: euclidean_space
  shows ball x d = cball y e ⟷ d ≤ 0 ∧ e < 0 (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
      apply (auto simp: set_eq_subset ball_subset_cball_iff cball_subset_ball_iff alge-
bra_simps)
   apply (metis add_increasing2 add_le_cancel_right add_less_same_cancel1 dist_not_less_zero
less_le_trans zero_le_dist)
    apply (metis add_less_same_cancel1 dist_not_less_zero less_le_trans not_le)
    using ‹?lhs› ball_eq_empty cball_eq_empty apply blast+
    done
next
  assume ?rhs then show ?lhs by auto
qed

lemma cball_eq_ball_iff:
  fixes x :: 'a :: euclidean_space
  shows cball x d = ball y e ⟷ d < 0 ∧ e ≤ 0
  using ball_eq_cball_iff by blast

lemma finite_ball_avoid:
  fixes S :: 'a :: euclidean_space set
  assumes open S finite X p ∈ S
  shows ∃ e>0. ∀ w∈ball p e. w∈S ∧ (w≠p ⟶ w∉X)
proof −
  obtain e1 where 0 < e1 and e1_b:ball p e1 ⊆ S
    using open_contains_ball_eq[OF ‹open S›] assms by auto
  obtain e2 where 0 < e2 and ∀ x∈X. x ≠ p ⟶ e2 ≤ dist p x
    using finite_set_avoid[OF ‹finite X›,of p] by auto
  hence ∀ w∈ball p (min e1 e2). w∈S ∧ (w≠p ⟶ w∉X) using e1_b by auto
  thus ∃ e>0. ∀ w∈ball p e. w ∈ S ∧ (w ≠ p ⟶ w ∉ X) using ‹e2>0› ‹e1>0›
```

  apply (*rule_tac x=min e1 e2* **in** *exI*)
  by *auto*
qed

lemma *finite_cball_avoid*:
 fixes *S* :: *'a* :: *euclidean_space set*
 assumes *open S finite X p* ∈ *S*
 shows ∃ *e*>*0*. ∀ *w*∈*cball p e*. *w*∈*S* ∧ (*w*≠*p* ⟶ *w*∉*X*)
proof −
 obtain *e1* **where** *e1*>*0* **and** *e1*: ∀ *w*∈*ball p e1*. *w*∈*S* ∧ (*w*≠*p* ⟶ *w*∉*X*)
  using *finite_ball_avoid*[*OF assms*] **by** *auto*
 define *e2* **where** *e2* ≡ *e1*/*2*
 have *e2*>*0* **and** *e2* < *e1* **unfolding** *e2_def* **using** ‹*e1*>*0*› **by** *auto*
 then have *cball p e2* ⊆ *ball p e1* **by** (*subst cball_subset_ball_iff*,*auto*)
 then show ∃ *e*>*0*. ∀ *w*∈*cball p e*. *w* ∈ *S* ∧ (*w* ≠ *p* ⟶ *w* ∉ *X*) **using** ‹*e2*>*0*›
e1 **by** *auto*
qed

lemma *dim_cball*:
 assumes *e* > *0*
 shows *dim* (*cball* (*0* :: *'n*::*euclidean_space*) *e*) = *DIM*(*'n*)
proof −
 {
  fix *x* :: *'n*::*euclidean_space*
  define *y* **where** *y* = (*e* / *norm x*) $*_R$ *x*
  then have *y* ∈ *cball 0 e*
   using *assms* **by** *auto*
  moreover have ∗: *x* = (*norm x* / *e*) $*_R$ *y*
   using *y_def assms* **by** *simp*
  moreover from ∗ **have** *x* = (*norm x*/*e*) $*_R$ *y*
   by *auto*
  ultimately have *x* ∈ *span* (*cball 0 e*)
   using *span_scale*[*of y cball 0 e norm x*/*e*]
    span_superset[*of cball 0 e*]
   by (*simp add*: *span_base*)
 }
 then have *span* (*cball 0 e*) = (*UNIV* :: *'n*::*euclidean_space set*)
  by *auto*
 then show *?thesis*
  using *dim_span*[*of cball* (*0* :: *'n*::*euclidean_space*) *e*] **by** (*auto*)
qed

4.1.3 Boxes

abbreviation *One* :: *'a*::*euclidean_space* **where**
One ≡ $\sum$ *Basis*

lemma *One_non_0*: **assumes** *One* = (*0*::*'a*::*euclidean_space*) **shows** *False*
proof −

have *dependent* (*Basis* :: *'a set*)
  apply (*simp add*: *dependent_finite*)
  apply (*rule_tac x*=*λi*. *1* **in** *exI*)
  using *SOME_Basis* **apply** (*auto simp*: *assms*)
  done
with *independent_Basis* **show** *False* **by** *force*
qed

corollary *One_neq_0*[*iff*]: *One* ≠ *0*
  by (*metis One_non_0*)

corollary *Zero_neq_One*[*iff*]: *0* ≠ *One*
  by (*metis One_non_0*)

definition (**in** *euclidean_space*) *eucl_less* (**infix** <*e 50*) **where**
eucl_less a b ⟷ (∀ *i*∈*Basis*. *a* · *i* < *b* · *i*)

definition *box_eucl_less*: *box a b* = {*x*. *a* <*e x* ∧ *x* <*e b*}
definition *cbox a b* = {*x*. ∀ *i*∈*Basis*. *a* · *i* ≤ *x* · *i* ∧ *x* · *i* ≤ *b* · *i*}

lemma *box_def*: *box a b* = {*x*. ∀ *i*∈*Basis*. *a* · *i* < *x* · *i* ∧ *x* · *i* < *b* · *i*}
  and *in_box_eucl_less*: *x* ∈ *box a b* ⟷ *a* <*e x* ∧ *x* <*e b*
  and *mem_box*: *x* ∈ *box a b* ⟷ (∀ *i*∈*Basis*. *a* · *i* < *x* · *i* ∧ *x* · *i* < *b* · *i*)
    x ∈ *cbox a b* ⟷ (∀ *i*∈*Basis*. *a* · *i* ≤ *x* · *i* ∧ *x* · *i* ≤ *b* · *i*)
  by (*auto simp*: *box_eucl_less eucl_less_def cbox_def*)

lemma *cbox_Pair_eq*: *cbox* (*a*, *c*) (*b*, *d*) = *cbox a b* × *cbox c d*
  by (*force simp*: *cbox_def Basis_prod_def*)

lemma *cbox_Pair_iff* [*iff*]: (*x*, *y*) ∈ *cbox* (*a*, *c*) (*b*, *d*) ⟷ *x* ∈ *cbox a b* ∧ *y* ∈ *cbox c d*
  by (*force simp*: *cbox_Pair_eq*)

lemma *cbox_Complex_eq*: *cbox* (*Complex a c*) (*Complex b d*) = (λ(*x*,*y*). *Complex x y*) ' (*cbox a b* × *cbox c d*)
  apply (*auto simp*: *cbox_def Basis_complex_def*)
  apply (*rule_tac x* = (*Re x*, *Im x*) **in** *image_eqI*)
  using *complex_eq* **by** *auto*

lemma *cbox_Pair_eq_0*: *cbox* (*a*, *c*) (*b*, *d*) = {} ⟷ *cbox a b* = {} ∨ *cbox c d* = {}
  by (*force simp*: *cbox_Pair_eq*)

lemma *swap_cbox_Pair* [*simp*]: *prod.swap* ' *cbox* (*c*, *a*) (*d*, *b*) = *cbox* (*a*,*c*) (*b*,*d*)
  by *auto*

lemma *mem_box_real*[*simp*]:
  (*x*::*real*) ∈ *box a b* ⟷ *a* < *x* ∧ *x* < *b*
  (*x*::*real*) ∈ *cbox a b* ⟷ *a* ≤ *x* ∧ *x* ≤ *b*

by (*auto simp*: *mem_box*)

lemma *box_real*[*simp*]:
fixes *a b*:: *real*
shows *box a b* = {*a* <..< *b*} *cbox a b* = {*a* .. *b*}
by *auto*

lemma *box_Int_box*:
fixes *a* :: ′*a*::*euclidean_space*
shows *box a b* ∩ *box c d* =
box (∑ *i*∈*Basis*. *max* (*a*•*i*) (*c*•*i*) $*_R$ *i*) (∑ *i*∈*Basis*. *min* (*b*•*i*) (*d*•*i*) $*_R$ *i*)
unfolding *set_eq_iff* **and** *Int_iff* **and** *mem_box* **by** *auto*

lemma *rational_boxes*:
fixes *x* :: ′*a*::*euclidean_space*
assumes *e* > *0*
shows ∃ *a b*. (∀ *i*∈*Basis*. *a* · *i* ∈ ℚ ∧ *b* · *i* ∈ ℚ) ∧ *x* ∈ *box a b* ∧ *box a b* ⊆ *ball x e*
proof −
define *e*′ **where** *e*′ = *e* / (*2* ∗ *sqrt* (*real* (*DIM* (′*a*))))
then have *e*: *e*′ > *0*
using *assms* **by** (*auto*)
have ∀ *i*. ∃ *y*. *y* ∈ ℚ ∧ *y* < *x* · *i* ∧ *x* · *i* − *y* < *e*′ (**is** ∀ *i*. *?th i*)
proof
fix *i*
from *Rats_dense_in_real*[*of x* · *i* − *e*′ *x* · *i*] *e*
show *?th i* **by** *auto*
qed
from *choice*[*OF this*] **obtain** *a* **where**
a: ∀ *xa*. *a xa* ∈ ℚ ∧ *a xa* < *x* · *xa* ∧ *x* · *xa* − *a xa* < *e*′ **..**
have ∀ *i*. ∃ *y*. *y* ∈ ℚ ∧ *x* · *i* < *y* ∧ *y* − *x* · *i* < *e*′ (**is** ∀ *i*. *?th i*)
proof
fix *i*
from *Rats_dense_in_real*[*of x* · *i x* · *i* + *e*′] *e*
show *?th i* **by** *auto*
qed
from *choice*[*OF this*] **obtain** *b* **where**
b: ∀ *xa*. *b xa* ∈ ℚ ∧ *x* · *xa* < *b xa* ∧ *b xa* − *x* · *xa* < *e*′ **..**
let *?a* = ∑ *i*∈*Basis*. *a i* $*_R$ *i* **and** *?b* = ∑ *i*∈*Basis*. *b i* $*_R$ *i*
show *?thesis*
proof (*rule exI*[*of _ ?a*], *rule exI*[*of _ ?b*], *safe*)
fix *y* :: ′*a*
assume ∗: *y* ∈ *box ?a ?b*
have *dist x y* = *sqrt* (∑ *i*∈*Basis*. (*dist* (*x* · *i*) (*y* · *i*))^2)
unfolding *L2_set_def*[*symmetric*] **by** (*rule euclidean_dist_l2*)
also have ... < *sqrt* (∑ (*i*::′*a*)∈*Basis*. *e*ˆ*2* / *real* (*DIM*(′*a*)))
proof (*rule real_sqrt_less_mono*, *rule sum_strict_mono*)
fix *i* :: ′*a*
assume *i*: *i* ∈ *Basis*

```
    have a i < y·i ∧ y·i < b i
      using ∗ i by (auto simp: box_def)
    moreover have a i < x·i x·i − a i < e'
      using a by auto
    moreover have x·i < b i b i − x·i < e'
      using b by auto
    ultimately have |x·i − y·i| < 2 ∗ e'
      by auto
    then have dist (x · i) (y · i) < e/sqrt (real (DIM('a)))
      unfolding e'_def by (auto simp: dist_real_def)
    then have (dist (x · i) (y · i))^2 < (e/sqrt (real (DIM('a))))^2
      by (rule power_strict_mono) auto
    then show (dist (x · i) (y · i))^2 < e^2 / real DIM('a)
      by (simp add: power_divide)
  qed auto
  also have ... = e
    using ‹0 < e› by simp
  finally show y ∈ ball x e
    by (auto simp: ball_def)
 qed (insert a b, auto simp: box_def)
qed

lemma open_UNION_box:
  fixes M :: 'a::euclidean_space set
  assumes open M
  defines a' ≡ λf :: 'a ⇒ real × real. (∑(i::'a)∈Basis. fst (f i) ∗R i)
  defines b' ≡ λf :: 'a ⇒ real × real. (∑(i::'a)∈Basis. snd (f i) ∗R i)
  defines I ≡ {f∈Basis →E ℚ × ℚ. box (a' f) (b' f) ⊆ M}
  shows M = (⋃f∈I. box (a' f) (b' f))
proof −
  have x ∈ (⋃f∈I. box (a' f) (b' f)) if x ∈ M for x
  proof −
    obtain e where e: e > 0 ball x e ⊆ M
      using openE[OF ‹open M› ‹x ∈ M›] by auto
    moreover obtain a b where ab:
      x ∈ box a b
      ∀ i ∈ Basis. a · i ∈ ℚ
      ∀ i∈Basis. b · i ∈ ℚ
      box a b ⊆ ball x e
      using rational_boxes[OF e(1)] by metis
    ultimately show ?thesis
       by (intro UN_I[of λi∈Basis. (a · i, b · i)])
          (auto simp: euclidean_representation I_def a'_def b'_def)
  qed
  then show ?thesis by (auto simp: I_def)
qed

corollary open_countable_Union_open_box:
  fixes S :: 'a :: euclidean_space set
```

```
  assumes open S
  obtains 𝒟 where countable 𝒟 𝒟 ⊆ Pow S ⋀X. X ∈ 𝒟 ⟹ ∃ a b. X = box a b
⋃𝒟 = S
proof −
  let ?a = λf. (∑(i::'a)∈Basis. fst (f i) *R i)
  let ?b = λf. (∑(i::'a)∈Basis. snd (f i) *R i)
  let ?I = {f∈Basis →E ℚ × ℚ. box (?a f) (?b f) ⊆ S}
  let ?𝒟 = (λf. box (?a f) (?b f)) ` ?I
  show ?thesis
  proof
    have countable ?I
      by (simp add: countable_PiE countable_rat)
    then show countable ?𝒟
      by blast
    show ⋃ ?𝒟 = S
      using open_UNION_box [OF assms] by metis
  qed auto
qed

lemma rational_cboxes:
  fixes x :: 'a::euclidean_space
  assumes e > 0
  shows ∃ a b. (∀ i∈Basis. a · i ∈ ℚ ∧ b · i ∈ ℚ) ∧ x ∈ cbox a b ∧ cbox a b ⊆
ball x e
proof −
  define e' where e' = e / (2 * sqrt (real (DIM ('a))))
  then have e: e' > 0
    using assms by auto
  have ∀ i. ∃ y. y ∈ ℚ ∧ y < x · i ∧ x · i − y < e' (is ∀ i. ?th i)
  proof
    fix i
    from Rats_dense_in_real[of x · i − e' x · i] e
    show ?th i by auto
  qed
  from choice[OF this] obtain a where
    a: ∀ u. a u ∈ ℚ ∧ a u < x · u ∧ x · u − a u < e' ..
  have ∀ i. ∃ y. y ∈ ℚ ∧ x · i < y ∧ y − x · i < e' (is ∀ i. ?th i)
  proof
    fix i
    from Rats_dense_in_real[of x · i x · i + e'] e
    show ?th i by auto
  qed
  from choice[OF this] obtain b where
    b: ∀ u. b u ∈ ℚ ∧ x · u < b u ∧ b u − x · u < e' ..
  let ?a = ∑ i∈Basis. a i *R i and ?b = ∑ i∈Basis. b i *R i
  show ?thesis
  proof (rule exI[of _ ?a], rule exI[of _ ?b], safe)
    fix y :: 'a
    assume *: y ∈ cbox ?a ?b
```

```
    have dist x y = sqrt (∑ i∈Basis. (dist (x · i) (y · i))^2)
      unfolding L2_set_def[symmetric] by (rule euclidean_dist_l2)
    also have ... < sqrt (∑ (i::'a)∈Basis. e^2 / real (DIM('a)))
    proof (rule real_sqrt_less_mono, rule sum_strict_mono)
      fix i :: 'a
      assume i: i ∈ Basis
      have a i ≤ y•i ∧ y•i ≤ b i
        using * i by (auto simp: cbox_def)
      moreover have a i < x•i x•i − a i < e'
        using a by auto
      moreover have x•i < b i b i − x•i < e'
        using b by auto
      ultimately have |x•i − y•i| < 2 * e'
        by auto
      then have dist (x · i) (y · i) < e/sqrt (real (DIM('a)))
        unfolding e'_def by (auto simp: dist_real_def)
      then have (dist (x · i) (y · i))^2 < (e/sqrt (real (DIM('a))))^2
        by (rule power_strict_mono) auto
      then show (dist (x · i) (y · i))^2 < e^2 / real DIM('a)
        by (simp add: power_divide)
    qed auto
    also have ... = e
      using ‹0 < e› by simp
    finally show y ∈ ball x e
      by (auto simp: ball_def)
  next
    show x ∈ cbox (∑ i∈Basis. a i *R i) (∑ i∈Basis. b i *R i)
      using a b less_imp_le by (auto simp: cbox_def)
  qed (use a b cbox_def in auto)
qed

lemma open_UNION_cbox:
  fixes M :: 'a::euclidean_space set
  assumes open M
  defines a' ≡ λf. (∑ (i::'a)∈Basis. fst (f i) *R i)
  defines b' ≡ λf. (∑ (i::'a)∈Basis. snd (f i) *R i)
  defines I ≡ {f∈Basis →E ℚ × ℚ. cbox (a' f) (b' f) ⊆ M}
  shows M = (⋃ f∈I. cbox (a' f) (b' f))
proof −
  have x ∈ (⋃ f∈I. cbox (a' f) (b' f)) if x ∈ M for x
  proof −
    obtain e where e: e > 0 ball x e ⊆ M
      using openE[OF ‹open M› ‹x ∈ M›] by auto
    moreover obtain a b where ab: x ∈ cbox a b ∀ i ∈ Basis. a · i ∈ ℚ
                                  ∀ i ∈ Basis. b · i ∈ ℚ cbox a b ⊆ ball x e
      using rational_cboxes[OF e(1)] by metis
    ultimately show ?thesis
       by (intro UN_I[of λi∈Basis. (a · i, b · i)])
          (auto simp: euclidean_representation I_def a'_def b'_def)
```

qed
then show *?thesis* **by** (*auto simp*: *I_def*)
qed

corollary *open_countable_Union_open_cbox*:
fixes S :: $'a$ :: *euclidean_space set*
assumes *open* S
obtains $\mathcal{D}$ **where** *countable* $\mathcal{D}$ $\mathcal{D} \subseteq$ *Pow* S $\bigwedge X.\ X \in \mathcal{D} \Longrightarrow \exists\, a\ b.\ X = cbox\ a\ b$ $\bigcup \mathcal{D} = S$
proof −
let $?a = \lambda f.\ (\sum (i::'a) \in Basis.\ fst\ (f\ i) *_R i)$
let $?b = \lambda f.\ (\sum (i::'a) \in Basis.\ snd\ (f\ i) *_R i)$
let $?I = \{f \in Basis \rightarrow_E \mathbb{Q} \times \mathbb{Q}.\ cbox\ (?a\ f)\ (?b\ f) \subseteq S\}$
let $?\mathcal{D} = (\lambda f.\ cbox\ (?a\ f)\ (?b\ f))$ ‘ $?I$
show *?thesis*
proof
have *countable ?I*
by (*simp add*: *countable_PiE countable_rat*)
then show *countable* $?\mathcal{D}$
by *blast*
show $\bigcup ?\mathcal{D} = S$
using *open_UNION_cbox* [*OF assms*] **by** *metis*
qed *auto*
qed

lemma *box_eq_empty*:
fixes a :: $'a$::*euclidean_space*
shows $(box\ a\ b = \{\} \longleftrightarrow (\exists\, i \in Basis.\ b \cdot i \le a \cdot i))$ (**is** *?th1*)
and $(cbox\ a\ b = \{\} \longleftrightarrow (\exists\, i \in Basis.\ b \cdot i < a \cdot i))$ (**is** *?th2*)
proof −
{
fix $i\ x$
assume i: $i \in Basis$ **and** *as*: $b \cdot i \le a \cdot i$ **and** x: $x \in box\ a\ b$
then have $a \cdot i < x \cdot i \wedge x \cdot i < b \cdot i$
unfolding *mem_box* **by** (*auto simp*: *box_def*)
then have $a \cdot i < b \cdot i$ **by** *auto*
then have *False* **using** *as* **by** *auto*
}
moreover
{
assume *as*: $\forall\, i \in Basis.\ \neg\ (b \cdot i \le a \cdot i)$
let $?x = (1/2) *_R (a + b)$
{
fix i :: $'a$
assume i: $i \in Basis$
have $a \cdot i < b \cdot i$
using *as*[*THEN bspec*[**where** $x=i$]] i **by** *auto*
then have $a \cdot i < ((1/2) *_R (a+b)) \cdot i$ $((1/2) *_R (a+b)) \cdot i < b \cdot i$
by (*auto simp*: *inner_add_left*)

```
    }
    then have box a b ≠ {}
      using mem_box(1)[of ?x a b] by auto
  }
  ultimately show ?th1 by blast

  {
    fix i x
    assume i: i ∈ Basis and as:b•i < a•i and x:x∈cbox a b
    then have a • i ≤ x • i ∧ x • i ≤ b • i
      unfolding mem_box by auto
    then have a•i ≤ b•i by auto
    then have False using as by auto
  }
  moreover
  {
    assume as:∀ i∈Basis. ¬ (b•i < a•i)
    let ?x = (1/2) *R (a + b)
    {
      fix i :: 'a
      assume i:i ∈ Basis
      have a•i ≤ b•i
        using as[THEN bspec[where x=i]] i by auto
      then have a•i ≤ ((1/2) *R (a+b)) • i ((1/2) *R (a+b)) • i ≤ b•i
        by (auto simp: inner_add_left)
    }
    then have cbox a b ≠ {}
      using mem_box(2)[of ?x a b] by auto
  }
  ultimately show ?th2 by blast
qed

lemma box_ne_empty:
  fixes a :: 'a::euclidean_space
  shows cbox a b ≠ {} ⟷ (∀ i∈Basis. a•i ≤ b•i)
  and box a b ≠ {} ⟷ (∀ i∈Basis. a•i < b•i)
  unfolding box_eq_empty[of a b] by fastforce+

lemma
  fixes a :: 'a::euclidean_space
  shows cbox_sing [simp]: cbox a a = {a}
    and box_sing [simp]: box a a = {}
  unfolding set_eq_iff mem_box eq_iff [symmetric]
  by (auto intro!: euclidean_eqI[where 'a='a])
     (metis all_not_in_conv nonempty_Basis)

lemma subset_box_imp:
  fixes a :: 'a::euclidean_space
  shows (∀ i∈Basis. a•i ≤ c•i ∧ d•i ≤ b•i) ⟹ cbox c d ⊆ cbox a b
```

and (∀ *i*∈*Basis*. *a*•*i* < *c*•*i* ∧ *d*•*i* < *b*•*i*) ⟹ *cbox c d* ⊆ *box a b*
and (∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*) ⟹ *box c d* ⊆ *cbox a b*
and (∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*) ⟹ *box c d* ⊆ *box a b*
unfolding *subset_eq*[*unfolded Ball_def*] **unfolding** *mem_box*
by (*best intro*: *order_trans less_le_trans le_less_trans less_imp_le*)+

lemma *box_subset_cbox*:
fixes *a* :: ′*a*::*euclidean_space*
shows *box a b* ⊆ *cbox a b*
unfolding *subset_eq* [*unfolded Ball_def*] *mem_box*
by (*fast intro*: *less_imp_le*)

lemma *subset_box*:
fixes *a* :: ′*a*::*euclidean_space*
shows *cbox c d* ⊆ *cbox a b* ⟷ (∀ *i*∈*Basis*. *c*•*i* ≤ *d*•*i*) ⟶ (∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*) (**is** *?th1*)
and *cbox c d* ⊆ *box a b* ⟷ (∀ *i*∈*Basis*. *c*•*i* ≤ *d*•*i*) ⟶ (∀ *i*∈*Basis*. *a*•*i* < *c*•*i* ∧ *d*•*i* < *b*•*i*) (**is** *?th2*)
and *box c d* ⊆ *cbox a b* ⟷ (∀ *i*∈*Basis*. *c*•*i* < *d*•*i*) ⟶ (∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*) (**is** *?th3*)
and *box c d* ⊆ *box a b* ⟷ (∀ *i*∈*Basis*. *c*•*i* < *d*•*i*) ⟶ (∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*) (**is** *?th4*)
proof −
let *?lesscd* = ∀ *i*∈*Basis*. *c*•*i* < *d*•*i*
let *?lerhs* = ∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*
show *?th1 ?th2*
by (*fastforce simp*: *mem_box*)+
have *acdb*: *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*
if *i*: *i* ∈ *Basis* **and** *box*: *box c d* ⊆ *cbox a b* **and** *cd*: ⋀*i*. *i* ∈ *Basis* ⟹ *c*•*i* < *d*•*i* **for** *i*
proof −
have *box c d* ≠ {}
using *that*
unfolding *box_eq_empty* **by** *force*
{ **let** *?x* = (∑*j*∈*Basis*. (*if j*=*i then* ((*min* (*a*•*j*) (*d*•*j*))+*c*•*j*)/*2 else* (*c*•*j*+*d*•*j*)/*2*) $*_R$ *j*)::′*a*
assume ∗: *a*•*i* > *c*•*i*
then have *c* • *j* < *?x* • *j* ∧ *?x* • *j* < *d* • *j* **if** *j* ∈ *Basis* **for** *j*
using *cd that* **by** (*fastforce simp add*: *i* ∗)
then have *?x* ∈ *box c d*
unfolding *mem_box* **by** *auto*
moreover have *?x* ∉ *cbox a b*
using *i cd* ∗ **by** (*force simp*: *mem_box*)
ultimately have *False* **using** *box* **by** *auto*
}
then have *a*•*i* ≤ *c*•*i* **by** *force*
moreover
{ **let** *?x* = (∑*j*∈*Basis*. (*if j*=*i then* ((*max* (*b*•*j*) (*c*•*j*))+*d*•*j*)/*2 else* (*c*•*j*+*d*•*j*)/*2*) $*_R$ *j*)::′*a*

```
      assume ∗: b·i < d·i
      then have d · j > ?x · j ∧ ?x · j > c · j if j ∈ Basis for j
        using cd that by (fastforce simp add: i ∗)
      then have ?x ∈ box c d
        unfolding mem_box by auto
      moreover have ?x ∉ cbox a b
        using i cd ∗ by (force simp: mem_box)
      ultimately have False using box by auto
    }
    then have b·i ≥ d·i by (rule ccontr) auto
    ultimately show ?thesis by auto
  qed
  show ?th3
    using acdb by (fastforce simp add: mem_box)
  have acdb': a·i ≤ c·i ∧ d·i ≤ b·i
    if i ∈ Basis box c d ⊆ box a b ⋀i. i ∈ Basis ⟹ c·i < d·i for i
      using box_subset_cbox[of a b] that acdb by auto
  show ?th4
    using acdb' by (fastforce simp add: mem_box)
qed

lemma eq_cbox: cbox a b = cbox c d ⟷ cbox a b = {} ∧ cbox c d = {} ∨ a = c
∧ b = d
      (is ?lhs = ?rhs)
proof
  assume ?lhs
  then have cbox a b ⊆ cbox c d cbox c d ⊆ cbox a b
    by auto
  then show ?rhs
    by (force simp: subset_box box_eq_empty intro: antisym euclidean_eqI)
next
  assume ?rhs
  then show ?lhs
    by force
qed

lemma eq_cbox_box [simp]: cbox a b = box c d ⟷ cbox a b = {} ∧ box c d = {}
  (is ?lhs ⟷ ?rhs)
proof
  assume L: ?lhs
  then have cbox a b ⊆ box c d box c d ⊆ cbox a b
    by auto
  then show ?rhs
    apply (simp add: subset_box)
    using L box_ne_empty box_sing apply (fastforce simp add:)
    done
qed force

lemma eq_box_cbox [simp]: box a b = cbox c d ⟷ box a b = {} ∧ cbox c d = {}
```

by (*metis eq_cbox_box*)

lemma *eq_box*: *box a b* = *box c d* ⟷ *box a b* = {} ∧ *box c d* = {} ∨ *a* = *c* ∧ *b* = *d*
(**is** *?lhs* ⟷ *?rhs*)
proof
assume *L*: *?lhs*
then have *box a b* ⊆ *box c d box c d* ⊆ *box a b*
by *auto*
then show *?rhs*
apply (*simp add*: *subset_box*)
using *box_ne_empty*(*2*) *L*
apply *auto*
apply (*meson euclidean_eqI less_eq_real_def not_less*)+
done
qed *force*

lemma *subset_box_complex*:
cbox a b ⊆ *cbox c d* ⟷
(*Re a* ≤ *Re b* ∧ *Im a* ≤ *Im b*) ⟶ *Re a* ≥ *Re c* ∧ *Im a* ≥ *Im c* ∧ *Re b* ≤ *Re d* ∧ *Im b* ≤ *Im d*
cbox a b ⊆ *box c d* ⟷
(*Re a* ≤ *Re b* ∧ *Im a* ≤ *Im b*) ⟶ *Re a* > *Re c* ∧ *Im a* > *Im c* ∧ *Re b* < *Re d* ∧ *Im b* < *Im d*
box a b ⊆ *cbox c d* ⟷
(*Re a* < *Re b* ∧ *Im a* < *Im b*) ⟶ *Re a* ≥ *Re c* ∧ *Im a* ≥ *Im c* ∧ *Re b* ≤ *Re d* ∧ *Im b* ≤ *Im d*
box a b ⊆ *box c d* ⟷
(*Re a* < *Re b* ∧ *Im a* < *Im b*) ⟶ *Re a* ≥ *Re c* ∧ *Im a* ≥ *Im c* ∧ *Re b* ≤ *Re d* ∧ *Im b* ≤ *Im d*
by (*subst subset_box*; *force simp*: *Basis_complex_def*)+

lemma *in_cbox_complex_iff*:
x ∈ *cbox a b* ⟷ *Re x* ∈ {*Re a*..*Re b*} ∧ *Im x* ∈ {*Im a*..*Im b*}
by (*cases x*; *cases a*; *cases b*) (*auto simp*: *cbox_Complex_eq*)

lemma *box_Complex_eq*:
box (*Complex a c*) (*Complex b d*) = (λ(*x*,*y*). *Complex x y*) ‘ (*box a b* × *box c d*)
by (*auto simp*: *box_def Basis_complex_def image_iff complex_eq_iff*)

lemma *in_box_complex_iff*:
x ∈ *box a b* ⟷ *Re x* ∈ {*Re a*<..<*Re b*} ∧ *Im x* ∈ {*Im a*<..<*Im b*}
by (*cases x*; *cases a*; *cases b*) (*auto simp*: *box_Complex_eq*)

lemma *Int_interval*:
fixes *a* :: ′*a*::*euclidean_space*
shows *cbox a b* ∩ *cbox c d* =
cbox (∑ *i*∈*Basis*. *max* (*a*·*i*) (*c*·*i*) $*_R$ *i*) (∑ *i*∈*Basis*. *min* (*b*·*i*) (*d*·*i*) $*_R$ *i*)
unfolding *set_eq_iff* **and** *Int_iff* **and** *mem_box*

```
  by auto

lemma disjoint_interval:
  fixes a::'a::euclidean_space
  shows cbox a b ∩ cbox c d = {} ⟷ (∃ i∈Basis. (b•i < a•i ∨ d•i < c•i ∨ b•i <
c•i ∨ d•i < a•i)) (is ?th1)
    and cbox a b ∩ box c d = {} ⟷ (∃ i∈Basis. (b•i < a•i ∨ d•i ≤ c•i ∨ b•i ≤
c•i ∨ d•i ≤ a•i)) (is ?th2)
    and box a b ∩ cbox c d = {} ⟷ (∃ i∈Basis. (b•i ≤ a•i ∨ d•i < c•i ∨ b•i ≤
c•i ∨ d•i ≤ a•i)) (is ?th3)
    and box a b ∩ box c d = {} ⟷ (∃ i∈Basis. (b•i ≤ a•i ∨ d•i ≤ c•i ∨ b•i ≤
c•i ∨ d•i ≤ a•i)) (is ?th4)
proof −
  let ?z = (∑ i∈Basis. (((max (a•i) (c•i)) + (min (b•i) (d•i))) / 2) *R i)::'a
  have **: ⋀P Q. (⋀i :: 'a. i ∈ Basis ⟹ Q ?z i ⟹ P i) ⟹
      (⋀i x :: 'a. i ∈ Basis ⟹ P i ⟹ Q x i) ⟹ (∀ x. ∃ i∈Basis. Q x i) ⟷
(∃ i∈Basis. P i)
    by blast
  note * = set_eq_iff Int_iff empty_iff mem_box ball_conj_distrib[symmetric] eq_False
ball_simps(10)
  show ?th1 unfolding * by (intro **) auto
  show ?th2 unfolding * by (intro **) auto
  show ?th3 unfolding * by (intro **) auto
  show ?th4 unfolding * by (intro **) auto
qed

lemma UN_box_eq_UNIV: (⋃ i::nat. box (− (real i *R One)) (real i *R One)) =
UNIV
proof −
  have |x · b| < real_of_int (⌈Max ((λb. |x · b|)`Basis)⌉ + 1)
    if [simp]: b ∈ Basis for x b :: 'a
  proof −
    have |x · b| ≤ real_of_int ⌈|x · b|⌉
      by (rule le_of_int_ceiling)
    also have ... ≤ real_of_int ⌈Max ((λb. |x · b|)`Basis)⌉
      by (auto intro!: ceiling_mono)
    also have ... < real_of_int (⌈Max ((λb. |x · b|)`Basis)⌉ + 1)
      by simp
    finally show ?thesis .
  qed
  then have ∃ n::nat. ∀ b∈Basis. |x · b| < real n for x :: 'a
    by (metis order.strict_trans reals_Archimedean2)
  moreover have ⋀x b::'a. ⋀n::nat. |x · b| < real n ⟷ − real n < x · b ∧ x ·
b < real n
    by auto
  ultimately show ?thesis
    by (auto simp: box_def inner_sum_left inner_Basis sum.If_cases)
qed
```

```
lemma image_affinity_cbox: fixes m::real
  fixes a b c :: 'a::euclidean_space
  shows (λx. m *R x + c) ` cbox a b =
    (if cbox a b = {} then {}
     else (if 0 ≤ m then cbox (m *R a + c) (m *R b + c)
     else cbox (m *R b + c) (m *R a + c)))
proof (cases m = 0)
  case True
  {
    fix x
    assume ∀ i∈Basis. x · i ≤ c · i ∀ i∈Basis. c · i ≤ x · i
    then have x = c
      by (simp add: dual_order.antisym euclidean_eqI)
  }
  moreover have c ∈ cbox (m *R a + c) (m *R b + c)
    unfolding True by (auto)
  ultimately show ?thesis using True by (auto simp: cbox_def)
next
  case False
  {
    fix y
    assume ∀ i∈Basis. a · i ≤ y · i ∀ i∈Basis. y · i ≤ b · i m > 0
    then have ∀ i∈Basis. (m *R a + c) · i ≤ (m *R y + c) · i and ∀ i∈Basis.
(m *R y + c) · i ≤ (m *R b + c) · i
      by (auto simp: inner_distrib)
  }
  moreover
  {
    fix y
    assume ∀ i∈Basis. a · i ≤ y · i ∀ i∈Basis. y · i ≤ b · i m < 0
    then have ∀ i∈Basis. (m *R b + c) · i ≤ (m *R y + c) · i and ∀ i∈Basis.
(m *R y + c) · i ≤ (m *R a + c) · i
      by (auto simp: mult_left_mono_neg inner_distrib)
  }
  moreover
  {
    fix y
    assume m > 0 and ∀ i∈Basis. (m *R a + c) · i ≤ y · i and ∀ i∈Basis. y · i
≤ (m *R b + c) · i
    then have y ∈ (λx. m *R x + c) ` cbox a b
      unfolding image_iff Bex_def mem_box
      apply (intro exI[where x=(1 / m) *R (y − c)])
     apply (auto simp: pos_le_divide_eq pos_divide_le_eq mult.commute inner_distrib
inner_diff_left)
      done
  }
  moreover
  {
    fix y
```

```
    assume ∀ i∈Basis. (m *R b + c) • i ≤ y • i ∀ i∈Basis. y • i ≤ (m *R a + c)
• i m < 0
    then have y ∈ (λx. m *R x + c) ` cbox a b
      unfolding image_iff Bex_def mem_box
      apply (intro exI[where x=(1 / m) *R (y − c)])
     apply (auto simp: neg_le_divide_eq neg_divide_le_eq mult.commute inner_distrib
inner_diff_left)
      done
  }
  ultimately show ?thesis using False by (auto simp: cbox_def)
qed

lemma image_smult_cbox:(λx. m *R (x::_::euclidean_space)) ` cbox a b =
  (if cbox a b = {} then {} else if 0 ≤ m then cbox (m *R a) (m *R b) else cbox
(m *R b) (m *R a))
  using image_affinity_cbox[of m 0 a b] by auto

lemma swap_continuous:
  assumes continuous_on (cbox (a,c) (b,d)) (λ(x,y). f x y)
    shows continuous_on (cbox (c,a) (d,b)) (λ(x, y). f y x)
proof −
  have (λ(x, y). f y x) = (λ(x, y). f x y) ◦ prod.swap
    by auto
  then show ?thesis
    apply (rule ssubst)
    apply (rule continuous_on_compose)
    apply (simp add: split_def)
    apply (rule continuous_intros | simp add: assms)+
    done
qed
```

4.1.4 General Intervals

```
definition is_interval (s::('a::euclidean_space) set) ⟷
  (∀ a∈s. ∀ b∈s. ∀ x. (∀ i∈Basis. ((a•i ≤ x•i ∧ x•i ≤ b•i) ∨ (b•i ≤ x•i ∧ x•i ≤
a•i))) ⟶ x ∈ s)

lemma is_interval_1:
  is_interval (s::real set) ⟷ (∀ a∈s. ∀ b∈s. ∀ x. a ≤ x ∧ x ≤ b ⟶ x ∈ s)
  unfolding is_interval_def by auto

lemma is_interval_Int: is_interval X ⟹ is_interval Y ⟹ is_interval (X ∩ Y)
  unfolding is_interval_def
  by blast

lemma is_interval_cbox [simp]: is_interval (cbox a (b::'a::euclidean_space)) (is ?th1)
  and is_interval_box [simp]: is_interval (box a b) (is ?th2)
  unfolding is_interval_def mem_box Ball_def atLeastAtMost_iff
  by (meson order_trans le_less_trans less_le_trans less_trans)+
```

lemma *is_interval_empty* [*iff*]: *is_interval* {}
  unfolding *is_interval_def* **by** *simp*

lemma *is_interval_univ* [*iff*]: *is_interval UNIV*
  unfolding *is_interval_def* **by** *simp*

lemma *mem_is_intervalI*:
  assumes *is_interval s*
    and $a \in s$ $b \in s$
    and $\bigwedge i.\ i \in Basis \Longrightarrow a \cdot i \le x \cdot i \wedge x \cdot i \le b \cdot i \vee b \cdot i \le x \cdot i \wedge x \cdot i \le a \cdot i$
  shows $x \in s$
  by (*rule assms*(*1*)[*simplified is_interval_def*, *rule_format*, *OF assms*(*2*,*3*,*4*)])

lemma *interval_subst*:
  fixes *S*::*'a*::*euclidean_space set*
  assumes *is_interval S*
    and $x \in S$ $y\ j \in S$
    and $j \in Basis$
  shows $(\sum i{\in}Basis.\ (if\ i = j\ then\ y\ i \cdot i\ else\ x \cdot i) *_R\ i) \in S$
  by (*rule mem_is_intervalI*[*OF assms*(*1*,*2*)]) (*auto simp*: *assms*)

lemma *mem_box_componentwiseI*:
  fixes *S*::*'a*::*euclidean_space set*
  assumes *is_interval S*
  assumes $\bigwedge i.\ i \in Basis \Longrightarrow x \cdot i \in ((\lambda x.\ x \cdot i)\ `\ S)$
  shows $x \in S$
proof −
  from *assms* **have** $\forall i \in Basis.\ \exists s \in S.\ x \cdot i = s \cdot i$
    by *auto*
  with *finite_Basis* **obtain** *s* **and** *bs*::*'a list*
    where *s*: $\bigwedge i.\ i \in Basis \Longrightarrow x \cdot i = s\ i \cdot i$ $\bigwedge i.\ i \in Basis \Longrightarrow s\ i \in S$
      and *bs*: *set bs* = *Basis distinct bs*
    by (*metis finite_distinct_list*)
  from *nonempty_Basis s* **obtain** *j* **where** *j*: $j \in Basis$ $s\ j \in S$
    by *blast*
  define *y* **where**
    $y = rec\_list\ (s\ j)\ (\lambda j\ \_\ Y.\ (\sum i{\in}Basis.\ (if\ i = j\ then\ s\ i \cdot i\ else\ Y \cdot i) *_R\ i))$
  have $x = (\sum i{\in}Basis.\ (if\ i \in set\ bs\ then\ s\ i \cdot i\ else\ s\ j \cdot i) *_R\ i)$
    using *bs* **by** (*auto simp*: *s*(*1*)[*symmetric*] *euclidean_representation*)
  also have [*symmetric*]: *y bs* = ...
    using *bs*(*2*) *bs*(*1*)[*THEN equalityD1*]
   by (*induct bs*) (*auto simp*: *y_def euclidean_representation intro*!: *euclidean_eqI*[**where** *'a*=*'a*])
  also have $y\ bs \in S$
    using *bs*(*1*)[*THEN equalityD1*]
    apply (*induct bs*)
     apply (*auto simp*: *y_def j*)

```
    apply (rule interval_subst[OF assms(1)])
      apply (auto simp: s)
    done
  finally show ?thesis .
qed

lemma cbox01_nonempty [simp]: cbox 0 One ≠ {}
  by (simp add: box_ne_empty inner_Basis inner_sum_left sum_nonneg)

lemma box01_nonempty [simp]: box 0 One ≠ {}
  by (simp add: box_ne_empty inner_Basis inner_sum_left)

lemma empty_as_interval: {} = cbox One (0::'a::euclidean_space)
  using nonempty_Basis box01_nonempty box_eq_empty(1) box_ne_empty(1) by
blast

lemma interval_subset_is_interval:
  assumes is_interval S
  shows cbox a b ⊆ S ⟷ cbox a b = {} ∨ a ∈ S ∧ b ∈ S (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs  using box_ne_empty(1) mem_box(2) by fastforce
next
  assume ?rhs
  have cbox a b ⊆ S if a ∈ S b ∈ S
    using assms unfolding is_interval_def
    apply (clarsimp simp add: mem_box)
    using that by blast
  with ‹?rhs› show ?lhs
    by blast
qed

lemma is_real_interval_union:
  is_interval (X ∪ Y)
  if X: is_interval X and Y: is_interval Y and I: (X ≠ {} ⟹ Y ≠ {} ⟹ X ∩
Y ≠ {})
  for X Y::real set
proof -
  consider X ≠ {} Y ≠ {} | X = {} | Y = {} by blast
  then show ?thesis
  proof cases
    case 1
    then obtain r where r ∈ X ∨ X ∩ Y = {} r ∈ Y ∨ X ∩ Y = {}
      by blast
    then show ?thesis
      using I 1 X Y unfolding is_interval_1
      by (metis (full_types) Un_iff le_cases)
  qed (use that in auto)
qed
```

lemma *is_interval_translationI*:
assumes *is_interval X*
shows *is_interval* ((+) *x* ‘ *X*)
unfolding *is_interval_def*
proof *safe*
fix *b d e*
assume $b \in X$ $d \in X$
$\forall i \in Basis.\ (x + b) \cdot i \leq e \cdot i \wedge e \cdot i \leq (x + d) \cdot i \vee$
$(x + d) \cdot i \leq e \cdot i \wedge e \cdot i \leq (x + b) \cdot i$
hence $e - x \in X$
by (*intro mem_is_intervalI*[*OF assms* ‹$b \in X$› ‹$d \in X$›, *of* $e - x$])
(*auto simp*: *algebra_simps*)
thus $e \in (+)\ x$ ‘ X **by** *force*
qed

lemma *is_interval_uminusI*:
assumes *is_interval X*
shows *is_interval* (*uminus* ‘ *X*)
unfolding *is_interval_def*
proof *safe*
fix *b d e*
assume $b \in X$ $d \in X$
$\forall i \in Basis.\ (-\ b) \cdot i \leq e \cdot i \wedge e \cdot i \leq (-\ d) \cdot i \vee$
$(-\ d) \cdot i \leq e \cdot i \wedge e \cdot i \leq (-\ b) \cdot i$
hence $-\ e \in X$
by (*intro mem_is_intervalI*[*OF assms* ‹$b \in X$› ‹$d \in X$›, *of* $-\ e$])
(*auto simp*: *algebra_simps*)
thus $e \in$ *uminus* ‘ *X* **by** *force*
qed

lemma *is_interval_uminus*[*simp*]: *is_interval* (*uminus* ‘ *x*) = *is_interval x*
using *is_interval_uminusI*[*of x*] *is_interval_uminusI*[*of uminus* ‘ *x*]
by (*auto simp*: *image_image*)

lemma *is_interval_neg_translationI*:
assumes *is_interval X*
shows *is_interval* ((−) *x* ‘ *X*)
proof −
have (−) *x* ‘ *X* = (+) *x* ‘ *uminus* ‘ *X*
by (*force simp*: *algebra_simps*)
also have *is_interval* ...
by (*metis is_interval_uminusI is_interval_translationI assms*)
finally show *?thesis* **.**
qed

lemma *is_interval_translation*[*simp*]:
is_interval ((+) *x* ‘ *X*) = *is_interval X*
using *is_interval_neg_translationI*[*of* (+) *x* ‘ *X x*]

by (*auto intro*!: *is_interval_translationI simp*: *image_image*)

lemma *is_interval_minus_translation*[*simp*]:
shows *is_interval* ((−) *x* ‘ *X*) = *is_interval X*
proof −
have (−) *x* ‘ *X* = (+) *x* ‘ *uminus* ‘ *X*
by (*force simp*: *algebra_simps*)
also have *is_interval* ... = *is_interval X*
by *simp*
finally show *?thesis* **.**
qed

lemma *is_interval_minus_translation'*[*simp*]:
shows *is_interval* ((λ*x*. *x* − *c*) ‘ *X*) = *is_interval X*
using *is_interval_translation*[*of* −*c X*]
by (*metis image_cong uminus_add_conv_diff*)

lemma *is_interval_cball_1*[*intro*, *simp*]: *is_interval* (*cball a b*) **for** *a b*::*real*
by (*simp add*: *cball_eq_atLeastAtMost is_interval_def*)

lemma *is_interval_ball_real*: *is_interval* (*ball a b*) **for** *a b*::*real*
by (*simp add*: *ball_eq_greaterThanLessThan is_interval_def*)

4.1.5 Bounded Projections

lemma *bounded_inner_imp_bdd_above*:
assumes *bounded s*
shows *bdd_above* ((λ*x*. *x* · *a*) ‘ *s*)
by (*simp add*: *assms bounded_imp_bdd_above bounded_linear_image bounded_linear_inner_left*)

lemma *bounded_inner_imp_bdd_below*:
assumes *bounded s*
shows *bdd_below* ((λ*x*. *x* · *a*) ‘ *s*)
by (*simp add*: *assms bounded_imp_bdd_below bounded_linear_image bounded_linear_inner_left*)

4.1.6 Structural rules for pointwise continuity

lemma *continuous_infnorm*[*continuous_intros*]:
continuous F f ⟹ *continuous F* (λ*x*. *infnorm* (*f x*))
unfolding *continuous_def* **by** (*rule tendsto_infnorm*)

lemma *continuous_inner*[*continuous_intros*]:
assumes *continuous F f*
and *continuous F g*
shows *continuous F* (λ*x*. *inner* (*f x*) (*g x*))
using *assms* **unfolding** *continuous_def* **by** (*rule tendsto_inner*)

4.1.7 Structural rules for setwise continuity

lemma *continuous_on_infnorm*[*continuous_intros*]:

continuous_on s f ⟹ *continuous_on s* (λ*x*. *infnorm* (*f x*))
unfolding *continuous_on* **by** (*fast intro*: *tendsto_infnorm*)

lemma *continuous_on_inner*[*continuous_intros*]:
fixes *g* :: ′*a*::*topological_space* ⇒ ′*b*::*real_inner*
assumes *continuous_on s f*
and *continuous_on s g*
shows *continuous_on s* (λ*x*. *inner* (*f x*) (*g x*))
using *bounded_bilinear_inner assms*
by (*rule bounded_bilinear.continuous_on*)

4.1.8 Openness of halfspaces.

lemma *open_halfspace_lt*: *open* {*x*. *inner a x* < *b*}
by (*simp add*: *open_Collect_less continuous_on_inner*)

lemma *open_halfspace_gt*: *open* {*x*. *inner a x* > *b*}
by (*simp add*: *open_Collect_less continuous_on_inner*)

lemma *open_halfspace_component_lt*: *open* {*x*::′*a*::*euclidean_space*. *x*·*i* < *a*}
by (*simp add*: *open_Collect_less continuous_on_inner*)

lemma *open_halfspace_component_gt*: *open* {*x*::′*a*::*euclidean_space*. *x*·*i* > *a*}
by (*simp add*: *open_Collect_less continuous_on_inner*)

lemma *eucl_less_eq_halfspaces*:
fixes *a* :: ′*a*::*euclidean_space*
shows {*x*. *x* <*e a*} = (⋂ *i*∈*Basis*. {*x*. *x* · *i* < *a* · *i*})
{*x*. *a* <*e x*} = (⋂ *i*∈*Basis*. {*x*. *a* · *i* < *x* · *i*})
by (*auto simp*: *eucl_less_def*)

lemma *open_Collect_eucl_less*[*simp*, *intro*]:
fixes *a* :: ′*a*::*euclidean_space*
shows *open* {*x*. *x* <*e a*} *open* {*x*. *a* <*e x*}
by (*auto simp*: *eucl_less_eq_halfspaces open_halfspace_component_lt open_halfspace_component_gt*)

4.1.9 Closure and Interior of halfspaces and hyperplanes

lemma *continuous_at_inner*: *continuous* (*at x*) (*inner a*)
unfolding *continuous_at* **by** (*intro tendsto_intros*)

lemma *closed_halfspace_le*: *closed* {*x*. *inner a x* ≤ *b*}
by (*simp add*: *closed_Collect_le continuous_on_inner*)

lemma *closed_halfspace_ge*: *closed* {*x*. *inner a x* ≥ *b*}
by (*simp add*: *closed_Collect_le continuous_on_inner*)

lemma *closed_hyperplane*: *closed* {*x*. *inner a x* = *b*}
by (*simp add*: *closed_Collect_eq continuous_on_inner*)

```
lemma closed_halfspace_component_le: closed {x::'a::euclidean_space. x•i ≤ a}
  by (simp add: closed_Collect_le continuous_on_inner)

lemma closed_halfspace_component_ge: closed {x::'a::euclidean_space. x•i ≥ a}
  by (simp add: closed_Collect_le continuous_on_inner)

lemma closed_interval_left:
  fixes b :: 'a::euclidean_space
  shows closed {x::'a. ∀ i∈Basis. x•i ≤ b•i}
  by (simp add: Collect_ball_eq closed_INT closed_Collect_le continuous_on_inner)

lemma closed_interval_right:
  fixes a :: 'a::euclidean_space
  shows closed {x::'a. ∀ i∈Basis. a•i ≤ x•i}
  by (simp add: Collect_ball_eq closed_INT closed_Collect_le continuous_on_inner)

lemma interior_halfspace_le [simp]:
  assumes a ≠ 0
    shows interior {x. a · x ≤ b} = {x. a · x < b}
proof −
  have ∗: a · x < b if x: x ∈ S and S: S ⊆ {x. a · x ≤ b} and open S for S x
  proof −
    obtain e where e>0 and e: cball x e ⊆ S
      using ‹open S› open_contains_cball x by blast
    then have x + (e / norm a) ∗R a ∈ cball x e
      by (simp add: dist_norm)
    then have x + (e / norm a) ∗R a ∈ S
      using e by blast
    then have x + (e / norm a) ∗R a ∈ {x. a · x ≤ b}
      using S by blast
    moreover have e ∗ (a · a) / norm a > 0
      by (simp add: ‹0 < e› assms)
    ultimately show ?thesis
      by (simp add: algebra_simps)
  qed
  show ?thesis
    by (rule interior_unique) (auto simp: open_halfspace_lt ∗)
qed

lemma interior_halfspace_ge [simp]:
   a ≠ 0 =⇒ interior {x. a · x ≥ b} = {x. a · x > b}
using interior_halfspace_le [of −a −b] by simp

lemma closure_halfspace_lt [simp]:
  assumes a ≠ 0
    shows closure {x. a · x < b} = {x. a · x ≤ b}
proof −
  have [simp]: −{x. a · x < b} = {x. a · x ≥ b}
    by (force simp:)
```

then show *?thesis*
using *interior_halfspace_ge* [*of a b*] *assms*
by (*force simp*: *closure_interior*)
qed

lemma *closure_halfspace_gt* [*simp*]:
$a \neq 0 \Longrightarrow closure\ \{x.\ a \cdot x > b\} = \{x.\ a \cdot x \geq b\}$
using *closure_halfspace_lt* [*of* $-a$ $-b$] **by** *simp*

lemma *interior_hyperplane* [*simp*]:
assumes $a \neq 0$
shows *interior* $\{x.\ a \cdot x = b\} = \{\}$
proof −
have [*simp*]: $\{x.\ a \cdot x = b\} = \{x.\ a \cdot x \leq b\} \cap \{x.\ a \cdot x \geq b\}$
by (*force simp*:)
then show *?thesis*
by (*auto simp*: *assms*)
qed

lemma *frontier_halfspace_le*:
assumes $a \neq 0 \vee b \neq 0$
shows *frontier* $\{x.\ a \cdot x \leq b\} = \{x.\ a \cdot x = b\}$
proof (*cases* $a = 0$)
case *True* **with** *assms* **show** *?thesis* **by** *simp*
next
case *False* **then show** *?thesis*
by (*force simp*: *frontier_def closed_halfspace_le*)
qed

lemma *frontier_halfspace_ge*:
assumes $a \neq 0 \vee b \neq 0$
shows *frontier* $\{x.\ a \cdot x \geq b\} = \{x.\ a \cdot x = b\}$
proof (*cases* $a = 0$)
case *True* **with** *assms* **show** *?thesis* **by** *simp*
next
case *False* **then show** *?thesis*
by (*force simp*: *frontier_def closed_halfspace_ge*)
qed

lemma *frontier_halfspace_lt*:
assumes $a \neq 0 \vee b \neq 0$
shows *frontier* $\{x.\ a \cdot x < b\} = \{x.\ a \cdot x = b\}$
proof (*cases* $a = 0$)
case *True* **with** *assms* **show** *?thesis* **by** *simp*
next
case *False* **then show** *?thesis*
by (*force simp*: *frontier_def interior_open open_halfspace_lt*)
qed

lemma *frontier_halfspace_gt*:
  assumes $a \neq 0 \lor b \neq 0$
    shows *frontier* $\{x.\ a \cdot x > b\} = \{x.\ a \cdot x = b\}$
proof (*cases a = 0*)
  case *True* **with** *assms* **show** *?thesis* **by** *simp*
next
  case *False* **then show** *?thesis*
    by (*force simp*: *frontier_def interior_open open_halfspace_gt*)
qed

4.1.10 Some more convenient intermediate-value theorem formulations

lemma *connected_ivt_hyperplane*:
  assumes *connected S* **and** *xy*: $x \in S$ $y \in S$ **and** *b*: *inner a x* $\leq$ *b b* $\leq$ *inner a y*
  shows $\exists z \in S.$ *inner a z* = *b*
proof (*rule ccontr*)
  assume *as*:$\neg$ ($\exists z \in S.$ *inner a z* = *b*)
  let *?A* = {*x. inner a x* < *b*}
  let *?B* = {*x. inner a x* > *b*}
  have *open ?A open ?B*
    using *open_halfspace_lt* **and** *open_halfspace_gt* **by** *auto*
  moreover have *?A* ∩ *?B* = {} **by** *auto*
  moreover have *S* ⊆ *?A* ∪ *?B* **using** *as* **by** *auto*
  ultimately show *False*
    using ‹*connected S*›[*unfolded connected_def not_ex*,
      THEN spec[**where** *x*=*?A*], *THEN spec*[**where** *x*=*?B*]]
    using *xy b* **by** *auto*
qed

lemma *connected_ivt_component*:
  fixes *x*::*'a*::*euclidean_space*
  shows *connected S* ⟹ *x* ∈ *S* ⟹ *y* ∈ *S* ⟹ *x*•*k* ≤ *a* ⟹ *a* ≤ *y*•*k* ⟹ (∃ *z*∈*S*. *z*•*k* = *a*)
  using *connected_ivt_hyperplane*[*of S x y k*::*'a a*]
  by (*auto simp*: *inner_commute*)

4.1.11 Limit Component Bounds

lemma *Lim_component_le*:
  fixes *f* :: *'a* ⇒ *'b*::*euclidean_space*
  assumes (*f* ⟶ *l*) *net*
    and ¬ (*trivial_limit net*)
    and *eventually* (λ*x*. *f*(*x*)•*i* ≤ *b*) *net*
  shows *l*•*i* ≤ *b*
  by (*rule tendsto_le*[*OF assms*(*2*) *tendsto_const tendsto_inner*[*OF assms*(*1*) *tendsto_const*] *assms*(*3*)]])

lemma *Lim_component_ge*:

fixes f :: $'a \Rightarrow 'b$::*euclidean_space*
assumes $(f \longrightarrow l)$ *net*
and $\neg$ (*trivial_limit net*)
and *eventually* $(\lambda x.\ b \leq (f\ x)\cdot i)$ *net*
shows $b \leq l\cdot i$
by (*rule tendsto_le*[*OF assms*(*2*) *tendsto_inner*[*OF assms*(*1*) *tendsto_const*] *tendsto_const assms*(*3*)])

lemma *Lim_component_eq*:
fixes f :: $'a \Rightarrow 'b$::*euclidean_space*
assumes *net*: $(f \longrightarrow l)$ *net* $\neg$ *trivial_limit net*
and *ev*:*eventually* $(\lambda x.\ f(x)\cdot i = b)$ *net*
shows $l\cdot i = b$
using *ev*[*unfolded order_eq_iff eventually_conj_iff*]
using *Lim_component_ge*[*OF net*, *of b i*]
using *Lim_component_le*[*OF net*, *of i b*]
by *auto*

lemma *open_box*[*intro*]: *open* (*box a b*)
proof –
have *open* $(\bigcap i\in Basis.\ ((\cdot)\ i)$ –' $\{a \cdot i <..< b \cdot i\})$
by (*auto intro*!: *continuous_open_vimage continuous_inner continuous_ident continuous_const*)
also have $(\bigcap i\in Basis.\ ((\cdot)\ i)$ –' $\{a \cdot i <..< b \cdot i\}) =$ *box a b*
by (*auto simp*: *box_def inner_commute*)
finally show *?thesis* **.**
qed

lemma *closed_cbox*[*intro*]:
fixes *a b* :: $'a$::*euclidean_space*
shows *closed* (*cbox a b*)
proof –
have *closed* $(\bigcap i\in Basis.\ (\lambda x.\ x\cdot i)$ –' $\{a\cdot i\ ..\ b\cdot i\})$
by (*intro closed_INT ballI continuous_closed_vimage allI linear_continuous_at closed_real_atLeastAtMost finite_Basis bounded_linear_inner_left*)
also have $(\bigcap i\in Basis.\ (\lambda x.\ x\cdot i)$ –' $\{a\cdot i\ ..\ b\cdot i\}) =$ *cbox a b*
by (*auto simp*: *cbox_def*)
finally show *closed* (*cbox a b*) **.**
qed

lemma *interior_cbox* [*simp*]:
fixes *a b* :: $'a$::*euclidean_space*
shows *interior* (*cbox a b*) = *box a b* (**is** *?L* = *?R*)
proof(*rule subset_antisym*)
show *?R* $\subseteq$ *?L*
using *box_subset_cbox open_box*
by (*rule interior_maximal*)
{
fix *x*

```
    assume x ∈ interior (cbox a b)
    then obtain s where s: open s x ∈ s s ⊆ cbox a b ..
    then obtain e where e>0 and e:∀ x'. dist x' x < e ⟶ x' ∈ cbox a b
      unfolding open_dist and subset_eq by auto
    {
      fix i :: 'a
      assume i: i ∈ Basis
      have dist (x − (e / 2) *R i) x < e
        and dist (x + (e / 2) *R i) x < e
        unfolding dist_norm
        apply auto
        unfolding norm_minus_cancel
        using norm_Basis[OF i] ‹e>0›
        apply auto
        done
      then have a · i ≤ (x − (e / 2) *R i) · i and (x + (e / 2) *R i) · i ≤ b · i
        using e[THEN spec[where x=x − (e/2) *R i]]
          and e[THEN spec[where x=x + (e/2) *R i]]
        unfolding mem_box
        using i
        by blast+
      then have a · i < x · i and x · i < b · i
        using ‹e>0› i
        by (auto simp: inner_diff_left inner_Basis inner_add_left)
    }
    then have x ∈ box a b
      unfolding mem_box by auto
  }
  then show ?L ⊆ ?R ..
qed

lemma bounded_cbox [simp]:
  fixes a :: 'a::euclidean_space
  shows bounded (cbox a b)
proof −
  let ?b = ∑ i∈Basis. |a·i| + |b·i|
  {
    fix x :: 'a
    assume ⋀i. i∈Basis ⟹ a · i ≤ x · i ∧ x · i ≤ b · i
    then have (∑ i∈Basis. |x · i|) ≤ ?b
      by (force simp: intro!: sum_mono)
    then have norm x ≤ ?b
      using norm_le_l1[of x] by auto
  }
  then show ?thesis
    unfolding cbox_def bounded_iff by force
qed

lemma bounded_box [simp]:
```

```
  fixes a :: 'a::euclidean_space
  shows bounded (box a b)
  using bounded_cbox[of a b] box_subset_cbox[of a b] bounded_subset[of cbox a b box
a b]
  by simp

lemma not_interval_UNIV [simp]:
  fixes a :: 'a::euclidean_space
  shows cbox a b ≠ UNIV box a b ≠ UNIV
  using bounded_box[of a b] bounded_cbox[of a b] by force+

lemma not_interval_UNIV2 [simp]:
  fixes a :: 'a::euclidean_space
  shows UNIV ≠ cbox a b UNIV ≠ box a b
  using bounded_box[of a b] bounded_cbox[of a b] by force+

lemma box_midpoint:
  fixes a :: 'a::euclidean_space
  assumes box a b ≠ {}
  shows ((1/2) *R (a + b)) ∈ box a b
proof -
  have a · i < ((1 / 2) *R (a + b)) · i ∧ ((1 / 2) *R (a + b)) · i < b · i if i ∈
Basis for i
    using assms that by (auto simp: inner_add_left box_ne_empty)
  then show ?thesis unfolding mem_box by auto
qed

lemma open_cbox_convex:
  fixes x :: 'a::euclidean_space
  assumes x: x ∈ box a b
    and y: y ∈ cbox a b
    and e: 0 < e e ≤ 1
  shows (e *R x + (1 - e) *R y) ∈ box a b
proof -
  {
    fix i :: 'a
    assume i: i ∈ Basis
    have a · i = e * (a · i) + (1 - e) * (a · i)
      unfolding left_diff_distrib by simp
    also have ... < e * (x · i) + (1 - e) * (y · i)
    proof (rule add_less_le_mono)
      show e * (a · i) < e * (x · i)
        using ‹0 < e› i mem_box(1) x by auto
      show (1 - e) * (a · i) ≤ (1 - e) * (y · i)
        by (meson diff_ge_0_iff_ge ‹e ≤ 1› i mem_box(2) mult_left_mono y)
    qed
    finally have a · i < (e *R x + (1 - e) *R y) · i
      unfolding inner_simps by auto
    moreover
```

```
    {
      have b · i = e * (b·i) + (1 − e) * (b·i)
        unfolding left_diff_distrib by simp
      also have . . . > e * (x · i) + (1 − e) * (y · i)
      proof (rule add_less_le_mono)
        show e * (x · i) < e * (b · i)
          using ‹0 < e› i mem_box(1) x by auto
        show (1 − e) * (y · i) ≤ (1 − e) * (b · i)
          by (meson diff_ge_0_iff_ge ‹e ≤ 1› i mem_box(2) mult_left_mono y)
      qed
      finally have (e *R x + (1 − e) *R y) · i < b · i
        unfolding inner_simps by auto
    }
    ultimately have a · i < (e *R x + (1 − e) *R y) · i ∧ (e *R x + (1 − e)
*R y) · i < b · i
      by auto
  }
  then show ?thesis
    unfolding mem_box by auto
qed

lemma closure_cbox [simp]: closure (cbox a b) = cbox a b
  by (simp add: closed_cbox)

lemma closure_box [simp]:
  fixes a :: 'a::euclidean_space
   assumes box a b ≠ {}
  shows closure (box a b) = cbox a b
proof −
  have ab: a <e b
    using assms by (simp add: eucl_less_def box_ne_empty)
  let ?c = (1 / 2) *R (a + b)
  {
    fix x
    assume as:x ∈ cbox a b
    define f where [abs_def]: f n = x + (inverse (real n + 1)) *R (?c − x) for n
    {
      fix n
      assume fn: f n <e b ⟶ a <e f n ⟶ f n = x and xc: x ≠ ?c
      have *: 0 < inverse (real n + 1) inverse (real n + 1) ≤ 1
        unfolding inverse_le_1_iff by auto
      have (inverse (real n + 1)) *R ((1 / 2) *R (a + b)) + (1 − inverse (real n
+ 1)) *R x =
        x + (inverse (real n + 1)) *R (((1 / 2) *R (a + b)) − x)
        by (auto simp: algebra_simps)
      then have f n <e b and a <e f n
        using open_cbox_convex[OF box_midpoint[OF assms] as *]
        unfolding f_def by (auto simp: box_def eucl_less_def)
      then have False
```

```
      using fn unfolding f_def using xc by auto
    }
    moreover
    {
      assume ¬ (f ⟶ x) sequentially
      {
        fix e :: real
        assume e > 0
        then obtain N :: nat where N: inverse (real (N + 1)) < e
          using reals_Archimedean by auto
        have inverse (real n + 1) < e if N ≤ n for n
          by (auto intro!: that le_less_trans [OF _ N])
        then have ∃N::nat. ∀n≥N. inverse (real n + 1) < e by auto
      }
      then have ((λn. inverse (real n + 1)) ⟶ 0) sequentially
        unfolding lim_sequentially by(auto simp: dist_norm)
      then have (f ⟶ x) sequentially
        unfolding f_def
        using tendsto_add[OF tendsto_const, of λn::nat. (inverse (real n + 1)) *R
((1 / 2) *R (a + b) − x) 0 sequentially x]
        using tendsto_scaleR [OF _ tendsto_const, of λn::nat. inverse (real n + 1)
0 sequentially ((1 / 2) *R (a + b) − x)]
        by auto
    }
    ultimately have x ∈ closure (box a b)
      using as box_midpoint[OF assms]
      unfolding closure_def islimpt_sequential
      by (cases x=?c) (auto simp: in_box_eucl_less)
  }
  then show ?thesis
    using closure_minimal[OF box_subset_cbox, of a b] by blast
qed

lemma bounded_subset_box_symmetric:
  fixes S :: ('a::euclidean_space) set
  assumes bounded S
  obtains a where S ⊆ box (−a) a
proof −
  obtain b where b>0 and b: ∀x∈S. norm x ≤ b
    using assms[unfolded bounded_pos] by auto
  define a :: 'a where a = (∑i∈Basis. (b + 1) *R i)
  have (−a)·i < x·i and x·i < a·i if x ∈ S and i: i ∈ Basis for x i
    using b Basis_le_norm[OF i, of x] that by (auto simp: a_def)
  then have S ⊆ box (−a) a
    by (auto simp: simp add: box_def)
  then show ?thesis ..
qed

lemma bounded_subset_cbox_symmetric:
```

```
  fixes S :: ('a::euclidean_space) set
  assumes bounded S
  obtains a where S ⊆ cbox (−a) a
proof −
  obtain a where S ⊆ box (−a) a
    using bounded_subset_box_symmetric[OF assms] by auto
  then show ?thesis
    by (meson box_subset_cbox dual_order.trans that)
qed

lemma frontier_cbox:
  fixes a b :: 'a::euclidean_space
  shows frontier (cbox a b) = cbox a b − box a b
 unfolding frontier_def unfolding interior_cbox and closure_closed[OF closed_cbox]
..

lemma frontier_box:
  fixes a b :: 'a::euclidean_space
  shows frontier (box a b) = (if box a b = {} then {} else cbox a b − box a b)
proof (cases box a b = {})
  case True
  then show ?thesis
    using frontier_empty by auto
next
  case False
  then show ?thesis
   unfolding frontier_def and closure_box[OF False] and interior_open[OF open_box]
    by auto
qed

lemma Int_interval_mixed_eq_empty:
  fixes a :: 'a::euclidean_space
   assumes box c d ≠ {}
  shows box a b ∩ cbox c d = {} ⟷ box a b ∩ box c d = {}
  unfolding closure_box[OF assms, symmetric]
  unfolding open_Int_closure_eq_empty[OF open_box] ..
```

4.1.12 Class Instances

```
lemma compact_lemma:
  fixes f :: nat ⇒ 'a::euclidean_space
  assumes bounded (range f)
  shows ∀ d⊆Basis. ∃ l::'a. ∃ r.
    strict_mono r ∧ (∀ e>0. eventually (λn. ∀ i∈d. dist (f (r n) · i) (l · i) < e)
sequentially)
  by (rule compact_lemma_general[where unproj=λe. ∑ i∈Basis. e i *R i])
     (auto intro!: assms bounded_linear_inner_left bounded_linear_image
       simp: euclidean_representation)
```

```
instance euclidean_space ⊆ heine_borel
proof
  fix f :: nat ⇒ 'a
  assume f: bounded (range f)
  then obtain l::'a and r where r: strict_mono r
   and l: ∀ e>0. eventually (λn. ∀ i∈Basis. dist (f (r n) · i) (l · i) < e) sequentially
    using compact_lemma [OF f] by blast
  {
    fix e::real
    assume e > 0
    hence e / real_of_nat DIM('a) > 0 by (simp)
   with l have eventually (λn. ∀ i∈Basis. dist (f (r n) · i) (l · i) < e / (real_of_nat
DIM('a))) sequentially
      by simp
    moreover
    {
      fix n
      assume n: ∀ i∈Basis. dist (f (r n) · i) (l · i) < e / (real_of_nat DIM('a))
      have dist (f (r n)) l ≤ (∑ i∈Basis. dist (f (r n) · i) (l · i))
        apply (subst euclidean_dist_l2)
        using zero_le_dist
        apply (rule L2_set_le_sum)
        done
      also have ... < (∑ i∈(Basis::'a set). e / (real_of_nat DIM('a)))
        apply (rule sum_strict_mono)
        using n
        apply auto
        done
      finally have dist (f (r n)) l < e
        by auto
    }
    ultimately have eventually (λn. dist (f (r n)) l < e) sequentially
      by (rule eventually_mono)
  }
  then have *: ((f ∘ r) ⟶ l) sequentially
    unfolding o_def tendsto_iff by simp
  with r show ∃ l r. strict_mono r ∧ ((f ∘ r) ⟶ l) sequentially
    by auto
qed

instance euclidean_space ⊆ banach ..

instance euclidean_space ⊆ second_countable_topology
proof
  define a where a f = (∑ i∈Basis. fst (f i) *R i) for f :: 'a ⇒ real × real
  then have a: ⋀f. (∑ i∈Basis. fst (f i) *R i) = a f
    by simp
  define b where b f = (∑ i∈Basis. snd (f i) *R i) for f :: 'a ⇒ real × real
  then have b: ⋀f. (∑ i∈Basis. snd (f i) *R i) = b f
```

  by *simp*
 define B **where** $B = (\lambda f.\ box\ (a\ f)\ (b\ f))$ ‘ $(Basis \rightarrow_E (\mathbb{Q} \times \mathbb{Q}))$

 have *Ball B open* **by** (*simp add*: *B_def open_box*)
 moreover have $(\forall A.\ open\ A \longrightarrow (\exists B'\subseteq B.\ \bigcup B' = A))$
 proof *safe*
  fix A::$'a$ *set*
  assume *open A*
  show $\exists B'\subseteq B.\ \bigcup B' = A$
   apply (*rule exI*[*of* _ $\{b\in B.\ b \subseteq A\}$])
   apply (*subst* (*3*) *open_UNION_box*[*OF* ‹*open A*›])
   apply (*auto simp*: *a b B_def*)
   done
 qed
 ultimately
 have *topological_basis B*
  unfolding *topological_basis_def* **by** *blast*
 moreover
 have *countable B*
  unfolding *B_def*
  by (*intro countable_image countable_PiE finite_Basis countable_SIGMA countable_rat*)
 ultimately show $\exists B$::$'a$ *set set*. *countable B* $\wedge$ *open* $=$ *generate_topology B*
  by (*blast intro*: *topological_basis_imp_subbasis*)
qed

instance *euclidean_space* $\subseteq$ *polish_space* **..**

4.1.13 Compact Boxes

lemma *compact_cbox* [*simp*]:
 fixes a :: $'a$::*euclidean_space*
 shows *compact* (*cbox a b*)
 using *bounded_closed_imp_seq_compact*[*of cbox a b*] **using** *bounded_cbox*[*of a b*]
 by (*auto simp*: *compact_eq_seq_compact_metric*)

proposition *is_interval_compact*:
  is_interval S $\wedge$ *compact S* $\longleftrightarrow$ $(\exists\ a\ b.\ S = cbox\ a\ b)$ (**is** *?lhs* $=$ *?rhs*)
proof (*cases* $S = \{\}$)
 case *True*
 with *empty_as_interval* **show** *?thesis* **by** *auto*
next
 case *False*
 show *?thesis*
 proof
  assume *L*: *?lhs*
  then have *is_interval S compact S* **by** *auto*
  define a **where** $a \equiv \sum i\in Basis.\ (INF\ x\in S.\ x \cdot i) *_R\ i$
  define b **where** $b \equiv \sum i\in Basis.\ (SUP\ x\in S.\ x \cdot i) *_R\ i$

```
  have 1: ⋀x i. ⟦x ∈ S; i ∈ Basis⟧ ⟹ (INF x∈S. x · i) ≤ x · i
    by (simp add: cInf_lower bounded_inner_imp_bdd_below compact_imp_bounded
L)
  have 2: ⋀x i. ⟦x ∈ S; i ∈ Basis⟧ ⟹ x · i ≤ (SUP x∈S. x · i)
    by (simp add: cSup_upper bounded_inner_imp_bdd_above compact_imp_bounded
L)
  have 3: x ∈ S if inf: ⋀i. i ∈ Basis ⟹ (INF x∈S. x · i) ≤ x · i
            and sup: ⋀i. i ∈ Basis ⟹ x · i ≤ (SUP x∈S. x · i) for x
  proof (rule mem_box_componentwiseI [OF ‹is_interval S›])
    fix i::'a
    assume i: i ∈ Basis
    have cont: continuous_on S (λx. x · i)
      by (intro continuous_intros)
    obtain a where a ∈ S and a: ⋀y. y∈S ⟹ a · i ≤ y · i
      using continuous_attains_inf [OF ‹compact S› False cont] by blast
    obtain b where b ∈ S and b: ⋀y. y∈S ⟹ y · i ≤ b · i
      using continuous_attains_sup [OF ‹compact S› False cont] by blast
    have a · i ≤ (INF x∈S. x · i)
      by (simp add: False a cINF_greatest)
    also have ... ≤ x · i
      by (simp add: i inf)
    finally have ai: a · i ≤ x · i .
    have x · i ≤ (SUP x∈S. x · i)
      by (simp add: i sup)
    also have (SUP x∈S. x · i) ≤ b · i
      by (simp add: False b cSUP_least)
    finally have bi: x · i ≤ b · i .
    show x · i ∈ (λx. x · i) ` S
      apply (rule_tac x=∑j∈Basis. (if j = i then x · i else a · j) *R j in
image_eqI)
      apply (simp add: i)
      apply (rule mem_is_intervalI [OF ‹is_interval S› ‹a ∈ S› ‹b ∈ S›])
      using i ai bi apply force
      done
  qed
  have S = cbox a b
    by (auto simp: a_def b_def mem_box intro: 1 2 3)
  then show ?rhs
    by blast
 next
  assume R: ?rhs
  then show ?lhs
    using compact_cbox is_interval_cbox by blast
 qed
qed
```

4.1.14 Componentwise limits and continuity

But is the premise really necessary? Need to generalise *dist ?x ?y = L2_set (λi. dist (?x · i) (?y · i)) Basis*

lemma *Euclidean_dist_upper*: *i ∈ Basis ⟹ dist (x · i) (y · i) ≤ dist x y*
  by (*metis* (*no_types*) *member_le_L2_set euclidean_dist_l2 finite_Basis*)

But is the premise *i ∈ Basis* really necessary?

```
lemma open_preimage_inner:
  assumes open S i ∈ Basis
    shows open {x. x · i ∈ S}
proof (rule openI, simp)
  fix x
  assume x: x · i ∈ S
  with assms obtain e where 0 < e and e: ball (x · i) e ⊆ S
    by (auto simp: open_contains_ball_eq)
  have ∃ e>0. ball (y · i) e ⊆ S if dxy: dist x y < e / 2 for y
  proof (intro exI conjI)
    have dist (x · i) (y · i) < e / 2
      by (meson ‹i ∈ Basis› dual_order.trans Euclidean_dist_upper not_le that)
    then have dist (x · i) z < e if dist (y · i) z < e / 2 for z
      by (metis dist_commute dist_triangle_half_l that)
    then have ball (y · i) (e / 2) ⊆ ball (x · i) e
      using mem_ball by blast
      with e show ball (y · i) (e / 2) ⊆ S
        by (metis order_trans)
  qed (simp add: ‹0 < e›)
  then show ∃ e>0. ball x e ⊆ {s. s · i ∈ S}
    by (metis (no_types, lifting) ‹0 < e› ‹open S› half_gt_zero_iff mem_Collect_eq
mem_ball open_contains_ball_eq subsetI)
qed

proposition tendsto_componentwise_iff:
  fixes f :: _ ⇒ 'b::euclidean_space
  shows (f ⟶ l) F ⟷ (∀ i ∈ Basis. ((λx. (f x · i)) ⟶ (l · i)) F)
        (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    unfolding tendsto_def
    apply clarify
    apply (drule_tac x={s. s · i ∈ S} in spec)
    apply (auto simp: open_preimage_inner)
    done
next
  assume R: ?rhs
  then have ⋀e. e > 0 ⟹ ∀ i∈Basis. ∀F x in F. dist (f x · i) (l · i) < e
    unfolding tendsto_iff by blast
  then have R': ⋀e. e > 0 ⟹ ∀F x in F. ∀ i∈Basis. dist (f x · i) (l · i) < e
```

```
    by (simp add: eventually_ball_finite_distrib [symmetric])
  show ?lhs
  unfolding tendsto_iff
  proof clarify
    fix e::real
    assume 0 < e
    have *: L2_set (λi. dist (f x · i) (l · i)) Basis < e
          if ∀ i∈Basis. dist (f x · i) (l · i) < e / real DIM('b) for x
    proof −
      have L2_set (λi. dist (f x · i) (l · i)) Basis ≤ sum (λi. dist (f x · i) (l · i))
Basis
        by (simp add: L2_set_le_sum)
      also have ... < DIM('b) ∗ (e / real DIM('b))
        apply (rule sum_bounded_above_strict)
        using that by auto
      also have ... = e
        by (simp add: field_simps)
      finally show L2_set (λi. dist (f x · i) (l · i)) Basis < e .
    qed
    have ∀F x in F. ∀ i∈Basis. dist (f x · i) (l · i) < e / DIM('b)
      apply (rule R')
      using ‹0 < e› by simp
    then show ∀F x in F. dist (f x) l < e
      apply (rule eventually_mono)
      apply (subst euclidean_dist_l2)
      using ∗ by blast
  qed
qed


corollary continuous_componentwise:
   continuous F f ⟷ (∀ i ∈ Basis. continuous F (λx. (f x · i)))
by (simp add: continuous_def tendsto_componentwise_iff [symmetric])

corollary continuous_on_componentwise:
  fixes S :: 'a :: t2_space set
  shows continuous_on S f ⟷ (∀ i ∈ Basis. continuous_on S (λx. (f x · i)))
  apply (simp add: continuous_on_eq_continuous_within)
  using continuous_componentwise by blast

lemma linear_componentwise_iff:
     (linear f') ⟷ (∀ i∈Basis. linear (λx. f' x · i))
  apply (auto simp: linear_iff inner_left_distrib)
   apply (metis inner_left_distrib euclidean_eq_iff)
  by (metis euclidean_eqI inner_scaleR_left)

lemma bounded_linear_componentwise_iff:
     (bounded_linear f') ⟷ (∀ i∈Basis. bounded_linear (λx. f' x · i))
     (is ?lhs = ?rhs)
```

proof
  assume *?lhs* **then show** *?rhs*
    by (*simp add*: *bounded_linear_inner_left_comp*)
next
  assume *?rhs*
  then have (∀ *i*∈*Basis*. ∃ *K*. ∀ *x*. |*f′ x* · *i*| ≤ *norm x* ∗ *K*) *linear f′*
    by (*auto simp*: *bounded_linear_def bounded_linear_axioms_def linear_componentwise_iff* [*symmetric*] *ball_conj_distrib*)
  then obtain *F* **where** *F*: ⋀*i x*. *i* ∈ *Basis* ⟹ |*f′ x* · *i*| ≤ *norm x* ∗ *F i*
    by *metis*
  have *norm* (*f′ x*) ≤ *norm x* ∗ *sum F Basis* **for** *x*
  proof −
    have *norm* (*f′ x*) ≤ (∑ *i*∈*Basis*. |*f′ x* · *i*|)
      by (*rule norm_le_l1*)
    also have ... ≤ (∑ *i*∈*Basis*. *norm x* ∗ *F i*)
      by (*metis F sum_mono*)
    also have ... = *norm x* ∗ *sum F Basis*
      by (*simp add*: *sum_distrib_left*)
    finally show *?thesis* **.**
  qed
  then show *?lhs*
    by (*force simp*: *bounded_linear_def bounded_linear_axioms_def* ‹*linear f′*›)
qed

4.1.15 Continuous Extension

definition *clamp* :: ′*a*::*euclidean_space* ⇒ ′*a* ⇒ ′*a* ⇒ ′*a* **where**
  clamp a b x = (*if* (∀ *i*∈*Basis*. *a* · *i* ≤ *b* · *i*)
    then (∑ *i*∈*Basis*. (*if x*·*i* < *a*·*i then a*·*i else if x*·*i* ≤ *b*·*i then x*·*i else b*·*i*) $*_R$ *i*)
    else a)

lemma *clamp_in_interval*[*simp*]:
  assumes ⋀*i*. *i* ∈ *Basis* ⟹ *a* · *i* ≤ *b* · *i*
  shows *clamp a b x* ∈ *cbox a b*
  unfolding *clamp_def*
  using *box_ne_empty*(*1*)[*of a b*] *assms* **by** (*auto simp*: *cbox_def*)

lemma *clamp_cancel_cbox*[*simp*]:
  fixes *x a b* :: ′*a*::*euclidean_space*
  assumes *x*: *x* ∈ *cbox a b*
  shows *clamp a b x* = *x*
  using *assms*
  by (*auto simp*: *clamp_def mem_box intro*!: *euclidean_eqI*[**where** ′*a*=′*a*])

lemma *clamp_empty_interval*:
  assumes *i* ∈ *Basis a* · *i* > *b* · *i*
  shows *clamp a b* = (λ_. *a*)
  using *assms*
  by (*force simp*: *clamp_def*[*abs_def*] *split*: *if_splits intro*!: *ext*)

```
lemma dist_clamps_le_dist_args:
  fixes x :: 'a::euclidean_space
  shows dist (clamp a b y) (clamp a b x) ≤ dist y x
proof cases
  assume le: (∀ i∈Basis. a · i ≤ b · i)
  then have (∑ i∈Basis. (dist (clamp a b y · i) (clamp a b x · i))^2) ≤
    (∑ i∈Basis. (dist (y · i) (x · i))^2)
   by (auto intro!: sum_mono simp: clamp_def dist_real_def abs_le_square_iff [symmetric])
  then show ?thesis
    by (auto intro: real_sqrt_le_mono
      simp: euclidean_dist_l2[where y=x] euclidean_dist_l2[where y=clamp a b x]
L2_set_def)
qed (auto simp: clamp_def)

lemma clamp_continuous_at:
  fixes f :: 'a::euclidean_space ⇒ 'b::metric_space
    and x :: 'a
  assumes f_cont: continuous_on (cbox a b) f
  shows continuous (at x) (λx. f (clamp a b x))
proof cases
  assume le: (∀ i∈Basis. a · i ≤ b · i)
  show ?thesis
    unfolding continuous_at_eps_delta
  proof safe
    fix x :: 'a
    fix e :: real
    assume e > 0
    moreover have clamp a b x ∈ cbox a b
      by (simp add: le)
    moreover note f_cont[simplified continuous_on_iff]
    ultimately
    obtain d where d: 0 < d
      ⋀x'. x' ∈ cbox a b ⟹ dist x' (clamp a b x) < d ⟹ dist (f x') (f (clamp a
b x)) < e
      by force
    show ∃ d>0. ∀ x'. dist x' x < d ⟶
      dist (f (clamp a b x')) (f (clamp a b x)) < e
      using le
     by (auto intro!: d clamp_in_interval dist_clamps_le_dist_args[THEN le_less_trans])
  qed
qed (auto simp: clamp_empty_interval)

lemma clamp_continuous_on:
  fixes f :: 'a::euclidean_space ⇒ 'b::metric_space
  assumes f_cont: continuous_on (cbox a b) f
  shows continuous_on S (λx. f (clamp a b x))
  using assms
  by (auto intro: continuous_at_imp_continuous_on clamp_continuous_at)
```

lemma *clamp_bounded*:
  fixes *f* :: *'a::euclidean_space* ⇒ *'b::metric_space*
  assumes *bounded*: *bounded* (*f* ‘ (*cbox a b*))
  shows *bounded* (*range* (λ*x*. *f* (*clamp a b x*)))
proof *cases*
  assume *le*: (∀ *i*∈*Basis*. *a* · *i* ≤ *b* · *i*)
  from *bounded* **obtain** *c* **where** *f_bound*: ∀ *x*∈*f* ‘ *cbox a b*. *dist undefined x* ≤ *c*
    by (*auto simp*: *bounded_any_center*[**where** *a*=*undefined*])
  then show *?thesis*
    by (*auto intro*!: *exI*[**where** *x*=*c*] *clamp_in_interval*[*OF le*[*rule_format*]]
        simp: *bounded_any_center*[**where** *a*=*undefined*])
qed (*auto simp*: *clamp_empty_interval image_def*)

definition *ext_cont* :: (*'a::euclidean_space* ⇒ *'b::metric_space*) ⇒ *'a* ⇒ *'a* ⇒ *'a* ⇒ *'b*
  where *ext_cont f a b* = (λ*x*. *f* (*clamp a b x*))

lemma *ext_cont_cancel_cbox*[*simp*]:
  fixes *x a b* :: *'a::euclidean_space*
  assumes *x*: *x* ∈ *cbox a b*
  shows *ext_cont f a b x* = *f x*
  using *assms*
  unfolding *ext_cont_def*
  by (*auto simp*: *clamp_def mem_box intro*!: *euclidean_eqI*[**where** *'a*=*'a*] *arg_cong*[**where** *f*=*f*])

lemma *continuous_on_ext_cont*[*continuous_intros*]:
  continuous_on (*cbox a b*) *f* ⟹ *continuous_on S* (*ext_cont f a b*)
  by (*auto intro*!: *clamp_continuous_on simp*: *ext_cont_def*)

4.1.16 Separability

lemma *univ_second_countable_sequence*:
  obtains *B* :: *nat* ⇒ *'a::euclidean_space set*
    where *inj B* ⋀*n*. *open*(*B n*) ⋀*S*. *open S* ⟹ ∃ *k*. *S* = ⋃{*B n* |*n*. *n* ∈ *k*}
proof −
  obtain *𝓑* :: *'a set set*
  where *countable 𝓑*
    and *opn*: ⋀*C*. *C* ∈ *𝓑* ⟹ *open C*
    and *Un*: ⋀*S*. *open S* ⟹ ∃ *U*. *U* ⊆ *𝓑* ∧ *S* = ⋃ *U*
    using *univ_second_countable* **by** *blast*
  have ∗: *infinite* (*range* (λ*n*. *ball* (*0*::*'a*) (*inverse*(*Suc n*))))
    apply (*rule Infinite_Set.range_inj_infinite*)
    apply (*simp add*: *inj_on_def ball_eq_ball_iff*)
    done
  have *infinite 𝓑*
  proof

```
    assume finite ℬ
    then have finite (Union ` (Pow ℬ))
      by simp
    then have finite (range (λn. ball (0::'a) (inverse(Suc n))))
      apply (rule rev_finite_subset)
     by (metis (no_types, lifting) PowI image_eqI image_subset_iff Un [OF open_ball])
    with * show False by simp
  qed
  obtain f :: nat ⇒ 'a set where ℬ = range f inj f
    by (blast intro: countable_as_injective_image [OF ‹countable ℬ› ‹infinite ℬ›])
  have *: ∃ k. S = ⋃{f n |n. n ∈ k} if open S for S
    using Un [OF that]
    apply clarify
    apply (rule_tac x=f-`U in exI)
    using ‹inj f› ‹ℬ = range f› apply force
    done
  show ?thesis
    apply (rule that [OF ‹inj f› _ *])
    apply (auto simp: ‹ℬ = range f› opn)
    done
qed

proposition separable:
  fixes S :: 'a::{metric_space, second_countable_topology} set
  obtains T where countable T T ⊆ S S ⊆ closure T
proof -
  obtain ℬ :: 'a set set
    where countable ℬ
      and {} ∉ ℬ
      and ope: ⋀C. C ∈ ℬ ⟹ openin(top_of_set S) C
      and if_ope: ⋀T. openin(top_of_set S) T ⟹ ∃𝒰. 𝒰 ⊆ ℬ ∧ T = ⋃𝒰
    by (meson subset_second_countable)
  then obtain f where f: ⋀C. C ∈ ℬ ⟹ f C ∈ C
    by (metis equals0I)
  show ?thesis
  proof
    show countable (f ` ℬ)
      by (simp add: ‹countable ℬ›)
    show f ` ℬ ⊆ S
      using ope f openin_imp_subset by blast
    show S ⊆ closure (f ` ℬ)
    proof (clarsimp simp: closure_approachable)
      fix x and e::real
      assume x ∈ S 0 < e
      have openin (top_of_set S) (S ∩ ball x e)
        by (simp add: openin_Int_open)
      with if_ope obtain 𝒰 where 𝒰: 𝒰 ⊆ ℬ S ∩ ball x e = ⋃𝒰
        by meson
      show ∃ C ∈ ℬ. dist (f C) x < e
```

```
    proof (cases 𝒰 = {})
      case True
      then show ?thesis
        using ‹0 < e› 𝒰 ‹x ∈ S› by auto
    next
      case False
      then obtain C where C ∈ 𝒰 by blast
      show ?thesis
      proof
        show dist (f C) x < e
         by (metis Int_iff Union_iff 𝒰 ‹C ∈ 𝒰› dist_commute f mem_ball subsetCE)
        show C ∈ ℬ
          using ‹𝒰 ⊆ ℬ› ‹C ∈ 𝒰› by blast
      qed
    qed
  qed
 qed
qed
```

4.1.17 Diameter

```
lemma diameter_cball [simp]:
  fixes a :: 'a::euclidean_space
  shows diameter(cball a r) = (if r < 0 then 0 else 2*r)
proof -
  have diameter(cball a r) = 2*r if r ≥ 0
  proof (rule order_antisym)
    show diameter (cball a r) ≤ 2*r
    proof (rule diameter_le)
      fix x y assume x ∈ cball a r y ∈ cball a r
      then have norm (x - a) ≤ r norm (a - y) ≤ r
        by (auto simp: dist_norm norm_minus_commute)
      then have norm (x - y) ≤ r+r
        using norm_diff_triangle_le by blast
      then show norm (x - y) ≤ 2*r by simp
    qed (simp add: that)
     have 2*r = dist (a + r *R (SOME i. i ∈ Basis)) (a - r *R (SOME i. i ∈
Basis))
      apply (simp add: dist_norm)
     by (metis abs_of_nonneg mult.right_neutral norm_numeral norm_scaleR norm_some_Basis
real_norm_def scaleR_2 that)
    also have ... ≤ diameter (cball a r)
      apply (rule diameter_bounded_bound)
      using that by (auto simp: dist_norm)
    finally show 2*r ≤ diameter (cball a r) .
  qed
  then show ?thesis by simp
qed
```

lemma *diameter_ball* [*simp*]:
fixes *a* :: *'a::euclidean_space*
shows *diameter*(*ball a r*) = (*if r* < *0 then 0 else 2*∗*r*)
proof −
have *diameter*(*ball a r*) = *2*∗*r* **if** *r* > *0*
by (*metis bounded_ball diameter_closure closure_ball diameter_cball less_eq_real_def linorder_not_less that*)
then show *?thesis*
by (*simp add*: *diameter_def*)
qed

lemma *diameter_closed_interval* [*simp*]: *diameter* {*a..b*} = (*if b* < *a then 0 else b*−*a*)
proof −
have {*a* .. *b*} = *cball* ((*a*+*b*)/*2*) ((*b*−*a*)/*2*)
by (*auto simp*: *dist_norm abs_if field_split_simps split*: *if_split_asm*)
then show *?thesis*
by *simp*
qed

lemma *diameter_open_interval* [*simp*]: *diameter* {*a*<..<*b*} = (*if b* < *a then 0 else b*−*a*)
proof −
have {*a* <..< *b*} = *ball* ((*a*+*b*)/*2*) ((*b*−*a*)/*2*)
by (*auto simp*: *dist_norm abs_if field_split_simps split*: *if_split_asm*)
then show *?thesis*
by *simp*
qed

lemma *diameter_cbox*:
fixes *a b::'a::euclidean_space*
shows (∀ *i* ∈ *Basis*. *a* · *i* ≤ *b* · *i*) ⟹ *diameter* (*cbox a b*) = *dist a b*
by (*force simp*: *diameter_def intro*!: *cSup_eq_maximum L2_set_mono*
simp: *euclidean_dist_l2*[**where** *'a*=*'a*] *cbox_def dist_norm*)

4.1.18 Relating linear images to open/closed/interior/closure/connected

proposition *open_surjective_linear_image*:
fixes *f* :: *'a::real_normed_vector* ⇒ *'b::euclidean_space*
assumes *open A linear f surj f*
shows *open*(*f* ‘ *A*)
unfolding *open_dist*
proof *clarify*
fix *x*
assume *x* ∈ *A*
have *bounded* (*inv f* ‘ *Basis*)
by (*simp add*: *finite_imp_bounded*)
with *bounded_pos* **obtain** *B* **where** *B* > *0* **and** *B*: ⋀*x*. *x* ∈ *inv f* ‘ *Basis* ⟹

```
norm x ≤ B
    by metis
  obtain e where e > 0 and e: ⋀z. dist z x < e ⟹ z ∈ A
    by (metis open_dist ‹x ∈ A› ‹open A›)
  define δ where δ ≡ e / B / DIM('b)
  show ∃e>0. ∀y. dist y (f x) < e ⟶ y ∈ f ` A
  proof (intro exI conjI)
    show δ > 0
      using ‹e > 0› ‹B > 0›  by (simp add: δ_def field_split_simps)
    have y ∈ f ` A if dist y (f x) * (B * real DIM('b)) < e for y
    proof −
      define u where u ≡ y − f x
      show ?thesis
      proof (rule image_eqI)
        show y = f (x + (∑i∈Basis. (u · i) *R inv f i))
          apply (simp add: linear_add linear_sum linear.scaleR ‹linear f› surj_f_inv_f
‹surj f›)
          apply (simp add: euclidean_representation u_def)
          done
        have dist (x + (∑i∈Basis. (u · i) *R inv f i)) x ≤ (∑i∈Basis. norm ((u
· i) *R inv f i))
          by (simp add: dist_norm sum_norm_le)
        also have ... = (∑i∈Basis. |u · i| * norm (inv f i))
          by simp
        also have ... ≤ (∑i∈Basis. |u · i|) * B
          by (simp add: B sum_distrib_right sum_mono mult_left_mono)
        also have ... ≤ DIM('b) * dist y (f x) * B
          apply (rule mult_right_mono [OF sum_bounded_above])
          using ‹0 < B› by (auto simp: Basis_le_norm dist_norm u_def)
        also have ... < e
          by (metis mult.commute mult.left_commute that)
        finally show x + (∑i∈Basis. (u · i) *R inv f i) ∈ A
          by (rule e)
      qed
    qed
    then show ∀y. dist y (f x) < δ ⟶ y ∈ f ` A
      using ‹e > 0› ‹B > 0›
      by (auto simp: δ_def field_split_simps)
  qed
qed

corollary open_bijective_linear_image_eq:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes linear f bij f
    shows open(f ` A) ⟷ open A
proof
  assume open(f ` A)
  then have open(f −` (f ` A))
     using assms by (force simp: linear_continuous_at linear_conv_bounded_linear
```

continuous_open_vimage)
  then show *open A*
    by (*simp add*: *assms bij_is_inj inj_vimage_image_eq*)
next
  assume *open A*
  then show *open*(*f* ‘ *A*)
    by (*simp add*: *assms bij_is_surj open_surjective_linear_image*)
qed

corollary *interior_bijective_linear_image*:
  fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
  assumes *linear f bij f*
  shows *interior* (*f* ‘ *S*) = *f* ‘ *interior S* (**is** *?lhs* = *?rhs*)
proof *safe*
  fix *x*
  assume *x*: *x* ∈ *?lhs*
  then obtain *T* **where** *open T* **and** *x* ∈ *T* **and** *T* ⊆ *f* ‘ *S*
    by (*metis interiorE*)
  then show *x* ∈ *?rhs*
   by (*metis* (*no_types*, *hide_lams*) *assms subsetD interior_maximal open_bijective_linear_image_eq subset_image_iff*)
next
  fix *x*
  assume *x*: *x* ∈ *interior S*
  then show *f x* ∈ *interior* (*f* ‘ *S*)
   by (*meson assms imageI image_mono interiorI interior_subset open_bijective_linear_image_eq open_interior*)
qed

lemma *interior_injective_linear_image*:
  fixes *f* :: *'a*::*euclidean_space* ⇒ *'a*::*euclidean_space*
  assumes *linear f inj f*
   shows *interior*(*f* ‘ *S*) = *f* ‘ (*interior S*)
 by (*simp add*: *linear_injective_imp_surjective assms bijI interior_bijective_linear_image*)

lemma *interior_surjective_linear_image*:
  fixes *f* :: *'a*::*euclidean_space* ⇒ *'a*::*euclidean_space*
  assumes *linear f surj f*
   shows *interior*(*f* ‘ *S*) = *f* ‘ (*interior S*)
 by (*simp add*: *assms interior_injective_linear_image linear_surjective_imp_injective*)

lemma *interior_negations*:
  fixes *S* :: *'a*::*euclidean_space set*
  shows *interior*(*uminus* ‘ *S*) = *image uminus* (*interior S*)
  by (*simp add*: *bij_uminus interior_bijective_linear_image linear_uminus*)

lemma *connected_linear_image*:
  fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*real_normed_vector*
  assumes *linear f* **and** *connected s*

```
  shows connected (f ` s)
using connected_continuous_image assms linear_continuous_on linear_conv_bounded_linear
by blast
```

4.1.19 "Isometry" (up to constant bounds) of Injective Linear Map

```
proposition injective_imp_isometric:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes s: closed s subspace s
    and f: bounded_linear f ∀x∈s. f x = 0 ⟶ x = 0
  shows ∃e>0. ∀x∈s. norm (f x) ≥ e * norm x
proof (cases s ⊆ {0::'a})
  case True
  have norm x ≤ norm (f x) if x ∈ s for x
  proof -
    from True that have x = 0 by auto
    then show ?thesis by simp
  qed
  then show ?thesis
    by (auto intro!: exI[where x=1])
next
  case False
  interpret f: bounded_linear f by fact
  from False obtain a where a: a ≠ 0 a ∈ s
    by auto
  from False have s ≠ {}
    by auto
  let ?S = {f x| x. x ∈ s ∧ norm x = norm a}
  let ?S' = {x::'a. x∈s ∧ norm x = norm a}
  let ?S'' = {x::'a. norm x = norm a}

  have ?S'' = frontier (cball 0 (norm a))
    by (simp add: sphere_def dist_norm)
  then have compact ?S'' by (metis compact_cball compact_frontier)
  moreover have ?S' = s ∩ ?S'' by auto
  ultimately have compact ?S'
    using closed_Int_compact[of s ?S''] using s(1) by auto
  moreover have *:f ` ?S' = ?S by auto
  ultimately have compact ?S
   using compact_continuous_image[OF linear_continuous_on[OF f(1)], of ?S'] by
auto
  then have closed ?S
    using compact_imp_closed by auto
  moreover from a have ?S ≠ {} by auto
  ultimately obtain b' where b'∈?S ∀y∈?S. norm b' ≤ norm y
    using distance_attains_inf[of ?S 0] unfolding dist_0_norm by auto
  then obtain b where b∈s
    and ba: norm b = norm a
```

and *b*: ∀ *x*∈{*x* ∈ *s*. *norm x* = *norm a*}. *norm* (*f b*) ≤ *norm* (*f x*)
unfolding ∗[*symmetric*] **unfolding** *image_iff* **by** *auto*

let *?e* = *norm* (*f b*) / *norm b*
have *norm b* > *0*
using *ba* **and** *a* **and** *norm_ge_zero* **by** *auto*
moreover have *norm* (*f b*) > *0*
using *f*(*2*)[*THEN bspec*[**where** *x*=*b*], *OF* ‹*b*∈*s*›]
using ‹*norm b* >*0*› **by** *simp*
ultimately have *0* < *norm* (*f b*) / *norm b* **by** *simp*
moreover
have *norm* (*f b*) / *norm b* ∗ *norm x* ≤ *norm* (*f x*) **if** *x*∈*s* **for** *x*
proof (*cases x* = *0*)
case *True*
then show *norm* (*f b*) / *norm b* ∗ *norm x* ≤ *norm* (*f x*)
by *auto*
next
case *False*
with ‹*a* ≠ *0*› **have** ∗: *0* < *norm a* / *norm x*
unfolding *zero_less_norm_iff*[*symmetric*] **by** *simp*
have ∀ *x*∈*s*. *c* ∗$_R$ *x* ∈ *s* **for** *c*
using *s*[*unfolded subspace_def*] **by** *simp*
with ‹*x* ∈ *s*› ‹*x* ≠ *0*› **have** (*norm a* / *norm x*) ∗$_R$ *x* ∈ {*x* ∈ *s*. *norm x* = *norm a*}
by *simp*
with ‹*x* ≠ *0*› ‹*a* ≠ *0*› **show** *norm* (*f b*) / *norm b* ∗ *norm x* ≤ *norm* (*f x*)
using *b*[*THEN bspec*[**where** *x*=(*norm a* / *norm x*) ∗$_R$ *x*]]
unfolding *f*.*scaleR* **and** *ba*
by (*auto simp*: *mult*.*commute pos_le_divide_eq pos_divide_le_eq*)
qed
ultimately show *?thesis* **by** *auto*
qed

proposition *closed_injective_image_subspace*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
assumes *subspace s bounded_linear f* ∀ *x*∈*s*. *f x* = *0* ⟶ *x* = *0 closed s*
shows *closed*(*f* ‘ *s*)
proof −
obtain *e* **where** *e* > *0* **and** *e*: ∀ *x*∈*s*. *e* ∗ *norm x* ≤ *norm* (*f x*)
using *injective_imp_isometric*[*OF assms*(*4*,*1*,*2*,*3*)] **by** *auto*
show *?thesis*
using *complete_isometric_image*[*OF* ‹*e*>*0*› *assms*(*1*,*2*) *e*] **and** *assms*(*4*)
unfolding *complete_eq_closed*[*symmetric*] **by** *auto*
qed

lemma *closure_bounded_linear_image_subset*:
assumes *f*: *bounded_linear f*
shows *f* ‘ *closure S* ⊆ *closure* (*f* ‘ *S*)

```
  using linear_continuous_on [OF f] closed_closure closure_subset
  by (rule image_closure_subset)

lemma closure_linear_image_subset:
  fixes f :: 'm::euclidean_space ⇒ 'n::real_normed_vector
  assumes linear f
  shows f ‘ (closure S) ⊆ closure (f ‘ S)
  using assms unfolding linear_conv_bounded_linear
  by (rule closure_bounded_linear_image_subset)

lemma closed_injective_linear_image:
    fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
    assumes S: closed S and f: linear f inj f
    shows closed (f ‘ S)
proof –
  obtain g where g: linear g g ∘ f = id
    using linear_injective_left_inverse [OF f] by blast
  then have confg: continuous_on (range f) g
    using linear_continuous_on linear_conv_bounded_linear by blast
  have [simp]: g ‘ f ‘ S = S
    using g by (simp add: image_comp)
  have cgf: closed (g ‘ f ‘ S)
    by (simp add: ‹g ∘ f = id› S image_comp)
  have [simp]: (range f ∩ g –‘ S) = f ‘ S
    using g unfolding o_def id_def image_def by auto metis+
  show ?thesis
  proof (rule closedin_closed_trans [of range f])
    show closedin (top_of_set (range f)) (f ‘ S)
      using continuous_closedin_preimage [OF confg cgf] by simp
    show closed (range f)
      apply (rule closed_injective_image_subspace)
      using f apply (auto simp: linear_linear linear_injective_0)
      done
  qed
qed

lemma closed_injective_linear_image_eq:
    fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
    assumes f: linear f inj f
      shows (closed(image f s) ⟷ closed s)
  by (metis closed_injective_linear_image closure_eq closure_linear_image_subset clo-
sure_subset_eq f(1) f(2) inj_image_subset_iff)

lemma closure_injective_linear_image:
    fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
    shows ⟦linear f; inj f⟧ ⟹ f ‘ (closure S) = closure (f ‘ S)
  apply (rule subset_antisym)
  apply (simp add: closure_linear_image_subset)
   by (simp add: closure_minimal closed_injective_linear_image closure_subset im-
```

age_mono)

lemma *closure_bounded_linear_image*:
  fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
  shows ⟦*linear f*; *bounded S*⟧ ⟹ *f* ‘ (*closure S*) = *closure* (*f* ‘ *S*)
 apply (*rule subset_antisym*, *simp add*: *closure_linear_image_subset*)
 apply (*rule closure_minimal*, *simp add*: *closure_subset image_mono*)
 by (*meson bounded_closure closed_closure compact_continuous_image compact_eq_bounded_closed linear_continuous_on linear_conv_bounded_linear*)

lemma *closure_scaleR*:
 fixes *S* :: *'a::real_normed_vector set*
 shows $((*_R)\ c)$ ‘ (*closure S*) = *closure* $(((*_R)\ c)$ ‘ *S*)
proof
 show $((*_R)\ c)$ ‘ (*closure S*) ⊆ *closure* $(((*_R)\ c)$ ‘ *S*)
  using *bounded_linear_scaleR_right*
  by (*rule closure_bounded_linear_image_subset*)
 show *closure* $(((*_R)\ c)$ ‘ *S*) ⊆ $((*_R)\ c)$ ‘ (*closure S*)
  by (*intro closure_minimal image_mono closure_subset closed_scaling closed_closure*)
qed

4.1.20 Some properties of a canonical subspace

lemma *closed_substandard*: *closed* {*x::'a::euclidean_space*. ∀ *i*∈*Basis*. *P i* ⟶ *x*•*i* = *0*}
 (**is** *closed ?A*)
proof –
 let *?D* = {*i*∈*Basis*. *P i*}
 have *closed* (⋂ *i*∈*?D*. {*x::'a*. *x*•*i* = *0*})
  by (*simp add*: *closed_INT closed_Collect_eq continuous_on_inner*)
 also have (⋂ *i*∈*?D*. {*x::'a*. *x*•*i* = *0*}) = *?A*
  by *auto*
 finally show *closed ?A* **.**
qed

lemma *closed_subspace*:
 fixes *s* :: *'a::euclidean_space set*
 assumes *subspace s*
 shows *closed s*
proof –
 have *dim s* ≤ *card* (*Basis* :: *'a set*)
  using *dim_subset_UNIV* **by** *auto*
 with *ex_card*[*OF this*] **obtain** *d* :: *'a set* **where** *t*: *card d* = *dim s* **and** *d*: *d* ⊆ *Basis*
  by *auto*
 let *?t* = {*x::'a*. ∀ *i*∈*Basis*. *i* ∉ *d* ⟶ *x*•*i* = *0*}
 have ∃ *f*. *linear f* ∧ *f* ‘ {*x::'a*. ∀ *i*∈*Basis*. *i* ∉ *d* ⟶ *x* • *i* = *0*} = *s* ∧
   inj_on f {*x::'a*. ∀ *i*∈*Basis*. *i* ∉ *d* ⟶ *x* • *i* = *0*}
  using *dim_substandard*[*of d*] *t d assms*

by (*intro subspace_isomorphism*[*OF subspace_substandard*[*of* $\lambda i.\ i \notin d$]]) (*auto simp*: *inner_Basis*)
then obtain *f* **where** *f*:
linear f
f ‘ $\{x.\ \forall i{\in}Basis.\ i \notin d \longrightarrow x \cdot i = 0\} = s$
inj_on f $\{x.\ \forall i{\in}Basis.\ i \notin d \longrightarrow x \cdot i = 0\}$
by *blast*
interpret *f*: *bounded_linear f*
using *f* **by** (*simp add*: *linear_conv_bounded_linear*)
have $x \in ?t \Longrightarrow f\ x = 0 \Longrightarrow x = 0$ **for** *x*
using *f.zero d f*(*3*)[*THEN inj_onD*, *of x 0*] **by** *auto*
moreover have *closed ?t* **by** (*rule closed_substandard*)
moreover have *subspace ?t* **by** (*rule subspace_substandard*)
ultimately show *?thesis*
using *closed_injective_image_subspace*[*of ?t f*]
unfolding *f*(*2*) **using** *f*(*1*) **unfolding** *linear_conv_bounded_linear* **by** *auto*
qed

lemma *complete_subspace*: *subspace s* $\Longrightarrow$ *complete s*
for *s* :: *'a::euclidean_space set*
using *complete_eq_closed closed_subspace* **by** *auto*

lemma *closed_span* [*iff*]: *closed* (*span s*)
for *s* :: *'a::euclidean_space set*
by (*simp add*: *closed_subspace*)

lemma *dim_closure* [*simp*]: *dim* (*closure s*) = *dim s* (**is** *?dc* = *?d*)
for *s* :: *'a::euclidean_space set*
proof –
have $?dc \le ?d$
using *closure_minimal*[*OF span_superset*, *of s*]
using *closed_subspace*[*OF subspace_span*, *of s*]
using *dim_subset*[*of closure s span s*]
by *simp*
then show *?thesis*
using *dim_subset*[*OF closure_subset*, *of s*]
by *simp*
qed

4.1.21 Set Distance

lemma *setdist_compact_closed*:
fixes *A* :: *'a::heine_borel set*
assumes *A*: *compact A* **and** *B*: *closed B*
and $A \neq \{\}$ $B \neq \{\}$
shows $\exists x \in A.\ \exists y \in B.$ *dist x y* = *setdist A B*
proof –
obtain *x* **where** $x \in A$ *setdist A B* = *infdist x B*
by (*metis A assms*(*3*) *setdist_attains_inf setdist_sym*)

```
  moreover
  obtain y where y ∈ B infdist x B = dist x y
    using B ‹B ≠ {}› infdist_attains_inf by blast
  ultimately show ?thesis
    using ‹x ∈ A› ‹y ∈ B› by auto
qed

lemma setdist_closed_compact:
  fixes S :: 'a::heine_borel set
  assumes S: closed S and T: compact T
      and S ≠ {} T ≠ {}
    shows ∃ x ∈ S. ∃ y ∈ T. dist x y = setdist S T
  using setdist_compact_closed [OF T S ‹T ≠ {}› ‹S ≠ {}›]
  by (metis dist_commute setdist_sym)

lemma setdist_eq_0_compact_closed:
  assumes S: compact S and T: closed T
    shows setdist S T = 0 ⟷ S = {} ∨ T = {} ∨ S ∩ T ≠ {}
proof (cases S = {} ∨ T = {})
  case True
  then show ?thesis
    by force
next
  case False
  then show ?thesis
     by (metis S T disjoint_iff_not_equal in_closed_iff_infdist_zero setdist_attains_inf
setdist_eq_0I setdist_sym)
qed

corollary setdist_gt_0_compact_closed:
  assumes S: compact S and T: closed T
    shows setdist S T > 0 ⟷ (S ≠ {} ∧ T ≠ {} ∧ S ∩ T = {})
  using setdist_pos_le [of S T] setdist_eq_0_compact_closed [OF assms] by linarith

lemma setdist_eq_0_closed_compact:
  assumes S: closed S and T: compact T
    shows setdist S T = 0 ⟷ S = {} ∨ T = {} ∨ S ∩ T ≠ {}
  using setdist_eq_0_compact_closed [OF T S]
  by (metis Int_commute setdist_sym)

lemma setdist_eq_0_bounded:
  fixes S :: 'a::heine_borel set
  assumes bounded S ∨ bounded T
  shows setdist S T = 0 ⟷ S = {} ∨ T = {} ∨ closure S ∩ closure T ≠ {}
proof (cases S = {} ∨ T = {})
  case False
  then show ?thesis
    using setdist_eq_0_compact_closed [of closure S closure T]
          setdist_eq_0_closed_compact [of closure S closure T] assms
```

```
  by (force simp: bounded_closure compact_eq_bounded_closed)
qed force

lemma setdist_eq_0_sing_1:
  setdist {x} S = 0 ⟷ S = {} ∨ x ∈ closure S
  by (metis in_closure_iff_infdist_zero infdist_def infdist_eq_setdist)

lemma setdist_eq_0_sing_2:
  setdist S {x} = 0 ⟷ S = {} ∨ x ∈ closure S
  by (metis setdist_eq_0_sing_1 setdist_sym)

lemma setdist_neq_0_sing_1:
  ⟦setdist {x} S = a; a ≠ 0⟧ ⟹ S ≠ {} ∧ x ∉ closure S
  by (metis setdist_closure_2 setdist_empty2 setdist_eq_0I singletonI)

lemma setdist_neq_0_sing_2:
  ⟦setdist S {x} = a; a ≠ 0⟧ ⟹ S ≠ {} ∧ x ∉ closure S
  by (simp add: setdist_neq_0_sing_1 setdist_sym)

lemma setdist_sing_in_set:
   x ∈ S ⟹ setdist {x} S = 0
  by (simp add: setdist_eq_0I)

lemma setdist_eq_0_closed:
   closed S ⟹ (setdist {x} S = 0 ⟷ S = {} ∨ x ∈ S)
by (simp add: setdist_eq_0_sing_1)

lemma setdist_eq_0_closedin:
  shows ⟦closedin (top_of_set U) S; x ∈ U⟧
        ⟹ (setdist {x} S = 0 ⟷ S = {} ∨ x ∈ S)
  by (auto simp: closedin_limpt setdist_eq_0_sing_1 closure_def)

lemma setdist_gt_0_closedin:
  shows ⟦closedin (top_of_set U) S; x ∈ U; S ≠ {}; x ∉ S⟧
        ⟹ setdist {x} S > 0
  using less_eq_real_def setdist_eq_0_closedin by fastforce

no_notation
  eucl_less (infix <e 50)

end
```

4.2 Convex Sets and Functions on (Normed) Euclidean Spaces

```
theory Convex_Euclidean_Space
imports
  Convex
```

Topology_Euclidean_Space
begin

4.2.1 Topological Properties of Convex Sets and Functions

lemma *aff_dim_cball*:
fixes *a* :: *'n::euclidean_space*
assumes *e* > *0*
shows *aff_dim* (*cball a e*) = *int* (*DIM*(*'n*))
proof −
have (λ*x*. *a* + *x*) ‘ (*cball 0 e*) ⊆ *cball a e*
unfolding *cball_def dist_norm* **by** *auto*
then have *aff_dim* (*cball* (*0* :: *'n::euclidean_space*) *e*) ≤ *aff_dim* (*cball a e*)
using *aff_dim_translation_eq*[*of a cball 0 e*]
aff_dim_subset[*of* (+) *a* ‘ *cball 0 e cball a e*]
by *auto*
moreover have *aff_dim* (*cball* (*0* :: *'n::euclidean_space*) *e*) = *int* (*DIM*(*'n*))
using *hull_inc*[*of* (*0* :: *'n::euclidean_space*) *cball 0 e*]
centre_in_cball[*of* (*0* :: *'n::euclidean_space*)] *assms*
by (*simp add*: *dim_cball*[*of e*] *aff_dim_zero*[*of cball 0 e*])
ultimately show *?thesis*
using *aff_dim_le_DIM*[*of cball a e*] **by** *auto*
qed

lemma *aff_dim_open*:
fixes *S* :: *'n::euclidean_space set*
assumes *open S*
and *S* ≠ {}
shows *aff_dim S* = *int* (*DIM*(*'n*))
proof −
obtain *x* **where** *x* ∈ *S*
using *assms* **by** *auto*
then obtain *e* **where** *e*: *e* > *0 cball x e* ⊆ *S*
using *open_contains_cball*[*of S*] *assms* **by** *auto*
then have *aff_dim* (*cball x e*) ≤ *aff_dim S*
using *aff_dim_subset* **by** *auto*
with *e* **show** *?thesis*
using *aff_dim_cball*[*of e x*] *aff_dim_le_DIM*[*of S*] **by** *auto*
qed

lemma *low_dim_interior*:
fixes *S* :: *'n::euclidean_space set*
assumes ¬ *aff_dim S* = *int* (*DIM*(*'n*))
shows *interior S* = {}
proof −
have *aff_dim*(*interior S*) ≤ *aff_dim S*
using *interior_subset aff_dim_subset*[*of interior S S*] **by** *auto*
then show *?thesis*
using *aff_dim_open*[*of interior S*] *aff_dim_le_DIM*[*of S*] *assms* **by** *auto*

qed

corollary *empty_interior_lowdim*:
  fixes *S* :: *'n::euclidean_space set*
  shows *dim S < DIM ('n) ⟹ interior S = {}*
by (*metis low_dim_interior affine_hull_UNIV dim_affine_hull less_not_refl dim_UNIV*)

corollary *aff_dim_nonempty_interior*:
  fixes *S* :: *'a::euclidean_space set*
  shows *interior S ≠ {} ⟹ aff_dim S = DIM('a)*
by (*metis low_dim_interior*)

4.2.2 Relative interior of a set

definition *rel_interior S* =
  {*x*. ∃ *T*. *openin* (*top_of_set* (*affine hull S*)) *T* ∧ *x* ∈ *T* ∧ *T* ⊆ *S*}

lemma *rel_interior_mono*:
   ⟦*S* ⊆ *T*; *affine hull S* = *affine hull T*⟧
   ⟹ (*rel_interior S*) ⊆ (*rel_interior T*)
  by (*auto simp*: *rel_interior_def*)

lemma *rel_interior_maximal*:
   ⟦*T* ⊆ *S*; *openin*(*top_of_set* (*affine hull S*)) *T*⟧ ⟹ *T* ⊆ (*rel_interior S*)
  by (*auto simp*: *rel_interior_def*)

lemma *rel_interior*: *rel_interior S* = {*x* ∈ *S*. ∃ *T*. *open T* ∧ *x* ∈ *T* ∧ *T* ∩ *affine hull S* ⊆ *S*}
      (**is** *?lhs* = *?rhs*)
proof
  show *?lhs* ⊆ *?rhs*
    by (*force simp add*: *rel_interior_def openin_open*)
  { **fix** *x T*
    assume ∗: *x* ∈ *S open T x* ∈ *T T* ∩ *affine hull S* ⊆ *S*
    then have ∗∗: *x* ∈ *T* ∩ *affine hull S*
      using *hull_inc* **by** *auto*
    with ∗ **have** ∃ *Tb*. (∃ *Ta*. *open Ta* ∧ *Tb* = *affine hull S* ∩ *Ta*) ∧ *x* ∈ *Tb* ∧ *Tb* ⊆ *S*
      by (*rule_tac x* = *T* ∩ (*affine hull S*) **in** *exI*) *auto*
  }
  then show *?rhs* ⊆ *?lhs*
    by (*force simp add*: *rel_interior_def openin_open*)
qed

lemma *mem_rel_interior*: *x* ∈ *rel_interior S* ⟷ (∃ *T*. *open T* ∧ *x* ∈ *T* ∩ *S* ∧ *T* ∩ *affine hull S* ⊆ *S*)
  by (*auto simp*: *rel_interior*)

lemma *mem_rel_interior_ball*:

x ∈ *rel_interior S* ⟷ *x* ∈ *S* ∧ (∃ *e*. *e* > *0* ∧ *ball x e* ∩ *affine hull S* ⊆ *S*)
(**is** *?lhs* = *?rhs*)
proof
assume *?rhs* **then show** *?lhs*
by (*simp add*: *rel_interior*) (*meson Elementary_Metric_Spaces.open_ball centre_in_ball*)
qed (*force simp*: *rel_interior open_contains_ball*)

lemma *rel_interior_ball*:
rel_interior S = {*x* ∈ *S*. ∃ *e*. *e* > *0* ∧ *ball x e* ∩ *affine hull S* ⊆ *S*}
using *mem_rel_interior_ball* [*of* _ *S*] **by** *auto*

lemma *mem_rel_interior_cball*:
x ∈ *rel_interior S* ⟷ *x* ∈ *S* ∧ (∃ *e*. *e* > *0* ∧ *cball x e* ∩ *affine hull S* ⊆ *S*)
(**is** *?lhs* = *?rhs*)
proof
assume *?rhs* **then obtain** *e* **where** *x* ∈ *S e* > *0 cball x e* ∩ *affine hull S* ⊆ *S*
by (*auto simp*: *rel_interior*)
then have *ball x e* ∩ *affine hull S* ⊆ *S*
by *auto*
then show *?lhs*
using ‹*0* < *e*› ‹*x* ∈ *S*› *rel_interior_ball* **by** *auto*
qed (*force simp*: *rel_interior open_contains_cball*)

lemma *rel_interior_cball*:
rel_interior S = {*x* ∈ *S*. ∃ *e*. *e* > *0* ∧ *cball x e* ∩ *affine hull S* ⊆ *S*}
using *mem_rel_interior_cball* [*of* _ *S*] **by** *auto*

lemma *rel_interior_empty* [*simp*]: *rel_interior* {} = {}
by (*auto simp*: *rel_interior_def*)

lemma *affine_hull_sing* [*simp*]: *affine hull* {*a* :: ′*n*::*euclidean_space*} = {*a*}
by (*metis affine_hull_eq affine_sing*)

lemma *rel_interior_sing* [*simp*]:
fixes *a* :: ′*n*::*euclidean_space* **shows** *rel_interior* {*a*} = {*a*}
proof −
have ∃ *x*::*real*. *0* < *x*
using *zero_less_one* **by** *blast*
then show *?thesis*
by (*auto simp*: *rel_interior_ball*)
qed

lemma *subset_rel_interior*:
fixes *S T* :: ′*n*::*euclidean_space set*
assumes *S* ⊆ *T*
and *affine hull S* = *affine hull T*
shows *rel_interior S* ⊆ *rel_interior T*
using *assms* **by** (*auto simp*: *rel_interior_def*)

```
lemma rel_interior_subset: rel_interior S ⊆ S
  by (auto simp: rel_interior_def)

lemma rel_interior_subset_closure: rel_interior S ⊆ closure S
  using rel_interior_subset by (auto simp: closure_def)

lemma interior_subset_rel_interior: interior S ⊆ rel_interior S
  by (auto simp: rel_interior interior_def)

lemma interior_rel_interior:
  fixes S :: 'n::euclidean_space set
  assumes aff_dim S = int(DIM('n))
  shows rel_interior S = interior S
proof -
  have affine hull S = UNIV
    using assms affine_hull_UNIV[of S] by auto
  then show ?thesis
    unfolding rel_interior interior_def by auto
qed

lemma rel_interior_interior:
  fixes S :: 'n::euclidean_space set
  assumes affine hull S = UNIV
  shows rel_interior S = interior S
  using assms unfolding rel_interior interior_def by auto

lemma rel_interior_open:
  fixes S :: 'n::euclidean_space set
  assumes open S
  shows rel_interior S = S
 by (metis assms interior_eq interior_subset_rel_interior rel_interior_subset set_eq_subset)

lemma interior_rel_interior_gen:
  fixes S :: 'n::euclidean_space set
  shows interior S = (if aff_dim S = int(DIM('n)) then rel_interior S else {})
  by (metis interior_rel_interior low_dim_interior)

lemma rel_interior_nonempty_interior:
  fixes S :: 'n::euclidean_space set
  shows interior S ≠ {} ⟹ rel_interior S = interior S
by (metis interior_rel_interior_gen)

lemma affine_hull_nonempty_interior:
  fixes S :: 'n::euclidean_space set
  shows interior S ≠ {} ⟹ affine hull S = UNIV
by (metis affine_hull_UNIV interior_rel_interior_gen)

lemma rel_interior_affine_hull [simp]:
  fixes S :: 'n::euclidean_space set
```

shows *rel_interior* (*affine hull S*) = *affine hull S*
proof −
have ∗: *rel_interior* (*affine hull S*) ⊆ *affine hull S*
using *rel_interior_subset* **by** *auto*
{
fix *x*
assume *x*: *x* ∈ *affine hull S*
define *e* :: *real* **where** *e* = *1*
then have *e* > *0 ball x e* ∩ *affine hull* (*affine hull S*) ⊆ *affine hull S*
using *hull_hull*[*of* _ *S*] **by** *auto*
then have *x* ∈ *rel_interior* (*affine hull S*)
using *x rel_interior_ball*[*of affine hull S*] **by** *auto*
}
then show *?thesis* **using** ∗ **by** *auto*
qed

lemma *rel_interior_UNIV* [*simp*]: *rel_interior* (*UNIV* :: (*'n*::*euclidean_space*) *set*) = *UNIV*
by (*metis open_UNIV rel_interior_open*)

lemma *rel_interior_convex_shrink*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *convex S*
and *c* ∈ *rel_interior S*
and *x* ∈ *S*
and *0* < *e*
and *e* ≤ *1*
shows *x* − *e* $*_R$ (*x* − *c*) ∈ *rel_interior S*
proof −
obtain *d* **where** *d* > *0* **and** *d*: *ball c d* ∩ *affine hull S* ⊆ *S*
using *assms*(*2*) **unfolding** *mem_rel_interior_ball* **by** *auto*
{
fix *y*
assume *as*: *dist* (*x* − *e* $*_R$ (*x* − *c*)) *y* < *e* ∗ *d y* ∈ *affine hull S*
have ∗: *y* = (*1* − (*1* − *e*)) $*_R$ ((*1* / *e*) $*_R$ *y* − ((*1* − *e*) / *e*) $*_R$ *x*) + (*1* − *e*) $*_R$ *x*
using ‹*e* > *0*› **by** (*auto simp*: *scaleR_left_diff_distrib scaleR_right_diff_distrib*)
have *x* ∈ *affine hull S*
using *assms hull_subset*[*of S*] **by** *auto*
moreover have *1* / *e* + − ((*1* − *e*) / *e*) = *1*
using ‹*e* > *0*› *left_diff_distrib*[*of 1* (*1*−*e*) *1*/*e*] **by** *auto*
ultimately have ∗∗: (*1* / *e*) $*_R$ *y* − ((*1* − *e*) / *e*) $*_R$ *x* ∈ *affine hull S*
using *as affine_affine_hull*[*of S*] *mem_affine*[*of affine hull S y x* (*1* / *e*) −((*1* − *e*) / *e*)]
by (*simp add*: *algebra_simps*)
have *c* − ((*1* / *e*) $*_R$ *y* − ((*1* − *e*) / *e*) $*_R$ *x*) = (*1* / *e*) $*_R$ (*e* $*_R$ *c* − *y* + (*1* − *e*) $*_R$ *x*)
using ‹*e* > *0*›
by (*auto simp*: *euclidean_eq_iff* [**where** *'a*=*'a*] *field_simps inner_simps*)

```
    then have dist c ((1 / e) *R y − ((1 − e) / e) *R x) = |1/e| * norm (e *R
c − y + (1 − e) *R x)
      unfolding dist_norm norm_scaleR[symmetric] by auto
    also have ... = |1/e| * norm (x − e *R (x − c) − y)
      by (auto intro!:arg_cong[where f=norm] simp add: algebra_simps)
    also have ... < d
      using as[unfolded dist_norm] and ‹e > 0›
      by (auto simp:pos_divide_less_eq[OF ‹e > 0›] mult.commute)
    finally have (1 / e) *R y − ((1 − e) / e) *R x ∈ S
      using **  d by auto
    then have y ∈ S
      using * convexD [OF ‹convex S›] assms(3−5)
      by (metis diff_add_cancel diff_ge_0_iff_ge le_add_same_cancel1 less_eq_real_def)
  }
  then have ball (x − e *R (x − c)) (e*d) ∩ affine hull S ⊆ S
    by auto
  moreover have e * d > 0
    using ‹e > 0› ‹d > 0› by simp
  moreover have c: c ∈ S
    using assms rel_interior_subset by auto
  moreover from c have x − e *R (x − c) ∈ S
    using convexD_alt[of S x c e] assms
     by (metis  diff_add_eq diff_diff_eq2 less_eq_real_def scaleR_diff_left scaleR_one
scale_right_diff_distrib)
  ultimately show ?thesis
    using mem_rel_interior_ball[of x − e *R (x − c) S] ‹e > 0› by auto
qed

lemma interior_real_atLeast [simp]:
  fixes a :: real
  shows interior {a..} = {a<..}
proof −
  {
    fix y
    have ball y (y − a) ⊆ {a..}
      by (auto simp: dist_norm)
    moreover assume a < y
    ultimately have y ∈ interior {a..}
      by (force simp add: mem_interior)
  }
  moreover
  {
    fix y
    assume y ∈ interior {a..}
    then obtain e where e: e > 0 cball y e ⊆ {a..}
      using mem_interior_cball[of y {a..}] by auto
    moreover from e have y − e ∈ cball y e
      by (auto simp: cball_def dist_norm)
    ultimately have a ≤ y − e by blast
```

```
    then have a < y using e by auto
  }
  ultimately show ?thesis by auto
qed

lemma continuous_ge_on_Ioo:
   assumes continuous_on {c..d} g ⋀x. x ∈ {c<..<d} ⟹ g x ≥ a c < d x ∈
{c..d}
  shows g (x::real) ≥ (a::real)
proof−
 from assms(3) have {c..d} = closure {c<..<d} by (rule closure_greaterThanLessThan[symmetric])
  also from assms(2) have {c<..<d} ⊆ (g −‘ {a..} ∩ {c..d}) by auto
  hence closure {c<..<d} ⊆ closure (g −‘ {a..} ∩ {c..d}) by (rule closure_mono)
  also from assms(1) have closed (g −‘ {a..} ∩ {c..d})
    by (auto simp: continuous_on_closed_vimage)
  hence closure (g −‘ {a..} ∩ {c..d}) = g −‘ {a..} ∩ {c..d} by simp
  finally show ?thesis using ‹x ∈ {c..d}› by auto
qed

lemma interior_real_atMost [simp]:
  fixes a :: real
  shows interior {..a} = {..<a}
proof −
  {
    fix y
    have ball y (a − y) ⊆ {..a}
      by (auto simp: dist_norm)
    moreover assume a > y
    ultimately have y ∈ interior {..a}
      by (force simp add: mem_interior)
  }
  moreover
  {
    fix y
    assume y ∈ interior {..a}
    then obtain e where e: e > 0 cball y e ⊆ {..a}
      using mem_interior_cball[of y {..a}] by auto
    moreover from e have y + e ∈ cball y e
      by (auto simp: cball_def dist_norm)
    ultimately have a ≥ y + e by auto
    then have a > y using e by auto
  }
  ultimately show ?thesis by auto
qed

lemma interior_atLeastAtMost_real [simp]: interior {a..b} = {a<..<b :: real}
proof−
  have {a..b} = {a..} ∩ {..b} by auto
  also have interior ... = {a<..} ∩ {..<b}
```

by (*simp*)
also have ... = {*a*<..<*b*} **by** *auto*
finally show *?thesis* **.**
qed

lemma *interior_atLeastLessThan* [*simp*]:
fixes *a*::*real* **shows** *interior* {*a*..<*b*} = {*a*<..<*b*}
by (*metis atLeastLessThan_def greaterThanLessThan_def interior_atLeastAtMost_real interior_Int interior_interior interior_real_atLeast*)

lemma *interior_lessThanAtMost* [*simp*]:
fixes *a*::*real* **shows** *interior* {*a*<..*b*} = {*a*<..<*b*}
by (*metis atLeastAtMost_def greaterThanAtMost_def interior_atLeastAtMost_real interior_Int*
interior_interior interior_real_atLeast)

lemma *interior_greaterThanLessThan_real* [*simp*]: *interior* {*a*<..<*b*} = {*a*<..<*b* :: *real*}
by (*metis interior_atLeastAtMost_real interior_interior*)

lemma *frontier_real_atMost* [*simp*]:
fixes *a* :: *real*
shows *frontier* {..*a*} = {*a*}
unfolding *frontier_def* **by** *auto*

lemma *frontier_real_atLeast* [*simp*]: *frontier* {*a*..} = {*a*::*real*}
by (*auto simp*: *frontier_def*)

lemma *frontier_real_greaterThan* [*simp*]: *frontier* {*a*<..} = {*a*::*real*}
by (*auto simp*: *interior_open frontier_def*)

lemma *frontier_real_lessThan* [*simp*]: *frontier* {..<*a*} = {*a*::*real*}
by (*auto simp*: *interior_open frontier_def*)

lemma *rel_interior_real_box* [*simp*]:
fixes *a b* :: *real*
assumes *a* < *b*
shows *rel_interior* {*a* .. *b*} = {*a* <..< *b*}
proof −
have *box a b* ≠ {}
using *assms*
unfolding *set_eq_iff*
by (*auto intro*!: *exI*[*of* _ (*a* + *b*) / *2*] *simp*: *box_def*)
then show *?thesis*
using *interior_rel_interior_gen*[*of cbox a b*, *symmetric*]
by (*simp split*: *if_split_asm del*: *box_real add*: *box_real*[*symmetric*])
qed

lemma *rel_interior_real_semiline* [*simp*]:

fixes *a* :: *real*
shows *rel_interior* {*a*..} = {*a*<..}
proof −
have ∗: {*a*<..} ≠ {}
unfolding *set_eq_iff* **by** (*auto intro*!: *exI*[*of* _ *a* + *1*])
then show *?thesis* **using** *interior_real_atLeast interior_rel_interior_gen*[*of* {*a*..}]
by (*auto split*: *if_split_asm*)
qed

Relative open sets

definition *rel_open S* ⟷ *rel_interior S* = *S*

lemma *rel_open*: *rel_open S* ⟷ *openin* (*top_of_set* (*affine hull S*)) *S* (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
unfolding *rel_open_def rel_interior_def*
using *openin_subopen*[*of top_of_set* (*affine hull S*) *S*] **by** *auto*
qed (*auto simp*: *rel_open_def rel_interior_def*)

lemma *openin_rel_interior*: *openin* (*top_of_set* (*affine hull S*)) (*rel_interior S*)
using *openin_subopen* **by** (*fastforce simp add*: *rel_interior_def*)

lemma *openin_set_rel_interior*:
openin (*top_of_set S*) (*rel_interior S*)
by (*rule openin_subset_trans* [*OF openin_rel_interior rel_interior_subset hull_subset*])

lemma *affine_rel_open*:
fixes *S* :: *'n*::*euclidean_space set*
assumes *affine S*
shows *rel_open S*
unfolding *rel_open_def*
using *assms rel_interior_affine_hull*[*of S*] *affine_hull_eq*[*of S*]
by *metis*

lemma *affine_closed*:
fixes *S* :: *'n*::*euclidean_space set*
assumes *affine S*
shows *closed S*
proof −
{
assume *S* ≠ {}
then obtain *L* **where** *L*: *subspace L affine_parallel S L*
using *assms affine_parallel_subspace*[*of S*] **by** *auto*
then obtain *a* **where** *a*: *S* = ((+) *a* ‘ *L*)
using *affine_parallel_def*[*of L S*] *affine_parallel_commut* **by** *auto*
from *L* **have** *closed L* **using** *closed_subspace* **by** *auto*

```
    then have closed S
      using closed_translation a by auto
  }
  then show ?thesis by auto
qed

lemma closure_affine_hull:
  fixes S :: 'n::euclidean_space set
  shows closure S ⊆ affine hull S
  by (intro closure_minimal hull_subset affine_closed affine_affine_hull)

lemma closed_affine_hull [iff]:
  fixes S :: 'n::euclidean_space set
  shows closed (affine hull S)
  by (metis affine_affine_hull affine_closed)

lemma closure_same_affine_hull [simp]:
  fixes S :: 'n::euclidean_space set
  shows affine hull (closure S) = affine hull S
proof -
  have affine hull (closure S) ⊆ affine hull S
    using hull_mono[of closure S affine hull S affine]
      closure_affine_hull[of S] hull_hull[of affine S]
    by auto
  moreover have affine hull (closure S) ⊇ affine hull S
    using hull_mono[of S closure S affine] closure_subset by auto
  ultimately show ?thesis by auto
qed

lemma closure_aff_dim [simp]:
  fixes S :: 'n::euclidean_space set
  shows aff_dim (closure S) = aff_dim S
proof -
  have aff_dim S ≤ aff_dim (closure S)
    using aff_dim_subset closure_subset by auto
  moreover have aff_dim (closure S) ≤ aff_dim (affine hull S)
    using aff_dim_subset closure_affine_hull by blast
  moreover have aff_dim (affine hull S) = aff_dim S
    using aff_dim_affine_hull by auto
  ultimately show ?thesis by auto
qed

lemma rel_interior_closure_convex_shrink:
  fixes S :: _::euclidean_space set
  assumes convex S
    and c ∈ rel_interior S
    and x ∈ closure S
    and e > 0
    and e ≤ 1
```

```
  shows x − e *R (x − c) ∈ rel_interior S
proof −
  obtain d where d > 0 and d: ball c d ∩ affine hull S ⊆ S
    using assms(2) unfolding mem_rel_interior_ball by auto
  have ∃ y ∈ S. norm (y − x) * (1 − e) < e * d
  proof (cases x ∈ S)
    case True
    then show ?thesis using ‹e > 0› ‹d > 0› by force
  next
    case False
    then have x: x islimpt S
      using assms(3)[unfolded closure_def] by auto
    show ?thesis
    proof (cases e = 1)
      case True
      obtain y where y ∈ S y ≠ x dist y x < 1
        using x[unfolded islimpt_approachable,THEN spec[where x=1]] by auto
      then show ?thesis
        unfolding True using ‹d > 0› by (force simp add: )
    next
      case False
      then have 0 < e * d / (1 − e) and *: 1 − e > 0
        using ‹e ≤ 1› ‹e > 0› ‹d > 0› by auto
      then obtain y where y ∈ S y ≠ x dist y x < e * d / (1 − e)
       using x[unfolded islimpt_approachable,THEN spec[where x=e*d / (1 − e)]]
by auto
      then show ?thesis
        unfolding dist_norm using pos_less_divide_eq[OF *] by force
    qed
  qed
  then obtain y where y ∈ S and y: norm (y − x) * (1 − e) < e * d
    by auto
  define z where z = c + ((1 − e) / e) *R (x − y)
  have *: x − e *R (x − c) = y − e *R (y − z)
    unfolding z_def using ‹e > 0›
   by (auto simp: scaleR_right_diff_distrib scaleR_right_distrib scaleR_left_diff_distrib)
  have zball: z ∈ ball c d
    using mem_ball z_def dist_norm[of c]
    using y and assms(4,5)
    by (simp add: norm_minus_commute) (simp add: field_simps)
  have x ∈ affine hull S
    using closure_affine_hull assms by auto
  moreover have y ∈ affine hull S
    using ‹y ∈ S› hull_subset[of S] by auto
  moreover have c ∈ affine hull S
    using assms rel_interior_subset hull_subset[of S] by auto
  ultimately have z ∈ affine hull S
    using z_def affine_affine_hull[of S]
      mem_affine_3_minus [of affine hull S c x y (1 − e) / e]
```

```
      assms
    by simp
  then have z ∈ S using d zball by auto
  obtain d1 where d1 > 0 and d1: ball z d1 ≤ ball c d
    using zball open_ball[of c d] openE[of ball c d z] by auto
  then have ball z d1 ∩ affine hull S ⊆ ball c d ∩ affine hull S
    by auto
  then have ball z d1 ∩ affine hull S ⊆ S
    using d by auto
  then have z ∈ rel_interior S
    using mem_rel_interior_ball using ‹d1 > 0› ‹z ∈ S› by auto
  then have y − e *R (y − z) ∈ rel_interior S
    using rel_interior_convex_shrink[of S z y e] assms ‹y ∈ S› by auto
  then show ?thesis using * by auto
qed

lemma rel_interior_eq:
   rel_interior s = s ⟷ openin(top_of_set (affine hull s)) s
using rel_open rel_open_def by blast

lemma rel_interior_openin:
   openin(top_of_set (affine hull s)) s ⟹ rel_interior s = s
by (simp add: rel_interior_eq)

lemma rel_interior_affine:
  fixes S :: 'n::euclidean_space set
  shows  affine S ⟹ rel_interior S = S
using affine_rel_open rel_open_def by auto

lemma rel_interior_eq_closure:
  fixes S :: 'n::euclidean_space set
  shows rel_interior S = closure S ⟷ affine S
proof (cases S = {})
  case True
 then show ?thesis
    by auto
next
  case False show ?thesis
  proof
    assume eq: rel_interior S = closure S
    have openin (top_of_set (affine hull S)) S
    by (metis eq closure_subset openin_rel_interior rel_interior_subset subset_antisym)
    moreover have closedin (top_of_set (affine hull S)) S
      by (metis closed_subset closure_subset_eq eq hull_subset rel_interior_subset)
    ultimately have S = {} ∨ S = affine hull S
      using convex_connected connected_clopen convex_affine_hull by metis
    with False have affine hull S = S
      by auto
    then show affine S
```

```
    by (metis affine_hull_eq)
  next
    assume affine S
    then show rel_interior S = closure S
      by (simp add: rel_interior_affine affine_closed)
  qed
qed
```

Relative interior preserves under linear transformations

```
lemma rel_interior_translation_aux:
  fixes a :: 'n::euclidean_space
  shows ((λx. a + x) ` rel_interior S) ⊆ rel_interior ((λx. a + x) ` S)
proof -
  {
    fix x
    assume x: x ∈ rel_interior S
    then obtain T where open T x ∈ T ∩ S T ∩ affine hull S ⊆ S
      using mem_rel_interior[of x S] by auto
    then have open ((λx. a + x) ` T)
      and a + x ∈ ((λx. a + x) ` T) ∩ ((λx. a + x) ` S)
      and ((λx. a + x) ` T) ∩ affine hull ((λx. a + x) ` S) ⊆ (λx. a + x) ` S
      using affine_hull_translation[of a S] open_translation[of T a] x by auto
    then have a + x ∈ rel_interior ((λx. a + x) ` S)
      using mem_rel_interior[of a+x ((λx. a + x) ` S)] by auto
  }
  then show ?thesis by auto
qed

lemma rel_interior_translation:
  fixes a :: 'n::euclidean_space
  shows rel_interior ((λx. a + x) ` S) = (λx. a + x) ` rel_interior S
proof -
  have (λx. (−a) + x) ` rel_interior ((λx. a + x) ` S) ⊆ rel_interior S
    using rel_interior_translation_aux[of −a (λx. a + x) ` S]
      translation_assoc[of −a a]
    by auto
  then have ((λx. a + x) ` rel_interior S) ⊇ rel_interior ((λx. a + x) ` S)
    using translation_inverse_subset[of a rel_interior ((+) a ` S) rel_interior S]
    by auto
  then show ?thesis
    using rel_interior_translation_aux[of a S] by auto
qed


lemma affine_hull_linear_image:
  assumes bounded_linear f
  shows f ` (affine hull s) = affine hull f ` s
proof -
```

```
  interpret f: bounded_linear f by fact
  have affine {x. f x ∈ affine hull f ‘ s}
    unfolding affine_def
   by (auto simp: f.scaleR f.add affine_affine_hull[unfolded affine_def, rule_format])
  moreover have affine {x. x ∈ f ‘ (affine hull s)}
    using affine_affine_hull[unfolded affine_def, of s]
    unfolding affine_def by (auto simp: f.scaleR [symmetric] f.add [symmetric])
  ultimately show ?thesis
    by (auto simp: hull_inc elim!: hull_induct)
qed


lemma rel_interior_injective_on_span_linear_image:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
    and S :: 'm::euclidean_space set
  assumes bounded_linear f
    and inj_on f (span S)
  shows rel_interior (f ‘ S) = f ‘ (rel_interior S)
proof –
  {
    fix z
    assume z: z ∈ rel_interior (f ‘ S)
    then have z ∈ f ‘ S
      using rel_interior_subset[of f ‘ S] by auto
    then obtain x where x: x ∈ S f x = z by auto
    obtain e2 where e2: e2 > 0 cball z e2 ∩ affine hull (f ‘ S) ⊆ (f ‘ S)
      using z rel_interior_cball[of f ‘ S] by auto
    obtain K where K: K > 0 ⋀x. norm (f x) ≤ norm x ∗ K
     using assms Real_Vector_Spaces.bounded_linear.pos_bounded[of f] by auto
    define e1 where e1 = 1 / K
    then have e1: e1 > 0 ⋀x. e1 ∗ norm (f x) ≤ norm x
      using K pos_le_divide_eq[of e1] by auto
    define e where e = e1 ∗ e2
    then have e > 0 using e1 e2 by auto
    {
      fix y
      assume y: y ∈ cball x e ∩ affine hull S
      then have h1: f y ∈ affine hull (f ‘ S)
        using affine_hull_linear_image[of f S] assms by auto
      from y have norm (x−y) ≤ e1 ∗ e2
        using cball_def[of x e] dist_norm[of x y] e_def by auto
      moreover have f x − f y = f (x − y)
        using assms linear_diff[of f x y] linear_conv_bounded_linear[of f] by auto
      moreover have e1 ∗ norm (f (x−y)) ≤ norm (x − y)
        using e1 by auto
      ultimately have e1 ∗ norm ((f x)−(f y)) ≤ e1 ∗ e2
        by auto
      then have f y ∈ cball z e2
        using cball_def[of f x e2] dist_norm[of f x f y] e1 x by auto
```

```
    then have f y ∈ f ‘ S
      using y e2 h1 by auto
    then have y ∈ S
      using assms y hull_subset[of S] affine_hull_subset_span
        inj_on_image_mem_iff [OF ‹inj_on f (span S)›]
      by (metis Int_iff span_superset subsetCE)
  }
  then have z ∈ f ‘ (rel_interior S)
    using mem_rel_interior_cball[of x S] ‹e > 0› x by auto
}
moreover
{
  fix x
  assume x: x ∈ rel_interior S
  then obtain e2 where e2: e2 > 0 cball x e2 ∩ affine hull S ⊆ S
    using rel_interior_cball[of S] by auto
  have x ∈ S using x rel_interior_subset by auto
  then have *: f x ∈ f ‘ S by auto
  have ∀ x∈span S. f x = 0 ⟶ x = 0
    using assms subspace_span linear_conv_bounded_linear[of f]
      linear_injective_on_subspace_0[of f span S]
    by auto
  then obtain e1 where e1: e1 > 0 ∀ x ∈ span S. e1 * norm x ≤ norm (f x)
    using assms injective_imp_isometric[of span S f]
      subspace_span[of S] closed_subspace[of span S]
    by auto
  define e where e = e1 * e2
  hence e > 0 using e1 e2 by auto
  {
    fix y
    assume y: y ∈ cball (f x) e ∩ affine hull (f ‘ S)
    then have y ∈ f ‘ (affine hull S)
      using affine_hull_linear_image[of f S] assms by auto
    then obtain xy where xy: xy ∈ affine hull S f xy = y by auto
    with y have norm (f x − f xy) ≤ e1 * e2
      using cball_def[of f x e] dist_norm[of f x y] e_def by auto
    moreover have f x − f xy = f (x − xy)
      using assms linear_diff[of f x xy] linear_conv_bounded_linear[of f] by auto
    moreover have *: x − xy ∈ span S
      using subspace_diff[of span S x xy] subspace_span ‹x ∈ S› xy
        affine_hull_subset_span[of S] span_superset
      by auto
    moreover from * have e1 * norm (x − xy) ≤ norm (f (x − xy))
      using e1 by auto
    ultimately have e1 * norm (x − xy) ≤ e1 * e2
      by auto
    then have xy ∈ cball x e2
      using cball_def[of x e2] dist_norm[of x xy] e1 by auto
    then have y ∈ f ‘ S
```

```
      using xy e2 by auto
    }
    then have f x ∈ rel_interior (f ‘ S)
      using mem_rel_interior_cball[of (f x) (f ‘ S)] ∗ ‹e > 0› by auto
  }
  ultimately show ?thesis by auto
qed

lemma rel_interior_injective_linear_image:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes bounded_linear f
    and inj f
  shows rel_interior (f ‘ S) = f ‘ (rel_interior S)
  using assms rel_interior_injective_on_span_linear_image[of f S]
    subset_inj_on[of f UNIV span S]
  by auto
```

4.2.3 Openness and compactness are preserved by convex hull operation

```
lemma open_convex_hull[intro]:
  fixes S :: 'a::real_normed_vector set
  assumes open S
  shows open (convex hull S)
proof (clarsimp simp: open_contains_cball convex_hull_explicit)
  fix T and u :: 'a⇒real
  assume obt: finite T T⊆S ∀x∈T. 0 ≤ u x sum u T = 1

  from assms[unfolded open_contains_cball] obtain b
    where b: ⋀x. x∈S =⇒ 0 < b x ∧ cball x (b x) ⊆ S by metis
  have b ‘ T ≠ {}
    using obt by auto
  define i where i = b ‘ T
  let ?Φ = λy. ∃F. finite F ∧ F ⊆ S ∧ (∃u. (∀x∈F. 0 ≤ u x) ∧ sum u F = 1
∧ (∑v∈F. u v ∗R v) = y)
  let ?a = ∑v∈T. u v ∗R v
  show ∃e > 0. cball ?a e ⊆ {y. ?Φ y}
  proof (intro exI subsetI conjI)
    show 0 < Min i
      unfolding i_def and Min_gr_iff[OF finite_imageI[OF obt(1)] ‹b ‘ T≠{}›]
      using b ‹T⊆S› by auto
  next
    fix y
    assume y ∈ cball ?a (Min i)
    then have y: norm (?a − y) ≤ Min i
      unfolding dist_norm[symmetric] by auto
    { fix x
      assume x ∈ T
      then have Min i ≤ b x
```

    by (*simp add*: *i_def obt*(*1*))
   then have $x + (y - ?a) \in$ *cball x* (*b x*)
    using *y* **unfolding** *mem_cball dist_norm* **by** *auto*
   moreover have $x \in S$
    using ‹$x \in T$› ‹$T \subseteq S$› **by** *auto*
   ultimately have $x + (y - ?a) \in S$
    using *y b* **by** *blast*
  }
  moreover
  have ∗: *inj_on* ($\lambda v.\ v + (y - ?a)$) *T*
   unfolding *inj_on_def* **by** *auto*
  have ($\sum v \in (\lambda v.\ v + (y - ?a))$ ‘ $T.\ u\ (v - (y - ?a)) *_R v) = y$
   unfolding *sum.reindex*[*OF* ∗] *o_def* **using** *obt*(*4*)
  by (*simp add*: *sum.distrib sum_subtractf scaleR_left.sum*[*symmetric*] *scaleR_right_distrib*)
  ultimately show $y \in \{y.\ ?\Phi\ y\}$
  proof (*intro CollectI exI conjI*)
   show *finite* (($\lambda v.\ v + (y - ?a)$) ‘ *T*)
    by (*simp add*: *obt*(*1*))
   show *sum* ($\lambda v.\ u\ (v - (y - ?a))$) (($\lambda v.\ v + (y - ?a)$) ‘ *T*) = *1*
    unfolding *sum.reindex*[*OF* ∗] *o_def* **using** *obt*(*4*) **by** *auto*
  qed (*use obt*(*1*, *3*) **in** *auto*)
 qed
qed

lemma *compact_convex_combinations*:
 fixes *S T* :: *'a::real_normed_vector set*
 assumes *compact S compact T*
 shows *compact* $\{ (1 - u) *_R x + u *_R y \mid x\ y\ u.\ 0 \le u \wedge u \le 1 \wedge x \in S \wedge y \in T\}$
proof −
 let $?X = \{0..1\} \times S \times T$
 let $?h = (\lambda z.\ (1 - fst\ z) *_R fst\ (snd\ z) + fst\ z *_R snd\ (snd\ z))$
 have ∗: $\{ (1 - u) *_R x + u *_R y \mid x\ y\ u.\ 0 \le u \wedge u \le 1 \wedge x \in S \wedge y \in T\} = ?h$ ‘ $?X$
  by *force*
 have *continuous_on* $?X$ ($\lambda z.\ (1 - fst\ z) *_R fst\ (snd\ z) + fst\ z *_R snd\ (snd\ z)$)
  unfolding *continuous_on* **by** (*rule ballI*) (*intro tendsto_intros*)
 with *assms* **show** *?thesis*
  by (*simp add*: ∗ *compact_Times compact_continuous_image*)
qed

lemma *finite_imp_compact_convex_hull*:
 fixes *S* :: *'a::real_normed_vector set*
 assumes *finite S*
 shows *compact* (*convex hull S*)
proof (*cases* $S = \{\}$)
 case *True*
 then show *?thesis* **by** *simp*
next

```
  case False
  with assms show ?thesis
  proof (induct rule: finite_ne_induct)
    case (singleton x)
    show ?case by simp
  next
    case (insert x A)
    let ?f = λ(u, y::'a). u *R x + (1 − u) *R y
    let ?T = {0..1::real} × (convex hull A)
    have continuous_on ?T ?f
      unfolding split_def continuous_on by (intro ballI tendsto_intros)
    moreover have compact ?T
      by (intro compact_Times compact_Icc insert)
    ultimately have compact (?f ` ?T)
      by (rule compact_continuous_image)
    also have ?f ` ?T = convex hull (insert x A)
      unfolding convex_hull_insert [OF ‹A ≠ {}›]
      apply safe
      apply (rule_tac x=a in exI, simp)
      apply (rule_tac x=1 − a in exI, simp, fast)
      apply (rule_tac x=(u, b) in image_eqI, simp_all)
      done
    finally show compact (convex hull (insert x A)) .
  qed
qed

lemma compact_convex_hull:
  fixes S :: 'a::euclidean_space set
  assumes compact S
  shows compact (convex hull S)
proof (cases S = {})
  case True
  then show ?thesis using compact_empty by simp
next
  case False
  then obtain w where w ∈ S by auto
  show ?thesis
    unfolding caratheodory[of S]
  proof (induct (DIM('a) + 1))
    case 0
    have *: {x.∃sa. finite sa ∧ sa ⊆ S ∧ card sa ≤ 0 ∧ x ∈ convex hull sa} = {}
      using compact_empty by auto
    from 0 show ?case unfolding * by simp
  next
    case (Suc n)
    show ?case
    proof (cases n = 0)
      case True
      have {x. ∃T. finite T ∧ T ⊆ S ∧ card T ≤ Suc n ∧ x ∈ convex hull T} = S
```

```
      unfolding set_eq_iff and mem_Collect_eq
    proof (rule, rule)
      fix x
      assume ∃ T. finite T ∧ T ⊆ S ∧ card T ≤ Suc n ∧ x ∈ convex hull T
      then obtain T where T: finite T T ⊆ S card T ≤ Suc n x ∈ convex hull
T
        by auto
      show x ∈ S
      proof (cases card T = 0)
        case True
        then show ?thesis
          using T(4) unfolding card_0_eq[OF T(1)] by simp
      next
        case False
        then have card T = Suc 0 using T(3) ‹n=0› by auto
        then obtain a where T = {a} unfolding card_Suc_eq by auto
        then show ?thesis using T(2,4) by simp
      qed
    next
      fix x assume x∈S
      then show ∃ T. finite T ∧ T ⊆ S ∧ card T ≤ Suc n ∧ x ∈ convex hull T
        by (rule_tac x={x} in exI) (use convex_hull_singleton in auto)
    qed
    then show ?thesis using assms by simp
  next
    case False
    have {x. ∃ T. finite T ∧ T ⊆ S ∧ card T ≤ Suc n ∧ x ∈ convex hull T} =
      {(1 − u) *R x + u *R y | x y u.
        0 ≤ u ∧ u ≤ 1 ∧ x ∈ S ∧ y ∈ {x. ∃ T. finite T ∧ T ⊆ S ∧ card T ≤ n
∧ x ∈ convex hull T}}
      unfolding set_eq_iff and mem_Collect_eq
    proof (rule, rule)
      fix x
      assume ∃ u v c. x = (1 − c) *R u + c *R v ∧
        0 ≤ c ∧ c ≤ 1 ∧ u ∈ S ∧ (∃ T. finite T ∧ T ⊆ S ∧ card T ≤ n ∧ v ∈
convex hull T)
      then obtain u v c T where obt: x = (1 − c) *R u + c *R v
        0 ≤ c ∧ c ≤ 1 u ∈ S finite T T ⊆ S card T ≤ n  v ∈ convex hull T
        by auto
      moreover have (1 − c) *R u + c *R v ∈ convex hull insert u T
          by (meson convexD_alt convex_convex_hull hull_inc hull_mono in_mono
insertCI obt(2) obt(7) subset_insertI)
      ultimately show ∃ T. finite T ∧ T ⊆ S ∧ card T ≤ Suc n ∧ x ∈ convex
hull T
        by (rule_tac x=insert u T in exI) (auto simp: card_insert_if)
    next
      fix x
      assume ∃ T. finite T ∧ T ⊆ S ∧ card T ≤ Suc n ∧ x ∈ convex hull T
      then obtain T where T: finite T T ⊆ S card T ≤ Suc n x ∈ convex hull
```

```
T
        by auto
      show ∃ u v c. x = (1 − c) *R u + c *R v ∧
        0 ≤ c ∧ c ≤ 1 ∧ u ∈ S ∧ (∃ T. finite T ∧ T ⊆ S ∧ card T ≤ n ∧ v ∈
convex hull T)
      proof (cases card T = Suc n)
        case False
        then have card T ≤ n using T(3) by auto
        then show ?thesis
          using ‹w∈S› and T
          by (rule_tac x=w in exI, rule_tac x=x in exI, rule_tac x=1 in exI) auto
      next
        case True
        then obtain a u where au: T = insert a u a∉u
          by (metis card_le_Suc_iff order_refl)
        show ?thesis
        proof (cases u = {})
          case True
          then have x = a using T(4)[unfolded au] by auto
          show ?thesis unfolding ‹x = a›
            using T ‹n ≠ 0› unfolding au
             by (rule_tac x=a in exI, rule_tac x=a in exI, rule_tac x=1 in exI)
force
        next
          case False
          obtain ux vx b where obt: ux≥0 vx≥0 ux + vx = 1
            b ∈ convex hull u x = ux *R a + vx *R b
            using T(4)[unfolded au convex_hull_insert[OF False]]
            by auto
          have *: 1 − vx = ux using obt(3) by auto
          show ?thesis
            using obt T(1−3) card_insert_disjoint[OF _ au(2)] unfolding au *
             by (rule_tac x=a in exI, rule_tac x=b in exI, rule_tac x=vx in exI)
force
        qed
      qed
    qed
    then show ?thesis
      using compact_convex_combinations[OF assms Suc] by simp
  qed
 qed
qed
```

4.2.4 Extremal points of a simplex are some vertices

```
lemma dist_increases_online:
  fixes a b d :: 'a::real_inner
  assumes d ≠ 0
  shows dist a (b + d) > dist a b ∨ dist a (b − d) > dist a b
```

proof (*cases inner a d* − *inner b d* > *0*)
  case *True*
  then have *0* < *inner d d* + (*inner a d* ∗ *2* − *inner b d* ∗ *2*)
    using *assms*
    by (*intro add_pos_pos*) *auto*
  then show *?thesis*
    unfolding *dist_norm* **and** *norm_eq_sqrt_inner* **and** *real_sqrt_less_iff*
    by (*simp add*: *algebra_simps inner_commute*)
next
  case *False*
  then have *0* < *inner d d* + (*inner b d* ∗ *2* − *inner a d* ∗ *2*)
    using *assms*
    by (*intro add_pos_nonneg*) *auto*
  then show *?thesis*
    unfolding *dist_norm* **and** *norm_eq_sqrt_inner* **and** *real_sqrt_less_iff*
    by (*simp add*: *algebra_simps inner_commute*)
qed

lemma *norm_increases_online*:
  fixes *d* :: *'a*::*real_inner*
  shows *d* ≠ *0* ⟹ *norm* (*a* + *d*) > *norm a* ∨ *norm*(*a* − *d*) > *norm a*
  using *dist_increases_online*[*of d a 0*] **unfolding** *dist_norm* **by** *auto*

lemma *simplex_furthest_lt*:
  fixes *S* :: *'a*::*real_inner set*
  assumes *finite S*
  shows ∀ *x* ∈ *convex hull S*. *x* ∉ *S* ⟶ (∃ *y* ∈ *convex hull S*. *norm* (*x* − *a*) < *norm*(*y* − *a*))
  using *assms*
proof *induct*
  fix *x S*
  assume *as*: *finite S x*∉*S* ∀ *x*∈*convex hull S*. *x* ∉ *S* ⟶ (∃ *y*∈*convex hull S*. *norm* (*x* − *a*) < *norm* (*y* − *a*))
  show ∀ *xa*∈*convex hull insert x S*. *xa* ∉ *insert x S* ⟶
    (∃ *y*∈*convex hull insert x S*. *norm* (*xa* − *a*) < *norm* (*y* − *a*))
  proof (*intro impI ballI*, *cases S* = {})
    case *False*
    fix *y*
    assume *y*: *y* ∈ *convex hull insert x S y* ∉ *insert x S*
    obtain *u v b* **where** *obt*: *u*≥*0 v*≥*0 u* + *v* = *1 b* ∈ *convex hull S y* = *u* $*_R$ *x* + *v* $*_R$ *b*
      using *y*(*1*)[*unfolded convex_hull_insert*[*OF False*]] **by** *auto*
    show ∃ *z*∈*convex hull insert x S*. *norm* (*y* − *a*) < *norm* (*z* − *a*)
    proof (*cases y* ∈ *convex hull S*)
      case *True*
      then obtain *z* **where** *z* ∈ *convex hull S norm* (*y* − *a*) < *norm* (*z* − *a*)
        using *as*(*3*)[*THEN bspec*[**where** *x*=*y*]] **and** *y*(*2*) **by** *auto*
      then show *?thesis*
        by (*meson hull_mono subsetD subset_insertI*)

```
next
  case False
  show ?thesis
  proof (cases u = 0 ∨ v = 0)
    case True
    with False show ?thesis
      using obt y by auto
  next
    case False
    then obtain w where w: w>0 w<u w<v
      using field_lbound_gt_zero[of u v] and obt(1,2) by auto
    have x ≠ b
    proof
      assume x = b
      then have y = b unfolding obt(5)
        using obt(3) by (auto simp: scaleR_left_distrib[symmetric])
      then show False using obt(4) and False
        using ‹x = b› y(2) by blast
    qed
    then have *: w *R (x − b) ≠ 0 using w(1) by auto
    show ?thesis
      using dist_increases_online[OF *, of a y]
    proof (elim disjE)
      assume dist a y < dist a (y + w *R (x − b))
      then have norm (y − a) < norm ((u + w) *R x + (v − w) *R b − a)
        unfolding dist_commute[of a]
        unfolding dist_norm obt(5)
        by (simp add: algebra_simps)
      moreover have (u + w) *R x + (v − w) *R b ∈ convex hull insert x S
        unfolding convex_hull_insert[OF ‹S≠{}›]
      proof (intro CollectI conjI exI)
        show u + w ≥ 0 v − w ≥ 0
          using obt(1) w by auto
      qed (use obt in auto)
      ultimately show ?thesis by auto
    next
      assume dist a y < dist a (y − w *R (x − b))
      then have norm (y − a) < norm ((u − w) *R x + (v + w) *R b − a)
        unfolding dist_commute[of a]
        unfolding dist_norm obt(5)
        by (simp add: algebra_simps)
      moreover have (u − w) *R x + (v + w) *R b ∈ convex hull insert x S
        unfolding convex_hull_insert[OF ‹S≠{}›]
      proof (intro CollectI conjI exI)
        show u − w ≥ 0 v + w ≥ 0
          using obt(1) w by auto
      qed (use obt in auto)
      ultimately show ?thesis by auto
    qed
```

qed
qed
qed *auto*
qed (*auto simp*: *assms*)

lemma *simplex_furthest_le*:
fixes *S* :: *'a::real_inner set*
assumes *finite S*
and $S \neq \{\}$
shows $\exists y \in S.\ \forall x \in$ *convex hull S*. *norm* $(x - a) \leq$ *norm* $(y - a)$
proof −
have *convex hull S* $\neq \{\}$
using *hull_subset*[*of S convex*] **and** *assms*(*2*) **by** *auto*
then obtain *x* **where** *x*: $x \in$ *convex hull S* $\forall y \in$*convex hull S*. *norm* $(y - a) \leq$ *norm* $(x - a)$
using *distance_attains_sup*[*OF finite_imp_compact_convex_hull*[*OF* ‹*finite S*›], *of a*]
unfolding *dist_commute*[*of a*]
unfolding *dist_norm*
by *auto*
show *?thesis*
proof (*cases* $x \in S$)
case *False*
then obtain *y* **where** $y \in$ *convex hull S norm* $(x - a) <$ *norm* $(y - a)$
using *simplex_furthest_lt*[*OF assms*(*1*), *THEN bspec*[**where** *x*=*x*]] **and** *x*(*1*)
by *auto*
then show *?thesis*
using *x*(*2*)[*THEN bspec*[**where** *x*=*y*]] **by** *auto*
next
case *True*
with *x* **show** *?thesis* **by** *auto*
qed
qed

lemma *simplex_furthest_le_exists*:
fixes *S* :: (*'a::real_inner*) *set*
shows *finite S* $\Longrightarrow \forall x \in$(*convex hull S*). $\exists y \in S$. *norm* $(x - a) \leq$ *norm* $(y - a)$
using *simplex_furthest_le*[*of S*] **by** (*cases* $S = \{\}$) *auto*

lemma *simplex_extremal_le*:
fixes *S* :: *'a::real_inner set*
assumes *finite S*
and $S \neq \{\}$
shows $\exists u \in S.\ \exists v \in S.\ \forall x \in$*convex hull S*. $\forall y \in$ *convex hull S*. *norm* $(x - y) \leq$ *norm* $(u - v)$
proof −
have *convex hull S* $\neq \{\}$
using *hull_subset*[*of S convex*] **and** *assms*(*2*) **by** *auto*
then obtain *u v* **where** *obt*: $u \in$ *convex hull S* $v \in$ *convex hull S*

```
    ∀x∈convex hull S. ∀y∈convex hull S. norm (x − y) ≤ norm (u − v)
   using compact_sup_maxdistance[OF finite_imp_compact_convex_hull[OF assms(1)]]
    by (auto simp: dist_norm)
  then show ?thesis
  proof (cases u∉S ∨ v∉S, elim disjE)
    assume u ∉ S
    then obtain y where y ∈ convex hull S norm (u − v) < norm (y − v)
     using simplex_furthest_lt[OF assms(1), THEN bspec[where x=u]] and obt(1)
      by auto
    then show ?thesis
     using obt(3)[THEN bspec[where x=y], THEN bspec[where x=v]] and obt(2)
      by auto
  next
    assume v ∉ S
    then obtain y where y ∈ convex hull S norm (v − u) < norm (y − u)
     using simplex_furthest_lt[OF assms(1), THEN bspec[where x=v]] and obt(2)
      by auto
    then show ?thesis
     using obt(3)[THEN bspec[where x=u], THEN bspec[where x=y]] and obt(1)
      by (auto simp: norm_minus_commute)
  qed auto
qed

lemma simplex_extremal_le_exists:
  fixes S :: 'a::real_inner set
  shows finite S ⟹ x ∈ convex hull S ⟹ y ∈ convex hull S ⟹
    ∃u∈S. ∃v∈S. norm (x − y) ≤ norm (u − v)
  using convex_hull_empty simplex_extremal_le[of S]
  by(cases S = {}) auto
```

4.2.5 Closest point of a convex set is unique, with a continuous projection

```
definition closest_point :: 'a::{real_inner,heine_borel} set ⇒ 'a ⇒ 'a
  where closest_point S a = (SOME x. x ∈ S ∧ (∀y∈S. dist a x ≤ dist a y))

lemma closest_point_exists:
  assumes closed S
    and S ≠ {}
  shows closest_point_in_set: closest_point S a ∈ S
    and ∀y∈S. dist a (closest_point S a) ≤ dist a y
  unfolding closest_point_def
  by (rule_tac someI2_ex, auto intro: distance_attains_inf[OF assms(1,2), of a])+

lemma closest_point_le: closed S ⟹ x ∈ S ⟹ dist a (closest_point S a) ≤ dist
a x
  using closest_point_exists[of S] by auto

lemma closest_point_self:
```

```
  assumes x ∈ S
  shows closest_point S x = x
  unfolding closest_point_def
  by (rule some1_equality, rule ex1I[of _ x]) (use assms in auto)

lemma closest_point_refl: closed S ⟹ S ≠ {} ⟹ closest_point S x = x ⟷ x
∈ S
  using closest_point_in_set[of S x] closest_point_self[of x S]
  by auto

lemma closer_points_lemma:
  assumes inner y z > 0
  shows ∃u>0. ∀v>0. v ≤ u ⟶ norm(v *R z − y) < norm y
proof −
  have z: inner z z > 0
    unfolding inner_gt_zero_iff using assms by auto
  have norm (v *R z − y) < norm y
    if 0 < v and v ≤ inner y z / inner z z for v
    unfolding norm_lt using z assms that
    by (simp add: field_simps inner_diff inner_commute mult_strict_left_mono[OF _
‹0<v›])
  then show ?thesis
    using assms z
    by (rule_tac x = inner y z / inner z z in exI) auto
qed

lemma closer_point_lemma:
  assumes inner (y − x) (z − x) > 0
  shows ∃u>0. u ≤ 1 ∧ dist (x + u *R (z − x)) y < dist x y
proof −
  obtain u where u > 0
    and u: ⋀v. ⟦0<v; v ≤ u⟧ ⟹ norm (v *R (z − x) − (y − x)) < norm (y −
x)
    using closer_points_lemma[OF assms] by auto
  show ?thesis
    using u[of min u 1] and ‹u > 0›
   by (metis diff_diff_add dist_commute dist_norm less_eq_real_def not_less u zero_less_one)
qed

lemma any_closest_point_dot:
  assumes convex S closed S x ∈ S y ∈ S ∀z∈S. dist a x ≤ dist a z
  shows inner (a − x) (y − x) ≤ 0
proof (rule ccontr)
  assume ¬ ?thesis
  then obtain u where u: u>0 u≤1 dist (x + u *R (y − x)) a < dist x a
    using closer_point_lemma[of a x y] by auto
  let ?z = (1 − u) *R x + u *R y
  have ?z ∈ S
    using convexD_alt[OF assms(1,3,4), of u] using u by auto
```

```
  then show False
    using assms(5)[THEN bspec[where x=?z]] and u(3)
    by (auto simp: dist_commute algebra_simps)
qed

lemma any_closest_point_unique:
  fixes x :: 'a::real_inner
  assumes convex S closed S x ∈ S y ∈ S
    ∀ z∈S. dist a x ≤ dist a z ∀ z∈S. dist a y ≤ dist a z
  shows x = y
   using any_closest_point_dot[OF assms(1−4,5)] and any_closest_point_dot[OF
assms(1−2,4,3,6)]
  unfolding norm_pths(1) and norm_le_square
  by (auto simp: algebra_simps)

lemma closest_point_unique:
  assumes convex S closed S x ∈ S ∀ z∈S. dist a x ≤ dist a z
  shows x = closest_point S a
  using any_closest_point_unique[OF assms(1−3) _ assms(4), of closest_point S a]
  using closest_point_exists[OF assms(2)] and assms(3) by auto

lemma closest_point_dot:
  assumes convex S closed S x ∈ S
  shows inner (a − closest_point S a) (x − closest_point S a) ≤ 0
  using any_closest_point_dot[OF assms(1,2) _ assms(3)]
  by (metis assms(2) assms(3) closest_point_in_set closest_point_le empty_iff)

lemma closest_point_lt:
  assumes convex S closed S x ∈ S x ≠ closest_point S a
  shows dist a (closest_point S a) < dist a x
   using closest_point_unique[where a=a] closest_point_le[where a=a] assms by
fastforce

lemma setdist_closest_point:
    ⟦closed S; S ≠ {}⟧ ⟹ setdist {a} S = dist a (closest_point S a)
 by (metis closest_point_exists(2) closest_point_in_set emptyE insert_iff setdist_unique)

lemma closest_point_lipschitz:
  assumes convex S
    and closed S S ≠ {}
  shows dist (closest_point S x) (closest_point S y) ≤ dist x y
proof −
  have inner (x − closest_point S x) (closest_point S y − closest_point S x) ≤ 0
    and inner (y − closest_point S y) (closest_point S x − closest_point S y) ≤ 0
    by (simp_all add: assms closest_point_dot closest_point_in_set)
  then show ?thesis unfolding dist_norm and norm_le
    using inner_ge_zero[of (x − closest_point S x) − (y − closest_point S y)]
    by (simp add: inner_add inner_diff inner_commute)
qed
```

lemma *continuous_at_closest_point*:
  assumes *convex S*
    and *closed S*
    and *S ≠ {}*
  shows *continuous* (*at x*) (*closest_point S*)
  unfolding *continuous_at_eps_delta*
  using *le_less_trans*[*OF closest_point_lipschitz*[*OF assms*]] **by** *auto*

lemma *continuous_on_closest_point*:
  assumes *convex S*
    and *closed S*
    and *S ≠ {}*
  shows *continuous_on t* (*closest_point S*)
  by (*metis continuous_at_imp_continuous_on continuous_at_closest_point*[*OF assms*])

proposition *closest_point_in_rel_interior*:
  assumes *closed S S ≠ {}* **and** *x*: *x ∈ affine hull S*
    shows *closest_point S x ∈ rel_interior S ⟷ x ∈ rel_interior S*
proof (*cases x ∈ S*)
  case *True*
  then show *?thesis*
    by (*simp add*: *closest_point_self*)
next
  case *False*
  then have *False* **if** *asm*: *closest_point S x ∈ rel_interior S*
  proof −
    obtain *e* **where** *e > 0* **and** *clox*: *closest_point S x ∈ S*
        and *e*: *cball* (*closest_point S x*) *e ∩ affine hull S ⊆ S*
      using *asm mem_rel_interior_cball* **by** *blast*
    then have *clo_notx*: *closest_point S x ≠ x*
      using ‹*x ∉ S*› **by** *auto*
    define *y* **where** *y ≡ closest_point S x* −
        (*min 1* (*e* / *norm*(*closest_point S x* − *x*))) $*_R$ (*closest_point S x* − *x*)
     have *x* − *y* = (*1* − *min 1* (*e* / *norm* (*closest_point S x* − *x*))) $*_R$ (*x* − *closest_point S x*)
      by (*simp add*: *y_def algebra_simps*)
    then have *norm* (*x* − *y*) = *abs* ((*1* − *min 1* (*e* / *norm* (*closest_point S x* − *x*)))) ∗ *norm*(*x* − *closest_point S x*)
      by *simp*
    also have . . . < *norm*(*x* − *closest_point S x*)
      using *clo_notx* ‹*e > 0*›
      by (*auto simp*: *mult_less_cancel_right2 field_split_simps*)
    finally have *no_less*: *norm* (*x* − *y*) < *norm* (*x* − *closest_point S x*) **.**
    have *y ∈ affine hull S*
      unfolding *y_def*
      by (*meson affine_affine_hull clox hull_subset mem_affine_3_minus2 subsetD x*)
    moreover have *dist* (*closest_point S x*) *y ≤ e*

```
      using ‹e > 0› by (auto simp: y_def min_mult_distrib_right)
    ultimately have y ∈ S
      using subsetD [OF e] by simp
    then have dist x (closest_point S x) ≤ dist x y
      by (simp add: closest_point_le ‹closed S›)
    with no_less show False
      by (simp add: dist_norm)
  qed
  moreover have x ∉ rel_interior S
    using rel_interior_subset False by blast
  ultimately show ?thesis by blast
qed
```

Various point-to-set separating/supporting hyperplane theorems

```
lemma supporting_hyperplane_closed_point:
  fixes z :: 'a::{real_inner,heine_borel}
  assumes convex S
    and closed S
    and S ≠ {}
    and z ∉ S
  shows ∃ a b. ∃ y∈S. inner a z < b ∧ inner a y = b ∧ (∀ x∈S. inner a x ≥ b)
proof −
  obtain y where y ∈ S and y: ∀ x∈S. dist z y ≤ dist z x
    by (metis distance_attains_inf[OF assms(2−3)])
  show ?thesis
  proof (intro exI bexI conjI ballI)
    show (y − z) · z < (y − z) · y
      by (metis ‹y ∈ S› assms(4) diff_gt_0_iff_gt inner_commute inner_diff_left
inner_gt_zero_iff right_minus_eq)
    show (y − z) · y ≤ (y − z) · x if x ∈ S for x
    proof (rule ccontr)
      have *: ⋀u. 0 ≤ u ∧ u ≤ 1 ⟶ dist z y ≤ dist z ((1 − u) *R y + u *R x)
        using assms(1)[unfolded convex_alt] and y and ‹x∈S› and ‹y∈S› by auto
      assume ¬ (y − z) · y ≤ (y − z) · x
      then obtain v where v > 0 v ≤ 1 dist (y + v *R (x − y)) z < dist y z
        using closer_point_lemma[of z y x] by (auto simp: inner_diff)
      then show False
        using *[of v] by (auto simp: dist_commute algebra_simps)
    qed
  qed (use ‹y ∈ S› in auto)
qed

lemma separating_hyperplane_closed_point:
  fixes z :: 'a::{real_inner,heine_borel}
  assumes convex S
    and closed S
    and z ∉ S
  shows ∃ a b. inner a z < b ∧ (∀ x∈S. inner a x > b)
```

proof (*cases* $S = \{\}$)
  case *True*
  then show *?thesis*
    by (*simp add*: *gt_ex*)
next
  case *False*
  obtain *y* **where** $y \in S$ **and** *y*: $\bigwedge x.\ x \in S \Longrightarrow$ *dist z y* $\leq$ *dist z x*
    by (*metis distance_attains_inf*[*OF assms*(*2*) *False*])
  show *?thesis*
  proof (*intro exI conjI ballI*)
    show $(y - z) \cdot z <$ *inner* $(y - z)\ z + ($*norm* $(y - z))^2$ / *2*
      using ‹y∈S› ‹z∉S› **by** *auto*
  next
    fix *x*
    assume $x \in S$
    have *False* **if** ∗: *0* < *inner* (*z* − *y*) (*x* − *y*)
    proof −
      obtain *u* **where** *u* > *0* *u* ≤ *1* *dist* $(y + u *_R (x - y))\ z <$ *dist y z*
        using ∗ *closer_point_lemma* **by** *blast*
      then show *False* **using** *y*[*of* $y + u *_R (x - y)$] *convexD_alt* [*OF* ‹*convex S*›]
        using ‹x∈S› ‹y∈S› **by** (*auto simp*: *dist_commute algebra_simps*)
    qed
    moreover have *0* < (*norm* (*y* − *z*))^2
      using ‹y∈S› ‹z∉S› **by** *auto*
    then have *0* < *inner* (*y* − *z*) (*y* − *z*)
      unfolding *power2_norm_eq_inner* **by** *simp*
    ultimately show $(y - z) \cdot z + ($*norm* $(y - z))^2$ / *2* $< (y - z) \cdot x$
      by (*force simp*: *field_simps power2_norm_eq_inner inner_commute inner_diff*)
  qed
qed

lemma *separating_hyperplane_closed_0*:
  assumes *convex* (*S*::(′*a*::*euclidean_space*) *set*)
    and *closed S*
    and *0* ∉ *S*
  shows ∃ *a b*. *a* ≠ *0* ∧ *0* < *b* ∧ (∀ *x*∈*S*. *inner a x* > *b*)
proof (*cases* $S = \{\}$)
  case *True*
  have (*SOME i*. *i*∈*Basis*) ≠ (*0*::′*a*)
    by (*metis Basis_zero SOME_Basis*)
  then show *?thesis*
    using *True zero_less_one* **by** *blast*
next
  case *False*
  then show *?thesis*
    using *False* **using** *separating_hyperplane_closed_point*[*OF assms*]
    by (*metis all_not_in_conv inner_zero_left inner_zero_right less_eq_real_def not_le*)
qed

Now set-to-set for closed/compact sets

```
lemma separating_hyperplane_closed_compact:
  fixes S :: 'a::euclidean_space set
  assumes convex S
    and closed S
    and convex T
    and compact T
    and T ≠ {}
    and S ∩ T = {}
  shows ∃ a b. (∀ x∈S. inner a x < b) ∧ (∀ x∈T. inner a x > b)
proof (cases S = {})
  case True
  obtain b where b: b > 0 ∀ x∈T. norm x ≤ b
    using compact_imp_bounded[OF assms(4)] unfolding bounded_pos by auto
  obtain z :: 'a where z: norm z = b + 1
    using vector_choose_size[of b + 1] and b(1) by auto
  then have z ∉ T using b(2)[THEN bspec[where x=z]] by auto
  then obtain a b where ab: inner a z < b ∀ x∈T. b < inner a x
    using separating_hyperplane_closed_point[OF assms(3) compact_imp_closed[OF
assms(4)], of z]
    by auto
  then show ?thesis
    using True by auto
next
  case False
  then obtain y where y ∈ S by auto
  obtain a b where 0 < b and §: ⋀x. x ∈ (⋃x∈ S. ⋃ y ∈ T. {x − y}) ⟹ b <
inner a x
   using separating_hyperplane_closed_point[OF convex_differences[OF assms(1,3)],
of 0]
    using closed_compact_differences assms by fastforce
  have ab: b + inner a y < inner a x if x∈S y∈T for x y
    using § [of x−y] that by (auto simp add: inner_diff_right less_diff_eq)
  define k where k = (SUP x∈T. a · x)
  have k + b / 2 < a · x if x ∈ S for x
  proof −
    have k ≤ inner a x − b
      unfolding k_def
      using ‹T ≠ {}› ab that by (fastforce intro: cSUP_least)
    then show ?thesis
      using ‹0 < b› by auto
  qed
  moreover
  have − (k + b / 2) < − a · x if x ∈ T for x
  proof −
    have inner a x − b / 2 < k
      unfolding k_def
    proof (subst less_cSUP_iff)
      show T ≠ {} by fact
```

show *bdd_above* ((·) *a* ‘ *T*)
using *ab*[*rule_format*, *of y*] ‹*y* ∈ *S*›
by (*intro bdd_aboveI2*[**where** *M*=*inner a y* − *b*]) (*auto simp*: *field_simps*
intro: *less_imp_le*)
show ∃ *y*∈*T*. *a* · *x* − *b* / *2* < *a* · *y*
using ‹*0* < *b*› *that* **by** *force*
qed
then show *?thesis*
by *auto*
qed
ultimately show *?thesis*
by (*metis inner_minus_left neg_less_iff_less*)
qed

lemma *separating_hyperplane_compact_closed*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *convex S*
and *compact S*
and *S* ≠ {}
and *convex T*
and *closed T*
and *S* ∩ *T* = {}
shows ∃ *a b*. (∀ *x*∈*S*. *inner a x* < *b*) ∧ (∀ *x*∈*T*. *inner a x* > *b*)
proof −
obtain *a b* **where** (∀ *x*∈*T*. *inner a x* < *b*) ∧ (∀ *x*∈*S*. *b* < *inner a x*)
by (*metis disjoint_iff_not_equal separating_hyperplane_closed_compact assms*)
then show *?thesis*
by (*metis inner_minus_left neg_less_iff_less*)
qed

General case without assuming closure and getting non-strict separation

lemma *separating_hyperplane_set_0*:
assumes *convex S* (*0*::′*a*::*euclidean_space*) ∉ *S*
shows ∃ *a*. *a* ≠ *0* ∧ (∀ *x*∈*S*. *0* ≤ *inner a x*)
proof −
let *?k* = λ*c*. {*x*::′*a*. *0* ≤ *inner c x*}
have ∗: *frontier* (*cball 0 1*) ∩ ⋂*f* ≠ {} **if** *as*: *f* ⊆ *?k* ‘ *S finite f* **for** *f*
proof −
obtain *c* **where** *c*: *f* = *?k* ‘ *c c* ⊆ *S finite c*
using *finite_subset_image*[*OF as*(*2*,*1*)] **by** *auto*
then obtain *a b* **where** *ab*: *a* ≠ *0 0* < *b* ∀ *x*∈*convex hull c*. *b* < *inner a x*
using *separating_hyperplane_closed_0*[*OF convex_convex_hull*, *of c*]
using *finite_imp_compact_convex_hull*[*OF c*(*3*), *THEN compact_imp_closed*]
and *assms*(*2*)
using *subset_hull*[*of convex*, *OF assms*(*1*), *symmetric*, *of c*]
by *force*
have *norm* (*a* $/_R$ *norm a*) = *1*

```
      by (simp add: ab(1))
    moreover have (∀ y∈c. 0 ≤ y · (a /R norm a))
      using hull_subset[of c convex] ab by (force simp: inner_commute)
    ultimately have ∃ x. norm x = 1 ∧ (∀ y∈c. 0 ≤ inner y x)
      by blast
    then show frontier (cball 0 1) ∩ ⋂f ≠ {}
      unfolding c(1) frontier_cball sphere_def dist_norm by auto
  qed
  have frontier (cball 0 1) ∩ (⋂(?k ‘ S)) ≠ {}
    by (rule compact_imp_fip) (use ∗ closed_halfspace_ge in auto)
  then obtain x where norm x = 1 ∀ y∈S. x∈?k y
    unfolding frontier_cball dist_norm sphere_def by auto
  then show ?thesis
    by (metis inner_commute mem_Collect_eq norm_eq_zero zero_neq_one)
qed

lemma separating_hyperplane_sets:
  fixes S T :: 'a::euclidean_space set
  assumes convex S
    and convex T
    and S ≠ {}
    and T ≠ {}
    and S ∩ T = {}
  shows ∃ a b. a ≠ 0 ∧ (∀ x∈S. inner a x ≤ b) ∧ (∀ x∈T. inner a x ≥ b)
proof −
  from separating_hyperplane_set_0[OF convex_differences[OF assms(2,1)]]
  obtain a where a ≠ 0 ∀ x∈{x − y |x y. x ∈ T ∧ y ∈ S}. 0 ≤ inner a x
    using assms(3−5) by force
  then have ∗: ⋀x y. x ∈ T =⇒ y ∈ S =⇒ inner a y ≤ inner a x
    by (force simp: inner_diff)
  then have bdd: bdd_above (((·) a)‘S)
    using ‹T ≠ {}› by (auto intro: bdd_aboveI2[OF ∗])
  show ?thesis
    using ‹a≠0›
    by (intro exI[of _ a] exI[of _ SUP x∈S. a · x])
       (auto intro!: cSUP_upper bdd cSUP_least ‹a ≠ 0› ‹S ≠ {}› ∗)
qed
```

4.2.6 More convexity generalities

```
lemma convex_closure [intro,simp]:
  fixes S :: 'a::real_normed_vector set
  assumes convex S
  shows convex (closure S)
  apply (rule convexI)
  unfolding closure_sequential
  apply (elim exE)
  subgoal for x y u v f g
    by (rule_tac x=λn. u ∗R f n + v ∗R g n in exI) (force intro: tendsto_intros
```

```
dest: convexD [OF assms])
  done

lemma convex_interior [intro,simp]:
  fixes S :: 'a::real_normed_vector set
  assumes convex S
  shows convex (interior S)
  unfolding convex_alt Ball_def mem_interior
proof clarify
  fix x y u
  assume u: 0 ≤ u u ≤ (1::real)
  fix e d
  assume ed: ball x e ⊆ S ball y d ⊆ S 0<d 0<e
  show ∃ e>0. ball ((1 − u) *R x + u *R y) e ⊆ S
  proof (intro exI conjI subsetI)
    fix z
    assume z: z ∈ ball ((1 − u) *R x + u *R y) (min d e)
    have (1− u) *R (z − u *R (y − x)) + u *R (z + (1 − u) *R (y − x)) ∈ S
    proof (rule_tac assms[unfolded convex_alt, rule_format])
      show z − u *R (y − x) ∈ S z + (1 − u) *R (y − x) ∈ S
        using ed z u by (auto simp add: algebra_simps dist_norm)
    qed (use u in auto)
    then show z ∈ S
      using u by (auto simp: algebra_simps)
  qed(use u ed in auto)
qed

lemma convex_hull_eq_empty[simp]: convex hull S = {} ⟷ S = {}
  using hull_subset[of S convex] convex_hull_empty by auto
```

4.2.7 Convex set as intersection of halfspaces

```
lemma convex_halfspace_intersection:
  fixes S :: ('a::euclidean_space) set
  assumes closed S convex S
  shows S = ⋂{h. S ⊆ h ∧ (∃ a b. h = {x. inner a x ≤ b})}
proof −
  { fix z
    assume ∀ T. S ⊆ T ∧ (∃ a b. T = {x. inner a x ≤ b}) ⟶ z ∈ T  z ∉ S
    then have §: ⋀a b. S ⊆ {x. inner a x ≤ b} ⟹ z ∈ {x. inner a x ≤ b}
      by blast
    obtain a b where inner a z < b (∀ x∈S. inner a x > b)
      using ‹z ∉ S› assms separating_hyperplane_closed_point by blast
    then have False
      using § [of −a −b] by fastforce
  }
  then show ?thesis
    by force
qed
```

4.2.8 Convexity of general and special intervals

```
lemma is_interval_convex:
  fixes S :: 'a::euclidean_space set
  assumes is_interval S
  shows convex S
proof (rule convexI)
  fix x y and u v :: real
  assume x ∈ S y ∈ S and uv: 0 ≤ u 0 ≤ v u + v = 1
  then have *: u = 1 − v 1 − v ≥ 0 and **: v = 1 − u 1 − u ≥ 0
    by auto
  {
    fix a b
    assume ¬ b ≤ u ∗ a + v ∗ b
    then have u ∗ a < (1 − v) ∗ b
      unfolding not_le using ‹0 ≤ v›by (auto simp: field_simps)
    then have a < b
      using ∗(1) less_eq_real_def uv(1) by auto
    then have a ≤ u ∗ a + v ∗ b
      unfolding ∗ using ‹0 ≤ v›
      by (auto simp: field_simps intro!:mult_right_mono)
  }
  moreover
  {
    fix a b
    assume ¬ u ∗ a + v ∗ b ≤ a
    then have v ∗ b > (1 − u) ∗ a
      unfolding not_le using ‹0 ≤ v› by (auto simp: field_simps)
    then have a < b
      unfolding ∗ using ‹0 ≤ v›
      by (rule_tac mult_left_less_imp_less) (auto simp: field_simps)
    then have u ∗ a + v ∗ b ≤ b
      unfolding ∗∗
      using ∗∗(2) ‹0 ≤ u› by (auto simp: algebra_simps mult_right_mono)
  }
  ultimately show u ∗R x + v ∗R y ∈ S
    using DIM_positive[where 'a='a]
   by (intro mem_is_intervalI [OF assms ‹x ∈ S› ‹y ∈ S›]) (auto simp: inner_simps)
qed

lemma is_interval_connected:
  fixes S :: 'a::euclidean_space set
  shows is_interval S ⟹ connected S
  using is_interval_convex convex_connected by auto

lemma convex_box [simp]: convex (cbox a b) convex (box a (b::'a::euclidean_space))
  by (auto simp add: is_interval_convex)
```

A non-singleton connected set is perfect (i.e. has no isolated points).

```
lemma connected_imp_perfect:
```

fixes *a* :: *'a::metric_space*
assumes *connected S a* ∈ *S* **and** *S*: ⋀*x*. *S* ≠ {*x*}
shows *a islimpt S*
proof −
have *False* **if** *a* ∈ *T open T* ⋀*y*. ⟦*y* ∈ *S*; *y* ∈ *T*⟧ ⟹ *y* = *a* **for** *T*
proof −
obtain *e* **where** *e* > *0* **and** *e*: *cball a e* ⊆ *T*
using ‹*open T*› ‹*a* ∈ *T*› **by** (*auto simp*: *open_contains_cball*)
have *openin* (*top_of_set S*) {*a*}
unfolding *openin_open* **using** *that* ‹*a* ∈ *S*› **by** *blast*
moreover have *closedin* (*top_of_set S*) {*a*}
by (*simp add*: *assms*)
ultimately show *False*
using ‹*connected S*› *connected_clopen S* **by** *blast*
qed
then show *?thesis*
unfolding *islimpt_def* **by** *blast*
qed

lemma *connected_imp_perfect_aff_dim*:
⟦*connected S*; *aff_dim S* ≠ *0*; *a* ∈ *S*⟧ ⟹ *a islimpt S*
using *aff_dim_sing connected_imp_perfect* **by** *blast*

4.2.9 On *real*, *is_interval*, *convex* and *connected* are all equivalent

lemma *mem_is_interval_1_I*:
fixes *a b c::real*
assumes *is_interval S*
assumes *a* ∈ *S c* ∈ *S*
assumes *a* ≤ *b b* ≤ *c*
shows *b* ∈ *S*
using *assms is_interval_1* **by** *blast*

lemma *is_interval_connected_1*:
fixes *S* :: *real set*
shows *is_interval S* ⟷ *connected S*
by (*meson connected_iff_interval is_interval_1*)

lemma *is_interval_convex_1*:
fixes *S* :: *real set*
shows *is_interval S* ⟷ *convex S*
by (*metis is_interval_convex convex_connected is_interval_connected_1*)

lemma *connected_compact_interval_1*:
connected S ∧ *compact S* ⟷ (∃ *a b*. *S* = {*a*..*b*::*real*})
by (*auto simp*: *is_interval_connected_1* [*symmetric*] *is_interval_compact*)

lemma *connected_convex_1*:
fixes *S* :: *real set*

shows *connected S* ⟷ *convex S*
by (*metis is_interval_convex convex_connected is_interval_connected_1*)

lemma *connected_convex_1_gen*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *DIM*(*'a*) = *1*
shows *connected S* ⟷ *convex S*
proof −
obtain *f*:: *'a* ⇒ *real* **where** *linf*: *linear f* **and** *inj f*
using *subspace_isomorphism*[*OF subspace_UNIV subspace_UNIV*,
where *'a='a* **and** *'b=real*]
unfolding *Euclidean_Space.dim_UNIV*
by (*auto simp*: *assms*)
then have *f* −' (*f* ' *S*) = *S*
by (*simp add*: *inj_vimage_image_eq*)
then show *?thesis*
by (*metis connected_convex_1 convex_linear_vimage linf convex_connected connected_linear_image*)
qed

lemma [*simp*]:
fixes *r s*::*real*
shows *is_interval_io*: *is_interval* {..<*r*}
and *is_interval_oi*: *is_interval* {*r*<..}
and *is_interval_oo*: *is_interval* {*r*<..<*s*}
and *is_interval_oc*: *is_interval* {*r*<..*s*}
and *is_interval_co*: *is_interval* {*r*..<*s*}
by (*simp_all add*: *is_interval_convex_1*)

4.2.10 Another intermediate value theorem formulation

lemma *ivt_increasing_component_on_1*:
fixes *f* :: *real* ⇒ *'a*::*euclidean_space*
assumes *a* ≤ *b*
and *continuous_on* {*a*..*b*} *f*
and (*f a*)·*k* ≤ *y y* ≤ (*f b*)·*k*
shows ∃ *x*∈{*a*..*b*}. (*f x*)·*k* = *y*
proof −
have *f a* ∈ *f* ' *cbox a b f b* ∈ *f* ' *cbox a b*
using ‹*a* ≤ *b*› **by** *auto*
then show *?thesis*
using *connected_ivt_component*[*of f* ' *cbox a b f a f b k y*]
by (*simp add*: *connected_continuous_image assms*)
qed

lemma *ivt_increasing_component_1*:
fixes *f* :: *real* ⇒ *'a*::*euclidean_space*
shows *a* ≤ *b* ⟹ ∀ *x*∈{*a*..*b*}. *continuous* (*at x*) *f* ⟹
f a·*k* ≤ *y* ⟹ *y* ≤ *f b*·*k* ⟹ ∃ *x*∈{*a*..*b*}. (*f x*)·*k* = *y*

by (*rule ivt_increasing_component_on_1*) (*auto simp*: *continuous_at_imp_continuous_on*)

lemma *ivt_decreasing_component_on_1*:
fixes *f* :: *real* ⇒ *'a::euclidean_space*
assumes *a* ≤ *b*
and *continuous_on* {*a..b*} *f*
and (*f b*)•*k* ≤ *y*
and *y* ≤ (*f a*)•*k*
shows ∃ *x*∈{*a..b*}. (*f x*)•*k* = *y*
using *ivt_increasing_component_on_1*[*of a b λx. − f x k − y*] *neg_equal_iff_equal*
using *assms continuous_on_minus* **by** *force*

lemma *ivt_decreasing_component_1*:
fixes *f* :: *real* ⇒ *'a::euclidean_space*
shows *a* ≤ *b* ⟹ ∀ *x*∈{*a..b*}. *continuous* (*at x*) *f* ⟹
f b•*k* ≤ *y* ⟹ *y* ≤ *f a*•*k* ⟹ ∃ *x*∈{*a..b*}. (*f x*)•*k* = *y*
by (*rule ivt_decreasing_component_on_1*) (*auto simp*: *continuous_at_imp_continuous_on*)

4.2.11 A bound within an interval

lemma *convex_hull_eq_real_cbox*:
fixes *x y* :: *real* **assumes** *x* ≤ *y*
shows *convex hull* {*x*, *y*} = *cbox x y*
proof (*rule hull_unique*)
show {*x*, *y*} ⊆ *cbox x y* **using** ‹*x* ≤ *y*› **by** *auto*
show *convex* (*cbox x y*)
by (*rule convex_box*)
next
fix *S* **assume** {*x*, *y*} ⊆ *S* **and** *convex S*
then show *cbox x y* ⊆ *S*
unfolding *is_interval_convex_1* [*symmetric*] *is_interval_def Basis_real_def*
by − (*clarify*, *simp* (*no_asm_use*), *fast*)
qed

lemma *unit_interval_convex_hull*:
cbox (*0::'a::euclidean_space*) *One* = *convex hull* {*x*. ∀ *i*∈*Basis*. (*x*•*i* = *0*) ∨ (*x*•*i* = *1*)}
(**is** *?int* = *convex hull ?points*)
proof −
have *One*[*simp*]: ⋀*i*. *i* ∈ *Basis* ⟹ *One* • *i* = *1*
by (*simp add*: *inner_sum_left sum.If_cases inner_Basis*)
have *?int* = {*x*. ∀ *i*∈*Basis*. *x* • *i* ∈ *cbox 0 1*}
by (*auto simp*: *cbox_def*)
also have ... = (∑ *i*∈*Basis*. (λ*x*. *x* $*_R$ *i*) ‘ *cbox 0 1*)
by (*simp only*: *box_eq_set_sum_Basis*)
also have ... = (∑ *i*∈*Basis*. (λ*x*. *x* $*_R$ *i*) ‘ (*convex hull* {*0*, *1*}))
by (*simp only*: *convex_hull_eq_real_cbox zero_le_one*)
also have ... = (∑ *i*∈*Basis*. *convex hull* ((λ*x*. *x* $*_R$ *i*) ‘ {*0*, *1*}))
by (*simp add*: *convex_hull_linear_image*)

```
  also have ... = convex hull (∑ i∈Basis. (λx. x *R i) ` {0, 1})
    by (simp only: convex_hull_set_sum)
  also have ... = convex hull {x. ∀ i∈Basis. x•i ∈ {0, 1}}
    by (simp only: box_eq_set_sum_Basis)
  also have convex hull {x. ∀ i∈Basis. x•i ∈ {0, 1}} = convex hull ?points
    by simp
  finally show ?thesis .
qed
```

And this is a finite set of vertices.

```
lemma unit_cube_convex_hull:
  obtains S :: 'a::euclidean_space set
  where finite S and cbox 0 (∑ Basis) = convex hull S
proof
  show finite {x::'a. ∀ i∈Basis. x • i = 0 ∨ x • i = 1}
  proof (rule finite_subset, clarify)
    show finite ((λS. ∑ i∈Basis. (if i ∈ S then 1 else 0) *R i) ` Pow Basis)
      using finite_Basis by blast
    fix x :: 'a
    assume x: ∀ i∈Basis. x • i = 0 ∨ x • i = 1
    show x ∈ (λS. ∑ i∈Basis. (if i∈S then 1 else 0) *R i) ` Pow Basis
      apply (rule image_eqI[where x={i. i ∈ Basis ∧ x•i = 1}])
      using x
      by (subst euclidean_eq_iff, auto)
  qed
  show cbox 0 One = convex hull {x. ∀ i∈Basis. x • i = 0 ∨ x • i = 1}
    using unit_interval_convex_hull by blast
qed
```

Hence any cube (could do any nonempty interval).

```
lemma cube_convex_hull:
  assumes d > 0
  obtains S :: 'a::euclidean_space set where
    finite S and cbox (x − (∑ i∈Basis. d*Ri)) (x + (∑ i∈Basis. d*Ri)) = convex
hull S
proof −
  let ?d = (∑ i∈Basis. d *R i)::'a
  have *: cbox (x − ?d) (x + ?d) = (λy. x − ?d + (2 * d) *R y) ` cbox 0
(∑ Basis)
  proof (intro set_eqI iffI)
    fix y
    assume y ∈ cbox (x − ?d) (x + ?d)
    then have inverse (2 * d) *R (y − (x − ?d)) ∈ cbox 0 (∑ Basis)
      using assms by (simp add: mem_box inner_simps) (simp add: field_simps)
    with ‹0 < d› show y ∈ (λy. x − sum ((*R) d) Basis + (2 * d) *R y) ` cbox
0 One
      by (auto intro: image_eqI[where x= inverse (2 * d) *R (y − (x − ?d))])
  next
    fix y
```

assume $y \in (\lambda y.\ x - ?d + (2 * d) *_R y)$ ‘ *cbox 0 One*
then obtain z **where** z: $z \in$ *cbox 0 One* $y = x - ?d + (2*d) *_R z$
by *auto*
then show $y \in$ *cbox* $(x - ?d)$ $(x + ?d)$
using *z assms* **by** (*auto simp*: *mem_box inner_simps*)
qed
obtain S **where** *finite S cbox 0* ($\sum$ *Basis*::′*a*) = *convex hull S*
using *unit_cube_convex_hull* **by** *auto*
then show *?thesis*
by (*rule_tac that*[*of* $(\lambda y.\ x - ?d + (2 * d) *_R y)$‘ *S*]) (*auto simp*: *convex_hull_affinity* ∗)
qed

4.2.12 Representation of any interval as a finite convex hull

lemma *image_stretch_interval*:
$(\lambda x.\ \sum k\in Basis.\ (m\ k * (x\cdot k)) *_R k)$ ‘ *cbox a* (*b*::′*a*::*euclidean_space*) =
(*if* (*cbox a b*) = {} *then* {} *else*
cbox ($\sum k\in Basis.\ (min\ (m\ k * (a\cdot k))\ (m\ k * (b\cdot k))) *_R k$::′*a*)
($\sum k\in Basis.\ (max\ (m\ k * (a\cdot k))\ (m\ k * (b\cdot k))) *_R k$))
proof *cases*
assume ∗: *cbox a b* ≠ {}
show *?thesis*
unfolding *box_ne_empty if_not_P*[*OF* ∗]
apply (*simp add*: *cbox_def image_Collect set_eq_iff euclidean_eq_iff*[**where** ′*a*=′*a*] *ball_conj_distrib*[*symmetric*])
apply (*subst choice_Basis_iff* [*symmetric*])
proof (*intro allI ball_cong refl*)
fix *x i* :: ′*a* **assume** $i \in Basis$
with ∗ **have** *a_le_b*: $a \cdot i \le b \cdot i$
unfolding *box_ne_empty* **by** *auto*
show $(\exists xa.\ x \cdot i = m\ i * xa \wedge a \cdot i \le xa \wedge xa \le b \cdot i) \longleftrightarrow$
$min\ (m\ i * (a \cdot i))\ (m\ i * (b \cdot i)) \le x \cdot i \wedge x \cdot i \le max\ (m\ i * (a \cdot i))\ (m\ i * (b \cdot i))$
proof (*cases* $m\ i = 0$)
case *True*
with *a_le_b* **show** *?thesis* **by** *auto*
next
case *False*
then have ∗: $\bigwedge a\ b.\ a = m\ i * b \longleftrightarrow b = a\ /\ m\ i$
by (*auto simp*: *field_simps*)
from *False* **have**
$min\ (m\ i * (a \cdot i))\ (m\ i * (b \cdot i)) = (if\ 0 < m\ i\ then\ m\ i * (a \cdot i)\ else\ m\ i * (b \cdot i))$
$max\ (m\ i * (a \cdot i))\ (m\ i * (b \cdot i)) = (if\ 0 < m\ i\ then\ m\ i * (b \cdot i)\ else\ m\ i * (a \cdot i))$
using *a_le_b* **by** (*auto simp*: *min_def max_def mult_le_cancel_left*)
with *False* **show** *?thesis* **using** *a_le_b* ∗
by (*simp add*: *le_divide_eq divide_le_eq*) (*simp add*: *ac_simps*)

```
    qed
  qed
qed simp

lemma interval_image_stretch_interval:
  ∃u v. (λx. ∑k∈Basis. (m k * (x·k))*R k) ` cbox a (b::'a::euclidean_space) =
cbox u (v::'a::euclidean_space)
  unfolding image_stretch_interval by auto

lemma cbox_translation: cbox (c + a) (c + b) = image (λx. c + x) (cbox a b)
  using image_affinity_cbox [of 1 c a b]
  using box_ne_empty [of a+c b+c]  box_ne_empty [of a b]
  by (auto simp: inner_left_distrib add.commute)

lemma cbox_image_unit_interval:
  fixes a :: 'a::euclidean_space
  assumes cbox a b ≠ {}
    shows cbox a b =
          (+) a ` (λx. ∑k∈Basis. ((b · k − a · k) * (x · k)) *R k) ` cbox 0 One
using assms
apply (simp add: box_ne_empty image_stretch_interval cbox_translation [symmetric])
apply (simp add: min_def max_def algebra_simps sum_subtractf euclidean_representation)
done

lemma closed_interval_as_convex_hull:
  fixes a :: 'a::euclidean_space
  obtains S where finite S cbox a b = convex hull S
proof (cases cbox a b = {})
  case True with convex_hull_empty that show ?thesis
    by blast
next
  case False
  obtain S::'a set where finite S and eq: cbox 0 One = convex hull S
    by (blast intro: unit_cube_convex_hull)
  let ?S = ((+) a ` (λx. ∑k∈Basis. ((b · k − a · k) * (x · k)) *R k) ` S)
  show thesis
  proof
    show finite ?S
      by (simp add: ‹finite S›)
    have lin: linear (λx. ∑k∈Basis. ((b · k − a · k) * (x · k)) *R k)
      by (rule linear_compose_sum) (auto simp: algebra_simps linearI)
    show cbox a b = convex hull ?S
      using convex_hull_linear_image [OF lin]
      by (simp add: convex_hull_translation eq cbox_image_unit_interval [OF False])
  qed
qed
```

4.2.13 Bounded convex function on open set is continuous

```
lemma convex_on_bounded_continuous:
  fixes S :: ('a::real_normed_vector) set
  assumes open S
    and convex_on S f
    and ∀x∈S. |f x| ≤ b
  shows continuous_on S f
proof -
  have ∃d>0. ∀x'. norm (x' - x) < d ⟶ |f x' - f x| < e if x ∈ S e > 0 for x
and e :: real
  proof -
    define B where B = |b| + 1
    then have B:  0 < B⋀x. x∈S ⟹ |f x| ≤ B
      using assms(3) by auto
    obtain k where k > 0 and k: cball x k ⊆ S
      using ‹x ∈ S› assms(1) open_contains_cball_eq by blast
    show ∃d>0. ∀x'. norm (x' - x) < d ⟶ |f x' - f x| < e
    proof (intro exI conjI allI impI)
      fix y
      assume as: norm (y - x) < min (k / 2) (e / (2 * B) * k)
      show |f y - f x| < e
      proof (cases y = x)
        case False
        define t where t = k / norm (y - x)
        have 2 < t 0<t
          unfolding t_def using as False and ‹k>0›
          by (auto simp:field_simps)
        have y ∈ S
          apply (rule k[THEN subsetD])
          unfolding mem_cball dist_norm
          apply (rule order_trans[of _ 2 * norm (x - y)])
          using as
          by (auto simp: field_simps norm_minus_commute)
        {
          define w where w = x + t *R (y - x)
          have w ∈ S
            using ‹k>0› by (auto simp: dist_norm t_def w_def k[THEN subsetD])
          have (1 / t) *R x + - x + ((t - 1) / t) *R x = (1 / t - 1 + (t - 1)
/ t) *R x
            by (auto simp: algebra_simps)
          also have ... = 0
            using ‹t > 0› by (auto simp:field_simps)
          finally have w: (1 / t) *R w + ((t - 1) / t) *R x = y
            unfolding w_def using False and ‹t > 0›
            by (auto simp: algebra_simps)
          have 2: 2 * B < e * t
            unfolding t_def using ‹0 < e› ‹0 < k› ‹B > 0› and as and False
            by (auto simp:field_simps)
          have f y - f x ≤ (f w - f x) / t
```

```
          using assms(2)[unfolded convex_on_def,rule_format,of w x 1/t (t − 1)/t,
unfolded w]
            using ‹0 < t› ‹2 < t› and ‹x ∈ S› ‹w ∈ S›
            by (auto simp:field_simps)
          also have ... < e
            using B(2)[OF ‹w∈S›] and B(2)[OF ‹x∈S›] 2 ‹t > 0› by (auto simp:
field_simps)
          finally have th1: f y − f x < e .
        }
        moreover
        {
          define w where w = x − t *R (y − x)
          have w ∈ S
            using ‹k > 0› by (auto simp: dist_norm t_def w_def k[THEN subsetD])
          have (1 / (1 + t)) *R x + (t / (1 + t)) *R x = (1 / (1 + t) + t / (1
+ t)) *R x
            by (auto simp: algebra_simps)
          also have . . . = x
            using ‹t > 0› by (auto simp:field_simps)
          finally have w: (1 / (1+t)) *R w + (t / (1 + t)) *R y = x
            unfolding w_def using False and ‹t > 0›
            by (auto simp: algebra_simps)
          have 2 * B < e * t
            unfolding t_def
            using ‹0 < e› ‹0 < k› ‹B > 0› and as and False
            by (auto simp:field_simps)
          then have *: (f w − f y) / t < e
            using B(2)[OF ‹w∈S›] and B(2)[OF ‹y∈S›]
            using ‹t > 0›
            by (auto simp:field_simps)
          have f x ≤ 1 / (1 + t) * f w + (t / (1 + t)) * f y
            using assms(2)[unfolded convex_on_def,rule_format,of w y 1/(1+t) t /
(1+t),unfolded w]
            using ‹0 < t› ‹2 < t› and ‹y ∈ S› ‹w ∈ S›
            by (auto simp:field_simps)
          also have . . . = (f w + t * f y) / (1 + t)
            using ‹t > 0› by (simp add: add_divide_distrib)
          also have . . . < e + f y
            using ‹t > 0› * ‹e > 0› by (auto simp: field_simps)
          finally have f x − f y < e by auto
        }
        ultimately show ?thesis by auto
      qed (use ‹0<e› in auto)
    qed (use ‹0<e› ‹0<k› ‹0<B› in ‹auto simp: field_simps›)
  qed
  then show ?thesis
    by (metis continuous_on_iff dist_norm real_norm_def)
qed
```

4.2.14 Upper bound on a ball implies upper and lower bounds

```
lemma convex_bounds_lemma:
  fixes x :: 'a::real_normed_vector
  assumes convex_on (cball x e) f
    and ∀ y ∈ cball x e. f y ≤ b and y: y ∈ cball x e
  shows |f y| ≤ b + 2 * |f x|
proof (cases 0 ≤ e)
  case True
  define z where z = 2 *R x − y
  have *: x − (2 *R x − y) = y − x
    by (simp add: scaleR_2)
  have z: z ∈ cball x e
   using y unfolding z_def mem_cball dist_norm * by (auto simp: norm_minus_commute)
  have (1 / 2) *R y + (1 / 2) *R z = x
    unfolding z_def by (auto simp: algebra_simps)
  then show |f y| ≤ b + 2 * |f x|
    using assms(1)[unfolded convex_on_def,rule_format, OF y z, of 1/2 1/2]
    using assms(2)[rule_format,OF y] assms(2)[rule_format,OF z]
    by (auto simp:field_simps)
next
  case False
  have dist x y < 0
   using False y unfolding mem_cball not_le by (auto simp del: dist_not_less_zero)
  then show |f y| ≤ b + 2 * |f x|
    using zero_le_dist[of x y] by auto
qed
```

Hence a convex function on an open set is continuous

```
lemma real_of_nat_ge_one_iff: 1 ≤ real (n::nat) ⟷ 1 ≤ n
  by auto

lemma convex_on_continuous:
  assumes open (s::('a::euclidean_space) set) convex_on s f
  shows continuous_on s f
  unfolding continuous_on_eq_continuous_at[OF assms(1)]
proof
  note dimge1 = DIM_positive[where 'a='a]
  fix x
  assume x ∈ s
  then obtain e where e: cball x e ⊆ s e > 0
    using assms(1) unfolding open_contains_cball by auto
  define d where d = e / real DIM('a)
  have 0 < d
    unfolding d_def using ‹e > 0› dimge1 by auto
  let ?d = (∑ i∈Basis. d *R i)::'a
  obtain c
    where c: finite c
    and c1: convex hull c ⊆ cball x e
```

```
    and c2: cball x d ⊆ convex hull c
  proof
    define c where c = (∑ i∈Basis. (λa. a *R i) ` {x·i − d, x·i + d})
    show finite c
      unfolding c_def by (simp add: finite_set_sum)
    have ⋀i. i ∈ Basis ⟹ convex hull {x · i − d, x · i + d} = cbox (x · i − d)
(x · i + d)
      using ‹0 < d› convex_hull_eq_real_cbox by auto
    then have 1: convex hull c = {a. ∀ i∈Basis. a · i ∈ cbox (x · i − d) (x · i +
d)}
      unfolding box_eq_set_sum_Basis c_def convex_hull_set_sum
      apply (subst convex_hull_linear_image [symmetric])
      by (force simp add: linear_iff scaleR_add_left)+
    then have 2: convex hull c = {a. ∀ i∈Basis. a · i ∈ cball (x · i) d}
      by (simp add: dist_norm abs_le_iff algebra_simps)
    show cball x d ⊆ convex hull c
      unfolding 2
     by (clarsimp simp: dist_norm) (metis inner_commute inner_diff_right norm_bound_Basis_le)
    have e': e = (∑ (i::'a)∈Basis. d)
      by (simp add: d_def)
    show convex hull c ⊆ cball x e
      unfolding 2
    proof clarsimp
      show dist x y ≤ e if ∀ i∈Basis. dist (x · i) (y · i) ≤ d for y
      proof −
        have ⋀i. i ∈ Basis ⟹ 0 ≤ dist (x · i) (y · i)
          by simp
        have (∑ i∈Basis. dist (x · i) (y · i)) ≤ e
          using e' sum_mono that by fastforce
        then show ?thesis
          by (metis (mono_tags) euclidean_dist_l2 order_trans [OF L2_set_le_sum]
zero_le_dist)
      qed
    qed
  qed
  define k where k = Max (f ` c)
  have convex_on (convex hull c) f
    using assms(2) c1 convex_on_subset e(1) by blast
  then have k: ∀ y∈convex hull c. f y ≤ k
    using c convex_on_convex_hull_bound k_def by fastforce
  have e ≤ e * real DIM('a)
    using e(2) real_of_nat_ge_one_iff by auto
  then have d ≤ e
    by (simp add: d_def field_split_simps)
  then have dsube: cball x d ⊆ cball x e
    by (rule subset_cball)
  have conv: convex_on (cball x d) f
    using ‹convex_on (convex hull c) f› c2 convex_on_subset by blast
  then have ⋀y. y∈cball x d ⟹ |f y| ≤ k + 2 * |f x|
```

by (*rule convex_bounds_lemma*) (*use c2 k* **in** *blast*)
then have *continuous_on* (*ball x d*) *f*
by (*meson Elementary_Metric_Spaces.open_ball ball_subset_cball conv convex_on_bounded_continuous convex_on_subset mem_ball_imp_mem_cball*)
then show *continuous* (*at x*) *f*
unfolding *continuous_on_eq_continuous_at*[*OF open_ball*]
using ‹*d* > *0*› **by** *auto*
qed

end

4.3 Operator Norm

theory *Operator_Norm*
imports *Complex_Main*
begin

This formulation yields zero if *'a* is the trivial vector space.

definition
onorm :: (*'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector*) ⇒ *real* **where**
onorm f = (*SUP x. norm* (*f x*) / *norm x*)

proposition *onorm_bound*:
assumes *0* ≤ *b* **and** ⋀*x. norm* (*f x*) ≤ *b* ∗ *norm x*
shows *onorm f* ≤ *b*
unfolding *onorm_def*
proof (*rule cSUP_least*)
fix *x*
show *norm* (*f x*) / *norm x* ≤ *b*
using *assms* **by** (*cases x* = *0*) (*simp_all add*: *pos_divide_le_eq*)
qed *simp*

In non-trivial vector spaces, the first assumption is redundant.

lemma *onorm_le*:
fixes *f* :: *'a*::{*real_normed_vector*, *perfect_space*} ⇒ *'b*::*real_normed_vector*
assumes ⋀*x. norm* (*f x*) ≤ *b* ∗ *norm x*
shows *onorm f* ≤ *b*
proof (*rule onorm_bound* [*OF _ assms*])
have {*0*::*'a*} ≠ *UNIV* **by** (*metis not_open_singleton open_UNIV*)
then obtain *a* :: *'a* **where** *a* ≠ *0* **by** *fast*
have *0* ≤ *b* ∗ *norm a*
by (*rule order_trans* [*OF norm_ge_zero assms*])
with ‹*a* ≠ *0*› **show** *0* ≤ *b*
by (*simp add*: *zero_le_mult_iff*)
qed

lemma *le_onorm*:

```
  assumes bounded_linear f
  shows norm (f x) / norm x ≤ onorm f
proof -
  interpret f: bounded_linear f by fact
  obtain b where 0 ≤ b and ∀x. norm (f x) ≤ norm x * b
    using f.nonneg_bounded by auto
  then have ∀x. norm (f x) / norm x ≤ b
    by (clarify, case_tac x = 0,
      simp_all add: f.zero pos_divide_le_eq mult.commute)
  then have bdd_above (range (λx. norm (f x) / norm x))
    unfolding bdd_above_def by fast
  with UNIV_I show ?thesis
    unfolding onorm_def by (rule cSUP_upper)
qed

lemma onorm:
  assumes bounded_linear f
  shows norm (f x) ≤ onorm f * norm x
proof -
  interpret f: bounded_linear f by fact
  show ?thesis
  proof (cases)
    assume x = 0
    then show ?thesis by (simp add: f.zero)
  next
    assume x ≠ 0
    have norm (f x) / norm x ≤ onorm f
      by (rule le_onorm [OF assms])
    then show norm (f x) ≤ onorm f * norm x
      by (simp add: pos_divide_le_eq ‹x ≠ 0›)
  qed
qed

lemma onorm_pos_le:
  assumes f: bounded_linear f
  shows 0 ≤ onorm f
  using le_onorm [OF f, where x=0] by simp

lemma onorm_zero: onorm (λx. 0) = 0
proof (rule order_antisym)
  show onorm (λx. 0) ≤ 0
    by (simp add: onorm_bound)
  show 0 ≤ onorm (λx. 0)
    using bounded_linear_zero by (rule onorm_pos_le)
qed

lemma onorm_eq_0:
  assumes f: bounded_linear f
  shows onorm f = 0 ⟷ (∀x. f x = 0)
```

using *onorm* [*OF f*] **by** (*auto simp*: *fun_eq_iff* [*symmetric*] *onorm_zero*)

lemma *onorm_pos_lt*:
assumes *f*: *bounded_linear f*
shows $0 < onorm\ f \longleftrightarrow \neg (\forall x.\ f\ x = 0)$
by (*simp add*: *less_le onorm_pos_le* [*OF f*] *onorm_eq_0* [*OF f*])

lemma *onorm_id_le*: $onorm\ (\lambda x.\ x) \leq 1$
by (*rule onorm_bound*) *simp_all*

lemma *onorm_id*: $onorm\ (\lambda x.\ x{::}'a{::}\{real\_normed\_vector,\ perfect\_space\}) = 1$
proof (*rule antisym*[*OF onorm_id_le*])
have $\{0{::}'a\} \neq UNIV$ **by** (*metis not_open_singleton open_UNIV*)
then obtain $x :: 'a$ **where** $x \neq 0$ **by** *fast*
hence $1 \leq norm\ x\ /\ norm\ x$
by *simp*
also have $\ldots \leq onorm\ (\lambda x{::}'a.\ x)$
by (*rule le_onorm*) (*rule bounded_linear_ident*)
finally show $1 \leq onorm\ (\lambda x{::}'a.\ x)$ **.**
qed

lemma *onorm_compose*:
assumes *f*: *bounded_linear f*
assumes *g*: *bounded_linear g*
shows $onorm\ (f \circ g) \leq onorm\ f * onorm\ g$
proof (*rule onorm_bound*)
show $0 \leq onorm\ f * onorm\ g$
by (*intro mult_nonneg_nonneg onorm_pos_le f g*)
next
fix *x*
have $norm\ (f\ (g\ x)) \leq onorm\ f * norm\ (g\ x)$
by (*rule onorm* [*OF f*])
also have $onorm\ f * norm\ (g\ x) \leq onorm\ f * (onorm\ g * norm\ x)$
by (*rule mult_left_mono* [*OF onorm* [*OF g*] *onorm_pos_le* [*OF f*]])
finally show $norm\ ((f \circ g)\ x) \leq onorm\ f * onorm\ g * norm\ x$
by (*simp add*: *mult.assoc*)
qed

lemma *onorm_scaleR_lemma*:
assumes *f*: *bounded_linear f*
shows $onorm\ (\lambda x.\ r *_R f\ x) \leq |r| * onorm\ f$
proof (*rule onorm_bound*)
show $0 \leq |r| * onorm\ f$
by (*intro mult_nonneg_nonneg onorm_pos_le abs_ge_zero f*)
next
fix *x*
have $|r| * norm\ (f\ x) \leq |r| * (onorm\ f * norm\ x)$
by (*intro mult_left_mono onorm abs_ge_zero f*)
then show $norm\ (r *_R f\ x) \leq |r| * onorm\ f * norm\ x$

by (*simp only*: *norm_scaleR mult.assoc*)
qed

lemma *onorm_scaleR*:
assumes *f*: *bounded_linear f*
shows *onorm* ($\lambda x.\ r *_R f\ x$) = $|r| *$ *onorm f*
proof (*cases r* = *0*)
assume $r \neq 0$
show *?thesis*
proof (*rule order_antisym*)
show *onorm* ($\lambda x.\ r *_R f\ x$) $\leq |r| *$ *onorm f*
using *f* **by** (*rule onorm_scaleR_lemma*)
next
have *bounded_linear* ($\lambda x.\ r *_R f\ x$)
using *bounded_linear_scaleR_right f* **by** (*rule bounded_linear_compose*)
then have *onorm* ($\lambda x.$ *inverse* $r *_R r *_R f\ x$) $\leq$ |*inverse* $r| *$ *onorm* ($\lambda x.\ r$ $*_R f\ x$)
by (*rule onorm_scaleR_lemma*)
with ‹$r \neq 0$› **show** $|r| *$ *onorm f* $\leq$ *onorm* ($\lambda x.\ r *_R f\ x$)
by (*simp add*: *inverse_eq_divide pos_le_divide_eq mult.commute*)
qed
qed (*simp add*: *onorm_zero*)

lemma *onorm_scaleR_left_lemma*:
assumes *r*: *bounded_linear r*
shows *onorm* ($\lambda x.\ r\ x *_R f$) $\leq$ *onorm r* $*$ *norm f*
proof (*rule onorm_bound*)
fix *x*
have *norm* ($r\ x *_R f$) = *norm* ($r\ x$) $*$ *norm f*
by *simp*
also have ... $\leq$ *onorm r* $*$ *norm x* $*$ *norm f*
by (*intro mult_right_mono onorm r norm_ge_zero*)
finally show *norm* ($r\ x *_R f$) $\leq$ *onorm r* $*$ *norm f* $*$ *norm x*
by (*simp add*: *ac_simps*)
qed (*intro mult_nonneg_nonneg norm_ge_zero onorm_pos_le r*)

lemma *onorm_scaleR_left*:
assumes *f*: *bounded_linear r*
shows *onorm* ($\lambda x.\ r\ x *_R f$) = *onorm r* $*$ *norm f*
proof (*cases f* = *0*)
assume $f \neq 0$
show *?thesis*
proof (*rule order_antisym*)
show *onorm* ($\lambda x.\ r\ x *_R f$) $\leq$ *onorm r* $*$ *norm f*
using *f* **by** (*rule onorm_scaleR_left_lemma*)
next
have *bl1*: *bounded_linear* ($\lambda x.\ r\ x *_R f$)
by (*metis bounded_linear_scaleR_const f*)
have *bounded_linear* ($\lambda x.\ r\ x *$ *norm f*)

```
      by (metis bounded_linear_mult_const f)
    from onorm_scaleR_left_lemma[OF this, of inverse (norm f)]
    have onorm r ≤ onorm (λx. r x * norm f) * inverse (norm f)
      using ‹f ≠ 0›
      by (simp add: inverse_eq_divide)
    also have onorm (λx. r x * norm f) ≤ onorm (λx. r x *R f)
      by (rule onorm_bound)
        (auto simp: abs_mult bl1 onorm_pos_le intro!: order_trans[OF _ onorm])
    finally show onorm r * norm f ≤ onorm (λx. r x *R f)
      using ‹f ≠ 0›
      by (simp add: inverse_eq_divide pos_le_divide_eq mult.commute)
  qed
qed (simp add: onorm_zero)

lemma onorm_neg:
  shows onorm (λx. − f x) = onorm f
  unfolding onorm_def by simp

lemma onorm_triangle:
  assumes f: bounded_linear f
  assumes g: bounded_linear g
  shows onorm (λx. f x + g x) ≤ onorm f + onorm g
proof (rule onorm_bound)
  show 0 ≤ onorm f + onorm g
    by (intro add_nonneg_nonneg onorm_pos_le f g)
next
  fix x
  have norm (f x + g x) ≤ norm (f x) + norm (g x)
    by (rule norm_triangle_ineq)
  also have norm (f x) + norm (g x) ≤ onorm f * norm x + onorm g * norm x
    by (intro add_mono onorm f g)
  finally show norm (f x + g x) ≤ (onorm f + onorm g) * norm x
    by (simp only: distrib_right)
qed

lemma onorm_triangle_le:
  assumes bounded_linear f
  assumes bounded_linear g
  assumes onorm f + onorm g ≤ e
  shows onorm (λx. f x + g x) ≤ e
  using assms by (rule onorm_triangle [THEN order_trans])

lemma onorm_triangle_lt:
  assumes bounded_linear f
  assumes bounded_linear g
  assumes onorm f + onorm g < e
  shows onorm (λx. f x + g x) < e
  using assms by (rule onorm_triangle [THEN order_le_less_trans])
```

```
lemma onorm_sum:
  assumes finite S
  assumes ⋀s. s ∈ S ⟹ bounded_linear (f s)
  shows onorm (λx. sum (λs. f s x) S) ≤ sum (λs. onorm (f s)) S
  using assms
 by (induction) (auto simp: onorm_zero intro!: onorm_triangle_le bounded_linear_sum)

lemmas onorm_sum_le = onorm_sum[THEN order_trans]

end
```

4.4 Line Segment

```
theory Line_Segment
imports
  Convex
  Topology_Euclidean_Space
begin
```

4.4.1 Topological Properties of Convex Sets, Metric Spaces and Functions

```
lemma convex_supp_sum:
  assumes convex S and 1: supp_sum u I = 1
      and ⋀i. i ∈ I ⟹ 0 ≤ u i ∧ (u i = 0 ∨ f i ∈ S)
    shows supp_sum (λi. u i *R f i) I ∈ S
proof -
  have fin: finite {i ∈ I. u i ≠ 0}
    using 1 sum.infinite by (force simp: supp_sum_def support_on_def)
  then have supp_sum (λi. u i *R f i) I = sum (λi. u i *R f i) {i ∈ I. u i ≠ 0}
    by (force intro: sum.mono_neutral_left simp: supp_sum_def support_on_def)
  also have ... ∈ S
    using 1 assms by (force simp: supp_sum_def support_on_def intro: convex_sum
[OF fin ‹convex S›])
  finally show ?thesis .
qed

lemma sphere_eq_empty [simp]:
  fixes a :: 'a::{real_normed_vector, perfect_space}
  shows sphere a r = {} ⟷ r < 0
by (auto simp: sphere_def dist_norm) (metis dist_norm le_less_linear vector_choose_dist)

lemma cone_closure:
  fixes S :: 'a::real_normed_vector set
  assumes cone S
  shows cone (closure S)
proof (cases S = {})
  case True
  then show ?thesis by auto
```

next
  case *False*
  then have $0 \in S \wedge (\forall c.\ c > 0 \longrightarrow (*_R)\ c\ `\ S = S)$
    using *cone_iff* [*of S*] *assms* **by** *auto*
  then have $0 \in closure\ S \wedge (\forall c.\ c > 0 \longrightarrow (*_R)\ c\ `\ closure\ S = closure\ S)$
    using *closure_subset* **by** (*auto simp*: *closure_scaleR*)
  then show *?thesis*
    using *False cone_iff* [*of closure S*] **by** *auto*
qed

corollary *component_complement_connected*:
  fixes *S* :: *'a*::*real_normed_vector set*
  assumes *connected S C* ∈ *components* (−*S*)
  shows *connected*(−*C*)
  using *component_diff_connected* [*of S UNIV*] *assms*
  by (*auto simp*: *Compl_eq_Diff_UNIV*)

proposition *clopen*:
  fixes *S* :: *'a* :: *real_normed_vector set*
  shows *closed S* ∧ *open S* ⟷ *S* = {} ∨ *S* = *UNIV*
  by (*force intro*!: *connected_UNIV* [*unfolded connected_clopen*, *rule_format*])

corollary *compact_open*:
  fixes *S* :: *'a* :: *euclidean_space set*
  shows *compact S* ∧ *open S* ⟷ *S* = {}
  by (*auto simp*: *compact_eq_bounded_closed clopen*)

corollary *finite_imp_not_open*:
  fixes *S* :: *'a*::{*real_normed_vector*, *perfect_space*} *set*
  shows ⟦*finite S*; *open S*⟧ ⟹ *S*={}
  using *clopen* [*of S*] *finite_imp_closed not_bounded_UNIV* **by** *blast*

corollary *empty_interior_finite*:
    fixes *S* :: *'a*::{*real_normed_vector*, *perfect_space*} *set*
    shows *finite S* ⟹ *interior S* = {}
  by (*metis interior_subset finite_subset open_interior* [*of S*] *finite_imp_not_open*)

Balls, being convex, are connected.

lemma *convex_local_global_minimum*:
  fixes *s* :: *'a*::*real_normed_vector set*
  assumes $e > 0$
    and *convex_on s f*
    and *ball x e* ⊆ *s*
    and ∀ *y*∈*ball x e*. *f x* ≤ *f y*
  shows ∀ *y*∈*s*. *f x* ≤ *f y*
proof (*rule ccontr*)
  have *x* ∈ *s* **using** *assms*(*1*,*3*) **by** *auto*
  assume ¬ *?thesis*

```
  then obtain y where y∈s and y: f x > f y by auto
  then have xy: 0 < dist x y  by auto
  then obtain u where 0 < u u ≤ 1 and u: u < e / dist x y
    using field_lbound_gt_zero[of 1 e / dist x y] xy ‹e>0› by auto
  then have f ((1−u) *R x + u *R y) ≤ (1−u) * f x + u * f y
    using ‹x∈s› ‹y∈s›
    using assms(2)[unfolded convex_on_def,
        THEN bspec[where x=x], THEN bspec[where x=y], THEN spec[where
x=1−u]]
    by auto
  moreover
  have *: x − ((1 − u) *R x + u *R y) = u *R (x − y)
    by (simp add: algebra_simps)
  have (1 − u) *R x + u *R y ∈ ball x e
    unfolding mem_ball dist_norm
    unfolding * and norm_scaleR and abs_of_pos[OF ‹0<u›]
    unfolding dist_norm[symmetric]
    using u
    unfolding pos_less_divide_eq[OF xy]
    by auto
  then have f x ≤ f ((1 − u) *R x + u *R y)
    using assms(4) by auto
  ultimately show False
    using mult_strict_left_mono[OF y ‹u>0›]
    unfolding left_diff_distrib
    by auto
qed

lemma convex_ball [iff]:
  fixes x :: 'a::real_normed_vector
  shows convex (ball x e)
proof (auto simp: convex_def)
  fix y z
  assume yz: dist x y < e dist x z < e
  fix u v :: real
  assume uv: 0 ≤ u 0 ≤ v u + v = 1
  have dist x (u *R y + v *R z) ≤ u * dist x y + v * dist x z
    using uv yz
    using convex_on_dist [of ball x e x, unfolded convex_on_def,
      THEN bspec[where x=y], THEN bspec[where x=z]]
    by auto
  then show dist x (u *R y + v *R z) < e
    using convex_bound_lt[OF yz uv] by auto
qed

lemma convex_cball [iff]:
  fixes x :: 'a::real_normed_vector
  shows convex (cball x e)
proof −
```

```
{
  fix y z
  assume yz: dist x y ≤ e dist x z ≤ e
  fix u v :: real
  assume uv: 0 ≤ u 0 ≤ v u + v = 1
  have dist x (u *R y + v *R z) ≤ u * dist x y + v * dist x z
    using uv yz
    using convex_on_dist [of cball x e x, unfolded convex_on_def,
      THEN bspec[where x=y], THEN bspec[where x=z]]
    by auto
  then have dist x (u *R y + v *R z) ≤ e
    using convex_bound_le[OF yz uv] by auto
}
then show ?thesis by (auto simp: convex_def Ball_def)
qed

lemma connected_ball [iff]:
  fixes x :: 'a::real_normed_vector
  shows connected (ball x e)
  using convex_connected convex_ball by auto

lemma connected_cball [iff]:
  fixes x :: 'a::real_normed_vector
  shows connected (cball x e)
  using convex_connected convex_cball by auto

lemma bounded_convex_hull:
  fixes s :: 'a::real_normed_vector set
  assumes bounded s
  shows bounded (convex hull s)
proof −
  from assms obtain B where B: ∀ x∈s. norm x ≤ B
    unfolding bounded_iff by auto
  show ?thesis
   by (simp add: bounded_subset[OF bounded_cball, of _ 0 B] B subsetI subset_hull)
qed

lemma finite_imp_bounded_convex_hull:
  fixes s :: 'a::real_normed_vector set
  shows finite s =⇒ bounded (convex hull s)
  using bounded_convex_hull finite_imp_bounded
  by auto
```

4.4.2 Midpoint

```
definition midpoint :: 'a::real_vector ⇒ 'a ⇒ 'a
  where midpoint a b = (inverse (2::real)) *R (a + b)

lemma midpoint_idem [simp]: midpoint x x = x
```

```
  unfolding midpoint_def by simp

lemma midpoint_sym: midpoint a b = midpoint b a
  unfolding midpoint_def by (auto simp add: scaleR_right_distrib)

lemma midpoint_eq_iff: midpoint a b = c ⟷ a + b = c + c
proof -
  have midpoint a b = c ⟷ scaleR 2 (midpoint a b) = scaleR 2 c
    by simp
  then show ?thesis
    unfolding midpoint_def scaleR_2 [symmetric] by simp
qed

lemma
  fixes a::real
  assumes a ≤ b shows ge_midpoint_1: a ≤ midpoint a b
                  and le_midpoint_1: midpoint a b ≤ b
  by (simp_all add: midpoint_def assms)

lemma dist_midpoint:
  fixes a b :: 'a::real_normed_vector shows
  dist a (midpoint a b) = (dist a b) / 2 (is ?t1)
  dist b (midpoint a b) = (dist a b) / 2 (is ?t2)
  dist (midpoint a b) a = (dist a b) / 2 (is ?t3)
  dist (midpoint a b) b = (dist a b) / 2 (is ?t4)
proof -
  have *: ⋀x y::'a. 2 *R x = - y ⟹ norm x = (norm y) / 2
    unfolding equation_minus_iff by auto
  have **: ⋀x y::'a. 2 *R x =   y ⟹ norm x = (norm y) / 2
    by auto
  note scaleR_right_distrib [simp]
  show ?t1
    unfolding midpoint_def dist_norm
    apply (rule **)
    apply (simp add: scaleR_right_diff_distrib)
    apply (simp add: scaleR_2)
    done
  show ?t2
    unfolding midpoint_def dist_norm
    apply (rule *)
    apply (simp add: scaleR_right_diff_distrib)
    apply (simp add: scaleR_2)
    done
  show ?t3
    unfolding midpoint_def dist_norm
    apply (rule *)
    apply (simp add: scaleR_right_diff_distrib)
    apply (simp add: scaleR_2)
    done
```

show *?t4*
unfolding *midpoint_def dist_norm*
apply (*rule* ∗∗)
apply (*simp add*: *scaleR_right_diff_distrib*)
apply (*simp add*: *scaleR_2*)
done
qed

lemma *midpoint_eq_endpoint* [*simp*]:
midpoint a b = a ⟷ *a = b*
midpoint a b = b ⟷ *a = b*
unfolding *midpoint_eq_iff* **by** *auto*

lemma *midpoint_plus_self* [*simp*]: *midpoint a b + midpoint a b = a + b*
using *midpoint_eq_iff* **by** *metis*

lemma *midpoint_linear_image*:
linear f ⟹ *midpoint(f a)(f b) = f(midpoint a b)*
by (*simp add*: *linear_iff midpoint_def*)

4.4.3 Open and closed segments

definition *closed_segment* :: *'a::real_vector* ⇒ *'a* ⇒ *'a set*
where *closed_segment a b* = {(*1* − *u*) $*_R$ *a* + *u* $*_R$ *b* | *u::real*. *0* ≤ *u* ∧ *u* ≤ *1*}

definition *open_segment* :: *'a::real_vector* ⇒ *'a* ⇒ *'a set* **where**
open_segment a b ≡ *closed_segment a b* − {*a*,*b*}

lemmas *segment* = *open_segment_def closed_segment_def*

lemma *in_segment*:
x ∈ *closed_segment a b* ⟷ (∃ *u*. *0* ≤ *u* ∧ *u* ≤ *1* ∧ *x* = (*1* − *u*) $*_R$ *a* + *u* $*_R$ *b*)
x ∈ *open_segment a b* ⟷ *a* ≠ *b* ∧ (∃ *u*. *0* < *u* ∧ *u* < *1* ∧ *x* = (*1* − *u*) $*_R$ *a* + *u* $*_R$ *b*)
using *less_eq_real_def* **by** (*auto simp*: *segment algebra_simps*)

lemma *closed_segment_linear_image*:
closed_segment (*f a*) (*f b*) = *f* ‘ (*closed_segment a b*) **if** *linear f*
proof −
interpret *linear f* **by** *fact*
show *?thesis*
by (*force simp add*: *in_segment add scale*)
qed

lemma *open_segment_linear_image*:
⟦*linear f*; *inj f*⟧ ⟹ *open_segment* (*f a*) (*f b*) = *f* ‘ (*open_segment a b*)
by (*force simp*: *open_segment_def closed_segment_linear_image inj_on_def*)

lemma *closed_segment_translation*:
closed_segment (*c* + *a*) (*c* + *b*) = *image* (λ*x*. *c* + *x*) (*closed_segment a b*)
apply *safe*
apply (*rule_tac x*=*x*−*c* **in** *image_eqI*)
apply (*auto simp*: *in_segment algebra_simps*)
done

lemma *open_segment_translation*:
open_segment (*c* + *a*) (*c* + *b*) = *image* (λ*x*. *c* + *x*) (*open_segment a b*)
by (*simp add*: *open_segment_def closed_segment_translation translation_diff*)

lemma *closed_segment_of_real*:
closed_segment (*of_real x*) (*of_real y*) = *of_real* ‘ *closed_segment x y*
apply (*auto simp*: *image_iff in_segment scaleR_conv_of_real*)
apply (*rule_tac x*=(*1*−*u*)∗*x* + *u*∗*y* **in** *bexI*)
apply (*auto simp*: *in_segment*)
done

lemma *open_segment_of_real*:
open_segment (*of_real x*) (*of_real y*) = *of_real* ‘ *open_segment x y*
apply (*auto simp*: *image_iff in_segment scaleR_conv_of_real*)
apply (*rule_tac x*=(*1*−*u*)∗*x* + *u*∗*y* **in** *bexI*)
apply (*auto simp*: *in_segment*)
done

lemma *closed_segment_Reals*:
⟦*x* ∈ *Reals*; *y* ∈ *Reals*⟧ ⟹ *closed_segment x y* = *of_real* ‘ *closed_segment* (*Re x*) (*Re y*)
by (*metis closed_segment_of_real of_real_Re*)

lemma *open_segment_Reals*:
⟦*x* ∈ *Reals*; *y* ∈ *Reals*⟧ ⟹ *open_segment x y* = *of_real* ‘ *open_segment* (*Re x*) (*Re y*)
by (*metis open_segment_of_real of_real_Re*)

lemma *open_segment_PairD*:
(*x*, *x*′) ∈ *open_segment* (*a*, *a*′) (*b*, *b*′)
⟹ (*x* ∈ *open_segment a b* ∨ *a* = *b*) ∧ (*x*′ ∈ *open_segment a*′ *b*′ ∨ *a*′ = *b*′)
by (*auto simp*: *in_segment*)

lemma *closed_segment_PairD*:
(*x*, *x*′) ∈ *closed_segment* (*a*, *a*′) (*b*, *b*′) ⟹ *x* ∈ *closed_segment a b* ∧ *x*′ ∈ *closed_segment a*′ *b*′
by (*auto simp*: *closed_segment_def*)

lemma *closed_segment_translation_eq* [*simp*]:
d + *x* ∈ *closed_segment* (*d* + *a*) (*d* + *b*) ⟷ *x* ∈ *closed_segment a b*
proof −
have ∗: ⋀*d x a b*. *x* ∈ *closed_segment a b* ⟹ *d* + *x* ∈ *closed_segment* (*d* + *a*)

(*d* + *b*)
  apply (*simp add*: *closed_segment_def*)
  apply (*erule ex_forward*)
  apply (*simp add*: *algebra_simps*)
  done
 show *?thesis*
 using ∗ [**where** *d* = −*d*] ∗
 by (*fastforce simp add*:)
qed

lemma *open_segment_translation_eq* [*simp*]:
  d + *x* ∈ *open_segment* (*d* + *a*) (*d* + *b*) ⟷ *x* ∈ *open_segment a b*
 by (*simp add*: *open_segment_def*)

lemma *of_real_closed_segment* [*simp*]:
 of_real x ∈ *closed_segment* (*of_real a*) (*of_real b*) ⟷ *x* ∈ *closed_segment a b*
 apply (*auto simp*: *in_segment scaleR_conv_of_real elim*!: *ex_forward*)
 using *of_real_eq_iff* **by** *fastforce*

lemma *of_real_open_segment* [*simp*]:
 of_real x ∈ *open_segment* (*of_real a*) (*of_real b*) ⟷ *x* ∈ *open_segment a b*
 apply (*auto simp*: *in_segment scaleR_conv_of_real elim*!: *ex_forward del*: *exE*)
 using *of_real_eq_iff* **by** *fastforce*

lemma *convex_contains_segment*:
 convex S ⟷ (∀ *a*∈*S*. ∀ *b*∈*S*. *closed_segment a b* ⊆ *S*)
 unfolding *convex_alt closed_segment_def* **by** *auto*

lemma *closed_segment_in_Reals*:
  ⟦*x* ∈ *closed_segment a b*; *a* ∈ *Reals*; *b* ∈ *Reals*⟧ ⟹ *x* ∈ *Reals*
 by (*meson subsetD convex_Reals convex_contains_segment*)

lemma *open_segment_in_Reals*:
  ⟦*x* ∈ *open_segment a b*; *a* ∈ *Reals*; *b* ∈ *Reals*⟧ ⟹ *x* ∈ *Reals*
 by (*metis Diff_iff closed_segment_in_Reals open_segment_def*)

lemma *closed_segment_subset*: ⟦*x* ∈ *S*; *y* ∈ *S*; *convex S*⟧ ⟹ *closed_segment x y* ⊆ *S*
 by (*simp add*: *convex_contains_segment*)

lemma *closed_segment_subset_convex_hull*:
  ⟦*x* ∈ *convex hull S*; *y* ∈ *convex hull S*⟧ ⟹ *closed_segment x y* ⊆ *convex hull S*
 using *convex_contains_segment* **by** *blast*

lemma *segment_convex_hull*:
 closed_segment a b = *convex hull* {*a*,*b*}
proof −
 have ∗: ⋀*x*. {*x*} ≠ {} **by** *auto*
 show *?thesis*

```
    unfolding segment convex_hull_insert[OF *] convex_hull_singleton
    by (safe; rule_tac x=1 − u in exI; force)
qed

lemma open_closed_segment: u ∈ open_segment w z ⟹ u ∈ closed_segment w z
  by (auto simp add: closed_segment_def open_segment_def)

lemma segment_open_subset_closed:
   open_segment a b ⊆ closed_segment a b
  by (auto simp: closed_segment_def open_segment_def)

lemma bounded_closed_segment:
  fixes a :: 'a::real_normed_vector shows bounded (closed_segment a b)
  by (rule boundedI[where B=max (norm a) (norm b)])
   (auto simp: closed_segment_def max_def convex_bound_le intro!: norm_triangle_le)

lemma bounded_open_segment:
    fixes a :: 'a::real_normed_vector shows bounded (open_segment a b)
 by (rule bounded_subset [OF bounded_closed_segment segment_open_subset_closed])

lemmas bounded_segment = bounded_closed_segment open_closed_segment

lemma ends_in_segment [iff]: a ∈ closed_segment a b b ∈ closed_segment a b
  unfolding segment_convex_hull
  by (auto intro!: hull_subset[unfolded subset_eq, rule_format])


lemma eventually_closed_segment:
  fixes x0::'a::real_normed_vector
  assumes open X0 x0 ∈ X0
  shows ∀F x in at x0 within U. closed_segment x0 x ⊆ X0
proof −
  from openE[OF assms]
  obtain e where e: 0 < e ball x0 e ⊆ X0 .
  then have ∀F x in at x0 within U. x ∈ ball x0 e
    by (auto simp: dist_commute eventually_at)
  then show ?thesis
  proof eventually_elim
    case (elim x)
    have x0 ∈ ball x0 e using ‹e > 0› by simp
    from convex_ball[unfolded convex_contains_segment, rule_format, OF this elim]
    have closed_segment x0 x ⊆ ball x0 e .
    also note ‹. . . ⊆ X0›
    finally show ?case .
  qed
qed

lemma closed_segment_commute: closed_segment a b = closed_segment b a
proof −
```

have {*a*, *b*} = {*b*, *a*} **by** *auto*
thus *?thesis*
by (*simp add*: *segment_convex_hull*)
qed

lemma *segment_bound1*:
assumes $x \in$ *closed_segment a b*
shows *norm* $(x - a) \leq$ *norm* $(b - a)$
proof −
obtain *u* **where** $x = (1 - u) *_R a + u *_R b$ $0 \leq u$ $u \leq 1$
using *assms* **by** (*auto simp add*: *closed_segment_def*)
then show *norm* $(x - a) \leq$ *norm* $(b - a)$
apply *clarify*
apply (*auto simp*: *algebra_simps*)
apply (*simp add*: *scaleR_diff_right* [*symmetric*] *mult_left_le_one_le*)
done
qed

lemma *segment_bound*:
assumes $x \in$ *closed_segment a b*
shows *norm* $(x - a) \leq$ *norm* $(b - a)$ *norm* $(x - b) \leq$ *norm* $(b - a)$
by (*metis assms closed_segment_commute dist_commute dist_norm segment_bound1*)+

lemma *open_segment_commute*: *open_segment a b* = *open_segment b a*
proof −
have {*a*, *b*} = {*b*, *a*} **by** *auto*
thus *?thesis*
by (*simp add*: *closed_segment_commute open_segment_def*)
qed

lemma *closed_segment_idem* [*simp*]: *closed_segment a a* = {*a*}
unfolding *segment* **by** (*auto simp add*: *algebra_simps*)

lemma *open_segment_idem* [*simp*]: *open_segment a a* = {}
by (*simp add*: *open_segment_def*)

lemma *closed_segment_eq_open*: *closed_segment a b* = *open_segment a b* ∪ {*a*,*b*}
using *open_segment_def* **by** *auto*

lemma *convex_contains_open_segment*:
convex s ⟷ (∀ *a*∈*s*. ∀ *b*∈*s*. *open_segment a b* ⊆ *s*)
by (*simp add*: *convex_contains_segment closed_segment_eq_open*)

lemma *closed_segment_eq_real_ivl1*:
fixes *a b*::*real*
assumes $a \leq b$
shows *closed_segment a b* = {*a* .. *b*}
proof *safe*
fix *x*

```
  assume x ∈ closed_segment a b
  then obtain u where u: 0 ≤ u u ≤ 1 and x_def: x = (1 − u) ∗ a + u ∗ b
    by (auto simp: closed_segment_def)
  have u ∗ a ≤ u ∗ b (1 − u) ∗ a ≤ (1 − u) ∗ b
    by (auto intro!: mult_left_mono u assms)
  then show x ∈ {a .. b}
    unfolding x_def by (auto simp: algebra_simps)
next
  show ⋀x. x ∈ {a..b} =⇒ x ∈ closed_segment a b
    by (force simp: closed_segment_def divide_simps algebra_simps
            intro: exI[where x=(x − a) / (b − a) for x])
qed

lemma closed_segment_eq_real_ivl:
  fixes a b::real
  shows closed_segment a b = (if a ≤ b then {a .. b} else {b .. a})
  using closed_segment_eq_real_ivl1[of a b] closed_segment_eq_real_ivl1[of b a]
  by (auto simp: closed_segment_commute)

lemma open_segment_eq_real_ivl:
  fixes a b::real
  shows open_segment a b = (if a ≤ b then {a<..<b} else {b<..<a})
by (auto simp: closed_segment_eq_real_ivl open_segment_def split: if_split_asm)

lemma closed_segment_real_eq:
  fixes u::real shows closed_segment u v = (λx. (v − u) ∗ x + u) ‘ {0..1}
  by (simp add: add.commute [of u] image_affinity_atLeastAtMost [where c=u]
closed_segment_eq_real_ivl)

lemma closed_segment_same_Re:
  assumes Re a = Re b
  shows  closed_segment a b = {z. Re z = Re a ∧ Im z ∈ closed_segment (Im a)
(Im b)}
proof safe
  fix z assume z ∈ closed_segment a b
  then obtain u where u: u ∈ {0..1} z = a + of_real u ∗ (b − a)
    by (auto simp: closed_segment_def scaleR_conv_of_real algebra_simps)
  from assms show Re z = Re a by (auto simp: u)
  from u(1) show Im z ∈ closed_segment (Im a) (Im b)
    by (force simp: u closed_segment_def algebra_simps)
next
  fix z assume [simp]: Re z = Re a and Im z ∈ closed_segment (Im a) (Im b)
  then obtain u where u: u ∈ {0..1} Im z = Im a + of_real u ∗ (Im b − Im a)
    by (auto simp: closed_segment_def scaleR_conv_of_real algebra_simps)
  from u(1) show z ∈ closed_segment a b using assms
     by (force simp: u closed_segment_def algebra_simps scaleR_conv_of_real com-
plex_eq_iff)
qed
```

lemma *closed_segment_same_Im*:
  assumes *Im a = Im b*
  shows *closed_segment a b = {z. Im z = Im a ∧ Re z ∈ closed_segment (Re a) (Re b)}*
proof *safe*
  fix *z* **assume** *z ∈ closed_segment a b*
  then obtain *u* **where** *u*: *u ∈ {0..1} z = a + of_real u ∗ (b − a)*
    by (*auto simp*: *closed_segment_def scaleR_conv_of_real algebra_simps*)
  from *assms* **show** *Im z = Im a* **by** (*auto simp*: *u*)
  from *u*(*1*) **show** *Re z ∈ closed_segment* (*Re a*) (*Re b*)
    by (*force simp*: *u closed_segment_def algebra_simps*)
next
  fix *z* **assume** [*simp*]: *Im z = Im a* **and** *Re z ∈ closed_segment* (*Re a*) (*Re b*)
  then obtain *u* **where** *u*: *u ∈ {0..1} Re z = Re a + of_real u ∗ (Re b − Re a)*
    by (*auto simp*: *closed_segment_def scaleR_conv_of_real algebra_simps*)
  from *u*(*1*) **show** *z ∈ closed_segment a b* **using** *assms*
    by (*force simp*: *u closed_segment_def algebra_simps scaleR_conv_of_real complex_eq_iff*)
qed

lemma *dist_in_closed_segment*:
  fixes *a* :: *'a* :: *euclidean_space*
  assumes *x ∈ closed_segment a b*
    shows *dist x a ≤ dist a b ∧ dist x b ≤ dist a b*
proof (*intro conjI*)
  obtain *u* **where** *u*: *0 ≤ u u ≤ 1* **and** *x*: *x = (1 − u) ∗R a + u ∗R b*
    using *assms* **by** (*force simp*: *in_segment algebra_simps*)
  have *dist x a = u ∗ dist a b*
    apply (*simp add*: *dist_norm algebra_simps x*)
   by (*metis ‹0 ≤ u› abs_of_nonneg norm_minus_commute norm_scaleR real_vector.scale_right_diff_distrib*)
  also have ... *≤ dist a b*
    by (*simp add*: *mult_left_le_one_le u*)
  finally show *dist x a ≤ dist a b* **.**
  have *dist x b = norm ((1−u) ∗R a − (1−u) ∗R b)*
    by (*simp add*: *dist_norm algebra_simps x*)
  also have ... *= (1−u) ∗ dist a b*
  proof −
    have *norm ((1 − 1 ∗ u) ∗R (a − b)) = (1 − 1 ∗ u) ∗ norm (a − b)*
      using *‹u ≤ 1›* **by** *force*
    then show *?thesis*
      by (*simp add*: *dist_norm real_vector.scale_right_diff_distrib*)
  qed
  also have ... *≤ dist a b*
    by (*simp add*: *mult_left_le_one_le u*)
  finally show *dist x b ≤ dist a b* **.**
qed

lemma *dist_in_open_segment*:
  fixes *a* :: *'a* :: *euclidean_space*

assumes $x \in$ *open_segment a b*
shows *dist x a* $<$ *dist a b* $\wedge$ *dist x b* $<$ *dist a b*
proof (*intro conjI*)
obtain *u* **where** *u*: *0* $<$ *u u* $<$ *1* **and** *x*: $x = (1 - u) *_R a + u *_R b$
using *assms* **by** (*force simp*: *in_segment algebra_simps*)
have *dist x a* $=$ *u* $*$ *dist a b*
apply (*simp add*: *dist_norm algebra_simps x*)
by (*metis abs_of_nonneg less_eq_real_def norm_minus_commute norm_scaleR real_vector.scale_right_diff_distrib* ‹*0* $<$ *u*›)
also have $*$: ... $<$ *dist a b*
using *assms mult_less_cancel_right2 u*(*2*) **by** *fastforce*
finally show *dist x a* $<$ *dist a b* **.**
have *ab_ne0*: *dist a b* $\neq$ *0*
using $*$ **by** *fastforce*
have *dist x b* $=$ *norm* $((1-u) *_R a - (1-u) *_R b)$
by (*simp add*: *dist_norm algebra_simps x*)
also have ... $= (1-u) *$ *dist a b*
proof −
have *norm* $((1 - 1 * u) *_R (a - b)) = (1 - 1 * u) *$ *norm* $(a - b)$
using ‹*u* $<$ *1*› **by** *force*
then show *?thesis*
by (*simp add*: *dist_norm real_vector.scale_right_diff_distrib*)
qed
also have ... $<$ *dist a b*
using *ab_ne0* ‹*0* $<$ *u*› **by** *simp*
finally show *dist x b* $<$ *dist a b* **.**
qed

lemma *dist_decreases_open_segment_0*:
fixes *x* :: *'a* :: *euclidean_space*
assumes *x* $\in$ *open_segment 0 b*
shows *dist c x* $<$ *dist c 0* $\vee$ *dist c x* $<$ *dist c b*
proof (*rule ccontr*, *clarsimp simp*: *not_less*)
obtain *u* **where** *u*: *0* $\neq$ *b 0* $<$ *u u* $<$ *1* **and** *x*: $x = u *_R b$
using *assms* **by** (*auto simp*: *in_segment*)
have *xb*: $x \cdot b < b \cdot b$
using *u x* **by** *auto*
assume *norm c* $\leq$ *dist c x*
then have $c \cdot c \leq (c - x) \cdot (c - x)$
by (*simp add*: *dist_norm norm_le*)
moreover have $0 < x \cdot b$
using *u x* **by** *auto*
ultimately have *less*: $c \cdot b < x \cdot b$
by (*simp add*: *x algebra_simps inner_commute u*)
assume *dist c b* $\leq$ *dist c x*
then have $(c - b) \cdot (c - b) \leq (c - x) \cdot (c - x)$
by (*simp add*: *dist_norm norm_le*)
then have $(b \cdot b) * (1 - u*u) \leq 2 * (b \cdot c) * (1-u)$
by (*simp add*: *x algebra_simps inner_commute*)

```
  then have (1+u) * (b · b) * (1−u) ≤ 2 * (b · c) * (1−u)
    by (simp add: algebra_simps)
  then have (1+u) * (b · b) ≤ 2 * (b · c)
    using ‹u < 1› by auto
  with xb have c · b ≥ x · b
    by (auto simp: x algebra_simps inner_commute)
  with less show False by auto
qed

proposition dist_decreases_open_segment:
  fixes a :: 'a :: euclidean_space
  assumes x ∈ open_segment a b
    shows dist c x < dist c a ∨ dist c x < dist c b
proof −
  have *: x − a ∈ open_segment 0 (b − a) using assms
    by (metis diff_self open_segment_translation_eq uminus_add_conv_diff)
  show ?thesis
    using dist_decreases_open_segment_0 [OF *, of c−a] assms
    by (simp add: dist_norm)
qed

corollary open_segment_furthest_le:
  fixes a b x y :: 'a::euclidean_space
  assumes x ∈ open_segment a b
  shows norm (y − x) < norm (y − a) ∨  norm (y − x) < norm (y − b)
  by (metis assms dist_decreases_open_segment dist_norm)

corollary dist_decreases_closed_segment:
  fixes a :: 'a :: euclidean_space
  assumes x ∈ closed_segment a b
    shows dist c x ≤ dist c a ∨ dist c x ≤ dist c b
apply (cases x ∈ open_segment a b)
 using dist_decreases_open_segment less_eq_real_def apply blast
by (metis DiffI assms empty_iff insertE open_segment_def order_refl)

corollary segment_furthest_le:
  fixes a b x y :: 'a::euclidean_space
  assumes x ∈ closed_segment a b
  shows norm (y − x) ≤ norm (y − a) ∨  norm (y − x) ≤ norm (y − b)
  by (metis assms dist_decreases_closed_segment dist_norm)

lemma convex_intermediate_ball:
  fixes a :: 'a :: euclidean_space
  shows ⟦ball a r ⊆ T; T ⊆ cball a r⟧ ⟹ convex T
apply (simp add: convex_contains_open_segment, clarify)
by (metis (no_types, hide_lams) less_le_trans mem_ball mem_cball subsetCE dist_decreases_open_segment)

lemma csegment_midpoint_subset: closed_segment (midpoint a b) b ⊆ closed_segment
a b
```

apply (*clarsimp simp*: *midpoint_def in_segment*)
apply (*rule_tac x*=(*1* + *u*) / *2* **in** *exI*)
apply (*auto simp*: *algebra_simps add_divide_distrib diff_divide_distrib*)
by (*metis field_sum_of_halves scaleR_left.add*)

lemma *notin_segment_midpoint*:
fixes *a* :: *'a* :: *euclidean_space*
shows $a \neq b \Longrightarrow a \notin$ *closed_segment* (*midpoint a b*) *b*
by (*auto simp*: *dist_midpoint dest*!: *dist_in_closed_segment*)

More lemmas, especially for working with the underlying formula

lemma *segment_eq_compose*:
fixes *a* :: *'a* :: *real_vector*
shows $(\lambda u.\ (1 - u) *_R a + u *_R b) = (\lambda x.\ a + x)\ o\ (\lambda u.\ u *_R (b - a))$
by (*simp add*: *o_def algebra_simps*)

lemma *segment_degen_1*:
fixes *a* :: *'a* :: *real_vector*
shows $(1 - u) *_R a + u *_R b = b \longleftrightarrow a{=}b \vee u{=}1$
proof –
{ **assume** $(1 - u) *_R a + u *_R b = b$
then have $(1 - u) *_R a = (1 - u) *_R b$
by (*simp add*: *algebra_simps*)
then have $a{=}b \vee u{=}1$
by *simp*
} **then show** *?thesis*
by (*auto simp*: *algebra_simps*)
qed

lemma *segment_degen_0*:
fixes *a* :: *'a* :: *real_vector*
shows $(1 - u) *_R a + u *_R b = a \longleftrightarrow a{=}b \vee u{=}0$
using *segment_degen_1* [*of 1*−*u b a*]
by (*auto simp*: *algebra_simps*)

lemma *add_scaleR_degen*:
fixes *a b* ::*'a*::*real_vector*
assumes $(u *_R b + v *_R a) = (u *_R a + v *_R b)$ $u \neq v$
shows *a*=*b*
by (*metis* (*no_types*, *hide_lams*) *add.commute add_diff_eq diff_add_cancel real_vector.scale_cancel_left real_vector.scale_left_diff_distrib assms*)

lemma *closed_segment_image_interval*:
closed_segment a b = $(\lambda u.\ (1 - u) *_R a + u *_R b)$ ‘ {*0*..*1*}
by (*auto simp*: *set_eq_iff image_iff closed_segment_def*)

lemma *open_segment_image_interval*:
open_segment a b = (*if a*=*b then* {} *else* $(\lambda u.\ (1 - u) *_R a + u *_R b)$ ‘

{*0*<..<*1*})
by (*auto simp*: *open_segment_def closed_segment_def segment_degen_0 segment_degen_1*)

lemmas *segment_image_interval* = *closed_segment_image_interval open_segment_image_interval*

lemma *closed_segment_neq_empty* [*simp*]: *closed_segment a b* ≠ {}
by *auto*

lemma *open_segment_eq_empty* [*simp*]: *open_segment a b* = {} ⟷ *a* = *b*
proof −
{ assume *a1*: *open_segment a b* = {}
have {} ≠ {*0*::*real*<..<*1*}
by *simp*
then have *a* = *b*
using *a1 open_segment_image_interval* **by** *fastforce*
} then show *?thesis* **by** *auto*
qed

lemma *open_segment_eq_empty'* [*simp*]: {} = *open_segment a b* ⟷ *a* = *b*
using *open_segment_eq_empty* **by** *blast*

lemmas *segment_eq_empty* = *closed_segment_neq_empty open_segment_eq_empty*

lemma *inj_segment*:
fixes *a* :: *'a* :: *real_vector*
assumes *a* ≠ *b*
shows *inj_on* (λ*u*. (*1* − *u*) $*_R$ *a* + *u* $*_R$ *b*) *I*
proof
fix *x y*
assume (*1* − *x*) $*_R$ *a* + *x* $*_R$ *b* = (*1* − *y*) $*_R$ *a* + *y* $*_R$ *b*
then have *x* $*_R$ (*b* − *a*) = *y* $*_R$ (*b* − *a*)
by (*simp add*: *algebra_simps*)
with *assms* **show** *x* = *y*
by (*simp add*: *real_vector.scale_right_imp_eq*)
qed

lemma *finite_closed_segment* [*simp*]: *finite*(*closed_segment a b*) ⟷ *a* = *b*
apply *auto*
apply (*rule ccontr*)
apply (*simp add*: *segment_image_interval*)
using *infinite_Icc* [*OF zero_less_one*] *finite_imageD* [*OF _ inj_segment*] **apply** *blast*
done

lemma *finite_open_segment* [*simp*]: *finite*(*open_segment a b*) ⟷ *a* = *b*
by (*auto simp*: *open_segment_def*)

lemmas *finite_segment* = *finite_closed_segment finite_open_segment*

lemma *closed_segment_eq_sing*: *closed_segment a b* = {*c*} ⟷ *a* = *c* ∧ *b* = *c*
  by *auto*

lemma *open_segment_eq_sing*: *open_segment a b* ≠ {*c*}
  by (*metis finite_insert finite_open_segment insert_not_empty open_segment_image_interval*)

lemmas *segment_eq_sing* = *closed_segment_eq_sing open_segment_eq_sing*

lemma *open_segment_bound1*:
  assumes *x* ∈ *open_segment a b*
  shows *norm* (*x* − *a*) < *norm* (*b* − *a*)
proof −
  obtain *u* **where** *x* = (*1* − *u*) $*_R$ *a* + *u* $*_R$ *b 0* < *u u* < *1 a* ≠ *b*
    using *assms* **by** (*auto simp add*: *open_segment_image_interval split*: *if_split_asm*)
  then show *norm* (*x* − *a*) < *norm* (*b* − *a*)
    apply *clarify*
    apply (*auto simp*: *algebra_simps*)
    apply (*simp add*: *scaleR_diff_right* [*symmetric*])
    done
qed

lemma *compact_segment* [*simp*]:
  fixes *a* :: *'a::real_normed_vector*
  shows *compact* (*closed_segment a b*)
  by (*auto simp*: *segment_image_interval intro*!: *compact_continuous_image continuous_intros*)

lemma *closed_segment* [*simp*]:
  fixes *a* :: *'a::real_normed_vector*
  shows *closed* (*closed_segment a b*)
  by (*simp add*: *compact_imp_closed*)

lemma *closure_closed_segment* [*simp*]:
  fixes *a* :: *'a::real_normed_vector*
  shows *closure*(*closed_segment a b*) = *closed_segment a b*
  by *simp*

lemma *open_segment_bound*:
  assumes *x* ∈ *open_segment a b*
  shows *norm* (*x* − *a*) < *norm* (*b* − *a*) *norm* (*x* − *b*) < *norm* (*b* − *a*)
apply (*simp add*: *assms open_segment_bound1*)
by (*metis assms norm_minus_commute open_segment_bound1 open_segment_commute*)

lemma *closure_open_segment* [*simp*]:
  closure (*open_segment a b*) = (*if a* = *b then* {} *else closed_segment a b*)
    for *a* :: *'a::euclidean_space*
proof (*cases a* = *b*)
  case *True*
  then show *?thesis*

by *simp*
next
case *False*
have *closure* $((\lambda u.\ u *_R (b - a))$ ‘ $\{0<..<1\}) = (\lambda u.\ u *_R (b - a))$ ‘ *closure* $\{0<..<1\}$
apply (*rule closure_injective_linear_image* [*symmetric*])
apply (*use False* **in** ‹*auto intro*!: *injI*›)
done
then have *closure*
$((\lambda u.\ (1 - u) *_R a + u *_R b)$ ‘ $\{0<..<1\}) =$
$(\lambda x.\ (1 - x) *_R a + x *_R b)$ ‘ *closure* $\{0<..<1\}$
using *closure_translation* [*of a* $((\lambda x.\ x *_R b - x *_R a)$ ‘ $\{0<..<1\})$]
by (*simp add*: *segment_eq_compose field_simps scaleR_diff_left scaleR_diff_right image_image*)
then show *?thesis*
by (*simp add*: *segment_image_interval closure_greaterThanLessThan* [*symmetric*] *del*: *closure_greaterThanLessThan*)
qed

lemma *closed_open_segment_iff* [*simp*]:
fixes *a* :: *'a::euclidean_space* **shows** *closed*(*open_segment a b*) $\longleftrightarrow$ $a = b$
by (*metis open_segment_def DiffE closure_eq closure_open_segment ends_in_segment*(*1*) *insert_iff segment_image_interval*(*2*))

lemma *compact_open_segment_iff* [*simp*]:
fixes *a* :: *'a::euclidean_space* **shows** *compact*(*open_segment a b*) $\longleftrightarrow$ $a = b$
by (*simp add*: *bounded_open_segment compact_eq_bounded_closed*)

lemma *convex_closed_segment* [*iff*]: *convex* (*closed_segment a b*)
unfolding *segment_convex_hull* **by**(*rule convex_convex_hull*)

lemma *convex_open_segment* [*iff*]: *convex* (*open_segment a b*)
proof −
have *convex* $((\lambda u.\ u *_R (b - a))$ ‘ $\{0<..<1\})$
by (*rule convex_linear_image*) *auto*
then have *convex* $((+)\ a$ ‘ $(\lambda u.\ u *_R (b - a))$ ‘ $\{0<..<1\})$
by (*rule convex_translation*)
then show *?thesis*
by (*simp add*: *image_image open_segment_image_interval segment_eq_compose field_simps scaleR_diff_left scaleR_diff_right*)
qed

lemmas *convex_segment* = *convex_closed_segment convex_open_segment*

lemma *subset_closed_segment*:
closed_segment a b $\subseteq$ *closed_segment c d* $\longleftrightarrow$
$a \in$ *closed_segment c d* $\land$ $b \in$ *closed_segment c d*
by *auto* (*meson contra_subsetD convex_closed_segment convex_contains_segment*)

```
lemma subset_co_segment:
  closed_segment a b ⊆ open_segment c d ⟷
   a ∈ open_segment c d ∧ b ∈ open_segment c d
using closed_segment_subset by blast

lemma subset_open_segment:
 fixes a :: 'a::euclidean_space
 shows open_segment a b ⊆ open_segment c d ⟷
     a = b ∨ a ∈ closed_segment c d ∧ b ∈ closed_segment c d
    (is ?lhs = ?rhs)
proof (cases a = b)
 case True then show ?thesis by simp
next
 case False show ?thesis
 proof
  assume rhs: ?rhs
  with ‹a ≠ b› have c ≠ d
   using closed_segment_idem singleton_iff by auto
  have ∃uc. (1 − u) *R ((1 − ua) *R c + ua *R d) + u *R ((1 − ub) *R c +
ub *R d) =
         (1 − uc) *R c + uc *R d ∧ 0 < uc ∧ uc < 1
    if neq: (1 − ua) *R c + ua *R d ≠ (1 − ub) *R c + ub *R d c ≠ d
     and a = (1 − ua) *R c + ua *R d b = (1 − ub) *R c + ub *R d
     and u: 0 < u u < 1 and uab: 0 ≤ ua ua ≤ 1 0 ≤ ub ub ≤ 1
    for u ua ub
  proof −
   have ua ≠ ub
    using neq by auto
   moreover have (u − 1) * ua ≤ 0 using u uab
    by (simp add: mult_nonpos_nonneg)
   ultimately have lt: (u − 1) * ua < u * ub using u uab
    by (metis antisym_conv diff_ge_0_iff_ge le_less_trans mult_eq_0_iff mult_le_0_iff
not_less)
   have p * ua + q * ub < p+q if p: 0 < p and q: 0 < q for p q
   proof −
    have ¬ p ≤ 0 ¬ q ≤ 0
     using p q not_less by blast+
    then show ?thesis
    by (metis ‹ua ≠ ub› add_less_cancel_left add_less_cancel_right add_mono_thms_linordered_field(5)
          less_eq_real_def mult_cancel_left1 mult_less_cancel_left2 uab(2) uab(4))
   qed
   then have (1 − u) * ua + u * ub < 1 using u ‹ua ≠ ub›
    by (metis diff_add_cancel diff_gt_0_iff_gt)
   with lt show ?thesis
    by (rule_tac x=ua + u*(ub−ua) in exI) (simp add: algebra_simps)
  qed
  with rhs ‹a ≠ b› ‹c ≠ d› show ?lhs
   unfolding open_segment_image_interval closed_segment_def
   by (fastforce simp add:)
```

next
assume *lhs*: *?lhs*
with ‹$a \neq b$› **have** $c \neq d$
by (*meson finite_open_segment rev_finite_subset*)
have *closure* (*open_segment a b*) ⊆ *closure* (*open_segment c d*)
using *lhs closure_mono* **by** *blast*
then have *closed_segment a b* ⊆ *closed_segment c d*
by (*simp add*: ‹$a \neq b$› ‹$c \neq d$›)
then show *?rhs*
by (*force simp*: ‹$a \neq b$›)
qed
qed

lemma *subset_oc_segment*:
fixes *a* :: *'a*::*euclidean_space*
shows *open_segment a b* ⊆ *closed_segment c d* ⟷
$a = b \vee a \in$ *closed_segment c d* ∧ $b \in$ *closed_segment c d*
apply (*simp add*: *subset_open_segment* [*symmetric*])
apply (*rule iffI*)
apply (*metis closure_closed_segment closure_mono closure_open_segment subset_closed_segment subset_open_segment*)
apply (*meson dual_order.trans segment_open_subset_closed*)
done

lemmas *subset_segment* = *subset_closed_segment subset_co_segment subset_oc_segment subset_open_segment*

lemma *dist_half_times2*:
fixes *a* :: *'a* :: *real_normed_vector*
shows *dist* $((1\ /\ 2) *_R (a + b))\ x * 2 =$ *dist* $(a+b)\ (2 *_R x)$
proof −
have *norm* $((1\ /\ 2) *_R (a + b) - x) * 2 =$ *norm* $(2 *_R ((1\ /\ 2) *_R (a + b) - x))$
by *simp*
also have ... = *norm* $((a + b) - 2 *_R x)$
by (*simp add*: *real_vector.scale_right_diff_distrib*)
finally show *?thesis*
by (*simp only*: *dist_norm*)
qed

lemma *closed_segment_as_ball*:
closed_segment a b = *affine hull* {*a*,*b*} ∩ *cball*(*inverse* $2 *_R (a + b)$)(*norm*$(b - a)\ /\ 2$)
proof (*cases* $b = a$)
case *True* **then show** *?thesis* **by** (*auto simp*: *hull_inc*)
next
case *False*
then have ∗: $((\exists\, u\ v.\ x = u *_R a + v *_R b \wedge u + v = 1) \wedge$
dist $((1\ /\ 2) *_R (a + b))\ x * 2 \leq$ *norm* $(b - a)) =$

```
                (∃u. x = (1 − u) *R a + u *R b ∧ 0 ≤ u ∧ u ≤ 1) for x
  proof −
    have ((∃u v. x = u *R a + v *R b ∧ u + v = 1) ∧
                dist ((1 / 2) *R (a + b)) x * 2 ≤ norm (b − a)) =
          ((∃u. x = (1 − u) *R a + u *R b) ∧
                dist ((1 / 2) *R (a + b)) x * 2 ≤ norm (b − a))
      unfolding eq_diff_eq [symmetric] by simp
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧
                      norm ((a+b) − (2 *R x)) ≤ norm (b − a))
      by (simp add: dist_half_times2) (simp add: dist_norm)
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧
              norm ((a+b) − (2 *R ((1 − u) *R a + u *R b))) ≤ norm (b − a))
      by auto
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧
                norm ((1 − u * 2) *R (b − a)) ≤ norm (b − a))
      by (simp add: algebra_simps scaleR_2)
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧
                      |1 − u * 2| * norm (b − a) ≤ norm (b − a))
      by simp
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧ |1 − u * 2| ≤ 1)
      by (simp add: mult_le_cancel_right2 False)
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧ 0 ≤ u ∧ u ≤ 1)
      by auto
    finally show ?thesis .
  qed
  show ?thesis
    by (simp add: affine_hull_2 Set.set_eq_iff closed_segment_def *)
qed

lemma open_segment_as_ball:
    open_segment a b =
     affine hull {a,b} ∩ ball(inverse 2 *R (a + b))(norm(b − a) / 2)
proof (cases b = a)
  case True then show ?thesis by (auto simp: hull_inc)
next
  case False
  then have *: ((∃u v. x = u *R a + v *R b ∧ u + v = 1) ∧
                dist ((1 / 2) *R (a + b)) x * 2 < norm (b − a)) =
                (∃u. x = (1 − u) *R a + u *R b ∧ 0 < u ∧ u < 1) for x
  proof −
    have ((∃u v. x = u *R a + v *R b ∧ u + v = 1) ∧
                dist ((1 / 2) *R (a + b)) x * 2 < norm (b − a)) =
          ((∃u. x = (1 − u) *R a + u *R b) ∧
                dist ((1 / 2) *R (a + b)) x * 2 < norm (b − a))
      unfolding eq_diff_eq [symmetric] by simp
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧
                      norm ((a+b) − (2 *R x)) < norm (b − a))
      by (simp add: dist_half_times2) (simp add: dist_norm)
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧
```

norm ((*a*+*b*) − (*2* $*_R$ ((*1* − *u*) $*_R$ *a* + *u* $*_R$ *b*))) < *norm* (*b* − *a*))
by *auto*
also have ... = (∃ *u*. *x* = (*1* − *u*) $*_R$ *a* + *u* $*_R$ *b* ∧
norm ((*1* − *u* ∗ *2*) $*_R$ (*b* − *a*)) < *norm* (*b* − *a*))
by (*simp add*: *algebra_simps scaleR_2*)
also have ... = (∃ *u*. *x* = (*1* − *u*) $*_R$ *a* + *u* $*_R$ *b* ∧
|*1* − *u* ∗ *2*| ∗ *norm* (*b* − *a*) < *norm* (*b* − *a*))
by *simp*
also have ... = (∃ *u*. *x* = (*1* − *u*) $*_R$ *a* + *u* $*_R$ *b* ∧ |*1* − *u* ∗ *2*| < *1*)
by (*simp add*: *mult_le_cancel_right2 False*)
also have ... = (∃ *u*. *x* = (*1* − *u*) $*_R$ *a* + *u* $*_R$ *b* ∧ *0* < *u* ∧ *u* < *1*)
by *auto*
finally show *?thesis* **.**
qed
show *?thesis*
using *False* **by** (*force simp*: *affine_hull_2 Set.set_eq_iff open_segment_image_interval* ∗)
qed

lemmas *segment_as_ball* = *closed_segment_as_ball open_segment_as_ball*

lemma *connected_segment* [*iff*]:
fixes *x* :: *'a* :: *real_normed_vector*
shows *connected* (*closed_segment x y*)
by (*simp add*: *convex_connected*)

lemma *is_interval_closed_segment_1*[*intro*, *simp*]: *is_interval* (*closed_segment a b*) **for** *a b*::*real*
unfolding *closed_segment_eq_real_ivl*
by (*auto simp*: *is_interval_def*)

lemma *IVT'_closed_segment_real*:
fixes *f* :: *real* ⇒ *real*
assumes *y* ∈ *closed_segment* (*f a*) (*f b*)
assumes *continuous_on* (*closed_segment a b*) *f*
shows ∃ *x* ∈ *closed_segment a b*. *f x* = *y*
using *IVT'*[*of f a y b*]
IVT'[*of* −*f a* −*y b*]
IVT'[*of f b y a*]
IVT'[*of* −*f b* −*y a*] *assms*
by (*cases a* ≤ *b*; *cases f b* ≥ *f a*) (*auto simp*: *closed_segment_eq_real_ivl continuous_on_minus*)

4.4.4 Betweenness

definition *between* = (λ(*a*,*b*) *x*. *x* ∈ *closed_segment a b*)

lemma *betweenI*:
assumes *0* ≤ *u u* ≤ *1 x* = (*1* − *u*) $*_R$ *a* + *u* $*_R$ *b*

```
  shows between (a, b) x
using assms unfolding between_def closed_segment_def by auto

lemma betweenE:
  assumes between (a, b) x
  obtains u where 0 ≤ u u ≤ 1 x = (1 − u) *R a + u *R b
using assms unfolding between_def closed_segment_def by auto

lemma between_implies_scaled_diff:
  assumes between (S, T) X between (S, T) Y S ≠ Y
  obtains c where (X − Y) = c *R (S − Y)
proof −
  from ‹between (S, T) X› obtain uX where X: X = uX *R S + (1 − uX) *R
T
    by (metis add.commute betweenE eq_diff_eq)
  from ‹between (S, T) Y› obtain uY where Y: Y = uY *R S + (1 − uY) *R
T
    by (metis add.commute betweenE eq_diff_eq)
  have X − Y = (uX − uY) *R (S − T)
  proof −
    from X Y have X − Y = uX *R S − uY *R S + ((1 − uX) *R T − (1 −
uY) *R T) by simp
      also have ... = (uX − uY) *R S − (uX − uY) *R T by (simp add:
scaleR_left.diff)
    finally show ?thesis by (simp add: real_vector.scale_right_diff_distrib)
  qed
  moreover from Y have S − Y = (1 − uY) *R (S − T)
   by (simp add: real_vector.scale_left_diff_distrib real_vector.scale_right_diff_distrib)
  moreover note ‹S ≠ Y›
  ultimately have (X − Y) = ((uX − uY) / (1 − uY)) *R (S − Y) by auto
  from this that show thesis by blast
qed

lemma between_mem_segment: between (a,b) x ⟷ x ∈ closed_segment a b
  unfolding between_def by auto

lemma between: between (a, b) (x::'a::euclidean_space) ⟷ dist a b = (dist a x)
+ (dist x b)
proof (cases a = b)
  case True
  then show ?thesis
    by (auto simp add: between_def dist_commute)
next
  case False
  then have Fal: norm (a − b) ≠ 0 and Fal2: norm (a − b) > 0
    by auto
  have *: ⋀u. a − ((1 − u) *R a + u *R b) = u *R (a − b)
    by (auto simp add: algebra_simps)
  have norm (a − x) *R (x − b) = norm (x − b) *R (a − x) if x = (1 − u) *R
```

a + *u* $*_R$ *b 0* ≤ *u u* ≤ *1* **for** *u*
  proof −
    have ∗: *a* − *x* = *u* $*_R$ (*a* − *b*) *x* − *b* = (*1* − *u*) $*_R$ (*a* − *b*)
      unfolding *that*(*1*) **by** (*auto simp add*:*algebra_simps*)
    show *norm* (*a* − *x*) $*_R$ (*x* − *b*) = *norm* (*x* − *b*) $*_R$ (*a* − *x*)
      unfolding *norm_minus_commute*[*of x a*] ∗ **using** ‹*0* ≤ *u*› ‹*u* ≤ *1*›
      by *simp*
  qed
  moreover have ∃ *u*. *x* = (*1* − *u*) $*_R$ *a* + *u* $*_R$ *b* ∧ *0* ≤ *u* ∧ *u* ≤ *1* **if** *dist a b* = *dist a x* + *dist x b*
  proof −
    let *?β* = *norm* (*a* − *x*) / *norm* (*a* − *b*)
    show ∃ *u*. *x* = (*1* − *u*) $*_R$ *a* + *u* $*_R$ *b* ∧ *0* ≤ *u* ∧ *u* ≤ *1*
    proof (*intro exI conjI*)
      show *?β* ≤ *1*
        using *Fal2* **unfolding** *that*[*unfolded dist_norm*] *norm_ge_zero* **by** *auto*
      show *x* = (*1* − *?β*) $*_R$ *a* + (*?β*) $*_R$ *b*
      proof (*subst euclidean_eq_iff*; *intro ballI*)
        fix *i* :: ′*a*
        assume *i*: *i* ∈ *Basis*
        have ((*1* − *?β*) $*_R$ *a* + (*?β*) $*_R$ *b*) · *i*
              = ((*norm* (*a* − *b*) − *norm* (*a* − *x*)) ∗ (*a* · *i*) + *norm* (*a* − *x*) ∗ (*b* · *i*)) / *norm* (*a* − *b*)
          using *Fal* **by** (*auto simp add*: *field_simps inner_simps*)
        also have . . . = *x*·*i*
          apply (*rule divide_eq_imp*[*OF Fal*])
          unfolding *that*[*unfolded dist_norm*]
          using *that*[*unfolded dist_triangle_eq*] *i*
          apply (*subst* (*asm*) *euclidean_eq_iff*)
           apply (*auto simp add*: *field_simps inner_simps*)
          done
        finally show *x* · *i* = ((*1* − *?β*) $*_R$ *a* + (*?β*) $*_R$ *b*) · *i*
          by *auto*
      qed
    qed (*use Fal2* **in** *auto*)
  qed
  ultimately show *?thesis*
    by (*force simp add*: *between_def closed_segment_def dist_triangle_eq*)
qed

lemma *between_midpoint*:
  fixes *a* :: ′*a*::*euclidean_space*
  shows *between* (*a*,*b*) (*midpoint a b*) (**is** *?t1*)
    and *between* (*b*,*a*) (*midpoint a b*) (**is** *?t2*)
proof −
  have ∗: ⋀*x y z*. *x* = (*1*/*2*::*real*) $*_R$ *z* ⟹ *y* = (*1*/*2*) $*_R$ *z* ⟹ *norm z* = *norm x* + *norm y*
    by *auto*
  show *?t1 ?t2*

unfolding *between midpoint_def dist_norm*
by (*auto simp add*: *field_simps inner_simps euclidean_eq_iff* [**where** ′*a*=′*a*] *intro*!: ∗)
qed

lemma *between_mem_convex_hull*:
between (*a*,*b*) *x* ⟷ *x* ∈ *convex hull* {*a*,*b*}
unfolding *between_mem_segment segment_convex_hull* ..

lemma *between_triv_iff* [*simp*]: *between* (*a*,*a*) *b* ⟷ *a*=*b*
by (*auto simp*: *between_def*)

lemma *between_triv1* [*simp*]: *between* (*a*,*b*) *a*
by (*auto simp*: *between_def*)

lemma *between_triv2* [*simp*]: *between* (*a*,*b*) *b*
by (*auto simp*: *between_def*)

lemma *between_commute*:
between (*a*,*b*) = *between* (*b*,*a*)
by (*auto simp*: *between_def closed_segment_commute*)

lemma *between_antisym*:
fixes *a* :: ′*a* :: *euclidean_space*
shows ⟦*between* (*b*,*c*) *a*; *between* (*a*,*c*) *b*⟧ ⟹ *a* = *b*
by (*auto simp*: *between dist_commute*)

lemma *between_trans*:
fixes *a* :: ′*a* :: *euclidean_space*
shows ⟦*between* (*b*,*c*) *a*; *between* (*a*,*c*) *d*⟧ ⟹ *between* (*b*,*c*) *d*
using *dist_triangle2* [*of b c d*] *dist_triangle3* [*of b d a*]
by (*auto simp*: *between dist_commute*)

lemma *between_norm*:
fixes *a* :: ′*a* :: *euclidean_space*
shows *between* (*a*,*b*) *x* ⟷ *norm*(*x* − *a*) $*_R$ (*b* − *x*) = *norm*(*b* − *x*) $*_R$ (*x* − *a*)
by (*auto simp*: *between dist_triangle_eq norm_minus_commute algebra_simps*)

lemma *between_swap*:
fixes *A B X Y* :: ′*a*::*euclidean_space*
assumes *between* (*A*, *B*) *X*
assumes *between* (*A*, *B*) *Y*
shows *between* (*X*, *B*) *Y* ⟷ *between* (*A*, *Y*) *X*
using *assms* **by** (*auto simp add*: *between*)

lemma *between_translation* [*simp*]: *between* (*a* + *y*,*a* + *z*) (*a* + *x*) ⟷ *between* (*y*,*z*) *x*
by (*auto simp*: *between_def*)

lemma *between_trans_2*:
fixes *a* :: *'a* :: *euclidean_space*
shows ⟦*between* (*b*,*c*) *a*; *between* (*a*,*b*) *d*⟧ ⟹ *between* (*c*,*d*) *a*
by (*metis between_commute between_swap between_trans*)

lemma *between_scaleR_lift* [*simp*]:
fixes *v* :: *'a*::*euclidean_space*
shows *between* ($a *_R v$, $b *_R v$) ($c *_R v$) ⟷ *v* = *0* ∨ *between* (*a*, *b*) *c*
by (*simp add*: *between dist_norm scaleR_left_diff_distrib* [*symmetric*] *distrib_right* [*symmetric*])

lemma *between_1*:
fixes *x*::*real*
shows *between* (*a*,*b*) *x* ⟷ (*a* ≤ *x* ∧ *x* ≤ *b*) ∨ (*b* ≤ *x* ∧ *x* ≤ *a*)
by (*auto simp*: *between_mem_segment closed_segment_eq_real_ivl*)

end

4.5 Limits on the Extended Real Number Line

theory *Extended_Real_Limits*
imports
Topology_Euclidean_Space
HOL−Library.Extended_Real
HOL−Library.Extended_Nonnegative_Real
HOL−Library.Indicator_Function
begin

lemma *compact_UNIV*:
compact (*UNIV* :: *'a*::{*complete_linorder*,*linorder_topology*,*second_countable_topology*} *set*)
using *compact_complete_linorder*
by (*auto simp*: *seq_compact_eq_compact*[*symmetric*] *seq_compact_def*)

lemma *compact_eq_closed*:
fixes *S* :: *'a*::{*complete_linorder*,*linorder_topology*,*second_countable_topology*} *set*
shows *compact S* ⟷ *closed S*
using *closed_Int_compact*[*of S*, *OF _ compact_UNIV*] *compact_imp_closed*
by *auto*

lemma *closed_contains_Sup_cl*:
fixes *S* :: *'a*::{*complete_linorder*,*linorder_topology*,*second_countable_topology*} *set*
assumes *closed S*
and *S* ≠ {}
shows *Sup S* ∈ *S*
proof −
from *compact_eq_closed*[*of S*] *compact_attains_sup*[*of S*] *assms*
obtain *s* **where** *S*: *s* ∈ *S* ∀ *t*∈*S*. *t* ≤ *s*

```
    by auto
  then have Sup S = s
    by (auto intro!: Sup_eqI)
  with S show ?thesis
    by simp
qed

lemma closed_contains_Inf_cl:
  fixes S :: 'a::{complete_linorder,linorder_topology,second_countable_topology} set
  assumes closed S
    and S ≠ {}
  shows Inf S ∈ S
proof −
  from compact_eq_closed[of S] compact_attains_inf[of S] assms
  obtain s where S: s ∈ S ∀ t∈S. s ≤ t
    by auto
  then have Inf S = s
    by (auto intro!: Inf_eqI)
  with S show ?thesis
    by simp
qed

instance enat :: second_countable_topology
proof
  show ∃ B::enat set set. countable B ∧ open = generate_topology B
  proof (intro exI conjI)
    show countable (range lessThan ∪ range greaterThan::enat set set)
      by auto
  qed (simp add: open_enat_def)
qed

instance ereal :: second_countable_topology
proof (standard, intro exI conjI)
  let ?B = (⋃r∈ℚ. {{..< r}, {r <..}} :: ereal set set)
  show countable ?B
    by (auto intro: countable_rat)
  show open = generate_topology ?B
  proof (intro ext iffI)
    fix S :: ereal set
    assume open S
    then show generate_topology ?B S
      unfolding open_generated_order
    proof induct
      case (Basis b)
      then obtain e where b = {..<e} ∨ b = {e<..}
        by auto
      moreover have {..<e} = ⋃{{..<x}|x. x ∈ ℚ ∧ x < e} {e<..} = ⋃{{x<..}|x.
x ∈ ℚ ∧ e < x}
        by (auto dest: ereal_dense3
```

```
          simp del: ex_simps
          simp add: ex_simps[symmetric] conj_commute Rats_def image_iff)
    ultimately show ?case
      by (auto intro: generate_topology.intros)
  qed (auto intro: generate_topology.intros)
next
  fix S
  assume generate_topology ?B S
  then show open S
    by induct auto
qed
qed
```

This is a copy from *ereal* :: *second_countable_topology*. Maybe find a common super class of topological spaces where the rational numbers are densely embedded ?

```
instance ennreal :: second_countable_topology
proof (standard, intro exI conjI)
  let ?B = (⋃r∈ℚ. {{..< r}, {r <..}} :: ennreal set set)
  show countable ?B
    by (auto intro: countable_rat)
  show open = generate_topology ?B
  proof (intro ext iffI)
    fix S :: ennreal set
    assume open S
    then show generate_topology ?B S
      unfolding open_generated_order
    proof induct
      case (Basis b)
      then obtain e where b = {..<e} ∨ b = {e<..}
        by auto
     moreover have {..<e} = ⋃{{..<x}|x. x ∈ ℚ ∧ x < e} {e<..} = ⋃{{x<..}|x.
x ∈ ℚ ∧ e < x}
        by (auto dest: ennreal_rat_dense
          simp del: ex_simps
          simp add: ex_simps[symmetric] conj_commute Rats_def image_iff)
      ultimately show ?case
        by (auto intro: generate_topology.intros)
    qed (auto intro: generate_topology.intros)
  next
    fix S
    assume generate_topology ?B S
    then show open S
      by induct auto
  qed
qed

lemma ereal_open_closed_aux:
  fixes S :: ereal set
```

assumes *open S*
and *closed S*
and *S*: $(-\infty) \notin S$
shows $S = \{\}$
proof (*rule ccontr*)
assume ¬ *?thesis*
then have ∗: *Inf S* ∈ *S*

by (*metis assms*(*2*) *closed_contains_Inf_cl*)
{
assume *Inf S* = −∞
then have *False*
using ∗ *assms*(*3*) **by** *auto*
}
moreover
{
assume *Inf S* = ∞
then have *S* = {∞}
by (*metis Inf_eq_PInfty* ‹*S* ≠ {}›)
then have *False*
by (*metis assms*(*1*) *not_open_singleton*)
}
moreover
{
assume *fin*: |*Inf S*| ≠ ∞
from *ereal_open_cont_interval*[*OF assms*(*1*) ∗ *fin*]
obtain *e* **where** *e*: *e* > *0* {*Inf S* − *e*<..<*Inf S* + *e*} ⊆ *S* **.**
then obtain *b* **where** *b*: *Inf S* − *e* < *b b* < *Inf S*
using *fin ereal_between*[*of Inf S e*] *dense*[*of Inf S* − *e*]
by *auto*
then have *b* ∈ {*Inf S* − *e* <..< *Inf S* + *e*}
using *e fin ereal_between*[*of Inf S e*]
by *auto*
then have *b* ∈ *S*
using *e* **by** *auto*
then have *False*
using *b* **by** (*metis complete_lattice_class.Inf_lower leD*)
}
ultimately show *False*
by *auto*
qed

lemma *ereal_open_closed*:
fixes *S* :: *ereal set*
shows *open S* ∧ *closed S* ⟷ *S* = {} ∨ *S* = *UNIV*
proof −
{
assume *lhs*: *open S* ∧ *closed S*
{

```
      assume −∞ ∉ S
      then have S = {}
        using lhs ereal_open_closed_aux by auto
    }
    moreover
    {
      assume −∞ ∈ S
      then have − S = {}
        using lhs ereal_open_closed_aux[of −S] by auto
    }
    ultimately have S = {} ∨ S = UNIV
      by auto
  }
  then show ?thesis
    by auto
qed

lemma ereal_open_atLeast:
  fixes x :: ereal
  shows open {x..} ⟷ x = −∞
proof
  assume x = −∞
  then have {x..} = UNIV
    by auto
  then show open {x..}
    by auto
next
  assume open {x..}
  then have open {x..} ∧ closed {x..}
    by auto
  then have {x..} = UNIV
    unfolding ereal_open_closed by auto
  then show x = −∞
    by (simp add: bot_ereal_def atLeast_eq_UNIV_iff)
qed

lemma mono_closed_real:
  fixes S :: real set
  assumes mono: ∀ y z. y ∈ S ∧ y ≤ z ⟶ z ∈ S
    and closed S
  shows S = {} ∨ S = UNIV ∨ (∃ a. S = {a..})
proof −
  {
    assume S ≠ {}
    { assume ex: ∃ B. ∀ x∈S. B ≤ x
      then have ∗: ∀ x∈S. Inf S ≤ x
        using cInf_lower[of _ S] ex by (metis bdd_below_def)
      then have Inf S ∈ S
        apply (subst closed_contains_Inf)
```

```
        using ex ‹S ≠ {}› ‹closed S›
        apply auto
        done
      then have ∀x. Inf S ≤ x ⟷ x ∈ S
        using mono[rule_format, of Inf S] *
        by auto
      then have S = {Inf S ..}
        by auto
      then have ∃a. S = {a ..}
        by auto
    }
    moreover
    {
      assume ¬ (∃B. ∀x∈S. B ≤ x)
      then have nex: ∀B. ∃x∈S. x < B
        by (simp add: not_le)
      {
        fix y
        obtain x where x∈S and x < y
          using nex by auto
        then have y ∈ S
          using mono[rule_format, of x y] by auto
      }
      then have S = UNIV
        by auto
    }
    ultimately have S = UNIV ∨ (∃a. S = {a ..})
      by blast
  }
  then show ?thesis
    by blast
qed

lemma mono_closed_ereal:
  fixes S :: real set
  assumes mono: ∀y z. y ∈ S ∧ y ≤ z ⟶ z ∈ S
    and closed S
  shows ∃a. S = {x. a ≤ ereal x}
proof –
  {
    assume S = {}
    then have ?thesis
      apply (rule_tac x=PInfty in exI)
      apply auto
      done
  }
  moreover
  {
    assume S = UNIV
```

then have *?thesis*
apply (*rule_tac x*=−∞ **in** *exI*)
apply *auto*
done
}
moreover
{
assume ∃ *a*. *S* = {*a* ..}
then obtain *a* **where** *S* = {*a* ..}
by *auto*
then have *?thesis*
apply (*rule_tac x*=*ereal a* **in** *exI*)
apply *auto*
done
}
ultimately show *?thesis*
using *mono_closed_real*[*of S*] *assms* **by** *auto*
qed

lemma *Liminf_within*:
fixes *f* :: ′*a*::*metric_space* ⇒ ′*b*::*complete_lattice*
shows *Liminf* (*at x within S*) *f* = (*SUP e*∈{*0*<..}. *INF y*∈(*S* ∩ *ball x e* − {*x*}). *f y*)
unfolding *Liminf_def eventually_at*
proof (*rule SUP_eq*, *simp_all add*: *Ball_def Bex_def*, *safe*)
fix *P d*
assume *0* < *d* **and** ∀ *y*. *y* ∈ *S* ⟶ *y* ≠ *x* ∧ *dist y x* < *d* ⟶ *P y*
then have *S* ∩ *ball x d* − {*x*} ⊆ {*x*. *P x*}
by (*auto simp*: *dist_commute*)
then show ∃ *r*>*0*. *Inf* (*f* ‘ (*Collect P*)) ≤ *Inf* (*f* ‘ (*S* ∩ *ball x r* − {*x*}))
by (*intro exI*[*of* _ *d*] *INF_mono conjI* ‹*0* < *d*›) *auto*
next
fix *d* :: *real*
assume *0* < *d*
then show ∃ *P*. (∃ *d*>*0*. ∀ *xa*. *xa* ∈ *S* ⟶ *xa* ≠ *x* ∧ *dist xa x* < *d* ⟶ *P xa*) ∧
Inf (*f* ‘ (*S* ∩ *ball x d* − {*x*})) ≤ *Inf* (*f* ‘ (*Collect P*))
by (*intro exI*[*of* _ λ*y*. *y* ∈ *S* ∩ *ball x d* − {*x*}])
(*auto intro*!: *INF_mono exI*[*of* _ *d*] *simp*: *dist_commute*)
qed

lemma *Limsup_within*:
fixes *f* :: ′*a*::*metric_space* ⇒ ′*b*::*complete_lattice*
shows *Limsup* (*at x within S*) *f* = (*INF e*∈{*0*<..}. *SUP y*∈(*S* ∩ *ball x e* − {*x*}). *f y*)
unfolding *Limsup_def eventually_at*
proof (*rule INF_eq*, *simp_all add*: *Ball_def Bex_def*, *safe*)
fix *P d*
assume *0* < *d* **and** ∀ *y*. *y* ∈ *S* ⟶ *y* ≠ *x* ∧ *dist y x* < *d* ⟶ *P y*
then have *S* ∩ *ball x d* − {*x*} ⊆ {*x*. *P x*}

by (*auto simp*: *dist_commute*)
then show $\exists r>0.$ *Sup* (*f* ‘ (*S* ∩ *ball x r* − {*x*})) ≤ *Sup* (*f* ‘ (*Collect P*))
by (*intro exI*[*of* _ *d*] *SUP_mono conjI* ‹0 < *d*›) *auto*
next
fix *d* :: *real*
assume 0 < *d*
then show ∃ *P*. (∃ *d*>0. ∀ *xa*. *xa* ∈ *S* ⟶ *xa* ≠ *x* ∧ *dist xa x* < *d* ⟶ *P xa*) ∧
Sup (*f* ‘ (*Collect P*)) ≤ *Sup* (*f* ‘ (*S* ∩ *ball x d* − {*x*}))
by (*intro exI*[*of* _ λ*y*. *y* ∈ *S* ∩ *ball x d* − {*x*}])
(*auto intro*!: *SUP_mono exI*[*of* _ *d*] *simp*: *dist_commute*)
qed

lemma *Liminf_at*:
fixes *f* :: ′*a*::*metric_space* ⇒ ′*b*::*complete_lattice*
shows *Liminf* (*at x*) *f* = (*SUP e*∈{0<..}. *INF y*∈(*ball x e* − {*x*}). *f y*)
using *Liminf_within*[*of x UNIV f*] **by** *simp*

lemma *Limsup_at*:
fixes *f* :: ′*a*::*metric_space* ⇒ ′*b*::*complete_lattice*
shows *Limsup* (*at x*) *f* = (*INF e*∈{0<..}. *SUP y*∈(*ball x e* − {*x*}). *f y*)
using *Limsup_within*[*of x UNIV f*] **by** *simp*

lemma *min_Liminf_at*:
fixes *f* :: ′*a*::*metric_space* ⇒ ′*b*::*complete_linorder*
shows *min* (*f x*) (*Liminf* (*at x*) *f*) = (*SUP e*∈{0<..}. *INF y*∈*ball x e*. *f y*)
apply (*simp add*: *inf_min* [*symmetric*] *Liminf_at*)
apply (*subst inf_commute*)
apply (*subst SUP_inf*)
apply *auto*
apply (*metis* (*no_types*, *lifting*) *INF_insert centre_in_ball greaterThan_iff image_cong inf_commute insert_Diff*)
done

4.5.1 Extended-Real.thy

lemma *sum_constant_ereal*:
fixes *a*::*ereal*
shows (∑ *i*∈*I*. *a*) = *a* ∗ *card I*
apply (*cases finite I*, *induct set*: *finite*, *simp_all*)
apply (*cases a*, *auto*, *metis* (*no_types*, *hide_lams*) *add.commute mult.commute semiring_normalization_rules*(3))
done

lemma *real_lim_then_eventually_real*:
assumes (*u* ⟶ *ereal l*) *F*
shows *eventually* (λ*n*. *u n* = *ereal*(*real_of_ereal*(*u n*))) *F*
proof −
have *ereal l* ∈ {−∞<..<(∞::*ereal*)} **by** *simp*
moreover have *open* {−∞<..<(∞::*ereal*)} **by** *simp*

ultimately have *eventually* ($\lambda n.\ u\ n \in \{-\infty<..<(\infty::ereal)\}$) *F* **using** *assms tendsto_def* **by** *blast*
moreover have $\bigwedge x.\ x \in \{-\infty<..<(\infty::ereal)\} \Longrightarrow x = ereal(real\_of\_ereal\ x)$ **using** *ereal_real* **by** *auto*
ultimately show *?thesis* **by** (*metis* (*mono_tags*, *lifting*) *eventually_mono*)
qed

lemma *ereal_Inf_cmult*:
assumes $c>(0::real)$
shows *Inf* $\{ereal\ c * x\ |x.\ P\ x\} = ereal\ c * Inf\ \{x.\ P\ x\}$
proof −
have $(\lambda x::ereal.\ c * x)\ (Inf\ \{x::ereal.\ P\ x\}) = Inf\ ((\lambda x::ereal.\ c * x)`\{x::ereal.\ P\ x\})$
apply (*rule mono_bij_Inf*)
apply (*simp add*: *assms ereal_mult_left_mono less_imp_le mono_def*)
apply (*rule bij_betw_byWitness*[*of* _ $\lambda x.\ (x::ereal) / c$], *auto simp add*: *assms ereal_mult_divide*)
using *assms ereal_divide_eq* **apply** *auto*
done
then show *?thesis* **by** (*simp only*: *setcompr_eq_image*[*symmetric*])
qed

Continuity of addition

The next few lemmas remove an unnecessary assumption in *tendsto_add_ereal*, culminating in *tendsto_add_ereal_general* which essentially says that the addition is continuous on ereal times ereal, except at $(-\infty, \infty)$ and $(\infty, -\infty)$. It is much more convenient in many situations, see for instance the proof of *tendsto_sum_ereal* below.

lemma *tendsto_add_ereal_PInf*:
fixes *y* :: *ereal*
assumes *y*: $y \neq -\infty$
assumes *f*: $(f \longrightarrow \infty)$ *F* **and** *g*: $(g \longrightarrow y)$ *F*
shows $((\lambda x.\ f\ x + g\ x) \longrightarrow \infty)$ *F*
proof −
have $\exists C.$ *eventually* ($\lambda x.\ g\ x > ereal\ C$) *F*
proof (*cases y*)
case (*real r*)
have $y > y-1$ **using** *y real* **by** (*simp add*: *ereal_between*(*1*))
then have *eventually* ($\lambda x.\ g\ x > y - 1$) *F* **using** *g y order_tendsto_iff* **by** *auto*
moreover have $y-1 = ereal(real\_of\_ereal(y-1))$
by (*metis real ereal_eq_1*(*1*) *ereal_minus*(*1*) *real_of_ereal.simps*(*1*))
ultimately have *eventually* ($\lambda x.\ g\ x > ereal(real\_of\_ereal(y - 1))$) *F* **by** *simp*
then show *?thesis* **by** *auto*
next
case (*PInf*)
have *eventually* ($\lambda x.\ g\ x > ereal\ 0$) *F* **using** *g PInf* **by** (*simp add*: *tendsto_PInfty*)

then show *?thesis* **by** *auto*
qed (*simp add*: *y*)
then obtain *C*::*real* **where** *ge*: *eventually* ($\lambda x.$ *g x* > *ereal C*) *F* **by** *auto*

{
fix *M*::*real*
have *eventually* ($\lambda x.$ *f x* > *ereal*(*M* − *C*)) *F* **using** *f* **by** (*simp add*: *tendsto_PInfty*)
then have *eventually* ($\lambda x.$ (*f x* > *ereal* (*M*−*C*)) ∧ (*g x* > *ereal C*)) *F*
by (*auto simp add*: *ge eventually_conj_iff*)
moreover have $\bigwedge x.$ ((*f x* > *ereal* (*M*−*C*)) ∧ (*g x* > *ereal C*)) ⟹ (*f x* + *g x* > *ereal M*)
using *ereal_add_strict_mono2* **by** *fastforce*
ultimately have *eventually* ($\lambda x.$ *f x* + *g x* > *ereal M*) *F* **using** *eventually_mono* **by** *force*
}
then show *?thesis* **by** (*simp add*: *tendsto_PInfty*)
qed

One would like to deduce the next lemma from the previous one, but the fact that − (*x* + *y*) is in general different from (− *x*) + (− *y*) in ereal creates difficulties, so it is more efficient to copy the previous proof.

lemma *tendsto_add_ereal_MInf*:
fixes *y* :: *ereal*
assumes *y*: *y* ≠ ∞
assumes *f*: (*f* ⟶ −∞) *F* **and** *g*: (*g* ⟶ *y*) *F*
shows (($\lambda x.$ *f x* + *g x*) ⟶ −∞) *F*
proof −
have ∃ *C*. *eventually* ($\lambda x.$ *g x* < *ereal C*) *F*
proof (*cases y*)
case (*real r*)
have *y* < *y*+*1* **using** *y real* **by** (*simp add*: *ereal_between*(*1*))
then have *eventually* ($\lambda x.$ *g x* < *y* + *1*) *F* **using** *g y order_tendsto_iff* **by** *force*
moreover have *y*+*1* = *ereal*(*real_of_ereal* (*y*+*1*)) **by** (*simp add*: *real*)
ultimately have *eventually* ($\lambda x.$ *g x* < *ereal*(*real_of_ereal*(*y* + *1*))) *F* **by** *simp*
then show *?thesis* **by** *auto*
next
case (*MInf*)
have *eventually* ($\lambda x.$ *g x* < *ereal 0*) *F* **using** *g MInf* **by** (*simp add*: *tendsto_MInfty*)
then show *?thesis* **by** *auto*
qed (*simp add*: *y*)
then obtain *C*::*real* **where** *ge*: *eventually* ($\lambda x.$ *g x* < *ereal C*) *F* **by** *auto*

{
fix *M*::*real*
have *eventually* ($\lambda x.$ *f x* < *ereal*(*M* − *C*)) *F* **using** *f* **by** (*simp add*: *tendsto_MInfty*)

then have *eventually* (λ*x*. (*f x* < *ereal* (*M*− *C*)) ∧ (*g x* < *ereal C*)) *F*
by (*auto simp add*: *ge eventually_conj_iff*)
moreover have ⋀*x*. ((*f x* < *ereal* (*M*−*C*)) ∧ (*g x* < *ereal C*)) ⟹ (*f x* + *g x* < *ereal M*)
using *ereal_add_strict_mono2* **by** *fastforce*
ultimately have *eventually* (λ*x*. *f x* + *g x* < *ereal M*) *F* **using** *eventually_mono* **by** *force*
}
then show *?thesis* **by** (*simp add*: *tendsto_MInfty*)
qed

lemma *tendsto_add_ereal_general1*:
fixes *x y* :: *ereal*
assumes *y*: |*y*| ≠ ∞
assumes *f*: (*f* ⟶ *x*) *F* **and** *g*: (*g* ⟶ *y*) *F*
shows ((λ*x*. *f x* + *g x*) ⟶ *x* + *y*) *F*
proof (*cases x*)
case (*real r*)
have *a*: |*x*| ≠ ∞ **by** (*simp add*: *real*)
show *?thesis* **by** (*rule tendsto_add_ereal*[*OF a*, *OF y*, *OF f*, *OF g*])
next
case *PInf*
then show *?thesis* **using** *tendsto_add_ereal_PInf assms* **by** *force*
next
case *MInf*
then show *?thesis* **using** *tendsto_add_ereal_MInf assms*
by (*metis abs_ereal.simps*(*3*) *ereal_MInfty_eq_plus*)
qed

lemma *tendsto_add_ereal_general2*:
fixes *x y* :: *ereal*
assumes *x*: |*x*| ≠ ∞
and *f*: (*f* ⟶ *x*) *F* **and** *g*: (*g* ⟶ *y*) *F*
shows ((λ*x*. *f x* + *g x*) ⟶ *x* + *y*) *F*
proof −
have ((λ*x*. *g x* + *f x*) ⟶ *x* + *y*) *F*
using *tendsto_add_ereal_general1*[*OF x*, *OF g*, *OF f*] *add.commute*[*of y*, *of x*] **by** *simp*
moreover have ⋀*x*. *g x* + *f x* = *f x* + *g x* **using** *add.commute* **by** *auto*
ultimately show *?thesis* **by** *simp*
qed

The next lemma says that the addition is continuous on *ereal*, except at the pairs (−∞, ∞) and (∞, −∞).

lemma *tendsto_add_ereal_general* [*tendsto_intros*]:
fixes *x y* :: *ereal*
assumes ¬((*x*=∞ ∧ *y*=−∞) ∨ (*x*=−∞ ∧ *y*=∞))
and *f*: (*f* ⟶ *x*) *F* **and** *g*: (*g* ⟶ *y*) *F*
shows ((λ*x*. *f x* + *g x*) ⟶ *x* + *y*) *F*

proof (*cases x*)
case (*real r*)
show *?thesis*
apply (*rule tendsto_add_ereal_general2*) **using** *real assms* **by** *auto*
next
case (*PInf*)
then have $y \neq -\infty$ **using** *assms* **by** *simp*
then show *?thesis* **using** *tendsto_add_ereal_PInf PInf assms* **by** *auto*
next
case (*MInf*)
then have $y \neq \infty$ **using** *assms* **by** *simp*
then show *?thesis* **using** *tendsto_add_ereal_MInf MInf f g* **by** (*metis ereal_MInfty_eq_plus*)
qed

Continuity of multiplication

In the same way as for addition, we prove that the multiplication is continuous on ereal times ereal, except at $(\infty,\ 0)$ and $(-\infty,\ 0)$ and $(0,\ \infty)$ and $(0,\ -\infty)$, starting with specific situations.

lemma *tendsto_mult_real_ereal*:
assumes $(u \longrightarrow ereal\ l)\ F$ $(v \longrightarrow ereal\ m)\ F$
shows $((\lambda n.\ u\ n * v\ n) \longrightarrow ereal\ l * ereal\ m)\ F$
proof −
have *ureal*: *eventually* $(\lambda n.\ u\ n = ereal(real\_of\_ereal(u\ n)))\ F$ **by** (*rule real_lim_then_eventually_real*[*OF assms*(*1*)])
then have $((\lambda n.\ ereal(real\_of\_ereal(u\ n))) \longrightarrow ereal\ l)\ F$ **using** *assms* **by** *auto*
then have *limu*: $((\lambda n.\ real\_of\_ereal(u\ n)) \longrightarrow l)\ F$ **by** *auto*
have *vreal*: *eventually* $(\lambda n.\ v\ n = ereal(real\_of\_ereal(v\ n)))\ F$ **by** (*rule real_lim_then_eventually_real*[*OF assms*(*2*)])
then have $((\lambda n.\ ereal(real\_of\_ereal(v\ n))) \longrightarrow ereal\ m)\ F$ **using** *assms* **by** *auto*
then have *limv*: $((\lambda n.\ real\_of\_ereal(v\ n)) \longrightarrow m)\ F$ **by** *auto*

{
fix n **assume** $u\ n = ereal(real\_of\_ereal(u\ n))$ $v\ n = ereal(real\_of\_ereal(v\ n))$
then have $ereal(real\_of\_ereal(u\ n) * real\_of\_ereal(v\ n)) = u\ n * v\ n$ **by** (*metis times_ereal.simps*(*1*))
}
then have $*$: *eventually* $(\lambda n.\ ereal(real\_of\_ereal(u\ n) * real\_of\_ereal(v\ n)) = u\ n * v\ n)\ F$
using *eventually_elim2*[*OF ureal vreal*] **by** *auto*

have $((\lambda n.\ real\_of\_ereal(u\ n) * real\_of\_ereal(v\ n)) \longrightarrow l * m)\ F$ **using** *tendsto_mult*[*OF limu limv*] **by** *auto*
then have $((\lambda n.\ ereal(real\_of\_ereal(u\ n)) * real\_of\_ereal(v\ n)) \longrightarrow ereal(l * m))\ F$ **by** *auto*
then show *?thesis* **using** $*$ *filterlim_cong* **by** *fastforce*
qed

lemma *tendsto_mult_ereal_PInf*:
fixes *f g*::_ ⇒ *ereal*
assumes $(f \longrightarrow l)$ *F* $l>0$ $(g \longrightarrow \infty)$ *F*
shows $((\lambda x.\ f\ x * g\ x) \longrightarrow \infty)$ *F*
proof −
obtain *a*::*real* **where** $0 <$ *ereal a a* $< l$ **using** *assms*(2) **using** *ereal_dense2* **by** *blast*
have ∗: *eventually* $(\lambda x.\ f\ x > a)$ *F* **using** ‹$a < l$› *assms*(1) **by** (*simp add*: *order_tendsto_iff*)
{
fix *K*::*real*
define *M* **where** $M = max\ K\ 1$
then have $M > 0$ **by** *simp*
then have *ereal*$(M/a) > 0$ **using** ‹*ereal a* > 0› **by** *simp*
then have $\bigwedge x.\ ((f\ x > a) \wedge (g\ x > M/a)) \Longrightarrow (f\ x * g\ x >$ *ereal a* ∗ *ereal*$(M/a))$
using *ereal_mult_mono_strict'*[**where** $?c = M/a$, *OF* ‹$0 <$ *ereal a*›] **by** *auto*
moreover have *ereal a* ∗ *ereal*$(M/a) = M$ **using** ‹*ereal a* > 0› **by** *simp*
ultimately have $\bigwedge x.\ ((f\ x > a) \wedge (g\ x > M/a)) \Longrightarrow (f\ x * g\ x > M)$ **by** *simp*
moreover have $M \geq K$ **unfolding** *M_def* **by** *simp*
ultimately have *Imp*: $\bigwedge x.\ ((f\ x > a) \wedge (g\ x > M/a)) \Longrightarrow (f\ x * g\ x > K)$
using *ereal_less_eq*(3) *le_less_trans* **by** *blast*

have *eventually* $(\lambda x.\ g\ x > M/a)$ *F* **using** *assms*(3) **by** (*simp add*: *tendsto_PInfty*)
then have *eventually* $(\lambda x.\ (f\ x > a) \wedge (g\ x > M/a))$ *F*
using ∗ **by** (*auto simp add*: *eventually_conj_iff*)
then have *eventually* $(\lambda x.\ f\ x * g\ x > K)$ *F* **using** *eventually_mono Imp* **by** *force*
}
then show *?thesis* **by** (*auto simp add*: *tendsto_PInfty*)
qed

lemma *tendsto_mult_ereal_pos*:
fixes *f g*::_ ⇒ *ereal*
assumes $(f \longrightarrow l)$ *F* $(g \longrightarrow m)$ *F* $l>0$ $m>0$
shows $((\lambda x.\ f\ x * g\ x) \longrightarrow l * m)$ *F*
proof (*cases*)
assume ∗: $l = \infty \vee m = \infty$
then show *?thesis*
proof (*cases*)
assume $m = \infty$
then show *?thesis* **using** *tendsto_mult_ereal_PInf assms* **by** *auto*
next
assume $\neg(m = \infty)$
then have $l = \infty$ **using** ∗ **by** *simp*
then have $((\lambda x.\ g\ x * f\ x) \longrightarrow l * m)$ *F* **using** *tendsto_mult_ereal_PInf assms* **by** *auto*

```
    moreover have ⋀x. g x * f x = f x * g x using mult.commute by auto
    ultimately show ?thesis by simp
  qed
next
  assume ¬(l = ∞ ∨ m = ∞)
  then have l < ∞ m < ∞ by auto
  then obtain lr mr where l = ereal lr m = ereal mr
    using ‹l>0› ‹m>0› by (metis ereal_cases ereal_less(6) not_less_iff_gr_or_eq)
  then show ?thesis using tendsto_mult_real_ereal assms by auto
qed
```

We reduce the general situation to the positive case by multiplying by suitable signs. Unfortunately, as ereal is not a ring, all the neat sign lemmas are not available there. We give the bare minimum we need.

```
lemma ereal_sgn_abs:
  fixes l::ereal
  shows sgn(l) * l = abs(l)
apply (cases l) by (auto simp add: sgn_if ereal_less_uminus_reorder)

lemma sgn_squared_ereal:
  assumes l ≠ (0::ereal)
  shows sgn(l) * sgn(l) = 1
apply (cases l) using assms by (auto simp add: one_ereal_def sgn_if)

lemma tendsto_mult_ereal [tendsto_intros]:
  fixes f g::_ ⇒ ereal
  assumes (f ⟶ l) F (g ⟶ m) F ¬((l=0 ∧ abs(m) = ∞) ∨ (m=0 ∧ abs(l)
= ∞))
  shows ((λx. f x * g x) ⟶ l * m) F
proof (cases)
  assume l=0 ∨ m=0
  then have abs(l) ≠ ∞ abs(m) ≠ ∞ using assms(3) by auto
  then obtain lr mr where l = ereal lr m = ereal mr by auto
  then show ?thesis using tendsto_mult_real_ereal assms by auto
next
  have sgn_finite: ⋀a::ereal. abs(sgn a) ≠ ∞
   by (metis MInfty_neq_ereal(2) PInfty_neq_ereal(2) abs_eq_infinity_cases ereal_times(1)
ereal_times(3) ereal_uminus_eq_reorder sgn_ereal.elims)
  then have sgn_finite2: ⋀a b::ereal. abs(sgn a * sgn b) ≠ ∞
   by (metis abs_eq_infinity_cases abs_ereal.simps(2) abs_ereal.simps(3) ereal_mult_eq_MInfty
ereal_mult_eq_PInfty)
  assume ¬(l=0 ∨ m=0)
  then have l ≠ 0 m ≠ 0 by auto
  then have abs(l) > 0 abs(m) > 0
   by (metis abs_ereal_ge0 abs_ereal_less0 abs_ereal_pos ereal_uminus_uminus ereal_uminus_zero
less_le not_less)+
  then have sgn(l) * l > 0 sgn(m) * m > 0 using ereal_sgn_abs by auto
  moreover have ((λx. sgn(l) * f x) ⟶ (sgn(l) * l)) F
    by (rule tendsto_cmult_ereal, auto simp add: sgn_finite assms(1))
```

moreover have $((\lambda x.\ sgn(m) * g\ x) \longrightarrow (sgn(m) * m))\ F$
by (*rule tendsto_cmult_ereal*, *auto simp add*: *sgn_finite assms*(*2*))
ultimately have ∗: $((\lambda x.\ (sgn(l) * f\ x) * (sgn(m) * g\ x)) \longrightarrow (sgn(l) * l) * (sgn(m) * m))\ F$
using *tendsto_mult_ereal_pos* **by** *force*
have $((\lambda x.\ (sgn(l) * sgn(m)) * ((sgn(l) * f\ x) * (sgn(m) * g\ x))) \longrightarrow (sgn(l) * sgn(m)) * ((sgn(l) * l) * (sgn(m) * m)))\ F$
by (*rule tendsto_cmult_ereal*, *auto simp add*: *sgn_finite2* ∗)
moreover have $\bigwedge x.\ (sgn(l) * sgn(m)) * ((sgn(l) * f\ x) * (sgn(m) * g\ x)) = f\ x * g\ x$
by (*metis mult.left_neutral sgn_squared_ereal*[*OF* ‹$l \neq 0$›] *sgn_squared_ereal*[*OF* ‹$m \neq 0$›] *mult.assoc mult.commute*)
moreover have $(sgn(l) * sgn(m)) * ((sgn(l) * l) * (sgn(m) * m)) = l * m$
by (*metis mult.left_neutral sgn_squared_ereal*[*OF* ‹$l \neq 0$›] *sgn_squared_ereal*[*OF* ‹$m \neq 0$›] *mult.assoc mult.commute*)
ultimately show *?thesis* **by** *auto*
qed

lemma *tendsto_cmult_ereal_general* [*tendsto_intros*]:
fixes *f*::_ ⇒ *ereal* **and** *c*::*ereal*
assumes $(f \longrightarrow l)\ F\ \neg\ (l{=}0 \wedge abs(c) = \infty)$
shows $((\lambda x.\ c * f\ x) \longrightarrow c * l)\ F$
by (*cases* $c = 0$, *auto simp add*: *assms tendsto_mult_ereal*)

Continuity of division

lemma *tendsto_inverse_ereal_PInf*:
fixes *u*::_ ⇒ *ereal*
assumes $(u \longrightarrow \infty)\ F$
shows $((\lambda x.\ 1/\ u\ x) \longrightarrow 0)\ F$
proof −
{
fix *e*::*real* **assume** $e{>}0$
have $1/e < \infty$ **by** *auto*
then have *eventually* $(\lambda n.\ u\ n > 1/e)\ F$ **using** *assms*(*1*) **by** (*simp add*: *tendsto_PInfty*)
moreover
{
fix *z*::*ereal* **assume** $z{>}1/e$
then have $z{>}0$ **using** ‹$e{>}0$› **using** *less_le_trans not_le* **by** *fastforce*
then have $1/z \geq 0$ **by** *auto*
moreover have $1/z < e$ **using** ‹$e{>}0$› ‹$z{>}1/e$›
apply (*cases z*) **apply** *auto*
by (*metis* (*mono_tags*, *hide_lams*) *less_ereal.simps*(*2*) *less_ereal.simps*(*4*) *divide_less_eq ereal_divide_less_pos ereal_less*(*4*)
ereal_less_eq(*4*) *less_le_trans mult_eq_0_iff not_le not_one_less_zero times_ereal.simps*(*1*))
ultimately have $1/z \geq 0\ 1/z < e$ **by** *auto*
}
ultimately have *eventually* $(\lambda n.\ 1/u\ n{<}e)\ F$ *eventually* $(\lambda n.\ 1/u\ n{\geq}0)\ F$ **by**

(*auto simp add*: *eventually_mono*)
} **note** ∗ = *this*
show *?thesis*
proof (*subst order_tendsto_iff*, *auto*)
fix *a*::*ereal* **assume** *a*<*0*
then show *eventually* (λ*n*. *1* / *u n* > *a*) *F* **using** ∗(*2*) *eventually_mono less_le_trans linordered_field_no_ub* **by** *fastforce*
next
fix *a*::*ereal* **assume** *a*>*0*
then obtain *e*::*real* **where** *e*>*0 a*>*e* **using** *ereal_dense2 ereal_less*(*2*) **by** *blast*
then have *eventually* (λ*n*. *1* / *u n* < *e*) *F* **using** ∗(*1*) **by** *auto*
then show *eventually* (λ*n*. *1* / *u n* < *a*) *F* **using** ‹*a*>*e*› **by** (*metis* (*mono_tags*, *lifting*) *eventually_mono less_trans*)
qed
qed

The next lemma deserves to exist by itself, as it is so common and useful.

lemma *tendsto_inverse_real* [*tendsto_intros*]:
fixes *u*::_ ⇒ *real*
shows (*u* ⟶ *l*) *F* ⟹ *l* ≠ *0* ⟹ ((λ*x*. *1*/ *u x*) ⟶ *1*/*l*) *F*
using *tendsto_inverse* **unfolding** *inverse_eq_divide* **.**

lemma *tendsto_inverse_ereal* [*tendsto_intros*]:
fixes *u*::_ ⇒ *ereal*
assumes (*u* ⟶ *l*) *F l* ≠ *0*
shows ((λ*x*. *1*/ *u x*) ⟶ *1*/*l*) *F*
proof (*cases l*)
case (*real r*)
then have *r* ≠ *0* **using** *assms*(*2*) **by** *auto*
then have *1*/*l* = *ereal*(*1*/*r*) **using** *real* **by** (*simp add*: *one_ereal_def*)
define *v* **where** *v* = (λ*n*. *real_of_ereal*(*u n*))
have *ureal*: *eventually* (λ*n*. *u n* = *ereal*(*v n*)) *F* **unfolding** *v_def* **using** *real_lim_then_eventually_real assms*(*1*) *real* **by** *auto*
then have ((λ*n*. *ereal*(*v n*)) ⟶ *ereal r*) *F* **using** *assms real v_def* **by** *auto*
then have ∗: ((λ*n*. *v n*) ⟶ *r*) *F* **by** *auto*
then have ((λ*n*. *1*/*v n*) ⟶ *1*/*r*) *F* **using** ‹*r* ≠ *0*› *tendsto_inverse_real* **by** *auto*
then have *lim*: ((λ*n*. *ereal*(*1*/*v n*)) ⟶ *1*/*l*) *F* **using** ‹*1*/*l* = *ereal*(*1*/*r*)› **by** *auto*

have *r* ∈ −{*0*} *open* (−{(*0*::*real*)}) **using** ‹*r* ≠ *0*› **by** *auto*
then have *eventually* (λ*n*. *v n* ∈ −{*0*}) *F* **using** ∗ **using** *topological_tendstoD* **by** *blast*
then have *eventually* (λ*n*. *v n* ≠ *0*) *F* **by** *auto*
moreover
{
fix *n* **assume** *H*: *v n* ≠ *0 u n* = *ereal*(*v n*)
then have *ereal*(*1*/*v n*) = *1*/*ereal*(*v n*) **by** (*simp add*: *one_ereal_def*)
then have *ereal*(*1*/*v n*) = *1*/*u n* **using** *H*(*2*) **by** *simp*

}
ultimately have *eventually* ($\lambda n.$ *ereal*($1/v\ n$) = $1/u\ n$) *F* **using** *ureal eventually_elim2* **by** *force*
with *Lim_transform_eventually*[*OF lim this*] **show** *?thesis* **by** *simp*
next
case (*PInf*)
then have $1/l = 0$ **by** *auto*
then show *?thesis* **using** *tendsto_inverse_ereal_PInf assms PInf* **by** *auto*
next
case (*MInf*)
then have $1/l = 0$ **by** *auto*
have $1/z = -1/ -z$ **if** $z < 0$ **for** *z::ereal*
apply (*cases z*) **using** *divide_ereal_def* ‹ $z < 0$ › **by** *auto*
moreover have *eventually* ($\lambda n.\ u\ n < 0$) *F* **by** (*metis* (*no_types*) *MInf assms*(*1*) *tendsto_MInfty zero_ereal_def*)
ultimately have ∗: *eventually* ($\lambda n.\ -1/-u\ n = 1/u\ n$) *F* **by** (*simp add*: *eventually_mono*)

define *v* **where** $v = (\lambda n.\ -\ u\ n)$
have ($v \longrightarrow \infty$) *F* **unfolding** *v_def* **using** *MInf assms*(*1*) *tendsto_uminus_ereal* **by** *fastforce*
then have (($\lambda n.\ 1/v\ n$) $\longrightarrow 0$) *F* **using** *tendsto_inverse_ereal_PInf* **by** *auto*
then have (($\lambda n.\ -1/v\ n$) $\longrightarrow 0$) *F* **using** *tendsto_uminus_ereal* **by** *fastforce*
then show *?thesis* **unfolding** *v_def* **using** *Lim_transform_eventually*[*OF _ ∗*] ‹ $1/l = 0$ › **by** *auto*
qed

lemma *tendsto_divide_ereal* [*tendsto_intros*]:
fixes *f g::_ ⇒ ereal*
assumes ($f \longrightarrow l$) *F* ($g \longrightarrow m$) *F* $m \neq 0$ $\neg(abs(l) = \infty \wedge abs(m) = \infty)$
shows (($\lambda x.\ f\ x\ /\ g\ x$) $\longrightarrow l\ /\ m$) *F*
proof −
define *h* **where** $h = (\lambda x.\ 1/\ g\ x)$
have ∗: ($h \longrightarrow 1/m$) *F* **unfolding** *h_def* **using** *assms*(*2*) *assms*(*3*) *tendsto_inverse_ereal* **by** *auto*
have (($\lambda x.\ f\ x * h\ x$) $\longrightarrow l * (1/m)$) *F*
apply (*rule tendsto_mult_ereal*[*OF assms*(*1*) ∗]) **using** *assms*(*3*) *assms*(*4*) **by** (*auto simp add*: *divide_ereal_def*)
moreover have $f\ x * h\ x = f\ x\ /\ g\ x$ **for** *x* **unfolding** *h_def* **by** (*simp add*: *divide_ereal_def*)
moreover have $l * (1/m) = l/m$ **by** (*simp add*: *divide_ereal_def*)
ultimately show *?thesis* **unfolding** *h_def* **using** *Lim_transform_eventually* **by** *auto*
qed

Further limits

The assumptions of $[\![|?x| \neq \infty; |?y| \neq \infty; (?f \longrightarrow ?x)\ ?F; (?g \longrightarrow ?y)\ ?F]\!] \Longrightarrow ((\lambda x.\ ?f\ x - ?g\ x) \longrightarrow ?x - ?y)\ ?F$ are too strong, we weaken them here.

```
lemma tendsto_diff_ereal_general [tendsto_intros]:
  fixes u v::'a ⇒ ereal
  assumes (u ⟶ l) F (v ⟶ m) F ¬((l = ∞ ∧ m = ∞) ∨ (l = −∞ ∧ m
= −∞))
  shows ((λn. u n − v n) ⟶ l − m) F
proof −
  have ((λn. u n + (−v n)) ⟶ l + (−m)) F
   apply (intro tendsto_intros assms) using assms by (auto simp add: ereal_uminus_eq_reorder)
  then show ?thesis by (simp add: minus_ereal_def)
qed

lemma id_nat_ereal_tendsto_PInf [tendsto_intros]:
  (λ n::nat. real n) ⟶ ∞
by (simp add: filterlim_real_sequentially tendsto_PInfty_eq_at_top)

lemma tendsto_at_top_pseudo_inverse [tendsto_intros]:
  fixes u::nat ⇒ nat
  assumes LIM n sequentially. u n :> at_top
  shows LIM n sequentially. Inf {N. u N ≥ n} :> at_top
proof −
  {
    fix C::nat
    define M where M = Max {u n| n. n ≤ C}+1
    {
      fix n assume n ≥ M
      have eventually (λN. u N ≥ n) sequentially using assms
        by (simp add: filterlim_at_top)
      then have ∗: {N. u N ≥ n} ≠ {} by force

      have N > C if u N ≥ n for N
      proof (rule ccontr)
        assume ¬(N > C)
        have u N ≤ Max {u n| n. n ≤ C}
          apply (rule Max_ge) using ‹¬(N > C)› by auto
        then show False using ‹u N ≥ n› ‹n ≥ M› unfolding M_def by auto
      qed
      then have ∗∗: {N. u N ≥ n} ⊆ {C..} by fastforce
      have Inf {N. u N ≥ n} ≥ C
        by (metis ∗ ∗∗ Inf_nat_def1 atLeast_iff subset_eq)
    }
    then have eventually (λn. Inf {N. u N ≥ n} ≥ C) sequentially
      using eventually_sequentially by auto
  }
  then show ?thesis using filterlim_at_top by auto
qed

lemma pseudo_inverse_finite_set:
  fixes u::nat ⇒ nat
```

assumes *LIM n sequentially. u n :> at_top*
shows *finite* {*N. u N* ≤ *n*}
proof –
fix *n*
have *eventually* (λ*N. u N* ≥ *n*+*1*) *sequentially* **using** *assms*
by (*simp add*: *filterlim_at_top*)
then obtain *N1* **where** *N1*: ⋀*N. N* ≥ *N1* ⟹ *u N* ≥ *n* + *1*
using *eventually_sequentially* **by** *auto*
have {*N. u N* ≤ *n*} ⊆ {..<*N1*}
apply *auto* **using** *N1* **by** (*metis Suc_eq_plus1 not_less not_less_eq_eq*)
then show *finite* {*N. u N* ≤ *n*} **by** (*simp add*: *finite_subset*)
qed

lemma *tendsto_at_top_pseudo_inverse2* [*tendsto_intros*]:
fixes *u*::*nat* ⇒ *nat*
assumes *LIM n sequentially. u n :> at_top*
shows *LIM n sequentially. Max* {*N. u N* ≤ *n*} *:> at_top*
proof –
{
fix *N0*::*nat*
have *N0* ≤ *Max* {*N. u N* ≤ *n*} **if** *n* ≥ *u N0* **for** *n*
apply (*rule Max.coboundedI*) **using** *pseudo_inverse_finite_set*[*OF assms*] *that*
by *auto*
then have *eventually* (λ*n. N0* ≤ *Max* {*N. u N* ≤ *n*}) *sequentially*
using *eventually_sequentially* **by** *blast*
}
then show *?thesis* **using** *filterlim_at_top* **by** *auto*
qed

lemma *ereal_truncation_top* [*tendsto_intros*]:
fixes *x*::*ereal*
shows (λ*n*::*nat. min x n*) ⟶ *x*
proof (*cases x*)
case (*real r*)
then obtain *K*::*nat* **where** *K*>*0 K* > *abs*(*r*) **using** *reals_Archimedean2 gr0I*
by *auto*
then have *min x n* = *x* **if** *n* ≥ *K* **for** *n* **apply** (*subst real*, *subst real*, *auto*)
using *that eq_iff* **by** *fastforce*
then have *eventually* (λ*n. min x n* = *x*) *sequentially* **using** *eventually_at_top_linorder*
by *blast*
then show *?thesis* **by** (*simp add*: *tendsto_eventually*)
next
case (*PInf*)
then have *min x n* = *n* **for** *n*::*nat* **by** (*auto simp add*: *min_def*)
then show *?thesis* **using** *id_nat_ereal_tendsto_PInf PInf* **by** *auto*
next
case (*MInf*)
then have *min x n* = *x* **for** *n*::*nat* **by** (*auto simp add*: *min_def*)
then show *?thesis* **by** *auto*

qed

lemma *ereal_truncation_real_top* [*tendsto_intros*]:
fixes *x*::*ereal*
assumes $x \neq -\infty$
shows (λ*n*::*nat*. *real_of_ereal*(*min x n*)) ⟶ *x*
proof (*cases x*)
case (*real r*)
then obtain *K*::*nat* **where** *K*>*0* *K* > *abs*(*r*) **using** *reals_Archimedean2 gr0I* **by** *auto*
then have *min x n* = *x* **if** *n* ≥ *K* **for** *n* **apply** (*subst real*, *subst real*, *auto*) **using** *that eq_iff* **by** *fastforce*
then have *real_of_ereal*(*min x n*) = *r* **if** *n* ≥ *K* **for** *n* **using** *real that* **by** *auto*
then have *eventually* (λ*n*. *real_of_ereal*(*min x n*) = *r*) *sequentially* **using** *eventually_at_top_linorder* **by** *blast*
then have (λ*n*. *real_of_ereal*(*min x n*)) ⟶ *r* **by** (*simp add*: *tendsto_eventually*)
then show *?thesis* **using** *real* **by** *auto*
next
case (*PInf*)
then have *real_of_ereal*(*min x n*) = *n* **for** *n*::*nat* **by** (*auto simp add*: *min_def*)
then show *?thesis* **using** *id_nat_ereal_tendsto_PInf PInf* **by** *auto*
qed (*simp add*: *assms*)

lemma *ereal_truncation_bottom* [*tendsto_intros*]:
fixes *x*::*ereal*
shows (λ*n*::*nat*. *max x* (− *real n*)) ⟶ *x*
proof (*cases x*)
case (*real r*)
then obtain *K*::*nat* **where** *K*>*0* *K* > *abs*(*r*) **using** *reals_Archimedean2 gr0I* **by** *auto*
then have *max x* (−*real n*) = *x* **if** *n* ≥ *K* **for** *n* **apply** (*subst real*, *subst real*, *auto*) **using** *that eq_iff* **by** *fastforce*
then have *eventually* (λ*n*. *max x* (−*real n*) = *x*) *sequentially* **using** *eventually_at_top_linorder* **by** *blast*
then show *?thesis* **by** (*simp add*: *tendsto_eventually*)
next
case (*MInf*)
then have *max x* (−*real n*) = (−*1*)∗ *ereal*(*real n*) **for** *n*::*nat* **by** (*auto simp add*: *max_def*)
moreover have (λ*n*. (−*1*)∗ *ereal*(*real n*)) ⟶ −∞
using *tendsto_cmult_ereal*[*of* −*1*, *OF* _ *id_nat_ereal_tendsto_PInf*] **by** (*simp add*: *one_ereal_def*)
ultimately show *?thesis* **using** *MInf* **by** *auto*
next
case (*PInf*)
then have *max x* (−*real n*) = *x* **for** *n*::*nat* **by** (*auto simp add*: *max_def*)
then show *?thesis* **by** *auto*
qed

lemma *ereal_truncation_real_bottom* [*tendsto_intros*]:
fixes x::*ereal*
assumes $x \neq \infty$
shows (λn::*nat*. *real_of_ereal*(*max* x ($-$ *real* n))) $\longlongrightarrow x$
proof (*cases* x)
case (*real* r)
then obtain K::*nat* **where** $K>0$ $K > abs(r)$ **using** *reals_Archimedean2 gr0I* **by** *auto*
then have *max* x ($-$*real* n) = x **if** $n \geq K$ **for** n **apply** (*subst real*, *subst real*, *auto*) **using** *that eq_iff* **by** *fastforce*
then have *real_of_ereal*(*max* x ($-$*real* n)) = r **if** $n \geq K$ **for** n **using** *real that* **by** *auto*
then have *eventually* (λn. *real_of_ereal*(*max* x ($-$*real* n)) = r) *sequentially* **using** *eventually_at_top_linorder* **by** *blast*
then have (λn. *real_of_ereal*(*max* x ($-$*real* n))) $\longlongrightarrow r$ **by** (*simp add*: *tendsto_eventually*)
then show *?thesis* **using** *real* **by** *auto*
next
case (*MInf*)
then have *real_of_ereal*(*max* x ($-$*real* n)) = (-1)$*$ *ereal*(*real* n) **for** n::*nat* **by** (*auto simp add*: *max_def*)
moreover have (λn. (-1)$*$ *ereal*(*real* n)) $\longlongrightarrow -\infty$
using *tendsto_cmult_ereal*[*of* -1, *OF* _ *id_nat_ereal_tendsto_PInf*] **by** (*simp add*: *one_ereal_def*)
ultimately show *?thesis* **using** *MInf* **by** *auto*
qed (*simp add*: *assms*)

the next one is copied from *tendsto_sum*.

lemma *tendsto_sum_ereal* [*tendsto_intros*]:
fixes f :: $'a \Rightarrow 'b \Rightarrow$ *ereal*
assumes $\bigwedge i.\ i \in S \Longrightarrow (f\ i \longrightarrow a\ i)\ F$
$\bigwedge i.\ abs(a\ i) \neq \infty$
shows $((\lambda x. \sum i{\in}S.\ f\ i\ x) \longrightarrow (\sum i{\in}S.\ a\ i))\ F$
proof (*cases finite S*)
assume *finite S* **then show** *?thesis* **using** *assms*
by (*induct*, *simp*, *simp add*: *tendsto_add_ereal_general2 assms*)
qed(*simp*)

lemma *continuous_ereal_abs*:
continuous_on (*UNIV*::*ereal set*) *abs*
proof –
have *continuous_on* ({..*0*} ∪ {(*0*::*ereal*)..}) *abs*
apply (*rule continuous_on_closed_Un*, *auto*)
apply (*rule iffD1*[*OF continuous_on_cong*, *of* {..*0*} _ $\lambda x.\ -x$])
using *less_eq_ereal_def* **apply** (*auto simp add*: *continuous_uminus_ereal*)
apply (*rule iffD1*[*OF continuous_on_cong*, *of* {*0*..} _ $\lambda x.\ x$])
apply (*auto*)
done

moreover have $(UNIV::ereal\ set) = \{..0\} \cup \{(0::ereal)..\}$ **by** *auto*
ultimately show *?thesis* **by** *auto*
qed

lemmas *continuous_on_compose_ereal_abs*[*continuous_intros*] =
continuous_on_compose2[*OF continuous_ereal_abs _ subset_UNIV*]

lemma *tendsto_abs_ereal* [*tendsto_intros*]:
assumes $(u \longrightarrow (l::ereal))\ F$
shows $((\lambda n.\ abs(u\ n)) \longrightarrow abs\ l)\ F$
using *continuous_ereal_abs assms* **by** (*metis UNIV_I continuous_on tendsto_compose*)

lemma *ereal_minus_real_tendsto_MInf* [*tendsto_intros*]:
$(\lambda x.\ ereal\ (-\ real\ x)) \longlongrightarrow -\ \infty$
by (*subst uminus_ereal.simps*(*1*)[*symmetric*], *intro tendsto_intros*)

4.5.2 Extended-Nonnegative-Real.thy

lemma *tendsto_diff_ennreal_general* [*tendsto_intros*]:
fixes $u\ v::'a \Rightarrow ennreal$
assumes $(u \longrightarrow l)\ F\ (v \longrightarrow m)\ F\ \neg(l = \infty \wedge m = \infty)$
shows $((\lambda n.\ u\ n\ -\ v\ n) \longrightarrow l\ -\ m)\ F$
proof −
have $((\lambda n.\ e2ennreal(enn2ereal(u\ n) - enn2ereal(v\ n))) \longrightarrow e2ennreal(enn2ereal\ l\ -\ enn2ereal\ m))\ F$
apply (*intro tendsto_intros*) **using** *assms* **by** *auto*
then show *?thesis* **by** *auto*
qed

lemma *tendsto_mult_ennreal* [*tendsto_intros*]:
fixes $l\ m::ennreal$
assumes $(u \longrightarrow l)\ F\ (v \longrightarrow m)\ F\ \neg((l = 0 \wedge m = \infty) \vee (l = \infty \wedge m = 0))$
shows $((\lambda n.\ u\ n * v\ n) \longrightarrow l * m)\ F$
proof −
have $((\lambda n.\ e2ennreal(enn2ereal\ (u\ n) * enn2ereal\ (v\ n))) \longrightarrow e2ennreal(enn2ereal\ l * enn2ereal\ m))\ F$
apply (*intro tendsto_intros*) **using** *assms* **apply** *auto*
using *enn2ereal_inject zero_ennreal.rep_eq* **by** *fastforce+*
moreover have $e2ennreal(enn2ereal\ (u\ n) * enn2ereal\ (v\ n)) = u\ n * v\ n$ **for** n
by (*subst times_ennreal.abs_eq*[*symmetric*], *auto simp add*: *eq_onp_same_args*)
moreover have $e2ennreal(enn2ereal\ l * enn2ereal\ m) = l * m$
by (*subst times_ennreal.abs_eq*[*symmetric*], *auto simp add*: *eq_onp_same_args*)
ultimately show *?thesis*
by *auto*
qed

4.5.3 monoset

definition (**in** *order*) *mono_set*:

```
  mono_set S ⟷ (∀ x y. x ≤ y ⟶ x ∈ S ⟶ y ∈ S)

lemma (in order) mono_greaterThan [intro, simp]: mono_set {B<..} unfolding
mono_set by auto
lemma (in order) mono_atLeast [intro, simp]: mono_set {B..} unfolding mono_set
by auto
lemma (in order) mono_UNIV [intro, simp]: mono_set UNIV unfolding mono_set
by auto
lemma (in order) mono_empty [intro, simp]: mono_set {} unfolding mono_set
by auto

lemma (in complete_linorder) mono_set_iff:
  fixes S :: 'a set
  defines a ≡ Inf S
  shows mono_set S ⟷ S = {a <..} ∨ S = {a..} (is _ = ?c)
proof
  assume mono_set S
  then have mono: ⋀x y. x ≤ y ⟹ x ∈ S ⟹ y ∈ S
    by (auto simp: mono_set)
  show ?c
  proof cases
    assume a ∈ S
    show ?c
      using mono[OF _ ‹a ∈ S›]
      by (auto intro: Inf_lower simp: a_def)
  next
    assume a ∉ S
    have S = {a <..}
    proof safe
      fix x assume x ∈ S
      then have a ≤ x
        unfolding a_def by (rule Inf_lower)
      then show a < x
        using ‹x ∈ S› ‹a ∉ S› by (cases a = x) auto
    next
      fix x assume a < x
      then obtain y where y < x y ∈ S
        unfolding a_def Inf_less_iff ..
      with mono[of y x] show x ∈ S
        by auto
    qed
    then show ?c ..
  qed
qed auto

lemma ereal_open_mono_set:
  fixes S :: ereal set
  shows open S ∧ mono_set S ⟷ S = UNIV ∨ S = {Inf S <..}
  by (metis Inf_UNIV atLeast_eq_UNIV_iff ereal_open_atLeast
```

ereal_open_closed mono_set_iff open_ereal_greaterThan)

lemma *ereal_closed_mono_set*:
fixes *S* :: *ereal set*
shows *closed S* ∧ *mono_set S* ⟷ *S* = {} ∨ *S* = {*Inf S* ..}
by (*metis Inf_UNIV atLeast_eq_UNIV_iff closed_ereal_atLeast*
ereal_open_closed mono_empty mono_set_iff open_ereal_greaterThan)

lemma *ereal_Liminf_Sup_monoset*:
fixes *f* :: *'a* ⇒ *ereal*
shows *Liminf net f* =
Sup {*l*. ∀ *S*. *open S* ⟶ *mono_set S* ⟶ *l* ∈ *S* ⟶ *eventually* (λ*x*. *f x* ∈ *S*)
net}
(**is** _ = *Sup ?A*)
proof (*safe intro*!: *Liminf_eqI complete_lattice_class.Sup_upper complete_lattice_class.Sup_least*)
fix *P*
assume *P*: *eventually P net*
fix *S*
assume *S*: *mono_set S Inf* (*f* ‘ (*Collect P*)) ∈ *S*
{
fix *x*
assume *P x*
then have *Inf* (*f* ‘ (*Collect P*)) ≤ *f x*
by (*intro complete_lattice_class.INF_lower*) *simp*
with *S* **have** *f x* ∈ *S*
by (*simp add*: *mono_set*)
}
with *P* **show** *eventually* (λ*x*. *f x* ∈ *S*) *net*
by (*auto elim*: *eventually_mono*)
next
fix *y l*
assume *S*: ∀ *S*. *open S* ⟶ *mono_set S* ⟶ *l* ∈ *S* ⟶ *eventually* (λ*x*. *f x* ∈ *S*)
net
assume *P*: ∀ *P*. *eventually P net* ⟶ *Inf* (*f* ‘ (*Collect P*)) ≤ *y*
show *l* ≤ *y*
proof (*rule dense_le*)
fix *B*
assume *B* < *l*
then have *eventually* (λ*x*. *f x* ∈ {*B* <..}) *net*
by (*intro S*[*rule_format*]) *auto*
then have *Inf* (*f* ‘ {*x*. *B* < *f x*}) ≤ *y*
using *P* **by** *auto*
moreover have *B* ≤ *Inf* (*f* ‘ {*x*. *B* < *f x*})
by (*intro INF_greatest*) *auto*
ultimately show *B* ≤ *y*
by *simp*
qed
qed

lemma *ereal_Limsup_Inf_monoset*:
fixes *f* :: *'a* ⇒ *ereal*
shows *Limsup net f* =
Inf {*l*. ∀ *S*. *open S* ⟶ *mono_set* (*uminus* ' *S*) ⟶ *l* ∈ *S* ⟶ *eventually* (λ*x*. *f x* ∈ *S*) *net*}
(**is** _ = *Inf ?A*)
proof (*safe intro*!: *Limsup_eqI complete_lattice_class.Inf_lower complete_lattice_class.Inf_greatest*)
fix *P*
assume *P*: *eventually P net*
fix *S*
assume *S*: *mono_set* (*uminus*'*S*) *Sup* (*f* ' (*Collect P*)) ∈ *S*
{
fix *x*
assume *P x*
then have *f x* ≤ *Sup* (*f* ' (*Collect P*))
by (*intro complete_lattice_class.SUP_upper*) *simp*
with *S*(*1*)[*unfolded mono_set*, *rule_format*, *of* − *Sup* (*f* ' (*Collect P*)) − *f x*] *S*(*2*)
have *f x* ∈ *S*
by (*simp add*: *inj_image_mem_iff*) **}**
with *P* **show** *eventually* (λ*x*. *f x* ∈ *S*) *net*
by (*auto elim*: *eventually_mono*)
next
fix *y l*
assume *S*: ∀ *S*. *open S* ⟶ *mono_set* (*uminus* ' *S*) ⟶ *l* ∈ *S* ⟶ *eventually* (λ*x*. *f x* ∈ *S*) *net*
assume *P*: ∀ *P*. *eventually P net* ⟶ *y* ≤ *Sup* (*f* ' (*Collect P*))
show *y* ≤ *l*
proof (*rule dense_ge*)
fix *B*
assume *l* < *B*
then have *eventually* (λ*x*. *f x* ∈ {..< *B*}) *net*
by (*intro S*[*rule_format*]) *auto*
then have *y* ≤ *Sup* (*f* ' {*x*. *f x* < *B*})
using *P* **by** *auto*
moreover have *Sup* (*f* ' {*x*. *f x* < *B*}) ≤ *B*
by (*intro SUP_least*) *auto*
ultimately show *y* ≤ *B*
by *simp*
qed
qed

lemma *liminf_bounded_open*:
fixes *x* :: *nat* ⇒ *ereal*
shows *x0* ≤ *liminf x* ⟷ (∀ *S*. *open S* ⟶ *mono_set S* ⟶ *x0* ∈ *S* ⟶ (∃ *N*. ∀ *n*≥*N*. *x n* ∈ *S*))
(**is** _ ⟷ *?P x0*)
proof
assume *?P x0*

```
  then show x0 ≤ liminf x
    unfolding ereal_Liminf_Sup_monoset eventually_sequentially
    by (intro complete_lattice_class.Sup_upper) auto
next
  assume x0 ≤ liminf x
  {
    fix S :: ereal set
    assume om: open S mono_set S x0 ∈ S
    {
      assume S = UNIV
      then have ∃N. ∀n≥N. x n ∈ S
        by auto
    }
    moreover
    {
      assume S ≠ UNIV
      then obtain B where B: S = {B<..}
        using om ereal_open_mono_set by auto
      then have B < x0
        using om by auto
      then have ∃N. ∀n≥N. x n ∈ S
        unfolding B
        using ‹x0 ≤ liminf x› liminf_bounded_iff
        by auto
    }
    ultimately have ∃N. ∀n≥N. x n ∈ S
      by auto
  }
  then show ?P x0
    by auto
qed

lemma limsup_finite_then_bounded:
  fixes u::nat ⇒ real
  assumes limsup u < ∞
  shows ∃C. ∀n. u n ≤ C
proof –
  obtain C where C: limsup u < C C < ∞ using assms ereal_dense2 by blast
  then have C = ereal(real_of_ereal C) using ereal_real by force
  have eventually (λn. u n < C) sequentially using C(1) unfolding Limsup_def
    apply (auto simp add: INF_less_iff)
    using SUP_lessD eventually_mono by fastforce
 then obtain N where N: ⋀n. n ≥ N ⟹ u n < C using eventually_sequentially
by auto
  define D where D = max (real_of_ereal C) (Max {u n |n. n ≤ N})
  have ⋀n. u n ≤ D
  proof –
    fix n show u n ≤ D
    proof (cases)
```

assume ∗: $n \leq N$
have $u\ n \leq Max\ \{u\ n\ |n.\ n \leq N\}$ **by** (*rule Max_ge*, *auto simp add*: ∗)
then show $u\ n \leq D$ **unfolding** *D_def* **by** *linarith*
next
assume $\neg(n \leq N)$
then have $n \geq N$ **by** *simp*
then have $u\ n < C$ **using** N **by** *auto*
then have $u\ n <$ *real_of_ereal* C **using** ‹C = *ereal*(*real_of_ereal* C)› *less_ereal.simps*(*1*) **by** *fastforce*
then show $u\ n \leq D$ **unfolding** *D_def* **by** *linarith*
qed
qed
then show *?thesis* **by** *blast*
qed

lemma *liminf_finite_then_bounded_below*:
fixes *u*::*nat* ⇒ *real*
assumes *liminf* $u > -\infty$
shows $\exists\, C.\ \forall\, n.\ u\ n \geq C$
proof −
obtain C **where** C: *liminf* $u > C$ $C > -\infty$ **using** *assms* **using** *ereal_dense2* **by** *blast*
then have C = *ereal*(*real_of_ereal* C) **using** *ereal_real* **by** *force*
have *eventually* ($\lambda n.\ u\ n > C$) *sequentially* **using** C(*1*) **unfolding** *Liminf_def*
apply (*auto simp add*: *less_SUP_iff*)
using *eventually_elim2 less_INF_D* **by** *fastforce*
then obtain N **where** N: $\bigwedge n.\ n \geq N \Longrightarrow u\ n > C$ **using** *eventually_sequentially* **by** *auto*
define D **where** D = *min* (*real_of_ereal* C) (*Min* $\{u\ n\ |n.\ n \leq N\}$)
have $\bigwedge n.\ u\ n \geq D$
proof −
fix n **show** $u\ n \geq D$
proof (*cases*)
assume ∗: $n \leq N$
have $u\ n \geq Min\ \{u\ n\ |n.\ n \leq N\}$ **by** (*rule Min_le*, *auto simp add*: ∗)
then show $u\ n \geq D$ **unfolding** *D_def* **by** *linarith*
next
assume $\neg(n \leq N)$
then have $n \geq N$ **by** *simp*
then have $u\ n > C$ **using** N **by** *auto*
then have $u\ n >$ *real_of_ereal* C **using** ‹C = *ereal*(*real_of_ereal* C)› *less_ereal.simps*(*1*) **by** *fastforce*
then show $u\ n \geq D$ **unfolding** *D_def* **by** *linarith*
qed
qed
then show *?thesis* **by** *blast*
qed

lemma *liminf_upper_bound*:

fixes *u*:: *nat* ⇒ *ereal*
assumes *liminf u* < *l*
shows ∃ *N*>*k*. *u N* < *l*
by (*metis assms gt_ex less_le_trans liminf_bounded_iff not_less*)

lemma *limsup_shift*:
limsup (λ*n*. *u* (*n*+*1*)) = *limsup u*
proof −
have (*SUP m*∈{*n*+*1*..}. *u m*) = (*SUP m*∈{*n*..}. *u* (*m* + *1*)) **for** *n*
apply (*rule SUP_eq*) **using** *Suc_le_D* **by** *auto*
then have *a*: (*INF n*. *SUP m*∈{*n*..}. *u* (*m* + *1*)) = (*INF n*. (*SUP m*∈{*n*+*1*..}. *u m*)) **by** *auto*
have *b*: (*INF n*. (*SUP m*∈{*n*+*1*..}. *u m*)) = (*INF n*∈{*1*..}. (*SUP m*∈{*n*..}. *u m*))
apply (*rule INF_eq*) **using** *Suc_le_D* **by** *auto*
have (*INF n*∈{*1*..}. *v n*) = (*INF n*. *v n*) **if** *decseq v* **for** *v*::*nat* ⇒ *'a*
apply (*rule INF_eq*) **using** ‹*decseq v*› *decseq_Suc_iff* **by** *auto*
moreover have *decseq* (λ*n*. (*SUP m*∈{*n*..}. *u m*)) **by** (*simp add*: *SUP_subset_mono decseq_def*)
ultimately have *c*: (*INF n*∈{*1*..}. (*SUP m*∈{*n*..}. *u m*)) = (*INF n*. (*SUP m*∈{*n*..}. *u m*)) **by** *simp*
have (*INF n*. *Sup* (*u* ‘ {*n*..})) = (*INF n*. *SUP m*∈{*n*..}. *u* (*m* + *1*)) **using** *a b c* **by** *simp*
then show *?thesis* **by** (*auto cong*: *limsup_INF_SUP*)
qed

lemma *limsup_shift_k*:
limsup (λ*n*. *u* (*n*+*k*)) = *limsup u*
proof (*induction k*)
case (*Suc k*)
have *limsup* (λ*n*. *u* (*n*+*k*+*1*)) = *limsup* (λ*n*. *u* (*n*+*k*)) **using** *limsup_shift*[**where** *?u*=λ*n*. *u*(*n*+*k*)] **by** *simp*
then show *?case* **using** *Suc.IH* **by** *simp*
qed (*auto*)

lemma *liminf_shift*:
liminf (λ*n*. *u* (*n*+*1*)) = *liminf u*
proof −
have (*INF m*∈{*n*+*1*..}. *u m*) = (*INF m*∈{*n*..}. *u* (*m* + *1*)) **for** *n*
apply (*rule INF_eq*) **using** *Suc_le_D* **by** (*auto*)
then have *a*: (*SUP n*. *INF m*∈{*n*..}. *u* (*m* + *1*)) = (*SUP n*. (*INF m*∈{*n*+*1*..}. *u m*)) **by** *auto*
have *b*: (*SUP n*. (*INF m*∈{*n*+*1*..}. *u m*)) = (*SUP n*∈{*1*..}. (*INF m*∈{*n*..}. *u m*))
apply (*rule SUP_eq*) **using** *Suc_le_D* **by** (*auto*)
have (*SUP n*∈{*1*..}. *v n*) = (*SUP n*. *v n*) **if** *incseq v* **for** *v*::*nat* ⇒ *'a*
apply (*rule SUP_eq*) **using** ‹*incseq v*› *incseq_Suc_iff* **by** *auto*
moreover have *incseq* (λ*n*. (*INF m*∈{*n*..}. *u m*)) **by** (*simp add*: *INF_superset_mono mono_def*)

ultimately have *c*: (*SUP n*∈{*1..*}. (*INF m*∈{*n..*}. *u m*)) = (*SUP n*. (*INF m*∈{*n..*}. *u m*)) **by** *simp*
have (*SUP n*. *Inf* (*u* ‘ {*n..*})) = (*SUP n*. *INF m*∈{*n..*}. *u* (*m* + *1*)) **using** *a b c* **by** *simp*
then show *?thesis* **by** (*auto cong*: *liminf_SUP_INF*)
qed

lemma *liminf_shift_k*:
liminf (λ*n*. *u* (*n*+*k*)) = *liminf u*
proof (*induction k*)
case (*Suc k*)
have *liminf* (λ*n*. *u* (*n*+*k*+*1*)) = *liminf* (λ*n*. *u* (*n*+*k*)) **using** *liminf_shift*[**where** *?u*=λ*n*. *u*(*n*+*k*)] **by** *simp*
then show *?case* **using** *Suc.IH* **by** *simp*
qed (*auto*)

lemma *Limsup_obtain*:
fixes *u*::_ ⇒ ′*a* :: *complete_linorder*
assumes *Limsup F u* > *c*
shows ∃ *i*. *u i* > *c*
proof −
have (*INF P*∈{*P*. *eventually P F*}. *SUP x*∈{*x*. *P x*}. *u x*) > *c* **using** *assms* **by** (*simp add*: *Limsup_def*)
then show *?thesis* **by** (*metis eventually_True mem_Collect_eq less_INF_D less_SUP_iff*)
qed

The next lemma is extremely useful, as it often makes it possible to reduce statements about limsups to statements about limits.

lemma *limsup_subseq_lim*:
fixes *u*::*nat* ⇒ ′*a* :: {*complete_linorder*, *linorder_topology*}
shows ∃ *r*::*nat*⇒*nat*. *strict_mono r* ∧ (*u o r*) ———→ *limsup u*
proof (*cases*)
assume ∀ *n*. ∃ *p*>*n*. ∀ *m*≥*p*. *u m* ≤ *u p*
then have ∃ *r*. ∀ *n*. (∀ *m*≥*r n*. *u m* ≤ *u* (*r n*)) ∧ *r n* < *r* (*Suc n*)
by (*intro dependent_nat_choice*) (*auto simp*: *conj_commute*)
then obtain *r* :: *nat* ⇒ *nat* **where** *strict_mono r* **and** *mono*: ⋀*n m*. *r n* ≤ *m* ⟹ *u m* ≤ *u* (*r n*)
by (*auto simp*: *strict_mono_Suc_iff*)
define *umax* **where** *umax* = (λ*n*. (*SUP m*∈{*n..*}. *u m*))
have *decseq umax* **unfolding** *umax_def* **by** (*simp add*: *SUP_subset_mono antimono_def*)
then have *umax* ———→ *limsup u* **unfolding** *umax_def* **by** (*metis LIMSEQ_INF limsup_INF_SUP*)
then have ∗: (*umax o r*) ———→ *limsup u* **by** (*simp add*: *LIMSEQ_subseq_LIMSEQ* ‹*strict_mono r*›)
have ⋀*n*. *umax*(*r n*) = *u*(*r n*) **unfolding** *umax_def* **using** *mono*
by (*metis SUP_le_iff antisym atLeast_def mem_Collect_eq order_refl*)
then have *umax o r* = *u o r* **unfolding** *o_def* **by** *simp*
then have (*u o r*) ———→ *limsup u* **using** ∗ **by** *simp*

```
  then show ?thesis using ‹strict_mono r› by blast
next
  assume ¬ (∀n. ∃p>n. (∀m≥p. u m ≤ u p))
  then obtain N where N: ⋀p. p > N ⟹ ∃m>p. u p < u m by (force simp:
not_le le_less)
  have ∃r. ∀n. N < r n ∧ r n < r (Suc n) ∧ (∀i∈ {N<..r (Suc n)}. u i ≤ u (r
(Suc n)))
  proof (rule dependent_nat_choice)
    fix x assume N < x
    then have a: finite {N<..x} {N<..x} ≠ {} by simp_all
    have Max {u i |i. i ∈ {N<..x}} ∈ {u i |i. i ∈ {N<..x}} apply (rule Max_in)
using a by (auto)
     then obtain p where p ∈ {N<..x} and upmax: u p = Max{u i |i. i ∈
{N<..x}} by auto
    define U where U = {m. m > p ∧ u p < u m}
    have U ≠ {} unfolding U_def using N[of p] ‹p ∈ {N<..x}› by auto
    define y where y = Inf U
    then have y ∈ U using ‹U ≠ {}› by (simp add: Inf_nat_def1)
    have a: ⋀i. i ∈ {N<..x} ⟹ u i ≤ u p
    proof –
      fix i assume i ∈ {N<..x}
      then have u i ∈ {u i |i. i ∈ {N<..x}} by blast
      then show u i ≤ u p using upmax by simp
    qed
    moreover have u p < u y using ‹y ∈ U› U_def by auto
    ultimately have y ∉ {N<..x} using not_le by blast
    moreover have y > N using ‹y ∈ U› U_def ‹p ∈ {N<..x}› by auto
    ultimately have y > x by auto

    have ⋀i. i ∈ {N<..y} ⟹ u i ≤ u y
    proof –
      fix i assume i ∈ {N<..y} show u i ≤ u y
      proof (cases)
        assume i = y
        then show ?thesis by simp
      next
        assume ¬(i=y)
        then have i:i ∈ {N<..<y} using ‹i ∈ {N<..y}› by simp
        have u i ≤ u p
        proof (cases)
          assume i ≤ x
          then have i ∈ {N<..x} using i by simp
          then show ?thesis using a by simp
        next
          assume ¬(i ≤ x)
          then have i > x by simp
          then have *: i > p using ‹p ∈ {N<..x}› by simp
          have i < Inf U using i y_def by simp
          then have i ∉ U using Inf_nat_def not_less_Least by auto
```

then show *?thesis* **using** *U_def* ∗ **by** *auto*
qed
then show *u i* ≤ *u y* **using** ‹*u p* < *u y*› **by** *auto*
qed
qed
then have *N* < *y* ∧ *x* < *y* ∧ (∀ *i*∈{*N*<..*y*}. *u i* ≤ *u y*) **using** ‹*y* > *x*› ‹*y* > *N*› **by** *auto*
then show ∃ *y*>*N*. *x* < *y* ∧ (∀ *i*∈{*N*<..*y*}. *u i* ≤ *u y*) **by** *auto*
qed (*auto*)
then obtain *r* **where** *r*: ∀ *n*. *N* < *r n* ∧ *r n* < *r* (*Suc n*) ∧ (∀ *i*∈ {*N*<..*r* (*Suc n*)}. *u i* ≤ *u* (*r* (*Suc n*))) **by** *auto*
have *strict_mono r* **using** *r* **by** (*auto simp*: *strict_mono_Suc_iff*)
have *incseq* (*u o r*) **unfolding** *o_def* **using** *r* **by** (*simp add*: *incseq_SucI order.strict_implies_order*)
then have (*u o r*) ⟶ (*SUP n*. (*u o r*) *n*) **using** *LIMSEQ_SUP* **by** *blast*
then have *limsup* (*u o r*) = (*SUP n*. (*u o r*) *n*) **by** (*simp add*: *lim_imp_Limsup*)
moreover have *limsup* (*u o r*) ≤ *limsup u* **using** ‹*strict_mono r*› **by** (*simp add*: *limsup_subseq_mono*)
ultimately have (*SUP n*. (*u o r*) *n*) ≤ *limsup u* **by** *simp*

{
fix *i* **assume** *i*: *i* ∈ {*N*<..}
obtain *n* **where** *i* < *r* (*Suc n*) **using** ‹*strict_mono r*› **using** *Suc_le_eq seq_suble* **by** *blast*
then have *i* ∈ {*N*<..*r*(*Suc n*)} **using** *i* **by** *simp*
then have *u i* ≤ *u* (*r*(*Suc n*)) **using** *r* **by** *simp*
then have *u i* ≤ (*SUP n*. (*u o r*) *n*) **unfolding** *o_def* **by** (*meson SUP_upper2 UNIV_I*)
}
then have (*SUP i*∈{*N*<..}. *u i*) ≤ (*SUP n*. (*u o r*) *n*) **using** *SUP_least* **by** *blast*
then have *limsup u* ≤ (*SUP n*. (*u o r*) *n*) **unfolding** *Limsup_def*
by (*metis* (*mono_tags*, *lifting*) *INF_lower2 atLeast_Suc_greaterThan atLeast_def eventually_ge_at_top mem_Collect_eq*)
then have *limsup u* = (*SUP n*. (*u o r*) *n*) **using** ‹(*SUP n*. (*u o r*) *n*) ≤ *limsup u*› **by** *simp*
then have (*u o r*) ⟶ *limsup u* **using** ‹(*u o r*) ⟶ (*SUP n*. (*u o r*) *n*)› **by** *simp*
then show *?thesis* **using** ‹*strict_mono r*› **by** *auto*
qed

lemma *liminf_subseq_lim*:
fixes *u*::*nat* ⇒ ′*a* :: {*complete_linorder*, *linorder_topology*}
shows ∃ *r*::*nat*⇒*nat*. *strict_mono r* ∧ (*u o r*) ⟶ *liminf u*
proof (*cases*)
assume ∀ *n*. ∃ *p*>*n*. ∀ *m*≥*p*. *u m* ≥ *u p*
then have ∃ *r*. ∀ *n*. (∀ *m*≥*r n*. *u m* ≥ *u* (*r n*)) ∧ *r n* < *r* (*Suc n*)
by (*intro dependent_nat_choice*) (*auto simp*: *conj_commute*)
then obtain *r* :: *nat* ⇒ *nat* **where** *strict_mono r* **and** *mono*: ⋀*n m*. *r n* ≤ *m* ⟹ *u m* ≥ *u* (*r n*)

by (*auto simp*: *strict_mono_Suc_iff*)
define *umin* **where** *umin* = (λ*n*. (*INF m*∈{*n*..}. *u m*))
have *incseq umin* **unfolding** *umin_def* **by** (*simp add*: *INF_superset_mono incseq_def*)
then have *umin* ⟶ *liminf u* **unfolding** *umin_def* **by** (*metis LIMSEQ_SUP liminf_SUP_INF*)
then have ∗: (*umin o r*) ⟶ *liminf u* **by** (*simp add*: *LIMSEQ_subseq_LIMSEQ* ‹*strict_mono r*›)
have ⋀*n*. *umin*(*r n*) = *u*(*r n*) **unfolding** *umin_def* **using** *mono*
by (*metis le_INF_iff antisym atLeast_def mem_Collect_eq order_refl*)
then have *umin o r* = *u o r* **unfolding** *o_def* **by** *simp*
then have (*u o r*) ⟶ *liminf u* **using** ∗ **by** *simp*
then show *?thesis* **using** ‹*strict_mono r*› **by** *blast*
next
assume ¬ (∀ *n*. ∃ *p*>*n*. (∀ *m*≥*p*. *u m* ≥ *u p*))
then obtain *N* **where** *N*: ⋀*p*. *p* > *N* ⟹ ∃ *m*>*p*. *u p* > *u m* **by** (*force simp*: *not_le le_less*)
have ∃ *r*. ∀ *n*. *N* < *r n* ∧ *r n* < *r* (*Suc n*) ∧ (∀ *i*∈ {*N*<..*r* (*Suc n*)}. *u i* ≥ *u* (*r* (*Suc n*)))
proof (*rule dependent_nat_choice*)
fix *x* **assume** *N* < *x*
then have *a*: *finite* {*N*<..*x*} {*N*<..*x*} ≠ {} **by** *simp_all*
have *Min* {*u i* |*i*. *i* ∈ {*N*<..*x*}} ∈ {*u i* |*i*. *i* ∈ {*N*<..*x*}} **apply** (*rule Min_in*) **using** *a* **by** (*auto*)
then obtain *p* **where** *p* ∈ {*N*<..*x*} **and** *upmin*: *u p* = *Min*{*u i* |*i*. *i* ∈ {*N*<..*x*}} **by** *auto*
define *U* **where** *U* = {*m*. *m* > *p* ∧ *u p* > *u m*}
have *U* ≠ {} **unfolding** *U_def* **using** *N*[*of p*] ‹*p* ∈ {*N*<..*x*}› **by** *auto*
define *y* **where** *y* = *Inf U*
then have *y* ∈ *U* **using** ‹*U* ≠ {}› **by** (*simp add*: *Inf_nat_def1*)
have *a*: ⋀*i*. *i* ∈ {*N*<..*x*} ⟹ *u i* ≥ *u p*
proof −
fix *i* **assume** *i* ∈ {*N*<..*x*}
then have *u i* ∈ {*u i* |*i*. *i* ∈ {*N*<..*x*}} **by** *blast*
then show *u i* ≥ *u p* **using** *upmin* **by** *simp*
qed
moreover have *u p* > *u y* **using** ‹*y* ∈ *U*› *U_def* **by** *auto*
ultimately have *y* ∉ {*N*<..*x*} **using** *not_le* **by** *blast*
moreover have *y* > *N* **using** ‹*y* ∈ *U*› *U_def* ‹*p* ∈ {*N*<..*x*}› **by** *auto*
ultimately have *y* > *x* **by** *auto*

have ⋀*i*. *i* ∈ {*N*<..*y*} ⟹ *u i* ≥ *u y*
proof −
fix *i* **assume** *i* ∈ {*N*<..*y*} **show** *u i* ≥ *u y*
proof (*cases*)
assume *i* = *y*
then show *?thesis* **by** *simp*
next
assume ¬(*i*=*y*)

then have *i*:*i* ∈ {*N*<..<*y*} **using** ‹*i* ∈ {*N*<..*y*}› **by** *simp*
have *u i* ≥ *u p*
proof (*cases*)
assume *i* ≤ *x*
then have *i* ∈ {*N*<..*x*} **using** *i* **by** *simp*
then show *?thesis* **using** *a* **by** *simp*
next
assume ¬(*i* ≤ *x*)
then have *i* > *x* **by** *simp*
then have ∗: *i* > *p* **using** ‹*p* ∈ {*N*<..*x*}› **by** *simp*
have *i* < *Inf U* **using** *i y_def* **by** *simp*
then have *i* ∉ *U* **using** *Inf_nat_def not_less_Least* **by** *auto*
then show *?thesis* **using** *U_def* ∗ **by** *auto*
qed
then show *u i* ≥ *u y* **using** ‹*u p* > *u y*› **by** *auto*
qed
qed
then have *N* < *y* ∧ *x* < *y* ∧ (∀ *i*∈{*N*<..*y*}. *u i* ≥ *u y*) **using** ‹*y* > *x*› ‹*y* > *N*› **by** *auto*
then show ∃ *y*>*N*. *x* < *y* ∧ (∀ *i*∈{*N*<..*y*}. *u i* ≥ *u y*) **by** *auto*
qed (*auto*)
then obtain *r* :: *nat* ⇒ *nat*
where *r*: ∀ *n*. *N* < *r n* ∧ *r n* < *r* (*Suc n*) ∧ (∀ *i*∈ {*N*<..*r* (*Suc n*)}. *u i* ≥ *u* (*r* (*Suc n*))) **by** *auto*
have *strict_mono r* **using** *r* **by** (*auto simp*: *strict_mono_Suc_iff*)
have *decseq* (*u o r*) **unfolding** *o_def* **using** *r* **by** (*simp add*: *decseq_SucI order.strict_implies_order*)
then have (*u o r*) ⟶ (*INF n*. (*u o r*) *n*) **using** *LIMSEQ_INF* **by** *blast*
then have *liminf* (*u o r*) = (*INF n*. (*u o r*) *n*) **by** (*simp add*: *lim_imp_Liminf*)
moreover have *liminf* (*u o r*) ≥ *liminf u* **using** ‹*strict_mono r*› **by** (*simp add*: *liminf_subseq_mono*)
ultimately have (*INF n*. (*u o r*) *n*) ≥ *liminf u* **by** *simp*

{
fix *i* **assume** *i*: *i* ∈ {*N*<..}
obtain *n* **where** *i* < *r* (*Suc n*) **using** ‹*strict_mono r*› **using** *Suc_le_eq seq_suble* **by** *blast*
then have *i* ∈ {*N*<..*r*(*Suc n*)} **using** *i* **by** *simp*
then have *u i* ≥ *u* (*r*(*Suc n*)) **using** *r* **by** *simp*
then have *u i* ≥ (*INF n*. (*u o r*) *n*) **unfolding** *o_def* **by** (*meson INF_lower2 UNIV_I*)
}
then have (*INF i*∈{*N*<..}. *u i*) ≥ (*INF n*. (*u o r*) *n*) **using** *INF_greatest* **by** *blast*
then have *liminf u* ≥ (*INF n*. (*u o r*) *n*) **unfolding** *Liminf_def*
by (*metis* (*mono_tags*, *lifting*) *SUP_upper2 atLeast_Suc_greaterThan atLeast_def eventually_ge_at_top mem_Collect_eq*)
then have *liminf u* = (*INF n*. (*u o r*) *n*) **using** ‹(*INF n*. (*u o r*) *n*) ≥ *liminf u*› **by** *simp*

```
  then have (u o r) ⟶ liminf u using ‹(u o r) ⟶ (INF n. (u o r) n)›
by simp
  then show ?thesis using ‹strict_mono r› by auto
qed
```

The following statement about limsups is reduced to a statement about limits using subsequences thanks to *limsup_subseq_lim*. The statement for limits follows for instance from *tendsto_add_ereal_general*.

```
lemma ereal_limsup_add_mono:
  fixes u v::nat ⇒ ereal
  shows limsup (λn. u n + v n) ≤ limsup u + limsup v
proof (cases)
  assume (limsup u = ∞) ∨ (limsup v = ∞)
  then have limsup u + limsup v = ∞ by simp
  then show ?thesis by auto
next
  assume ¬((limsup u = ∞) ∨ (limsup v = ∞))
  then have limsup u < ∞ limsup v < ∞ by auto

  define w where w = (λn. u n + v n)
 obtain r where r: strict_mono r (w o r) ⟶ limsup w using limsup_subseq_lim
by auto
  obtain s where s: strict_mono s (u o r o s) ⟶ limsup (u o r) using
limsup_subseq_lim by auto
  obtain t where t: strict_mono t (v o r o s o t) ⟶ limsup (v o r o s) using
limsup_subseq_lim by auto

  define a where a = r o s o t
  have strict_mono a using r s t by (simp add: a_def strict_mono_o)
  have l:(w o a) ⟶ limsup w
        (u o a) ⟶ limsup (u o r)
        (v o a) ⟶ limsup (v o r o s)
  apply (metis (no_types, lifting) r(2) s(1) t(1) LIMSEQ_subseq_LIMSEQ a_def
comp_assoc)
 apply (metis (no_types, lifting) s(2) t(1) LIMSEQ_subseq_LIMSEQ a_def comp_assoc)
  apply (metis (no_types, lifting) t(2) a_def comp_assoc)
  done

  have limsup (u o r) ≤ limsup u by (simp add: limsup_subseq_mono r(1))
  then have a: limsup (u o r) ≠ ∞ using ‹limsup u < ∞› by auto
  have limsup (v o r o s) ≤ limsup v
    by (simp add: comp_assoc limsup_subseq_mono r(1) s(1) strict_mono_o)
  then have b: limsup (v o r o s) ≠ ∞ using ‹limsup v < ∞› by auto

  have (λn. (u o a) n + (v o a) n) ⟶ limsup (u o r) + limsup (v o r o s)
    using l tendsto_add_ereal_general a b by fastforce
  moreover have (λn. (u o a) n + (v o a) n) = (w o a) unfolding w_def by
auto
  ultimately have (w o a) ⟶ limsup (u o r) + limsup (v o r o s) by simp
```

then have *limsup w* = *limsup* (*u o r*) + *limsup* (*v o r o s*) **using** *l*(*1*) *LIMSEQ_unique* **by** *blast*
then have *limsup w* ≤ *limsup u* + *limsup v*
using ‹*limsup* (*u o r*) ≤ *limsup u*› ‹*limsup* (*v o r o s*) ≤ *limsup v*› *add_mono* **by** *simp*
then show *?thesis* **unfolding** *w_def* **by** *simp*
qed

There is an asymmetry between liminfs and limsups in *ereal*, as ∞ + (−∞) = ∞. This explains why there are more assumptions in the next lemma dealing with liminfs that in the previous one about limsups.

lemma *ereal_liminf_add_mono*:
fixes *u v*::*nat* ⇒ *ereal*
assumes ¬((*liminf u* = ∞ ∧ *liminf v* = −∞) ∨ (*liminf u* = −∞ ∧ *liminf v* = ∞))
shows *liminf* (λ*n*. *u n* + *v n*) ≥ *liminf u* + *liminf v*
proof (*cases*)
assume (*liminf u* = −∞) ∨ (*liminf v* = −∞)
then have ∗: *liminf u* + *liminf v* = −∞ **using** *assms* **by** *auto*
show *?thesis* **by** (*simp add*: ∗)
next
assume ¬((*liminf u* = −∞) ∨ (*liminf v* = −∞))
then have *liminf u* > −∞ *liminf v* > −∞ **by** *auto*

define *w* **where** *w* = (λ*n*. *u n* + *v n*)
obtain *r* **where** *r*: *strict_mono r* (*w o r*) ⟶ *liminf w* **using** *liminf_subseq_lim* **by** *auto*
obtain *s* **where** *s*: *strict_mono s* (*u o r o s*) ⟶ *liminf* (*u o r*) **using** *liminf_subseq_lim* **by** *auto*
obtain *t* **where** *t*: *strict_mono t* (*v o r o s o t*) ⟶ *liminf* (*v o r o s*) **using** *liminf_subseq_lim* **by** *auto*

define *a* **where** *a* = *r o s o t*
have *strict_mono a* **using** *r s t* **by** (*simp add*: *a_def strict_mono_o*)
have *l*:(*w o a*) ⟶ *liminf w*
(*u o a*) ⟶ *liminf* (*u o r*)
(*v o a*) ⟶ *liminf* (*v o r o s*)
apply (*metis* (*no_types*, *lifting*) *r*(*2*) *s*(*1*) *t*(*1*) *LIMSEQ_subseq_LIMSEQ a_def comp_assoc*)
apply (*metis* (*no_types*, *lifting*) *s*(*2*) *t*(*1*) *LIMSEQ_subseq_LIMSEQ a_def comp_assoc*)
apply (*metis* (*no_types*, *lifting*) *t*(*2*) *a_def comp_assoc*)
done

have *liminf* (*u o r*) ≥ *liminf u* **by** (*simp add*: *liminf_subseq_mono r*(*1*))
then have *a*: *liminf* (*u o r*) ≠ −∞ **using** ‹*liminf u* > −∞› **by** *auto*
have *liminf* (*v o r o s*) ≥ *liminf v*
by (*simp add*: *comp_assoc liminf_subseq_mono r*(*1*) *s*(*1*) *strict_mono_o*)
then have *b*: *liminf* (*v o r o s*) ≠ −∞ **using** ‹*liminf v* > −∞› **by** *auto*

have (λ*n*. (*u o a*) *n* + (*v o a*) *n*) ⟶ *liminf* (*u o r*) + *liminf* (*v o r o s*)
using *l tendsto_add_ereal_general a b* **by** *fastforce*
moreover have (λ*n*. (*u o a*) *n* + (*v o a*) *n*) = (*w o a*) **unfolding** *w_def* **by** *auto*
ultimately have (*w o a*) ⟶ *liminf* (*u o r*) + *liminf* (*v o r o s*) **by** *simp*
then have *liminf w* = *liminf* (*u o r*) + *liminf* (*v o r o s*) **using** *l*(*1*) *LIMSEQ_unique* **by** *blast*
then have *liminf w* ≥ *liminf u* + *liminf v*
using ‹*liminf* (*u o r*) ≥ *liminf u*› ‹*liminf* (*v o r o s*) ≥ *liminf v*› *add_mono* **by** *simp*
then show *?thesis* **unfolding** *w_def* **by** *simp*
qed

lemma *ereal_limsup_lim_add*:
fixes *u v*::*nat* ⇒ *ereal*
assumes *u* ⟶ *a abs*(*a*) ≠ ∞
shows *limsup* (λ*n*. *u n* + *v n*) = *a* + *limsup v*
proof −
have *limsup u* = *a* **using** *assms*(*1*) **using** *tendsto_iff_Liminf_eq_Limsup trivial_limit_at_top_linorder* **by** *blast*
have (λ*n*. −*u n*) ⟶ −*a* **using** *assms*(*1*) **by** *auto*
then have *limsup* (λ*n*. −*u n*) = −*a* **using** *tendsto_iff_Liminf_eq_Limsup trivial_limit_at_top_linorder* **by** *blast*

have *limsup* (λ*n*. *u n* + *v n*) ≤ *limsup u* + *limsup v*
by (*rule ereal_limsup_add_mono*)
then have *up*: *limsup* (λ*n*. *u n* + *v n*) ≤ *a* + *limsup v* **using** ‹*limsup u* = *a*› **by** *simp*

have *a*: *limsup* (λ*n*. (*u n* + *v n*) + (−*u n*)) ≤ *limsup* (λ*n*. *u n* + *v n*) + *limsup* (λ*n*. −*u n*)
by (*rule ereal_limsup_add_mono*)
have *eventually* (λ*n*. *u n* = *ereal*(*real_of_ereal*(*u n*))) *sequentially* **using** *assms real_lim_then_eventually_real* **by** *auto*
moreover have ⋀*x*. *x* = *ereal*(*real_of_ereal*(*x*)) ⟹ *x* + (−*x*) = *0*
by (*metis plus_ereal.simps*(*1*) *right_minus uminus_ereal.simps*(*1*) *zero_ereal_def*)
ultimately have *eventually* (λ*n*. *u n* + (−*u n*) = *0*) *sequentially*
by (*metis* (*mono_tags*, *lifting*) *eventually_mono*)
moreover have ⋀*n*. *u n* + (−*u n*) = *0* ⟹ *u n* + *v n* + (−*u n*) = *v n*
by (*metis add.commute add.left_commute add.left_neutral*)
ultimately have *eventually* (λ*n*. *u n* + *v n* + (−*u n*) = *v n*) *sequentially*
using *eventually_mono* **by** *force*
then have *limsup v* = *limsup* (λ*n*. *u n* + *v n* + (−*u n*)) **using** *Limsup_eq* **by** *force*
then have *limsup v* ≤ *limsup* (λ*n*. *u n* + *v n*) −*a* **using** *a* ‹*limsup* (λ*n*. −*u n*) = −*a*› **by** (*simp add*: *minus_ereal_def*)
then have *limsup* (λ*n*. *u n* + *v n*) ≥ *a* + *limsup v* **using** *assms*(*2*) **by** (*metis add.commute ereal_le_minus*)
then show *?thesis* **using** *up* **by** *simp*

qed

lemma *ereal_limsup_lim_mult*:
fixes *u v*::*nat* ⇒ *ereal*
assumes $u \longrightarrow a$ $a>0$ $a \neq \infty$
shows *limsup* $(\lambda n.\ u\ n * v\ n) = a *$ *limsup v*
proof −
define *w* **where** $w = (\lambda n.\ u\ n * v\ n)$
obtain *r* **where** *r*: *strict_mono r* $(v\ o\ r) \longrightarrow$ *limsup v* **using** *limsup_subseq_lim* **by** *auto*
have $(u\ o\ r) \longrightarrow a$ **using** *assms*(*1*) *LIMSEQ_subseq_LIMSEQ r* **by** *auto*
with *tendsto_mult_ereal*[*OF this r*(*2*)] **have** $(\lambda n.\ (u\ o\ r)\ n * (v\ o\ r)\ n) \longrightarrow a *$ *limsup v* **using** *assms*(*2*) *assms*(*3*) **by** *auto*
moreover have $\bigwedge n.\ (w\ o\ r)\ n = (u\ o\ r)\ n * (v\ o\ r)\ n$ **unfolding** *w_def* **by** *auto*
ultimately have $(w\ o\ r) \longrightarrow a *$ *limsup v* **unfolding** *w_def* **by** *presburger*
then have *limsup* $(w\ o\ r) = a *$ *limsup v* **by** (*simp add*: *tendsto_iff_Liminf_eq_Limsup*)
then have *I*: *limsup w* $\geq a *$ *limsup v* **by** (*metis limsup_subseq_mono r*(*1*))

obtain *s* **where** *s*: *strict_mono s* $(w\ o\ s) \longrightarrow$ *limsup w* **using** *limsup_subseq_lim* **by** *auto*
have $*$: $(u\ o\ s) \longrightarrow a$ **using** *assms*(*1*) *LIMSEQ_subseq_LIMSEQ s* **by** *auto*
have *eventually* $(\lambda n.\ (u\ o\ s)\ n > 0)$ *sequentially* **using** *assms*(*2*) $*$ *order_tendsto_iff* **by** *blast*
moreover have *eventually* $(\lambda n.\ (u\ o\ s)\ n < \infty)$ *sequentially* **using** *assms*(*3*) $*$ *order_tendsto_iff* **by** *blast*
moreover have $(w\ o\ s)\ n\ /\ (u\ o\ s)\ n = (v\ o\ s)\ n$ **if** $(u\ o\ s)\ n > 0$ $(u\ o\ s)\ n < \infty$ **for** *n*
unfolding *w_def* **using** *that* **by** (*auto simp add*: *ereal_divide_eq*)
ultimately have *eventually* $(\lambda n.\ (w\ o\ s)\ n\ /\ (u\ o\ s)\ n = (v\ o\ s)\ n)$ *sequentially* **using** *eventually_elim2* **by** *force*
moreover have $(\lambda n.\ (w\ o\ s)\ n\ /\ (u\ o\ s)\ n) \longrightarrow$ (*limsup w*) / *a*
apply (*rule tendsto_divide_ereal*[*OF s*(*2*) $*$]) **using** *assms*(*2*) *assms*(*3*) **by** *auto*
ultimately have $(v\ o\ s) \longrightarrow$ (*limsup w*) / *a* **using** *Lim_transform_eventually* **by** *fastforce*
then have *limsup* $(v\ o\ s) =$ (*limsup w*) / *a* **by** (*simp add*: *tendsto_iff_Liminf_eq_Limsup*)
then have *limsup v* $\geq$ (*limsup w*) / *a* **by** (*metis limsup_subseq_mono s*(*1*))
then have $a *$ *limsup v* $\geq$ *limsup w* **using** *assms*(*2*) *assms*(*3*) **by** (*simp add*: *ereal_divide_le_pos*)
then show *?thesis* **using** *I* **unfolding** *w_def* **by** *auto*
qed

lemma *ereal_liminf_lim_mult*:
fixes *u v*::*nat* ⇒ *ereal*
assumes $u \longrightarrow a$ $a>0$ $a \neq \infty$
shows *liminf* $(\lambda n.\ u\ n * v\ n) = a *$ *liminf v*
proof −
define *w* **where** $w = (\lambda n.\ u\ n * v\ n)$
obtain *r* **where** *r*: *strict_mono r* $(v\ o\ r) \longrightarrow$ *liminf v* **using** *liminf_subseq_lim*

by *auto*
have $(u\ o\ r) \longrightarrow a$ **using** *assms(1) LIMSEQ_subseq_LIMSEQ r* **by** *auto*
with *tendsto_mult_ereal[OF this r(2)]* **have** $(\lambda n.\ (u\ o\ r)\ n * (v\ o\ r)\ n) \longrightarrow a * liminf\ v$ **using** *assms(2) assms(3)* **by** *auto*
moreover have $\bigwedge n.\ (w\ o\ r)\ n = (u\ o\ r)\ n * (v\ o\ r)\ n$ **unfolding** *w_def* **by** *auto*
ultimately have $(w\ o\ r) \longrightarrow a * liminf\ v$ **unfolding** *w_def* **by** *presburger*
then have $liminf\ (w\ o\ r) = a * liminf\ v$ **by** *(simp add: tendsto_iff_Liminf_eq_Limsup)*
then have *I*: $liminf\ w \leq a * liminf\ v$ **by** *(metis liminf_subseq_mono r(1))*

obtain *s* **where** *s*: *strict_mono s* $(w\ o\ s) \longrightarrow liminf\ w$ **using** *liminf_subseq_lim* **by** *auto*
have $*$: $(u\ o\ s) \longrightarrow a$ **using** *assms(1) LIMSEQ_subseq_LIMSEQ s* **by** *auto*
have *eventually* $(\lambda n.\ (u\ o\ s)\ n > 0)$ *sequentially* **using** *assms(2)* $*$ *order_tendsto_iff* **by** *blast*
moreover have *eventually* $(\lambda n.\ (u\ o\ s)\ n < \infty)$ *sequentially* **using** *assms(3)* $*$ *order_tendsto_iff* **by** *blast*
moreover have $(w\ o\ s)\ n\ /\ (u\ o\ s)\ n = (v\ o\ s)\ n$ **if** $(u\ o\ s)\ n > 0$ $(u\ o\ s)\ n < \infty$ **for** *n*
unfolding *w_def* **using** *that* **by** *(auto simp add: ereal_divide_eq)*
ultimately have *eventually* $(\lambda n.\ (w\ o\ s)\ n\ /\ (u\ o\ s)\ n = (v\ o\ s)\ n)$ *sequentially* **using** *eventually_elim2* **by** *force*
moreover have $(\lambda n.\ (w\ o\ s)\ n\ /\ (u\ o\ s)\ n) \longrightarrow (liminf\ w)\ /\ a$
apply *(rule tendsto_divide_ereal[OF s(2) *])* **using** *assms(2) assms(3)* **by** *auto*
ultimately have $(v\ o\ s) \longrightarrow (liminf\ w)\ /\ a$ **using** *Lim_transform_eventually* **by** *fastforce*
then have $liminf\ (v\ o\ s) = (liminf\ w)\ /\ a$ **by** *(simp add: tendsto_iff_Liminf_eq_Limsup)*
then have $liminf\ v \leq (liminf\ w)\ /\ a$ **by** *(metis liminf_subseq_mono s(1))*
then have $a * liminf\ v \leq liminf\ w$ **using** *assms(2) assms(3)* **by** *(simp add: ereal_le_divide_pos)*
then show *?thesis* **using** *I* **unfolding** *w_def* **by** *auto*
qed

lemma *ereal_liminf_lim_add*:
fixes *u v::nat* $\Rightarrow$ *ereal*
assumes $u \longrightarrow a$ $abs(a) \neq \infty$
shows $liminf\ (\lambda n.\ u\ n + v\ n) = a + liminf\ v$
proof −
have $liminf\ u = a$ **using** *assms(1) tendsto_iff_Liminf_eq_Limsup trivial_limit_at_top_linorder* **by** *blast*
then have $*$: $abs(liminf\ u) \neq \infty$ **using** *assms(2)* **by** *auto*
have $(\lambda n.\ -u\ n) \longrightarrow -a$ **using** *assms(1)* **by** *auto*
then have $liminf\ (\lambda n.\ -u\ n) = -a$ **using** *tendsto_iff_Liminf_eq_Limsup trivial_limit_at_top_linorder* **by** *blast*
then have $**$: $abs(liminf\ (\lambda n.\ -u\ n)) \neq \infty$ **using** *assms(2)* **by** *auto*

have $liminf\ (\lambda n.\ u\ n + v\ n) \geq liminf\ u + liminf\ v$
apply *(rule ereal_liminf_add_mono)* **using** $*$ **by** *auto*
then have *up*: $liminf\ (\lambda n.\ u\ n + v\ n) \geq a + liminf\ v$ **using** ‹$liminf\ u = a$› **by**

simp

have *a*: *liminf* (λ*n*. (*u n* + *v n*) + (−*u n*)) ≥ *liminf* (λ*n*. *u n* + *v n*) + *liminf* (λ*n*. −*u n*)
apply (*rule ereal_liminf_add_mono*) **using** ∗∗ **by** *auto*
have *eventually* (λ*n*. *u n* = *ereal*(*real_of_ereal*(*u n*))) *sequentially* **using** *assms real_lim_then_eventually_real* **by** *auto*
moreover have ⋀*x*. *x* = *ereal*(*real_of_ereal*(*x*)) ⟹ *x* + (−*x*) = *0*
by (*metis plus_ereal.simps*(*1*) *right_minus uminus_ereal.simps*(*1*) *zero_ereal_def*)
ultimately have *eventually* (λ*n*. *u n* + (−*u n*) = *0*) *sequentially*
by (*metis* (*mono_tags*, *lifting*) *eventually_mono*)
moreover have ⋀*n*. *u n* + (−*u n*) = *0* ⟹ *u n* + *v n* + (−*u n*) = *v n*
by (*metis add.commute add.left_commute add.left_neutral*)
ultimately have *eventually* (λ*n*. *u n* + *v n* + (−*u n*) = *v n*) *sequentially*
using *eventually_mono* **by** *force*
then have *liminf v* = *liminf* (λ*n*. *u n* + *v n* + (−*u n*)) **using** *Liminf_eq* **by** *force*
then have *liminf v* ≥ *liminf* (λ*n*. *u n* + *v n*) −*a* **using** *a* ‹*liminf* (λ*n*. −*u n*) = −*a*› **by** (*simp add*: *minus_ereal_def*)
then have *liminf* (λ*n*. *u n* + *v n*) ≤ *a* + *liminf v* **using** *assms*(*2*) **by** (*metis add.commute ereal_minus_le*)
then show *?thesis* **using** *up* **by** *simp*
qed

lemma *ereal_liminf_limsup_add*:
fixes *u v*::*nat* ⇒ *ereal*
shows *liminf* (λ*n*. *u n* + *v n*) ≤ *liminf u* + *limsup v*
proof (*cases*)
assume *limsup v* = ∞ ∨ *liminf u* = ∞
then show *?thesis* **by** *auto*
next
assume ¬(*limsup v* = ∞ ∨ *liminf u* = ∞)
then have *limsup v* < ∞ *liminf u* < ∞ **by** *auto*

define *w* **where** *w* = (λ*n*. *u n* + *v n*)
obtain *r* **where** *r*: *strict_mono r* (*u o r*) ⟶ *liminf u* **using** *liminf_subseq_lim* **by** *auto*
obtain *s* **where** *s*: *strict_mono s* (*w o r o s*) ⟶ *liminf* (*w o r*) **using** *liminf_subseq_lim* **by** *auto*
obtain *t* **where** *t*: *strict_mono t* (*v o r o s o t*) ⟶ *limsup* (*v o r o s*) **using** *limsup_subseq_lim* **by** *auto*

define *a* **where** *a* = *r o s o t*
have *strict_mono a* **using** *r s t* **by** (*simp add*: *a_def strict_mono_o*)
have *l*:(*u o a*) ⟶ *liminf u*
(*w o a*) ⟶ *liminf* (*w o r*)
(*v o a*) ⟶ *limsup* (*v o r o s*)
apply (*metis* (*no_types*, *lifting*) *r*(*2*) *s*(*1*) *t*(*1*) *LIMSEQ_subseq_LIMSEQ a_def comp_assoc*)

```
 apply (metis (no_types, lifting) s(2) t(1) LIMSEQ_subseq_LIMSEQ a_def comp_assoc)
 apply (metis (no_types, lifting) t(2) a_def comp_assoc)
 done

 have liminf (w o r) ≥ liminf w by (simp add: liminf_subseq_mono r(1))
 have limsup (v o r o s) ≤ limsup v
   by (simp add: comp_assoc limsup_subseq_mono r(1) s(1) strict_mono_o)
 then have b: limsup (v o r o s) < ∞ using ‹limsup v < ∞› by auto

 have (λn. (u o a) n + (v o a) n) ⟶ liminf u + limsup (v o r o s)
   apply (rule tendsto_add_ereal_general) using b ‹liminf u < ∞› l(1) l(3) by force+
 moreover have (λn. (u o a) n + (v o a) n) = (w o a) unfolding w_def by auto
 ultimately have (w o a) ⟶ liminf u + limsup (v o r o s) by simp
 then have liminf (w o r) = liminf u + limsup (v o r o s) using l(2) using LIMSEQ_unique by blast
 then have liminf w ≤ liminf u + limsup v
   using ‹liminf (w o r) ≥ liminf w› ‹limsup (v o r o s) ≤ limsup v›
   by (metis add_mono_thms_linordered_semiring(2) le_less_trans not_less)
 then show ?thesis unfolding w_def by simp
qed

lemma ereal_liminf_limsup_minus:
 fixes u v::nat ⇒ ereal
 shows liminf (λn. u n − v n) ≤ limsup u − limsup v
 unfolding minus_ereal_def
 apply (subst add.commute)
 apply (rule order_trans[OF ereal_liminf_limsup_add])
 using ereal_Limsup_uminus[of sequentially λn. − v n]
 apply (simp add: add.commute)
 done


lemma liminf_minus_ennreal:
 fixes u v::nat ⇒ ennreal
 shows (⋀n. v n ≤ u n) ⟹ liminf (λn. u n − v n) ≤ limsup u − limsup v
 unfolding liminf_SUP_INF limsup_INF_SUP
 including ennreal.lifting
proof (transfer, clarsimp)
 fix v u :: nat ⇒ ereal assume *: ∀x. 0 ≤ v x ∀x. 0 ≤ u x ⋀n. v n ≤ u n
 moreover have 0 ≤ limsup u − limsup v
   using * by (intro ereal_diff_positive Limsup_mono always_eventually) simp
 moreover have 0 ≤ Sup (u ‘ {x..}) for x
   using * by (intro SUP_upper2[of x]) auto
 moreover have 0 ≤ Sup (v ‘ {x..}) for x
   using * by (intro SUP_upper2[of x]) auto
 ultimately show (SUP n. INF n∈{n..}. max 0 (u n − v n))
         ≤ max 0 ((INF x. max 0 (Sup (u ‘ {x..}))) − (INF x. max 0 (Sup (v ‘
```

{*x*..}))))
by (*auto simp*: ∗ *ereal_diff_positive max.absorb2 liminf_SUP_INF*[*symmetric*] *limsup_INF_SUP*[*symmetric*] *ereal_liminf_limsup_minus*)
qed

4.5.4 Relate extended reals and the indicator function

lemma *ereal_indicator_le_0*: (*indicator S x*::*ereal*) ≤ *0* ⟷ *x* ∉ *S*
by (*auto split*: *split_indicator simp*: *one_ereal_def*)

lemma *ereal_indicator*: *ereal* (*indicator A x*) = *indicator A x*
by (*auto simp*: *indicator_def one_ereal_def*)

lemma *ereal_mult_indicator*: *ereal* (*x* ∗ *indicator A y*) = *ereal x* ∗ *indicator A y*
by (*simp split*: *split_indicator*)

lemma *ereal_indicator_mult*: *ereal* (*indicator A y* ∗ *x*) = *indicator A y* ∗ *ereal x*
by (*simp split*: *split_indicator*)

lemma *ereal_indicator_nonneg*[*simp*, *intro*]: *0* ≤ (*indicator A x* ::*ereal*)
unfolding *indicator_def* **by** *auto*

lemma *indicator_inter_arith_ereal*: *indicator A x* ∗ *indicator B x* = (*indicator* (*A* ∩ *B*) *x* :: *ereal*)
by (*simp split*: *split_indicator*)

end

4.6 Radius of Convergence and Summation Tests

theory *Summation_Tests*
imports
Complex_Main
HOL−Library.Discrete
HOL−Library.Extended_Real
HOL−Library.Liminf_Limsup
Extended_Real_Limits
begin

The definition of the radius of convergence of a power series, various summability tests, lemmas to compute the radius of convergence etc.

4.6.1 Convergence tests for infinite sums

Root test

lemma *limsup_root_powser*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real_normed_div_algebra*}
shows *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n* ∗ *z* ˆ *n*)))) =

limsup (λ*n*. *ereal* (*root n* (*norm* (*f n*)))) ∗ *ereal* (*norm z*)
proof −
have *A*: (λ*n*. *ereal* (*root n* (*norm* (*f n* ∗ *z* ˆ *n*)))) =
(λ*n*. *ereal* (*root n* (*norm* (*f n*))) ∗ *ereal* (*norm z*)) (**is** *?g* = *?h*)
proof
fix *n* **show** *?g n* = *?h n*
by (*cases n* = *0*) (*simp_all add*: *norm_mult real_root_mult real_root_pos2 norm_power*)
qed
show *?thesis* **by** (*subst A*, *subst limsup_ereal_mult_right*) *simp_all*
qed

lemma *limsup_root_limit*:
assumes (λ*n*. *ereal* (*root n* (*norm* (*f n*)))) ⟶ *l* (**is** *?g* ⟶ _)
shows *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*)))) = *l*
proof −
from *assms* **have** *convergent ?g lim ?g* = *l*
unfolding *convergent_def* **by** (*blast intro*: *limI*)+
with *convergent_limsup_cl* **show** *?thesis* **by** *force*
qed

lemma *limsup_root_limit′*:
assumes (λ*n*. *root n* (*norm* (*f n*))) ⟶ *l*
shows *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*)))) = *ereal l*
by (*intro limsup_root_limit tendsto_ereal assms*)

theorem *root_test_convergence′*:
fixes *f* :: *nat* ⇒ *'a* :: *banach*
defines *l* ≡ *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*))))
assumes *l*: *l* < *1*
shows *summable f*
proof −
have *0* = *limsup* (λ*n*. *0*) **by** (*simp add*: *Limsup_const*)
also have ... ≤ *l* **unfolding** *l_def* **by** (*intro Limsup_mono*) (*simp_all add*: *real_root_ge_zero*)
finally have *l* ≥ *0* **by** *simp*
with *l* **obtain** *l′* **where** *l′*: *l* = *ereal l′* **by** (*cases l*) *simp_all*

define *c* **where** *c* = (*1* − *l′*) / *2*
from *l* **and** ‹*l* ≥ *0*› **have** *c*: *l* + *c* > *l l′* + *c* ≥ *0 l′* + *c* < *1* **unfolding** *c_def*
by (*simp_all add*: *field_simps l′*)
have ∀ *C*>*l*. *eventually* (λ*n*. *ereal* (*root n* (*norm* (*f n*))) < *C*) *sequentially*
by (*subst Limsup_le_iff* [*symmetric*]) (*simp add*: *l_def*)
with *c* **have** *eventually* (λ*n*. *ereal* (*root n* (*norm* (*f n*))) < *l* + *ereal c*) *sequentially* **by** *simp*
with *eventually_gt_at_top*[*of 0*::*nat*]
have *eventually* (λ*n*. *norm* (*f n*) ≤ (*l′* + *c*) ˆ *n*) *sequentially*
proof *eventually_elim*
fix *n* :: *nat* **assume** *n*: *n* > *0*
assume *ereal* (*root n* (*norm* (*f n*))) < *l* + *ereal c*

```
    hence root n (norm (f n)) ≤ l′ + c by (simp add: l′)
    with c n have root n (norm (f n)) ^ n ≤ (l′ + c) ^ n
      by (intro power_mono) (simp_all add: real_root_ge_zero)
    also from n have root n (norm (f n)) ^ n = norm (f n) by simp
    finally show norm (f n) ≤ (l′ + c) ^ n by simp
  qed
  thus ?thesis
    by (rule summable_comparison_test_ev[OF _ summable_geometric]) (simp add: c)
qed

theorem root_test_divergence:
  fixes f :: nat ⇒ 'a :: banach
  defines l ≡ limsup (λn. ereal (root n (norm (f n))))
  assumes l: l > 1
  shows ¬summable f
proof
  assume summable f
  hence bounded: Bseq f by (simp add: summable_imp_Bseq)

  have 0 = limsup (λn. 0) by (simp add: Limsup_const)
  also have ... ≤ l unfolding l_def by (intro Limsup_mono) (simp_all add: real_root_ge_zero)
  finally have l_nonneg: l ≥ 0 by simp

  define c where c = (if l = ∞ then 2 else 1 + (real_of_ereal l − 1) / 2)
  from l l_nonneg consider l = ∞ | ∃ l′. l = ereal l′ by (cases l) simp_all
  hence c: c > 1 ∧ ereal c < l by cases (insert l, auto simp: c_def field_simps)

  have unbounded: ¬bdd_above {n. root n (norm (f n)) > c}
  proof
    assume bdd_above {n. root n (norm (f n)) > c}
    then obtain N where ∀ n. root n (norm (f n)) > c ⟶ n ≤ N unfolding bdd_above_def by blast
    hence ∃ N. ∀ n≥N. root n (norm (f n)) ≤ c
      by (intro exI[of _ N + 1]) (force simp: not_less_eq_eq[symmetric])
    hence eventually (λn. root n (norm (f n)) ≤ c) sequentially
      by (auto simp: eventually_at_top_linorder)
    hence l ≤ c unfolding l_def by (intro Limsup_bounded) simp_all
    with c show False by auto
  qed

  from bounded obtain K where K: K > 0 ⋀n. norm (f n) ≤ K using BseqE by blast
  define n where n = nat ⌈log c K⌉
  from unbounded have ∃ m>n. c < root m (norm (f m)) unfolding bdd_above_def
    by (auto simp: not_le)
  then guess m by (elim exE conjE) note m = this
  from c K have K = c powr log c K by (simp add: powr_def log_def)
```

also from *c* **have** *c powr log c K* ≤ *c powr real n* **unfolding** *n_def*
by (*intro powr_mono, linarith, simp*)
finally have $K \leq c$ ˆ *n* **using** *c* **by** (*simp add: powr_realpow*)
also from *c m* **have** *c* ˆ *n* < *c* ˆ *m* **by** *simp*
also from *c m* **have** *c* ˆ *m* < *root m* (*norm* (*f m*)) ˆ *m* **by** (*intro power_strict_mono*) *simp_all*
also from *m* **have** ... = *norm* (*f m*) **by** *simp*
finally show *False* **using** *K(2)[of m]* **by** *simp*
qed

Cauchy's condensation test

context
fixes *f* :: *nat* ⇒ *real*
begin

private lemma *condensation_inequality*:
assumes *mono*: ⋀*m n. 0* < *m* ⟹ *m* ≤ *n* ⟹ *f n* ≤ *f m*
shows (∑*k=1..<n. f k*) ≥ (∑*k=1..<n. f* (*2* ∗ *2* ˆ *Discrete.log k*)) (**is** *?thesis1*)
(∑*k=1..<n. f k*) ≤ (∑*k=1..<n. f* (*2* ˆ *Discrete.log k*)) (**is** *?thesis2*)
by (*intro sum_mono mono Discrete.log_exp2_ge Discrete.log_exp2_le, simp, simp*)+

private lemma *condensation_condense1*: (∑*k=1..<2ˆn. f* (*2* ˆ *Discrete.log k*)) = (∑*k<n. 2ˆk* ∗ *f* (*2* ˆ *k*))
proof (*induction n*)
case (*Suc n*)
have {*1..<2ˆSuc n*} = {*1..<2ˆn*} ∪ {*2ˆn..<*(*2ˆSuc n* :: *nat*)} **by** *auto*
also have (∑*k*∈.... *f* (*2* ˆ *Discrete.log k*)) =
(∑*k<n. 2ˆk* ∗ *f* (*2ˆk*)) + (∑*k* = *2ˆn..<2ˆSuc n. f* (*2ˆDiscrete.log k*))
by (*subst sum.union_disjoint*) (*insert Suc, auto*)
also have *Discrete.log k* = *n* **if** *k* ∈ {*2ˆn..<2ˆSuc n*} **for** *k* **using** *that* **by** (*intro Discrete.log_eqI*) *simp_all*
hence (∑*k* = *2ˆn..<2ˆSuc n. f* (*2ˆDiscrete.log k*)) = (∑(_::*nat*) = *2ˆn..<2ˆSuc n. f* (*2ˆn*))
by (*intro sum.cong*) *simp_all*
also have ... = *2ˆn* ∗ *f* (*2ˆn*) **by** (*simp*)
finally show *?case* **by** *simp*
qed *simp*

private lemma *condensation_condense2*: (∑*k=1..<2ˆn. f* (*2* ∗ *2* ˆ *Discrete.log k*)) = (∑*k<n. 2ˆk* ∗ *f* (*2* ˆ *Suc k*))
proof (*induction n*)
case (*Suc n*)
have {*1..<2ˆSuc n*} = {*1..<2ˆn*} ∪ {*2ˆn..<*(*2ˆSuc n* :: *nat*)} **by** *auto*
also have (∑*k*∈.... *f* (*2* ∗ *2* ˆ *Discrete.log k*)) =
(∑*k<n. 2ˆk* ∗ *f* (*2ˆSuc k*)) + (∑*k* = *2ˆn..<2ˆSuc n. f* (*2* ∗ *2ˆDiscrete.log k*))
by (*subst sum.union_disjoint*) (*insert Suc, auto*)

also have *Discrete.log* $k = n$ **if** $k \in \{2\hat{}n..<2\hat{}Suc\ n\}$ **for** k **using** *that* **by** (*intro Discrete.log_eqI*) *simp_all*
hence $(\sum k = 2\hat{}n..<2\hat{}Suc\ n.\ f\ (2*2\hat{}Discrete.log\ k)) = (\sum (\_::nat) = 2\hat{}n..<2\hat{}Suc\ n.\ f\ (2\hat{}Suc\ n))$
by (*intro sum.cong*) *simp_all*
also have $\ldots = 2\hat{}n * f\ (2\hat{}Suc\ n)$ **by** (*simp*)
finally show *?case* **by** *simp*
qed *simp*

theorem *condensation_test*:
assumes *mono*: $\bigwedge m.\ 0 < m \Longrightarrow f\ (Suc\ m) \leq f\ m$
assumes *nonneg*: $\bigwedge n.\ f\ n \geq 0$
shows *summable f* $\longleftrightarrow$ *summable* $(\lambda n.\ 2\hat{}n * f\ (2\hat{}n))$
proof −
define f' **where** $f'\ n = (if\ n = 0\ then\ 0\ else\ f\ n)$ **for** n
from *mono* **have** *mono'*: *decseq* $(\lambda n.\ f\ (Suc\ n))$ **by** (*intro decseq_SucI*) *simp*
hence *mono'*: $f\ n \leq f\ m$ **if** $m \leq n\ m > 0$ **for** $m\ n$
using *that decseqD*[*OF mono'*, *of* $m - 1\ n - 1$] **by** *simp*

have $(\lambda n.\ f\ (Suc\ n)) = (\lambda n.\ f'\ (Suc\ n))$ **by** (*intro ext*) (*simp add*: *f'_def*)
hence *summable f* $\longleftrightarrow$ *summable f'*
by (*subst* (*1 2*) *summable_Suc_iff* [*symmetric*]) (*simp only*:)
also have $\ldots \longleftrightarrow$ *convergent* $(\lambda n.\ \sum k<n.\ f'\ k)$ **unfolding** *summable_iff_convergent* ..
also have *monoseq* $(\lambda n.\ \sum k<n.\ f'\ k)$ **unfolding** *f'_def*
by (*intro mono_SucI1*) (*auto intro*!: *mult_nonneg_nonneg nonneg*)
hence *convergent* $(\lambda n.\ \sum k<n.\ f'\ k) \longleftrightarrow$ *Bseq* $(\lambda n.\ \sum k<n.\ f'\ k)$
by (*rule monoseq_imp_convergent_iff_Bseq*)
also have $\ldots \longleftrightarrow$ *Bseq* $(\lambda n.\ \sum k=1..<n.\ f'\ k)$ **unfolding** *One_nat_def*
by (*subst sum_shift_lb_Suc0_0_upt*) (*simp_all add*: *f'_def atLeast0LessThan*)
also have $\ldots \longleftrightarrow$ *Bseq* $(\lambda n.\ \sum k=1..<n.\ f\ k)$ **unfolding** *f'_def* **by** *simp*
also have $\ldots \longleftrightarrow$ *Bseq* $(\lambda n.\ \sum k=1..<2\hat{}n.\ f\ k)$
by (*rule nonneg_incseq_Bseq_subseq_iff*[*symmetric*])
(*auto intro*!: *sum_nonneg incseq_SucI nonneg simp*: *strict_mono_def*)
also have $\ldots \longleftrightarrow$ *Bseq* $(\lambda n.\ \sum k<n.\ 2\hat{}k * f\ (2\hat{}k))$
proof (*intro iffI*)
assume *A*: *Bseq* $(\lambda n.\ \sum k=1..<2\hat{}n.\ f\ k)$
have *eventually* $(\lambda n.\ norm\ (\sum k<n.\ 2\hat{}k * f\ (2\hat{}Suc\ k)) \leq norm\ (\sum k=1..<2\hat{}n.\ f\ k))$ *sequentially*
proof (*intro always_eventually allI*)
fix n :: *nat*
have $norm\ (\sum k<n.\ 2\hat{}k * f\ (2\hat{}Suc\ k)) = (\sum k<n.\ 2\hat{}k * f\ (2\hat{}Suc\ k))$ **unfolding** *real_norm_def*
by (*intro abs_of_nonneg sum_nonneg ballI mult_nonneg_nonneg nonneg*) *simp_all*
also have $\ldots \leq (\sum k=1..<2\hat{}n.\ f\ k)$
by (*subst condensation_condense2* [*symmetric*]) (*intro condensation_inequality mono'*)
also have $\ldots = norm \ldots$ **unfolding** *real_norm_def*

by (*intro abs_of_nonneg*[*symmetric*] *sum_nonneg ballI mult_nonneg_nonneg nonneg*)
finally show *norm* (∑*k*<*n*. *2* ^ *k* ∗ *f* (*2* ^ *Suc k*)) ≤ *norm* (∑*k*=*1*..<*2*^*n*. *f k*) **.**
qed
from *this* **and** *A* **have** *Bseq* (λ*n*. ∑*k*<*n*. *2*^*k* ∗ *f* (*2*^*Suc k*)) **by** (*rule Bseq_eventually_mono*)
from *Bseq_mult*[*OF Bfun_const*[*of 2*] *this*] **have** *Bseq* (λ*n*. ∑*k*<*n*. *2*^*Suc k* ∗ *f* (*2*^*Suc k*))
by (*simp add*: *sum_distrib_left sum_distrib_right mult_ac*)
hence *Bseq* (λ*n*. (∑*k*=*Suc 0*..<*Suc n*. *2*^*k* ∗ *f* (*2*^*k*)) + *f 1*)
by (*intro Bseq_add*, *subst sum.shift_bounds_Suc_ivl*) (*simp add*: *atLeast0LessThan*)
hence *Bseq* (λ*n*. (∑*k*=*0*..<*Suc n*. *2*^*k* ∗ *f* (*2*^*k*)))
by (*subst sum.atLeast_Suc_lessThan*) (*simp_all add*: *add_ac*)
thus *Bseq* (λ*n*. (∑*k*<*n*. *2*^*k* ∗ *f* (*2*^*k*)))
by (*subst* (*asm*) *Bseq_Suc_iff*) (*simp add*: *atLeast0LessThan*)
next
assume *A*: *Bseq* (λ*n*. (∑*k*<*n*. *2*^*k* ∗ *f* (*2*^*k*)))
have *eventually* (λ*n*. *norm* (∑*k*=*1*..<*2*^*n*. *f k*) ≤ *norm* (∑*k*<*n*. *2*^*k* ∗ *f* (*2*^*k*))) *sequentially*
proof (*intro always_eventually allI*)
fix *n* :: *nat*
have *norm* (∑*k*=*1*..<*2*^*n*. *f k*) = (∑*k*=*1*..<*2*^*n*. *f k*) **unfolding** *real_norm_def*
by (*intro abs_of_nonneg sum_nonneg ballI mult_nonneg_nonneg nonneg*)
also have ... ≤ (∑*k*<*n*. *2*^*k* ∗ *f* (*2*^*k*))
by (*subst condensation_condense1* [*symmetric*]) (*intro condensation_inequality mono'*)
also have ... = *norm* ... **unfolding** *real_norm_def*
by (*intro abs_of_nonneg* [*symmetric*] *sum_nonneg ballI mult_nonneg_nonneg nonneg*) *simp_all*
finally show *norm* (∑*k*=*1*..<*2*^*n*. *f k*) ≤ *norm* (∑*k*<*n*. *2*^*k* ∗ *f* (*2*^*k*)) **.**
qed
from *this* **and** *A* **show** *Bseq* (λ*n*. ∑*k*=*1*..<*2*^*n*. *f k*) **by** (*rule Bseq_eventually_mono*)
qed
also have *monoseq* (λ*n*. (∑*k*<*n*. *2*^*k* ∗ *f* (*2*^*k*)))
by (*intro mono_SucI1*) (*auto intro*!: *mult_nonneg_nonneg nonneg*)
hence *Bseq* (λ*n*. (∑*k*<*n*. *2*^*k* ∗ *f* (*2*^*k*))) ⟷ *convergent* (λ*n*. (∑*k*<*n*. *2*^*k* ∗ *f* (*2*^*k*)))
by (*rule monoseq_imp_convergent_iff_Bseq* [*symmetric*])
also have ... ⟷ *summable* (λ*k*. *2*^*k* ∗ *f* (*2*^*k*)) **by** (*simp only*: *summable_iff_convergent*)
finally show *?thesis* **.**
qed

end

Summability of powers

lemma *abs_summable_complex_powr_iff*:
summable (λ*n*. *norm* (*exp* (*of_real* (*ln* (*of_nat n*)) ∗ *s*))) ⟷ *Re s* < −*1*

proof (*cases Re s* ≤ *0*)
 let *?l* = λ*n*. *complex_of_real* (*ln* (*of_nat n*))
 case *False*
 have *eventually* (λ*n*. *norm* (*1* :: *real*) ≤ *norm* (*exp* (*?l n* ∗ *s*))) *sequentially*
  apply (*rule eventually_mono* [*OF eventually_gt_at_top*[*of 0*::*nat*]])
  using *False ge_one_powr_ge_zero* **by** *auto*
 from *summable_comparison_test_ev*[*OF this*] *False* **show** *?thesis* **by** (*auto simp*: *summable_const_iff*)
next
 let *?l* = λ*n*. *complex_of_real* (*ln* (*of_nat n*))
 case *True*
 hence *summable* (λ*n*. *norm* (*exp* (*?l n* ∗ *s*))) ⟷ *summable* (λ*n*. *2*ˆ*n* ∗ *norm* (*exp* (*?l* (*2*ˆ*n*) ∗ *s*)))
  by (*intro condensation_test*) (*auto intro*!: *mult_right_mono_neg*)
 also have (λ*n*. *2*ˆ*n* ∗ *norm* (*exp* (*?l* (*2*ˆ*n*) ∗ *s*))) = (λ*n*. (*2 powr* (*Re s* + *1*)) ˆ *n*)
 proof
  fix *n* :: *nat*
  have *2*ˆ*n* ∗ *norm* (*exp* (*?l* (*2*ˆ*n*) ∗ *s*)) = *exp* (*real n* ∗ *ln 2*) ∗ *exp* (*real n* ∗ *ln 2* ∗ *Re s*)
   using *True* **by** (*subst exp_of_nat_mult*) (*simp add*: *ln_realpow algebra_simps*)
  also have ... = *exp* (*real n* ∗ (*ln 2* ∗ (*Re s* + *1*)))
   by (*simp add*: *algebra_simps exp_add*)
  also have ... = *exp* (*ln 2* ∗ (*Re s* + *1*)) ˆ *n* **by** (*subst exp_of_nat_mult*) *simp*
  also have *exp* (*ln 2* ∗ (*Re s* + *1*)) = *2 powr* (*Re s* + *1*) **by** (*simp add*: *powr_def*)
  finally show *2*ˆ*n* ∗ *norm* (*exp* (*?l* (*2*ˆ*n*) ∗ *s*)) = (*2 powr* (*Re s* + *1*)) ˆ *n* **.**
 qed
 also have *summable* ... ⟷ *2 powr* (*Re s* + *1*) < *2 powr 0*
  by (*subst summable_geometric_iff*) *simp*
 also have ... ⟷ *Re s* < −*1* **by** (*subst powr_less_cancel_iff*) (*simp*, *linarith*)
 finally show *?thesis* **.**
qed

theorem *summable_complex_powr_iff*:
 assumes *Re s* < −*1*
 shows *summable* (λ*n*. *exp* (*of_real* (*ln* (*of_nat n*)) ∗ *s*))
 by (*rule summable_norm_cancel*, *subst abs_summable_complex_powr_iff*) *fact*

lemma *summable_real_powr_iff*: *summable* (λ*n*. *of_nat n powr s* :: *real*) ⟷ *s* < −*1*
proof −
 from *eventually_gt_at_top*[*of 0*::*nat*]
  have *summable* (λ*n*. *of_nat n powr s*) ⟷ *summable* (λ*n*. *exp* (*ln* (*of_nat n*) ∗ *s*))
  by (*intro summable_cong*) (*auto elim*!: *eventually_mono simp*: *powr_def*)
 also have ... ⟷ *s* < −*1* **using** *abs_summable_complex_powr_iff*[*of of_real s*] **by** *simp*
 finally show *?thesis* **.**

qed

lemma *inverse_power_summable*:
  assumes *s*: $s \geq 2$
  shows *summable* (*λn. inverse* (*of_nat n ˆ s* :: *'a* :: {*real_normed_div_algebra*,*banach*}))
proof (*rule summable_norm_cancel*, *subst summable_cong*)
  from *eventually_gt_at_top*[*of 0*::*nat*]
    show *eventually* (*λn. norm* (*inverse* (*of_nat n ˆ s*:: *'a*)) = *real_of_nat n powr* (−*real s*)) *at_top*
    by *eventually_elim* (*simp add*: *norm_inverse norm_power powr_minus powr_realpow*)
qed (*insert s summable_real_powr_iff* [*of* −*s*], *simp_all*)

lemma *not_summable_harmonic*: ¬*summable* (*λn. inverse* (*of_nat n*) :: *'a* :: *real_normed_field*)
proof
  assume *summable* (*λn. inverse* (*of_nat n*) :: *'a*)
  hence *convergent* (*λn. norm* (*of_real* (∑*k*<*n. inverse* (*of_nat k*)) :: *'a*))
    by (*simp add*: *summable_iff_convergent convergent_norm*)
  hence *convergent* (*λn. abs* (∑*k*<*n. inverse* (*of_nat k*)) :: *real*) **by** (*simp only*: *norm_of_real*)
  also have (*λn. abs* (∑*k*<*n. inverse* (*of_nat k*)) :: *real*) = (*λn.* ∑*k*<*n. inverse* (*of_nat k*))
    by (*intro ext abs_of_nonneg sum_nonneg*) *auto*
  also have *convergent* ... ⟷ *summable* (*λk. inverse* (*of_nat k*) :: *real*)
    by (*simp add*: *summable_iff_convergent*)
  finally show *False* **using** *summable_real_powr_iff* [*of* −*1*] **by** (*simp add*: *powr_minus*)
qed

Kummer's test

theorem *kummers_test_convergence*:
  fixes *f p* :: *nat* ⇒ *real*
  assumes *pos_f*: *eventually* (*λn. f n* > *0*) *sequentially*
  assumes *nonneg_p*: *eventually* (*λn. p n* ≥ *0*) *sequentially*
  defines *l* ≡ *liminf* (*λn. ereal* (*p n* ∗ *f n* / *f* (*Suc n*) − *p* (*Suc n*)))
  assumes *l*: *l* > *0*
  shows *summable f*
  unfolding *summable_iff_convergent'*
proof −
  define *r* **where** *r* = (*if l* = ∞ *then 1 else real_of_ereal l* / *2*)
  from *l* **have** *r* > *0* ∧ *of_real r* < *l* **by** (*cases l*) (*simp_all add*: *r_def*)
  hence *r*: *r* > *0 of_real r* < *l* **by** *simp_all*
  hence *eventually* (*λn. p n* ∗ *f n* / *f* (*Suc n*) − *p* (*Suc n*) > *r*) *sequentially*
    unfolding *l_def* **by** (*force dest*: *less_LiminfD*)
  moreover from *pos_f* **have** *eventually* (*λn. f* (*Suc n*) > *0*) *sequentially*
    by (*subst eventually_sequentially_Suc*)
  ultimately have *eventually* (*λn. p n* ∗ *f n* − *p* (*Suc n*) ∗ *f* (*Suc n*) > *r* ∗ *f* (*Suc n*)) *sequentially*
    by *eventually_elim* (*simp add*: *field_simps*)
  from *eventually_conj*[*OF pos_f eventually_conj*[*OF nonneg_p this*]]

obtain *m* **where** *m*: $\bigwedge n.\ n \geq m \Longrightarrow f\ n > 0$ $\bigwedge n.\ n \geq m \Longrightarrow p\ n \geq 0$
$\bigwedge n.\ n \geq m \Longrightarrow p\ n * f\ n - p\ (Suc\ n) * f\ (Suc\ n) > r * f\ (Suc\ n)$
unfolding *eventually_at_top_linorder* **by** *blast*

let $?c = (norm\ (\sum k{\leq}m.\ r * f\ k) + p\ m * f\ m)\ /\ r$
have *Bseq* $(\lambda n.\ (\sum k{\leq}n + Suc\ m.\ f\ k))$
proof (*rule BseqI'*)
fix *k* :: *nat*
define *n* **where** $n = k + Suc\ m$
have *n*: $n > m$ **by** (*simp add*: *n_def*)

from *r* **have** $r * norm\ (\sum k{\leq}n.\ f\ k) = norm\ (\sum k{\leq}n.\ r * f\ k)$
by (*simp add*: *sum_distrib_left*[*symmetric*] *abs_mult*)
also from *n* **have** $\{..n\} = \{..m\} \cup \{Suc\ m..n\}$ **by** *auto*
hence $(\sum k{\leq}n.\ r * f\ k) = (\sum k{\in}\{..m\} \cup \{Suc\ m..n\}.\ r * f\ k)$ **by** (*simp only*:)
also have $\ldots = (\sum k{\leq}m.\ r * f\ k) + (\sum k{=}Suc\ m..n.\ r * f\ k)$
by (*subst sum.union_disjoint*) *auto*
also have $norm\ \ldots \leq norm\ (\sum k{\leq}m.\ r * f\ k) + norm\ (\sum k{=}Suc\ m..n.\ r * f\ k)$
by (*rule norm_triangle_ineq*)
also from *r less_imp_le*[*OF m*(*1*)] **have** $(\sum k{=}Suc\ m..n.\ r * f\ k) \geq 0$
by (*intro sum_nonneg*) *auto*
hence $norm\ (\sum k{=}Suc\ m..n.\ r * f\ k) = (\sum k{=}Suc\ m..n.\ r * f\ k)$ **by** *simp*
also have $(\sum k{=}Suc\ m..n.\ r * f\ k) = (\sum k{=}m..{<}n.\ r * f\ (Suc\ k))$
by (*subst sum.shift_bounds_Suc_ivl* [*symmetric*])
(*simp only*: *atLeastLessThanSuc_atLeastAtMost*)
also from *m* **have** $\ldots \leq (\sum k{=}m..{<}n.\ p\ k * f\ k - p\ (Suc\ k) * f\ (Suc\ k))$
by (*intro sum_mono*[*OF less_imp_le*]) *simp_all*
also have $\ldots = -(\sum k{=}m..{<}n.\ p\ (Suc\ k) * f\ (Suc\ k) - p\ k * f\ k)$
by (*simp add*: *sum_negf* [*symmetric*] *algebra_simps*)
also from *n* **have** $\ldots = p\ m * f\ m - p\ n * f\ n$
by (*cases n*, *simp*, *simp only*: *atLeastLessThanSuc_atLeastAtMost*, *subst sum_Suc_diff*) *simp_all*
also from *less_imp_le*[*OF m*(*1*)] *m*(*2*) *n* **have** $\ldots \leq p\ m * f\ m$ **by** *simp*
finally show $norm\ (\sum k{\leq}n.\ f\ k) \leq (norm\ (\sum k{\leq}m.\ r * f\ k) + p\ m * f\ m)\ /\ r$ **using** *r*
by (*subst pos_le_divide_eq*[*OF r*(*1*)]) (*simp only*: *mult_ac*)
qed
moreover have $(\sum k{\leq}n.\ f\ k) \leq (\sum k{\leq}n'.\ f\ k)$ **if** $Suc\ m \leq n$ $n \leq n'$ **for** $n\ n'$
using *less_imp_le*[*OF m*(*1*)] *that* **by** (*intro sum_mono2*) *auto*
ultimately show *convergent* $(\lambda n.\ \sum k{\leq}n.\ f\ k)$ **by** (*rule Bseq_monoseq_convergent'_inc*)
qed

theorem *kummers_test_divergence*:
fixes *f p* :: $nat \Rightarrow real$
assumes *pos_f*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
assumes *pos_p*: *eventually* $(\lambda n.\ p\ n > 0)$ *sequentially*
assumes *divergent_p*: $\neg summable\ (\lambda n.\ inverse\ (p\ n))$

defines $l \equiv limsup\ (\lambda n.\ ereal\ (p\ n * f\ n\ /\ f\ (Suc\ n) - p\ (Suc\ n)))$
assumes *l*: $l < 0$
shows ¬*summable f*
proof
assume *summable f*
from *eventually_conj*[*OF pos_f eventually_conj*[*OF pos_p Limsup_lessD*[*OF l*[*unfolded l_def*]]]]
obtain *N* **where** *N*: $\bigwedge n.\ n \geq N \Longrightarrow p\ n > 0$ $\bigwedge n.\ n \geq N \Longrightarrow f\ n > 0$
$\bigwedge n.\ n \geq N \Longrightarrow p\ n * f\ n\ /\ f\ (Suc\ n) - p\ (Suc\ n) < 0$
by (*auto simp*: *eventually_at_top_linorder*)
hence *A*: $p\ n * f\ n < p\ (Suc\ n) * f\ (Suc\ n)$ **if** $n \geq N$ **for** *n* **using** *that N*[*of n*] *N*[*of Suc n*]
by (*simp add*: *field_simps*)
have *B*: $p\ n * f\ n \geq p\ N * f\ N$ **if** $n \geq N$ **for** *n* **using** *that* **and** *A*
by (*induction n rule*: *dec_induct*) (*auto intro*!: *less_imp_le elim*!: *order.trans*)
have *eventually* $(\lambda n.\ norm\ (p\ N * f\ N * inverse\ (p\ n)) \leq f\ n)$ *sequentially*
apply (*rule eventually_mono* [*OF eventually_ge_at_top*[*of N*]])
using *B N* **by** (*auto simp*: *field_simps abs_of_pos*)
from *this* **and** ‹*summable f*› **have** *summable* $(\lambda n.\ p\ N * f\ N * inverse\ (p\ n))$
by (*rule summable_comparison_test_ev*)
from *summable_mult*[*OF this*, *of inverse* $(p\ N * f\ N)$] *N*[*OF le_refl*]
have *summable* $(\lambda n.\ inverse\ (p\ n))$ **by** (*simp add*: *field_split_simps*)
with *divergent_p* **show** *False* **by** *contradiction*
qed

Ratio test

theorem *ratio_test_convergence*:
fixes $f :: nat \Rightarrow real$
assumes *pos_f*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
defines $l \equiv liminf\ (\lambda n.\ ereal\ (f\ n\ /\ f\ (Suc\ n)))$
assumes *l*: $l > 1$
shows *summable f*
proof (*rule kummers_test_convergence*[*OF pos_f*])
note *l*
also have $l = liminf\ (\lambda n.\ ereal\ (f\ n\ /\ f\ (Suc\ n) - 1)) + 1$
by (*subst Liminf_add_ereal_right*[*symmetric*]) (*simp_all add*: *minus_ereal_def l_def one_ereal_def*)
finally show $liminf\ (\lambda n.\ ereal\ (1 * f\ n\ /\ f\ (Suc\ n) - 1)) > 0$
by (*cases* $liminf\ (\lambda n.\ ereal\ (1 * f\ n\ /\ f\ (Suc\ n) - 1))$) *simp_all*
qed *simp*

theorem *ratio_test_divergence*:
fixes $f :: nat \Rightarrow real$
assumes *pos_f*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
defines $l \equiv limsup\ (\lambda n.\ ereal\ (f\ n\ /\ f\ (Suc\ n)))$
assumes *l*: $l < 1$
shows ¬*summable f*
proof (*rule kummers_test_divergence*[*OF pos_f*])

have *limsup* (λ*n*. *ereal* (*f n* / *f* (*Suc n*) − *1*)) + *1* = *l*
by (*subst Limsup_add_ereal_right*[*symmetric*]) (*simp_all add*: *minus_ereal_def l_def one_ereal_def*)
also note *l*
finally show *limsup* (λ*n*. *ereal* (*1* ∗ *f n* / *f* (*Suc n*) − *1*)) < *0*
by (*cases limsup* (λ*n*. *ereal* (*1* ∗ *f n* / *f* (*Suc n*) − *1*))) *simp_all*
qed (*simp_all add*: *summable_const_iff*)

Raabe's test

theorem *raabes_test_convergence*:
fixes *f* :: *nat* ⇒ *real*
assumes *pos*: *eventually* (λ*n*. *f n* > *0*) *sequentially*
defines *l* ≡ *liminf* (λ*n*. *ereal* (*of_nat n* ∗ (*f n* / *f* (*Suc n*) − *1*)))
assumes *l*: *l* > *1*
shows *summable f*
proof (*rule kummers_test_convergence*)
let *?l′* = *liminf* (λ*n*. *ereal* (*of_nat n* ∗ *f n* / *f* (*Suc n*) − *of_nat* (*Suc n*)))
have *1* < *l* **by** *fact*
also have *l* = *liminf* (λ*n*. *ereal* (*of_nat n* ∗ *f n* / *f* (*Suc n*) − *of_nat* (*Suc n*)) + *1*)
by (*simp add*: *l_def algebra_simps*)
also have ... = *?l′* + *1* **by** (*subst Liminf_add_ereal_right*) *simp_all*
finally show *?l′* > *0* **by** (*cases ?l′*) (*simp_all add*: *algebra_simps*)
qed (*simp_all add*: *pos*)

theorem *raabes_test_divergence*:
fixes *f* :: *nat* ⇒ *real*
assumes *pos*: *eventually* (λ*n*. *f n* > *0*) *sequentially*
defines *l* ≡ *limsup* (λ*n*. *ereal* (*of_nat n* ∗ (*f n* / *f* (*Suc n*) − *1*)))
assumes *l*: *l* < *1*
shows ¬*summable f*
proof (*rule kummers_test_divergence*)
let *?l′* = *limsup* (λ*n*. *ereal* (*of_nat n* ∗ *f n* / *f* (*Suc n*) − *of_nat* (*Suc n*)))
note *l*
also have *l* = *limsup* (λ*n*. *ereal* (*of_nat n* ∗ *f n* / *f* (*Suc n*) − *of_nat* (*Suc n*)) + *1*)
by (*simp add*: *l_def algebra_simps*)
also have ... = *?l′* + *1* **by** (*subst Limsup_add_ereal_right*) *simp_all*
finally show *?l′* < *0* **by** (*cases ?l′*) (*simp_all add*: *algebra_simps*)
qed (*insert pos eventually_gt_at_top*[*of 0*::*nat*] *not_summable_harmonic*, *simp_all*)

4.6.2 Radius of convergence

The radius of convergence of a power series. This value always exists, ranges from *0* to ∞, and the power series is guaranteed to converge for all inputs with a norm that is smaller than that radius and to diverge for all inputs with a norm that is greater.

definition *conv_radius* :: (*nat* ⇒ ′*a* :: *banach*) ⇒ *ereal* **where**

conv_radius f = inverse (limsup (λn. ereal (root n (norm (f n)))))

lemma *conv_radius_cong_weak [cong]*: (⋀n. f n = g n) ⟹ *conv_radius f = conv_radius g*
by (*drule ext*) *simp_all*

lemma *conv_radius_nonneg*: *conv_radius f ≥ 0*
proof −
have *0 = limsup (λn. 0)* **by** (*subst Limsup_const*) *simp_all*
also have ... ≤ *limsup (λn. ereal (root n (norm (f n))))*
by (*intro Limsup_mono*) (*simp_all add: real_root_ge_zero*)
finally show *?thesis*
unfolding *conv_radius_def* **by** (*auto simp: ereal_inverse_nonneg_iff*)
qed

lemma *conv_radius_zero [simp]*: *conv_radius (λ_. 0) = ∞*
by (*auto simp: conv_radius_def zero_ereal_def [symmetric] Limsup_const*)

lemma *conv_radius_altdef*:
conv_radius f = liminf (λn. inverse (ereal (root n (norm (f n)))))
by (*subst Liminf_inverse_ereal*) (*simp_all add: real_root_ge_zero conv_radius_def*)

lemma *conv_radius_cong'*:
assumes *eventually (λx. f x = g x) sequentially*
shows *conv_radius f = conv_radius g*
unfolding *conv_radius_altdef* **by** (*intro Liminf_eq eventually_mono [OF assms]*) *auto*

lemma *conv_radius_cong*:
assumes *eventually (λx. norm (f x) = norm (g x)) sequentially*
shows *conv_radius f = conv_radius g*
unfolding *conv_radius_altdef* **by** (*intro Liminf_eq eventually_mono [OF assms]*) *auto*

theorem *abs_summable_in_conv_radius*:
fixes *f :: nat ⇒ 'a :: {banach, real_normed_div_algebra}*
assumes *ereal (norm z) < conv_radius f*
shows *summable (λn. norm (f n * z ^ n))*
proof (*rule root_test_convergence'*)
define *l* **where** *l = limsup (λn. ereal (root n (norm (f n))))*
have *0 = limsup (λn. 0)* **by** (*simp add: Limsup_const*)
also have ... ≤ *l* **unfolding** *l_def* **by** (*intro Limsup_mono*) (*simp_all add: real_root_ge_zero*)
finally have *l_nonneg*: *l ≥ 0* **.**

have *limsup (λn. root n (norm (f n * z^n))) = l * ereal (norm z)* **unfolding** *l_def*
by (*rule limsup_root_powser*)
also from *l_nonneg* **consider** *l = 0 | l = ∞ | ∃l'. l = ereal l' ∧ l' > 0*

by (*cases l*) (*auto simp*: *less_le*)
hence *l* ∗ *ereal* (*norm z*) < *1*
proof *cases*
assume *l* = ∞
hence *conv_radius f* = *0* **unfolding** *conv_radius_def l_def* **by** *simp*
with *assms* **show** *?thesis* **by** *simp*
next
assume ∃ *l'*. *l* = *ereal l'* ∧ *l'* > *0*
then guess *l'* **by** (*elim exE conjE*) **note** *l'* = *this*
hence *l* ≠ ∞ **by** *auto*
have *l* ∗ *ereal* (*norm z*) < *l* ∗ *conv_radius f*
by (*intro ereal_mult_strict_left_mono*) (*simp_all add*: *l' assms*)
also have *conv_radius f* = *inverse l* **by** (*simp add*: *conv_radius_def l_def*)
also from *l'* **have** *l* ∗ *inverse l* = *1* **by** *simp*
finally show *?thesis* **.**
qed *simp_all*
finally show *limsup* (λ*n*. *ereal* (*root n* (*norm* (*norm* (*f n* ∗ *z* ˆ *n*))))) < *1* **by** *simp*
qed

lemma *summable_in_conv_radius*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real_normed_div_algebra*}
assumes *ereal* (*norm z*) < *conv_radius f*
shows *summable* (λ*n*. *f n* ∗ *z* ˆ *n*)
by (*rule summable_norm_cancel*, *rule abs_summable_in_conv_radius*) *fact+*

theorem *not_summable_outside_conv_radius*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real_normed_div_algebra*}
assumes *ereal* (*norm z*) > *conv_radius f*
shows ¬*summable* (λ*n*. *f n* ∗ *z* ˆ *n*)
proof (*rule root_test_divergence*)
define *l* **where** *l* = *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*))))
have *0* = *limsup* (λ*n*. *0*) **by** (*simp add*: *Limsup_const*)
also have ... ≤ *l* **unfolding** *l_def* **by** (*intro Limsup_mono*) (*simp_all add*: *real_root_ge_zero*)
finally have *l_nonneg*: *l* ≥ *0* **.**
from *assms* **have** *l_nz*: *l* ≠ *0* **unfolding** *conv_radius_def l_def* **by** *auto*

have *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n* ∗ *z*ˆ*n*)))) = *l* ∗ *ereal* (*norm z*)
unfolding *l_def* **by** (*rule limsup_root_powser*)
also have ... > *1*
proof (*cases l*)
assume *l* = ∞
with *assms conv_radius_nonneg*[*of f*] **show** *?thesis*
by (*auto simp*: *zero_ereal_def*[*symmetric*])
next
fix *l'* **assume** *l'*: *l* = *ereal l'*
from *l_nonneg l_nz* **have** *1* = *l* ∗ *inverse l* **by** (*auto simp*: *l' field_simps*)
also from *l_nz* **have** *inverse l* = *conv_radius f*

unfolding *l_def conv_radius_def* **by** *auto*
also from *l′ l_nz l_nonneg assms* **have** *l* ∗ . . . < *l* ∗ *ereal* (*norm z*)
by (*intro ereal_mult_strict_left_mono*) (*auto simp*: *l′*)
finally show *?thesis* **.**
qed (*insert l_nonneg*, *simp_all*)
finally show *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n* ∗ *z*ˆ*n*)))) > *1* **.**
qed

lemma *conv_radius_geI*:
assumes *summable* (λ*n*. *f n* ∗ *z* ˆ *n* :: ′*a* :: {*banach*, *real_normed_div_algebra*})
shows *conv_radius f* ≥ *norm z*
using *not_summable_outside_conv_radius*[*of f z*] *assms* **by** (*force simp*: *not_le*[*symmetric*])

lemma *conv_radius_leI*:
assumes ¬*summable* (λ*n*. *norm* (*f n* ∗ *z* ˆ *n* :: ′*a* :: {*banach*, *real_normed_div_algebra*}))
shows *conv_radius f* ≤ *norm z*
using *abs_summable_in_conv_radius*[*of z f*] *assms* **by** (*force simp*: *not_le*[*symmetric*])

lemma *conv_radius_leI′*:
assumes ¬*summable* (λ*n*. *f n* ∗ *z* ˆ *n* :: ′*a* :: {*banach*, *real_normed_div_algebra*})
shows *conv_radius f* ≤ *norm z*
using *summable_in_conv_radius*[*of z f*] *assms* **by** (*force simp*: *not_le*[*symmetric*])

lemma *conv_radius_geI_ex*:
fixes *f* :: *nat* ⇒ ′*a* :: {*banach*, *real_normed_div_algebra*}
assumes ⋀*r*. *0* < *r* ⟹ *ereal r* < *R* ⟹ ∃ *z*. *norm z* = *r* ∧ *summable* (λ*n*. *f n* ∗ *z*ˆ*n*)
shows *conv_radius f* ≥ *R*
proof (*rule linorder_cases*[*of conv_radius f R*])
assume *R*: *conv_radius f* < *R*
with *conv_radius_nonneg*[*of f*] **obtain** *conv_radius′*
where [*simp*]: *conv_radius f* = *ereal conv_radius′*
by (*cases conv_radius f*) *simp_all*
define *r* **where** *r* = (*if R* = ∞ *then conv_radius′* + *1 else* (*real_of_ereal R* + *conv_radius′*) / *2*)
from *R conv_radius_nonneg*[*of f*] **have** *0* < *r* ∧ *ereal r* < *R* ∧ *ereal r* > *conv_radius f*
unfolding *r_def* **by** (*cases R*) (*auto simp*: *r_def field_simps*)
with *assms*(*1*)[*of r*] **obtain** *z* **where** *norm z* > *conv_radius f summable* (λ*n*. *f n* ∗ *z*ˆ*n*) **by** *auto*
with *not_summable_outside_conv_radius*[*of f z*] **show** *?thesis* **by** *simp*
qed *simp_all*

lemma *conv_radius_geI_ex′*:
fixes *f* :: *nat* ⇒ ′*a* :: {*banach*, *real_normed_div_algebra*}
assumes ⋀*r*. *0* < *r* ⟹ *ereal r* < *R* ⟹ *summable* (λ*n*. *f n* ∗ *of_real r*ˆ*n*)
shows *conv_radius f* ≥ *R*
proof (*rule conv_radius_geI_ex*)

fix *r* **assume** *0* < *r ereal r* < *R*
with *assms*[*of r*] **show** ∃ *z*. *norm z* = *r* ∧ *summable* (λ*n*. *f n* ∗ *z* ˆ *n*)
by (*intro exI*[*of* _ *of_real r* :: *'a*]) *auto*
qed

lemma *conv_radius_leI_ex*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real_normed_div_algebra*}
assumes *R* ≥ *0*
assumes ⋀*r*. *0* < *r* ⟹ *ereal r* > *R* ⟹ ∃ *z*. *norm z* = *r* ∧ ¬*summable* (λ*n*. *norm* (*f n* ∗ *z*ˆ*n*))
shows *conv_radius f* ≤ *R*
proof (*rule linorder_cases*[*of conv_radius f R*])
assume *R*: *conv_radius f* > *R*
from *R assms*(*1*) **obtain** *R'* **where** *R'*: *R* = *ereal R'* **by** (*cases R*) *simp_all*
define *r* **where**
r = (*if conv_radius f* = ∞ *then R'* + *1 else* (*R'* + *real_of_ereal* (*conv_radius f*)) / *2*)
from *R conv_radius_nonneg*[*of f*] **have** *r* > *R* ∧ *r* < *conv_radius f* **unfolding** *r_def*
by (*cases conv_radius f*) (*auto simp*: *r_def field_simps R'*)
with *assms*(*1*) *assms*(*2*)[*of r*] *R'*
obtain *z* **where** *norm z* < *conv_radius f* ¬*summable* (λ*n*. *norm* (*f n* ∗ *z*ˆ*n*)) **by** *auto*
with *abs_summable_in_conv_radius*[*of z f*] **show** *?thesis* **by** *auto*
qed *simp_all*

lemma *conv_radius_leI_ex'*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real_normed_div_algebra*}
assumes *R* ≥ *0*
assumes ⋀*r*. *0* < *r* ⟹ *ereal r* > *R* ⟹ ¬*summable* (λ*n*. *f n* ∗ *of_real r*ˆ*n*)
shows *conv_radius f* ≤ *R*
proof (*rule conv_radius_leI_ex*)
fix *r* **assume** *0* < *r ereal r* > *R*
with *assms*(*2*)[*of r*] **show** ∃ *z*. *norm z* = *r* ∧ ¬*summable* (λ*n*. *norm* (*f n* ∗ *z* ˆ *n*))
by (*intro exI*[*of* _ *of_real r* :: *'a*]) (*auto dest*: *summable_norm_cancel*)
qed *fact+*

lemma *conv_radius_eqI*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real_normed_div_algebra*}
assumes *R* ≥ *0*
assumes ⋀*r*. *0* < *r* ⟹ *ereal r* < *R* ⟹ ∃ *z*. *norm z* = *r* ∧ *summable* (λ*n*. *f n* ∗ *z*ˆ*n*)
assumes ⋀*r*. *0* < *r* ⟹ *ereal r* > *R* ⟹ ∃ *z*. *norm z* = *r* ∧ ¬*summable* (λ*n*. *norm* (*f n* ∗ *z*ˆ*n*))
shows *conv_radius f* = *R*
by (*intro antisym conv_radius_geI_ex conv_radius_leI_ex assms*)

lemma *conv_radius_eqI'*:

```
  fixes f :: nat ⇒ 'a :: {banach, real_normed_div_algebra}
  assumes R ≥ 0
  assumes ⋀r. 0 < r ⟹ ereal r < R ⟹ summable (λn. f n * (of_real r) ^n)
  assumes ⋀r. 0 < r ⟹ ereal r > R ⟹ ¬summable (λn. norm (f n * (of_real
r) ^n))
  shows   conv_radius f = R
proof (intro conv_radius_eqI[OF assms(1)])
  fix r assume 0 < r ereal r < R with assms(2)[OF this]
    show ∃z. norm z = r ∧ summable (λn. f n * z ^ n) by force
next
  fix r assume 0 < r ereal r > R with assms(3)[OF this]
    show ∃z. norm z = r ∧ ¬summable (λn. norm (f n * z ^ n)) by force
qed

lemma conv_radius_zeroI:
  fixes f :: nat ⇒ 'a :: {banach,real_normed_div_algebra}
  assumes ⋀z. z ≠ 0 ⟹ ¬summable (λn. f n * z^n)
  shows   conv_radius f = 0
proof (rule ccontr)
  assume conv_radius f ≠ 0
  with conv_radius_nonneg[of f] have pos: conv_radius f > 0 by simp
  define r where r = (if conv_radius f = ∞ then 1 else real_of_ereal (conv_radius
f) / 2)
  from pos have r: ereal r > 0 ∧ ereal r < conv_radius f
    by (cases conv_radius f) (simp_all add: r_def)
  hence summable (λn. f n * of_real r ^ n) by (intro summable_in_conv_radius)
simp
  moreover from r and assms[of of_real r] have ¬summable (λn. f n * of_real r
^ n) by simp
  ultimately show False by contradiction
qed

lemma conv_radius_inftyI':
  fixes f :: nat ⇒ 'a :: {banach,real_normed_div_algebra}
  assumes ⋀r. r > c ⟹ ∃z. norm z = r ∧ summable (λn. f n * z^n)
  shows   conv_radius f = ∞
proof -
  {
    fix r :: real
    have max r (c + 1) > c by (auto simp: max_def)
     from assms[OF this] obtain z where norm z = max r (c + 1) summable
(λn. f n * z^n) by blast
    from conv_radius_geI[OF this(2)] this(1) have conv_radius f ≥ r by simp
  }
  from this[of real_of_ereal (conv_radius f + 1)] show conv_radius f = ∞
    by (cases conv_radius f) simp_all
qed

lemma conv_radius_inftyI:
```

```
  fixes f :: nat ⇒ 'a :: {banach,real_normed_div_algebra}
  assumes ⋀r. ∃z. norm z = r ∧ summable (λn. f n * z^n)
  shows   conv_radius f = ∞
  using assms by (rule conv_radius_inftyI')

lemma conv_radius_inftyI'':
  fixes f :: nat ⇒ 'a :: {banach,real_normed_div_algebra}
  assumes ⋀z. summable (λn. f n * z^n)
  shows   conv_radius f = ∞
proof (rule conv_radius_inftyI')
  fix r :: real assume r > 0
  with assms show ∃z. norm z = r ∧ summable (λn. f n * z^n)
    by (intro exI[of _ of_real r]) simp
qed

lemma conv_radius_conv_Sup:
  fixes f :: nat ⇒ 'a :: {banach, real_normed_div_algebra}
  shows conv_radius f = Sup {r. ∀z. ereal (norm z) < r ⟶ summable (λn. f n
* z ^ n)}
proof (rule Sup_eqI [symmetric], goal_cases)
  case (1 r)
  thus ?case
    by (intro conv_radius_geI_ex') auto
next
  case (2 r)
  from 2[of 0] have r: r ≥ 0 by auto
  show ?case
  proof (intro conv_radius_leI_ex' r)
    fix R assume R: R > 0 ereal R > r
    with r obtain r' where [simp]: r = ereal r' by (cases r) auto
    show ¬summable (λn. f n * of_real R ^ n)
    proof
      assume *: summable (λn. f n * of_real R ^ n)
      define R' where R' = (R + r') / 2
      from R have R': R' > r' R' < R by (simp_all add: R'_def)
      hence ∀z. norm z < R' ⟶ summable (λn. f n * z ^ n)
        using powser_inside[OF *] by auto
      from 2[of R'] and this have R' ≤ r' by auto
      with ‹R' > r'› show False by simp
    qed
  qed
qed

lemma conv_radius_shift:
  fixes f :: nat ⇒ 'a :: {banach, real_normed_div_algebra}
  shows   conv_radius (λn. f (n + m)) = conv_radius f
  unfolding conv_radius_conv_Sup summable_powser_ignore_initial_segment ..

lemma conv_radius_norm [simp]: conv_radius (λx. norm (f x)) = conv_radius f
```

by (*simp add*: *conv_radius_def*)

lemma *conv_radius_ratio_limit_ereal*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*,*real_normed_div_algebra*}
assumes *nz*: *eventually* (λ*n*. *f n* ≠ *0*) *sequentially*
assumes *lim*: (λ*n*. *ereal* (*norm* (*f n*) / *norm* (*f* (*Suc n*)))) ⟶ *c*
shows *conv_radius f* = *c*
proof (*rule conv_radius_eqI'*)
show *c* ≥ *0* **by** (*intro Lim_bounded2*[*OF lim*]) *simp_all*
next
fix *r* **assume** *r*: *0* < *r ereal r* < *c*
let *?l* = *liminf* (λ*n*. *ereal* (*norm* (*f n* ∗ *of_real r* ^ *n*) / *norm* (*f* (*Suc n*) ∗ *of_real r* ^ *Suc n*)))
have *?l* = *liminf* (λ*n*. *ereal* (*norm* (*f n*) / (*norm* (*f* (*Suc n*)))) ∗ *ereal* (*inverse r*))
using *r* **by** (*simp add*: *norm_mult norm_power field_split_simps*)
also from *r* **have** ... = *liminf* (λ*n*. *ereal* (*norm* (*f n*) / (*norm* (*f* (*Suc n*))))) ∗ *ereal* (*inverse r*)
by (*intro Liminf_ereal_mult_right*) *simp_all*
also have *liminf* (λ*n*. *ereal* (*norm* (*f n*) / (*norm* (*f* (*Suc n*))))) = *c*
by (*intro lim_imp_Liminf lim*) *simp*
finally have *l*: *?l* = *c* ∗ *ereal* (*inverse r*) **by** *simp*
from *r* **have** *l'*: *c* ∗ *ereal* (*inverse r*) > *1* **by** (*cases c*) (*simp_all add*: *field_simps*)
show *summable* (λ*n*. *f n* ∗ *of_real r*^*n*)
by (*rule summable_norm_cancel*, *rule ratio_test_convergence*)
(*insert r nz l l'*, *auto elim*!: *eventually_mono*)
next
fix *r* **assume** *r*: *0* < *r ereal r* > *c*
let *?l* = *limsup* (λ*n*. *ereal* (*norm* (*f n* ∗ *of_real r* ^ *n*) / *norm* (*f* (*Suc n*) ∗ *of_real r* ^ *Suc n*)))
have *?l* = *limsup* (λ*n*. *ereal* (*norm* (*f n*) / (*norm* (*f* (*Suc n*)))) ∗ *ereal* (*inverse r*))
using *r* **by** (*simp add*: *norm_mult norm_power field_split_simps*)
also from *r* **have** ... = *limsup* (λ*n*. *ereal* (*norm* (*f n*) / (*norm* (*f* (*Suc n*))))) ∗ *ereal* (*inverse r*)
by (*intro Limsup_ereal_mult_right*) *simp_all*
also have *limsup* (λ*n*. *ereal* (*norm* (*f n*) / (*norm* (*f* (*Suc n*))))) = *c*
by (*intro lim_imp_Limsup lim*) *simp*
finally have *l*: *?l* = *c* ∗ *ereal* (*inverse r*) **by** *simp*
from *r* **have** *l'*: *c* ∗ *ereal* (*inverse r*) < *1* **by** (*cases c*) (*simp_all add*: *field_simps*)
show ¬*summable* (λ*n*. *norm* (*f n* ∗ *of_real r*^*n*))
by (*rule ratio_test_divergence*) (*insert r nz l l'*, *auto elim*!: *eventually_mono*)
qed

lemma *conv_radius_ratio_limit_ereal_nonzero*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*,*real_normed_div_algebra*}
assumes *nz*: *c* ≠ *0*
assumes *lim*: (λ*n*. *ereal* (*norm* (*f n*) / *norm* (*f* (*Suc n*)))) ⟶ *c*
shows *conv_radius f* = *c*

proof (*rule conv_radius_ratio_limit_ereal*[*OF _ lim*], *rule ccontr*)
assume ¬*eventually* (λ*n*. *f n* ≠ *0*) *sequentially*
hence *frequently* (λ*n*. *f n* = *0*) *sequentially* **by** (*simp add*: *frequently_def*)
hence *frequently* (λ*n*. *ereal* (*norm* (*f n*) / *norm* (*f* (*Suc n*))) = *0*) *sequentially*
by (*force elim*!: *frequently_elim1*)
hence *c* = *0* **by** (*intro limit_frequently_eq*[*OF _ _ lim*]) *auto*
with *nz* **show** *False* **by** *contradiction*
qed

lemma *conv_radius_ratio_limit*:
fixes *f* :: *nat* ⇒ ′*a* :: {*banach*,*real_normed_div_algebra*}
assumes *c*′ = *ereal c*
assumes *nz*: *eventually* (λ*n*. *f n* ≠ *0*) *sequentially*
assumes *lim*: (λ*n*. *norm* (*f n*) / *norm* (*f* (*Suc n*))) ⟶ *c*
shows *conv_radius f* = *c*′
using *assms* **by** (*intro conv_radius_ratio_limit_ereal*) *simp_all*

lemma *conv_radius_ratio_limit_nonzero*:
fixes *f* :: *nat* ⇒ ′*a* :: {*banach*,*real_normed_div_algebra*}
assumes *c*′ = *ereal c*
assumes *nz*: *c* ≠ *0*
assumes *lim*: (λ*n*. *norm* (*f n*) / *norm* (*f* (*Suc n*))) ⟶ *c*
shows *conv_radius f* = *c*′
using *assms* **by** (*intro conv_radius_ratio_limit_ereal_nonzero*) *simp_all*

lemma *conv_radius_cmult_left*:
assumes *c* ≠ (*0* :: ′*a* :: {*banach*, *real_normed_div_algebra*})
shows *conv_radius* (λ*n*. *c* ∗ *f n*) = *conv_radius f*
proof −
have *conv_radius* (λ*n*. *c* ∗ *f n*) =
inverse (*limsup* (λ*n*. *ereal* (*root n* (*norm* (*c* ∗ *f n*)))))
unfolding *conv_radius_def* **..**
also have (λ*n*. *ereal* (*root n* (*norm* (*c* ∗ *f n*)))) =
(λ*n*. *ereal* (*root n* (*norm c*)) ∗ *ereal* (*root n* (*norm* (*f n*))))
by (*rule ext*) (*auto simp*: *norm_mult real_root_mult*)
also have *limsup* ... = *ereal 1* ∗ *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*))))
using *assms* **by** (*intro ereal_limsup_lim_mult tendsto_ereal LIMSEQ_root_const*)
auto
also have *inverse* ... = *conv_radius f* **by** (*simp add*: *conv_radius_def*)
finally show *?thesis* **.**
qed

lemma *conv_radius_cmult_right*:
assumes *c* ≠ (*0* :: ′*a* :: {*banach*, *real_normed_div_algebra*})
shows *conv_radius* (λ*n*. *f n* ∗ *c*) = *conv_radius f*
proof −
have *conv_radius* (λ*n*. *f n* ∗ *c*) = *conv_radius* (λ*n*. *c* ∗ *f n*)
by (*simp add*: *conv_radius_def norm_mult mult.commute*)
with *conv_radius_cmult_left*[*OF assms*, *of f*] **show** *?thesis* **by** *simp*

qed

lemma *conv_radius_mult_power*:
  assumes $c \neq (0 :: 'a :: \{real\_normed\_div\_algebra,banach\})$
  shows $conv\_radius\ (\lambda n.\ c\ \hat{}\ n * f\ n) = conv\_radius\ f\ /\ ereal\ (norm\ c)$
proof –
  have $limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (c\ \hat{}\ n * f\ n)))) =$
      $limsup\ (\lambda n.\ ereal\ (norm\ c) * ereal\ (root\ n\ (norm\ (f\ n))))$
    by (*intro Limsup_eq eventually_mono* [*OF eventually_gt_at_top*[*of 0::nat*]])
      (*auto simp*: *norm_mult norm_power real_root_mult real_root_power*)
  also have ... $= ereal\ (norm\ c) * limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n))))$
    using *assms* **by** (*subst Limsup_ereal_mult_left*[*symmetric*]) *simp_all*
  finally have *A*: $limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (c\ \hat{}\ n * f\ n)))) =$
      $ereal\ (norm\ c) * limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n))))$ **.**
  show *?thesis* **using** *assms*
    apply (*cases* $limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n)))) = 0$)
    apply (*simp add*: *A conv_radius_def*)
    apply (*unfold conv_radius_def A divide_ereal_def*, *simp add*: *mult.commute ereal_inverse_mult*)
    done
qed

lemma *conv_radius_mult_power_right*:
  assumes $c \neq (0 :: 'a :: \{real\_normed\_div\_algebra,banach\})$
  shows $conv\_radius\ (\lambda n.\ f\ n * c\ \hat{}\ n) = conv\_radius\ f\ /\ ereal\ (norm\ c)$
  using *conv_radius_mult_power*[*OF assms*, *of f*]
  unfolding *conv_radius_def* **by** (*simp add*: *mult.commute norm_mult*)

lemma *conv_radius_divide_power*:
  assumes $c \neq (0 :: 'a :: \{real\_normed\_div\_algebra,banach\})$
  shows $conv\_radius\ (\lambda n.\ f\ n\ /\ c\hat{}n) = conv\_radius\ f * ereal\ (norm\ c)$
proof –
  from *assms* **have** $inverse\ c \neq 0$ **by** *simp*
  from *conv_radius_mult_power_right*[*OF this*, *of f*] **show** *?thesis*
    by (*simp add*: *divide_inverse divide_ereal_def assms norm_inverse power_inverse*)
qed

lemma *conv_radius_add_ge*:
  $min\ (conv\_radius\ f)\ (conv\_radius\ g) \leq$
    $conv\_radius\ (\lambda x.\ f\ x + g\ x :: 'a :: \{banach,real\_normed\_div\_algebra\})$
  by (*rule conv_radius_geI_ex'*)
    (*auto simp*: *algebra_simps intro!*: *summable_add summable_in_conv_radius*)

lemma *conv_radius_mult_ge*:
  fixes $f\ g :: nat \Rightarrow ('a :: \{banach,real\_normed\_div\_algebra\})$
  shows $conv\_radius\ (\lambda x.\ \sum i{\leq}x.\ f\ i * g\ (x - i)) \geq min\ (conv\_radius\ f)\ (conv\_radius\ g)$
proof (*rule conv_radius_geI_ex'*)

fix *r* **assume** *r*: $r > 0$ *ereal* $r <$ *min* (*conv_radius f*) (*conv_radius g*)
from *r* **have** *summable* ($\lambda n.$ ($\sum i \le n.$ (*f i* $*$ *of_real* $r\hat{}i$) $*$ (*g* ($n - i$) $*$ *of_real* $r\hat{}(n - i)$)))
by (*intro summable_Cauchy_product abs_summable_in_conv_radius*) *simp_all*
thus *summable* ($\lambda n.$ ($\sum i \le n.$ *f i* $*$ *g* ($n - i$)) $*$ *of_real* $r \hat{}\ n$)
by (*simp add*: *algebra_simps of_real_def power_add* [*symmetric*] *scaleR_sum_right*)
qed

lemma *le_conv_radius_iff*:
fixes *a* :: *nat* $\Rightarrow$ *'a*::{*real_normed_div_algebra*,*banach*}
shows $r \le$ *conv_radius a* $\longleftrightarrow$ ($\forall x.$ *norm* ($x-\xi$) $< r \longrightarrow$ *summable* ($\lambda i.$ *a i* $*$ ($x - \xi$) $\hat{}\ i$))
apply (*intro iffI allI impI summable_in_conv_radius conv_radius_geI_ex*)
apply (*meson less_ereal.simps*(*1*) *not_le order_trans*)
apply (*rule_tac x*=*of_real ra* **in** *exI*, *simp*)
apply (*metis abs_of_nonneg add_diff_cancel_left' less_eq_real_def norm_of_real*)
done

end

4.7 Uniform Limit and Uniform Convergence

theory *Uniform_Limit*
imports *Connected Summation_Tests*
begin

4.7.1 Definition

definition *uniformly_on* :: *'a set* $\Rightarrow$ (*'a* $\Rightarrow$ *'b*::*metric_space*) $\Rightarrow$ (*'a* $\Rightarrow$ *'b*) *filter*
where *uniformly_on S l* = (*INF* $e \in \{0 <..\}$. *principal* {*f*. $\forall x \in S.$ *dist* (*f x*) (*l x*) $< e$})

abbreviation
uniform_limit S f l $\equiv$ *filterlim f* (*uniformly_on S l*)

definition *uniformly_convergent_on* **where**
uniformly_convergent_on X f $\longleftrightarrow$ ($\exists l.$ *uniform_limit X f l sequentially*)

definition *uniformly_Cauchy_on* **where**
uniformly_Cauchy_on X f $\longleftrightarrow$ ($\forall e>0.\ \exists M.\ \forall x \in X.\ \forall (m::nat) \ge M.\ \forall n \ge M.$ *dist* (*f m x*) (*f n x*) $< e$)

proposition *uniform_limit_iff*:
uniform_limit S f l F $\longleftrightarrow$ ($\forall e>0.\ \forall_F\ n$ *in F*. $\forall x \in S.$ *dist* (*f n x*) (*l x*) $< e$)
unfolding *filterlim_iff uniformly_on_def*
by (*subst eventually_INF_base*)
(*fastforce*
simp: *eventually_principal uniformly_on_def*
intro: *bexI*[**where** *x*=*min a b* **for** *a b*]

elim: *eventually_mono*)+

lemma *uniform_limitD*:
uniform_limit S f l F ⟹ *e* > *0* ⟹ ∀$_F$ *n in F*. ∀*x*∈*S*. *dist* (*f n x*) (*l x*) < *e*
by (*simp add*: *uniform_limit_iff*)

lemma *uniform_limitI*:
(⋀*e*. *e* > *0* ⟹ ∀$_F$ *n in F*. ∀*x*∈*S*. *dist* (*f n x*) (*l x*) < *e*) ⟹ *uniform_limit S f l F*
by (*simp add*: *uniform_limit_iff*)

lemma *uniform_limit_sequentially_iff*:
uniform_limit S f l sequentially ⟷ (∀ *e*>*0*. ∃ *N*. ∀ *n*≥*N*. ∀ *x* ∈ *S*. *dist* (*f n x*) (*l x*) < *e*)
unfolding *uniform_limit_iff eventually_sequentially* **..**

lemma *uniform_limit_at_iff*:
uniform_limit S f l (*at x*) ⟷
(∀ *e*>*0*. ∃ *d*>*0*. ∀ *z*. *0* < *dist z x* ∧ *dist z x* < *d* ⟶ (∀ *x*∈*S*. *dist* (*f z x*) (*l x*) < *e*))
unfolding *uniform_limit_iff eventually_at* **by** *simp*

lemma *uniform_limit_at_le_iff*:
uniform_limit S f l (*at x*) ⟷
(∀ *e*>*0*. ∃ *d*>*0*. ∀ *z*. *0* < *dist z x* ∧ *dist z x* < *d* ⟶ (∀ *x*∈*S*. *dist* (*f z x*) (*l x*) ≤ *e*))
unfolding *uniform_limit_iff eventually_at*
by (*fastforce dest*: *spec*[**where** *x* = *e* / *2* **for** *e*])

lemma *metric_uniform_limit_imp_uniform_limit*:
assumes *f*: *uniform_limit S f a F*
assumes *le*: *eventually* (λ*x*. ∀ *y*∈*S*. *dist* (*g x y*) (*b y*) ≤ *dist* (*f x y*) (*a y*)) *F*
shows *uniform_limit S g b F*
proof (*rule uniform_limitI*)
fix *e* :: *real* **assume** *0* < *e*
from *uniform_limitD*[*OF f this*] *le*
show ∀$_F$ *x in F*. ∀ *y*∈*S*. *dist* (*g x y*) (*b y*) < *e*
by *eventually_elim force*
qed

4.7.2 Exchange limits

proposition *swap_uniform_limit*:
assumes *f*: ∀$_F$ *n in F*. (*f n* ⟶ *g n*) (*at x within S*)
assumes *g*: (*g* ⟶ *l*) *F*
assumes *uc*: *uniform_limit S f h F*
assumes ¬*trivial_limit F*
shows (*h* ⟶ *l*) (*at x within S*)
proof (*rule tendstoI*)

fix *e* :: *real*
define e' **where** $e' = e/3$
assume $0 < e$
then have $0 < e'$ **by** (*simp add*: e'*_def*)
from *uniform_limitD*[*OF uc* ‹$0 < e'$›]
have $\forall_F$ *n in F*. $\forall x \in S$. *dist* (*h x*) (*f n x*) $< e'$
  by (*simp add*: *dist_commute*)
moreover
from *f*
have $\forall_F$ *n in F*. $\forall_F$ *x in at x within S*. *dist* (*g n*) (*f n x*) $< e'$
  by *eventually_elim* (*auto dest*!: *tendstoD*[*OF* _ ‹$0 < e'$›] *simp*: *dist_commute*)
moreover
from *tendstoD*[*OF g* ‹$0 < e'$›] **have** $\forall_F$ *x in F*. *dist l* (*g x*) $< e'$
  by (*simp add*: *dist_commute*)
ultimately
have $\forall_F$ _ *in F*. $\forall_F$ *x in at x within S*. *dist* (*h x*) *l* $< e$
proof *eventually_elim*
  case (*elim n*)
  note *fh* = *elim*(*1*)
  note *gl* = *elim*(*3*)
  have $\forall_F$ *x in at x within S*. $x \in S$
    by (*auto simp*: *eventually_at_filter*)
  with *elim*(*2*)
  show *?case*
  proof *eventually_elim*
    case (*elim x*)
    from *fh*[*rule_format*, *OF* ‹$x \in S$›] *elim*(*1*)
    have *dist* (*h x*) (*g n*) $< e' + e'$
      by (*rule dist_triangle_lt*[*OF add_strict_mono*])
    from *dist_triangle_lt*[*OF add_strict_mono*, *OF this gl*]
    show *?case* **by** (*simp add*: e'*_def*)
  qed
qed
thus $\forall_F$ *x in at x within S*. *dist* (*h x*) *l* $< e$
  using *eventually_happens* **by** (*metis* ‹¬*trivial_limit F*›)
qed

4.7.3 Uniform limit theorem

lemma *tendsto_uniform_limitI*:
  assumes *uniform_limit S f l F*
  assumes $x \in S$
  shows ((λy. *f y x*) ⟶ *l x*) *F*
  using *assms*
  by (*auto intro*!: *tendstoI simp*: *eventually_mono dest*!: *uniform_limitD*)

theorem *uniform_limit_theorem*:
  assumes *c*: $\forall_F$ *n in F*. *continuous_on A* (*f n*)
  assumes *ul*: *uniform_limit A f l F*

assumes ¬ *trivial_limit F*
shows *continuous_on A l*
unfolding *continuous_on_def*
proof *safe*
fix *x* **assume** *x* ∈ *A*
then have ∀$_F$ *n in F*. (*f n* ⟶ *f n x*) (*at x within A*) ((λ*n*. *f n x*) ⟶ *l x*) *F*
using *c ul*
by (*auto simp*: *continuous_on_def eventually_mono tendsto_uniform_limitI*)
then show (*l* ⟶ *l x*) (*at x within A*)
by (*rule swap_uniform_limit*) *fact*+
qed

lemma *uniformly_Cauchy_onI*:
assumes ⋀*e*. *e* > *0* ⟹ ∃ *M*. ∀ *x*∈*X*. ∀ *m*≥*M*. ∀ *n*≥*M*. *dist* (*f m x*) (*f n x*) < *e*
shows *uniformly_Cauchy_on X f*
using *assms* **unfolding** *uniformly_Cauchy_on_def* **by** *blast*

lemma *uniformly_Cauchy_onI'*:
assumes ⋀*e*. *e* > *0* ⟹ ∃ *M*. ∀ *x*∈*X*. ∀ *m*≥*M*. ∀ *n*>*m*. *dist* (*f m x*) (*f n x*) < *e*
shows *uniformly_Cauchy_on X f*
proof (*rule uniformly_Cauchy_onI*)
fix *e* :: *real* **assume** *e*: *e* > *0*
from *assms*[*OF this*] **obtain** *M*
where *M*: ⋀*x m n*. *x* ∈ *X* ⟹ *m* ≥ *M* ⟹ *n* > *m* ⟹ *dist* (*f m x*) (*f n x*) < *e* **by** *fast*
{
fix *x m n* **assume** *x*: *x* ∈ *X* **and** *m*: *m* ≥ *M* **and** *n*: *n* ≥ *M*
with *M*[*OF this*(*1*,*2*), *of n*] *M*[*OF this*(*1*,*3*), *of m*] *e* **have** *dist* (*f m x*) (*f n x*) < *e*
by (*cases m n rule*: *linorder_cases*) (*simp_all add*: *dist_commute*)
}
thus ∃ *M*. ∀ *x*∈*X*. ∀ *m*≥*M*. ∀ *n*≥*M*. *dist* (*f m x*) (*f n x*) < *e* **by** *fast*
qed

lemma *uniformly_Cauchy_imp_Cauchy*:
uniformly_Cauchy_on X f ⟹ *x* ∈ *X* ⟹ *Cauchy* (λ*n*. *f n x*)
unfolding *Cauchy_def uniformly_Cauchy_on_def* **by** *fast*

lemma *uniform_limit_cong*:
fixes *f g* :: *'a* ⇒ *'b* ⇒ (*'c* :: *metric_space*) **and** *h i* :: *'b* ⇒ *'c*
assumes *eventually* (λ*y*. ∀ *x*∈*X*. *f y x* = *g y x*) *F*
assumes ⋀*x*. *x* ∈ *X* ⟹ *h x* = *i x*
shows *uniform_limit X f h F* ⟷ *uniform_limit X g i F*
proof −
{
fix *f g* :: *'a* ⇒ *'b* ⇒ *'c* **and** *h i* :: *'b* ⇒ *'c*
assume *C*: *uniform_limit X f h F* **and** *A*: *eventually* (λ*y*. ∀ *x*∈*X*. *f y x* = *g y x*) *F*

and B: $\bigwedge x.\ x \in X \Longrightarrow h\ x = i\ x$
{
fix e ::*real* **assume** $e > 0$
with C **have** *eventually* $(\lambda y.\ \forall x{\in}X.\ dist\ (f\ y\ x)\ (h\ x) < e)\ F$
unfolding *uniform_limit_iff* **by** *blast*
with A **have** *eventually* $(\lambda y.\ \forall x{\in}X.\ dist\ (g\ y\ x)\ (i\ x) < e)\ F$
by *eventually_elim* (*insert* B, *simp_all*)
}
hence *uniform_limit X g i F* **unfolding** *uniform_limit_iff* **by** *blast*
} **note** $A = this$
show *?thesis* **by** (*rule iffI*) (*erule A*; *insert assms*; *simp add*: *eq_commute*)+
qed

lemma *uniform_limit_cong'*:
fixes $f\ g$ :: $'a \Rightarrow 'b \Rightarrow ('c$ :: *metric_space*) **and** $h\ i$ :: $'b \Rightarrow 'c$
assumes $\bigwedge y\ x.\ x \in X \Longrightarrow f\ y\ x = g\ y\ x$
assumes $\bigwedge x.\ x \in X \Longrightarrow h\ x = i\ x$
shows *uniform_limit X f h F* $\longleftrightarrow$ *uniform_limit X g i F*
using *assms* **by** (*intro uniform_limit_cong always_eventually*) *blast+*

lemma *uniformly_convergent_cong*:
assumes *eventually* $(\lambda x.\ \forall y{\in}A.\ f\ x\ y = g\ x\ y)$ *sequentially* $A = B$
shows *uniformly_convergent_on A f* $\longleftrightarrow$ *uniformly_convergent_on B g*
unfolding *uniformly_convergent_on_def assms*(*2*) [*symmetric*]
by (*intro iff_exI uniform_limit_cong eventually_mono* [*OF assms*(*1*)]) *auto*

lemma *uniformly_convergent_uniform_limit_iff*:
uniformly_convergent_on X f $\longleftrightarrow$ *uniform_limit X f* $(\lambda x.\ lim\ (\lambda n.\ f\ n\ x))$ *sequentially*
proof
assume *uniformly_convergent_on X f*
then obtain l **where** l: *uniform_limit X f l sequentially*
unfolding *uniformly_convergent_on_def* **by** *blast*
from l **have** *uniform_limit X f* $(\lambda x.\ lim\ (\lambda n.\ f\ n\ x))$ *sequentially* $\longleftrightarrow$
uniform_limit X f l sequentially
by (*intro uniform_limit_cong' limI tendsto_uniform_limitI*[*of f X l*]) *simp_all*
also note l
finally show *uniform_limit X f* $(\lambda x.\ lim\ (\lambda n.\ f\ n\ x))$ *sequentially* **.**
qed (*auto simp*: *uniformly_convergent_on_def*)

lemma *uniformly_convergentI*: *uniform_limit X f l sequentially* $\Longrightarrow$ *uniformly_convergent_on X f*
unfolding *uniformly_convergent_on_def* **by** *blast*

lemma *uniformly_convergent_on_empty* [*iff*]: *uniformly_convergent_on* {} *f*
by (*simp add*: *uniformly_convergent_on_def uniform_limit_sequentially_iff*)

lemma *uniformly_convergent_on_const* [*simp*,*intro*]:
uniformly_convergent_on A $(\lambda\_.\ c)$
by (*auto simp*: *uniformly_convergent_on_def uniform_limit_iff intro*!: *exI*[*of _ c*])

Cauchy-type criteria for uniform convergence.

```
lemma Cauchy_uniformly_convergent:
  fixes f :: nat ⇒ 'a ⇒ 'b :: complete_space
  assumes uniformly_Cauchy_on X f
  shows   uniformly_convergent_on X f
unfolding uniformly_convergent_uniform_limit_iff uniform_limit_iff
proof safe
  let ?f = λx. lim (λn. f n x)
  fix e :: real assume e: e > 0
  hence e/2 > 0 by simp
  with assms obtain N where N: ⋀x m n. x ∈ X ⟹ m ≥ N ⟹ n ≥ N ⟹
dist (f m x) (f n x) < e/2
    unfolding uniformly_Cauchy_on_def by fast
  show eventually (λn. ∀x∈X. dist (f n x) (?f x) < e) sequentially
    using eventually_ge_at_top[of N]
  proof eventually_elim
    fix n assume n: n ≥ N
    show ∀x∈X. dist (f n x) (?f x) < e
    proof
      fix x assume x: x ∈ X
      with assms have (λn. f n x) ⟶ ?f x
          by (auto dest!: Cauchy_convergent uniformly_Cauchy_imp_Cauchy simp:
convergent_LIMSEQ_iff)
      with ‹e/2 > 0› have eventually (λm. m ≥ N ∧ dist (f m x) (?f x) < e/2)
sequentially
        by (intro tendstoD eventually_conj eventually_ge_at_top)
      then obtain m where m: m ≥ N dist (f m x) (?f x) < e/2
        unfolding eventually_at_top_linorder by blast
      have dist (f n x) (?f x) ≤ dist (f n x) (f m x) + dist (f m x) (?f x)
          by (rule dist_triangle)
      also from x n have ... < e/2 + e/2 by (intro add_strict_mono N m)
      finally show dist (f n x) (?f x) < e by simp
    qed
  qed
qed

lemma uniformly_convergent_Cauchy:
  assumes uniformly_convergent_on X f
  shows uniformly_Cauchy_on X f
proof (rule uniformly_Cauchy_onI)
  fix e::real assume e > 0
  then have 0 < e / 2 by simp
  with assms[unfolded uniformly_convergent_on_def uniform_limit_sequentially_iff]
  obtain l N where l:x ∈ X ⟹ n ≥ N ⟹ dist (f n x) (l x) < e / 2 for n x
    by metis
  from l l have x ∈ X ⟹ n ≥ N ⟹ m ≥ N ⟹ dist (f n x) (f m x) < e for
n m x
    by (rule dist_triangle_half_l)
  then show ∃M. ∀x∈X. ∀m≥M. ∀n≥M. dist (f m x) (f n x) < e by blast
```

qed

lemma *uniformly_convergent_eq_Cauchy*:
uniformly_convergent_on X f = *uniformly_Cauchy_on X f* **for** *f*::*nat* ⇒ *'b* ⇒ *'a*::*complete_space*
using *Cauchy_uniformly_convergent uniformly_convergent_Cauchy* **by** *blast*

lemma *uniformly_convergent_eq_cauchy*:
fixes *s*::*nat* ⇒ *'b* ⇒ *'a*::*complete_space*
shows
(∃ *l*. ∀ *e*>*0*. ∃ *N*. ∀ *n x*. *N* ≤ *n* ∧ *P x* ⟶ *dist*(*s n x*)(*l x*) < *e*) ⟷
(∀ *e*>*0*. ∃ *N*. ∀ *m n x*. *N* ≤ *m* ∧ *N* ≤ *n* ∧ *P x* ⟶ *dist* (*s m x*) (*s n x*) < *e*)
proof −
have ∗: (∀ *n*≥*N*. ∀ *x*. *Q x* ⟶ *R n x*) ⟷ (∀ *n x*. *N* ≤ *n* ∧ *Q x* ⟶ *R n x*)
(∀ *x*. *Q x* ⟶ (∀ *m*≥*N*. ∀ *n*≥*N*. *S n m x*)) ⟷ (∀ *m n x*. *N* ≤ *m* ∧ *N* ≤ *n* ∧ *Q x* ⟶ *S n m x*)
for *N*::*nat* **and** *Q*::*'b* ⇒ *bool* **and** *R S*
by *blast*+
show *?thesis*
using *uniformly_convergent_eq_Cauchy*[*of Collect P s*]
unfolding *uniformly_convergent_on_def uniformly_Cauchy_on_def uniform_limit_sequentially_iff*
by (*simp add*: ∗)
qed

lemma *uniformly_cauchy_imp_uniformly_convergent*:
fixes *s* :: *nat* ⇒ *'a* ⇒ *'b*::*complete_space*
assumes ∀ *e*>*0*.∃ *N*. ∀ *m* (*n*::*nat*) *x*. *N* ≤ *m* ∧ *N* ≤ *n* ∧ *P x* −−> *dist*(*s m x*)(*s n x*) < *e*
and ∀ *x*. *P x* −−> (∀ *e*>*0*. ∃ *N*. ∀ *n*. *N* ≤ *n* ⟶ *dist*(*s n x*)(*l x*) < *e*)
shows ∀ *e*>*0*. ∃ *N*. ∀ *n x*. *N* ≤ *n* ∧ *P x* ⟶ *dist*(*s n x*)(*l x*) < *e*
proof −
obtain *l'* **where** *l*:∀ *e*>*0*. ∃ *N*. ∀ *n x*. *N* ≤ *n* ∧ *P x* ⟶ *dist* (*s n x*) (*l' x*) < *e*
using *assms*(*1*) **unfolding** *uniformly_convergent_eq_cauchy*[*symmetric*] **by** *auto*
moreover
{
fix *x*
assume *P x*
then have *l x* = *l' x*
using *tendsto_unique*[*OF trivial_limit_sequentially*, *of* λ*n*. *s n x l x l' x*]
using *l* **and** *assms*(*2*) **unfolding** *lim_sequentially* **by** *blast*
}
ultimately show *?thesis* **by** *auto*
qed

TODO: remove explicit formulations (∃ *l*. ∀ *e*>*0*. ∃ *N*. ∀ *n x*. *N* ≤ *n* ∧ *?P x* ⟶ *dist* (*?s n x*) (*l x*) < *e*) = (∀ *e*>*0*. ∃ *N*. ∀ *m n x*. *N* ≤ *m* ∧ *N* ≤ *n* ∧ *?P x* ⟶ *dist* (*?s m x*) (*?s n x*) < *e*)

⟦∀ *e*>*0*. ∃ *N*. ∀ *m n x*. *N* ≤ *m* ∧ *N* ≤ *n* ∧ *?P x* ⟶ *dist* (*?s m x*) (*?s n x*) < *e*; ∀ *x*. *?P x* ⟶ (∀ *e*>*0*. ∃ *N*. ∀ *n*≥*N*. *dist* (*?s n x*) (*?l x*) < *e*)⟧ ⟹

```
∀ e>0. ∃ N. ∀ n x. N ≤ n ∧ ?P x ⟶ dist (?s n x) (?l x) < e?!

lemma uniformly_convergent_imp_convergent:
  uniformly_convergent_on X f ⟹ x ∈ X ⟹ convergent (λn. f n x)
  unfolding uniformly_convergent_on_def convergent_def
  by (auto dest: tendsto_uniform_limitI)
```

4.7.4 Weierstrass M-Test

```
proposition Weierstrass_m_test_ev:
fixes f :: _ ⇒ _ ⇒ _ :: banach
assumes eventually (λn. ∀ x∈A. norm (f n x) ≤ M n) sequentially
assumes summable M
shows uniform_limit A (λn x. ∑ i<n. f i x) (λx. suminf (λi. f i x)) sequentially
proof (rule uniform_limitI)
  fix e :: real
  assume 0 < e
  from suminf_exist_split[OF ‹0 < e› ‹summable M›]
  have ∀F k in sequentially. norm (∑ i. M (i + k)) < e
    by (auto simp: eventually_sequentially)
  with eventually_all_ge_at_top[OF assms(1)]
    show ∀F n in sequentially. ∀ x∈A. dist (∑ i<n. f i x) (∑ i. f i x) < e
  proof eventually_elim
    case (elim k)
    show ?case
    proof safe
      fix x assume x ∈ A
      have ∃ N. ∀ n≥N. norm (f n x) ≤ M n
        using assms(1) ‹x ∈ A› by (force simp: eventually_at_top_linorder)
      hence summable_norm_f: summable (λn. norm (f n x))
        by(rule summable_norm_comparison_test[OF _ ‹summable M›])
      have summable_f: summable (λn. f n x)
        using summable_norm_cancel[OF summable_norm_f] .
      have summable_norm_f_plus_k: summable (λi. norm (f (i + k) x))
        using summable_ignore_initial_segment[OF summable_norm_f]
        by auto
      have summable_M_plus_k: summable (λi. M (i + k))
        using summable_ignore_initial_segment[OF ‹summable M›]
        by auto

      have dist (∑ i<k. f i x) (∑ i. f i x) = norm ((∑ i. f i x) − (∑ i<k. f i x))
        using dist_norm dist_commute by (subst dist_commute)
      also have ... = norm (∑ i. f (i + k) x)
        using suminf_minus_initial_segment[OF summable_f, where k=k] by simp
      also have ... ≤ (∑ i. norm (f (i + k) x))
        using summable_norm[OF summable_norm_f_plus_k] .
      also have ... ≤ (∑ i. M (i + k))
        by (rule suminf_le[OF _ summable_norm_f_plus_k summable_M_plus_k])
          (insert elim(1) ‹x ∈ A›, simp)
      finally show dist (∑ i<k. f i x) (∑ i. f i x) < e
```

using *elim* **by** *auto*
qed
qed
qed

Alternative version, formulated as in HOL Light

corollary *series_comparison_uniform*:
fixes f :: _ ⇒ *nat* ⇒ _ :: *banach*
assumes g: *summable g* **and** *le*: $\bigwedge n\ x.\ N \le n \wedge x \in A \Longrightarrow$ *norm*$(f\ x\ n) \le g\ n$
shows $\exists l.\ \forall e.\ 0 < e \longrightarrow (\exists N.\ \forall n\ x.\ N \le n \wedge x \in A \longrightarrow$ *dist*(*sum* $(f\ x)$ $\{..<n\})\ (l\ x) < e)$
proof −
have *1*: $\forall_F$ *n in sequentially*. $\forall x \in A$. *norm* $(f\ x\ n) \le g\ n$
using *le eventually_sequentially* **by** *auto*
show *?thesis*
apply (*rule_tac x*=($\lambda x.\ \sum i.\ f\ x\ i$) **in** *exI*)
apply (*metis* (*no_types*, *lifting*) *eventually_sequentially uniform_limitD* [*OF Weierstrass_m_test_ev* [*OF 1 g*]])
done
qed

corollary *Weierstrass_m_test*:
fixes f :: _ ⇒ _ ⇒ _ :: *banach*
assumes $\bigwedge n\ x.\ x \in A \Longrightarrow$ *norm* $(f\ n\ x) \le M\ n$
assumes *summable M*
shows *uniform_limit A* ($\lambda n\ x.\ \sum i<n.\ f\ i\ x$) ($\lambda x.$ *suminf* ($\lambda i.\ f\ i\ x$)) *sequentially*
using *assms* **by** (*intro Weierstrass_m_test_ev always_eventually*) *auto*

corollary *Weierstrass_m_test′_ev*:
fixes f :: _ ⇒ _ ⇒ _ :: *banach*
assumes *eventually* ($\lambda n.\ \forall x \in A.$ *norm* $(f\ n\ x) \le M\ n$) *sequentially summable M*
shows *uniformly_convergent_on A* ($\lambda n\ x.\ \sum i<n.\ f\ i\ x$)
unfolding *uniformly_convergent_on_def* **by** (*rule exI*, *rule Weierstrass_m_test_ev*[*OF assms*])

corollary *Weierstrass_m_test′*:
fixes f :: _ ⇒ _ ⇒ _ :: *banach*
assumes $\bigwedge n\ x.\ x \in A \Longrightarrow$ *norm* $(f\ n\ x) \le M\ n$ *summable M*
shows *uniformly_convergent_on A* ($\lambda n\ x.\ \sum i<n.\ f\ i\ x$)
unfolding *uniformly_convergent_on_def* **by** (*rule exI*, *rule Weierstrass_m_test*[*OF assms*])

lemma *uniform_limit_eq_rhs*: *uniform_limit X f l F* $\Longrightarrow l = m \Longrightarrow$ *uniform_limit X f m F*
by *simp*

4.7.5 Structural introduction rules

named_theorems *uniform_limit_intros introduction rules for uniform_limit*

```
setup ‹
  Global_Theory.add_thms_dynamic (binding‹uniform_limit_eq_intros›,
    fn context =>
     Named_Theorems.get (Context.proof_of context) named_theorems‹uniform_limit_intros›
       |> map_filter (try (fn thm => @{thm uniform_limit_eq_rhs} OF [thm])))
›

lemma (in bounded_linear) uniform_limit[uniform_limit_intros]:
  assumes uniform_limit X g l F
  shows uniform_limit X (λa b. f (g a b)) (λa. f (l a)) F
proof (rule uniform_limitI)
  fix e::real
  from pos_bounded obtain K
    where K: ⋀x y. dist (f x) (f y) ≤ K * dist x y K > 0
    by (auto simp: ac_simps dist_norm diff[symmetric])
  assume 0 < e with ‹K > 0› have e / K > 0 by simp
  from uniform_limitD[OF assms this]
  show ∀F n in F. ∀x∈X. dist (f (g n x)) (f (l x)) < e
    by eventually_elim (metis le_less_trans mult.commute pos_less_divide_eq K)
qed

lemma (in bounded_linear) uniformly_convergent_on:
  assumes uniformly_convergent_on A g
  shows   uniformly_convergent_on A (λx y. f (g x y))
proof -
  from assms obtain l where uniform_limit A g l sequentially
    unfolding uniformly_convergent_on_def by blast
  hence uniform_limit A (λx y. f (g x y)) (λx. f (l x)) sequentially
    by (rule uniform_limit)
  thus ?thesis unfolding uniformly_convergent_on_def by blast
qed

lemmas bounded_linear_uniform_limit_intros[uniform_limit_intros] =
  bounded_linear.uniform_limit[OF bounded_linear_Im]
  bounded_linear.uniform_limit[OF bounded_linear_Re]
  bounded_linear.uniform_limit[OF bounded_linear_cnj]
  bounded_linear.uniform_limit[OF bounded_linear_fst]
  bounded_linear.uniform_limit[OF bounded_linear_snd]
  bounded_linear.uniform_limit[OF bounded_linear_zero]
  bounded_linear.uniform_limit[OF bounded_linear_of_real]
  bounded_linear.uniform_limit[OF bounded_linear_inner_left]
  bounded_linear.uniform_limit[OF bounded_linear_inner_right]
  bounded_linear.uniform_limit[OF bounded_linear_divide]
  bounded_linear.uniform_limit[OF bounded_linear_scaleR_right]
  bounded_linear.uniform_limit[OF bounded_linear_mult_left]
  bounded_linear.uniform_limit[OF bounded_linear_mult_right]
  bounded_linear.uniform_limit[OF bounded_linear_scaleR_left]
```

lemmas *uniform_limit_uminus*[*uniform_limit_intros*] =
bounded_linear.*uniform_limit*[*OF bounded_linear_minus*[*OF bounded_linear_ident*]]

lemma *uniform_limit_const*[*uniform_limit_intros*]: *uniform_limit S* (λx. *c*) *c f*
by (*auto intro*!: *uniform_limitI*)

lemma *uniform_limit_add*[*uniform_limit_intros*]:
fixes *f g*::'*a* ⇒ '*b* ⇒ '*c*::*real_normed_vector*
assumes *uniform_limit X f l F*
assumes *uniform_limit X g m F*
shows *uniform_limit X* (λa *b*. *f a b* + *g a b*) (λa. *l a* + *m a*) *F*
proof (*rule uniform_limitI*)
fix *e*::*real*
assume *0* < *e*
hence *0* < *e* / *2* **by** *simp*
from
uniform_limitD[*OF assms*(*1*) *this*]
uniform_limitD[*OF assms*(*2*) *this*]
show $\forall_F$ *n in F*. ∀ *x*∈*X*. *dist* (*f n x* + *g n x*) (*l x* + *m x*) < *e*
by *eventually_elim* (*simp add*: *dist_triangle_add_half*)
qed

lemma *uniform_limit_minus*[*uniform_limit_intros*]:
fixes *f g*::'*a* ⇒ '*b* ⇒ '*c*::*real_normed_vector*
assumes *uniform_limit X f l F*
assumes *uniform_limit X g m F*
shows *uniform_limit X* (λa *b*. *f a b* − *g a b*) (λa. *l a* − *m a*) *F*
unfolding *diff_conv_add_uminus*
by (*rule uniform_limit_intros assms*)+

lemma *uniform_limit_norm*[*uniform_limit_intros*]:
assumes *uniform_limit S g l f*
shows *uniform_limit S* (λx *y*. *norm* (*g x y*)) (λx. *norm* (*l x*)) *f*
using *assms*
apply (*rule metric_uniform_limit_imp_uniform_limit*)
apply (*rule eventuallyI*)
by (*metis dist_norm norm_triangle_ineq3 real_norm_def*)

lemma (**in** *bounded_bilinear*) *bounded_uniform_limit*[*uniform_limit_intros*]:
assumes *uniform_limit X f l F*
assumes *uniform_limit X g m F*
assumes *bounded* (*m* ‘ *X*)
assumes *bounded* (*l* ‘ *X*)
shows *uniform_limit X* (λa *b*. *prod* (*f a b*) (*g a b*)) (λa. *prod* (*l a*) (*m a*)) *F*
proof (*rule uniform_limitI*)
fix *e*::*real*
from *pos_bounded* **obtain** *K* **where** *K*:
0 < *K* ⋀*a b*. *norm* (*prod a b*) ≤ *norm a* ∗ *norm b* ∗ *K*
by *auto*

```
hence sqrt (K*4) > 0 by simp

from assms obtain Km Kl
where Km: Km > 0 ⋀x. x ∈ X ⟹ norm (m x) ≤ Km
  and Kl: Kl > 0 ⋀x. x ∈ X ⟹ norm (l x) ≤ Kl
  by (auto simp: bounded_pos)
hence K * Km * 4 > 0 K * Kl * 4 > 0
  using ‹K > 0›
  by simp_all
assume 0 < e

hence sqrt e > 0 by simp
from uniform_limitD[OF assms(1) divide_pos_pos[OF this ‹sqrt (K*4) > 0›]]
  uniform_limitD[OF assms(2) divide_pos_pos[OF this ‹sqrt (K*4) > 0›]]
  uniform_limitD[OF assms(1) divide_pos_pos[OF ‹e > 0› ‹K * Km * 4 > 0›]]
  uniform_limitD[OF assms(2) divide_pos_pos[OF ‹e > 0› ‹K * Kl * 4 > 0›]]
show ∀F n in F. ∀x∈X. dist (prod (f n x) (g n x)) (prod (l x) (m x)) < e
proof eventually_elim
  case (elim n)
  show ?case
  proof safe
    fix x assume x ∈ X
    have dist (prod (f n x) (g n x)) (prod (l x) (m x)) ≤
      norm (prod (f n x − l x) (g n x − m x)) +
      norm (prod (f n x − l x) (m x)) +
      norm (prod (l x) (g n x − m x))
     by (auto simp: dist_norm prod_diff_prod intro: order_trans norm_triangle_ineq
add_mono)
    also note K(2)[of f n x − l x g n x − m x]
    also from elim(1)[THEN bspec, OF ‹_ ∈ X›, unfolded dist_norm]
    have norm (f n x − l x) ≤ sqrt e / sqrt (K * 4)
      by simp
    also from elim(2)[THEN bspec, OF ‹_ ∈ X›, unfolded dist_norm]
    have norm (g n x − m x) ≤ sqrt e / sqrt (K * 4)
      by simp
    also have sqrt e / sqrt (K * 4) * (sqrt e / sqrt (K * 4)) * K = e / 4
      using ‹K > 0› ‹e > 0› by auto
    also note K(2)[of f n x − l x m x]
    also note K(2)[of l x g n x − m x]
    also from elim(3)[THEN bspec, OF ‹_ ∈ X›, unfolded dist_norm]
    have norm (f n x − l x) ≤ e / (K * Km * 4)
      by simp
    also from elim(4)[THEN bspec, OF ‹_ ∈ X›, unfolded dist_norm]
    have norm (g n x − m x) ≤ e / (K * Kl * 4)
      by simp
    also note Kl(2)[OF ‹_ ∈ X›]
    also note Km(2)[OF ‹_ ∈ X›]
    also have e / (K * Km * 4) * Km * K = e / 4
      using ‹K > 0› ‹Km > 0› by simp
```

```
    also have  Kl * (e / (K * Kl * 4)) * K = e / 4
      using ‹K > 0› ‹Kl > 0› by simp
    also have e / 4 + e / 4 + e / 4 < e using ‹e > 0› by simp
    finally show dist (prod (f n x) (g n x)) (prod (l x) (m x)) < e
      using ‹K > 0› ‹Kl > 0› ‹Km > 0› ‹e > 0›
      by (simp add: algebra_simps mult_right_mono divide_right_mono)
  qed
 qed
qed

lemmas bounded_bilinear_bounded_uniform_limit_intros[uniform_limit_intros] =
 bounded_bilinear.bounded_uniform_limit[OF Inner_Product.bounded_bilinear_inner]
 bounded_bilinear.bounded_uniform_limit[OF Real_Vector_Spaces.bounded_bilinear_mult]
 bounded_bilinear.bounded_uniform_limit[OF Real_Vector_Spaces.bounded_bilinear_scaleR]

lemma uniform_lim_mult:
 fixes f :: 'a ⇒ 'b ⇒ 'c::real_normed_algebra
 assumes f: uniform_limit S f l F
    and g: uniform_limit S g m F
    and l: bounded (l ` S)
    and m: bounded (m ` S)
   shows uniform_limit S (λa b. f a b * g a b) (λa. l a * m a) F
 by (intro bounded_bilinear_bounded_uniform_limit_intros assms)

lemma uniform_lim_inverse:
 fixes f :: 'a ⇒ 'b ⇒ 'c::real_normed_field
 assumes f: uniform_limit S f l F
    and b: ⋀x. x ∈ S ⟹ b ≤ norm(l x)
    and b > 0
   shows uniform_limit S (λx y. inverse (f x y)) (inverse ∘ l) F
proof (rule uniform_limitI)
 fix e::real
 assume e > 0
 have lte: dist (inverse (f x y)) ((inverse ∘ l) y) < e
       if b/2 ≤ norm (f x y) norm (f x y − l y) < e * b^2 / 2 y ∈ S
       for x y
 proof −
   have [simp]: l y ≠ 0 f x y ≠ 0
     using ‹b > 0› that b [OF ‹y ∈ S›] by fastforce+
   have norm (l y − f x y) <  e * b^2 / 2
     by (metis norm_minus_commute that(2))
   also have ... ≤ e * (norm (f x y) * norm (l y))
     using ‹e > 0› that b [OF ‹y ∈ S›] apply (simp add: power2_eq_square)
     by (metis ‹b > 0› less_eq_real_def mult.left_commute mult_mono')
   finally show ?thesis
     by (auto simp: dist_norm field_split_simps norm_mult norm_divide)
 qed
 have ∀F n in F. ∀x∈S. dist (f n x) (l x) < b/2
   using uniform_limitD [OF f, of b/2] by (simp add: ‹b > 0›)
```

```
  then have ∀F x in F. ∀y∈S. b/2 ≤ norm (f x y)
    apply (rule eventually_mono)
    using b apply (simp only: dist_norm)
   by (metis (no_types, hide_lams) diff_zero dist_commute dist_norm norm_triangle_half_l
not_less)
  then have ∀F x in F. ∀y∈S. b/2 ≤ norm (f x y) ∧ norm (f x y − l y) < e ∗
b² / 2
    apply (simp only: ball_conj_distrib dist_norm [symmetric])
    apply (rule eventually_conj, assumption)
      apply (rule uniform_limitD [OF f, of e ∗ b ˆ 2 / 2])
    using ‹b > 0› ‹e > 0› by auto
  then show ∀F x in F. ∀y∈S. dist (inverse (f x y)) ((inverse ∘ l) y) < e
    using lte by (force intro: eventually_mono)
qed

lemma uniform_lim_divide:
  fixes f :: 'a ⇒ 'b ⇒ 'c::real_normed_field
  assumes f: uniform_limit S f l F
      and g: uniform_limit S g m F
      and l: bounded (l ‘ S)
      and b: ⋀x. x ∈ S ⟹ b ≤ norm(m x)
      and b > 0
    shows uniform_limit S (λa b. f a b / g a b) (λa. l a / m a) F
proof −
  have m: bounded ((inverse ∘ m) ‘ S)
    using b ‹b > 0›
    apply (simp add: bounded_iff)
    by (metis le_imp_inverse_le norm_inverse)
  have uniform_limit S (λa b. f a b ∗ inverse (g a b))
         (λa. l a ∗ (inverse ∘ m) a) F
    by (rule uniform_lim_mult [OF f uniform_lim_inverse [OF g b ‹b > 0›] l m])
  then show ?thesis
    by (simp add: field_class.field_divide_inverse)
qed

lemma uniform_limit_null_comparison:
  assumes ∀F x in F. ∀a∈S. norm (f x a) ≤ g x a
  assumes uniform_limit S g (λ_. 0) F
  shows uniform_limit S f (λ_. 0) F
  using assms(2)
proof (rule metric_uniform_limit_imp_uniform_limit)
  show ∀F x in F. ∀y∈S. dist (f x y) 0 ≤ dist (g x y) 0
    using assms(1) by (rule eventually_mono) (force simp add: dist_norm)
qed

lemma uniform_limit_on_Un:
  uniform_limit I f g F ⟹ uniform_limit J f g F ⟹ uniform_limit (I ∪ J) f g F
  by (auto intro!: uniform_limitI dest!: uniform_limitD elim: eventually_elim2)
```

```
lemma uniform_limit_on_empty [iff]:
  uniform_limit {} f g F
  by (auto intro!: uniform_limitI)

lemma uniform_limit_on_UNION:
  assumes finite S
  assumes ⋀s. s ∈ S ⟹ uniform_limit (h s) f g F
  shows uniform_limit (⋃(h ‘ S)) f g F
  using assms
  by induct (auto intro: uniform_limit_on_empty uniform_limit_on_Un)

lemma uniform_limit_on_Union:
  assumes finite I
  assumes ⋀J. J ∈ I ⟹ uniform_limit J f g F
  shows uniform_limit (Union I) f g F
  by (metis SUP_identity_eq assms uniform_limit_on_UNION)

lemma uniform_limit_on_subset:
  uniform_limit J f g F ⟹ I ⊆ J ⟹ uniform_limit I f g F
  by (auto intro!: uniform_limitI dest!: uniform_limitD intro: eventually_mono)

lemma uniform_limit_bounded:
  fixes f::'i ⇒ 'a::topological_space ⇒ 'b::metric_space
  assumes l: uniform_limit S f l F
  assumes bnd: ∀F i in F. bounded (f i ‘ S)
  assumes F ≠ bot
  shows bounded (l ‘ S)
proof −
  from l have ∀F n in F. ∀x∈S. dist (l x) (f n x) < 1
    by (auto simp: uniform_limit_iff dist_commute dest!: spec[where x=1])
  with bnd
  have ∀F n in F. ∃M. ∀x∈S. dist undefined (l x) ≤ M + 1
    by eventually_elim
      (auto intro!: order_trans[OF dist_triangle2 add_mono] intro: less_imp_le
        simp: bounded_any_center[where a=undefined])
  then show ?thesis using assms
   by (auto simp: bounded_any_center[where a=undefined] dest!: eventually_happens)
qed

lemma uniformly_convergent_add:
  uniformly_convergent_on A f ⟹ uniformly_convergent_on A g⟹
      uniformly_convergent_on A (λk x. f k x + g k x :: 'a :: {real_normed_algebra})
  unfolding uniformly_convergent_on_def by (blast dest: uniform_limit_add)

lemma uniformly_convergent_minus:
  uniformly_convergent_on A f ⟹ uniformly_convergent_on A g⟹
      uniformly_convergent_on A (λk x. f k x − g k x :: 'a :: {real_normed_algebra})
  unfolding uniformly_convergent_on_def by (blast dest: uniform_limit_minus)
```

lemma *uniformly_convergent_mult*:
uniformly_convergent_on A f $\Longrightarrow$
uniformly_convergent_on A ($\lambda k\ x.\ c * f\ k\ x$ :: $'a$ :: {*real_normed_algebra*})
unfolding *uniformly_convergent_on_def*
by (*blast dest*: *bounded_linear_uniform_limit_intros*(*13*))

4.7.6 Power series and uniform convergence

proposition *powser_uniformly_convergent*:
fixes *a* :: *nat* $\Rightarrow$ $'a$::{*real_normed_div_algebra*,*banach*}
assumes $r <$ *conv_radius a*
shows *uniformly_convergent_on* (*cball* ξ *r*) ($\lambda n\ x.\ \sum i{<}n.\ a\ i * (x - \xi)$ ^ *i*)
proof (*cases* $0 \le r$)
case *True*
then have $*$: *summable* ($\lambda n.$ *norm* (*a n*) $*$ *r* ^ *n*)
using *abs_summable_in_conv_radius* [*of of_real r a*] *assms*
by (*simp add*: *norm_mult norm_power*)
show *?thesis*
by (*simp add*: *Weierstrass_m_test'_ev* [*OF _ $*$*] *norm_mult norm_power*
mult_left_mono power_mono dist_norm norm_minus_commute)
next
case *False* **then show** *?thesis* **by** (*simp add*: *not_le*)
qed

lemma *powser_uniform_limit*:
fixes *a* :: *nat* $\Rightarrow$ $'a$::{*real_normed_div_algebra*,*banach*}
assumes $r <$ *conv_radius a*
shows *uniform_limit* (*cball* ξ *r*) ($\lambda n\ x.\ \sum i{<}n.\ a\ i * (x - \xi)$ ^ *i*) ($\lambda x.$ *suminf* ($\lambda i.\ a\ i * (x - \xi)$ ^ *i*)) *sequentially*
using *powser_uniformly_convergent* [*OF assms*]
by (*simp add*: *Uniform_Limit.uniformly_convergent_uniform_limit_iff Series.suminf_eq_lim*)

lemma *powser_continuous_suminf*:
fixes *a* :: *nat* $\Rightarrow$ $'a$::{*real_normed_div_algebra*,*banach*}
assumes $r <$ *conv_radius a*
shows *continuous_on* (*cball* ξ *r*) ($\lambda x.$ *suminf* ($\lambda i.\ a\ i * (x - \xi)$ ^ *i*))
apply (*rule uniform_limit_theorem* [*OF _ powser_uniform_limit*])
apply (*rule eventuallyI continuous_intros assms*)+
apply (*simp add*:)
done

lemma *powser_continuous_sums*:
fixes *a* :: *nat* $\Rightarrow$ $'a$::{*real_normed_div_algebra*,*banach*}
assumes *r*: $r <$ *conv_radius a*
and *sm*: $\bigwedge x.\ x \in$ *cball* ξ *r* $\Longrightarrow$ ($\lambda n.\ a\ n * (x - \xi)$ ^ *n*) *sums* (*f x*)
shows *continuous_on* (*cball* ξ *r*) *f*
apply (*rule continuous_on_cong* [*THEN iffD1*, *OF refl _ powser_continuous_suminf* [*OF r*]])
using *sm sums_unique* **by** *fastforce*

lemmas *uniform_limit_subset_union* = *uniform_limit_on_subset*[*OF uniform_limit_on_Union*]

end

theory *Function_Topology*
 imports
  Elementary_Topology
  Abstract_Limits
  Connected
begin

4.8 Function Topology

We want to define the general product topology.

The product topology on a product of topological spaces is generated by the sets which are products of open sets along finitely many coordinates, and the whole space along the other coordinates. This is the coarsest topology for which the projection to each factor is continuous.

To form a product of objects in Isabelle/HOL, all these objects should be subsets of a common type 'a. The product is then Pi_E *I X*, the set of elements from $'i$ to $'a$ such that the *i*-th coordinate belongs to *X i* for all *i* ∈ *I*.

Hence, to form a product of topological spaces, all these spaces should be subsets of a common type. This means that type classes can not be used to define such a product if one wants to take the product of different topological spaces (as the type 'a can only be given one structure of topological space using type classes). On the other hand, one can define different topologies (as introduced in *thy*) on one type, and these topologies do not need to share the same maximal open set. Hence, one can form a product of topologies in this sense, and this works well. The big caveat is that it does not interact well with the main body of topology in Isabelle/HOL defined in terms of type classes... For instance, continuity of maps is not defined in this setting.

As the product of different topological spaces is very important in several areas of mathematics (for instance adeles), I introduce below the product topology in terms of topologies, and reformulate afterwards the consequences in terms of type classes (which are of course very handy for applications).

Given this limitation, it looks to me that it would be very beneficial to revamp the theory of topological spaces in Isabelle/HOL in terms of topologies, and keep the statements involving type classes as consequences of more general statements in terms of topologies (but I am probably too naive here).

Here is an example of a reformulation using topologies. Let

$continuous\_map\ T1\ T2\ f =$
  $((\forall\ U.\ openin\ T2\ U \longrightarrow openin\ T1\ (f-`U \cap topspace(T1)))$
    $\wedge\ (f`(topspace\ T1) \subseteq (topspace\ T2)))$

be the natural continuity definition of a map from the topology *T1* to the topology *T2*. Then the current *continuous_on* (with type classes) can be redefined as

$continuous\_on\ s\ f =$
  $continuous\_map\ (top\_of\_set\ s)\ (topology\ euclidean)\ f$

In fact, I need *continuous_map* to express the continuity of the projection on subfactors for the product topology, in Lemma *continuous_on_restrict_product_topology*, and I show the above equivalence in Lemma *continuous_map_iff_continuous*.

I only develop the basics of the product topology in this theory. The most important missing piece is Tychonov theorem, stating that a product of compact spaces is always compact for the product topology, even when the product is not finite (or even countable).

I realized afterwards that this theory has a lot in common with `~~/src/HOL/Library/Finite_Map.thy`.

4.8.1 The product topology

We can now define the product topology, as generated by the sets which are products of open sets along finitely many coordinates, and the whole space along the other coordinates. Equivalently, it is generated by sets which are one open set along one single coordinate, and the whole space along other coordinates. In fact, this is only equivalent for nonempty products, but for the empty product the first formulation is better (the second one gives an empty product space, while an empty product should have exactly one point, equal to *undefined* along all coordinates.

So, we use the first formulation, which moreover seems to give rise to more straightforward proofs.

definition $product\_topology::('i \Rightarrow ('a\ topology)) \Rightarrow ('i\ set) \Rightarrow (('i \Rightarrow 'a)\ topology)$
  where $product\_topology\ T\ I =$
    $topology\_generated\_by\ \{(\Pi_E\ i{\in}I.\ X\ i)\ |X.\ (\forall i.\ openin\ (T\ i)\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq topspace\ (T\ i)\}\}$

abbreviation $powertop\_real :: 'a\ set \Rightarrow ('a \Rightarrow real)\ topology$
  where $powertop\_real \equiv product\_topology\ (\lambda i.\ euclideanreal)$

The total set of the product topology is the product of the total sets along each coordinate.

proposition *product_topology*:
  $product\_topology\ X\ I =$

topology
(*arbitrary union_of*
((*finite intersection_of*
($\lambda F.\ \exists\, i\ U.\ F = \{f.\ f\ i \in U\} \wedge i \in I \wedge$ *openin* $(X\ i)\ U$))
relative_to ($\Pi_E\ i{\in}I.$ *topspace* $(X\ i)$))))
(**is** _ = *topology* (_ *union_of* ((_ *intersection_of* ?Ψ) *relative_to* ?*TOP*)))
proof −
let ?Ω = ($\lambda F.\ \exists\, Y.\ F = Pi_E\ I\ Y \wedge (\forall\, i.$ *openin* $(X\ i)\ (Y\ i)) \wedge$ *finite* $\{i.\ Y\ i \neq$ *topspace* $(X\ i)\}$)
have ∗: (*finite′ intersection_of* ?Ω) *A* = (*finite intersection_of* ?Ψ *relative_to* ?*TOP*) *A* **for** *A*
proof −
have *1*: $\exists\, U.\ (\exists\mathcal{U}.$ *finite* $\mathcal{U} \wedge \mathcal{U} \subseteq$ *Collect* ?Ψ $\wedge \bigcap\mathcal{U} = U) \wedge$?*TOP* $\cap\ U = \bigcap\mathcal{U}$
if $\mathcal{U}$: $\mathcal{U} \subseteq$ *Collect* ?Ω **and** *finite′* $\mathcal{U}$ $A = \bigcap\mathcal{U}$ $\mathcal{U} \neq \{\}$ **for** $\mathcal{U}$
proof −
have $\forall\, U \in \mathcal{U}.\ \exists\, Y.\ U = Pi_E\ I\ Y \wedge (\forall\, i.$ *openin* $(X\ i)\ (Y\ i)) \wedge$ *finite* $\{i.\ Y\ i \neq$ *topspace* $(X\ i)\}$
using $\mathcal{U}$ **by** *auto*
then obtain *Y* **where** *Y*: $\bigwedge U.\ U \in \mathcal{U} \Longrightarrow U = Pi_E\ I\ (Y\ U) \wedge (\forall\, i.$ *openin* $(X\ i)\ (Y\ U\ i)) \wedge$ *finite* $\{i.\ (Y\ U)\ i \neq$ *topspace* $(X\ i)\}$
by *metis*
obtain *U* **where** $U \in \mathcal{U}$
using ‹$\mathcal{U} \neq \{\}$› **by** *blast*
let ?*F* = $\lambda U.\ (\lambda i.\ \{f.\ f\ i \in Y\ U\ i\})$ ‘ $\{i \in I.\ Y\ U\ i \neq$ *topspace* $(X\ i)\}$
show ?*thesis*
proof (*intro conjI exI*)
show *finite* ($\bigcup U{\in}\mathcal{U}.$?*F U*)
using *Y* ‹*finite′* $\mathcal{U}$› **by** *auto*
show ?*TOP* $\cap \bigcap(\bigcup U{\in}\mathcal{U}.$?*F U*) $= \bigcap\mathcal{U}$
proof
have ∗: $f \in U$
if $U \in \mathcal{U}$ **and** $\forall\, V{\in}\mathcal{U}.\ \forall\, i.\ i \in I \wedge Y\ V\ i \neq$ *topspace* $(X\ i) \longrightarrow f\ i \in Y\ V\ i$
and $\forall\, i{\in}I.\ f\ i \in$ *topspace* $(X\ i)$ **and** $f \in$ *extensional I* **for** *f U*
proof −
have $Pi_E\ I\ (Y\ U) = U$
using *Y* ‹$U \in \mathcal{U}$› **by** *blast*
then show $f \in U$
using *that* **unfolding** *PiE_def Pi_def* **by** *blast*
qed
show ?*TOP* $\cap \bigcap(\bigcup U{\in}\mathcal{U}.$?*F U*) $\subseteq \bigcap\mathcal{U}$
by (*auto simp*: *PiE_iff* ∗)
show $\bigcap\mathcal{U} \subseteq$?*TOP* $\cap \bigcap(\bigcup U{\in}\mathcal{U}.$?*F U*)
using *Y openin_subset* ‹*finite′* $\mathcal{U}$› **by** *fastforce*
qed
qed (*use Y openin_subset* **in** ‹*blast*+›)
qed
have *2*: $\exists\mathcal{U}'.$ *finite′* $\mathcal{U}' \wedge \mathcal{U}' \subseteq$ *Collect* ?Ω $\wedge \bigcap\mathcal{U}' =$?*TOP* $\cap \bigcap\mathcal{U}$

if $\mathcal{U}$: $\mathcal{U} \subseteq$ *Collect ?Ψ* **and** *finite* $\mathcal{U}$ **for** $\mathcal{U}$
proof (*cases* $\mathcal{U}$={})
case *True*
then show *?thesis*
apply (*rule_tac x*={*?TOP*} **in** *exI*, *simp*)
apply (*rule_tac x*=λ*i*. *topspace* (*X i*) **in** *exI*)
apply *force*
done
next
case *False*
then obtain *U* **where** $U \in \mathcal{U}$
by *blast*
have $\forall U \in \mathcal{U}. \exists i\ Y.\ U = \{f.\ f\ i \in Y\} \wedge i \in I \wedge$ *openin* (*X i*) *Y*
using $\mathcal{U}$ **by** *auto*
then obtain *J Y* **where**
Y: $\bigwedge U.\ U \in \mathcal{U} \Longrightarrow U = \{f.\ f\ (J\ U) \in Y\ U\} \wedge J\ U \in I \wedge$ *openin* (*X* (*J U*)) (*Y U*)
by *metis*
let *?G* = λ*U*. Π_E *i*∈*I*. *if i* = *J U then Y U else topspace* (*X i*)
show *?thesis*
proof (*intro conjI exI*)
show *finite* (*?G* ‘ $\mathcal{U}$) *?G* ‘ $\mathcal{U} \neq$ {}
using ‹*finite* $\mathcal{U}$› ‹$U \in \mathcal{U}$› **by** *blast+*
have ∗: $\bigwedge U.\ U \in \mathcal{U} \Longrightarrow$ *openin* (*X* (*J U*)) (*Y U*)
using *Y* **by** *force*
show *?G* ‘ $\mathcal{U} \subseteq$ {*Pi*$_E$ *I Y* |*Y*. (∀ *i*. *openin* (*X i*) (*Y i*)) ∧ *finite* {*i*. *Y i* ≠ *topspace* (*X i*)}}
apply *clarsimp*
apply (*rule_tac x*= (λ*i*. *if i* = *J U then Y U else topspace* (*X i*)) **in** *exI*)
apply (*auto simp*: ∗)
done
next
show ($\bigcap U \in \mathcal{U}$. *?G U*) = *?TOP* ∩ $\bigcap \mathcal{U}$
proof
have (Π_E *i*∈*I*. *if i* = *J U then Y U else topspace* (*X i*)) ⊆ (Π_E *i*∈*I*. *topspace* (*X i*))
apply (*clarsimp simp*: *PiE_iff Ball_def all_conj_distrib split*: *if_split_asm*)
using *Y* ‹$U \in \mathcal{U}$› *openin_subset* **by** *blast+*
then have ($\bigcap U \in \mathcal{U}$. *?G U*) ⊆ *?TOP*
using ‹$U \in \mathcal{U}$›
by *fastforce*
moreover have ($\bigcap U \in \mathcal{U}$. *?G U*) ⊆ $\bigcap \mathcal{U}$
using *PiE_mem Y* **by** *fastforce*
ultimately show ($\bigcap U \in \mathcal{U}$. *?G U*) ⊆ *?TOP* ∩ $\bigcap \mathcal{U}$
by *auto*
qed (*use Y* **in** *fastforce*)
qed
qed
show *?thesis*

unfolding *relative_to_def intersection_of_def*
by (*safe*; *blast dest*!: *1 2*)
qed
show *?thesis*
unfolding *product_topology_def generate_topology_on_eq*
apply (*rule arg_cong* [**where** *f* = *topology*])
apply (*rule arg_cong* [**where** *f* = (*union_of*)*arbitrary*])
apply (*force simp*: ∗)
done
qed

lemma *topspace_product_topology* [*simp*]:
topspace (*product_topology T I*) = (Π_E *i*∈*I*. *topspace*(*T i*))
proof
show *topspace* (*product_topology T I*) ⊆ (Π_E *i*∈*I*. *topspace* (*T i*))
unfolding *product_topology_def topology_generated_by_topspace*
unfolding *topspace_def* **by** *auto*
have (Π_E *i*∈*I*. *topspace* (*T i*)) ∈ {(Π_E *i*∈*I*. *X i*) |*X*. (∀ *i*. *openin* (*T i*) (*X i*)) ∧ *finite* {*i*. *X i* ≠ *topspace* (*T i*)}}
using *openin_topspace not_finite_existsD* **by** *auto*
then show (Π_E *i*∈*I*. *topspace* (*T i*)) ⊆ *topspace* (*product_topology T I*)
unfolding *product_topology_def* **using** *PiE_def* **by** (*auto*)
qed

lemma *product_topology_basis*:
assumes ⋀*i*. *openin* (*T i*) (*X i*) *finite* {*i*. *X i* ≠ *topspace* (*T i*)}
shows *openin* (*product_topology T I*) (Π_E *i*∈*I*. *X i*)
unfolding *product_topology_def*
by (*rule topology_generated_by_Basis*) (*use assms* **in** *auto*)

proposition *product_topology_open_contains_basis*:
assumes *openin* (*product_topology T I*) *U x* ∈ *U*
shows ∃ *X*. *x* ∈ (Π_E *i*∈*I*. *X i*) ∧ (∀ *i*. *openin* (*T i*) (*X i*)) ∧ *finite* {*i*. *X i* ≠ *topspace* (*T i*)} ∧ (Π_E *i*∈*I*. *X i*) ⊆ *U*
proof −
have *generate_topology_on* {(Π_E *i*∈*I*. *X i*) |*X*. (∀ *i*. *openin* (*T i*) (*X i*)) ∧ *finite* {*i*. *X i* ≠ *topspace* (*T i*)}} *U*
using *assms* **unfolding** *product_topology_def* **by** (*intro openin_topology_generated_by*) *auto*
then have ⋀*x*. *x*∈*U* ⟹ ∃ *X*. *x* ∈ (Π_E *i*∈*I*. *X i*) ∧ (∀ *i*. *openin* (*T i*) (*X i*)) ∧ *finite* {*i*. *X i* ≠ *topspace* (*T i*)} ∧ (Π_E *i*∈*I*. *X i*) ⊆ *U*
proof *induction*
case (*Int U V x*)
then obtain *XU XV* **where** *H*:
x ∈ Pi_E *I XU* (∀ *i*. *openin* (*T i*) (*XU i*)) *finite* {*i*. *XU i* ≠ *topspace* (*T i*)} Pi_E *I XU* ⊆ *U*
x ∈ Pi_E *I XV* (∀ *i*. *openin* (*T i*) (*XV i*)) *finite* {*i*. *XV i* ≠ *topspace* (*T i*)} Pi_E *I XV* ⊆ *V*
by *auto meson*

define X **where** $X = (\lambda i.\ XU\ i \cap XV\ i)$
have $Pi_E\ I\ X \subseteq Pi_E\ I\ XU \cap Pi_E\ I\ XV$
unfolding *X_def* **by** (*auto simp add*: *PiE_iff*)
then have $Pi_E\ I\ X \subseteq U \cap V$ **using** H **by** *auto*
moreover have $\forall i.\ openin\ (T\ i)\ (X\ i)$
unfolding *X_def* **using** H **by** *auto*
moreover have *finite* $\{i.\ X\ i \neq topspace\ (T\ i)\}$
apply (*rule rev_finite_subset*[*of* $\{i.\ XU\ i \neq topspace\ (T\ i)\} \cup \{i.\ XV\ i \neq topspace\ (T\ i)\}$])
unfolding *X_def* **using** H **by** *auto*
moreover have $x \in Pi_E\ I\ X$
unfolding *X_def* **using** H **by** *auto*
ultimately show *?case*
by *auto*
next
case (*UN K x*)
then obtain k **where** $k \in K\ x \in k$ **by** *auto*
with *UN* **have** $\exists X.\ x \in Pi_E\ I\ X \wedge (\forall i.\ openin\ (T\ i)\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq topspace\ (T\ i)\} \wedge Pi_E\ I\ X \subseteq k$
by *simp*
then obtain X **where** $x \in Pi_E\ I\ X \wedge (\forall i.\ openin\ (T\ i)\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq topspace\ (T\ i)\} \wedge Pi_E\ I\ X \subseteq k$
by *blast*
then have $x \in Pi_E\ I\ X \wedge (\forall i.\ openin\ (T\ i)\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq topspace\ (T\ i)\} \wedge Pi_E\ I\ X \subseteq (\bigcup K)$
using ‹$k \in K$› **by** *auto*
then show *?case*
by *auto*
qed *auto*
then show *?thesis* **using** ‹$x \in U$› **by** *auto*
qed

lemma *product_topology_empty_discrete*:
product_topology T {} = *discrete_topology* $\{(\lambda x.\ undefined)\}$
by (*simp add*: *subtopology_eq_discrete_topology_sing*)

lemma *openin_product_topology*:
openin (*product_topology* X I) =
arbitrary union_of
((*finite intersection_of* $(\lambda F.\ (\exists i\ U.\ F = \{f.\ f\ i \in U\} \wedge i \in I \wedge openin\ (X\ i)\ U)))$
relative_to topspace (*product_topology* X I))
apply (*subst product_topology*)
apply (*simp add*: *topology_inverse'* [*OF istopology_subbase*])
done

lemma *subtopology_PiE*:
subtopology (*product_topology* X I) $(\Pi_E\ i{\in}I.\ (S\ i))$ = *product_topology* $(\lambda i.\ subtopology\ (X\ i)\ (S\ i))\ I$

```
proof −
  let ?P = λF. ∃ i U. F = {f. f i ∈ U} ∧ i ∈ I ∧ openin (X i) U
  let ?X = Π_E i∈I. topspace (X i)
  have finite intersection_of ?P relative_to ?X ∩ Pi_E I S =
        finite intersection_of (?P relative_to ?X ∩ Pi_E I S) relative_to ?X ∩ Pi_E I
S
    by (rule finite_intersection_of_relative_to)
  also have ... = finite intersection_of
                  ((λF. ∃ i U. F = {f. f i ∈ U} ∧ i ∈ I ∧ (openin (X i) relative_to
S i) U)
                    relative_to ?X ∩ Pi_E I S)
                  relative_to ?X ∩ Pi_E I S
    apply (rule arg_cong2 [where f = (relative_to)])
     apply (rule arg_cong [where f = (intersection_of)finite])
     apply (rule ext)
     apply (auto simp: relative_to_def intersection_of_def)
    done
  finally
  have finite intersection_of ?P relative_to ?X ∩ Pi_E I S =
        finite intersection_of
          (λF. ∃ i U. F = {f. f i ∈ U} ∧ i ∈ I ∧ (openin (X i) relative_to S i) U)
          relative_to ?X ∩ Pi_E I S
    by (metis finite_intersection_of_relative_to)
  then show ?thesis
  unfolding topology_eq
  apply clarify
  apply (simp add: openin_product_topology flip: openin_relative_to)
  apply (simp add: arbitrary_union_of_relative_to flip: PiE_Int)
  done
qed

lemma product_topology_base_alt:
   finite intersection_of (λF. (∃ i U. F = {f. f i ∈ U} ∧ i ∈ I ∧ openin (X i) U))
    relative_to (Π_E i∈I. topspace (X i)) =
    (λF. (∃ U. F =  Pi_E I U ∧ finite {i ∈ I. U i ≠ topspace(X i)} ∧ (∀ i ∈ I.
openin (X i) (U i))))
   (is ?lhs = ?rhs)
proof −
  have (∀ F. ?lhs F ⟶ ?rhs F)
    unfolding all_relative_to all_intersection_of topspace_product_topology
  proof clarify
    fix 𝓕
    assume finite 𝓕 and 𝓕 ⊆ {{f. f i ∈ U} |i U. i ∈ I ∧ openin (X i) U}
    then show ∃ U. (Π_E i∈I. topspace (X i)) ∩ ⋂𝓕 = Pi_E I U ∧
          finite {i ∈ I. U i ≠ topspace (X i)} ∧ (∀ i∈I. openin (X i) (U i))
    proof (induction)
      case (insert F 𝓕)
      then obtain U where eq: (Π_E i∈I. topspace (X i)) ∩ ⋂𝓕 = Pi_E I U
        and fin: finite {i ∈ I. U i ≠ topspace (X i)}
```

```
    and ope: ⋀i. i ∈ I ⟹ openin (X i) (U i)
    by auto
  obtain i V where F = {f. f i ∈ V} i ∈ I openin (X i) V
    using insert by auto
  let ?U = λj. U j ∩ (if j = i then V else topspace(X j))
  show ?case
  proof (intro exI conjI)
    show (Π_E i∈I. topspace (X i)) ∩ ⋂(insert F ℱ) = Pi_E I ?U
    using eq PiE_mem ‹i ∈ I› by (auto simp: ‹F = {f. f i ∈ V}›) fastforce
  next
    show finite {i ∈ I. ?U i ≠ topspace (X i)}
      by (rule rev_finite_subset [OF finite.insertI [OF fin]]) auto
  next
    show ∀i∈I. openin (X i) (?U i)
      by (simp add: ‹openin (X i) V› ope openin_Int)
  qed
 qed (auto intro: dest: not_finite_existsD)
qed
moreover have (∀F. ?rhs F ⟶ ?lhs F)
proof clarify
  fix U :: 'a ⇒ 'b set
  assume fin: finite {i ∈ I. U i ≠ topspace (X i)} and ope: ∀i∈I. openin (X i)
(U i)
  let ?U = ⋂i∈{i ∈ I. U i ≠ topspace (X i)}. {x. x i ∈ U i}
  show ?lhs (Pi_E I U)
    unfolding relative_to_def topspace_product_topology
  proof (intro exI conjI)
    show (finite intersection_of (λF. ∃i U. F = {f. f i ∈ U} ∧ i ∈ I ∧ openin
(X i) U)) ?U
      using fin ope by (intro finite_intersection_of_Inter finite_intersection_of_inc)
auto
    show (Π_E i∈I. topspace (X i)) ∩ ?U = Pi_E I U
      using ope openin_subset by fastforce
  qed
qed
ultimately show ?thesis
  by meson
qed

corollary openin_product_topology_alt:
 openin (product_topology X I) S ⟷
  (∀x ∈ S. ∃U. finite {i ∈ I. U i ≠ topspace(X i)} ∧
          (∀i ∈ I. openin (X i) (U i)) ∧ x ∈ Pi_E I U ∧ Pi_E I U ⊆ S)
 unfolding openin_product_topology arbitrary_union_of_alt product_topology_base_alt
topspace_product_topology
 apply safe
 apply (drule bspec; blast)+
 done
```

lemma *closure_of_product_topology*:
(*product_topology X I*) *closure_of* (*PiE I S*) = *PiE I* (λ*i*. (*X i*) *closure_of* (*S i*))
proof −
have ∗: (∀ *T*. *f* ∈ *T* ∧ *openin* (*product_topology X I*) *T* ⟶ (∃ *y*∈*Pi*$_E$ *I S*. *y* ∈ *T*))
⟷ (∀ *i* ∈ *I*. ∀ *T*. *f i* ∈ *T* ∧ *openin* (*X i*) *T* ⟶ *S i* ∩ *T* ≠ {})
(**is** *?lhs* = *?rhs*)
if *top*: ⋀*i*. *i* ∈ *I* ⟹ *f i* ∈ *topspace* (*X i*) **and** *ext*: *f* ∈ *extensional I* **for** *f*
proof
assume *L*[*rule_format*]: *?lhs*
show *?rhs*
proof *clarify*
fix *i T*
assume *i* ∈ *I f i* ∈ *T openin* (*X i*) *T S i* ∩ *T* = {}
then have *openin* (*product_topology X I*) ((Π$_E$ *i*∈*I*. *topspace* (*X i*)) ∩ {*x*. *x i* ∈ *T*})
by (*force simp*: *openin_product_topology intro*: *arbitrary_union_of_inc relative_to_inc finite_intersection_of_inc*)
then show *False*
using *L* [*of topspace* (*product_topology X I*) ∩ {*f*. *f i* ∈ *T*}] ‹*S i* ∩ *T* = {}› ‹*f i* ∈ *T*› ‹*i* ∈ *I*›
by (*auto simp*: *top ext PiE_iff*)
qed
next
assume *R* [*rule_format*]: *?rhs*
show *?lhs*
proof (*clarsimp simp*: *openin_product_topology union_of_def arbitrary_def*)
fix 𝒰 *U*
assume
𝒰: 𝒰 ⊆ *Collect*
(*finite intersection_of* (λ*F*. ∃ *i U*. *F* = {*x*. *x i* ∈ *U*} ∧ *i* ∈ *I* ∧ *openin* (*X i*) *U*) *relative_to*
(Π$_E$ *i*∈*I*. *topspace* (*X i*))) **and**
f ∈ *U U* ∈ 𝒰
then have (*finite intersection_of* (λ*F*. ∃ *i U*. *F* = {*x*. *x i* ∈ *U*} ∧ *i* ∈ *I* ∧ *openin* (*X i*) *U*)
relative_to (Π$_E$ *i*∈*I*. *topspace* (*X i*))) *U*
by *blast*
with ‹*f* ∈ *U*› ‹*U* ∈ 𝒰›
obtain 𝒯 **where** *finite* 𝒯
and 𝒯: ⋀*C*. *C* ∈ 𝒯 ⟹ ∃ *i* ∈ *I*. ∃ *V*. *openin* (*X i*) *V* ∧ *C* = {*x*. *x i* ∈ *V*}
and *topspace* (*product_topology X I*) ∩ ⋂ 𝒯 ⊆ *U f* ∈ *topspace* (*product_topology X I*) ∩ ⋂ 𝒯
apply (*clarsimp simp add*: *relative_to_def intersection_of_def*)
apply (*rule that*, *auto dest*!: *subsetD*)
done
then have *f* ∈ *PiE I* (*topspace* ∘ *X*) *f* ∈ ⋂𝒯 **and** *subU*: *PiE I* (*topspace* ∘ *X*) ∩ ⋂𝒯 ⊆ *U*
by (*auto simp*: *PiE_iff*)

have ∗: *f i* ∈ *topspace* (*X i*) ∩ ⋂{*U*. *openin* (*X i*) *U* ∧ {*x*. *x i* ∈ *U*} ∈ 𝒯}
∧ *openin* (*X i*) (*topspace* (*X i*) ∩ ⋂{*U*. *openin* (*X i*) *U* ∧ {*x*. *x i* ∈ *U*} ∈ 𝒯})
if *i* ∈ *I* **for** *i*
proof −
have *finite* ((λ*U*. {*x*. *x i* ∈ *U*}) −' 𝒯)
proof (*rule finite_vimageI* [*OF* ‹*finite* 𝒯›])
show *inj* (λ*U*. {*x*. *x i* ∈ *U*})
by (*auto simp*: *inj_on_def*)
qed
then have *fin*: *finite* {*U*. *openin* (*X i*) *U* ∧ {*x*. *x i* ∈ *U*} ∈ 𝒯}
by (*rule rev_finite_subset*) *auto*
have *openin* (*X i*) (⋂ (*insert* (*topspace* (*X i*)) {*U*. *openin* (*X i*) *U* ∧ {*x*. *x i* ∈ *U*} ∈ 𝒯}))
by (*rule openin_Inter*) (*auto simp*: *fin*)
then show *?thesis*
using ‹*f* ∈ ⋂ 𝒯› **by** (*fastforce simp*: *that top*)
qed
define Φ **where** Φ ≡ λ*i*. *topspace* (*X i*) ∩ ⋂{*U*. *openin* (*X i*) *U* ∧ {*f*. *f i* ∈ *U*} ∈ 𝒯}
have ∀ *i* ∈ *I*. ∃ *x*. *x* ∈ *S i* ∩ Φ *i*
using *R* [*OF* _ ∗] **unfolding** Φ*_def* **by** *blast*
then obtain ϑ **where** ϑ [*rule_format*]: ∀ *i* ∈ *I*. ϑ *i* ∈ *S i* ∩ Φ *i*
by *metis*
show ∃ *y*∈Pi_E *I S*. ∃ *x*∈𝒰. *y* ∈ *x*
proof
show ∃ *U* ∈ 𝒰. (λ*i* ∈ *I*. ϑ *i*) ∈ *U*
proof
have *restrict* ϑ *I* ∈ *PiE I* (*topspace* ∘ *X*) ∩ ⋂𝒯
using 𝒯 **by** (*fastforce simp*: Φ*_def PiE_def dest*: ϑ)
then show *restrict* ϑ *I* ∈ *U*
using *subU* **by** *blast*
qed (*rule* ‹*U* ∈ 𝒰›)
next
show (λ*i* ∈ *I*. ϑ *i*) ∈ Pi_E *I S*
using ϑ **by** *simp*
qed
qed
qed
show *?thesis*
apply (*simp add*: ∗ *closure_of_def PiE_iff set_eq_iff cong*: *conj_cong*)
by *metis*
qed

corollary *closedin_product_topology*:
closedin (*product_topology X I*) (*PiE I S*) ⟷ *PiE I S* = {} ∨ (∀ *i* ∈ *I*. *closedin* (*X i*) (*S i*))
apply (*simp add*: *PiE_eq PiE_eq_empty_iff closure_of_product_topology flip*: *closure_of_eq*)

apply (*metis closure_of_empty*)
done

corollary *closedin_product_topology_singleton*:
f ∈ *extensional I* ⟹ *closedin* (*product_topology X I*) {*f*} ⟷ (∀ *i* ∈ *I*. *closedin* (*X i*) {*f i*})
using *PiE_singleton closedin_product_topology* [*of X I*]
by (*metis* (*no_types*, *lifting*) *all_not_in_conv insertI1*)

lemma *product_topology_empty*:
product_topology X {} = *topology* (λ*S*. *S* ∈ {{},{λ*k*. *undefined*}})
unfolding *product_topology union_of_def intersection_of_def arbitrary_def relative_to_def*
by (*auto intro*: *arg_cong* [**where** *f*=*topology*])

lemma *openin_product_topology_empty*: *openin* (*product_topology X* {}) *S* ⟷ *S* ∈ {{},{λ*k*. *undefined*}}
unfolding *union_of_def intersection_of_def arbitrary_def relative_to_def openin_product_topology*
by *auto*

The basic property of the product topology is the continuity of projections:

lemma *continuous_map_product_coordinates* [*simp*]:
assumes *i* ∈ *I*
shows *continuous_map* (*product_topology T I*) (*T i*) (λ*x*. *x i*)
proof −
{
fix *U* **assume** *openin* (*T i*) *U*
define *X* **where** *X* = (λ*j*. *if j* = *i then U else topspace* (*T j*))
then have ∗: (λ*x*. *x i*) −‘ *U* ∩ (Π_E *i*∈*I*. *topspace* (*T i*)) = (Π_E *j*∈*I*. *X j*)
unfolding *X_def* **using** *assms openin_subset*[*OF* ‹*openin* (*T i*) *U*›]
by (*auto simp add*: *PiE_iff*, *auto*, *metis subsetCE*)
have ∗∗: (∀ *i*. *openin* (*T i*) (*X i*)) ∧ *finite* {*i*. *X i* ≠ *topspace* (*T i*)}
unfolding *X_def* **using** ‹*openin* (*T i*) *U*› **by** *auto*
have *openin* (*product_topology T I*) ((λ*x*. *x i*) −‘ *U* ∩ (Π_E *i*∈*I*. *topspace* (*T i*)))
unfolding *product_topology_def*
apply (*rule topology_generated_by_Basis*)
apply (*subst* ∗)
using ∗∗ **by** *auto*
}
then show *?thesis* **unfolding** *continuous_map_alt*
by (*auto simp add*: *assms PiE_iff*)
qed

lemma *continuous_map_coordinatewise_then_product* [*intro*]:
assumes ⋀*i*. *i* ∈ *I* ⟹ *continuous_map T1* (*T i*) (λ*x*. *f x i*)
⋀*i x*. *i* ∉ *I* ⟹ *x* ∈ *topspace T1* ⟹ *f x i* = *undefined*

shows *continuous_map T1* (*product_topology T I*) *f*
unfolding *product_topology_def*
proof (*rule continuous_on_generated_topo*)
fix *U* **assume** $U \in \{Pi_E\ I\ X\ |X.\ (\forall i.\ openin\ (T\ i)\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq topspace\ (T\ i)\}\}$
then obtain *X* **where** *H*: $U = Pi_E\ I\ X\ \bigwedge i.\ openin\ (T\ i)\ (X\ i)\ finite\ \{i.\ X\ i \neq topspace\ (T\ i)\}$
by *blast*
define *J* **where** $J = \{i \in I.\ X\ i \neq topspace\ (T\ i)\}$
have *finite J* $J \subseteq I$ **unfolding** *J_def* **using** *H*(*3*) **by** *auto*
have $(\lambda x.\ f\ x\ i)-`(topspace(T\ i)) \cap topspace\ T1 = topspace\ T1$ **if** $i \in I$ **for** *i*
using *that assms*(*1*) **by** (*simp add*: *continuous_map_preimage_topspace*)
then have ∗: $(\lambda x.\ f\ x\ i)-`(X\ i) \cap topspace\ T1 = topspace\ T1$ **if** $i \in I-J$ **for** *i*
using *that* **unfolding** *J_def* **by** *auto*
have $f-`U \cap topspace\ T1 = (\bigcap i\in I.\ (\lambda x.\ f\ x\ i)-`(X\ i) \cap topspace\ T1) \cap (topspace\ T1)$
by (*subst H*(*1*), *auto simp add*: *PiE_iff assms*)
also have ... $= (\bigcap i\in J.\ (\lambda x.\ f\ x\ i)-`(X\ i) \cap topspace\ T1) \cap (topspace\ T1)$
using ∗ ‹$J \subseteq I$› **by** *auto*
also have *openin T1* (...)
apply (*rule openin_INT*)
apply (*simp add*: ‹*finite J*›)
using *H*(*2*) *assms*(*1*) ‹$J \subseteq I$› **by** *auto*
ultimately show *openin T1* $(f-`U \cap topspace\ T1)$ **by** *simp*
next
show $f\ `topspace\ T1 \subseteq \bigcup\{Pi_E\ I\ X\ |X.\ (\forall i.\ openin\ (T\ i)\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq topspace\ (T\ i)\}\}$
apply (*subst topology_generated_by_topspace*[*symmetric*])
apply (*subst product_topology_def*[*symmetric*])
apply (*simp only*: *topspace_product_topology*)
apply (*auto simp add*: *PiE_iff*)
using *assms* **unfolding** *continuous_map_def* **by** *auto*
qed

lemma *continuous_map_product_then_coordinatewise* [*intro*]:
assumes *continuous_map T1* (*product_topology T I*) *f*
shows $\bigwedge i.\ i \in I \Longrightarrow continuous\_map\ T1\ (T\ i)\ (\lambda x.\ f\ x\ i)$
$\bigwedge i\ x.\ i \notin I \Longrightarrow x \in topspace\ T1 \Longrightarrow f\ x\ i = undefined$
proof −
fix *i* **assume** $i \in I$
have $(\lambda x.\ f\ x\ i) = (\lambda y.\ y\ i)\ o\ f$ **by** *auto*
also have *continuous_map T1* (*T i*) (...)
apply (*rule continuous_map_compose*[*of _ product_topology T I*])
using *assms* ‹$i \in I$› **by** *auto*
ultimately show *continuous_map T1* (*T i*) $(\lambda x.\ f\ x\ i)$
by *simp*
next
fix *i x* **assume** $i \notin I$ $x \in topspace\ T1$
have $f\ x \in topspace\ (product\_topology\ T\ I)$

using *assms* ‹$x \in$ *topspace T1*› **unfolding** *continuous_map_def* **by** *auto*
then have $f\ x \in (\Pi_E\ i{\in}I.\ topspace\ (T\ i))$
using *topspace_product_topology* **by** *metis*
then show $f\ x\ i =$ *undefined*
using ‹$i \notin I$› **by** (*auto simp add*: *PiE_iff extensional_def*)
qed

lemma *continuous_on_restrict*:
assumes $J \subseteq I$
shows *continuous_map* (*product_topology T I*) (*product_topology T J*) ($\lambda x.$ *restrict x J*)
proof (*rule continuous_map_coordinatewise_then_product*)
fix *i* **assume** $i \in J$
then have ($\lambda x.$ *restrict x J i*) = ($\lambda x.$ *x i*) **unfolding** *restrict_def* **by** *auto*
then show *continuous_map* (*product_topology T I*) (*T i*) ($\lambda x.$ *restrict x J i*)
using ‹$i \in J$› ‹$J \subseteq I$› **by** *auto*
next
fix *i* **assume** $i \notin J$
then show *restrict x J i* = *undefined* **for** *x*::'*a* ⇒ '*b*
unfolding *restrict_def* **by** *auto*
qed

Powers of a single topological space as a topological space, using type classes

instantiation *fun* :: (*type*, *topological_space*) *topological_space*
begin

definition *open_fun_def*:
open U = *openin* (*product_topology* ($\lambda i.$ *euclidean*) *UNIV*) *U*

instance proof
have *topspace* (*product_topology* (λ(*i*::'*a*). *euclidean*::('*b topology*)) *UNIV*) = *UNIV*
unfolding *topspace_product_topology topspace_euclidean* **by** *auto*
then show *open* (*UNIV*::('*a* ⇒ '*b*) *set*)
unfolding *open_fun_def* **by** (*metis openin_topspace*)
qed (*auto simp add*: *open_fun_def*)

end

lemma *open_PiE* [*intro?*]:
fixes *X*::'*i* ⇒ ('*b*::*topological_space*) *set*
assumes $\bigwedge i.$ *open* (*X i*) *finite* {*i*. *X i* ≠ *UNIV*}
shows *open* (Pi_E *UNIV X*)
by (*simp add*: *assms open_fun_def product_topology_basis*)

lemma *euclidean_product_topology*:
product_topology ($\lambda i.$ *euclidean*::('*b*::*topological_space*) *topology*) *UNIV* = *euclidean*

by (*metis open_openin topology_eq open_fun_def*)

proposition *product_topology_basis'*:
  fixes *x*::*'i* ⇒ *'a* **and** *U*::*'i* ⇒ (*'b*::*topological_space*) *set*
  assumes *finite I* ⋀*i*. *i* ∈ *I* ⟹ *open* (*U i*)
  shows *open* {*f*. ∀*i*∈*I*. *f* (*x i*) ∈ *U i*}
proof −
  define *J* **where** *J* = *x*'*I*
  define *V* **where** *V* = (λ*y*. *if y* ∈ *J then* ⋂{*U i*|*i*. *i*∈*I* ∧ *x i* = *y*} *else UNIV*)
  define *X* **where** *X* = (λ*y*. *if y* ∈ *J then V y else UNIV*)
  have ∗: *open* (*X i*) **for** *i*
    unfolding *X_def V_def* **using** *assms* **by** *auto*
  have ∗∗: *finite* {*i*. *X i* ≠ *UNIV*}
    unfolding *X_def V_def J_def* **using** *assms*(*1*) **by** *auto*
  have *open* (Pi_E *UNIV X*)
    by (*simp add*: ∗ ∗∗ *open_PiE*)
  moreover have Pi_E *UNIV X* = {*f*. ∀*i*∈*I*. *f* (*x i*) ∈ *U i*}
    apply (*auto simp add*: *PiE_iff*) **unfolding** *X_def V_def J_def*
    proof (*auto*)
      fix *f* :: *'a* ⇒ *'b* **and** *i* :: *'i*
      assume *a1*: *i* ∈ *I*
      assume *a2*: ∀*i*. *f i* ∈ (*if i* ∈ *x*'*I then if i* ∈ *x*'*I then* ⋂{*U ia* |*ia*. *ia* ∈ *I* ∧ *x ia* = *i*} *else UNIV else UNIV*)
      have *f3*: *x i* ∈ *x*'*I*
        using *a1* **by** *blast*
      have *U i* ∈ {*U ia* |*ia*. *ia* ∈ *I* ∧ *x ia* = *x i*}
        using *a1* **by** *blast*
      then show *f* (*x i*) ∈ *U i*
        using *f3 a2* **by** (*meson Inter_iff*)
    qed
  ultimately show *?thesis* **by** *simp*
qed

The results proved in the general situation of products of possibly different spaces have their counterparts in this simpler setting.

lemma *continuous_on_product_coordinates* [*simp*]:
  continuous_on UNIV (λ*x*. *x i*::(*'b*::*topological_space*))
  using *continuous_map_product_coordinates* [*of _ UNIV* λ*i*. *euclidean*]
    by (*metis* (*no_types*) *continuous_map_iff_continuous euclidean_product_topology iso_tuple_UNIV_I subtopology_UNIV*)

lemma *continuous_on_coordinatewise_then_product* [*continuous_intros*]:
  fixes *f* :: *'a*::*topological_space* ⇒ *'b* ⇒ *'c*::*topological_space*
  assumes ⋀*i*. *continuous_on S* (λ*x*. *f x i*)
  shows *continuous_on S f*
  using *continuous_map_coordinatewise_then_product* [*of UNIV*, **where** *T* = λ*i*. *euclidean*]
  by (*metis UNIV_I assms continuous_map_iff_continuous euclidean_product_topology*)

lemma *continuous_on_product_then_coordinatewise_UNIV*:
assumes *continuous_on UNIV f*
shows *continuous_on UNIV* ($\lambda x.\ f\ x\ i$)
unfolding *continuous_map_iff_continuous2* [*symmetric*]
by (*rule continuous_map_product_then_coordinatewise* [**where** I=*UNIV*]) (*use assms* **in** ‹*auto simp*: *euclidean_product_topology*›)

lemma *continuous_on_product_then_coordinatewise*:
assumes *continuous_on S f*
shows *continuous_on S* ($\lambda x.\ f\ x\ i$)
proof −
have *continuous_on S* (($\lambda q.\ q\ i$) ∘ f)
by (*metis assms continuous_on_compose continuous_on_id continuous_on_product_then_coordinatewise_UNIV continuous_on_subset subset_UNIV*)
then show *?thesis*
by *auto*
qed

lemma *continuous_on_coordinatewise_iff*:
fixes f :: ($'a \Rightarrow real$) $\Rightarrow$ $'b \Rightarrow real$
shows *continuous_on* ($A \cap S$) $f \longleftrightarrow$ ($\forall\, i.$ *continuous_on* ($A \cap S$) ($\lambda x.\ f\ x\ i$))
by (*auto simp*: *continuous_on_product_then_coordinatewise continuous_on_coordinatewise_then_product*)

lemma *continuous_map_span_sum*:
fixes B :: $'a$::*real_normed_vector set*
assumes *biB*: $\bigwedge i.\ i \in I \Longrightarrow b\ i \in B$
shows *continuous_map euclidean* (*top_of_set* (*span B*)) ($\lambda x.\ \sum i{\in}I.\ x\ i *_R\ b\ i$)
proof (*rule continuous_map_euclidean_top_of_set*)
show ($\lambda x.\ \sum i{\in}I.\ x\ i *_R\ b\ i$) −‘ *span B* = *UNIV*
by *auto* (*meson biB lessThan_iff span_base span_scale span_sum*)
show *continuous_on UNIV* ($\lambda x.\ \sum i{\in}\ I.\ x\ i *_R\ b\ i$)
by (*intro continuous_intros*) *auto*
qed

Topological countability for product spaces

The next two lemmas are useful to prove first or second countability of product spaces, but they have more to do with countability and could be put in the corresponding theory.

lemma *countable_nat_product_event_const*:
fixes F::$'a$ *set* **and** a::$'a$
assumes $a \in F$ *countable F*
shows *countable* $\{x$::(*nat* $\Rightarrow$ $'a$). ($\forall\, i.\ x\ i \in F$) $\wedge$ *finite* $\{i.\ x\ i \neq a\}\}$
proof −
have $*$: $\{x$::(*nat* $\Rightarrow$ $'a$). ($\forall\, i.\ x\ i \in F$) $\wedge$ *finite* $\{i.\ x\ i \neq a\}\}$
$\subseteq$ ($\bigcup N.\ \{x.\ (\forall\, i.\ x\ i \in F) \wedge (\forall\, i{\geq}N.\ x\ i = a)\}$)
using *infinite_nat_iff_unbounded_le* **by** *fastforce*
have *countable* $\{x.\ (\forall\, i.\ x\ i \in F) \wedge (\forall\, i{\geq}N.\ x\ i = a)\}$ **for** N::*nat*

proof (*induction N*)
  case *0*
  have $\{x. (\forall i. x\ i \in F) \wedge (\forall i \geq (0::nat). x\ i = a)\} = \{(\lambda i. a)\}$
    using ‹$a \in F$› **by** *auto*
  then show *?case* **by** *auto*
next
  case (*Suc N*)
  define $f::((nat \Rightarrow 'a) \times 'a) \Rightarrow (nat \Rightarrow 'a)$
    where $f = (\lambda(x, b). (\lambda i.\ if\ i = N\ then\ b\ else\ x\ i))$
  have $\{x. (\forall i. x\ i \in F) \wedge (\forall i \geq Suc\ N. x\ i = a)\} \subseteq f`(\{x. (\forall i. x\ i \in F) \wedge (\forall i \geq N. x\ i = a)\} \times F)$
  proof (*auto*)
    fix x **assume** H: $\forall i::nat. x\ i \in F\ \forall i \geq Suc\ N. x\ i = a$
    define y **where** $y = (\lambda i.\ if\ i = N\ then\ a\ else\ x\ i)$
    have $f\ (y, x\ N) = x$
      unfolding *f_def y_def* **by** *auto*
    moreover have $(y, x\ N) \in \{x. (\forall i. x\ i \in F) \wedge (\forall i \geq N. x\ i = a)\} \times F$
      unfolding *y_def* **using** H ‹$a \in F$› **by** *auto*
    ultimately show $x \in f`(\{x. (\forall i. x\ i \in F) \wedge (\forall i \geq N. x\ i = a)\} \times F)$
      by (*metis* (*no_types*, *lifting*) *image_eqI*)
  qed
  moreover have *countable* $(\{x. (\forall i. x\ i \in F) \wedge (\forall i \geq N. x\ i = a)\} \times F)$
    using *Suc.IH assms*(*2*) **by** *auto*
  ultimately show *?case*
    by (*meson countable_image countable_subset*)
qed
then show *?thesis* **using** *countable_subset*[*OF* ∗] **by** *auto*
qed

lemma *countable_product_event_const*:
  fixes $F::('a::countable) \Rightarrow 'b\ set$ **and** $b::'b$
  assumes $\bigwedge i.\ countable\ (F\ i)$
  shows *countable* $\{f::('a \Rightarrow 'b). (\forall i. f\ i \in F\ i) \wedge (finite\ \{i. f\ i \neq b\})\}$
proof −
  define G **where** $G = (\bigcup i. F\ i) \cup \{b\}$
  have *countable* G **unfolding** *G_def* **using** *assms* **by** *auto*
  have $b \in G$ **unfolding** *G_def* **by** *auto*
  define *pi* **where** $pi = (\lambda(x::(nat \Rightarrow 'b)). (\lambda\ i::'a.\ x\ ((to\_nat::('a \Rightarrow nat))\ i)))$
  have $\{f::('a \Rightarrow 'b). (\forall i. f\ i \in F\ i) \wedge (finite\ \{i. f\ i \neq b\})\}$
      $\subseteq pi`\{g::(nat \Rightarrow 'b). (\forall j. g\ j \in G) \wedge (finite\ \{j. g\ j \neq b\})\}$
  proof (*auto*)
    fix f **assume** H: $\forall i. f\ i \in F\ i\ finite\ \{i. f\ i \neq b\}$
    define I **where** $I = \{i. f\ i \neq b\}$
    define g **where** $g = (\lambda j.\ if\ j \in to\_nat`I\ then\ f\ (from\_nat\ j)\ else\ b)$
    have $\{j. g\ j \neq b\} \subseteq to\_nat`I$ **unfolding** *g_def* **by** *auto*
    then have *finite* $\{j. g\ j \neq b\}$
      unfolding *I_def* **using** $H(2)$ **using** *finite_surj* **by** *blast*
    moreover have $g\ j \in G$ **for** j
      unfolding *g_def G_def* **using** H **by** *auto*

ultimately have $g \in \{g::(nat \Rightarrow 'b).\ (\forall j.\ g\ j \in G) \wedge (finite\ \{j.\ g\ j \neq b\})\}$
by *auto*
moreover have $f = pi\ g$
unfolding *pi_def g_def I_def* **using** *H* **by** *fastforce*
ultimately show $f \in pi`\{g.\ (\forall j.\ g\ j \in G) \wedge finite\ \{j.\ g\ j \neq b\}\}$
by *auto*
qed
then show *?thesis*
using *countable_nat_product_event_const*[*OF* ‹$b \in G$› ‹*countable G*›]
by (*meson countable_image countable_subset*)
qed

instance *fun* :: (*countable*, *first_countable_topology*) *first_countable_topology*
proof
fix $x::'a \Rightarrow 'b$
have $\exists A::('b \Rightarrow nat \Rightarrow 'b\ set).\ \forall x.\ (\forall i.\ x \in A\ x\ i \wedge open\ (A\ x\ i)) \wedge (\forall S.\ open\ S \wedge x \in S \longrightarrow (\exists i.\ A\ x\ i \subseteq S))$
apply (*rule choice*) **using** *first_countable_basis* **by** *auto*
then obtain $A::('b \Rightarrow nat \Rightarrow 'b\ set)$ **where** A: $\bigwedge x\ i.\ x \in A\ x\ i$
$\bigwedge x\ i.\ open\ (A\ x\ i)$
$\bigwedge x\ S.\ open\ S \Longrightarrow x \in S \Longrightarrow (\exists i.\ A\ x\ i \subseteq S)$
by *metis*

$B\ i$ is a countable basis of neighborhoods of x_i.

define B **where** $B = (\lambda i.\ (A\ (x\ i))`UNIV \cup \{UNIV\})$
have *countable* $(B\ i)$ **for** i **unfolding** *B_def* **by** *auto*
have *open_B*: $\bigwedge X\ i.\ X \in B\ i \Longrightarrow open\ X$
by (*auto simp*: *B_def A*)
define K **where** $K = \{Pi_E\ UNIV\ X \mid X.\ (\forall i.\ X\ i \in B\ i) \wedge finite\ \{i.\ X\ i \neq UNIV\}\}$
have $Pi_E\ UNIV\ (\lambda i.\ UNIV) \in K$
unfolding *K_def B_def* **by** *auto*
then have $K \neq \{\}$ **by** *auto*
have *countable* $\{X.\ (\forall i.\ X\ i \in B\ i) \wedge finite\ \{i.\ X\ i \neq UNIV\}\}$
apply (*rule countable_product_event_const*) **using** ‹$\bigwedge i.\ countable\ (B\ i)$› **by** *auto*
moreover have $K = (\lambda X.\ Pi_E\ UNIV\ X)`\{X.\ (\forall i.\ X\ i \in B\ i) \wedge finite\ \{i.\ X\ i \neq UNIV\}\}$
unfolding *K_def* **by** *auto*
ultimately have *countable K* **by** *auto*
have $x \in k$ **if** $k \in K$ **for** k
using *that* **unfolding** *K_def B_def* **apply** *auto* **using** *A*(*1*) **by** *auto*
have *open k* **if** $k \in K$ **for** k
using *that* **unfolding** *K_def* **by** (*blast intro*: *open_B open_PiE elim*:)
have *Inc*: $\exists k{\in}K.\ k \subseteq U$ **if** *open* $U \wedge x \in U$ **for** U
proof −
have *openin* (*product_topology* ($\lambda i.$ *euclidean*) *UNIV*) $U\ x \in U$
using ‹*open* $U \wedge x \in U$› **unfolding** *open_fun_def* **by** *auto*
with *product_topology_open_contains_basis*[*OF this*]
have $\exists X.\ x \in (\Pi_E\ i{\in}UNIV.\ X\ i) \wedge (\forall i.\ open\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq UNIV\}$

```
∧ (Π_E i∈UNIV. X i) ⊆ U
    by simp
  then obtain X where H: x ∈ (Π_E i∈UNIV. X i)
                      ⋀i. open (X i)
                      finite {i. X i ≠ UNIV}
                      (Π_E i∈UNIV. X i) ⊆ U
    by auto
  define I where I = {i. X i ≠ UNIV}
  define Y where Y = (λi. if i ∈ I then (SOME y. y ∈ B i ∧ y ⊆ X i) else
UNIV)
  have *: ∃ y. y ∈ B i ∧ y ⊆ X i for i
   unfolding B_def using A(3)[OF H(2)] H(1) by (metis PiE_E UNIV_I UnCI
image_iff)
  have **: Y i ∈ B i ∧ Y i ⊆ X i for i
    apply (cases i ∈ I)
    unfolding Y_def using * that apply (auto)
      apply (metis (no_types, lifting) someI, metis (no_types, lifting) someI_ex
subset_iff)
    unfolding B_def apply simp
    unfolding I_def apply auto
    done
  have {i. Y i ≠ UNIV} ⊆ I
    unfolding Y_def by auto
  then have ***: finite {i. Y i ≠ UNIV}
    unfolding I_def using H(3) rev_finite_subset by blast
  have (∀ i. Y i ∈ B i) ∧ finite {i. Y i ≠ UNIV}
    using ** *** by auto
  then have Pi_E UNIV Y ∈ K
    unfolding K_def by auto

  have Y i ⊆ X i for i
    apply (cases i ∈ I) using ** apply simp unfolding Y_def I_def by auto
  then have Pi_E UNIV Y ⊆ Pi_E UNIV X by auto
  then have Pi_E UNIV Y ⊆ U using H(4) by auto
  then show ?thesis using ‹Pi_E UNIV Y ∈ K› by auto
 qed

 show ∃ L. (∀ (i::nat). x ∈ L i ∧ open (L i)) ∧ (∀ U. open U ∧ x ∈ U ⟶ (∃ i.
L i ⊆ U))
  apply (rule first_countableI[of K])
  using ‹countable K› ‹⋀k. k ∈ K ⟹ x ∈ k› ‹⋀k. k ∈ K ⟹ open k› Inc by
auto
qed

proposition product_topology_countable_basis:
 shows ∃ K::(('a::countable ⇒ 'b::second_countable_topology) set set).
        topological_basis K ∧ countable K ∧
        (∀ k∈K. ∃ X. (k = Pi_E UNIV X) ∧ (∀ i. open (X i)) ∧ finite {i. X i ≠
UNIV})
```

proof −
  obtain *B*::*'b set set* **where** *B*: *countable B* ∧ *topological_basis B*
    using *ex_countable_basis* **by** *auto*
  then have *B* ≠ {} **by** (*meson UNIV_I empty_iff open_UNIV topological_basisE*)
  define *B2* **where** *B2* = *B* ∪ {*UNIV*}
  have *countable B2*
    unfolding *B2_def* **using** *B* **by** *auto*
  have *open U* **if** *U* ∈ *B2* **for** *U*
    using *that* **unfolding** *B2_def* **using** *B topological_basis_open* **by** *auto*

  define *K* **where** *K* = {Pi_E *UNIV X* | *X*. (∀ *i*::*'a*. *X i* ∈ *B2*) ∧ *finite* {*i*. *X i* ≠ *UNIV*}}
  have *i*: ∀ *k*∈*K*. ∃ *X*. (*k* = Pi_E *UNIV X*) ∧ (∀ *i*. *open* (*X i*)) ∧ *finite* {*i*. *X i* ≠ *UNIV*}
    unfolding *K_def* **using** ‹⋀*U*. *U* ∈ *B2* ⟹ *open U*› **by** *auto*

  have *countable* {*X*. (∀ (*i*::*'a*). *X i* ∈ *B2*) ∧ *finite* {*i*. *X i* ≠ *UNIV*}}
    apply (*rule countable_product_event_const*) **using** ‹*countable B2*› **by** *auto*
  moreover have *K* = (λ*X*. Pi_E *UNIV X*)‘{*X*. (∀ *i*. *X i* ∈ *B2*) ∧ *finite* {*i*. *X i* ≠ *UNIV*}}
    unfolding *K_def* **by** *auto*
  ultimately have *ii*: *countable K* **by** *auto*

  have *iii*: *topological_basis K*
  proof (*rule topological_basisI*)
    fix *U* **and** *x*::*'a*⇒*'b* **assume** *open U x* ∈ *U*
    then have *openin* (*product_topology* (λ*i*. *euclidean*) *UNIV*) *U*
      unfolding *open_fun_def* **by** *auto*
    with *product_topology_open_contains_basis*[*OF this* ‹*x* ∈ *U*›]
    have ∃ *X*. *x* ∈ (Π_E *i*∈*UNIV*. *X i*) ∧ (∀ *i*. *open* (*X i*)) ∧ *finite* {*i*. *X i* ≠ *UNIV*} ∧ (Π_E *i*∈*UNIV*. *X i*) ⊆ *U*
      by *simp*
    then obtain *X* **where** *H*: *x* ∈ (Π_E *i*∈*UNIV*. *X i*)
                    ⋀*i*. *open* (*X i*)
                    finite {*i*. *X i* ≠ *UNIV*}
                    (Π_E *i*∈*UNIV*. *X i*) ⊆ *U*
      by *auto*
    then have *x i* ∈ *X i* **for** *i* **by** *auto*
    define *I* **where** *I* = {*i*. *X i* ≠ *UNIV*}
    define *Y* **where** *Y* = (λ*i*. *if i* ∈ *I then* (*SOME y*. *y* ∈ *B2* ∧ *y* ⊆ *X i* ∧ *x i* ∈ *y*) *else UNIV*)
    have ∗: ∃ *y*. *y* ∈ *B2* ∧ *y* ⊆ *X i* ∧ *x i* ∈ *y* **for** *i*
        unfolding *B2_def* **using** *B* ‹*open* (*X i*)› ‹*x i* ∈ *X i*› **by** (*meson UnCI topological_basisE*)
    have ∗∗: *Y i* ∈ *B2* ∧ *Y i* ⊆ *X i* ∧ *x i* ∈ *Y i* **for** *i*
      using *someI_ex*[*OF* ∗]
      apply (*cases i* ∈ *I*)
      unfolding *Y_def* **using** ∗ **apply** (*auto*)
      unfolding *B2_def I_def* **by** *auto*

```
    have {i. Y i ≠ UNIV} ⊆ I
      unfolding Y_def by auto
    then have ***: finite {i. Y i ≠ UNIV}
      unfolding I_def using H(3) rev_finite_subset by blast
    have (∀ i. Y i ∈ B2) ∧ finite {i. Y i ≠ UNIV}
      using ** *** by auto
    then have Pi_E UNIV Y ∈ K
      unfolding K_def by auto

    have Y i ⊆ X i for i
      apply (cases i ∈ I) using ** apply simp unfolding Y_def I_def by auto
    then have Pi_E UNIV Y ⊆ Pi_E UNIV X by auto
    then have Pi_E UNIV Y ⊆ U using H(4) by auto

    have x ∈ Pi_E UNIV Y
      using ** by auto

    show ∃ V∈K. x ∈ V ∧ V ⊆ U
      using ‹Pi_E UNIV Y ∈ K› ‹Pi_E UNIV Y ⊆ U› ‹x ∈ Pi_E UNIV Y› by auto
  next
    fix U assume U ∈ K
    show open U
      using ‹U ∈ K› unfolding open_fun_def K_def by clarify (metis ‹U ∈ K› i
open_PiE open_fun_def)
  qed

  show ?thesis using i ii iii by auto
qed

instance fun :: (countable, second_countable_topology) second_countable_topology
  apply standard
  using product_topology_countable_basis topological_basis_imp_subbasis by auto
```

4.8.2 The Alexander subbase theorem

```
theorem Alexander_subbase:
  assumes X: topology (arbitrary union_of (finite intersection_of (λx. x ∈ ℬ)
relative_to ⋃ℬ)) = X
    and fin: ⋀C. ⟦C ⊆ ℬ; ⋃C = topspace X⟧ ⟹ ∃ C'. finite C' ∧ C' ⊆ C ∧
⋃C' = topspace X
  shows compact_space X
proof −
  have Uℬ: ⋃ℬ = topspace X
    by (simp flip: X)
  have False if 𝒰: ∀ U∈𝒰. openin X U and sub: topspace X ⊆ ⋃𝒰
    and neg: ⋀ℱ. ⟦ℱ ⊆ 𝒰; finite ℱ⟧ ⟹ ¬ topspace X ⊆ ⋃ℱ for 𝒰
  proof −
    define 𝒜 where 𝒜 ≡ {𝒞. (∀ U ∈ 𝒞. openin X U) ∧ topspace X ⊆ ⋃𝒞 ∧ (∀ ℱ.
finite ℱ ⟶ ℱ ⊆ 𝒞 ⟶ ~(topspace X ⊆ ⋃ℱ))}
```

have *1*: $\mathcal{A} \neq \{\}$
  unfolding *$\mathcal{A}$_def* **using** *sub $\mathcal{U}$ neg* **by** *force*
have *2*: $\bigcup\mathcal{C} \in \mathcal{A}$ **if** $\mathcal{C}\neq\{\}$ **and** $\mathcal{C}$: *subset.chain* $\mathcal{A}$ $\mathcal{C}$ **for** $\mathcal{C}$
  unfolding *$\mathcal{A}$_def*
proof (*intro CollectI conjI ballI allI impI notI*)
  show *openin X U* **if** *U*: $U \in \bigcup\mathcal{C}$ **for** *U*
    using *U* $\mathcal{C}$ **unfolding** *$\mathcal{A}$_def subset_chain_def* **by** *force*
  have $\mathcal{C} \subseteq \mathcal{A}$
    using *subset_chain_def* $\mathcal{C}$ **by** *blast*
  with *that $\mathcal{A}$_def* **show** *UUC*: *topspace X* $\subseteq \bigcup(\bigcup\mathcal{C})$
    by *blast*
  show *False* **if** *finite* $\mathcal{F}$ **and** $\mathcal{F} \subseteq \bigcup\mathcal{C}$ **and** *topspace X* $\subseteq \bigcup\mathcal{F}$ **for** $\mathcal{F}$
  proof −
    obtain $\mathcal{B}$ **where** $\mathcal{B} \in \mathcal{C}$ $\mathcal{F} \subseteq \mathcal{B}$
    by (*metis Sup_empty* $\mathcal{C}$ ‹$\mathcal{F} \subseteq \bigcup\mathcal{C}$› ‹*finite* $\mathcal{F}$› *UUC empty_subsetI finite.emptyI finite_subset_Union_chain neg*)
    then show *False*
      using *$\mathcal{A}$_def* ‹$\mathcal{C} \subseteq \mathcal{A}$› ‹*finite* $\mathcal{F}$› ‹*topspace X* $\subseteq \bigcup\mathcal{F}$› **by** *blast*
  qed
qed
obtain $\mathcal{K}$ **where** $\mathcal{K} \in \mathcal{A}$ **and** $\bigwedge X.$ $[\![X\in\mathcal{A};\ \mathcal{K} \subseteq X]\!] \Longrightarrow X = \mathcal{K}$
  using *subset_Zorn_nonempty* [*OF 1 2*] **by** *metis*
then have ∗: $\bigwedge\mathcal{W}.$ $[\![\bigwedge W.\ W\in\mathcal{W} \Longrightarrow$ *openin X W*; *topspace X* $\subseteq \bigcup\mathcal{W}$; $\mathcal{K} \subseteq \mathcal{W}$;
          $\bigwedge\mathcal{F}.$ $[\![$*finite* $\mathcal{F}$; $\mathcal{F} \subseteq \mathcal{W}$; *topspace X* $\subseteq \bigcup\mathcal{F}]\!] \Longrightarrow$ *False*$]\!]$
      $\Longrightarrow \mathcal{W} = \mathcal{K}$
  and *ope*: $\forall U\in\mathcal{K}.$ *openin X U* **and** *top*: *topspace X* $\subseteq \bigcup\mathcal{K}$
  and *non*: $\bigwedge\mathcal{F}.$ $[\![$*finite* $\mathcal{F}$; $\mathcal{F} \subseteq \mathcal{K}$; *topspace X* $\subseteq \bigcup\mathcal{F}]\!] \Longrightarrow$ *False*
  unfolding *$\mathcal{A}$_def* **by** *simp_all metis+*
then obtain *x* **where** $x \in$ *topspace X* $x \notin \bigcup(\mathcal{B} \cap \mathcal{K})$
proof −
  have $\bigcup(\mathcal{B} \cap \mathcal{K}) \neq \bigcup\mathcal{B}$
    by (*metis* ‹$\bigcup\mathcal{B}$ = *topspace X*› *fin inf.bounded_iff non order_refl*)
  then have $\exists a.\ a \notin \bigcup(\mathcal{B} \cap \mathcal{K}) \wedge a \in \bigcup\mathcal{B}$
    by *blast*
  then show *?thesis*
    using *that* **by** (*metis U$\mathcal{B}$*)
qed
obtain *C* **where** *C*: *openin X C* $C \in \mathcal{K}$ $x \in C$
  using ‹$x \in$ *topspace X*› *ope top* **by** *auto*
then have $C \subseteq$ *topspace X*
  by (*metis openin_subset*)
then have (*arbitrary union_of* (*finite intersection_of* $(\lambda x.\ x \in \mathcal{B})$ *relative_to* $\bigcup\mathcal{B}$)) *C*
  using *openin_subbase C* **unfolding** *X* [*symmetric*] **by** *blast*
moreover have $C \neq$ *topspace X*
  using ‹$\mathcal{K} \in \mathcal{A}$› ‹$C \in \mathcal{K}$› **unfolding** *$\mathcal{A}$_def* **by** *blast*
ultimately obtain $\mathcal{V}$ *W* **where** *W*: (*finite intersection_of* $(\lambda x.\ x \in \mathcal{B})$ *relative_to topspace X*) *W*

```
    and x ∈ W W ∈ 𝒱 ⋃𝒱 ≠ topspace X C = ⋃𝒱
    using C by (auto simp: union_of_def Uℬ)
  then have ⋃𝒱 ⊆ topspace X
    by (metis ‹C ⊆ topspace X›)
  then have topspace X ∉ 𝒱
    using ‹⋃𝒱 ≠ topspace X› by blast
  then obtain ℬ' where ℬ': finite ℬ' ℬ' ⊆ ℬ x ∈ ⋂ℬ' W = topspace X ∩ ⋂ℬ'
    using W ‹x ∈ W› unfolding relative_to_def intersection_of_def by auto
  then have ⋂ℬ' ⊆ ⋃ℬ
    using ‹W ∈ 𝒱› ‹⋃𝒱 ≠ topspace X› ‹⋃𝒱 ⊆ topspace X› by blast
  then have ⋂ℬ' ⊆ C
    using Uℬ ‹C = ⋃𝒱› ‹W = topspace X ∩ ⋂ℬ'› ‹W ∈ 𝒱› by auto
  have ∀ b ∈ ℬ'. ∃ C'. finite C' ∧ C' ⊆ 𝒦 ∧ topspace X ⊆ ⋃(insert b C')
  proof
    fix b
    assume b ∈ ℬ'
     have insert b 𝒦 = 𝒦 if neg: ¬ (∃ C'. finite C' ∧ C' ⊆ 𝒦 ∧ topspace X ⊆
⋃(insert b C'))
    proof (rule *)
      show openin X W if W ∈ insert b 𝒦 for W
        using that
      proof
        have b ∈ ℬ
          using ‹b ∈ ℬ'› ‹ℬ' ⊆ ℬ› by blast
        then have ∃𝒰. finite 𝒰 ∧ 𝒰 ⊆ ℬ ∧ ⋂𝒰 = b
          by (rule_tac x={b} in exI) auto
        moreover have ⋃ℬ ∩ b = b
          using ℬ'(2) ‹b ∈ ℬ'› by auto
        ultimately show openin X W if W = b
          using  that ‹b ∈ ℬ'›
          apply (simp add: openin_subbase flip: X)
          apply (auto simp: arbitrary_def intersection_of_def relative_to_def intro!:
union_of_inc)
          done
        show openin X W if W ∈ 𝒦
          by (simp add: ‹W ∈ 𝒦› ope)
      qed
    next
      show topspace X ⊆ ⋃ (insert b 𝒦)
        using top by auto
    next
      show False if finite ℱ and ℱ ⊆ insert b 𝒦 topspace X ⊆ ⋃ℱ for ℱ
      proof -
        have insert b (ℱ ∩ 𝒦) = ℱ
          using non that by blast
        then show False
          by (metis Int_lower2 finite_insert neg that(1) that(3))
      qed
    qed auto
```

then show $\exists C'$. *finite* $C' \wedge C' \subseteq \mathcal{K} \wedge$ *topspace* $X \subseteq \bigcup$(*insert* b C')
using ‹$b \in \mathcal{B}'$› ‹$x \notin \bigcup(\mathcal{B} \cap \mathcal{K})$› $\mathcal{B}'$
by (*metis IntI InterE Union_iff subsetD insertI1*)
qed
then obtain F **where** F: $\forall b \in \mathcal{B}'$. *finite* ($F$ b) $\wedge$ F $b \subseteq \mathcal{K} \wedge$ *topspace* $X \subseteq \bigcup$(*insert* b (F b))
by *metis*
let *?*$\mathcal{D}$ = *insert* C ($\bigcup$(F ‘ $\mathcal{B}'$))
show *False*
proof (*rule non*)
have *topspace* $X \subseteq$ ($\bigcap b \in \mathcal{B}'$. $\bigcup$(*insert* b (F b)))
using F **by** (*simp add*: *INT_greatest*)
also have ... $\subseteq \bigcup$ *?*$\mathcal{D}$
using ‹$\bigcap \mathcal{B}' \subseteq C$› **by** *force*
finally show *topspace* $X \subseteq \bigcup$ *?*$\mathcal{D}$ **.**
show *?*$\mathcal{D} \subseteq \mathcal{K}$
using ‹$C \in \mathcal{K}$› F **by** *auto*
show *finite ?*$\mathcal{D}$
using ‹*finite* $\mathcal{B}'$› F **by** *auto*
qed
qed
then show *?thesis*
by (*force simp*: *compact_space_def compactin_def*)
qed

corollary *Alexander_subbase_alt*:
assumes $U \subseteq \bigcup\mathcal{B}$
and *fin*: $\bigwedge C$. $[\![C \subseteq \mathcal{B}; U \subseteq \bigcup C]\!] \Longrightarrow \exists C'$. *finite* $C' \wedge C' \subseteq C \wedge U \subseteq \bigcup C'$
and X: *topology*
(*arbitrary union_of*
(*finite intersection_of* (λx. $x \in \mathcal{B}$) *relative_to* U)) = X
shows *compact_space* X
proof −
have *topspace* $X = U$
using X *topspace_subbase* **by** *fastforce*
have *eq*: $\bigcup$ (*Collect* ((λx. $x \in \mathcal{B}$) *relative_to* U)) = U
unfolding *relative_to_def*
using ‹$U \subseteq \bigcup\mathcal{B}$› **by** *blast*
have $*$: $\exists\mathcal{F}$. *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{C} \wedge \bigcup\mathcal{F}$ = *topspace* X
if $\mathcal{C} \subseteq$ *Collect* ((λx. $x \in \mathcal{B}$) *relative_to topspace* X) **and** *UC*: $\bigcup\mathcal{C}$ = *topspace* X **for** $\mathcal{C}$
proof −
have $\mathcal{C} \subseteq$ (λU. *topspace* $X \cap U$) ‘ $\mathcal{B}$
using *that* **by** (*auto simp*: *relative_to_def*)
then obtain $\mathcal{B}'$ **where** $\mathcal{B}' \subseteq \mathcal{B}$ **and** $\mathcal{B}'$: $\mathcal{C}$ = ($\cap$) (*topspace* X) ‘ $\mathcal{B}'$
by (*auto simp*: *subset_image_iff*)
moreover have $U \subseteq \bigcup\mathcal{B}'$
using $\mathcal{B}'$ ‹*topspace* $X = U$› *UC* **by** *auto*

ultimately obtain $\mathcal{C}'$ **where** *finite* $\mathcal{C}'$ $\mathcal{C}' \subseteq \mathcal{B}'$ $U \subseteq \bigcup \mathcal{C}'$
using *fin* [*of* $\mathcal{B}'$] ‹*topspace* $X = U$› ‹$U \subseteq \bigcup \mathcal{B}$› **by** *blast*
then show *?thesis*
unfolding $\mathcal{B}'$ *ex_finite_subset_image* ‹*topspace* $X = U$› **by** *auto*
qed
show *?thesis*
apply (*rule Alexander_subbase* [**where** $\mathcal{B}$ = *Collect* (($\lambda x.\ x \in \mathcal{B}$) *relative_to* (*topspace* X))])
apply (*simp flip*: X)
apply (*metis finite_intersection_of_relative_to eq*)
apply (*blast intro*: ∗)
done
qed

proposition *continuous_map_componentwise*:
continuous_map X (*product_topology* Y I) f ⟷
f ‘ (*topspace* X) ⊆ *extensional* I ∧ (∀ k ∈ I. *continuous_map* X (Y k) ($\lambda x.$ f x k))
(**is** *?lhs* ⟷ _ ∧ *?rhs*)
proof (*cases* ∀ x ∈ *topspace* X. f x ∈ *extensional* I)
case *True*
then have f ‘ (*topspace* X) ⊆ *extensional* I
by *force*
moreover have *?rhs* **if** L: *?lhs*
proof −
have *openin* X {x ∈ *topspace* X. f x k ∈ U} **if** k ∈ I **and** *openin* (Y k) U **for** k U
proof −
have *openin* (*product_topology* Y I) ({Y. Y k ∈ U} ∩ (Π_E i∈I. *topspace* (Y i)))
apply (*simp add*: *openin_product_topology flip*: *arbitrary_union_of_relative_to*)
apply (*simp add*: *relative_to_def*)
using *that* **apply** (*blast intro*: *arbitrary_union_of_inc finite_intersection_of_inc*)
done
with *that* **have** *openin* X {x ∈ *topspace* X. f x ∈ ({Y. Y k ∈ U} ∩ (Π_E i∈I. *topspace* (Y i)))}
using L **unfolding** *continuous_map_def* **by** *blast*
moreover have {x ∈ *topspace* X. f x ∈ ({Y. Y k ∈ U} ∩ (Π_E i∈I. *topspace* (Y i)))} = {x ∈ *topspace* X. f x k ∈ U}
using L **by** (*auto simp*: *continuous_map_def*)
ultimately show *?thesis*
by *metis*
qed
with *that*
show *?thesis*
by (*auto simp*: *continuous_map_def*)
qed
moreover have *?lhs* **if** *?rhs*
proof −

have *1*: $\bigwedge x.\ x \in topspace\ X \Longrightarrow f\ x \in (\Pi_E\ i{\in}I.\ topspace\ (Y\ i))$
using *that True* **by** (*auto simp*: *continuous_map_def PiE_iff*)
have *2*: $\{x \in S.\ \exists\ T{\in}\mathcal{T}.\ f\ x \in T\} = (\bigcup\ T{\in}\mathcal{T}.\ \{x \in S.\ f\ x \in T\})$ **for** $S\ \mathcal{T}$
by *blast*
have *3*: $\{x \in S.\ \forall\ U{\in}\mathcal{U}.\ f\ x \in U\} = (\bigcap\ (insert\ S\ ((\lambda U.\ \{x \in S.\ f\ x \in U\})\ `\ \mathcal{U})))$ **for** $S\ \mathcal{U}$
by *blast*
show *?thesis*
unfolding *continuous_map_def openin_product_topology arbitrary_def*
proof (*clarsimp simp*: *all_union_of 1 2*)
fix $\mathcal{T}$
assume $\mathcal{T}$: $\mathcal{T} \subseteq Collect\ (finite\ intersection\_of\ (\lambda F.\ \exists\ i\ U.\ F = \{f.\ f\ i \in U\} \land i \in I \land openin\ (Y\ i)\ U)$
relative_to $(\Pi_E\ i{\in}I.\ topspace\ (Y\ i)))$
show *openin X* $(\bigcup\ T{\in}\mathcal{T}.\ \{x \in topspace\ X.\ f\ x \in T\})$
proof (*rule openin_Union*; *clarify*)
fix $S\ T$
assume $T \in \mathcal{T}$
obtain $\mathcal{U}$ **where** $T = (\Pi_E\ i{\in}I.\ topspace\ (Y\ i)) \cap \bigcap\mathcal{U}$ **and** *finite* $\mathcal{U}$
$\mathcal{U} \subseteq \{\{f.\ f\ i \in U\}\ |i\ U.\ i \in I \land openin\ (Y\ i)\ U\}$
using *subsetD* [*OF* $\mathcal{T}$ ‹$T \in \mathcal{T}$›] **by** (*auto simp*: *intersection_of_def relative_to_def*)
with *that* **show** *openin X* $\{x \in topspace\ X.\ f\ x \in T\}$
apply (*simp add*: *continuous_map_def 1 cong*: *conj_cong*)
unfolding *3*
apply (*rule openin_Inter*; *auto*)
done
qed
qed
qed
ultimately show *?thesis*
by *metis*
next
case *False*
then show *?thesis*
by (*auto simp*: *continuous_map_def PiE_def*)
qed

lemma *continuous_map_componentwise_UNIV*:
continuous_map X (*product_topology Y UNIV*) $f \longleftrightarrow (\forall\ k.\ continuous\_map\ X\ (Y\ k)\ (\lambda x.\ f\ x\ k))$
by (*simp add*: *continuous_map_componentwise*)

lemma *continuous_map_product_projection* [*continuous_intros*]:
$k \in I \Longrightarrow continuous\_map\ (product\_topology\ X\ I)\ (X\ k)\ (\lambda x.\ x\ k)$
using *continuous_map_componentwise* [*of product_topology X I X I id*] **by** *simp*

declare *continuous_map_from_subtopology* [*OF continuous_map_product_projection*,

```
continuous_intros]

proposition open_map_product_projection:
  assumes i ∈ I
  shows open_map (product_topology Y I) (Y i) (λf. f i)
   unfolding openin_product_topology all_union_of arbitrary_def open_map_def im-
age_Union
proof clarify
  fix 𝒱
  assume 𝒱: 𝒱 ⊆ Collect
              (finite intersection_of
               (λF. ∃i U. F = {f. f i ∈ U} ∧ i ∈ I ∧ openin (Y i) U) relative_to
                          topspace (product_topology Y I))
  show openin (Y i) (⋃x∈𝒱. (λf. f i) ‘ x)
  proof (rule openin_Union, clarify)
    fix S V
    assume V ∈ 𝒱
    obtain ℱ where finite ℱ
      and V: V = (Π_E i∈I. topspace (Y i)) ∩ ⋂ℱ
      and ℱ: ℱ ⊆ {{f. f i ∈ U} |i U. i ∈ I ∧ openin (Y i) U}
      using subsetD [OF 𝒱 ‹V ∈ 𝒱›]
      by (auto simp: intersection_of_def relative_to_def)
    show openin (Y i) ((λf. f i) ‘ V)
    proof (subst openin_subopen; clarify)
      fix x f
      assume f ∈ V
      let ?T = {a ∈ topspace(Y i).
                (λj. if j = i then a
                     else if j ∈ I then f j else undefined) ∈ (Π_E i∈I. topspace (Y
i)) ∩ ⋂ℱ}
      show ∃T. openin (Y i) T ∧ f i ∈ T ∧ T ⊆ (λf. f i) ‘ V
      proof (intro exI conjI)
        show openin (Y i) ?T
        proof (rule openin_continuous_map_preimage)
           have continuous_map (Y i) (Y k) (λx. if k = i then x else f k) if k ∈ I
for k
          proof (cases k=i)
            case True
            then show ?thesis
              by (metis (mono_tags) continuous_map_id eq_id_iff)
          next
            case False
            then show ?thesis
              by simp (metis IntD1 PiE_iff V ‹f ∈ V› that)
          qed
          then show continuous_map (Y i) (product_topology Y I)
                  (λx j. if j = i then x else if j ∈ I then f j else undefined)
            by (auto simp: continuous_map_componentwise assms extensional_def)
        next
```

have *openin* (*product_topology Y I*) (Π_E *i*∈*I*. *topspace* (*Y i*))
by (*metis openin_topspace topspace_product_topology*)
moreover have *openin* (*product_topology Y I*) (⋂*B*∈$\mathcal{F}$. (Π_E *i*∈*I*. *topspace* (*Y i*)) ∩ *B*)
if $\mathcal{F}$ ≠ {}
proof −
show *?thesis*
proof (*rule openin_Inter*)
show ⋀*X*. *X* ∈ (∩) (Π_E *i*∈*I*. *topspace* (*Y i*)) ' $\mathcal{F}$ ⟹ *openin* (*product_topology Y I*) *X*
unfolding *openin_product_topology relative_to_def*
apply (*clarify intro*!: *arbitrary_union_of_inc*)
apply (*rename_tac F*)
apply (*rule_tac x*=*F* **in** *exI*)
using *subsetD* [*OF* $\mathcal{F}$]
apply (*force intro*: *finite_intersection_of_inc*)
done
qed (*use* ‹*finite* $\mathcal{F}$› ‹$\mathcal{F}$ ≠ {}› **in** *auto*)
qed
ultimately show *openin* (*product_topology Y I*) ((Π_E *i*∈*I*. *topspace* (*Y i*)) ∩ ⋂$\mathcal{F}$)
by (*auto simp only*: *Int_Inter_eq split*: *if_split*)
qed
next
have *eqf*: (λ*j*. *if j* = *i then f i else if j* ∈ *I then f j else undefined*) = *f*
using *PiE_arb V* ‹*f* ∈ *V*› **by** *force*
show *f i* ∈ *?T*
using *V assms* ‹*f* ∈ *V*› **by** (*auto simp*: *PiE_iff eqf*)
next
show *?T* ⊆ (λ*f*. *f i*) ' *V*
unfolding *V* **by** (*auto simp*: *intro*!: *rev_image_eqI*)
qed
qed
qed
qed

lemma *retraction_map_product_projection*:
assumes *i* ∈ *I*
shows (*retraction_map* (*product_topology X I*) (*X i*) (λ*x*. *x i*) ⟷
(*topspace* (*product_topology X I*) = {}) ⟶ *topspace* (*X i*) = {})
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
using *retraction_imp_surjective_map* **by** *force*
next
assume *R*: *?rhs*
show *?lhs*
proof (*cases topspace* (*product_topology X I*) = {})

```
    case True
    then show ?thesis
      using R by (auto simp: retraction_map_def retraction_maps_def continuous_map_on_empty)
  next
    case False
    have *: ∃g. continuous_map (X i) (product_topology X I) g ∧ (∀x∈topspace (X i). g x i = x)
      if z: z ∈ (Π_E i∈I. topspace (X i)) for z
    proof -
      have cm: continuous_map (X i) (X j) (λx. if j = i then x else z j) if j ∈ I for j
        using ‹j ∈ I› z by (case_tac j = i) auto
      show ?thesis
        using ‹i ∈ I› that
        by (rule_tac x=λx j. if j = i then x else z j in exI) (auto simp: continuous_map_componentwise PiE_iff extensional_def cm)
    qed
    show ?thesis
      using ‹i ∈ I› False
      by (auto simp: retraction_map_def retraction_maps_def assms continuous_map_product_projection *)
  qed
qed
```

4.8.3 Open Pi-sets in the product topology

```
proposition openin_PiE_gen:
  openin (product_topology X I) (PiE I S) ⟷
      PiE I S = {} ∨
      finite {i ∈ I. ~(S i = topspace(X i))} ∧ (∀i ∈ I. openin (X i) (S i))
  (is ?lhs ⟷ _ ∨ ?rhs)
proof (cases PiE I S = {})
  case False
  moreover have ?lhs = ?rhs
  proof
    assume L: ?lhs
    moreover
    obtain z where z: z ∈ PiE I S
      using False by blast
    ultimately obtain U where fin: finite {i ∈ I. U i ≠ topspace (X i)}
      and Pi_E I U ≠ {}
      and sub: Pi_E I U ⊆ Pi_E I S
      by (fastforce simp add: openin_product_topology_alt)
    then have *: ⋀i. i ∈ I ⟹ U i ⊆ S i
      by (simp add: subset_PiE)
    show ?rhs
    proof (intro conjI ballI)
      show finite {i ∈ I. S i ≠ topspace (X i)}
```

```
        apply (rule finite_subset [OF _ fin], clarify)
        using ∗
        by (metis False L openin_subset topspace_product_topology subset_PiE subset_antisym)
    next
      fix i :: 'a
      assume i ∈ I
      then show openin (X i) (S i)
        using open_map_product_projection [of i I X] L
        apply (simp add: open_map_def)
        apply (drule_tac x=PiE I S in spec)
        apply (simp add: False image_projection_PiE split: if_split_asm)
        done
    qed
  next
    assume ?rhs
    then show ?lhs
      apply (simp only: openin_product_topology)
      apply (rule arbitrary_union_of_inc)
      apply (auto simp: product_topology_base_alt)
      done
  qed
  ultimately show ?thesis
    by simp
qed simp


corollary openin_PiE:
  finite I ⟹ openin (product_topology X I) (PiE I S) ⟷ PiE I S = {} ∨ (∀ i ∈ I. openin (X i) (S i))
  by (simp add: openin_PiE_gen)

proposition compact_space_product_topology:
  compact_space(product_topology X I) ⟷
   topspace(product_topology X I) = {} ∨ (∀ i ∈ I. compact_space(X i))
   (is ?lhs = ?rhs)
proof (cases topspace(product_topology X I) = {})
  case False
  then obtain z where z: z ∈ (Π_E i∈I. topspace(X i))
    by auto
  show ?thesis
  proof
    assume L: ?lhs
    show ?rhs
    proof (clarsimp simp add: False compact_space_def)
      fix i
      assume i ∈ I
      with L have continuous_map (product_topology X I) (X i) (λf. f i)
        by (simp add: continuous_map_product_projection)
```

moreover
have $\bigwedge x.\ x \in topspace\ (X\ i) \Longrightarrow x \in (\lambda f.\ f\ i)\ `\ (\Pi_E\ i{\in}I.\ topspace\ (X\ i))$
using ‹$i \in I$› z
apply (*rule_tac* $x{=}\lambda j.$ *if* $j = i$ *then* x *else if* $j \in I$ *then* $z\ j$ *else undefined* **in** *image_eqI*, *auto*)
done
then have $(\lambda f.\ f\ i)\ `\ (\Pi_E\ i{\in}I.\ topspace\ (X\ i)) = topspace\ (X\ i)$
using ‹$i \in I$› z **by** *auto*
ultimately show *compactin* $(X\ i)$ (*topspace* $(X\ i)$)
by (*metis L compact_space_def image_compactin topspace_product_topology*)
qed
next
assume *R*: *?rhs*
show *?lhs*
proof (*cases* $I = \{\}$)
case *True*
with *R* **show** *?thesis*
by (*simp add*: *compact_space_def*)
next
case *False*
then obtain i **where** $i \in I$
by *blast*
show *?thesis*
using *R*
proof
assume *com* [*rule_format*]: $\forall i{\in}I.$ *compact_space* $(X\ i)$
let $?\mathcal{C} = \{\{f.\ f\ i \in U\}\ |i\ U.\ i \in I \wedge openin\ (X\ i)\ U\}$
show *compact_space* (*product_topology* $X\ I$)
proof (*rule Alexander_subbase_alt*)
show *topspace* (*product_topology* $X\ I$) $\subseteq \bigcup ?\mathcal{C}$
unfolding *topspace_product_topology* **using** ‹$i \in I$› **by** *blast*
next
fix C
assume *Csub*: $C \subseteq ?\mathcal{C}$ **and** *UC*: *topspace* (*product_topology* $X\ I$) $\subseteq \bigcup C$
define $\mathcal{D}$ **where** $\mathcal{D} \equiv \lambda i.\ \{U.\ openin\ (X\ i)\ U \wedge \{f.\ f\ i \in U\} \in C\}$
show $\exists C'.\ finite\ C' \wedge C' \subseteq C \wedge$ *topspace* (*product_topology* $X\ I$) $\subseteq \bigcup C'$
proof (*cases* $\exists i.\ i \in I \wedge topspace\ (X\ i) \subseteq \bigcup(\mathcal{D}\ i)$)
case *True*
then obtain i **where** $i \in I$
and i: *topspace* $(X\ i) \subseteq \bigcup(\mathcal{D}\ i)$
unfolding *$\mathcal{D}$_def* **by** *blast*
then have $*$: $\bigwedge\mathcal{U}.\ [\![Ball\ \mathcal{U}\ (openin\ (X\ i));\ topspace\ (X\ i) \subseteq \bigcup\mathcal{U}]\!] \Longrightarrow$
$\exists\mathcal{F}.\ finite\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge topspace\ (X\ i) \subseteq \bigcup\mathcal{F}$
using *com* [*OF* ‹$i \in I$›] **by** (*auto simp*: *compact_space_def compactin_def*)
have *topspace* $(X\ i) \subseteq \bigcup(\mathcal{D}\ i)$
using i **by** *auto*
with $*$ **obtain** $\mathcal{F}$ **where** *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq (\mathcal{D}\ i) \wedge$ *topspace* $(X\ i) \subseteq \bigcup\mathcal{F}$
unfolding *$\mathcal{D}$_def* **by** *fastforce*
with ‹$i \in I$› **show** *?thesis*

unfolding $\mathcal{D}$_*def*
by (*rule_tac x*=(λU. {*x*. *x i* ∈ *U*}) ‘ $\mathcal{F}$ **in** *exI*) *auto*
next
case *False*
then have ∀ *i* ∈ *I*. ∃ *y*. *y* ∈ *topspace* (*X i*) ∧ *y* ∉ ⋃(𝒟 *i*)
by *force*
then obtain *g* **where** *g*: ⋀*i*. *i* ∈ *I* ⟹ *g i* ∈ *topspace* (*X i*) ∧ *g i* ∉ ⋃(𝒟 *i*)
by *metis*
then have (λ*i*. *if i* ∈ *I then g i else undefined*) ∈ *topspace* (*product_topology X I*)
by (*simp add*: *PiE_I*)
moreover have (λ*i*. *if i* ∈ *I then g i else undefined*) ∉ ⋃ *C*
using *Csub g* **unfolding** 𝒟_*def* **by** *force*
ultimately show *?thesis*
using *UC* **by** *blast*
qed
qed (*simp add*: *product_topology*)
qed (*simp add*: *compact_space_topspace_empty*)
qed
qed
qed (*simp add*: *compact_space_topspace_empty*)

corollary *compactin_PiE*:
compactin (*product_topology X I*) (*PiE I S*) ⟷
PiE I S = {} ∨ (∀ *i* ∈ *I*. *compactin* (*X i*) (*S i*))
by (*auto simp*: *compactin_subspace subtopology_PiE subset_PiE compact_space_product_topology PiE_eq_empty_iff*)

lemma *in_product_topology_closure_of*:
z ∈ (*product_topology X I*) *closure_of S*
⟹ *i* ∈ *I* ⟹ *z i* ∈ ((*X i*) *closure_of* ((λ*x*. *x i*) ‘ *S*))
using *continuous_map_product_projection*
by (*force simp*: *continuous_map_eq_image_closure_subset image_subset_iff*)

lemma *homeomorphic_space_singleton_product*:
product_topology X {*k*} *homeomorphic_space* (*X k*)
unfolding *homeomorphic_space*
apply (*rule_tac x*=λ*x*. *x k* **in** *exI*)
apply (*rule bijective_open_imp_homeomorphic_map*)
apply (*simp_all add*: *continuous_map_product_projection open_map_product_projection*)
unfolding *PiE_over_singleton_iff*
apply (*auto simp*: *image_iff inj_on_def*)
done

4.8.4 Relationship with connected spaces, paths, etc.

proposition *connected_space_product_topology*:
connected_space(*product_topology X I*) ⟷

```
   (ΠE i∈I. topspace (X i)) = {} ∨ (∀ i ∈ I. connected_space(X i))
  (is ?lhs ⟷ ?eq ∨ ?rhs)
proof (cases ?eq)
  case False
  moreover have ?lhs = ?rhs
  proof
    assume ?lhs
    moreover
    have connectedin(X i) (topspace(X i))
      if i ∈ I and ci: connectedin(product_topology X I) (topspace(product_topology
X I)) for i
    proof −
      have cm: continuous_map (product_topology X I) (X i) (λf. f i)
        by (simp add: ‹i ∈ I› continuous_map_product_projection)
      show ?thesis
        using connectedin_continuous_map_image [OF cm ci] ‹i ∈ I›
        by (simp add: False image_projection_PiE)
    qed
    ultimately show ?rhs
      by (meson connectedin_topspace)
  next
    assume cs [rule_format]: ?rhs
    have False
      if disj: U ∩ V = {} and subUV: (ΠE i∈I. topspace (X i)) ⊆ U ∪ V
        and U: openin (product_topology X I) U
        and V: openin (product_topology X I) V
        and U ≠ {} V ≠ {}
      for U V
    proof −
      obtain f where f ∈ U
        using ‹U ≠ {}› by blast
      then have f: f ∈ (ΠE i∈I. topspace (X i))
        using U openin_subset by fastforce
       have U ⊆ topspace(product_topology X I) V ⊆ topspace(product_topology X
I)
        using U V openin_subset by blast+
      moreover have (ΠE i∈I. topspace (X i)) ⊆ U
      proof −
        obtain C where (finite intersection_of (λF. ∃ i U. F = {x. x i ∈ U} ∧ i
∈ I ∧ openin (X i) U) relative_to
            (ΠE i∈I. topspace (X i))) C C ⊆ U f ∈ C
         using U ‹f ∈ U› unfolding openin_product_topology union_of_def by auto
        then obtain 𝒯 where finite 𝒯
          and t: ⋀C. C ∈ 𝒯 ⟹ ∃ i u. (i ∈ I ∧ openin (X i) u) ∧ C = {x. x i ∈
u}
          and subU: topspace (product_topology X I) ∩ ⋂𝒯 ⊆ U
          and ftop: f ∈ topspace (product_topology X I)
          and fint: f ∈ ⋂ 𝒯
          by (fastforce simp: relative_to_def intersection_of_def subset_iff)
```

let *?L* = $\bigcup$ *C*∈$\mathcal{T}$. {*i*. (λ*x*. *x i*) ' *C* ⊂ *topspace* (*X i*)}
obtain *L* **where** *finite L*
and *L*: $\bigwedge$*i U*. ⟦*i* ∈ *I*; *openin* (*X i*) *U*; *U* ⊂ *topspace*(*X i*); {*x*. *x i* ∈ *U*} ∈ $\mathcal{T}$⟧ ⟹ *i* ∈ *L*
proof
show *finite ?L*
proof (*rule finite_Union*)
fix *M*
assume *M* ∈ (λ*C*. {*i*. (λ*x*. *x i*) ' *C* ⊂ *topspace* (*X i*)}) ' $\mathcal{T}$
then obtain *C* **where** *C* ∈ $\mathcal{T}$ **and** *C*: *M* = {*i*. (λ*x*. *x i*) ' *C* ⊂ *topspace* (*X i*)}
by *blast*
then obtain *j V* **where** *j* ∈ *I* **and** *ope*: *openin* (*X j*) *V* **and** *Ceq*: *C* = {*x*. *x j* ∈ *V*}
using *t* **by** *meson*
then have *f j* ∈ *V*
using ‹*C* ∈ $\mathcal{T}$› *fint* **by** *force*
then have (λ*x*. *x k*) ' {*x*. *x j* ∈ *V*} = *UNIV* **if** *k* ≠ *j* **for** *k*
using *that*
apply (*clarsimp simp add*: *set_eq_iff*)
apply (*rule_tac x*=λ*m*. *if m* = *k then x else f m* **in** *image_eqI*, *auto*)
done
then have {*i*. (λ*x*. *x i*) ' *C* ⊂ *topspace* (*X i*)} ⊆ {*j*}
using *Ceq* **by** *auto*
then show *finite M*
using *C finite_subset* **by** *fastforce*
qed (*use* ‹*finite* $\mathcal{T}$› **in** *blast*)
next
fix *i U*
assume *i* ∈ *I* **and** *ope*: *openin* (*X i*) *U* **and** *psub*: *U* ⊂ *topspace* (*X i*) **and** *int*: {*x*. *x i* ∈ *U*} ∈ $\mathcal{T}$
then show *i* ∈ *?L*
by (*rule_tac a*={*x*. *x i* ∈ *U*} **in** *UN_I*) (*force*+)
qed
show *?thesis*
proof
fix *h*
assume *h*: *h* ∈ (Π_E *i*∈*I*. *topspace* (*X i*))
define *g* **where** *g* ≡ λ*i*. *if i* ∈ *L then f i else h i*
have *gin*: *g* ∈ (Π_E *i*∈*I*. *topspace* (*X i*))
unfolding *g_def* **using** *f h* **by** *auto*
moreover have *g* ∈ *X* **if** *X* ∈ $\mathcal{T}$ **for** *X*
using *fint openin_subset t* [*OF that*] *L g_def h that* **by** *fastforce*
ultimately have *g* ∈ *U*
using *subU* **by** *auto*
have *h* ∈ *U* **if** *finite M h* ∈ *PiE I* (*topspace* ∘ *X*) {*i* ∈ *I*. *h i* ≠ *g i*} ⊆ *M* **for** *M h*
using *that*
proof (*induction arbitrary*: *h*)

```
      case empty
      then show ?case
        using PiE_ext ‹g ∈ U› gin by force
    next
      case (insert i M)
      define f where f ≡ λj. if j = i then g i else h j
      have fin: f ∈ PiE I (topspace ∘ X)
        unfolding f_def using gin insert.prems(1) by auto
      have subM: {j ∈ I. f j ≠ g j} ⊆ M
        unfolding f_def using insert.prems(2) by auto
      have f ∈ U
        using insert.IH [OF fin subM] .
      show ?case
      proof (cases h ∈ V)
        case True
        show ?thesis
        proof (cases i ∈ I)
          case True
          let ?U = {x ∈ topspace(X i). (λj. if j = i then x else h j) ∈ U}
          let ?V = {x ∈ topspace(X i). (λj. if j = i then x else h j) ∈ V}
          have False
          proof (rule connected_spaceD [OF cs [OF ‹i ∈ I›]])
           have ⋀k. k ∈ I ⟹ continuous_map (X i) (X k) (λx. if k = i then
x else h k)
                   using continuous_map_eq_topcontinuous_at insert.prems(1)
topcontinuous_at_def by fastforce
           then have cm: continuous_map (X i) (product_topology X I) (λx j.
if j = i then x else h j)
              using ‹i ∈ I› insert.prems(1)
              by (auto simp: continuous_map_componentwise extensional_def)
            show openin (X i) ?U
              by (rule openin_continuous_map_preimage [OF cm U])
            show openin (X i) ?V
              by (rule openin_continuous_map_preimage [OF cm V])
            show topspace (X i) ⊆ ?U ∪ ?V
            proof clarsimp
              fix x
             assume x ∈ topspace (X i) and (λj. if j = i then x else h j) ∉ V
              with True subUV ‹h ∈ Pi_E I (topspace ∘ X)›
              show (λj. if j = i then x else h j) ∈ U
                by (drule_tac c=(λj. if j = i then x else h j) in subsetD) auto
            qed
            show ?U ∩ ?V = {}
              using disj by blast
            show ?U ≠ {}
              using ‹U ≠ {}› f_def
            proof -
              have (λj. if j = i then g i else h j) ∈ U
                using ‹f ∈ U› f_def by blast
```

moreover have *f i* ∈ *topspace* (*X i*)
by (*metis PiE_iff True comp_apply fin*)
ultimately have ∃ *b*. *b* ∈ *topspace* (*X i*) ∧ (λ*a*. *if a* = *i then b else h a*) ∈ *U*
using *f_def* **by** *auto*
then show *?thesis*
by *blast*
qed
have (λ*j*. *if j* = *i then h i else h j*) = *h*
by *force*
moreover have *h i* ∈ *topspace* (*X i*)
using *True insert.prems*(*1*) **by** *auto*
ultimately show *?V* ≠ {}
using ‹*h* ∈ *V*› **by** *force*
qed
then show *?thesis* **..**
next
case *False*
show *?thesis*
proof (*cases h* = *f*)
case *True*
show *?thesis*
by (*rule insert.IH* [*OF insert.prems*(*1*)]) (*simp add*: *True subM*)
next
case *False*
then show *?thesis*
using *gin insert.prems*(*1*) ‹*i* ∉ *I*› **unfolding** *f_def* **by** *fastforce*
qed
qed
next
case *False*
then show *?thesis*
using *subUV insert.prems*(*1*) **by** *auto*
qed
qed
then show *h* ∈ *U*
unfolding *g_def* **using** *PiE_iff* ‹*finite L*› *h* **by** *fastforce*
qed
qed
ultimately show *?thesis*
using *disj inf_absorb2* ‹*V* ≠ {}› **by** *fastforce*
qed
then show *?lhs*
unfolding *connected_space_def*
by *auto*
qed
ultimately show *?thesis*
by *simp*
qed (*simp add*: *connected_space_topspace_empty*)

lemma *connectedin_PiE*:
  connectedin (product_topology X I) (PiE I S) ⟷
    PiE I S = {} ∨ (∀ *i* ∈ *I*. *connectedin (X i) (S i)*)
 by (*fastforce simp add*: *connectedin_def subtopology_PiE connected_space_product_topology subset_PiE PiE_eq_empty_iff*)

lemma *path_connected_space_product_topology*:
  path_connected_space(*product_topology X I*) ⟷
   topspace(*product_topology X I*) = {} ∨ (∀ *i* ∈ *I*. *path_connected_space*(*X i*))
 (**is** *?lhs* ⟷ *?eq* ∨ *?rhs*)
proof (*cases ?eq*)
 case *False*
 moreover have *?lhs* = *?rhs*
 proof
  assume *L*: *?lhs*
  show *?rhs*
  proof (*clarsimp simp flip*: *path_connectedin_topspace*)
   fix *i* :: *'a*
   assume *i* ∈ *I*
   have *cm*: *continuous_map* (*product_topology X I*) (*X i*) (λ*f*. *f i*)
    by (*simp add*: ‹*i* ∈ *I*› *continuous_map_product_projection*)
   show *path_connectedin* (*X i*) (*topspace* (*X i*))
   using *path_connectedin_continuous_map_image* [*OF cm L* [*unfolded path_connectedin_topspace* [*symmetric*]]]
   by (*metis* ‹*i* ∈ *I*› *False retraction_imp_surjective_map retraction_map_product_projection*)
  qed
 next
  assume *R* [*rule_format*]: *?rhs*
  show *?lhs*
   unfolding *path_connected_space_def topspace_product_topology*
  proof *clarify*
   fix *x y*
   assume *x*: *x* ∈ (Π$_E$ *i*∈*I*. *topspace* (*X i*)) **and** *y*: *y* ∈ (Π$_E$ *i*∈*I*. *topspace* (*X i*))
   have ∀ *i*. ∃ *g*. *i* ∈ *I* ⟶ *pathin* (*X i*) *g* ∧ *g 0* = *x i* ∧ *g 1* = *y i*
    using *PiE_mem R path_connected_space_def x y* **by** *force*
   then obtain *g* **where** *g*: ⋀*i*. *i* ∈ *I* ⟹ *pathin* (*X i*) (*g i*) ∧ *g i 0* = *x i* ∧ *g i 1* = *y i*
    by *metis*
   with *x y* **show** ∃ *g*. *pathin* (*product_topology X I*) *g* ∧ *g 0* = *x* ∧ *g 1* = *y*
    apply (*rule_tac x*=λ*a*. λ*i* ∈ *I*. *g i a* **in** *exI*)
    apply (*force simp*: *pathin_def continuous_map_componentwise*)
    done
  qed
 qed
 ultimately show *?thesis*
  by *simp*

qed (*simp add*: *path_connected_space_topspace_empty*)

lemma *path_connectedin_PiE*:
path_connectedin (*product_topology X I*) (*PiE I S*) $\longleftrightarrow$
PiE I S = {} ∨ (∀ *i* ∈ *I*. *path_connectedin* (*X i*) (*S i*))
by (*fastforce simp add*: *path_connectedin_def subtopology_PiE path_connected_space_product_topology subset_PiE PiE_eq_empty_iff topspace_subtopology_subset*)

4.8.5 Projections from a function topology to a component

lemma *quotient_map_product_projection*:
assumes *i* ∈ *I*
shows *quotient_map*(*product_topology X I*) (*X i*) (λ*x*. *x i*) $\longleftrightarrow$
(*topspace*(*product_topology X I*) = {} ⟶ *topspace*(*X i*) = {})
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **with** *assms* **show** *?rhs*
by (*auto simp*: *continuous_open_quotient_map open_map_product_projection*)
next
assume *?rhs* **with** *assms* **show** *?lhs*
by (*auto simp*: *Abstract_Topology.retraction_imp_quotient_map retraction_map_product_projection*)
qed

lemma *product_topology_homeomorphic_component*:
assumes *i* ∈ *I* ⋀*j*. ⟦*j* ∈ *I*; *j* ≠ *i*⟧ ⟹ ∃ *a*. *topspace*(*X j*) = {*a*}
shows *product_topology X I homeomorphic_space* (*X i*)
proof –
have *quotient_map* (*product_topology X I*) (*X i*) (λ*x*. *x i*)
using *assms* **by** (*force simp add*: *quotient_map_product_projection PiE_eq_empty_iff*)
moreover
have *inj_on* (λ*x*. *x i*) (Π_E *i*∈*I*. *topspace* (*X i*))
using *assms* **by** (*auto simp*: *inj_on_def PiE_iff*) (*metis extensionalityI singletonD*)
ultimately show *?thesis*
unfolding *homeomorphic_space_def*
by (*rule_tac x*=λ*x*. *x i* **in** *exI*) (*simp add*: *homeomorphic_map_def flip*: *homeomorphic_map_maps*)
qed

lemma *topological_property_of_product_component*:
assumes *major*: *P* (*product_topology X I*)
and *minor*: ⋀*z i*. ⟦*z* ∈ (Π_E *i*∈*I*. *topspace*(*X i*)); *P*(*product_topology X I*); *i* ∈ *I*⟧
⟹ *P*(*subtopology* (*product_topology X I*) (*PiE I* (λ*j*. *if j* = *i then topspace*(*X i*) *else* {*z j*})))
(**is** ⋀*z i*. ⟦_; _; _⟧ ⟹ *P* (*?SX z i*))
and *PQ*: ⋀*X X'*. *X homeomorphic_space X'* ⟹ (*P X* $\longleftrightarrow$ *Q X'*)
shows (Π_E *i*∈*I*. *topspace*(*X i*)) = {} ∨ (∀ *i* ∈ *I*. *Q*(*X i*))
proof –

```
  have Q(X i) if (Π_E i∈I. topspace(X i)) ≠ {} i ∈ I for i
  proof -
    from that obtain f where f: f ∈ (Π_E i∈I. topspace (X i))
      by force
    have ?SX f i homeomorphic_space X i
      apply (simp add: subtopology_PiE )
     using product_topology_homeomorphic_component [OF ‹i ∈ I›, of λj. subtopol-
ogy (X j) (if j = i then topspace (X i) else {f j})]
      using f by fastforce
    then show ?thesis
      using minor [OF f major ‹i ∈ I›] PQ by auto
  qed
  then show ?thesis by metis
qed

end
```

4.9 Bounded Linear Function

```
theory Bounded_Linear_Function
imports
  Topology_Euclidean_Space
  Operator_Norm
  Uniform_Limit
  Function_Topology

begin

lemma onorm_componentwise:
  assumes bounded_linear f
  shows onorm f ≤ (∑ i∈Basis. norm (f i))
proof -
  {
    fix i::'a
    assume i ∈ Basis
    hence onorm (λx. (x · i) *_R f i) ≤ onorm (λx. (x · i)) * norm (f i)
      by (auto intro!: onorm_scaleR_left_lemma bounded_linear_inner_left)
    also have ... ≤  norm i * norm (f i)
      by (rule mult_right_mono)
        (auto simp: ac_simps Cauchy_Schwarz_ineq2 intro!: onorm_le)
    finally have onorm (λx. (x · i) *_R f i) ≤ norm (f i) using ‹i ∈ Basis›
      by simp
  } hence onorm (λx. ∑ i∈Basis. (x · i) *_R f i) ≤ (∑ i∈Basis. norm (f i))
    by (auto intro!: order_trans[OF onorm_sum_le] bounded_linear_scaleR_const
      sum_mono bounded_linear_inner_left)
  also have (λx. ∑ i∈Basis. (x · i) *_R f i) = (λx. f (∑ i∈Basis. (x · i) *_R i))
    by (simp add: linear_sum bounded_linear.linear assms linear_simps)
  also have ... = f
    by (simp add: euclidean_representation)
```

finally show *?thesis* **.**
qed

lemmas *onorm_componentwise_le* = *order_trans*[*OF onorm_componentwise*]

4.9.1 Intro rules for *bounded_linear*

named_theorems *bounded_linear_intros*

lemma *onorm_inner_left*:
  assumes *bounded_linear r*
  shows *onorm* (λ*x*. *r x* • *f*) ≤ *onorm r* ∗ *norm f*
proof (*rule onorm_bound*)
  fix *x*
  have *norm* (*r x* • *f*) ≤ *norm* (*r x*) ∗ *norm f*
    by (*simp add*: *Cauchy_Schwarz_ineq2*)
  also have . . . ≤ *onorm r* ∗ *norm x* ∗ *norm f*
    by (*intro mult_right_mono onorm assms norm_ge_zero*)
  finally show *norm* (*r x* • *f*) ≤ *onorm r* ∗ *norm f* ∗ *norm x*
    by (*simp add*: *ac_simps*)
qed (*intro mult_nonneg_nonneg norm_ge_zero onorm_pos_le assms*)

lemma *onorm_inner_right*:
  assumes *bounded_linear r*
  shows *onorm* (λ*x*. *f* • *r x*) ≤ *norm f* ∗ *onorm r*
  apply (*subst inner_commute*)
  apply (*rule onorm_inner_left*[*OF assms*, *THEN order_trans*])
  apply *simp*
  done

lemmas [*bounded_linear_intros*] =
  bounded_linear_zero
  bounded_linear_add
  bounded_linear_const_mult
  bounded_linear_mult_const
  bounded_linear_scaleR_const
  bounded_linear_const_scaleR
  bounded_linear_ident
  bounded_linear_sum
  bounded_linear_Pair
  bounded_linear_sub
  bounded_linear_fst_comp
  bounded_linear_snd_comp
  bounded_linear_inner_left_comp
  bounded_linear_inner_right_comp

4.9.2 declaration of derivative/continuous/tendsto introduction rules for bounded linear functions

attribute_setup *bounded_linear* =
‹*Scan.succeed* (*Thm.declaration_attribute* (*fn thm* =>
fold (*fn* (*r*, *s*) => *Named_Theorems.add_thm s* (*thm RS r*))
[
(@{*thm bounded_linear.has_derivative*}, ***named_theorems*** ‹*derivative_intros*›),
(@{*thm bounded_linear.tendsto*}, ***named_theorems*** ‹*tendsto_intros*›),
(@{*thm bounded_linear.continuous*}, ***named_theorems*** ‹*continuous_intros*›),
(@{*thm bounded_linear.continuous_on*}, ***named_theorems*** ‹*continuous_intros*›),
(@{*thm bounded_linear.uniformly_continuous_on*}, ***named_theorems*** ‹*continuous_intros*›),
(@{*thm bounded_linear_compose*}, ***named_theorems*** ‹*bounded_linear_intros*›)
]))›

attribute_setup *bounded_bilinear* =
‹*Scan.succeed* (*Thm.declaration_attribute* (*fn thm* =>
fold (*fn* (*r*, *s*) => *Named_Theorems.add_thm s* (*thm RS r*))
[
(@{*thm bounded_bilinear.FDERIV*}, ***named_theorems*** ‹*derivative_intros*›),
(@{*thm bounded_bilinear.tendsto*}, ***named_theorems*** ‹*tendsto_intros*›),
(@{*thm bounded_bilinear.continuous*}, ***named_theorems*** ‹*continuous_intros*›),
(@{*thm bounded_bilinear.continuous_on*}, ***named_theorems*** ‹*continuous_intros*›),
(@{*thm bounded_linear_compose*[*OF bounded_bilinear.bounded_linear_left*]},
named_theorems ‹*bounded_linear_intros*›),
(@{*thm bounded_linear_compose*[*OF bounded_bilinear.bounded_linear_right*]},
named_theorems ‹*bounded_linear_intros*›),
(@{*thm bounded_linear.uniformly_continuous_on*[*OF bounded_bilinear.bounded_linear_left*]},
named_theorems ‹*continuous_intros*›),
(@{*thm bounded_linear.uniformly_continuous_on*[*OF bounded_bilinear.bounded_linear_right*]},
named_theorems ‹*continuous_intros*›)
]))›

4.9.3 Type of bounded linear functions

typedef (overloaded) ('*a*, '*b*) *blinfun* (($\_ \Rightarrow_L /\_$) [*22*, *21*] *21*) =
{*f*::'*a*::*real_normed_vector*⇒'*b*::*real_normed_vector*. *bounded_linear f*}
morphisms *blinfun_apply Blinfun*
by (*blast intro*: *bounded_linear_intros*)

declare [[*coercion*
blinfun_apply :: ('*a*::*real_normed_vector* $\Rightarrow_L$'*b*::*real_normed_vector*) ⇒ '*a* ⇒ '*b*]]

lemma *bounded_linear_blinfun_apply*[*bounded_linear_intros*]:
bounded_linear g ⟹ *bounded_linear* (λ*x*. *blinfun_apply f* (*g x*))
by (*metis blinfun_apply mem_Collect_eq bounded_linear_compose*)

setup_lifting *type_definition_blinfun*

lemma *blinfun_eqI*: (⋀*i*. *blinfun_apply x i* = *blinfun_apply y i*) ⟹ *x* = *y*

by *transfer auto*

lemma *bounded_linear_Blinfun_apply*: *bounded_linear f* $\Longrightarrow$ *blinfun_apply* (*Blinfun f*) = *f*
by (*auto simp*: *Blinfun_inverse*)

4.9.4 Type class instantiations

instantiation *blinfun* :: (*real_normed_vector*, *real_normed_vector*) *real_normed_vector*
begin

lift_definition *norm_blinfun* :: *'a* $\Rightarrow_L$ *'b* $\Rightarrow$ *real* **is** *onorm* **.**

lift_definition *minus_blinfun* :: *'a* $\Rightarrow_L$ *'b* $\Rightarrow$ *'a* $\Rightarrow_L$ *'b* $\Rightarrow$ *'a* $\Rightarrow_L$ *'b*
is $\lambda f\ g\ x.\ f\ x - g\ x$
by (*rule bounded_linear_sub*)

definition *dist_blinfun* :: *'a* $\Rightarrow_L$ *'b* $\Rightarrow$ *'a* $\Rightarrow_L$ *'b* $\Rightarrow$ *real*
where *dist_blinfun a b* = *norm* (*a* − *b*)

definition [*code del*]:
(*uniformity* :: ((*'a* $\Rightarrow_L$ *'b*) × (*'a* $\Rightarrow_L$ *'b*)) *filter*) = (*INF e*∈{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

definition *open_blinfun* :: (*'a* $\Rightarrow_L$ *'b*) *set* $\Rightarrow$ *bool*
where [*code del*]: *open_blinfun S* = (∀ *x*∈*S*. $\forall_F$ (*x'*, *y*) *in uniformity*. *x'* = *x* ⟶ *y* ∈ *S*)

lift_definition *uminus_blinfun* :: *'a* $\Rightarrow_L$ *'b* $\Rightarrow$ *'a* $\Rightarrow_L$ *'b* **is** $\lambda f\ x.\ -\ f\ x$
by (*rule bounded_linear_minus*)

lift_definition *zero_blinfun* :: *'a* $\Rightarrow_L$ *'b* **is** $\lambda x.\ 0$
by (*rule bounded_linear_zero*)

lift_definition *plus_blinfun* :: *'a* $\Rightarrow_L$ *'b* $\Rightarrow$ *'a* $\Rightarrow_L$ *'b* $\Rightarrow$ *'a* $\Rightarrow_L$ *'b*
is $\lambda f\ g\ x.\ f\ x + g\ x$
by (*metis bounded_linear_add*)

lift_definition *scaleR_blinfun*::*real* $\Rightarrow$ *'a* $\Rightarrow_L$ *'b* $\Rightarrow$ *'a* $\Rightarrow_L$ *'b* **is** $\lambda r\ f\ x.\ r *_R f\ x$
by (*metis bounded_linear_compose bounded_linear_scaleR_right*)

definition *sgn_blinfun* :: *'a* $\Rightarrow_L$ *'b* $\Rightarrow$ *'a* $\Rightarrow_L$ *'b*
where *sgn_blinfun x* = *scaleR* (*inverse* (*norm x*)) *x*

instance
apply *standard*
unfolding *dist_blinfun_def open_blinfun_def sgn_blinfun_def uniformity_blinfun_def*
apply (*rule refl* | (*transfer*, *force simp*: *onorm_triangle onorm_scaleR onorm_eq_0 algebra_simps*))+

```
  done

end

declare uniformity_Abort[where 'a=('a :: real_normed_vector) ⇒L ('b :: real_normed_vector),
code]

lemma norm_blinfun_eqI:
  assumes n ≤ norm (blinfun_apply f x) / norm x
  assumes ⋀x. norm (blinfun_apply f x) ≤ n * norm x
  assumes 0 ≤ n
  shows norm f = n
  by (auto simp: norm_blinfun_def
    intro!: antisym onorm_bound assms order_trans[OF _ le_onorm]
    bounded_linear_intros)

lemma norm_blinfun: norm (blinfun_apply f x) ≤ norm f * norm x
  by transfer (rule onorm)

lemma norm_blinfun_bound: 0 ≤ b ⟹ (⋀x. norm (blinfun_apply f x) ≤ b * norm
x) ⟹ norm f ≤ b
  by transfer (rule onorm_bound)

lemma bounded_bilinear_blinfun_apply[bounded_bilinear]: bounded_bilinear blinfun_apply
proof
  fix f g::'a ⇒L 'b and a b::'a and r::real
  show (f + g) a = f a + g a (r *R f) a = r *R f a
    by (transfer, simp)+
  interpret bounded_linear f for f::'a ⇒L 'b
    by (auto intro!: bounded_linear_intros)
  show f (a + b) = f a + f b f (r *R a) = r *R f a
    by (simp_all add: add scaleR)
  show ∃K. ∀a b. norm (blinfun_apply a b) ≤ norm a * norm b * K
    by (auto intro!: exI[where x=1] norm_blinfun)
qed

interpretation blinfun: bounded_bilinear blinfun_apply
  by (rule bounded_bilinear_blinfun_apply)

lemmas bounded_linear_apply_blinfun[intro, simp] = blinfun.bounded_linear_left

declare blinfun.zero_left [simp] blinfun.zero_right [simp]


context bounded_bilinear
begin

named_theorems bilinear_simps
```

```
lemmas [bilinear_simps] =
  add_left
  add_right
  diff_left
  diff_right
  minus_left
  minus_right
  scaleR_left
  scaleR_right
  zero_left
  zero_right
  sum_left
  sum_right

end


instance blinfun :: (real_normed_vector, banach) banach
proof
  fix X::nat ⇒ 'a ⇒L 'b
  assume Cauchy X
  {
    fix x::'a
    {
      fix x::'a
      assume norm x ≤ 1
      have Cauchy (λn. X n x)
      proof (rule CauchyI)
        fix e::real
        assume 0 < e
        from CauchyD[OF ‹Cauchy X› ‹0 < e›] obtain M
          where M: ⋀m n. m ≥ M ⟹ n ≥ M ⟹ norm (X m − X n) < e
          by auto
        show ∃ M. ∀ m≥M. ∀ n≥M. norm (X m x − X n x) < e
        proof (safe intro!: exI[where x=M])
          fix m n
          assume le: M ≤ m M ≤ n
          have norm (X m x − X n x) = norm ((X m − X n) x)
            by (simp add: blinfun.bilinear_simps)
          also have ... ≤ norm (X m − X n) ∗ norm x
             by (rule norm_blinfun)
          also have ... ≤ norm (X m − X n) ∗ 1
            using ‹norm x ≤ 1› norm_ge_zero by (rule mult_left_mono)
          also have ... = norm (X m − X n) by simp
          also have ... < e using le by fact
          finally show norm (X m x − X n x) < e .
        qed
      qed
      hence convergent (λn. X n x)
```

```
        by (metis Cauchy_convergent_iff)
    } note convergent_norm1 = this
    define y where y = x /R norm x
    have y: norm y ≤ 1 and xy: x = norm x *R y
      by (simp_all add: y_def inverse_eq_divide)
    have convergent (λn. norm x *R X n y)
        by (intro bounded_bilinear.convergent[OF bounded_bilinear_scaleR] conver-
gent_const
        convergent_norm1 y)
    also have (λn. norm x *R X n y) = (λn. X n x)
      by (subst xy) (simp add: blinfun.bilinear_simps)
    finally have convergent (λn. X n x) .
  }
  then obtain v where v: ⋀x. (λn. X n x) ⟶ v x
    unfolding convergent_def
    by metis

  have Cauchy (λn. norm (X n))
  proof (rule CauchyI)
    fix e::real
    assume e > 0
    from CauchyD[OF ‹Cauchy X› ‹0 < e›] obtain M
      where M: ⋀m n. m ≥ M ⟹ n ≥ M ⟹ norm (X m − X n) < e
      by auto
    show ∃M. ∀m≥M. ∀n≥M. norm (norm (X m) − norm (X n)) < e
    proof (safe intro!: exI[where x=M])
      fix m n assume mn: m ≥ M n ≥ M
      have norm (norm (X m) − norm (X n)) ≤ norm (X m − X n)
        by (metis norm_triangle_ineq3 real_norm_def)
      also have ... < e using mn by fact
      finally show norm (norm (X m) − norm (X n)) < e .
    qed
  qed
  then obtain K where K: (λn. norm (X n)) ⟶ K
    unfolding Cauchy_convergent_iff convergent_def
    by metis

  have bounded_linear v
  proof
    fix x y and r::real
   from tendsto_add[OF v[of x] v [of y]] v[of x + y, unfolded blinfun.bilinear_simps]
        tendsto_scaleR[OF tendsto_const[of r] v[of x]] v[of r *R x, unfolded blin-
fun.bilinear_simps]
    show v (x + y) = v x + v y v (r *R x) = r *R v x
      by (metis (poly_guards_query) LIMSEQ_unique)+
    show ∃K. ∀x. norm (v x) ≤ norm x * K
    proof (safe intro!: exI[where x=K])
      fix x
      have norm (v x) ≤ K * norm x
```

```
    by (rule tendsto_le[OF _ tendsto_mult[OF K tendsto_const] tendsto_norm[OF v]])
      (auto simp: norm_blinfun)
   thus norm (v x) ≤ norm x * K
    by (simp add: ac_simps)
  qed
 qed
 hence Bv: ⋀x. (λn. X n x) ⟶ Blinfun v x
  by (auto simp: bounded_linear_Blinfun_apply v)

 have X ⟶ Blinfun v
 proof (rule LIMSEQ_I)
  fix r::real assume r > 0
  define r' where r' = r / 2
  have 0 < r' r' < r using ‹r > 0› by (simp_all add: r'_def)
  from CauchyD[OF ‹Cauchy X› ‹r' > 0›]
  obtain M where M: ⋀m n. m ≥ M ⟹ n ≥ M ⟹ norm (X m − X n) < r'
   by metis
  show ∃ no. ∀ n≥no. norm (X n − Blinfun v) < r
  proof (safe intro!: exI[where x=M])
   fix n assume n: M ≤ n
   have norm (X n − Blinfun v) ≤ r'
   proof (rule norm_blinfun_bound)
    fix x
    have eventually (λm. m ≥ M) sequentially
     by (metis eventually_ge_at_top)
     hence ev_le: eventually (λm. norm (X n x − X m x) ≤ r' * norm x) sequentially
    proof eventually_elim
     case (elim m)
     have norm (X n x − X m x) = norm ((X n − X m) x)
      by (simp add: blinfun.bilinear_simps)
     also have ... ≤ norm ((X n − X m)) * norm x
      by (rule norm_blinfun)
     also have ... ≤ r' * norm x
      using M[OF n elim] by (simp add: mult_right_mono)
     finally show ?case .
    qed
     have tendsto_v: (λm. norm (X n x − X m x)) ⟶ norm (X n x − Blinfun v x)
      by (auto intro!: tendsto_intros Bv)
    show norm ((X n − Blinfun v) x) ≤ r' * norm x
     by (auto intro!: tendsto_upperbound tendsto_v ev_le simp: blinfun.bilinear_simps)
   qed (simp add: ‹0 < r'› less_imp_le)
   thus norm (X n − Blinfun v) < r
    by (metis ‹r' < r› le_less_trans)
  qed
 qed
```

thus *convergent X*
  by (*rule convergentI*)
qed

4.9.5 On Euclidean Space

lemma *Zfun_sum*:
  assumes *finite s*
  assumes *f*: ⋀*i*. *i* ∈ *s* ⟹ *Zfun* (*f i*) *F*
  shows *Zfun* (λ*x*. *sum* (λ*i*. *f i x*) *s*) *F*
  using *assms* **by** *induct* (*auto intro*!: *Zfun_zero Zfun_add*)

lemma *norm_blinfun_euclidean_le*:
  fixes *a*::′*a*::*euclidean_space* ⇒$_L$ ′*b*::*real_normed_vector*
  shows *norm a* ≤ *sum* (λ*x*. *norm* (*a x*)) *Basis*
  apply (*rule norm_blinfun_bound*)
   apply (*simp add*: *sum_nonneg*)
  apply (*subst euclidean_representation*[*symmetric*, **where** ′*a*=′*a*])
  apply (*simp only*: *blinfun.bilinear_simps sum_distrib_right*)
  apply (*rule order.trans*[*OF norm_sum sum_mono*])
  apply (*simp add*: *abs_mult mult_right_mono ac_simps Basis_le_norm*)
  done

lemma *tendsto_componentwise1*:
  fixes *a*::′*a*::*euclidean_space* ⇒$_L$ ′*b*::*real_normed_vector*
    and *b*::′*c* ⇒ ′*a* ⇒$_L$ ′*b*
  assumes (⋀*j*. *j* ∈ *Basis* ⟹ ((λ*n*. *b n j*) ⟶ *a j*) *F*)
  shows (*b* ⟶ *a*) *F*
proof −
  have ⋀*j*. *j* ∈ *Basis* ⟹ *Zfun* (λ*x*. *norm* (*b x j* − *a j*)) *F*
    using *assms* **unfolding** *tendsto_Zfun_iff Zfun_norm_iff* **.**
  hence *Zfun* (λ*x*. ∑*j*∈*Basis*. *norm* (*b x j* − *a j*)) *F*
    by (*auto intro*!: *Zfun_sum*)
  thus *?thesis*
    unfolding *tendsto_Zfun_iff*
    by (*rule Zfun_le*)
      (*auto intro*!: *order_trans*[*OF norm_blinfun_euclidean_le*] *simp*: *blinfun.bilinear_simps*)
qed

lift_definition
  blinfun_of_matrix::(′*b*::*euclidean_space* ⇒ ′*a*::*euclidean_space* ⇒ *real*) ⇒ ′*a* ⇒$_L$ ′*b*
  is λ*a x*. ∑*i*∈*Basis*. ∑*j*∈*Basis*. ((*x* · *j*) ∗ *a i j*) ∗$_R$ *i*
  by (*intro bounded_linear_intros*)

lemma *blinfun_of_matrix_works*:
  fixes *f*::′*a*::*euclidean_space* ⇒$_L$ ′*b*::*euclidean_space*
  shows *blinfun_of_matrix* (λ*i j*. (*f j*) · *i*) = *f*
proof (*transfer*, *rule*, *rule euclidean_eqI*)
  fix *f*::′*a* ⇒ ′*b* **and** *x*::′*a* **and** *b*::′*b* **assume** *bounded_linear f* **and** *b*: *b* ∈ *Basis*

then interpret *bounded_linear f* **by** *simp*
have $(\sum j\in Basis.\ \sum i\in Basis.\ (x \cdot i * (f\ i \cdot j)) *_R j) \cdot b$
$= (\sum j\in Basis.\ if\ j = b\ then\ (\sum i\in Basis.\ (x \cdot i * (f\ i \cdot j)))\ else\ 0)$
using *b*
by (*simp add*: *inner_sum_left inner_Basis if_distrib cong*: *if_cong*) (*simp add*: *sum.swap*)
also have ... $= (\sum i\in Basis.\ (x \cdot i * (f\ i \cdot b)))$
using *b* **by** (*simp*)
also have ... $= f\ x \cdot b$
by (*metis* (*mono_tags*, *lifting*) *Linear_Algebra.linear_componentwise linear_axioms*)
finally show $(\sum j\in Basis.\ \sum i\in Basis.\ (x \cdot i * (f\ i \cdot j)) *_R j) \cdot b = f\ x \cdot b$ **.**
qed

lemma *blinfun_of_matrix_apply*:
blinfun_of_matrix a x $= (\sum i\in Basis.\ \sum j\in Basis.\ ((x \cdot j) * a\ i\ j) *_R i)$
by *transfer simp*

lemma *blinfun_of_matrix_minus*: *blinfun_of_matrix x* − *blinfun_of_matrix y* = *blinfun_of_matrix* (*x* − *y*)
by *transfer* (*auto simp*: *algebra_simps sum_subtractf*)

lemma *norm_blinfun_of_matrix*:
norm (*blinfun_of_matrix a*) $\leq (\sum i\in Basis.\ \sum j\in Basis.\ |a\ i\ j|)$
apply (*rule norm_blinfun_bound*)
apply (*simp add*: *sum_nonneg*)
apply (*simp only*: *blinfun_of_matrix_apply sum_distrib_right*)
apply (*rule order_trans*[*OF norm_sum sum_mono*])
apply (*rule order_trans*[*OF norm_sum sum_mono*])
apply (*simp add*: *abs_mult mult_right_mono ac_simps Basis_le_norm*)
done

lemma *tendsto_blinfun_of_matrix*:
assumes $\bigwedge i\ j.\ i \in Basis \Longrightarrow j \in Basis \Longrightarrow ((\lambda n.\ b\ n\ i\ j) \longrightarrow a\ i\ j)\ F$
shows ((λ*n*. *blinfun_of_matrix* (*b n*)) ⟶ *blinfun_of_matrix a*) *F*
proof −
have $\bigwedge i\ j.\ i \in Basis \Longrightarrow j \in Basis \Longrightarrow Zfun\ (\lambda x.\ norm\ (b\ x\ i\ j - a\ i\ j))\ F$
using *assms* **unfolding** *tendsto_Zfun_iff Zfun_norm_iff* **.**
hence $Zfun\ (\lambda x.\ (\sum i\in Basis.\ \sum j\in Basis.\ |b\ x\ i\ j - a\ i\ j|))\ F$
by (*auto intro*!: *Zfun_sum*)
thus *?thesis*
unfolding *tendsto_Zfun_iff blinfun_of_matrix_minus*
by (*rule Zfun_le*) (*auto intro*!: *order_trans*[*OF norm_blinfun_of_matrix*])
qed

lemma *tendsto_componentwise*:
fixes $a::'a::euclidean\_space \Rightarrow_L 'b::euclidean\_space$
and $b::'c \Rightarrow 'a \Rightarrow_L 'b$
shows $(\bigwedge i\ j.\ i \in Basis \Longrightarrow j \in Basis \Longrightarrow ((\lambda n.\ b\ n\ j \cdot i) \longrightarrow a\ j \cdot i)\ F) \Longrightarrow (b \longrightarrow a)\ F$

apply (*subst blinfun_of_matrix_works*[*of a*, *symmetric*])
apply (*subst blinfun_of_matrix_works*[*of b x* **for** *x*, *symmetric*, *abs_def*])
by (*rule tendsto_blinfun_of_matrix*)

lemma
continuous_blinfun_componentwiseI:
fixes *f*:: *'b*::*t2_space* ⇒ *'a*::*euclidean_space* $\Rightarrow_L$ *'c*::*euclidean_space*
assumes ⋀*i j*. *i* ∈ *Basis* ⟹ *j* ∈ *Basis* ⟹ *continuous F* (λ*x*. (*f x*) *j* · *i*)
shows *continuous F f*
using *assms* **by** (*auto simp*: *continuous_def intro*!: *tendsto_componentwise*)

lemma
continuous_blinfun_componentwiseI1:
fixes *f*:: *'b*::*t2_space* ⇒ *'a*::*euclidean_space* $\Rightarrow_L$ *'c*::*real_normed_vector*
assumes ⋀*i*. *i* ∈ *Basis* ⟹ *continuous F* (λ*x*. *f x i*)
shows *continuous F f*
using *assms* **by** (*auto simp*: *continuous_def intro*!: *tendsto_componentwise1*)

lemma
continuous_on_blinfun_componentwise:
fixes *f*:: *'d*::*t2_space* ⇒ *'e*::*euclidean_space* $\Rightarrow_L$ *'f*::*real_normed_vector*
assumes ⋀*i*. *i* ∈ *Basis* ⟹ *continuous_on s* (λ*x*. *f x i*)
shows *continuous_on s f*
using *assms*
by (*auto intro*!: *continuous_at_imp_continuous_on intro*!: *tendsto_componentwise1*
simp: *continuous_on_eq_continuous_within continuous_def*)

lemma *bounded_linear_blinfun_matrix*: *bounded_linear* (λ*x*. (*x*::_$\Rightarrow_L$ _) *j* · *i*)
by (*auto intro*!: *bounded_linearI' bounded_linear_intros*)

lemma *continuous_blinfun_matrix*:
fixes *f*:: *'b*::*t2_space* ⇒ *'a*::*real_normed_vector* $\Rightarrow_L$ *'c*::*real_inner*
assumes *continuous F f*
shows *continuous F* (λ*x*. (*f x*) *j* · *i*)
by (*rule bounded_linear.continuous*[*OF bounded_linear_blinfun_matrix assms*])

lemma *continuous_on_blinfun_matrix*:
fixes *f*::*'a*::*t2_space* ⇒ *'b*::*real_normed_vector* $\Rightarrow_L$ *'c*::*real_inner*
assumes *continuous_on S f*
shows *continuous_on S* (λ*x*. (*f x*) *j* · *i*)
using *assms*
by (*auto simp*: *continuous_on_eq_continuous_within continuous_blinfun_matrix*)

lemma *continuous_on_blinfun_of_matrix*[*continuous_intros*]:
assumes ⋀*i j*. *i* ∈ *Basis* ⟹ *j* ∈ *Basis* ⟹ *continuous_on S* (λ*s*. *g s i j*)
shows *continuous_on S* (λ*s*. *blinfun_of_matrix* (*g s*))
using *assms*
by (*auto simp*: *continuous_on intro*!: *tendsto_blinfun_of_matrix*)

lemma *mult_if_delta*:
(*if P then* (*1*::′*a*::*comm_semiring_1*) *else 0*) ∗ *q* = (*if P then q else 0*)
by *auto*

lemma *compact_blinfun_lemma*:
fixes *f* :: *nat* ⇒ ′*a*::*euclidean_space* $\Rightarrow_L$ ′*b*::*euclidean_space*
assumes *bounded* (*range f*)
shows ∀ *d*⊆*Basis*. ∃ *l*::′*a* $\Rightarrow_L$ ′*b*. ∃ *r*::*nat*⇒*nat*.
strict_mono r ∧ (∀ *e*>*0*. *eventually* (λ*n*. ∀ *i*∈*d*. *dist* (*f* (*r n*) *i*) (*l i*) < *e*) *sequentially*)
by (*rule compact_lemma_general*[**where** *unproj* = λ*e*. *blinfun_of_matrix* (λ*i j*. *e j* · *i*)])
(*auto intro*!: *euclidean_eqI*[**where** ′*a*=′*b*] *bounded_linear_image assms*
simp: *blinfun_of_matrix_works blinfun_of_matrix_apply inner_Basis mult_if_delta sum.delta*′
scaleR_sum_left[*symmetric*])

lemma *blinfun_euclidean_eqI*: (⋀*i*. *i* ∈ *Basis* ⟹ *blinfun_apply x i* = *blinfun_apply y i*) ⟹ *x* = *y*
apply (*auto intro*!: *blinfun_eqI*)
apply (*subst* (*2*) *euclidean_representation*[*symmetric*, **where** ′*a*=′*a*])
apply (*subst* (*1*) *euclidean_representation*[*symmetric*, **where** ′*a*=′*a*])
apply (*simp add*: *blinfun.bilinear_simps*)
done

lemma *Blinfun_eq_matrix*: *bounded_linear f* ⟹ *Blinfun f* = *blinfun_of_matrix* (λ*i j*. *f j* · *i*)
by (*intro blinfun_euclidean_eqI*)
(*auto simp*: *blinfun_of_matrix_apply bounded_linear_Blinfun_apply inner_Basis if_distrib*
if_distribR sum.delta′ *euclidean_representation*
cong: *if_cong*)

TODO: generalize (via *compact_cball*)?

instance *blinfun* :: (*euclidean_space*, *euclidean_space*) *heine_borel*
proof
fix *f* :: *nat* ⇒ ′*a* $\Rightarrow_L$ ′*b*
assume *f*: *bounded* (*range f*)
then obtain *l*::′*a* $\Rightarrow_L$ ′*b* **and** *r* **where** *r*: *strict_mono r*
and *l*: ∀ *e*>*0*. *eventually* (λ*n*. ∀ *i*∈*Basis*. *dist* (*f* (*r n*) *i*) (*l i*) < *e*) *sequentially*
using *compact_blinfun_lemma* [*OF f*] **by** *blast*
{
fix *e*::*real*
let *?d* = *real_of_nat DIM*(′*a*) ∗ *real_of_nat DIM*(′*b*)
assume *e* > *0*
hence *e* / *?d* > *0* **by** (*simp*)
with *l* **have** *eventually* (λ*n*. ∀ *i*∈*Basis*. *dist* (*f* (*r n*) *i*) (*l i*) < *e* / *?d*) *sequentially*
by *simp*

```
    moreover
    {
      fix n
      assume n: ∀ i∈Basis. dist (f (r n) i) (l i) < e / ?d
      have norm (f (r n) − l) = norm (blinfun_of_matrix (λi j. (f (r n) − l) j ·
i))
        unfolding blinfun_of_matrix_works ..
      also note norm_blinfun_of_matrix
      also have (∑ i∈Basis. ∑ j∈Basis. |(f (r n) − l) j · i|) <
        (∑ i∈(Basis::'b set). e / real_of_nat DIM('b))
      proof (rule sum_strict_mono)
        fix i::'b assume i: i ∈ Basis
        have (∑ j::'a∈Basis. |(f (r n) − l) j · i|) < (∑ j::'a∈Basis. e / ?d)
        proof (rule sum_strict_mono)
          fix j::'a assume j: j ∈ Basis
          have |(f (r n) − l) j · i| ≤ norm ((f (r n) − l) j)
            by (simp add: Basis_le_norm i)
          also have ... < e / ?d
            using n i j by (auto simp: dist_norm blinfun.bilinear_simps)
          finally show |(f (r n) − l) j · i| < e / ?d by simp
        qed simp_all
        also have ... ≤ e / real_of_nat DIM('b)
          by simp
        finally show (∑ j∈Basis. |(f (r n) − l) j · i|) < e / real_of_nat DIM('b)
          by simp
      qed simp_all
      also have ... ≤ e by simp
      finally have dist (f (r n)) l < e
        by (auto simp: dist_norm)
    }
    ultimately have eventually (λn. dist (f (r n)) l < e) sequentially
      using eventually_elim2 by force
  }
  then have *: ((f ∘ r) ⟶ l) sequentially
    unfolding o_def tendsto_iff by simp
  with r show ∃ l r. strict_mono r ∧ ((f ∘ r) ⟶ l) sequentially
    by auto
qed
```

4.9.6 concrete bounded linear functions

```
lemma transfer_bounded_bilinear_bounded_linearI:
  assumes g = (λi x. (blinfun_apply (f i) x))
  shows bounded_bilinear g = bounded_linear f
proof
  assume bounded_bilinear g
  then interpret bounded_bilinear f by (simp add: assms)
  show bounded_linear f
  proof (unfold_locales, safe intro!: blinfun_eqI)
```

fix *i*
show $f\ (x + y)\ i = (f\,x + f\,y)\ i\ f\ (r *_R x)\ i = (r *_R f\,x)\ i$ **for** *r x y*
by (*auto intro*!: *blinfun_eqI simp*: *blinfun.bilinear_simps*)
from _ *nonneg_bounded* **show** $\exists K.\ \forall x.\ norm\ (f\,x) \leq norm\ x * K$
by (*rule ex_reg*) (*auto intro*!: *onorm_bound simp*: *norm_blinfun.rep_eq ac_simps*)
qed
qed (*auto simp*: *assms intro*!: *blinfun.comp*)

lemma *transfer_bounded_bilinear_bounded_linear*[*transfer_rule*]:
(*rel_fun* (*rel_fun* (=) (*pcr_blinfun* (=) (=))) (=)) *bounded_bilinear bounded_linear*
by (*auto simp*: *pcr_blinfun_def cr_blinfun_def rel_fun_def OO_def*
intro!: *transfer_bounded_bilinear_bounded_linearI*)

context *bounded_bilinear*
begin

lift_definition *prod_left*::$'b \Rightarrow 'a \Rightarrow_L 'c$ **is** ($\lambda b\ a.\ prod\ a\ b$)
by (*rule bounded_linear_left*)
declare *prod_left.rep_eq*[*simp*]

lemma *bounded_linear_prod_left*[*bounded_linear*]: *bounded_linear prod_left*
by *transfer* (*rule flip*)

lift_definition *prod_right*::$'a \Rightarrow 'b \Rightarrow_L 'c$ **is** ($\lambda a\ b.\ prod\ a\ b$)
by (*rule bounded_linear_right*)
declare *prod_right.rep_eq*[*simp*]

lemma *bounded_linear_prod_right*[*bounded_linear*]: *bounded_linear prod_right*
by *transfer* (*rule bounded_bilinear_axioms*)

end

lift_definition *id_blinfun*::$'a$::*real_normed_vector* $\Rightarrow_L 'a$ **is** $\lambda x.\ x$
by (*rule bounded_linear_ident*)

lemmas *blinfun_apply_id_blinfun*[*simp*] = *id_blinfun.rep_eq*

lemma *norm_blinfun_id*[*simp*]:
norm (*id_blinfun*::$'a$::{*real_normed_vector*, *perfect_space*} $\Rightarrow_L 'a$) = *1*
by *transfer* (*auto simp*: *onorm_id*)

lemma *norm_blinfun_id_le*:
norm (*id_blinfun*::$'a$::*real_normed_vector* $\Rightarrow_L 'a$) $\leq$ *1*
by *transfer* (*auto simp*: *onorm_id_le*)

lift_definition *fst_blinfun*::($'a$::*real_normed_vector* $\times$ $'b$::*real_normed_vector*) $\Rightarrow_L$
$'a$ **is** *fst*
by (*rule bounded_linear_fst*)

lemma *blinfun_apply_fst_blinfun*[*simp*]: *blinfun_apply fst_blinfun* = *fst*
  by *transfer* (*rule refl*)

lift_definition *snd_blinfun*::('*a*::*real_normed_vector* × '*b*::*real_normed_vector*) ⇒$_L$ '*b* **is** *snd*
  by (*rule bounded_linear_snd*)

lemma *blinfun_apply_snd_blinfun*[*simp*]: *blinfun_apply snd_blinfun* = *snd*
  by *transfer* (*rule refl*)

lift_definition *blinfun_compose*::
  '*a*::*real_normed_vector* ⇒$_L$ '*b*::*real_normed_vector* ⇒
    '*c*::*real_normed_vector* ⇒$_L$ '*a* ⇒
    '*c* ⇒$_L$ '*b* (**infixl** *o*$_L$ *55*) **is** (*o*)
  parametric *comp_transfer*
  unfolding *o_def*
  by (*rule bounded_linear_compose*)

lemma *blinfun_apply_blinfun_compose*[*simp*]: (*a o*$_L$ *b*) *c* = *a* (*b c*)
  by (*simp add*: *blinfun_compose.rep_eq*)

lemma *norm_blinfun_compose*:
  norm (*f o*$_L$ *g*) ≤ *norm f* ∗ *norm g*
  by *transfer* (*rule onorm_compose*)

lemma *bounded_bilinear_blinfun_compose*[*bounded_bilinear*]: *bounded_bilinear* (*o*$_L$)
  by *unfold_locales*
   (*auto intro*!: *blinfun_eqI exI*[**where** *x*=*1*] *simp*: *blinfun.bilinear_simps norm_blinfun_compose*)

lemma *blinfun_compose_zero*[*simp*]:
  blinfun_compose 0 = (λ_. *0*)
  blinfun_compose x 0 = *0*
  by (*auto simp*: *blinfun.bilinear_simps intro*!: *blinfun_eqI*)

lemma *blinfun_bij2*:
  fixes *f*::'*a* ⇒$_L$ '*a*::*euclidean_space*
  assumes *f o*$_L$ *g* = *id_blinfun*
  shows *bij* (*blinfun_apply g*)
proof (*rule bijI*)
  show *inj g*
    using *assms*
    by (*metis blinfun_apply_id_blinfun blinfun_compose.rep_eq injI inj_on_imageI2*)
  then show *surj g*
      using *blinfun.bounded_linear_right bounded_linear_def linear_inj_imp_surj* **by**
blast
qed

lemma *blinfun_bij1*:
fixes *f*::'*a* $\Rightarrow_L$ '*a*::*euclidean_space*
assumes *f* o_L *g* = *id_blinfun*
shows *bij* (*blinfun_apply f*)
proof (*rule bijI*)
show *surj* (*blinfun_apply f*)
by (*metis assms blinfun_apply_blinfun_compose blinfun_apply_id_blinfun surjI*)
then show *inj* (*blinfun_apply f*)
using *blinfun.bounded_linear_right bounded_linear_def linear_surj_imp_inj* **by** *blast*
qed

lift_definition *blinfun_inner_right*::'*a*::*real_inner* ⇒ '*a* $\Rightarrow_L$ *real* **is** (·)
by (*rule bounded_linear_inner_right*)
declare *blinfun_inner_right.rep_eq*[*simp*]

lemma *bounded_linear_blinfun_inner_right*[*bounded_linear*]: *bounded_linear blinfun_inner_right*
by *transfer* (*rule bounded_bilinear_inner*)

lift_definition *blinfun_inner_left*::'*a*::*real_inner* ⇒ '*a* $\Rightarrow_L$ *real* **is** λ*x y*. *y* · *x*
by (*rule bounded_linear_inner_left*)
declare *blinfun_inner_left.rep_eq*[*simp*]

lemma *bounded_linear_blinfun_inner_left*[*bounded_linear*]: *bounded_linear blinfun_inner_left*
by *transfer* (*rule bounded_bilinear.flip*[*OF bounded_bilinear_inner*])

lift_definition *blinfun_scaleR_right*::*real* ⇒ '*a* $\Rightarrow_L$ '*a*::*real_normed_vector* **is** ($*_R$)
by (*rule bounded_linear_scaleR_right*)
declare *blinfun_scaleR_right.rep_eq*[*simp*]

lemma *bounded_linear_blinfun_scaleR_right*[*bounded_linear*]: *bounded_linear blinfun_scaleR_right*
by *transfer* (*rule bounded_bilinear_scaleR*)

lift_definition *blinfun_scaleR_left*::'*a*::*real_normed_vector* ⇒ *real* $\Rightarrow_L$ '*a* **is** λ*x y*. *y* $*_R$ *x*
by (*rule bounded_linear_scaleR_left*)
lemmas [*simp*] = *blinfun_scaleR_left.rep_eq*

lemma *bounded_linear_blinfun_scaleR_left*[*bounded_linear*]: *bounded_linear blinfun_scaleR_left*
by *transfer* (*rule bounded_bilinear.flip*[*OF bounded_bilinear_scaleR*])

lift_definition *blinfun_mult_right*::'*a* ⇒ '*a* $\Rightarrow_L$ '*a*::*real_normed_algebra* **is** (*)
by (*rule bounded_linear_mult_right*)
declare *blinfun_mult_right.rep_eq*[*simp*]

lemma *bounded_linear_blinfun_mult_right*[*bounded_linear*]: *bounded_linear blinfun_mult_right*
by *transfer* (*rule bounded_bilinear_mult*)

lift_definition *blinfun_mult_left*::'*a*::*real_normed_algebra* ⇒ '*a* ⇒$_L$ '*a* **is** $\lambda x\ y.\ y * x$
by (*rule bounded_linear_mult_left*)
lemmas [*simp*] = *blinfun_mult_left.rep_eq*

lemma *bounded_linear_blinfun_mult_left*[*bounded_linear*]: *bounded_linear blinfun_mult_left*
by *transfer* (*rule bounded_bilinear.flip*[*OF bounded_bilinear_mult*])

lemmas *bounded_linear_function_uniform_limit_intros*[*uniform_limit_intros*] =
bounded_linear.uniform_limit[*OF bounded_linear_apply_blinfun*]
bounded_linear.uniform_limit[*OF bounded_linear_blinfun_apply*]
bounded_linear.uniform_limit[*OF bounded_linear_blinfun_matrix*]

4.9.7 The strong operator topology on continuous linear operators

Let '*a* and '*b* be two normed real vector spaces. Then the space of linear continuous operators from '*a* to '*b* has a canonical norm, and therefore a canonical corresponding topology (the type classes instantiation are given in `Bounded_Linear_Function.thy`).

However, there is another topology on this space, the strong operator topology, where T_n tends to T iff, for all x in '*a*, then $T_n\ x$ tends to $T\ x$. This is precisely the product topology where the target space is endowed with the norm topology. It is especially useful when '*b* is the set of real numbers, since then this topology is compact.

We can not implement it using type classes as there is already a topology, but at least we can define it as a topology.

Note that there is yet another (common and useful) topology on operator spaces, the weak operator topology, defined analogously using the product topology, but where the target space is given the weak-* topology, i.e., the pullback of the weak topology on the bidual of the space under the canonical embedding of a space into its bidual. We do not define it there, although it could also be defined analogously.

definition *strong_operator_topology*::('*a*::*real_normed_vector* ⇒$_L$'*b*::*real_normed_vector*) *topology*
where *strong_operator_topology* = *pullback_topology UNIV blinfun_apply euclidean*

lemma *strong_operator_topology_topspace*:
topspace strong_operator_topology = *UNIV*
unfolding *strong_operator_topology_def topspace_pullback_topology topspace_euclidean*
by *auto*

lemma *strong_operator_topology_basis*:
 fixes *f*::(*'a*::*real_normed_vector* $\Rightarrow_L$ *'b*::*real_normed_vector*) **and** *U*::*'i* ⇒ *'b set*
and *x*::*'i* ⇒ *'a*
 assumes *finite I* ⋀*i*. *i* ∈ *I* ⟹ *open* (*U i*)
 shows *openin strong_operator_topology* {*f*. ∀ *i*∈*I*. *blinfun_apply f* (*x i*) ∈ *U i*}
proof −
 have *open* {*g*::(*'a*⇒*'b*). ∀ *i*∈*I*. *g* (*x i*) ∈ *U i*}
  by (*rule product_topology_basis'*[*OF assms*])
 moreover have {*f*. ∀ *i*∈*I*. *blinfun_apply f* (*x i*) ∈ *U i*}
     = *blinfun_apply*− ‘{*g*::(*'a*⇒*'b*). ∀ *i*∈*I*. *g* (*x i*) ∈ *U i*} ∩ *UNIV*
  by *auto*
 ultimately show *?thesis*
  unfolding *strong_operator_topology_def* **by** (*subst openin_pullback_topology*) *auto*
qed

lemma *strong_operator_topology_continuous_evaluation*:
 continuous_map strong_operator_topology euclidean (λ*f*. *blinfun_apply f x*)
proof −
 have *continuous_map strong_operator_topology euclidean* ((λ*f*. *f x*) *o blinfun_apply*)
  unfolding *strong_operator_topology_def* **apply** (*rule continuous_map_pullback*)
  using *continuous_on_product_coordinates* **by** *fastforce*
 then show *?thesis* **unfolding** *comp_def* **by** *simp*
qed

lemma *continuous_on_strong_operator_topo_iff_coordinatewise*:
 continuous_map T strong_operator_topology f
  ⟷ (∀ *x*. *continuous_map T euclidean* (λ*y*. *blinfun_apply* (*f y*) *x*))
proof (*auto*)
 fix *x*::*'b*
 assume *continuous_map T strong_operator_topology f*
 with *continuous_map_compose*[*OF this strong_operator_topology_continuous_evaluation*]
 have *continuous_map T euclidean* ((λ*z*. *blinfun_apply z x*) *o f*)
  by *simp*
 then show *continuous_map T euclidean* (λ*y*. *blinfun_apply* (*f y*) *x*)
  unfolding *comp_def* **by** *auto*
next
 assume ∗: ∀ *x*. *continuous_map T euclidean* (λ*y*. *blinfun_apply* (*f y*) *x*)
 have ⋀*i*. *continuous_map T euclidean* (λ*x*. *blinfun_apply* (*f x*) *i*)
  using ∗ **unfolding** *comp_def* **by** *auto*
 then have *continuous_map T euclidean* (*blinfun_apply o f*)
  unfolding *o_def*
  by (*metis* (*no_types*) *continuous_map_componentwise_UNIV euclidean_product_topology*)
 show *continuous_map T strong_operator_topology f*
  unfolding *strong_operator_topology_def*
  apply (*rule continuous_map_pullback'*)
  by (*auto simp add*: ‹*continuous_map T euclidean* (*blinfun_apply o f*)›)
qed

lemma *strong_operator_topology_weaker_than_euclidean*:
continuous_map euclidean strong_operator_topology ($\lambda f.\ f$)
by (*subst continuous_on_strong_operator_topo_iff_coordinatewise*,
auto simp add: *linear_continuous_on*)

end

4.10 Derivative

theory *Derivative*
imports
Bounded_Linear_Function
Line_Segment
Convex_Euclidean_Space
begin

declare *bounded_linear_inner_left* [*intro*]

declare *has_derivative_bounded_linear*[*dest*]

4.10.1 Derivatives

lemma *has_derivative_add_const*:
(f *has_derivative* f') *net* $\Longrightarrow$ (($\lambda x.\ f\ x + c$) *has_derivative* f') *net*
by (*intro derivative_eq_intros*) *auto*

4.10.2 Derivative with composed bilinear function

More explicit epsilon-delta forms.

proposition *has_derivative_within'*:
(f *has_derivative* f')(*at* x *within* s) $\longleftrightarrow$
bounded_linear f' $\wedge$
($\forall e>0.\ \exists d>0.\ \forall x'\in s.\ 0 < norm\ (x' - x) \wedge norm\ (x' - x) < d \longrightarrow$
$norm\ (f\ x' - f\ x - f'(x' - x))\ /\ norm\ (x' - x) < e$)
unfolding *has_derivative_within Lim_within dist_norm*
by (*simp add*: *diff_diff_eq*)

lemma *has_derivative_at'*:
(f *has_derivative* f') (*at* x)
$\longleftrightarrow$ *bounded_linear* f' $\wedge$
($\forall e>0.\ \exists d>0.\ \forall x'.\ 0 < norm\ (x' - x) \wedge norm\ (x' - x) < d \longrightarrow$
$norm\ (f\ x' - f\ x - f'(x' - x))\ /\ norm\ (x' - x) < e$)
using *has_derivative_within'* [*of f f' x UNIV*] **by** *simp*

lemma *has_derivative_componentwise_within*:
(f *has_derivative* f') (*at a within S*) $\longleftrightarrow$
($\forall i \in Basis.$ (($\lambda x.\ f\ x \cdot i$) *has_derivative* ($\lambda x.\ f'\ x \cdot i$)) (*at a within S*))
apply (*simp add*: *has_derivative_within*)

apply (*subst tendsto_componentwise_iff*)
apply (*simp add*: *bounded_linear_componentwise_iff* [*symmetric*] *ball_conj_distrib*)
apply (*simp add*: *algebra_simps*)
done

lemma *has_derivative_at_withinI*:
(*f has_derivative f'*) (*at x*) $\Longrightarrow$ (*f has_derivative f'*) (*at x within s*)
unfolding *has_derivative_within' has_derivative_at'*
by *blast*

lemma *has_derivative_right*:
fixes *f* :: *real* $\Rightarrow$ *real*
and *y* :: *real*
shows (*f has_derivative* ((∗) *y*)) (*at x within* ({*x* <..} ∩ *I*)) $\longleftrightarrow$
((λ*t*. (*f x* − *f t*) / (*x* − *t*)) ⟶ *y*) (*at x within* ({*x* <..} ∩ *I*))
proof −
have ((λ*t*. (*f t* − (*f x* + *y* ∗ (*t* − *x*))) / |*t* − *x*|) ⟶ *0*) (*at x within* ({*x*<..} ∩ *I*)) $\longleftrightarrow$
((λ*t*. (*f t* − *f x*) / (*t* − *x*) − *y*) ⟶ *0*) (*at x within* ({*x*<..} ∩ *I*))
by (*intro Lim_cong_within*) (*auto simp add*: *diff_divide_distrib add_divide_distrib*)
also have ... $\longleftrightarrow$ ((λ*t*. (*f t* − *f x*) / (*t* − *x*)) ⟶ *y*) (*at x within* ({*x*<..} ∩ *I*))
by (*simp add*: *Lim_null*[*symmetric*])
also have ... $\longleftrightarrow$ ((λ*t*. (*f x* − *f t*) / (*x* − *t*)) ⟶ *y*) (*at x within* ({*x*<..} ∩ *I*))
by (*intro Lim_cong_within*) (*simp_all add*: *field_simps*)
finally show *?thesis*
by (*simp add*: *bounded_linear_mult_right has_derivative_within*)
qed

Caratheodory characterization

lemma *DERIV_caratheodory_within*:
(*f has_field_derivative l*) (*at x within S*) $\longleftrightarrow$
(∃ *g*. (∀ *z*. *f z* − *f x* = *g z* ∗ (*z* − *x*)) ∧ *continuous* (*at x within S*) *g* ∧ *g x* = *l*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
show *?rhs*
proof (*intro exI conjI*)
let *?g* = (%*z*. *if z* = *x then l else* (*f z* − *f x*) / (*z*−*x*))
show ∀ *z*. *f z* − *f x* = *?g z* ∗ (*z*−*x*) **by** *simp*
show *continuous* (*at x within S*) *?g* **using** ‹*?lhs*›
by (*auto simp add*: *continuous_within has_field_derivative_iff cong*: *Lim_cong_within*)
show *?g x* = *l* **by** *simp*
qed
next
assume *?rhs*
then obtain *g* **where**

(∀ z. f z − f x = g z ∗ (z−x)) **and** *continuous* (*at x within S*) g **and** g x = l
by *blast*
thus *?lhs*
by (*auto simp add*: *continuous_within has_field_derivative_iff cong*: *Lim_cong_within*)
qed

4.10.3 Differentiability

definition
differentiable_on :: ('a::*real_normed_vector* ⇒ 'b::*real_normed_vector*) ⇒ 'a *set* ⇒ *bool*
(**infix** *differentiable'_on 50*)
where f *differentiable_on* s ⟷ (∀ x∈s. f *differentiable* (*at x within s*))

lemma *differentiableI*: (f *has_derivative* f') *net* ⟹ f *differentiable net*
unfolding *differentiable_def*
by *auto*

lemma *differentiable_onD*: ⟦f *differentiable_on* S; x ∈ S⟧ ⟹ f *differentiable* (*at x within S*)
using *differentiable_on_def* **by** *blast*

lemma *differentiable_at_withinI*: f *differentiable* (*at x*) ⟹ f *differentiable* (*at x within s*)
unfolding *differentiable_def*
using *has_derivative_at_withinI*
by *blast*

lemma *differentiable_at_imp_differentiable_on*:
(⋀x. x ∈ s ⟹ f *differentiable at* x) ⟹ f *differentiable_on* s
by (*metis differentiable_at_withinI differentiable_on_def*)

corollary *differentiable_iff_scaleR*:
fixes f :: *real* ⇒ 'a::*real_normed_vector*
shows f *differentiable* F ⟷ (∃ d. (f *has_derivative* (λx. x $*_R$ d)) F)
by (*auto simp*: *differentiable_def dest*: *has_derivative_linear linear_imp_scaleR*)

lemma *differentiable_on_eq_differentiable_at*:
open s ⟹ f *differentiable_on* s ⟷ (∀ x∈s. f *differentiable at* x)
unfolding *differentiable_on_def*
by (*metis at_within_interior interior_open*)

lemma *differentiable_transform_within*:
assumes f *differentiable* (*at x within s*)
and *0* < d
and x ∈ s
and ⋀x'. ⟦x'∈s; *dist* x' x < d⟧ ⟹ f x' = g x'
shows g *differentiable* (*at x within s*)
using *assms has_derivative_transform_within* **unfolding** *differentiable_def*

by *blast*

lemma *differentiable_on_ident* [*simp*, *derivative_intros*]: ($\lambda x.\ x$) *differentiable_on S*
by (*simp add*: *differentiable_at_imp_differentiable_on*)

lemma *differentiable_on_id* [*simp*, *derivative_intros*]: *id differentiable_on S*
by (*simp add*: *id_def*)

lemma *differentiable_on_const* [*simp*, *derivative_intros*]: ($\lambda z.\ c$) *differentiable_on S*
by (*simp add*: *differentiable_on_def*)

lemma *differentiable_on_mult* [*simp*, *derivative_intros*]:
fixes *f* :: *'M::real_normed_vector* ⇒ *'a::real_normed_algebra*
shows ⟦*f differentiable_on S*; *g differentiable_on S*⟧ ⟹ ($\lambda z.\ f\ z * g\ z$) *differentiable_on S*
unfolding *differentiable_on_def differentiable_def*
using *differentiable_def differentiable_mult* **by** *blast*

lemma *differentiable_on_compose*:
⟦*g differentiable_on S*; *f differentiable_on* (*g* ‘ *S*)⟧ ⟹ ($\lambda x.\ f\ (g\ x)$) *differentiable_on S*
by (*simp add*: *differentiable_in_compose differentiable_on_def*)

lemma *bounded_linear_imp_differentiable_on*: *bounded_linear f* ⟹ *f differentiable_on S*
by (*simp add*: *differentiable_on_def bounded_linear_imp_differentiable*)

lemma *linear_imp_differentiable_on*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::real_normed_vector*
shows *linear f* ⟹ *f differentiable_on S*
by (*simp add*: *differentiable_on_def linear_imp_differentiable*)

lemma *differentiable_on_minus* [*simp*, *derivative_intros*]:
f differentiable_on S ⟹ ($\lambda z.\ -(f\ z)$) *differentiable_on S*
by (*simp add*: *differentiable_on_def*)

lemma *differentiable_on_add* [*simp*, *derivative_intros*]:
⟦*f differentiable_on S*; *g differentiable_on S*⟧ ⟹ ($\lambda z.\ f\ z + g\ z$) *differentiable_on S*
by (*simp add*: *differentiable_on_def*)

lemma *differentiable_on_diff* [*simp*, *derivative_intros*]:
⟦*f differentiable_on S*; *g differentiable_on S*⟧ ⟹ ($\lambda z.\ f\ z - g\ z$) *differentiable_on S*
by (*simp add*: *differentiable_on_def*)

lemma *differentiable_on_inverse* [*simp*, *derivative_intros*]:
fixes *f* :: *'a* :: *real_normed_vector* ⇒ *'b* :: *real_normed_field*
shows *f differentiable_on S* ⟹ ($\bigwedge x.\ x \in S \Longrightarrow f\ x \neq 0$) ⟹ ($\lambda x.$ *inverse* ($f\ x$))

differentiable_on S
by (*simp add*: *differentiable_on_def*)

lemma *differentiable_on_scaleR* [*derivative_intros*, *simp*]:
⟦*f differentiable_on S*; *g differentiable_on S*⟧ ⟹ (λ*x*. *f x* $*_R$ *g x*) *differentiable_on S*
unfolding *differentiable_on_def*
by (*blast intro*: *differentiable_scaleR*)

lemma *has_derivative_sqnorm_at* [*derivative_intros*, *simp*]:
((λ*x*. (*norm x*)^2) *has_derivative* (λ*x*. *2* $*_R$ (*a* · *x*))) (*at a*)
using *bounded_bilinear.FDERIV* [*of* (·) *id id a _ id id*]
by (*auto simp*: *inner_commute dot_square_norm bounded_bilinear_inner*)

lemma *differentiable_sqnorm_at* [*derivative_intros*, *simp*]:
fixes *a* :: *'a* :: {*real_normed_vector*,*real_inner*}
shows (λ*x*. (*norm x*)^2) *differentiable* (*at a*)
by (*force simp add*: *differentiable_def intro*: *has_derivative_sqnorm_at*)

lemma *differentiable_on_sqnorm* [*derivative_intros*, *simp*]:
fixes *S* :: *'a* :: {*real_normed_vector*,*real_inner*} *set*
shows (λ*x*. (*norm x*)^2) *differentiable_on S*
by (*simp add*: *differentiable_at_imp_differentiable_on*)

lemma *differentiable_norm_at* [*derivative_intros*, *simp*]:
fixes *a* :: *'a* :: {*real_normed_vector*,*real_inner*}
shows *a* ≠ *0* ⟹ *norm differentiable* (*at a*)
using *differentiableI has_derivative_norm* **by** *blast*

lemma *differentiable_on_norm* [*derivative_intros*, *simp*]:
fixes *S* :: *'a* :: {*real_normed_vector*,*real_inner*} *set*
shows *0* ∉ *S* ⟹ *norm differentiable_on S*
by (*metis differentiable_at_imp_differentiable_on differentiable_norm_at*)

4.10.4 Frechet derivative and Jacobian matrix

definition *frechet_derivative f net* = (*SOME f'*. (*f has_derivative f'*) *net*)

proposition *frechet_derivative_works*:
f differentiable net ⟷ (*f has_derivative* (*frechet_derivative f net*)) *net*
unfolding *frechet_derivative_def differentiable_def*
unfolding *some_eq_ex*[*of* λ *f'* . (*f has_derivative f'*) *net*] **..**

lemma *linear_frechet_derivative*: *f differentiable net* ⟹ *linear* (*frechet_derivative f net*)
unfolding *frechet_derivative_works has_derivative_def*
by (*auto intro*: *bounded_linear.linear*)

lemma *frechet_derivative_const* [*simp*]: *frechet_derivative* (λ*x*. *c*) (*at a*) = (λ*x*. *0*)

using *differentiable_const frechet_derivative_works has_derivative_const has_derivative_unique*
by *blast*

lemma *frechet_derivative_id* [*simp*]: *frechet_derivative id* (*at a*) = *id*
using *differentiable_def frechet_derivative_works has_derivative_id has_derivative_unique*
by *blast*

lemma *frechet_derivative_ident* [*simp*]: *frechet_derivative* ($\lambda x.\ x$) (*at a*) = ($\lambda x.\ x$)
by (*metis eq_id_iff frechet_derivative_id*)

4.10.5 Differentiability implies continuity

proposition *differentiable_imp_continuous_within*:
f differentiable (*at x within s*) $\Longrightarrow$ *continuous* (*at x within s*) *f*
by (*auto simp*: *differentiable_def intro*: *has_derivative_continuous*)

lemma *differentiable_imp_continuous_on*:
f differentiable_on s $\Longrightarrow$ *continuous_on s f*
unfolding *differentiable_on_def continuous_on_eq_continuous_within*
using *differentiable_imp_continuous_within* **by** *blast*

lemma *differentiable_on_subset*:
f differentiable_on t $\Longrightarrow$ $s \subseteq t$ $\Longrightarrow$ *f differentiable_on s*
unfolding *differentiable_on_def*
using *differentiable_within_subset*
by *blast*

lemma *differentiable_on_empty*: *f differentiable_on* {}
unfolding *differentiable_on_def*
by *auto*

lemma *has_derivative_continuous_on*:
($\bigwedge x.\ x \in s \Longrightarrow$ (*f has_derivative f′ x*) (*at x within s*)) $\Longrightarrow$ *continuous_on s f*
by (*auto intro*!: *differentiable_imp_continuous_on differentiableI simp*: *differentiable_on_def*)

Results about neighborhoods filter.

lemma *eventually_nhds_metric_le*:
eventually P (*nhds a*) = ($\exists\, d>0.\ \forall x.\ dist\ x\ a \le d \longrightarrow P\ x$)
unfolding *eventually_nhds_metric* **by** (*safe*, *rule_tac x=d / 2* **in** *exI*, *auto*)

lemma *le_nhds*: $F \le nhds\ a \longleftrightarrow (\forall S.\ open\ S \wedge a \in S \longrightarrow eventually\ (\lambda x.\ x \in S)\ F)$
unfolding *le_filter_def eventually_nhds* **by** (*fast elim*: *eventually_mono*)

lemma *le_nhds_metric*: $F \le nhds\ a \longleftrightarrow (\forall\, e>0.\ eventually\ (\lambda x.\ dist\ x\ a < e)\ F)$
unfolding *le_filter_def eventually_nhds_metric* **by** (*fast elim*: *eventually_mono*)

lemma *le_nhds_metric_le*: $F \le nhds\ a \longleftrightarrow (\forall\, e>0.\ eventually\ (\lambda x.\ dist\ x\ a \le e)$

F)
unfolding *le_filter_def eventually_nhds_metric_le* **by** (*fast elim*: *eventually_mono*)

Several results are easier using a "multiplied-out" variant. (I got this idea from Dieudonne's proof of the chain rule).

lemma *has_derivative_within_alt*:
(*f has_derivative f′*) (*at x within s*) ⟷ *bounded_linear f′* ∧
(∀ *e*>*0*. ∃ *d*>*0*. ∀ *y*∈*s*. *norm*(*y* − *x*) < *d* ⟶ *norm* (*f y* − *f x* − *f′* (*y* − *x*)) ≤ *e* ∗ *norm* (*y* − *x*))
unfolding *has_derivative_within filterlim_def le_nhds_metric_le eventually_filtermap eventually_at dist_norm diff_diff_eq*
by (*force simp add*: *linear_0 bounded_linear.linear pos_divide_le_eq*)

lemma *has_derivative_within_alt2*:
(*f has_derivative f′*) (*at x within s*) ⟷ *bounded_linear f′* ∧
(∀ *e*>*0*. *eventually* (λ*y*. *norm* (*f y* − *f x* − *f′* (*y* − *x*)) ≤ *e* ∗ *norm* (*y* − *x*)) (*at x within s*))
unfolding *has_derivative_within filterlim_def le_nhds_metric_le eventually_filtermap eventually_at dist_norm diff_diff_eq*
by (*force simp add*: *linear_0 bounded_linear.linear pos_divide_le_eq*)

lemma *has_derivative_at_alt*:
(*f has_derivative f′*) (*at x*) ⟷
bounded_linear f′ ∧
(∀ *e*>*0*. ∃ *d*>*0*. ∀ *y*. *norm*(*y* − *x*) < *d* ⟶ *norm* (*f y* − *f x* − *f′*(*y* − *x*)) ≤ *e* ∗ *norm* (*y* − *x*))
using *has_derivative_within_alt*[**where** *s*=*UNIV*]
by *simp*

4.10.6 The chain rule

proposition *diff_chain_within*[*derivative_intros*]:
assumes (*f has_derivative f′*) (*at x within s*)
and (*g has_derivative g′*) (*at* (*f x*) *within* (*f* ‘ *s*))
shows ((*g* ∘ *f*) *has_derivative* (*g′* ∘ *f′*))(*at x within s*)
using *has_derivative_in_compose*[*OF assms*]
by (*simp add*: *comp_def*)

lemma *diff_chain_at*[*derivative_intros*]:
(*f has_derivative f′*) (*at x*) ⟹
(*g has_derivative g′*) (*at* (*f x*)) ⟹ ((*g* ∘ *f*) *has_derivative* (*g′* ∘ *f′*)) (*at x*)
using *has_derivative_compose*[*of f f′ x UNIV g g′*]
by (*simp add*: *comp_def*)

lemma *has_vector_derivative_within_open*:
a ∈ *S* ⟹ *open S* ⟹
(*f has_vector_derivative f′*) (*at a within S*) ⟷ (*f has_vector_derivative f′*) (*at a*)
by (*simp only*: *at_within_interior interior_open*)

lemma *field_vector_diff_chain_within*:
assumes *Df*: (*f has_vector_derivative f'*) (*at x within S*)
and *Dg*: (*g has_field_derivative g'*) (*at* (*f x*) *within f ` S*)
shows ((*g* ∘ *f*) *has_vector_derivative* (*f' * g'*)) (*at x within S*)
using *diff_chain_within*[*OF Df*[*unfolded has_vector_derivative_def*]
Dg [*unfolded has_field_derivative_def*]]
by (*auto simp*: *o_def mult.commute has_vector_derivative_def*)

lemma *vector_derivative_diff_chain_within*:
assumes *Df*: (*f has_vector_derivative f'*) (*at x within S*)
and *Dg*: (*g has_derivative g'*) (*at* (*f x*) *within f`S*)
shows ((*g* ∘ *f*) *has_vector_derivative* (*g' f'*)) (*at x within S*)
using *diff_chain_within*[*OF Df*[*unfolded has_vector_derivative_def*] *Dg*]
linear.scaleR[*OF has_derivative_linear*[*OF Dg*]]
unfolding *has_vector_derivative_def o_def*
by (*auto simp*: *o_def mult.commute has_vector_derivative_def*)

4.10.7 Composition rules stated just for differentiability

lemma *differentiable_chain_at*:
f differentiable (*at x*) ⟹
g differentiable (*at* (*f x*)) ⟹ (*g* ∘ *f*) *differentiable* (*at x*)
unfolding *differentiable_def*
by (*meson diff_chain_at*)

lemma *differentiable_chain_within*:
f differentiable (*at x within S*) ⟹
g differentiable (*at*(*f x*) *within* (*f ` S*)) ⟹ (*g* ∘ *f*) *differentiable* (*at x within S*)
unfolding *differentiable_def*
by (*meson diff_chain_within*)

4.10.8 Uniqueness of derivative

The general result is a bit messy because we need approachability of the limit point from any direction. But OK for nontrivial intervals etc.

proposition *frechet_derivative_unique_within*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::real_normed_vector*
assumes *1*: (*f has_derivative f'*) (*at x within S*)
and *2*: (*f has_derivative f''*) (*at x within S*)
and *S*: ⋀*i e*. ⟦*i*∈*Basis*; *e*>*0*⟧ ⟹ ∃ *d*. *0* < |*d*| ∧ |*d*| < *e* ∧ (*x* + *d* $*_R$ *i*) ∈ *S*
shows *f'* = *f''*
proof −
note *as* = *assms*(*1*,*2*)[*unfolded has_derivative_def*]
then interpret *f'*: *bounded_linear f'* **by** *auto*
from *as* **interpret** *f''*: *bounded_linear f''* **by** *auto*
have *x islimpt S* **unfolding** *islimpt_approachable*
proof (*intro allI impI*)
fix *e* :: *real*

assume $e > 0$
obtain d **where** $0 < |d|$ **and** $|d| < e$ **and** $x + d *_R (SOME\ i.\ i \in Basis) \in S$
using *assms(3) SOME_Basis* ‹$e>0$› **by** *blast*
then show $\exists x'\in S.\ x' \neq x \wedge dist\ x'\ x < e$
by (*rule_tac* $x=x + d *_R (SOME\ i.\ i \in Basis)$ **in** *bexI*) (*auto simp*: *dist_norm SOME_Basis nonzero_Basis*) **qed**
then have ∗: *netlimit* (*at x within S*) = x
by (*simp add*: *Lim_ident_at trivial_limit_within*)
show *?thesis*
proof (*rule linear_eq_stdbasis*)
show *linear* f' *linear* f''
unfolding *linear_conv_bounded_linear* **using** *as* **by** *auto*
next
fix i :: $'a$
assume i: $i \in Basis$
define e **where** $e = norm\ (f'\ i - f''\ i)$
show $f'\ i = f''\ i$
proof (*rule ccontr*)
assume $f'\ i \neq f''\ i$
then have $e > 0$
unfolding *e_def* **by** *auto*
obtain d **where** d:
$0 < d$
$(\bigwedge y.\ y\in S \longrightarrow 0 < dist\ y\ x \wedge dist\ y\ x < d \longrightarrow$
$dist\ ((f\ y - f\ x - f'\ (y - x))\ /_R\ norm\ (y - x) -$
$(f\ y - f\ x - f''\ (y - x))\ /_R\ norm\ (y - x))\ (0 - 0) < e)$
using *tendsto_diff* [*OF as*(*1*,*2*)[*THEN conjunct2*]]
unfolding ∗ *Lim_within*
using ‹$e>0$› **by** *blast*
obtain c **where** c: $0 < |c|\ |c| < d \wedge x + c *_R i \in S$
using *assms(3)* i $d(1)$ **by** *blast*
have ∗: $norm\ (- ((1\ /\ |c|) *_R f'\ (c *_R i)) + (1\ /\ |c|) *_R f''\ (c *_R i)) =$
$norm\ ((1\ /\ |c|) *_R (- (f'\ (c *_R i)) + f''\ (c *_R i)))$
unfolding *scaleR_right_distrib* **by** *auto*
also have $\ldots = norm\ ((1\ /\ |c|) *_R (c *_R (- (f'\ i) + f''\ i)))$
unfolding f'.*scaleR* f''.*scaleR*
unfolding *scaleR_right_distrib scaleR_minus_right*
by *auto*
also have $\ldots = e$
unfolding *e_def*
using $c(1)$
using *norm_minus_cancel*[*of* $f'\ i - f''\ i$]
by *auto*
finally show *False*
using c
using $d(2)$[*of* $x + c *_R i$]
unfolding *dist_norm*
unfolding f'.*scaleR* f''.*scaleR* f'.*add* f''.*add* f'.*diff* f''.*diff*
scaleR_scaleR scaleR_right_diff_distrib scaleR_right_distrib

```
    using i
    by (auto simp: inverse_eq_divide)
  qed
 qed
qed

proposition frechet_derivative_unique_within_closed_interval:
 fixes f::'a::euclidean_space ⇒ 'b::real_normed_vector
 assumes ab: ⋀i. i∈Basis ⟹ a•i < b•i
  and x: x ∈ cbox a b
  and (f has_derivative f' ) (at x within cbox a b)
  and (f has_derivative f'') (at x within cbox a b)
 shows f' = f''
proof (rule frechet_derivative_unique_within)
 fix e :: real
 fix i :: 'a
 assume e > 0 and i: i ∈ Basis
 then show ∃ d. 0 < |d| ∧ |d| < e ∧ x + d *R i ∈ cbox a b
 proof (cases x•i = a•i)
  case True
  with ab[of i] ‹e>0› x i show ?thesis
   by (rule_tac x=(min (b•i − a•i) e) / 2 in exI)
     (auto simp add: mem_box field_simps inner_simps inner_Basis)
 next
  case False
  moreover have a • i < x • i
   using False i mem_box(2) x by force
  moreover {
   have a • i * 2 + min (x • i − a • i) e ≤ a•i *2 + x•i − a•i
    by auto
   also have ... = a•i + x•i
    by auto
   also have ... ≤ 2 * (x•i)
    using ‹a • i < x • i› by auto
   finally have a • i * 2 + min (x • i − a • i) e ≤ x • i * 2
    by auto
  }
  moreover have min (x • i − a • i) e ≥ 0
   by (simp add: ‹0 < e› ‹a • i < x • i› less_eq_real_def)
  then have x • i * 2 ≤ b • i * 2 + min (x • i − a • i) e
   using i mem_box(2) x by force
  ultimately show ?thesis
  using ab[of i] ‹e>0› x i
   by (rule_tac x=− (min (x•i − a•i) e) / 2 in exI)
     (auto simp add: mem_box field_simps inner_simps inner_Basis)
 qed
qed (use assms in auto)

lemma frechet_derivative_unique_within_open_interval:
```

fixes *f*::'*a*::*euclidean_space* ⇒ '*b*::*real_normed_vector*
assumes *x*: $x \in$ *box a b*
and *f*: (*f has_derivative f'*) (*at x within box a b*) (*f has_derivative f''*) (*at x within box a b*)
shows $f' = f''$
proof −
have *at x within box a b* = *at x*
by (*metis x at_within_interior interior_open open_box*)
with *f* **show** $f' = f''$
by (*simp add*: *has_derivative_unique*)
qed

lemma *frechet_derivative_at*:
(*f has_derivative f'*) (*at x*) ⟹ *f'* = *frechet_derivative f* (*at x*)
using *differentiable_def frechet_derivative_works has_derivative_unique* **by** *blast*

lemma *frechet_derivative_compose*:
frechet_derivative (*f o g*) (*at x*) = *frechet_derivative* (*f*) (*at* (*g x*)) *o frechet_derivative g* (*at x*)
if *g differentiable at x f differentiable at* (*g x*)
by (*metis diff_chain_at frechet_derivative_at frechet_derivative_works that*)

lemma *frechet_derivative_within_cbox*:
fixes *f* :: '*a*::*euclidean_space* ⇒ '*b*::*real_normed_vector*
assumes ⋀*i*. *i*∈*Basis* ⟹ *a*•*i* < *b*•*i*
and $x \in$ *cbox a b*
and (*f has_derivative f'*) (*at x within cbox a b*)
shows *frechet_derivative f* (*at x within cbox a b*) = *f'*
using *assms*
by (*metis Derivative.differentiableI frechet_derivative_unique_within_closed_interval frechet_derivative_works*)

lemma *frechet_derivative_transform_within_open*:
frechet_derivative f (*at x*) = *frechet_derivative g* (*at x*)
if *f differentiable at x open X x* ∈ *X* ⋀*x*. *x* ∈ *X* ⟹ *f x* = *g x*
by (*meson frechet_derivative_at frechet_derivative_works has_derivative_transform_within_open that*)

4.10.9 Derivatives of local minima and maxima are zero

lemma *has_derivative_local_min*:
fixes *f* :: '*a*::*real_normed_vector* ⇒ *real*
assumes *deriv*: (*f has_derivative f'*) (*at x*)
assumes *min*: *eventually* (λ*y*. *f x* ≤ *f y*) (*at x*)
shows *f'* = (λ*h*. *0*)
proof
fix *h* :: '*a*
interpret *f'*: *bounded_linear f'*
using *deriv* **by** (*rule has_derivative_bounded_linear*)

show $f'\ h = 0$
proof (*cases* $h = 0$)
case *False*
from *min* **obtain** d **where** *d1*: $0 < d$ **and** *d2*: $\forall y \in ball\ x\ d.\ f\ x \le f\ y$
unfolding *eventually_at* **by** (*force simp*: *dist_commute*)
have $FDERIV\ (\lambda r.\ x + r *_R h)\ 0 :> (\lambda r.\ r *_R h)$
by (*intro derivative_eq_intros*) *auto*
then have $FDERIV\ (\lambda r.\ f\ (x + r *_R h))\ 0 :> (\lambda k.\ f'\ (k *_R h))$
by (*rule has_derivative_compose*, *simp add*: *deriv*)
then have $DERIV\ (\lambda r.\ f\ (x + r *_R h))\ 0 :> f'\ h$
unfolding *has_field_derivative_def* **by** (*simp add*: *f'.scaleR mult_commute_abs*)
moreover have $0 < d\ /\ norm\ h$ **using** *d1* **and** ‹$h \neq 0$› **by** *simp*
moreover have $\forall y.\ |0 - y| < d\ /\ norm\ h \longrightarrow f\ (x + 0 *_R h) \le f\ (x + y *_R h)$
using ‹$h \neq 0$› **by** (*auto simp add*: *d2 dist_norm pos_less_divide_eq*)
ultimately show $f'\ h = 0$
by (*rule DERIV_local_min*)
qed *simp*
qed

lemma *has_derivative_local_max*:
fixes f :: $'a$::*real_normed_vector* $\Rightarrow$ *real*
assumes (f *has_derivative* f') (*at* x)
assumes *eventually* $(\lambda y.\ f\ y \le f\ x)$ (*at* x)
shows $f' = (\lambda h.\ 0)$
using *has_derivative_local_min* [*of* $\lambda x.\ -\ f\ x\ \lambda h.\ -\ f'\ h\ x$]
using *assms* **unfolding** *fun_eq_iff* **by** *simp*

lemma *differential_zero_maxmin*:
fixes f::$'a$::*real_normed_vector* $\Rightarrow$ *real*
assumes $x \in S$
and *open* S
and *deriv*: (f *has_derivative* f') (*at* x)
and *mono*: $(\forall y \in S.\ f\ y \le f\ x) \vee (\forall y \in S.\ f\ x \le f\ y)$
shows $f' = (\lambda v.\ 0)$
using *mono*
proof
assume $\forall y \in S.\ f\ y \le f\ x$
with ‹$x \in S$› **and** ‹*open* S› **have** *eventually* $(\lambda y.\ f\ y \le f\ x)$ (*at* x)
unfolding *eventually_at_topological* **by** *auto*
with *deriv* **show** *?thesis*
by (*rule has_derivative_local_max*)
next
assume $\forall y \in S.\ f\ x \le f\ y$
with ‹$x \in S$› **and** ‹*open* S› **have** *eventually* $(\lambda y.\ f\ x \le f\ y)$ (*at* x)
unfolding *eventually_at_topological* **by** *auto*
with *deriv* **show** *?thesis*
by (*rule has_derivative_local_min*)
qed

lemma *differential_zero_maxmin_component*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *k*: *k* ∈ *Basis*
and *ball*: *0* < *e* (∀ *y* ∈ *ball x e*. (*f y*)•*k* ≤ (*f x*)•*k*) ∨ (∀ *y*∈*ball x e*. (*f x*)•*k* ≤ (*f y*)•*k*)
and *diff*: *f differentiable* (*at x*)
shows (∑ *j*∈*Basis*. (*frechet_derivative f* (*at x*) *j* • *k*) $*_R$ *j*) = (*0*::*'a*) (**is** *?D k* = *0*)
proof −
let *?f'* = *frechet_derivative f* (*at x*)
have *x* ∈ *ball x e* **using** ‹*0* < *e*› **by** *simp*
moreover have *open* (*ball x e*) **by** *simp*
moreover have ((λ*x*. *f x* • *k*) *has_derivative* (λ*h*. *?f' h* • *k*)) (*at x*)
using *bounded_linear_inner_left diff*[*unfolded frechet_derivative_works*]
by (*rule bounded_linear.has_derivative*)
ultimately have (λ*h*. *frechet_derivative f* (*at x*) *h* • *k*) = (λ*v*. *0*)
using *ball*(*2*) **by** (*rule differential_zero_maxmin*)
then show *?thesis*
unfolding *fun_eq_iff* **by** *simp*
qed

4.10.10 One-dimensional mean value theorem

lemma *mvt_simple*:
fixes *f* :: *real* ⇒ *real*
assumes *a* < *b*
and *derf*: ⋀*x*. ⟦*a* ≤ *x*; *x* ≤ *b*⟧ ⟹ (*f has_derivative f' x*) (*at x within* {*a*..*b*})
shows ∃ *x*∈{*a*<..<*b*}. *f b* − *f a* = *f' x* (*b* − *a*)
proof (*rule mvt*)
have *f differentiable_on* {*a*..*b*}
using *derf* **unfolding** *differentiable_on_def differentiable_def* **by** *force*
then show *continuous_on* {*a*..*b*} *f*
by (*rule differentiable_imp_continuous_on*)
show (*f has_derivative f' x*) (*at x*) **if** *a* < *x x* < *b* **for** *x*
by (*metis at_within_Icc_at derf leI order.asym that*)
qed (*use assms* **in** *auto*)

lemma *mvt_very_simple*:
fixes *f* :: *real* ⇒ *real*
assumes *a* ≤ *b*
and *derf*: ⋀*x*. ⟦*a* ≤ *x*; *x* ≤ *b*⟧ ⟹ (*f has_derivative f' x*) (*at x within* {*a*..*b*})
shows ∃ *x*∈{*a*..*b*}. *f b* − *f a* = *f' x* (*b* − *a*)
proof (*cases a* = *b*)
interpret *bounded_linear f' b*
using *assms*(*2*) *assms*(*1*) **by** *auto*
case *True*
then show *?thesis*
by *force*

next
  case *False*
  then show *?thesis*
    using *mvt_simple*[*OF* _ *derf*]
   by (*metis* ‹$a \leq b$› *atLeastAtMost_iff dual_order.order_iff_strict greaterThanLessThan_iff*)
qed

A nice generalization (see Havin's proof of 5.19 from Rudin's book).

lemma *mvt_general*:
  fixes f :: *real* $\Rightarrow$ *'a::real_inner*
  assumes $a < b$
    and *contf*: *continuous_on* $\{a..b\}$ f
    and *derf*: $\bigwedge x.$ ⟦$a < x$; $x < b$⟧ $\Longrightarrow$ (f *has_derivative* f' x) (*at* x)
  shows $\exists x \in \{a<..<b\}$. *norm* (f b $-$ f a) $\leq$ *norm* (f' x (b $-$ a))
proof $-$
  have $\exists x \in \{a<..<b\}$. (f b $-$ f a) $\cdot$ f b $-$ (f b $-$ f a) $\cdot$ f a $=$ (f b $-$ f a) $\cdot$ f' x (b
$-$ a)
    apply (*rule mvt* [*OF* ‹$a < b$›, **where** $f = \lambda x.$ (f b $-$ f a) $\cdot$ f x])
    apply (*intro continuous_intros contf*)
    using *derf* **apply** (*auto intro*: *has_derivative_inner_right*)
    done
  then obtain x **where** x: $x \in \{a<..<b\}$
    (f b $-$ f a) $\cdot$ f b $-$ (f b $-$ f a) $\cdot$ f a $=$ (f b $-$ f a) $\cdot$ f' x (b $-$ a) **..**
  show *?thesis*
  proof (*cases* f a $=$ f b)
    case *False*
    have *norm* (f b $-$ f a) $*$ *norm* (f b $-$ f a) $=$ (*norm* (f b $-$ f a))^2
      by (*simp add*: *power2_eq_square*)
    also have ... $=$ (f b $-$ f a) $\cdot$ (f b $-$ f a)
      unfolding *power2_norm_eq_inner* **..**
    also have ... $=$ (f b $-$ f a) $\cdot$ f' x (b $-$ a)
      using $x(2)$ **by** (*simp only*: *inner_diff_right*)
    also have ... $\leq$ *norm* (f b $-$ f a) $*$ *norm* (f' x (b $-$ a))
      by (*rule norm_cauchy_schwarz*)
    finally show *?thesis*
      using *False* $x(1)$
      by (*auto simp add*: *mult_left_cancel*)
  next
    case *True*
    then show *?thesis*
      using ‹$a < b$› **by** (*rule_tac* x=(a $+$ b) /*2* **in** *bexI*) *auto*
  qed
qed

4.10.11 More general bound theorems

proposition *differentiable_bound_general*:
  fixes f :: *real* $\Rightarrow$ *'a::real_normed_vector*
  assumes $a < b$

```
    and f_cont: continuous_on {a..b} f
    and phi_cont: continuous_on {a..b} φ
    and f': ⋀x. a < x ⟹ x < b ⟹ (f has_vector_derivative f' x) (at x)
    and phi': ⋀x. a < x ⟹ x < b ⟹ (φ has_vector_derivative φ' x) (at x)
    and bnd: ⋀x. a < x ⟹ x < b ⟹ norm (f' x) ≤ φ' x
  shows norm (f b − f a) ≤ φ b − φ a
proof −
  {
    fix x assume x: a < x x < b
    have 0 ≤ norm (f' x) by simp
    also have ... ≤ φ' x using x by (auto intro!: bnd)
    finally have 0 ≤ φ' x .
  } note phi'_nonneg = this
  note f_tendsto = assms(2)[simplified continuous_on_def, rule_format]
  note phi_tendsto = assms(3)[simplified continuous_on_def, rule_format]
  {
    fix e::real assume e > 0
    define e2 where e2 = e / 2
    with ‹e > 0› have e2 > 0 by simp
    let ?le = λx1. norm (f x1 − f a) ≤ φ x1 − φ a + e ∗ (x1 − a) + e
    define A where A = {x2. a ≤ x2 ∧ x2 ≤ b ∧ (∀x1∈{a ..< x2}. ?le x1)}
    have A_subset: A ⊆ {a..b} by (auto simp: A_def)
    {
      fix x2
      assume a: a ≤ x2 x2 ≤ b and le: ∀x1∈{a..<x2}. ?le x1
      have ?le x2 using ‹e > 0›
      proof cases
        assume x2 ≠ a with a have a < x2 by simp
        have at x2 within {a <..<x2}≠ bot
          using ‹a < x2›
          by (auto simp: trivial_limit_within islimpt_in_closure)
        moreover
        have ((λx1. (φ x1 − φ a) + e ∗ (x1 − a) + e) ⟶ (φ x2 − φ a) + e ∗
(x2 − a) + e) (at x2 within {a <..<x2})
          ((λx1. norm (f x1 − f a)) ⟶ norm (f x2 − f a)) (at x2 within {a
<..<x2})
          using a
          by (auto intro!: tendsto_eq_intros f_tendsto phi_tendsto
            intro: tendsto_within_subset[where S={a..b}])
        moreover
        have eventually (λx. x > a) (at x2 within {a <..<x2})
          by (auto simp: eventually_at_filter)
        hence eventually ?le (at x2 within {a <..<x2})
          unfolding eventually_at_filter
          by eventually_elim (insert le, auto)
        ultimately
        show ?thesis
          by (rule tendsto_le)
      qed simp
```

} **note** le_cont = this
have a ∈ A
  using assms **by** (auto simp: A_def)
hence [simp]: A ≠ {} **by** auto
have A_ivl: ⋀x1 x2. x2 ∈ A ⟹ x1 ∈ {a ..x2} ⟹ x1 ∈ A
  by (simp add: A_def)
have [simp]: bdd_above A **by** (auto simp: A_def)
define y **where** y = Sup A
have y ≤ b
  unfolding y_def
  by (simp add: cSup_le_iff) (simp add: A_def)
 have leI: ⋀x x1. a ≤ x1 ⟹ x ∈ A ⟹ x1 < x ⟹ ?le x1
   by (auto simp: A_def intro!: le_cont)
have y_all_le: ∀ x1∈{a..<y}. ?le x1
  by (auto simp: y_def less_cSup_iff leI)
have a ≤ y
  by (metis ‹a ∈ A› ‹bdd_above A› cSup_upper y_def)
have y ∈ A
  using y_all_le ‹a ≤ y› ‹y ≤ b›
  by (auto simp: A_def)
hence A = {a .. y}
  using A_subset **by** (auto simp: subset_iff y_def cSup_upper intro: A_ivl)
from le_cont[OF ‹a ≤ y› ‹y ≤ b› y_all_le] **have** le_y: ?le y **.**
have y = b
proof (cases a = y)
  case True
  with ‹a < b› **have** y < b **by** simp
  with ‹a = y› f_cont phi_cont ‹e2 > 0›
  have 1: $\forall_F$ x in at y within {y..b}. dist (f x) (f y) < e2
   and 2: $\forall_F$ x in at y within {y..b}. dist (φ x) (φ y) < e2
    by (auto simp: continuous_on_def tendsto_iff)
  have 3: eventually (λx. y < x) (at y within {y..b})
    by (auto simp: eventually_at_filter)
  have 4: eventually (λx::real. x < b) (at y within {y..b})
    using _ ‹y < b›
    by (rule order_tendstoD) (auto intro!: tendsto_eq_intros)
  from 1 2 3 4
  have eventually_le: eventually (λx. ?le x) (at y within {y .. b})
  proof eventually_elim
    case (elim x1)
    have norm (f x1 − f a) = norm (f x1 − f y)
      by (simp add: ‹a = y›)
    also have norm (f x1 − f y) ≤ e2
      using elim ‹a = y› **by** (auto simp : dist_norm intro!: less_imp_le)
    also have ... ≤ e2 + (φ x1 − φ a + e2 + e ∗ (x1 − a))
      using ‹0 < e› elim
      by (intro add_increasing2[OF add_nonneg_nonneg order.refl])
        (auto simp: ‹a = y› dist_norm intro!: mult_nonneg_nonneg)
    also have ... = φ x1 − φ a + e ∗ (x1 − a) + e

```
      by (simp add: e2_def)
    finally show ?le x1 .
  qed
  from this[unfolded eventually_at_topological] ‹?le y›
  obtain S where S: open S y ∈ S ⋀x. x∈S ⟹ x ∈ {y..b} ⟹ ?le x
    by metis
  from ‹open S› obtain d where d: ⋀x. dist x y < d ⟹ x ∈ S d > 0
    by (force simp: dist_commute open_dist ball_def dest!: bspec[OF _ ‹y ∈ S›])
  define d' where d' = min b (y + (d/2))
  have d' ∈ A
    unfolding A_def
  proof safe
    show a ≤ d' using ‹a = y› ‹0 < d› ‹y < b› by (simp add: d'_def)
    show d' ≤ b by (simp add: d'_def)
    fix x1
    assume x1 ∈ {a..<d'}
    hence x1 ∈ S x1 ∈ {y..b}
      by (auto simp: ‹a = y› d'_def dist_real_def intro!: d )
    thus ?le x1
      by (rule S)
  qed
  hence d' ≤ y
    unfolding y_def
    by (rule cSup_upper) simp
  then show y = b using ‹d > 0› ‹y < b›
    by (simp add: d'_def)
next
  case False
  with ‹a ≤ y› have a < y by simp
  show y = b
  proof (rule ccontr)
    assume y ≠ b
    hence y < b using ‹y ≤ b› by simp
    let ?F = at y within {y..<b}
    from f' phi'
    have (f has_vector_derivative f' y) ?F
      and (φ has_vector_derivative φ' y) ?F
      using ‹a < y› ‹y < b›
     by (auto simp add: at_within_open[of _ {a<..<b}] has_vector_derivative_def
        intro!: has_derivative_subset[where s={a<..<b} and t={y..<b}])
   hence ∀F x1 in ?F. norm (f x1 − f y − (x1 − y) *R f' y) ≤ e2 * |x1 − y|
       ∀F x1 in ?F. norm (φ x1 − φ y − (x1 − y) *R φ' y) ≤ e2 * |x1 − y|
      using ‹e2 > 0›
      by (auto simp: has_derivative_within_alt2 has_vector_derivative_def)
    moreover
    have ∀F x1 in ?F. y ≤ x1 ∀F x1 in ?F. x1 < b
      by (auto simp: eventually_at_filter)
    ultimately
    have ∀F x1 in ?F. norm (f x1 − f y) ≤ (φ x1 − φ y) + e * |x1 − y|
```

```
    (is ∀F x1 in ?F. ?le' x1)
  proof eventually_elim
    case (elim x1)
    from norm_triangle_ineq2[THEN order_trans, OF elim(1)]
    have norm (f x1 − f y) ≤ norm (f' y) ∗ |x1 − y| + e2 ∗ |x1 − y|
      by (simp add: ac_simps)
    also have norm (f' y) ≤ φ' y using bnd ‹a < y› ‹y < b› by simp
    also have φ' y ∗ |x1 − y| ≤ φ x1 − φ y + e2 ∗ |x1 − y|
      using elim by (simp add: ac_simps)
    finally
    have norm (f x1 − f y) ≤ φ x1 − φ y + e2 ∗ |x1 − y| + e2 ∗ |x1 − y|
      by (auto simp: mult_right_mono)
    thus ?case by (simp add: e2_def)
  qed
  moreover have ?le' y by simp
  ultimately obtain S
  where S: open S y ∈ S ⋀x. x∈S ⟹ x ∈ {y..<b} ⟹ ?le' x
    unfolding eventually_at_topological
    by metis
  from ‹open S› obtain d where d: ⋀x. dist x y < d ⟹ x ∈ S d > 0
   by (force simp: dist_commute open_dist ball_def dest!: bspec[OF _ ‹y ∈ S›])
  define d' where d' = min ((y + b)/2) (y + (d/2))
  have d' ∈ A
    unfolding A_def
  proof safe
    show a ≤ d' using ‹a < y› ‹0 < d› ‹y < b› by (simp add: d'_def)
    show d' ≤ b using ‹y < b› by (simp add: d'_def min_def)
    fix x1
    assume x1: x1 ∈ {a..<d'}
    show ?le x1
    proof (cases x1 < y)
      case True
      then show ?thesis
        using ‹y ∈ A› local.leI x1 by auto
    next
      case False
      hence x1': x1 ∈ S x1 ∈ {y..<b} using x1
        by (auto simp: d'_def dist_real_def intro!: d)
      have norm (f x1 − f a) ≤ norm (f x1 − f y) + norm (f y − f a)
        by (rule order_trans[OF _ norm_triangle_ineq]) simp
      also note S(3)[OF x1']
      also note le_y
      finally show ?le x1
        using False by (auto simp: algebra_simps)
    qed
  qed
  hence d' ≤ y
    unfolding y_def by (rule cSup_upper) simp
  thus False using ‹d > 0› ‹y < b›
```

```
        by (simp add: d'_def min_def split: if_split_asm)
      qed
    qed
    with le_y have norm (f b − f a) ≤ φ b − φ a + e ∗ (b − a + 1)
      by (simp add: algebra_simps)
  } note ∗ = this
  show ?thesis
  proof (rule field_le_epsilon)
    fix e::real assume e > 0
    then show norm (f b − f a) ≤ φ b − φ a + e
      using ∗[of e / (b − a + 1)] ‹a < b› by simp
  qed
qed

lemma differentiable_bound:
  fixes f :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
  assumes convex S
    and derf: ⋀x. x∈S ⟹ (f has_derivative f' x) (at x within S)
    and B: ⋀x. x ∈ S ⟹ onorm (f' x) ≤ B
    and x: x ∈ S
    and y: y ∈ S
  shows norm (f x − f y) ≤ B ∗ norm (x − y)
proof −
  let ?p = λu. x + u ∗R (y − x)
  let ?φ = λh. h ∗ B ∗ norm (x − y)
  have ∗: x + u ∗R (y − x) ∈ S if u ∈ {0..1} for u
  proof −
    have u ∗R y = u ∗R (y − x) + u ∗R x
      by (simp add: scale_right_diff_distrib)
    then show x + u ∗R (y − x) ∈ S
      using that ‹convex S› x y by (simp add: convex_alt)
        (metis pth_b(2) pth_c(1) scaleR_collapse)
  qed
  have ⋀z. z ∈ (λu. x + u ∗R (y − x)) ‘ {0..1} ⟹
          (f has_derivative f' z) (at z within (λu. x + u ∗R (y − x)) ‘ {0..1})
    by (auto intro: ∗ has_derivative_subset [OF derf])
  then have continuous_on (?p ‘ {0..1}) f
    unfolding continuous_on_eq_continuous_within
    by (meson has_derivative_continuous)
  with ∗ have 1: continuous_on {0 .. 1} (f ∘ ?p)
    by (intro continuous_intros)+
  {
    fix u::real assume u: u ∈{0 <..< 1}
    let ?u = ?p u
    interpret linear (f' ?u)
      using u by (auto intro!: has_derivative_linear derf ∗)
   have (f ∘ ?p has_derivative (f' ?u) ∘ (λu. 0 + u ∗R (y − x))) (at u within box
0 1)
      by (intro derivative_intros has_derivative_subset [OF derf]) (use u ∗ in auto)
```

hence ((*f* ∘ *?p*) *has_vector_derivative f ′ ?u* (*y* − *x*)) (*at u*)
by (*simp add*: *at_within_open*[*OF u open_greaterThanLessThan*] *scaleR has_vector_derivative_def o_def*)
} **note** *2* = *this*
have *3*: *continuous_on* {*0..1*} *?φ*
by (*rule continuous_intros*)+
have *4*: (*?φ has_vector_derivative B* ∗ *norm* (*x* − *y*)) (*at u*) **for** *u*
by (*auto simp*: *has_vector_derivative_def intro*!: *derivative_eq_intros*)
{
fix *u*::*real* **assume** *u*: *u* ∈{*0* <..< *1*}
let *?u* = *?p u*
interpret *bounded_linear* (*f ′ ?u*)
using *u* **by** (*auto intro*!: *has_derivative_bounded_linear derf* ∗)
have *norm* (*f ′ ?u* (*y* − *x*)) ≤ *onorm* (*f ′ ?u*) ∗ *norm* (*y* − *x*)
by (*rule onorm*) (*rule bounded_linear*)
also have *onorm* (*f ′ ?u*) ≤ *B*
using *u* **by** (*auto intro*!: *assms*(*3*)[*rule_format*] ∗)
finally have *norm* ((*f ′ ?u*) (*y* − *x*)) ≤ *B* ∗ *norm* (*x* − *y*)
by (*simp add*: *mult_right_mono norm_minus_commute*)
} **note** *5* = *this*
have *norm* (*f x* − *f y*) = *norm* ((*f* ∘ (λ*u*. *x* + *u* $*_R$ (*y* − *x*))) *1* − (*f* ∘ (λ*u*. *x* + *u* $*_R$ (*y* − *x*))) *0*)
by (*auto simp add*: *norm_minus_commute*)
also
from *differentiable_bound_general*[*OF zero_less_one 1*, *OF 3 2 4 5*]
have *norm* ((*f* ∘ *?p*) *1* − (*f* ∘ *?p*) *0*) ≤ *B* ∗ *norm* (*x* − *y*)
by *simp*
finally show *?thesis* **.**
qed

lemma *field_differentiable_bound*:
fixes *S* :: *′a*::*real_normed_field set*
assumes *cvs*: *convex S*
and *df*: ⋀*z*. *z* ∈ *S* ⟹ (*f has_field_derivative f ′ z*) (*at z within S*)
and *dn*: ⋀*z*. *z* ∈ *S* ⟹ *norm* (*f ′ z*) ≤ *B*
and *x* ∈ *S* *y* ∈ *S*
shows *norm*(*f x* − *f y*) ≤ *B* ∗ *norm*(*x* − *y*)
apply (*rule differentiable_bound* [*OF cvs*])
apply (*erule df* [*unfolded has_field_derivative_def*])
apply (*rule onorm_le*, *simp_all add*: *norm_mult mult_right_mono assms*)
done

lemma
differentiable_bound_segment:
fixes *f*::*′a*::*real_normed_vector* ⇒ *′b*::*real_normed_vector*
assumes ⋀*t*. *t* ∈ {*0..1*} ⟹ *x0* + *t* $*_R$ *a* ∈ *G*
assumes *f ′*: ⋀*x*. *x* ∈ *G* ⟹ (*f has_derivative f ′ x*) (*at x within G*)
assumes *B*: ⋀*x*. *x* ∈ {*0..1*} ⟹ *onorm* (*f ′* (*x0* + *x* $*_R$ *a*)) ≤ *B*
shows *norm* (*f* (*x0* + *a*) − *f x0*) ≤ *norm a* ∗ *B*

```
proof -
  let ?G = (λx. x0 + x *R a) ` {0..1}
  have ?G = (+) x0 ` (λx. x *R a) ` {0..1} by auto
  also have convex ...
    by (intro convex_translation convex_scaled convex_real_interval)
  finally have convex ?G .
  moreover have ?G ⊆ G x0 ∈ ?G x0 + a ∈ ?G using assms by (auto intro:
image_eqI[where x=1])
  ultimately show ?thesis
    using has_derivative_subset[OF f' ‹?G ⊆ G›] B
      differentiable_bound[of (λx. x0 + x *R a) ` {0..1} f f' B x0 + a x0]
    by (force simp: ac_simps)
qed

lemma differentiable_bound_linearization:
  fixes f::'a::real_normed_vector ⇒ 'b::real_normed_vector
  assumes S: ⋀t. t ∈ {0..1} ⟹ a + t *R (b - a) ∈ S
  assumes f'[derivative_intros]: ⋀x. x ∈ S ⟹ (f has_derivative f' x) (at x within
S)
  assumes B: ⋀x. x ∈ S ⟹ onorm (f' x - f' x0) ≤ B
  assumes x0 ∈ S
  shows norm (f b - f a - f' x0 (b - a)) ≤ norm (b - a) * B
proof -
  define g where [abs_def]: g x = f x - f' x0 x for x
  have g: ⋀x. x ∈ S ⟹ (g has_derivative (λi. f' x i - f' x0 i)) (at x within S)
    unfolding g_def using assms
    by (auto intro!: derivative_eq_intros
      bounded_linear.has_derivative[OF has_derivative_bounded_linear, OF f'])
  from B have ∀x∈{0..1}. onorm (λi. f' (a + x *R (b - a)) i - f' x0 i) ≤ B
    using assms by (auto simp: fun_diff_def)
  with differentiable_bound_segment[OF S g] ‹x0 ∈ S›
  show ?thesis
    by (simp add: g_def field_simps linear_diff[OF has_derivative_linear[OF f']])
qed

lemma vector_differentiable_bound_linearization:
  fixes f::real ⇒ 'b::real_normed_vector
  assumes f': ⋀x. x ∈ S ⟹ (f has_vector_derivative f' x) (at x within S)
  assumes closed_segment a b ⊆ S
  assumes B: ⋀x. x ∈ S ⟹ norm (f' x - f' x0) ≤ B
  assumes x0 ∈ S
  shows norm (f b - f a - (b - a) *R f' x0) ≤ norm (b - a) * B
  using assms
  by (intro differentiable_bound_linearization[of a b S f λx h. h *R f' x x0 B])
    (force simp: closed_segment_real_eq has_vector_derivative_def
      scaleR_diff_right[symmetric] mult.commute[of B]
      intro!: onorm_le mult_left_mono)+
```

In particular.

lemma *has_derivative_zero_constant*:
  fixes *f* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
  assumes *convex s*
    and $\bigwedge x.\ x \in s \Longrightarrow$ (*f has_derivative* (*λh. 0*)) (*at x within s*)
  shows $\exists\, c.\ \forall\, x{\in}s.\ f\, x = c$
proof –
  { fix *x y* **assume** $x \in s\ y \in s$
    then have *norm* $(f\, x - f\, y) \le 0 * $ *norm* $(x - y)$
      using *assms* **by** (*intro differentiable_bound*[*of s*]) (*auto simp*: *onorm_zero*)
    then have $f\, x = f\, y$
      by *simp* **}**
  then show *?thesis*
    by *metis*
qed

lemma *has_field_derivative_zero_constant*:
  assumes *convex s* $\bigwedge x.\ x \in s \Longrightarrow$ (*f has_field_derivative 0*) (*at x within s*)
  shows $\exists\, c.\ \forall\, x{\in}s.\ f\ (x) = (c$ :: *'a* :: *real_normed_field*)
proof (*rule has_derivative_zero_constant*)
  have *A*: (∗) *0* = (λ_. *0* :: *'a*) **by** (*intro ext*) *simp*
  fix *x* **assume** $x \in s$ **thus** (*f has_derivative* (*λh. 0*)) (*at x within s*)
    using *assms*(*2*)[*of x*] **by** (*simp add*: *has_field_derivative_def A*)
qed *fact*

lemma
  has_vector_derivative_zero_constant:
  assumes *convex s*
  assumes $\bigwedge x.\ x \in s \Longrightarrow$ (*f has_vector_derivative 0*) (*at x within s*)
  obtains *c* **where** $\bigwedge x.\ x \in s \Longrightarrow f\, x = c$
  using *has_derivative_zero_constant*[*of s f*] *assms*
  by (*auto simp*: *has_vector_derivative_def*)

lemma *has_derivative_zero_unique*:
  fixes *f* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
  assumes *convex s*
    and $\bigwedge x.\ x \in s \Longrightarrow$ (*f has_derivative* (*λh. 0*)) (*at x within s*)
    and $x \in s\ y \in s$
  shows $f\, x = f\, y$
  using *has_derivative_zero_constant*[*OF assms*(*1*,*2*)] *assms*(*3*−) **by** *force*

lemma *has_derivative_zero_unique_connected*:
  fixes *f* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
  assumes *open s connected s*
  assumes *f*: $\bigwedge x.\ x \in s \Longrightarrow$ (*f has_derivative* (*λx. 0*)) (*at x*)
  assumes $x \in s\ y \in s$
  shows $f\, x = f\, y$
proof (*rule connected_local_const*[**where** *f*=*f*, *OF* ‹*connected s*› ‹*x*∈*s*› ‹*y*∈*s*›])
  show ∀ *a*∈*s*. *eventually* (*λb. f a = f b*) (*at a within s*)
  proof

```
    fix a assume a ∈ s
    with ‹open s› obtain e where 0 < e ball a e ⊆ s
      by (rule openE)
    then have ∃ c. ∀ x∈ball a e. f x = c
      by (intro has_derivative_zero_constant)
        (auto simp: at_within_open[OF _ open_ball] f)
    with ‹0<e› have ∀ x∈ball a e. f a = f x
      by auto
    then show eventually (λb. f a = f b) (at a within s)
      using ‹0<e› unfolding eventually_at_topological
      by (intro exI[of _ ball a e]) auto
  qed
qed
```

4.10.12 Differentiability of inverse function (most basic form)

```
lemma has_derivative_inverse_basic:
  fixes f :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
  assumes derf: (f has_derivative f') (at (g y))
    and ling': bounded_linear g'
    and g' ∘ f' = id
    and contg: continuous (at y) g
    and open T
    and y ∈ T
    and fg: ⋀z. z ∈ T ⟹ f (g z) = z
  shows (g has_derivative g') (at y)
proof −
  interpret f': bounded_linear f'
    using assms unfolding has_derivative_def by auto
  interpret g': bounded_linear g'
    using assms by auto
  obtain C where C: 0 < C ⋀x. norm (g' x) ≤ norm x ∗ C
    using bounded_linear.pos_bounded[OF assms(2)] by blast
  have lem1: ∀ e>0. ∃ d>0. ∀ z.
    norm (z − y) < d ⟶ norm (g z − g y − g'(z − y)) ≤ e ∗ norm (g z − g y)
  proof (intro allI impI)
    fix e :: real
    assume e > 0
    with C(1) have ∗: e / C > 0 by auto
    obtain d0 where 0 < d0 and d0:
        ⋀u. norm (u − g y) < d0 ⟹ norm (f u − f (g y) − f' (u − g y)) ≤ e /
C ∗ norm (u − g y)
      using derf ∗ unfolding has_derivative_at_alt by blast
    obtain d1 where 0 < d1 and d1: ⋀x. ⟦0 < dist x y; dist x y < d1⟧ ⟹ dist
(g x) (g y) < d0
      using contg ‹0 < d0› unfolding continuous_at Lim_at by blast
    obtain d2 where 0 < d2 and d2: ⋀u. dist u y < d2 ⟹ u ∈ T
      using ‹open T› ‹y ∈ T› unfolding open_dist by blast
    obtain d where d: 0 < d d < d1 d < d2
```

using *field_lbound_gt_zero*[*OF* ‹*0* < *d1*› ‹*0* < *d2*›] **by** *blast*
show ∃ *d*>*0*. ∀ *z*. *norm* (*z* − *y*) < *d* ⟶ *norm* (*g z* − *g y* − *g*′ (*z* − *y*)) ≤ *e* ∗ *norm* (*g z* − *g y*)
proof (*intro exI allI impI conjI*)
fix *z*
assume *as*: *norm* (*z* − *y*) < *d*
then have *z* ∈ *T*
using *d2 d* **unfolding** *dist_norm* **by** *auto*
have *norm* (*g z* − *g y* − *g*′ (*z* − *y*)) ≤ *norm* (*g*′ (*f* (*g z*) − *y* − *f*′ (*g z* − *g y*)))
unfolding *g*′.*diff f*′.*diff*
unfolding *assms*(*3*)[*unfolded o_def id_def*, *THEN fun_cong*] *fg*[*OF* ‹*z*∈*T*›]
by (*simp add*: *norm_minus_commute*)
also have ... ≤ *norm* (*f* (*g z*) − *y* − *f*′ (*g z* − *g y*)) ∗ *C*
by (*rule C*(*2*))
also have ... ≤ (*e* / *C*) ∗ *norm* (*g z* − *g y*) ∗ *C*
proof −
have *norm* (*g z* − *g y*) < *d0*
by (*metis as cancel_comm_monoid_add_class.diff_cancel d*(*2*) ‹*0* < *d0*› *d1 diff_gt_0_iff_gt diff_strict_mono dist_norm dist_self zero_less_dist_iff*)
then show *?thesis*
by (*metis C*(*1*) ‹*y* ∈ *T*› *d0 fg mult_le_cancel_iff1*)
qed
also have ... ≤ *e* ∗ *norm* (*g z* − *g y*)
using *C* **by** (*auto simp add*: *field_simps*)
finally show *norm* (*g z* − *g y* − *g*′ (*z* − *y*)) ≤ *e* ∗ *norm* (*g z* − *g y*)
by *simp*
qed (*use d* **in** *auto*)
qed
have ∗: (*0*::*real*) < *1* / *2*
by *auto*
obtain *d* **where** *0* < *d* **and** *d*:
⋀*z*. *norm* (*z* − *y*) < *d* ⟹ *norm* (*g z* − *g y* − *g*′ (*z* − *y*)) ≤ *1*/*2* ∗ *norm* (*g z* − *g y*)
using *lem1* ∗ **by** *blast*
define *B* **where** *B* = *C* ∗ *2*
have *B* > *0*
unfolding *B_def* **using** *C* **by** *auto*
have *lem2*: *norm* (*g z* − *g y*) ≤ *B* ∗ *norm* (*z* − *y*) **if** *z*: *norm*(*z* − *y*) < *d* **for** *z*
proof −
have *norm* (*g z* − *g y*) ≤ *norm*(*g*′ (*z* − *y*)) + *norm* ((*g z* − *g y*) − *g*′(*z* − *y*))
by (*rule norm_triangle_sub*)
also have ... ≤ *norm* (*g*′ (*z* − *y*)) + *1* / *2* ∗ *norm* (*g z* − *g y*)
by (*rule add_left_mono*) (*use d z* **in** *auto*)
also have ... ≤ *norm* (*z* − *y*) ∗ *C* + *1* / *2* ∗ *norm* (*g z* − *g y*)
by (*rule add_right_mono*) (*use C* **in** *auto*)
finally show *norm* (*g z* − *g y*) ≤ *B* ∗ *norm* (*z* − *y*)
unfolding *B_def*
by (*auto simp add*: *field_simps*)

```
  qed
  show ?thesis
    unfolding has_derivative_at_alt
  proof (intro conjI assms allI impI)
    fix e :: real
    assume e > 0
    then have *: e / B > 0 by (metis ‹B > 0› divide_pos_pos)
    obtain d' where 0 < d' and d':
        ⋀z. norm (z − y) < d' ⟹ norm (g z − g y − g' (z − y)) ≤ e / B * norm
(g z − g y)
      using lem1 * by blast
    obtain k where k: 0 < k k < d k < d'
      using field_lbound_gt_zero[OF ‹0 < d› ‹0 < d'›] by blast
    show ∃d>0. ∀ya. norm (ya − y) < d ⟶ norm (g ya − g y − g' (ya − y))
≤ e * norm (ya − y)
    proof (intro exI allI impI conjI)
      fix z
      assume as: norm (z − y) < k
      then have norm (g z − g y − g' (z − y)) ≤ e / B * norm(g z − g y)
        using d' k by auto
      also have ... ≤ e * norm (z − y)
        unfolding times_divide_eq_left pos_divide_le_eq[OF ‹B>0›]
        using lem2[of z] k as ‹e > 0›
        by (auto simp add: field_simps)
      finally show norm (g z − g y − g' (z − y)) ≤ e * norm (z − y)
        by simp
    qed (use k in auto)
  qed
qed
```

Inverse function theorem for complex derivatives

```
lemma has_field_derivative_inverse_basic:
  shows DERIV f (g y) :> f' ⟹
        f' ≠ 0 ⟹
        continuous (at y) g ⟹
        open t ⟹
        y ∈ t ⟹
        (⋀z. z ∈ t ⟹ f (g z) = z)
        ⟹ DERIV g y :> inverse (f')
  unfolding has_field_derivative_def
  apply (rule has_derivative_inverse_basic)
  apply (auto simp: bounded_linear_mult_right)
  done
```

Simply rewrite that based on the domain point x.

```
lemma has_derivative_inverse_basic_x:
  fixes f :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
  assumes (f has_derivative f') (at x)
    and bounded_linear g'
```

and $g' \circ f' = id$
and *continuous* (*at* (*f x*)) *g*
and *g* (*f x*) = *x*
and *open T*
and *f x* ∈ *T*
and ⋀*y*. *y* ∈ *T* ⟹ *f* (*g y*) = *y*
shows (*g has_derivative g'*) (*at* (*f x*))
by (*rule has_derivative_inverse_basic*) (*use assms* **in** *auto*)

This is the version in Dieudonne', assuming continuity of f and g.

lemma *has_derivative_inverse_dieudonne*:
fixes *f* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *open S*
and *open* (*f ` S*)
and *continuous_on S f*
and *continuous_on* (*f ` S*) *g*
and ⋀*x*. *x* ∈ *S* ⟹ *g* (*f x*) = *x*
and *x* ∈ *S*
and (*f has_derivative f'*) (*at x*)
and *bounded_linear g'*
and $g' \circ f' = id$
shows (*g has_derivative g'*) (*at* (*f x*))
apply (*rule has_derivative_inverse_basic_x*[*OF assms*(*7*−*9*) _ _ *assms*(*2*)])
using *assms*(*3*−*6*)
unfolding *continuous_on_eq_continuous_at*[*OF assms*(*1*)] *continuous_on_eq_continuous_at*[*OF assms*(*2*)]
apply *auto*
done

Here's the simplest way of not assuming much about g.

proposition *has_derivative_inverse*:
fixes *f* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *compact S*
and *x* ∈ *S*
and *fx*: *f x* ∈ *interior* (*f ` S*)
and *continuous_on S f*
and *gf*: ⋀*y*. *y* ∈ *S* ⟹ *g* (*f y*) = *y*
and (*f has_derivative f'*) (*at x*)
and *bounded_linear g'*
and $g' \circ f' = id$
shows (*g has_derivative g'*) (*at* (*f x*))
proof −
have ∗: ⋀*y*. *y* ∈ *interior* (*f ` S*) ⟹ *f* (*g y*) = *y*
by (*metis gf image_iff interior_subset subsetCE*)
show *?thesis*
apply (*rule has_derivative_inverse_basic_x*[*OF assms*(*6*−*8*), **where** *T* = *interior* (*f ` S*)])
apply (*rule continuous_on_interior*[*OF _ fx*])
apply (*rule continuous_on_inv*)

```
  apply (simp_all add: assms *)
  done
qed
```

Invertible derivative continuous at a point implies local injectivity. It's only for this we need continuity of the derivative, except of course if we want the fact that the inverse derivative is also continuous. So if we know for some other reason that the inverse function exists, it's OK.

```
proposition has_derivative_locally_injective:
 fixes f :: 'n::euclidean_space ⇒ 'm::euclidean_space
 assumes a ∈ S
     and open S
     and bling: bounded_linear g'
     and g' ∘ f' a = id
     and derf: ⋀x. x ∈ S ⟹ (f has_derivative f' x) (at x)
     and ⋀e. e > 0 ⟹ ∃ d>0. ∀ x. dist a x < d ⟶ onorm (λv. f' x v − f' a
v) < e
 obtains r where r > 0 ball a r ⊆ S inj_on f (ball a r)
proof −
 interpret bounded_linear g'
  using assms by auto
 note f'g' = assms(4)[unfolded id_def o_def,THEN cong]
 have g' (f' a (∑ Basis)) = (∑ Basis) (∑ Basis) ≠ (0::'n)
  using f'g' by auto
 then have *: 0 < onorm g'
  unfolding onorm_pos_lt[OF assms(3)]
  by fastforce
 define k where k = 1 / onorm g' / 2
 have *: k > 0
  unfolding k_def using * by auto
 obtain d1 where d1:
    0 < d1
    ⋀x. dist a x < d1 ⟹ onorm (λv. f' x v − f' a v) < k
  using assms(6) * by blast
 from ‹open S› obtain d2 where d2 > 0 ball a d2 ⊆ S
  using ‹a∈S› ..
 obtain d2 where d2: 0 < d2 ball a d2 ⊆ S
  using ‹0 < d2› ‹ball a d2 ⊆ S› by blast
 obtain d where d: 0 < d d < d1 d < d2
  using field_lbound_gt_zero[OF d1(1) d2(1)] by blast
 show ?thesis
 proof
  show 0 < d by (fact d)
  show ball a d ⊆ S
    using ‹d < d2› ‹ball a d2 ⊆ S› by auto
  show inj_on f (ball a d)
  unfolding inj_on_def
  proof (intro strip)
    fix x y
```

assume *as*: $x \in$ *ball a d* $y \in$ *ball a d* $f\ x = f\ y$
define *ph* **where** [*abs_def*]: *ph w* $= w - g'\ (f\ w - f\ x)$ **for** *w*
have *ph'*:*ph* $= g' \circ (\lambda w.\ f'\ a\ w - (f\ w - f\ x))$
unfolding *ph_def o_def* **by** (*simp add*: *diff f'g'*)
have *norm* (*ph x* $-$ *ph y*) $\leq$ (*1* / *2*) $*$ *norm* ($x - y$)
proof (*rule differentiable_bound*[*OF convex_ball _ _ as*(*1*−*2*)])
fix *u*
assume *u*: $u \in$ *ball a d*
then have $u \in S$
using *d d2* **by** *auto*
have $*$: ($\lambda v.\ v - g'\ (f'\ u\ v)) = g' \circ (\lambda w.\ f'\ a\ w - f'\ u\ w)$
unfolding *o_def* **and** *diff*
using *f'g'* **by** *auto*
have *blin*: *bounded_linear* ($f'\ a$)
using ‹$a \in S$› *derf* **by** *blast*
show (*ph has_derivative* ($\lambda v.\ v - g'\ (f'\ u\ v)$)) (*at u within ball a d*)
unfolding *ph'* $*$ *comp_def*
by (*rule* ‹$u \in S$› *derivative_eq_intros has_derivative_at_withinI* [*OF derf*] *bounded_linear.has_derivative* [*OF blin*] *bounded_linear.has_derivative* [*OF bling*] |*simp*)+
have $**$: *bounded_linear* ($\lambda x.\ f'\ u\ x - f'\ a\ x$) *bounded_linear* ($\lambda x.\ f'\ a\ x - f'\ u\ x$)
using ‹$u \in S$› *blin bounded_linear_sub derf* **by** *auto*
then have *onorm* ($\lambda v.\ v - g'\ (f'\ u\ v)$) $\leq$ *onorm* g' $*$ *onorm* ($\lambda w.\ f'\ a\ w - f'\ u\ w$)
by (*simp add*: $*$ *bounded_linear_axioms onorm_compose*)
also have ... $\leq$ *onorm* g' $*$ *k*
apply (*rule mult_left_mono*)
using *d1*(*2*)[*of u*]
using *onorm_neg*[**where** $f = \lambda x.\ f'\ u\ x - f'\ a\ x$] *d u onorm_pos_le*[*OF bling*] **apply** (*auto simp*: *algebra_simps*)
done
also have ... $\leq$ *1* / *2*
unfolding *k_def* **by** *auto*
finally show *onorm* ($\lambda v.\ v - g'\ (f'\ u\ v)$) $\leq$ *1* / *2* **.**
qed
moreover have *norm* (*ph y* $-$ *ph x*) $=$ *norm* ($y - x$)
by (*simp add*: *as*(*3*) *ph_def*)
ultimately show $x = y$
unfolding *norm_minus_commute* **by** *auto*
qed
qed
qed

4.10.13 Uniformly convergent sequence of derivatives

lemma *has_derivative_sequence_lipschitz_lemma*:
fixes *f* :: *nat* ⇒ *'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector*
assumes *convex S*

```
    and derf: ⋀n x. x ∈ S ⟹ ((f n) has_derivative (f' n x)) (at x within S)
    and nle: ⋀n x h. ⟦n≥N; x ∈ S⟧ ⟹ norm (f' n x h − g' x h) ≤ e ∗ norm h
    and 0 ≤ e
  shows ∀m≥N. ∀n≥N. ∀x∈S. ∀y∈S. norm ((f m x − f n x) − (f m y − f n
y)) ≤ 2 ∗ e ∗ norm (x − y)
proof clarify
  fix m n x y
  assume as: N ≤ m N ≤ n x ∈ S y ∈ S
  show norm ((f m x − f n x) − (f m y − f n y)) ≤ 2 ∗ e ∗ norm (x − y)
  proof (rule differentiable_bound[where f'=λx h. f' m x h − f' n x h, OF ‹convex
S› _ _ as(3−4)])
    fix x
    assume x ∈ S
    show ((λa. f m a − f n a) has_derivative (λh. f' m x h − f' n x h)) (at x
within S)
      by (rule derivative_intros derf ‹x∈S›)+
    show onorm (λh. f' m x h − f' n x h) ≤ 2 ∗ e
    proof (rule onorm_bound)
      fix h
      have norm (f' m x h − f' n x h) ≤ norm (f' m x h − g' x h) + norm (f' n
x h − g' x h)
        using norm_triangle_ineq[of f' m x h − g' x h − f' n x h + g' x h]
        by (auto simp add: algebra_simps norm_minus_commute)
      also have . . . ≤ e ∗ norm h + e ∗ norm h
        using nle[OF ‹N ≤ m› ‹x ∈ S›, of h] nle[OF ‹N ≤ n› ‹x ∈ S›, of h]
        by (auto simp add: field_simps)
      finally show norm (f' m x h − f' n x h) ≤ 2 ∗ e ∗ norm h
        by auto
    qed (simp add: ‹0 ≤ e›)
  qed
qed

lemma has_derivative_sequence_Lipschitz:
  fixes f :: nat ⇒ 'a::real_normed_vector ⇒ 'b::real_normed_vector
  assumes convex S
    and ⋀n x. x ∈ S ⟹ ((f n) has_derivative (f' n x)) (at x within S)
    and nle: ⋀e. e > 0 ⟹ ∀F n in sequentially. ∀x∈S. ∀h. norm (f' n x h − g'
x h) ≤ e ∗ norm h
    and e > 0
  shows ∃N. ∀m≥N. ∀n≥N. ∀x∈S. ∀y∈S.
    norm ((f m x − f n x) − (f m y − f n y)) ≤ e ∗ norm (x − y)
proof −
  have ∗: 2 ∗ (e/2) = e
    using ‹e > 0› by auto
  obtain N where ∀n≥N. ∀x∈S. ∀h. norm (f' n x h − g' x h) ≤ (e/2) ∗ norm
h
    using nle ‹e > 0›
    unfolding eventually_sequentially
    by (metis less_divide_eq_numeral1(1) mult_zero_left)
```

then show $\exists N. \forall m\geq N. \forall n\geq N. \forall x\in S. \forall y\in S.$ *norm* $(f\ m\ x - f\ n\ x - (f\ m\ y - f\ n\ y)) \leq e * $ *norm* $(x - y)$
apply (*rule_tac x=N* **in** *exI*)
apply (*rule has_derivative_sequence_lipschitz_lemma*[**where** *e=e/2*, *unfolded* *])
using *assms* ‹*e* > *0*›
apply *auto*
done
qed

proposition *has_derivative_sequence*:
fixes f :: *nat* ⇒ *'a::real_normed_vector* ⇒ *'b::banach*
assumes *convex S*
and *derf*: $\bigwedge n\ x.\ x \in S \Longrightarrow ((f\ n)$ *has_derivative* $(f'\ n\ x))$ (*at x within S*)
and *nle*: $\bigwedge e.\ e > 0 \Longrightarrow \forall_F$ *n in sequentially*. $\forall x\in S. \forall h.$ *norm* $(f'\ n\ x\ h - g'\ x\ h) \leq e *$ *norm h*
and $x0 \in S$
and *lim*: $((\lambda n.\ f\ n\ x0) \longrightarrow l)$ *sequentially*
shows $\exists g. \forall x\in S. (\lambda n.\ f\ n\ x) \longrightarrow g\ x \wedge (g$ *has_derivative* $g'(x))$ (*at x within S*)
proof −
have *lem1*: $\bigwedge e.\ e > 0 \Longrightarrow \exists N. \forall m\geq N. \forall n\geq N. \forall x\in S. \forall y\in S.$
norm $((f\ m\ x - f\ n\ x) - (f\ m\ y - f\ n\ y)) \leq e *$ *norm* $(x - y)$
using *assms*(*1*,*2*,*3*) **by** (*rule has_derivative_sequence_Lipschitz*)
have $\exists g. \forall x\in S. ((\lambda n.\ f\ n\ x) \longrightarrow g\ x)$ *sequentially*
proof (*intro ballI bchoice*)
fix *x*
assume $x \in S$
show $\exists y. (\lambda n.\ f\ n\ x) \longrightarrow y$
unfolding *convergent_eq_Cauchy*
proof (*cases x = x0*)
case *True*
then show *Cauchy* $(\lambda n.\ f\ n\ x)$
using *LIMSEQ_imp_Cauchy*[*OF lim*] **by** *auto*
next
case *False*
show *Cauchy* $(\lambda n.\ f\ n\ x)$
unfolding *Cauchy_def*
proof (*intro allI impI*)
fix *e* :: *real*
assume $e > 0$
hence *: $e / 2 > 0\ e / 2 /$ *norm* $(x - x0) > 0$ **using** *False* **by** *auto*
obtain *M* **where** *M*: $\forall m\geq M. \forall n\geq M.$ *dist* $(f\ m\ x0)\ (f\ n\ x0) < e / 2$
using *LIMSEQ_imp_Cauchy*[*OF lim*] * **unfolding** *Cauchy_def* **by** *blast*
obtain *N* **where** *N*:
$\forall m\geq N. \forall n\geq N.$
$\forall u\in S. \forall y\in S.$ *norm* $(f\ m\ u - f\ n\ u - (f\ m\ y - f\ n\ y)) \leq$
$e / 2 /$ *norm* $(x - x0) *$ *norm* $(u - y)$
using *lem1* *(*2*) **by** *blast*

```
        show ∃M. ∀m≥M. ∀n≥M. dist (f m x) (f n x) < e
        proof (intro exI allI impI)
          fix m n
          assume as: max M N ≤m max M N≤n
          have dist (f m x) (f n x) ≤ norm (f m x0 − f n x0) + norm (f m x − f
n x − (f m x0 − f n x0))
            unfolding dist_norm
            by (rule norm_triangle_sub)
          also have ... ≤ norm (f m x0 − f n x0) + e / 2
            using N ‹x∈S› ‹x0∈S› as False by fastforce
          also have ... < e / 2 + e / 2
            by (rule add_strict_right_mono) (use as M in ‹auto simp: dist_norm›)
          finally show dist (f m x) (f n x) < e
            by auto
        qed
      qed
    qed
  qed
  then obtain g where g: ∀x∈S. (λn. f n x) ⟶ g x ..
  have lem2: ∃N. ∀n≥N. ∀x∈S. ∀y∈S. norm ((f n x − f n y) − (g x − g y)) ≤
e ∗ norm (x − y) if e > 0 for e
  proof −
    obtain N where
      N: ∀m≥N. ∀n≥N. ∀x∈S. ∀y∈S. norm (f m x − f n x − (f m y − f n y))
≤ e ∗ norm (x − y)
      using lem1 ‹e > 0› by blast
    show ∃N. ∀n≥N. ∀x∈S. ∀y∈S. norm (f n x − f n y − (g x − g y)) ≤ e ∗
norm (x − y)
    proof (intro exI ballI allI impI)
      fix n x y
      assume as: N ≤ n x ∈ S y ∈ S
      have ((λm. norm (f n x − f n y − (f m x − f m y))) ⟶ norm (f n x − f
n y − (g x − g y))) sequentially
        by (intro tendsto_intros g[rule_format] as)
      moreover have eventually (λm. norm (f n x − f n y − (f m x − f m y)) ≤
e ∗ norm (x − y)) sequentially
        unfolding eventually_sequentially
      proof (intro exI allI impI)
        fix m
        assume N ≤ m
        then show norm (f n x − f n y − (f m x − f m y)) ≤ e ∗ norm (x − y)
          using N as by (auto simp add: algebra_simps)
      qed
      ultimately show norm (f n x − f n y − (g x − g y)) ≤ e ∗ norm (x − y)
        by (simp add: tendsto_upperbound)
    qed
  qed
  have ∀x∈S. ((λn. f n x) ⟶ g x) sequentially ∧ (g has_derivative g' x) (at x
within S)
```

unfolding *has_derivative_within_alt2*
proof (*intro ballI conjI allI impI*)
fix *x*
assume $x \in S$
then show ($\lambda n.$ *f n x*) ⟶ *g x*
by (*simp add*: *g*)
have *tog′*: ($\lambda n.$ *f′ n x u*) ⟶ *g′ x u* **for** *u*
unfolding *filterlim_def le_nhds_metric_le eventually_filtermap dist_norm*
proof (*intro allI impI*)
fix *e* :: *real*
assume $e > 0$
show *eventually* ($\lambda n.$ *norm* (*f′ n x u* − *g′ x u*) ≤ *e*) *sequentially*
proof (*cases u* = *0*)
case *True*
have *eventually* ($\lambda n.$ *norm* (*f′ n x u* − *g′ x u*) ≤ *e* ∗ *norm u*) *sequentially*
using *nle* ‹*0* < *e*› ‹*x* ∈ *S*› **by** (*fast elim*: *eventually_mono*)
then show *?thesis*
using ‹*u* = *0*› ‹*0* < *e*› **by** (*auto elim*: *eventually_mono*)
next
case *False*
with ‹*0* < *e*› **have** *0* < *e* / *norm u* **by** *simp*
then have *eventually* ($\lambda n.$ *norm* (*f′ n x u* − *g′ x u*) ≤ *e* / *norm u* ∗ *norm u*) *sequentially*
using *nle* ‹*x* ∈ *S*› **by** (*fast elim*: *eventually_mono*)
then show *?thesis*
using ‹*u* ≠ *0*› **by** *simp*
qed
qed
show *bounded_linear* (*g′ x*)
proof
fix *x′ y z* :: *′a*
fix *c* :: *real*
note *lin* = *assms*(*2*)[*rule_format*,*OF* ‹*x*∈*S*›,*THEN has_derivative_bounded_linear*]
show *g′ x* (*c* ∗$_R$ *x′*) = *c* ∗$_R$ *g′ x x′*
apply (*rule tendsto_unique*[*OF trivial_limit_sequentially tog′*])
unfolding *lin*[*THEN bounded_linear.linear*, *THEN linear_cmul*]
apply (*intro tendsto_intros tog′*)
done
show *g′ x* (*y* + *z*) = *g′ x y* + *g′ x z*
apply (*rule tendsto_unique*[*OF trivial_limit_sequentially tog′*])
unfolding *lin*[*THEN bounded_linear.linear*, *THEN linear_add*]
apply (*rule tendsto_add*)
apply (*rule tog′*)+
done
obtain *N* **where** *N*: ∀ *h*. *norm* (*f′ N x h* − *g′ x h*) ≤ *1* ∗ *norm h*
using *nle* ‹*x* ∈ *S*› **unfolding** *eventually_sequentially* **by** (*fast intro*: *zero_less_one*)
have *bounded_linear* (*f′ N x*)
using *derf* ‹*x* ∈ *S*› **by** *fast*

```
      from bounded_linear.bounded [OF this]
      obtain K where K: ∀h. norm (f' N x h) ≤ norm h * K ..
      {
        fix h
        have norm (g' x h) = norm (f' N x h − (f' N x h − g' x h))
          by simp
        also have ... ≤ norm (f' N x h) + norm (f' N x h − g' x h)
          by (rule norm_triangle_ineq4)
        also have ... ≤ norm h * K + 1 * norm h
          using N K by (fast intro: add_mono)
        finally have norm (g' x h) ≤ norm h * (K + 1)
          by (simp add: ring_distribs)
      }
      then show ∃K. ∀h. norm (g' x h) ≤ norm h * K by fast
    qed
    show eventually (λy. norm (g y − g x − g' x (y − x)) ≤ e * norm (y − x))
(at x within S)
      if e > 0 for e
    proof −
      have *: e / 3 > 0
        using that by auto
      obtain N1 where N1: ∀n≥N1. ∀x∈S. ∀h. norm (f' n x h − g' x h) ≤ e /
3 * norm h
        using nle * unfolding eventually_sequentially by blast
      obtain N2 where
          N2[rule_format]: ∀n≥N2. ∀x∈S. ∀y∈S. norm (f n x − f n y − (g x − g
y)) ≤ e / 3 * norm (x − y)
        using lem2 * by blast
      let ?N = max N1 N2
      have eventually (λy. norm (f ?N y − f ?N x − f' ?N x (y − x)) ≤ e / 3 *
norm (y − x)) (at x within S)
        using derf[unfolded has_derivative_within_alt2] and ‹x ∈ S› and * by fast
      moreover have eventually (λy. y ∈ S) (at x within S)
        unfolding eventually_at by (fast intro: zero_less_one)
      ultimately show ∀F y in at x within S. norm (g y − g x − g' x (y − x))
≤ e * norm (y − x)
      proof (rule eventually_elim2)
        fix y
        assume y ∈ S
        assume norm (f ?N y − f ?N x − f' ?N x (y − x)) ≤ e / 3 * norm (y −
x)
        moreover have norm (g y − g x − (f ?N y − f ?N x)) ≤ e / 3 * norm (y
− x)
          using N2[OF _ ‹y ∈ S› ‹x ∈ S›]
          by (simp add: norm_minus_commute)
        ultimately have norm (g y − g x − f' ?N x (y − x)) ≤ 2 * e / 3 * norm
(y − x)
          using norm_triangle_le[of g y − g x − (f ?N y − f ?N x) f ?N y − f ?N x
− f' ?N x (y − x) 2 * e / 3 * norm (y − x)]
```

```
          by (auto simp add: algebra_simps)
        moreover
        have  norm (f' ?N x (y − x) − g' x (y − x)) ≤ e / 3 ∗ norm (y − x)
          using N1 ‹x ∈ S› by auto
        ultimately show norm (g y − g x − g' x (y − x)) ≤ e ∗ norm (y − x)
         using norm_triangle_le[of g y − g x − f' (max N1 N2) x (y − x) f' (max
N1 N2) x (y − x) − g' x (y − x)]
          by (auto simp add: algebra_simps)
      qed
    qed
  qed
  then show ?thesis by fast
qed
```

Can choose to line up antiderivatives if we want.

```
lemma has_antiderivative_sequence:
  fixes f :: nat ⇒ 'a::real_normed_vector ⇒ 'b::banach
  assumes convex S
    and der: ⋀n x. x ∈ S ⟹ ((f n) has_derivative (f' n x)) (at x within S)
    and no: ⋀e. e > 0 ⟹ ∀F n in sequentially.
       ∀x∈S. ∀h. norm (f' n x h − g' x h) ≤ e ∗ norm h
  shows ∃g. ∀x∈S. (g has_derivative g' x) (at x within S)
proof (cases S = {})
  case False
  then obtain a where a ∈ S
    by auto
  have ∗: ⋀P Q. ∃g. ∀x∈S. P g x ∧ Q g x ⟹ ∃g. ∀x∈S. Q g x
    by auto
  show ?thesis
    apply (rule ∗)
     apply (rule has_derivative_sequence [OF ‹convex S› _ no, of λn x. f n x + (f
0 a − f n a)])
      apply (metis assms(2) has_derivative_add_const)
    using ‹a ∈ S›
     apply auto
    done
qed auto

lemma has_antiderivative_limit:
  fixes g' :: 'a::real_normed_vector ⇒ 'a ⇒ 'b::banach
  assumes convex S
    and ⋀e. e>0 ⟹ ∃f f'. ∀x∈S.
          (f has_derivative (f' x)) (at x within S) ∧ (∀h. norm (f' x h − g' x h) ≤
e ∗ norm h)
  shows ∃g. ∀x∈S. (g has_derivative g' x) (at x within S)
proof −
  have ∗: ∀n. ∃f f'. ∀x∈S.
    (f has_derivative (f' x)) (at x within S) ∧
    (∀h. norm(f' x h − g' x h) ≤ inverse (real (Suc n)) ∗ norm h)
```

```
    by (simp add: assms(2))
  obtain f where
    *: ⋀x. ∃f'. ∀xa∈S. (f x has_derivative f' xa) (at xa within S) ∧
        (∀h. norm (f' xa h − g' xa h) ≤ inverse (real (Suc x)) ∗ norm h)
    using ∗ by metis
  obtain f' where
    f': ⋀x. ∀z∈S. (f x has_derivative f' x z) (at z within S) ∧
          (∀h. norm (f' x z h − g' z h) ≤ inverse (real (Suc x)) ∗ norm h)
    using ∗ by metis
  show ?thesis
  proof (rule has_antiderivative_sequence[OF ‹convex S›, of f f'])
    fix e :: real
    assume e > 0
    obtain N where N: inverse (real (Suc N)) < e
      using reals_Archimedean[OF ‹e>0›] ..
    show ∀F n in sequentially. ∀x∈S.  ∀h. norm (f' n x h − g' x h) ≤ e ∗ norm
h
        unfolding eventually_sequentially
    proof (intro exI allI ballI impI)
      fix n x h
      assume n: N ≤ n and x: x ∈ S
      have ∗: inverse (real (Suc n)) ≤ e
        apply (rule order_trans[OF _ N[THEN less_imp_le]])
        using n apply (auto simp add: field_simps)
        done
      show norm (f' n x h − g' x h) ≤ e ∗ norm h
        by (meson ∗ mult_right_mono norm_ge_zero order.trans x f')
    qed
  qed (use f' in auto)
qed
```

4.10.14 Differentiation of a series

```
proposition has_derivative_series:
  fixes f :: nat ⇒ 'a::real_normed_vector ⇒ 'b::banach
  assumes convex S
    and ⋀n x. x ∈ S ⟹ ((f n) has_derivative (f' n x)) (at x within S)
    and ⋀e. e>0 ⟹ ∀F n in sequentially. ∀x∈S. ∀h. norm (sum (λi. f' i x h)
{..<n} − g' x h) ≤ e ∗ norm h
    and x ∈ S
    and (λn. f n x) sums l
  shows ∃g. ∀x∈S. (λn. f n x) sums (g x) ∧ (g has_derivative g' x) (at x within
S)
  unfolding sums_def
  apply (rule has_derivative_sequence[OF assms(1) _ assms(3)])
  apply (metis assms(2) has_derivative_sum)
  using assms(4−5)
  unfolding sums_def
  apply auto
```

done

lemma *has_field_derivative_series*:
fixes f :: *nat* ⇒ (′*a* :: {*real_normed_field*,*banach*}) ⇒ ′*a*
assumes *convex S*
assumes $\bigwedge n\ x.\ x \in S \Longrightarrow$ (*f n has_field_derivative f′ n x*) (*at x within S*)
assumes *uniform_limit S* ($\lambda n\ x.\ \sum i{<}n.\ f'\ i\ x$) *g′ sequentially*
assumes *x0* ∈ *S summable* (λ*n. f n x0*)
shows ∃ *g*. ∀ *x*∈*S*. (λ*n. f n x*) *sums g x* ∧ (*g has_field_derivative g′ x*) (*at x within S*)
unfolding *has_field_derivative_def*
proof (*rule has_derivative_series*)
show ∀$_F$ *n in sequentially*.
∀ *x*∈*S*. ∀ *h*. *norm* ((∑ *i*<*n*. *f′ i x* ∗ *h*) − *g′ x* ∗ *h*) ≤ *e* ∗ *norm h* **if** *e* > *0* **for** *e*
unfolding *eventually_sequentially*
proof −
from *that assms*(*3*) **obtain** *N* **where** *N*: ⋀*n x*. *n* ≥ *N* ⟹ *x* ∈ *S* ⟹ *norm* ((∑ *i*<*n*. *f′ i x*) − *g′ x*) < *e*
unfolding *uniform_limit_iff eventually_at_top_linorder dist_norm* **by** *blast*
{
fix *n* :: *nat* **and** *x h* :: ′*a* **assume** *nx*: *n* ≥ *N x* ∈ *S*
have *norm* ((∑ *i*<*n*. *f′ i x* ∗ *h*) − *g′ x* ∗ *h*) = *norm* ((∑ *i*<*n*. *f′ i x*) − *g′ x*) ∗ *norm h*
by (*simp add*: *norm_mult* [*symmetric*] *ring_distribs sum_distrib_right*)
also from *N*[*OF nx*] **have** *norm* ((∑ *i*<*n*. *f′ i x*) − *g′ x*) ≤ *e* **by** *simp*
hence *norm* ((∑ *i*<*n*. *f′ i x*) − *g′ x*) ∗ *norm h* ≤ *e* ∗ *norm h*
by (*intro mult_right_mono*) *simp_all*
finally have *norm* ((∑ *i*<*n*. *f′ i x* ∗ *h*) − *g′ x* ∗ *h*) ≤ *e* ∗ *norm h* **.**
}
thus ∃ *N*. ∀ *n*≥*N*. ∀ *x*∈*S*. ∀ *h*. *norm* ((∑ *i*<*n*. *f′ i x* ∗ *h*) − *g′ x* ∗ *h*) ≤ *e* ∗ *norm h* **by** *blast*
qed
qed (*use assms* **in** ‹*auto simp*: *has_field_derivative_def*›)

lemma *has_field_derivative_series′*:
fixes f :: *nat* ⇒ (′*a* :: {*real_normed_field*,*banach*}) ⇒ ′*a*
assumes *convex S*
assumes ⋀*n x*. *x* ∈ *S* ⟹ (*f n has_field_derivative f′ n x*) (*at x within S*)
assumes *uniformly_convergent_on S* (λ*n x*. ∑ *i*<*n*. *f′ i x*)
assumes *x0* ∈ *S summable* (λ*n. f n x0*) *x* ∈ *interior S*
shows *summable* (λ*n. f n x*) ((λ*x*. ∑ *n. f n x*) *has_field_derivative* (∑ *n. f′ n x*)) (*at x*)
proof −
from ‹*x* ∈ *interior S*› **have** *x* ∈ *S* **using** *interior_subset* **by** *blast*
define *g′* **where** [*abs_def*]: *g′ x* = (∑ *i. f′ i x*) **for** *x*
from *assms*(*3*) **have** *uniform_limit S* (λ*n x*. ∑ *i*<*n*. *f′ i x*) *g′ sequentially*
by (*simp add*: *uniformly_convergent_uniform_limit_iff suminf_eq_lim g′_def*)
from *has_field_derivative_series*[*OF assms*(*1*,*2*) *this assms*(*4*,*5*)] **obtain** *g* **where**

```
g:
    ⋀x. x ∈ S ⟹ (λn. f n x) sums g x
    ⋀x. x ∈ S ⟹ (g has_field_derivative g' x) (at x within S) by blast
  from g(1)[OF ‹x ∈ S›] show summable (λn. f n x) by (simp add: sums_iff)
  from g(2)[OF ‹x ∈ S›] ‹x ∈ interior S› have (g has_field_derivative g' x) (at x)
    by (simp add: at_within_interior[of x S])
  also have (g has_field_derivative g' x) (at x) ⟷
                ((λx. ∑n. f n x) has_field_derivative g' x) (at x)
    using eventually_nhds_in_nhd[OF ‹x ∈ interior S›] interior_subset[of S] g(1)
    by (intro DERIV_cong_ev) (auto elim!: eventually_mono simp: sums_iff)
  finally show ((λx. ∑n. f n x) has_field_derivative g' x) (at x) .
qed

lemma differentiable_series:
  fixes f :: nat ⇒ ('a :: {real_normed_field,banach}) ⇒ 'a
  assumes convex S open S
  assumes ⋀n x. x ∈ S ⟹ (f n has_field_derivative f' n x) (at x)
  assumes uniformly_convergent_on S (λn x. ∑i<n. f' i x)
  assumes x0 ∈ S summable (λn. f n x0) and x: x ∈ S
  shows   summable (λn. f n x) and (λx. ∑n. f n x) differentiable (at x)
proof –
  from assms(4) obtain g' where A: uniform_limit S (λn x. ∑i<n. f' i x) g'
sequentially
    unfolding uniformly_convergent_on_def by blast
  from x and ‹open S› have S: at x within S = at x by (rule at_within_open)
  have ∃g. ∀x∈S. (λn. f n x) sums g x ∧ (g has_field_derivative g' x) (at x within
S)
   by (intro has_field_derivative_series[of S f f' g' x0] assms A has_field_derivative_at_within)
  then obtain g where g: ⋀x. x ∈ S ⟹ (λn. f n x) sums g x
    ⋀x. x ∈ S ⟹ (g has_field_derivative g' x) (at x within S) by blast
  from g[OF x] show summable (λn. f n x) by (auto simp: summable_def)
  from g(2)[OF x] have g': (g has_derivative (∗) (g' x)) (at x)
    by (simp add: has_field_derivative_def S)
  have ((λx. ∑n. f n x) has_derivative (∗) (g' x)) (at x)
    by (rule has_derivative_transform_within_open[OF g' ‹open S› x])
       (insert g, auto simp: sums_iff)
  thus (λx. ∑n. f n x) differentiable (at x) unfolding differentiable_def
    by (auto simp: summable_def differentiable_def has_field_derivative_def)
qed

lemma differentiable_series':
  fixes f :: nat ⇒ ('a :: {real_normed_field,banach}) ⇒ 'a
  assumes convex S open S
  assumes ⋀n x. x ∈ S ⟹ (f n has_field_derivative f' n x) (at x)
  assumes uniformly_convergent_on S (λn x. ∑i<n. f' i x)
  assumes x0 ∈ S summable (λn. f n x0)
  shows   (λx. ∑n. f n x) differentiable (at x0)
  using differentiable_series[OF assms, of x0] ‹x0 ∈ S› by blast+
```

4.10.15 Derivative as a vector

Considering derivative *real* ⇒ *'b* as a vector.

definition *vector_derivative f net* = (*SOME f'*. (*f has_vector_derivative f'*) *net*)

lemma *vector_derivative_unique_within*:
  assumes *not_bot*: *at x within S* ≠ *bot*
    and *f'*: (*f has_vector_derivative f'*) (*at x within S*)
    and *f''*: (*f has_vector_derivative f''*) (*at x within S*)
  shows *f'* = *f''*
proof −
  have (λ*x*. *x* $*_R$ *f'*) = (λ*x*. *x* $*_R$ *f''*)
  proof (*rule frechet_derivative_unique_within*, *simp_all*)
    show ∃ *d*. *d* ≠ *0* ∧ |*d*| < *e* ∧ *x* + *d* ∈ *S* **if** *0* < *e* **for** *e*
    proof −
      from *that*
      obtain *x'* **where** *x'* ∈ *S* *x'* ≠ *x* |*x'* − *x*| < *e*
        using *islimpt_approachable_real*[*of x S*] *not_bot*
        by (*auto simp add*: *trivial_limit_within*)
      then show *?thesis*
        using *eq_iff_diff_eq_0* **by** *fastforce*
    qed
  qed (*use f' f''* **in** ‹*auto simp*: *has_vector_derivative_def*›)
  then show *?thesis*
    unfolding *fun_eq_iff* **by** (*metis scaleR_one*)
qed

lemma *vector_derivative_unique_at*:
  (*f has_vector_derivative f'*) (*at x*) ⟹ (*f has_vector_derivative f''*) (*at x*) ⟹ *f'* = *f''*
  by (*rule vector_derivative_unique_within*) *auto*

lemma *differentiableI_vector*: (*f has_vector_derivative y*) *F* ⟹ *f differentiable F*
  by (*auto simp*: *differentiable_def has_vector_derivative_def*)

proposition *vector_derivative_works*:
  f differentiable net ⟷ (*f has_vector_derivative* (*vector_derivative f net*)) *net*
    (**is** *?l* = *?r*)
proof
  assume *?l*
  obtain *f'* **where** *f'*: (*f has_derivative f'*) *net*
    using ‹*?l*› **unfolding** *differentiable_def* **..**
  then interpret *bounded_linear f'*
    by *auto*
  show *?r*
    unfolding *vector_derivative_def has_vector_derivative_def*
    by (*rule someI*[*of _ f' 1*]) (*simp add*: *scaleR*[*symmetric*] *f'*)
qed (*auto simp*: *vector_derivative_def has_vector_derivative_def differentiable_def*)

```
lemma vector_derivative_within:
  assumes not_bot: at x within S ≠ bot and y: (f has_vector_derivative y) (at x within S)
  shows vector_derivative f (at x within S) = y
  using y
  by (intro vector_derivative_unique_within[OF not_bot vector_derivative_works[THEN iffD1] y])
     (auto simp: differentiable_def has_vector_derivative_def)

lemma frechet_derivative_eq_vector_derivative:
  assumes f differentiable (at x)
    shows (frechet_derivative f (at x)) = (λr. r *R vector_derivative f (at x))
using assms
by (auto simp: differentiable_iff_scaleR vector_derivative_def has_vector_derivative_def
         intro: someI frechet_derivative_at [symmetric])

lemma has_real_derivative:
  fixes f :: real ⇒ real
  assumes (f has_derivative f′) F
  obtains c where (f has_real_derivative c) F
proof −
  obtain c where f′ = (λx. x * c)
    by (metis assms has_derivative_bounded_linear real_bounded_linear)
  then show ?thesis
    by (metis assms that has_field_derivative_def mult_commute_abs)
qed

lemma has_real_derivative_iff:
  fixes f :: real ⇒ real
  shows (∃ c. (f has_real_derivative c) F) = (∃ D. (f has_derivative D) F)
  by (metis has_field_derivative_def has_real_derivative)

lemma has_vector_derivative_cong_ev:
  assumes *: eventually (λx. x ∈ S ⟶ f x = g x) (nhds x) f x = g x
  shows (f has_vector_derivative f′) (at x within S) = (g has_vector_derivative f′) (at x within S)
  unfolding has_vector_derivative_def has_derivative_def
  using *
  apply (cases at x within S ≠ bot)
  apply (intro refl conj_cong filterlim_cong)
  apply (auto simp: Lim_ident_at eventually_at_filter elim: eventually_mono)
  done

lemma islimpt_closure_open:
  fixes s :: ′a::perfect_space set
  assumes open s and t: t = closure s x ∈ t
  shows x islimpt t
proof cases
  assume x ∈ s
```

{ fix *T* **assume** $x \in T$ *open T*
then have *open* $(s \cap T)$
using ‹*open s*› **by** *auto*
then have $s \cap T \neq \{x\}$
using *not_open_singleton*[*of x*] **by** *auto*
with ‹$x \in T$› ‹$x \in s$› **have** $\exists y \in t.\ y \in T \wedge y \neq x$
using *closure_subset*[*of s*] **by** (*auto simp*: *t*) **}**
then show *?thesis*
by (*auto intro*!: *islimptI*)
next
assume $x \notin s$ **with** *t* **show** *?thesis*
unfolding *t closure_def* **by** (*auto intro*: *islimpt_subset*)
qed

lemma *vector_derivative_unique_within_closed_interval*:
assumes *ab*: $a < b$ $x \in$ *cbox a b*
assumes *D*: (*f has_vector_derivative f'*) (*at x within cbox a b*) (*f has_vector_derivative f''*) (*at x within cbox a b*)
shows $f' = f''$
using *ab*
by (*intro vector_derivative_unique_within*[*OF _ D*])
(*auto simp*: *trivial_limit_within intro*!: *islimpt_closure_open*[**where** *s*={*a* <..< *b*}])

lemma *vector_derivative_at*:
(*f has_vector_derivative f'*) (*at x*) $\Longrightarrow$ *vector_derivative f* (*at x*) = *f'*
by (*intro vector_derivative_within at_neq_bot*)

lemma *has_vector_derivative_id_at* [*simp*]: *vector_derivative* ($\lambda x.\ x$) (*at a*) = *1*
by (*simp add*: *vector_derivative_at*)

lemma *vector_derivative_minus_at* [*simp*]:
f differentiable at a
$\Longrightarrow$ *vector_derivative* ($\lambda x.\ -\ f\ x$) (*at a*) = − *vector_derivative f* (*at a*)
by (*simp add*: *vector_derivative_at has_vector_derivative_minus vector_derivative_works* [*symmetric*])

lemma *vector_derivative_add_at* [*simp*]:
⟦*f differentiable at a*; *g differentiable at a*⟧
$\Longrightarrow$ *vector_derivative* ($\lambda x.\ f\ x + g\ x$) (*at a*) = *vector_derivative f* (*at a*) + *vector_derivative g* (*at a*)
by (*simp add*: *vector_derivative_at has_vector_derivative_add vector_derivative_works* [*symmetric*])

lemma *vector_derivative_diff_at* [*simp*]:
⟦*f differentiable at a*; *g differentiable at a*⟧
$\Longrightarrow$ *vector_derivative* ($\lambda x.\ f\ x - g\ x$) (*at a*) = *vector_derivative f* (*at a*) − *vector_derivative g* (*at a*)
by (*simp add*: *vector_derivative_at has_vector_derivative_diff vector_derivative_works*

[*symmetric*])

lemma *vector_derivative_mult_at* [*simp*]:
fixes *f g* :: *real* ⇒ *'a* :: *real_normed_algebra*
shows ⟦*f differentiable at a*; *g differentiable at a*⟧
⟹ *vector_derivative* (λ*x*. *f x* ∗ *g x*) (*at a*) = *f a* ∗ *vector_derivative g* (*at a*) + *vector_derivative f* (*at a*) ∗ *g a*
by (*simp add*: *vector_derivative_at has_vector_derivative_mult vector_derivative_works* [*symmetric*])

lemma *vector_derivative_scaleR_at* [*simp*]:
⟦*f differentiable at a*; *g differentiable at a*⟧
⟹ *vector_derivative* (λ*x*. *f x* $*_R$ *g x*) (*at a*) = *f a* $*_R$ *vector_derivative g* (*at a*) + *vector_derivative f* (*at a*) $*_R$ *g a*
apply (*rule vector_derivative_at*)
apply (*rule has_vector_derivative_scaleR*)
apply (*auto simp*: *vector_derivative_works has_vector_derivative_def has_field_derivative_def mult_commute_abs*)
done

lemma *vector_derivative_within_cbox*:
assumes *ab*: *a* < *b x* ∈ *cbox a b*
assumes *f*: (*f has_vector_derivative f'*) (*at x within cbox a b*)
shows *vector_derivative f* (*at x within cbox a b*) = *f'*
by (*intro vector_derivative_unique_within_closed_interval*[*OF ab* _ *f*]
vector_derivative_works[*THEN iffD1*] *differentiableI_vector*)
fact

lemma *vector_derivative_within_closed_interval*:
fixes *f*::*real* ⇒ *'a*::*euclidean_space*
assumes *a* < *b* **and** *x* ∈ {*a*..*b*}
assumes (*f has_vector_derivative f'*) (*at x within* {*a*..*b*})
shows *vector_derivative f* (*at x within* {*a*..*b*}) = *f'*
using *assms vector_derivative_within_cbox*
by *fastforce*

lemma *has_vector_derivative_within_subset*:
(*f has_vector_derivative f'*) (*at x within S*) ⟹ *T* ⊆ *S* ⟹ (*f has_vector_derivative f'*) (*at x within T*)
by (*auto simp*: *has_vector_derivative_def intro*: *has_derivative_subset*)

lemma *has_vector_derivative_at_within*:
(*f has_vector_derivative f'*) (*at x*) ⟹ (*f has_vector_derivative f'*) (*at x within S*)
unfolding *has_vector_derivative_def*
by (*rule has_derivative_at_withinI*)

lemma *has_vector_derivative_weaken*:
fixes *x D* **and** *f g S T*
assumes *f*: (*f has_vector_derivative D*) (*at x within T*)

and $x \in S$ $S \subseteq T$
and $\bigwedge x.\ x \in S \Longrightarrow f\ x = g\ x$
shows (*g has_vector_derivative D*) (*at x within S*)
proof −
have (*f has_vector_derivative D*) (*at x within S*) $\longleftrightarrow$ (*g has_vector_derivative D*) (*at x within S*)
unfolding *has_vector_derivative_def has_derivative_iff_norm*
using *assms* **by** (*intro conj_cong Lim_cong_within refl*) *auto*
then show *?thesis*
using *has_vector_derivative_within_subset*[*OF f* ‹$S \subseteq T$›] **by** *simp*
qed

lemma *has_vector_derivative_transform_within*:
assumes (*f has_vector_derivative f'*) (*at x within S*)
and $0 < d$
and $x \in S$
and $\bigwedge x'.\ [\![x' \in S;\ dist\ x'\ x < d]\!] \Longrightarrow f\ x' = g\ x'$
shows (*g has_vector_derivative f'*) (*at x within S*)
using *assms*
unfolding *has_vector_derivative_def*
by (*rule has_derivative_transform_within*)

lemma *has_vector_derivative_transform_within_open*:
assumes (*f has_vector_derivative f'*) (*at x*)
and *open S*
and $x \in S$
and $\bigwedge y.\ y \in S \Longrightarrow f\ y = g\ y$
shows (*g has_vector_derivative f'*) (*at x*)
using *assms*
unfolding *has_vector_derivative_def*
by (*rule has_derivative_transform_within_open*)

lemma *has_vector_derivative_transform*:
assumes $x \in S$ $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
assumes *f'*: (*f has_vector_derivative f'*) (*at x within S*)
shows (*g has_vector_derivative f'*) (*at x within S*)
using *assms*
unfolding *has_vector_derivative_def*
by (*rule has_derivative_transform*)

lemma *vector_diff_chain_at*:
assumes (*f has_vector_derivative f'*) (*at x*)
and (*g has_vector_derivative g'*) (*at* (*f x*))
shows (($g \circ f$) *has_vector_derivative* ($f' *_R g'$)) (*at x*)
using *assms has_vector_derivative_at_within has_vector_derivative_def vector_derivative_diff_chain_within*
by *blast*

lemma *vector_diff_chain_within*:
assumes (*f has_vector_derivative f'*) (*at x within s*)

```
    and (g has_vector_derivative g′) (at (f x) within f ‘ s)
  shows ((g ∘ f) has_vector_derivative (f′ *R g′)) (at x within s)
  using assms has_vector_derivative_def vector_derivative_diff_chain_within by blast

lemma vector_derivative_const_at [simp]: vector_derivative (λx. c) (at a) = 0
  by (simp add: vector_derivative_at)

lemma vector_derivative_at_within_ivl:
  (f has_vector_derivative f′) (at x) =⇒
    a ≤ x =⇒ x ≤ b =⇒ a<b =⇒ vector_derivative f (at x within {a..b}) = f′
  using has_vector_derivative_at_within vector_derivative_within_cbox by fastforce

lemma vector_derivative_chain_at:
  assumes f differentiable at x (g differentiable at (f x))
  shows vector_derivative (g ∘ f) (at x) =
        vector_derivative f (at x) *R vector_derivative g (at (f x))
by (metis vector_diff_chain_at vector_derivative_at vector_derivative_works assms)

lemma field_vector_diff_chain_at:
 assumes Df: (f has_vector_derivative f′) (at x)
     and Dg: (g has_field_derivative g′) (at (f x))
 shows ((g ∘ f) has_vector_derivative (f′ * g′)) (at x)
using diff_chain_at[OF Df[unfolded has_vector_derivative_def]
                       Dg [unfolded has_field_derivative_def]]
 by (auto simp: o_def mult.commute has_vector_derivative_def)

lemma vector_derivative_chain_within:
  assumes at x within S ≠ bot f differentiable (at x within S)
    (g has_derivative g′) (at (f x) within f ‘ S)
  shows vector_derivative (g ∘ f) (at x within S) =
        g′ (vector_derivative f (at x within S))
  apply (rule vector_derivative_within [OF ‹at x within S ≠ bot›])
  apply (rule vector_derivative_diff_chain_within)
  using assms(2−3) vector_derivative_works
  by auto
```

4.10.16 Field differentiability

```
definition field_differentiable :: ['a ⇒ 'a::real_normed_field, 'a filter] ⇒ bool
           (infixr (field′_differentiable) 50)
  where f field_differentiable F ≡ ∃f′. (f has_field_derivative f′) F

lemma field_differentiable_imp_differentiable:
  f field_differentiable F =⇒ f differentiable F
  unfolding field_differentiable_def differentiable_def
  using has_field_derivative_imp_has_derivative by auto

lemma field_differentiable_imp_continuous_at:
    f field_differentiable (at x within S) =⇒ continuous (at x within S) f
```

by (*metis DERIV_continuous field_differentiable_def*)

lemma *field_differentiable_within_subset*:
⟦*f field_differentiable* (*at x within S*); $T \subseteq S$⟧ ⟹ *f field_differentiable* (*at x within T*)
by (*metis DERIV_subset field_differentiable_def*)

lemma *field_differentiable_at_within*:
⟦*f field_differentiable* (*at x*)⟧
⟹ *f field_differentiable* (*at x within S*)
unfolding *field_differentiable_def*
by (*metis DERIV_subset top_greatest*)

lemma *field_differentiable_linear* [*simp*,*derivative_intros*]: ((∗) *c*) *field_differentiable F*
unfolding *field_differentiable_def has_field_derivative_def mult_commute_abs*
by (*force intro*: *has_derivative_mult_right*)

lemma *field_differentiable_const* [*simp*,*derivative_intros*]: (λ*z*. *c*) *field_differentiable F*
unfolding *field_differentiable_def has_field_derivative_def*
using *DERIV_const has_field_derivative_imp_has_derivative* **by** *blast*

lemma *field_differentiable_ident* [*simp*,*derivative_intros*]: (λ*z*. *z*) *field_differentiable F*
unfolding *field_differentiable_def has_field_derivative_def*
using *DERIV_ident has_field_derivative_def* **by** *blast*

lemma *field_differentiable_id* [*simp*,*derivative_intros*]: *id field_differentiable F*
unfolding *id_def* **by** (*rule field_differentiable_ident*)

lemma *field_differentiable_minus* [*derivative_intros*]:
f field_differentiable F ⟹ (λ*z*. − (*f z*)) *field_differentiable F*
unfolding *field_differentiable_def*
by (*metis field_differentiable_minus*)

lemma *field_differentiable_add* [*derivative_intros*]:
assumes *f field_differentiable F g field_differentiable F*
shows (λ*z*. *f z* + *g z*) *field_differentiable F*
using *assms* **unfolding** *field_differentiable_def*
by (*metis field_differentiable_add*)

lemma *field_differentiable_add_const* [*simp*,*derivative_intros*]:
(+) *c field_differentiable F*
by (*simp add*: *field_differentiable_add*)

lemma *field_differentiable_sum* [*derivative_intros*]:
(⋀*i*. *i* ∈ *I* ⟹ (*f i*) *field_differentiable F*) ⟹ (λ*z*. ∑ *i*∈*I*. *f i z*) *field_differentiable F*

by (*induct I rule*: *infinite_finite_induct*)
(*auto intro*: *field_differentiable_add field_differentiable_const*)

lemma *field_differentiable_diff* [*derivative_intros*]:
assumes *f field_differentiable F g field_differentiable F*
shows (*λz. f z − g z*) *field_differentiable F*
using *assms* **unfolding** *field_differentiable_def*
by (*metis field_differentiable_diff*)

lemma *field_differentiable_inverse* [*derivative_intros*]:
assumes *f field_differentiable* (*at a within S*) *f a* ≠ *0*
shows (*λz. inverse* (*f z*)) *field_differentiable* (*at a within S*)
using *assms* **unfolding** *field_differentiable_def*
by (*metis DERIV_inverse_fun*)

lemma *field_differentiable_mult* [*derivative_intros*]:
assumes *f field_differentiable* (*at a within S*)
g field_differentiable (*at a within S*)
shows (*λz. f z ∗ g z*) *field_differentiable* (*at a within S*)
using *assms* **unfolding** *field_differentiable_def*
by (*metis DERIV_mult* [*of f _ a S g*])

lemma *field_differentiable_divide* [*derivative_intros*]:
assumes *f field_differentiable* (*at a within S*)
g field_differentiable (*at a within S*)
g a ≠ *0*
shows (*λz. f z / g z*) *field_differentiable* (*at a within S*)
using *assms* **unfolding** *field_differentiable_def*
by (*metis DERIV_divide* [*of f _ a S g*])

lemma *field_differentiable_power* [*derivative_intros*]:
assumes *f field_differentiable* (*at a within S*)
shows (*λz. f z ^ n*) *field_differentiable* (*at a within S*)
using *assms* **unfolding** *field_differentiable_def*
by (*metis DERIV_power*)

lemma *field_differentiable_transform_within*:
0 < *d* ⟹
x ∈ *S* ⟹
(⋀*x'*. *x'* ∈ *S* ⟹ *dist x' x* < *d* ⟹ *f x'* = *g x'*) ⟹
f field_differentiable (*at x within S*)
⟹ *g field_differentiable* (*at x within S*)
unfolding *field_differentiable_def has_field_derivative_def*
by (*blast intro*: *has_derivative_transform_within*)

lemma *field_differentiable_compose_within*:
assumes *f field_differentiable* (*at a within S*)
g field_differentiable (*at* (*f a*) *within f'S*)
shows (*g o f*) *field_differentiable* (*at a within S*)

using *assms* **unfolding** *field_differentiable_def*
by (*metis DERIV_image_chain*)

lemma *field_differentiable_compose*:
f field_differentiable at z $\Longrightarrow$ *g field_differentiable at* (*f z*)
$\Longrightarrow$ (*g o f*) *field_differentiable at z*
by (*metis field_differentiable_at_within field_differentiable_compose_within*)

lemma *field_differentiable_within_open*:
$[\![a \in S;\ open\ S]\!] \Longrightarrow$ *f field_differentiable at a within S* $\longleftrightarrow$
f field_differentiable at a
unfolding *field_differentiable_def*
by (*metis at_within_open*)

lemma *exp_scaleR_has_vector_derivative_right*:
((λt. *exp* ($t *_R A$)) *has_vector_derivative exp* ($t *_R A$) $*$ *A*) (*at t within T*)
unfolding *has_vector_derivative_def*
proof (*rule has_derivativeI*)
let *?F* = *at t within* ($T \cap \{t - 1 <..< t + 1\}$)
have $*$: *at t within T* = *?F*
by (*rule at_within_nhd*[**where** *S*=$\{t - 1 <..< t + 1\}$]) *auto*
let *?e* = $\lambda i\ x$. (*inverse* (*1* + *real i*) $*$ *inverse* (*fact i*) $*$ ($x - t$) ^ *i*) $*_R$ (*A* $*$ *A*
^ *i*)
have $\forall_F$ *n in sequentially*.
$\forall x \in T \cap \{t - 1 <..< t + 1\}$. *norm* (*?e n x*) $\le$ *norm* (*A* ^ (*n* + *1*) $/_R$ *fact* (*n*
+ *1*))
apply (*auto simp*: *algebra_split_simps intro*!: *eventuallyI*)
apply (*rule mult_left_mono*)
apply (*auto simp add*: *field_simps power_abs intro*!: *divide_right_mono power_le_one*)
done
then have *uniform_limit* ($T \cap \{t - 1 <..< t + 1\}$) ($\lambda n\ x$. $\sum i<n$. *?e i x*) (λx.
$\sum i$. *?e i x*) *sequentially*
by (*rule Weierstrass_m_test_ev*) (*intro summable_ignore_initial_segment summable_norm_exp*)
moreover
have $\forall_F$ *x in sequentially*. $x > 0$
by (*metis eventually_gt_at_top*)
then have
$\forall_F$ *n in sequentially*. ((λx. $\sum i<n$. *?e i x*) $\longrightarrow$ *A*) *?F*
by *eventually_elim*
(*auto intro*!: *tendsto_eq_intros*
simp: *power_0_left if_distrib if_distribR*
cong: *if_cong*)
ultimately
have [*tendsto_intros*]: ((λx. $\sum i$. *?e i x*) $\longrightarrow$ *A*) *?F*
by (*auto intro*!: *swap_uniform_limit*[**where** *f*=$\lambda n\ x$. $\sum i < n$. *?e i x* **and** *F* =
sequentially])
have [*tendsto_intros*]: ((λx. *if x* = *t then 0 else 1*) $\longrightarrow$ *1*) *?F*
by (*rule tendsto_eventually*) (*simp add*: *eventually_at_filter*)
have ((λy. (($y - t$) / *abs* ($y - t$)) $*_R$ (($\sum n$. *?e n y*) $-$ *A*)) $\longrightarrow$ *0*) (*at t within*

T)
unfolding ∗
by (*rule tendsto_norm_zero_cancel*) (*auto intro*!: *tendsto_eq_intros*)

moreover have $\forall_F$ *x in at t within T*. $x \neq t$
by (*simp add*: *eventually_at_filter*)
then have $\forall_F$ *x in at t within T*. $((x - t) / |x - t|) *_R ((\sum n.\ ?e\ n\ x) - A) =$
$(exp\ ((x - t) *_R A) - 1 - (x - t) *_R A) /_R norm\ (x - t)$
proof *eventually_elim*
case (*elim x*)
have $(exp\ ((x - t) *_R A) - 1 - (x - t) *_R A) /_R norm\ (x - t) =$
$((\sum n.\ (x - t) *_R\ ?e\ n\ x) - (x - t) *_R A) /_R norm\ (x - t)$
unfolding *exp_first_term*
by (*simp add*: *ac_simps*)
also
have *summable* (λ*n*. *?e n x*)
proof −
from *elim* **have** $?e\ n\ x = (((x - t) *_R A)\ \hat{}\ (n + 1)) /_R fact\ (n + 1) /_R$
$(x - t)$ **for** *n*
by *simp*
then show *?thesis*
by (*auto simp only*:
intro!: *summable_scaleR_right summable_ignore_initial_segment summable_exp_generic*)
qed
then have $(\sum n.\ (x - t) *_R\ ?e\ n\ x) = (x - t) *_R (\sum n.\ ?e\ n\ x)$
by (*rule suminf_scaleR_right*[*symmetric*])
also have $(\ldots - (x - t) *_R A) /_R norm\ (x - t) = (x - t) *_R ((\sum n.\ ?e\ n$
$x) - A) /_R norm\ (x - t)$
by (*simp add*: *algebra_simps*)
finally show *?case*
by *simp* (*simp add*: *field_simps*)
qed

ultimately have $((\lambda y.\ (exp\ ((y - t) *_R A) - 1 - (y - t) *_R A) /_R norm\ (y$
$- t)) \longrightarrow 0)$ (*at t within T*)
by (*rule Lim_transform_eventually*)
from *tendsto_mult_right_zero*[*OF this*, **where** $c=exp\ (t *_R A)$]
show $((\lambda y.\ (exp\ (y *_R A) - exp\ (t *_R A) - (y - t) *_R (exp\ (t *_R A) * A)) /_R$
$norm\ (y - t)) \longrightarrow 0)$
(*at t within T*)
by (*rule Lim_transform_eventually*)
(*auto simp*: *field_split_simps exp_add_commuting*[*symmetric*])
qed (*rule bounded_linear_scaleR_left*)

lemma *exp_times_scaleR_commute*: $exp\ (t *_R A) * A = A * exp\ (t *_R A)$
using *exp_times_arg_commute*[*symmetric*, *of* $t *_R A$]
by (*auto simp*: *algebra_simps*)

lemma *exp_scaleR_has_vector_derivative_left*: $((\lambda t.\ exp\ (t *_R A))$ *has_vector_derivative*

$A * exp\ (t *_R A))\ (at\ t)$
  using *exp_scaleR_has_vector_derivative_right*[*of A t*]
  by (*simp add*: *exp_times_scaleR_commute*)

lemma *field_differentiable_series*:
  fixes f :: *nat* ⇒ *'a*::{*real_normed_field*,*banach*} ⇒ *'a*
  assumes *convex S open S*
  assumes $\bigwedge n\ x.\ x \in S \Longrightarrow (f\ n$ *has_field_derivative* $f'\ n\ x)\ (at\ x)$
  assumes *uniformly_convergent_on* $S\ (\lambda n\ x.\ \sum i{<}n.\ f'\ i\ x)$
  assumes $x0 \in S$ *summable* $(\lambda n.\ f\ n\ x0)$ **and** x: $x \in S$
  shows $(\lambda x.\ \sum n.\ f\ n\ x)$ *field_differentiable* $(at\ x)$
proof −
  from *assms*(*4*) **obtain** g' **where** A: *uniform_limit* $S\ (\lambda n\ x.\ \sum i{<}n.\ f'\ i\ x)\ g'$ *sequentially*
    unfolding *uniformly_convergent_on_def* **by** *blast*
  from x **and** ‹*open S*› **have** S: *at x within S* = *at x* **by** (*rule at_within_open*)
  have $\exists g.\ \forall x{\in}S.\ (\lambda n.\ f\ n\ x)$ *sums* $g\ x \wedge (g$ *has_field_derivative* $g'\ x)\ (at\ x\ within\ S)$
   by (*intro has_field_derivative_series*[*of S f f′ g′ x0*] *assms A has_field_derivative_at_within*)
  then obtain g **where** g: $\bigwedge x.\ x \in S \Longrightarrow (\lambda n.\ f\ n\ x)$ *sums* $g\ x$
    $\bigwedge x.\ x \in S \Longrightarrow (g$ *has_field_derivative* $g'\ x)\ (at\ x\ within\ S)$ **by** *blast*
  from $g(2)[OF\ x]$ **have** g': $(g$ *has_derivative* $(*)\ (g'\ x))\ (at\ x)$
    by (*simp add*: *has_field_derivative_def S*)
  have $((\lambda x.\ \sum n.\ f\ n\ x)$ *has_derivative* $(*)\ (g'\ x))\ (at\ x)$
    by (*rule has_derivative_transform_within_open*[*OF g′* ‹*open S*› *x*])
      (*insert g*, *auto simp*: *sums_iff*)
  thus $(\lambda x.\ \sum n.\ f\ n\ x)$ *field_differentiable* $(at\ x)$ **unfolding** *differentiable_def*
    by (*auto simp*: *summable_def field_differentiable_def has_field_derivative_def*)
qed

Caratheodory characterization

lemma *field_differentiable_caratheodory_at*:
  f field_differentiable $(at\ z) \longleftrightarrow$
      $(\exists g.\ (\forall w.\ f(w) - f(z) = g(w) * (w - z)) \wedge continuous\ (at\ z)\ g)$
  using *CARAT_DERIV* [*of f*]
  by (*simp add*: *field_differentiable_def has_field_derivative_def*)

lemma *field_differentiable_caratheodory_within*:
  f field_differentiable $(at\ z\ within\ s) \longleftrightarrow$
      $(\exists g.\ (\forall w.\ f(w) - f(z) = g(w) * (w - z)) \wedge continuous\ (at\ z\ within\ s)\ g)$
  using *DERIV_caratheodory_within* [*of f*]
  by (*simp add*: *field_differentiable_def has_field_derivative_def*)

4.10.17 Field derivative

definition *deriv* :: (*'a* ⇒ *'a*::*real_normed_field*) ⇒ *'a* ⇒ *'a* **where**
  deriv f x ≡ *SOME D. DERIV f x* :> *D*

lemma *DERIV_imp_deriv*: *DERIV f x* :> $f' \Longrightarrow$ *deriv f x* = f'

unfolding *deriv_def* **by** (*metis some_equality DERIV_unique*)

lemma *DERIV_deriv_iff_has_field_derivative*:
DERIV f x :> deriv f x ⟷ (∃*f'*. (*f has_field_derivative f'*) (*at x*))
by (*auto simp*: *has_field_derivative_def DERIV_imp_deriv*)

lemma *DERIV_deriv_iff_real_differentiable*:
fixes *x* :: *real*
shows *DERIV f x :> deriv f x* ⟷ *f differentiable at x*
unfolding *differentiable_def* **by** (*metis DERIV_imp_deriv has_real_derivative_iff*)

lemma *deriv_cong_ev*:
assumes *eventually* (λ*x*. *f x* = *g x*) (*nhds x*) *x* = *y*
shows *deriv f x* = *deriv g y*
proof −
have (λ*D*. (*f has_field_derivative D*) (*at x*)) = (λ*D*. (*g has_field_derivative D*) (*at y*))
by (*intro ext DERIV_cong_ev refl assms*)
thus *?thesis* **by** (*simp add*: *deriv_def assms*)
qed

lemma *higher_deriv_cong_ev*:
assumes *eventually* (λ*x*. *f x* = *g x*) (*nhds x*) *x* = *y*
shows (*deriv* ^^ *n*) *f x* = (*deriv* ^^ *n*) *g y*
proof −
from *assms*(*1*) **have** *eventually* (λ*x*. (*deriv* ^^ *n*) *f x* = (*deriv* ^^ *n*) *g x*) (*nhds x*)
proof (*induction n arbitrary*: *f g*)
case (*Suc n*)
from *Suc.prems* **have** *eventually* (λ*y*. *eventually* (λ*z*. *f z* = *g z*) (*nhds y*)) (*nhds x*)
by (*simp add*: *eventually_eventually*)
hence *eventually* (λ*x*. *deriv f x* = *deriv g x*) (*nhds x*)
by *eventually_elim* (*rule deriv_cong_ev*, *simp_all*)
thus *?case* **by** (*auto intro*!: *deriv_cong_ev Suc simp*: *funpow_Suc_right simp del*: *funpow.simps*)
qed *auto*
from *eventually_nhds_x_imp_x*[*OF this*] *assms*(*2*) **show** *?thesis* **by** *simp*
qed

lemma *real_derivative_chain*:
fixes *x* :: *real*
shows *f differentiable at x* ⟹ *g differentiable at* (*f x*)
⟹ *deriv* (*g o f*) *x* = *deriv g* (*f x*) ∗ *deriv f x*
by (*metis DERIV_deriv_iff_real_differentiable DERIV_chain DERIV_imp_deriv*)
lemma *field_derivative_eq_vector_derivative*:
(*deriv f x*) = *vector_derivative f* (*at x*)
by (*simp add*: *mult.commute deriv_def vector_derivative_def has_vector_derivative_def has_field_derivative_def*)

proposition *field_differentiable_derivI*:
f field_differentiable (*at x*) ⟹ (*f has_field_derivative deriv f x*) (*at x*)
by (*simp add*: *field_differentiable_def DERIV_deriv_iff_has_field_derivative*)

lemma *vector_derivative_chain_at_general*:
assumes *f differentiable at x g field_differentiable at* (*f x*)
shows *vector_derivative* (*g* ∘ *f*) (*at x*) = *vector_derivative f* (*at x*) ∗ *deriv g* (*f x*)
apply (*rule vector_derivative_at* [*OF field_vector_diff_chain_at*])
using *assms vector_derivative_works* **by** (*auto simp*: *field_differentiable_derivI*)

lemma *DERIV_deriv_iff_field_differentiable*:
DERIV f x :> *deriv f x* ⟷ *f field_differentiable at x*
unfolding *field_differentiable_def* **by** (*metis DERIV_imp_deriv*)

lemma *deriv_chain*:
f field_differentiable at x ⟹ *g field_differentiable at* (*f x*)
⟹ *deriv* (*g o f*) *x* = *deriv g* (*f x*) ∗ *deriv f x*
by (*metis DERIV_deriv_iff_field_differentiable DERIV_chain DERIV_imp_deriv*)

lemma *deriv_linear* [*simp*]: *deriv* (λ*w*. *c* ∗ *w*) = (λ*z*. *c*)
by (*metis DERIV_imp_deriv DERIV_cmult_Id*)

lemma *deriv_uminus* [*simp*]: *deriv* (λ*w*. −*w*) = (λ*z*. −*1*)
using *deriv_linear*[*of* −*1*] **by** (*simp del*: *deriv_linear*)

lemma *deriv_ident* [*simp*]: *deriv* (λ*w*. *w*) = (λ*z*. *1*)
by (*metis DERIV_imp_deriv DERIV_ident*)

lemma *deriv_id* [*simp*]: *deriv id* = (λ*z*. *1*)
by (*simp add*: *id_def*)

lemma *deriv_const* [*simp*]: *deriv* (λ*w*. *c*) = (λ*z*. *0*)
by (*metis DERIV_imp_deriv DERIV_const*)

lemma *deriv_add* [*simp*]:
⟦*f field_differentiable at z*; *g field_differentiable at z*⟧
⟹ *deriv* (λ*w*. *f w* + *g w*) *z* = *deriv f z* + *deriv g z*
unfolding *DERIV_deriv_iff_field_differentiable*[*symmetric*]
by (*auto intro*!: *DERIV_imp_deriv derivative_intros*)

lemma *deriv_diff* [*simp*]:
⟦*f field_differentiable at z*; *g field_differentiable at z*⟧
⟹ *deriv* (λ*w*. *f w* − *g w*) *z* = *deriv f z* − *deriv g z*
unfolding *DERIV_deriv_iff_field_differentiable*[*symmetric*]
by (*auto intro*!: *DERIV_imp_deriv derivative_intros*)

lemma *deriv_mult* [*simp*]:

⟦f field_differentiable at z; g field_differentiable at z⟧
⟹ deriv (λw. f w * g w) z = f z * deriv g z + deriv f z * g z
unfolding DERIV_deriv_iff_field_differentiable[symmetric]
by (auto intro!: DERIV_imp_deriv derivative_eq_intros)

lemma deriv_cmult:
f field_differentiable at z ⟹ deriv (λw. c * f w) z = c * deriv f z
by simp

lemma deriv_cmult_right:
f field_differentiable at z ⟹ deriv (λw. f w * c) z = deriv f z * c
by simp

lemma deriv_inverse [simp]:
⟦f field_differentiable at z; f z ≠ 0⟧
⟹ deriv (λw. inverse (f w)) z = − deriv f z / f z ^ 2
unfolding DERIV_deriv_iff_field_differentiable[symmetric]
by (safe intro!: DERIV_imp_deriv derivative_eq_intros) (auto simp: field_split_simps power2_eq_square)

lemma deriv_divide [simp]:
⟦f field_differentiable at z; g field_differentiable at z; g z ≠ 0⟧
⟹ deriv (λw. f w / g w) z = (deriv f z * g z − f z * deriv g z) / g z ^ 2
by (simp add: field_class.field_divide_inverse field_differentiable_inverse)
(simp add: field_split_simps power2_eq_square)

lemma deriv_cdivide_right:
f field_differentiable at z ⟹ deriv (λw. f w / c) z = deriv f z / c
by (simp add: field_class.field_divide_inverse)

lemma deriv_compose_linear:
f field_differentiable at (c * z) ⟹ deriv (λw. f (c * w)) z = c * deriv f (c * z)
apply (rule DERIV_imp_deriv)
unfolding DERIV_deriv_iff_field_differentiable [symmetric]
by (metis (full_types) DERIV_chain2 DERIV_cmult_Id mult.commute)

lemma nonzero_deriv_nonconstant:
assumes df: DERIV f ξ :> df **and** S: open S ξ ∈ S **and** df ≠ 0
shows ¬ f constant_on S
unfolding constant_on_def
by (metis ‹df ≠ 0› has_field_derivative_transform_within_open [OF df S] DERIV_const DERIV_unique)

4.10.18 Relation between convexity and derivative

proposition convex_on_imp_above_tangent:
assumes convex: convex_on A f **and** connected: connected A
assumes c: c ∈ interior A **and** x : x ∈ A

assumes *deriv*: (*f has_field_derivative f′*) (*at c within A*)
shows $f\,x - f\,c \geq f' * (x - c)$
proof (*cases x c rule*: *linorder_cases*)
assume *xc*: $x > c$
let *?A′* = *interior A* ∩ {*c*<..}
from *c* **have** *c* ∈ *interior A* ∩ *closure* {*c*<..} **by** *auto*
also have … ⊆ *closure* (*interior A* ∩ {*c*<..}) **by** (*intro open_Int_closure_subset*) *auto*
finally have *at c within ?A′* ≠ *bot* **by** (*subst at_within_eq_bot_iff*) *auto*
moreover from *deriv* **have** ((λ*y*. (*f y* − *f c*) / (*y* − *c*)) ⟶ *f′*) (*at c within ?A′*)
unfolding *has_field_derivative_iff* **using** *interior_subset*[*of A*] **by** (*blast intro*: *tendsto_mono at_le*)
moreover from *eventually_at_right_real*[*OF xc*]
have *eventually* (λ*y*. (*f y* − *f c*) / (*y* − *c*) ≤ (*f x* − *f c*) / (*x* − *c*)) (*at_right c*)
proof *eventually_elim*
fix *y* **assume** *y*: *y* ∈ {*c*<..<*x*}
with *convex connected x c* **have** *f y* ≤ (*f x* − *f c*) / (*x* − *c*) ∗ (*y* − *c*) + *f c*
using *interior_subset*[*of A*]
by (*intro convex_onD_Icc′ convex_on_subset*[*OF convex*] *connected_contains_Icc*) *auto*
hence *f y* − *f c* ≤ (*f x* − *f c*) / (*x* − *c*) ∗ (*y* − *c*) **by** *simp*
thus (*f y* − *f c*) / (*y* − *c*) ≤ (*f x* − *f c*) / (*x* − *c*) **using** *y xc* **by** (*simp add*: *field_split_simps*)
qed
hence *eventually* (λ*y*. (*f y* − *f c*) / (*y* − *c*) ≤ (*f x* − *f c*) / (*x* − *c*)) (*at c within ?A′*)
by (*blast intro*: *filter_leD at_le*)
ultimately have *f′* ≤ (*f x* − *f c*) / (*x* − *c*) **by** (*simp add*: *tendsto_upperbound*)
thus *?thesis* **using** *xc* **by** (*simp add*: *field_simps*)
next
assume *xc*: $x < c$
let *?A′* = *interior A* ∩ {..<*c*}
from *c* **have** *c* ∈ *interior A* ∩ *closure* {..<*c*} **by** *auto*
also have … ⊆ *closure* (*interior A* ∩ {..<*c*}) **by** (*intro open_Int_closure_subset*) *auto*
finally have *at c within ?A′* ≠ *bot* **by** (*subst at_within_eq_bot_iff*) *auto*
moreover from *deriv* **have** ((λ*y*. (*f y* − *f c*) / (*y* − *c*)) ⟶ *f′*) (*at c within ?A′*)
unfolding *has_field_derivative_iff* **using** *interior_subset*[*of A*] **by** (*blast intro*: *tendsto_mono at_le*)
moreover from *eventually_at_left_real*[*OF xc*]
have *eventually* (λ*y*. (*f y* − *f c*) / (*y* − *c*) ≥ (*f x* − *f c*) / (*x* − *c*)) (*at_left c*)
proof *eventually_elim*
fix *y* **assume** *y*: *y* ∈ {*x*<..<*c*}
with *convex connected x c* **have** *f y* ≤ (*f x* − *f c*) / (*c* − *x*) ∗ (*c* − *y*) + *f c*
using *interior_subset*[*of A*]
by (*intro convex_onD_Icc″ convex_on_subset*[*OF convex*] *connected_contains_Icc*) *auto*

hence $f\ y - f\ c \le (f\ x - f\ c) * ((c - y) / (c - x))$ **by** *simp*
also have $(c - y) / (c - x) = (y - c) / (x - c)$ **using** *y xc* **by** (*simp add*: *field_simps*)
finally show $(f\ y - f\ c) / (y - c) \ge (f\ x - f\ c) / (x - c)$ **using** *y xc*
by (*simp add*: *field_split_simps*)
qed
hence *eventually* $(\lambda y.\ (f\ y - f\ c) / (y - c) \ge (f\ x - f\ c) / (x - c))$ (*at c within ?A'*)
by (*blast intro*: *filter_leD at_le*)
ultimately have $f' \ge (f\ x - f\ c) / (x - c)$ **by** (*simp add*: *tendsto_lowerbound*)
thus *?thesis* **using** *xc* **by** (*simp add*: *field_simps*)
qed *simp_all*

4.10.19 Partial derivatives

lemma *eventually_at_Pair_within_TimesI1*:
fixes *x*::*'a*::*metric_space*
assumes $\forall_F\ x'$ *in at x within X*. $P\ x'$
assumes $P\ x$
shows $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y$. $P\ x'$
proof −
from *assms*[*unfolded eventually_at_topological*]
obtain *S* **where** *S*: *open S* $x \in S$ $\bigwedge x'.\ x' \in X \Longrightarrow x' \in S \Longrightarrow P\ x'$
by *metis*
show $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y$. $P\ x'$
unfolding *eventually_at_topological*
by (*auto intro*!: *exI*[**where** $x=S \times UNIV$] *S open_Times*)
qed

lemma *eventually_at_Pair_within_TimesI2*:
fixes *x*::*'a*::*metric_space*
assumes $\forall_F\ y'$ *in at y within Y*. $P\ y'$ $P\ y$
shows $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y$. $P\ y'$
proof −
from *assms*[*unfolded eventually_at_topological*]
obtain *S* **where** *S*: *open S* $y \in S$ $\bigwedge y'.\ y' \in Y \Longrightarrow y' \in S \Longrightarrow P\ y'$
by *metis*
show $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y$. $P\ y'$
unfolding *eventually_at_topological*
by (*auto intro*!: *exI*[**where** $x=UNIV \times S$] *S open_Times*)
qed

proposition *has_derivative_partialsI*:
fixes *f*::*'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector* ⇒ *'c*::*real_normed_vector*
assumes *fx*: $((\lambda x.\ f\ x\ y)$ *has_derivative fx*) (*at x within X*)
assumes *fy*: $\bigwedge x\ y.\ x \in X \Longrightarrow y \in Y \Longrightarrow ((\lambda y.\ f\ x\ y)$ *has_derivative blinfun_apply* (*fy x y*)) (*at y within Y*)
assumes *fy_cont*[*unfolded continuous_within*]: *continuous* (*at* (x, y) *within* $X \times Y$) $(\lambda(x, y).\ fy\ x\ y)$

assumes $y \in Y$ *convex* Y
shows $((\lambda(x, y).\ f\ x\ y)$ *has_derivative* $(\lambda(tx, ty).\ fx\ tx + fy\ x\ y\ ty))$ $(at\ (x, y)$ *within* $X \times Y)$
proof (*safe intro*!: *has_derivativeI tendstoI*, *goal_cases*)
case $(2\ e')$
interpret *fx*: *bounded_linear fx* **using** *fx* **by** (*rule has_derivative_bounded_linear*)
define e **where** $e = e' / 9$
have $e > 0$ **using** ‹$e' > 0$› **by** (*simp add*: *e_def*)

from *fy_cont*[*THEN tendstoD*, *OF* ‹$e > 0$›]
have $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y$. *dist* $(fy\ x'\ y')\ (fy\ x\ y) < e$
by (*auto simp*: *split_beta'*)
from *this*[*unfolded eventually_at*] **obtain** d' **where**
$d' > 0$
$\bigwedge x'\ y'.\ x' \in X \Longrightarrow y' \in Y \Longrightarrow (x', y') \neq (x, y) \Longrightarrow dist\ (x', y')\ (x, y) < d'$
$\Longrightarrow$
dist $(fy\ x'\ y')\ (fy\ x\ y) < e$
by *auto*
then
have d': $x' \in X \Longrightarrow y' \in Y \Longrightarrow dist\ (x', y')\ (x, y) < d' \Longrightarrow dist\ (fy\ x'\ y')\ (fy$
$x\ y) < e$
for $x'\ y'$
using ‹$0 < e$›
by (*cases* $(x', y') = (x, y)$) *auto*
define d **where** $d = d' /$ *sqrt 2*
have $d > 0$ **using** ‹$0 < d'$› **by** (*simp add*: *d_def*)
have d: $x' \in X \Longrightarrow y' \in Y \Longrightarrow dist\ x'\ x < d \Longrightarrow dist\ y'\ y < d \Longrightarrow dist\ (fy\ x'$
$y')\ (fy\ x\ y) < e$
for $x'\ y'$
by (*auto simp*: *dist_prod_def d_def intro*!: d' *real_sqrt_sum_squares_less*)

let $?S$ = *ball* $y\ d \cap Y$
have *convex* $?S$
by (*auto intro*!: *convex_Int* ‹*convex* Y›)
{
fix x'::$'a$ **and** y'::$'b$
assume x': $x' \in X$ **and** y': $y' \in Y$
assume dx': *dist* $x'\ x < d$ **and** dy': *dist* $y'\ y < d$
have *norm* $(fy\ x'\ y' - fy\ x'\ y) \leq dist\ (fy\ x'\ y')\ (fy\ x\ y) + dist\ (fy\ x'\ y)\ (fy\ x$
$y)$
by *norm*
also have *dist* $(fy\ x'\ y')\ (fy\ x\ y) < e$
by (*rule d*; *fact*)
also have *dist* $(fy\ x'\ y)\ (fy\ x\ y) < e$
by (*auto intro*!: *d simp*: *dist_prod_def* x' ‹$d > 0$› ‹$y \in Y$› dx')
finally
have *norm* $(fy\ x'\ y' - fy\ x'\ y) < e + e$
by *arith*
then have *onorm* (*blinfun_apply* $(fy\ x'\ y')$ − *blinfun_apply* $(fy\ x'\ y)) < e + e$

by (*auto simp*: *norm_blinfun.rep_eq blinfun.diff_left*[*abs_def*] *fun_diff_def*)
} **note** *onorm* = *this*

have *ev_mem*: $\forall_F$ (*x'*, *y'*) *in at* (*x*, *y*) *within X* × *Y*. (*x'*, *y'*) ∈ *X* × *Y*
using ‹*y* ∈ *Y*›
by (*auto simp*: *eventually_at intro*!: *zero_less_one*)
moreover
have *ev_dist*: $\forall_F$ *xy in at* (*x*, *y*) *within X* × *Y*. *dist xy* (*x*, *y*) < *d* **if** *d* > *0* **for** *d*
using *eventually_at_ball*[*OF that*]
by (*rule eventually_elim2*) (*auto simp*: *dist_commute intro*!: *eventually_True*)
note *ev_dist*[*OF* ‹*0* < *d*›]
ultimately
have $\forall_F$ (*x'*, *y'*) *in at* (*x*, *y*) *within X* × *Y*.
norm (*f x' y'* − *f x' y* − (*fy x' y*) (*y'* − *y*)) ≤ *norm* (*y'* − *y*) ∗ (*e* + *e*)
proof (*eventually_elim*, *safe*)
fix *x' y'*
assume *x'* ∈ *X* **and** *y'*: *y'* ∈ *Y*
assume *dist*: *dist* (*x'*, *y'*) (*x*, *y*) < *d*
then have *dx*: *dist x' x* < *d* **and** *dy*: *dist y' y* < *d*
unfolding *dist_prod_def fst_conv snd_conv atomize_conj*
by (*metis le_less_trans real_sqrt_sum_squares_ge1 real_sqrt_sum_squares_ge2*)
{
fix *t*::*real*
assume *t* ∈ {*0* .. *1*}
then have *y* + *t* $*_R$ (*y'* − *y*) ∈ *closed_segment y y'*
by (*auto simp*: *closed_segment_def algebra_simps intro*!: *exI*[**where** *x*=*t*])
also
have . . . ⊆ *ball y d* ∩ *Y*
using ‹*y* ∈ *Y*› ‹*0* < *d*› *dy y'*
by (*intro* ‹*convex ?S*›[*unfolded convex_contains_segment*, *rule_format*, *of y y'*])
(*auto simp*: *dist_commute*)
finally have *y* + *t* $*_R$ (*y'* − *y*) ∈ *?S* .
} **note** *seg* = *this*

have ⋀*x*. *x* ∈ *ball y d* ∩ *Y* ⟹ *onorm* (*blinfun_apply* (*fy x' x*) − *blinfun_apply* (*fy x' y*)) ≤ *e* + *e*
by (*safe intro*!: *onorm less_imp_le* ‹*x'* ∈ *X*› *dx*) (*auto simp*: *dist_commute* ‹*0* < *d*› ‹*y* ∈ *Y*›)
with *seg has_derivative_subset*[*OF assms*(*2*)[*OF* ‹*x'* ∈ *X*›]]
show *norm* (*f x' y'* − *f x' y* − (*fy x' y*) (*y'* − *y*)) ≤ *norm* (*y'* − *y*) ∗ (*e* + *e*)
by (*rule differentiable_bound_linearization*[**where** *S*=*?S*])
(*auto intro*!: ‹*0* < *d*› ‹*y* ∈ *Y*›)
qed
moreover
let *?le* = λ*x'*. *norm* (*f x' y* − *f x y* − (*fx*) (*x'* − *x*)) ≤ *norm* (*x'* − *x*) ∗ *e*
from *fx*[*unfolded has_derivative_within*, *THEN conjunct2*, *THEN tendstoD*, *OF* ‹*0* < *e*›]

have $\forall_F$ *x′ in at x within X*. *?le x′*
by *eventually_elim* (*simp*,
simp add: *dist_norm field_split_simps split*: *if_split_asm*)
then have $\forall_F$ (*x′*, *y′*) *in at* (*x*, *y*) *within X* × *Y*. *?le x′*
by (*rule eventually_at_Pair_within_TimesI1*)
(*simp add*: *blinfun.bilinear_simps*)
moreover have $\forall_F$ (*x′*, *y′*) *in at* (*x*, *y*) *within X* × *Y*. *norm* ((*x′*, *y′*) − (*x*, *y*)) ≠ *0*
unfolding *norm_eq_zero right_minus_eq*
by (*auto simp*: *eventually_at intro*!: *zero_less_one*)
moreover
from *fy_cont*[*THEN tendstoD*, *OF* ‹*0* < *e*›]
have $\forall_F$ *x′ in at x within X*. *norm* (*fy x′ y* − *fy x y*) < *e*
unfolding *eventually_at*
using ‹*y* ∈ *Y*›
by (*auto simp*: *dist_prod_def dist_norm*)
then have $\forall_F$ (*x′*, *y′*) *in at* (*x*, *y*) *within X* × *Y*. *norm* (*fy x′ y* − *fy x y*) < *e*
by (*rule eventually_at_Pair_within_TimesI1*)
(*simp add*: *blinfun.bilinear_simps* ‹*0* < *e*›)
ultimately
have $\forall_F$ (*x′*, *y′*) *in at* (*x*, *y*) *within X* × *Y*.
norm ((*f x′ y′* − *f x y* − (*fx* (*x′* − *x*) + *fy x y* (*y′* − *y*))) $/_R$
norm ((*x′*, *y′*) − (*x*, *y*)))
< *e′*
apply *eventually_elim*
proof *safe*
fix *x′ y′*
have *norm* (*f x′ y′* − *f x y* − (*fx* (*x′* − *x*) + *fy x y* (*y′* − *y*))) ≤
norm (*f x′ y′* − *f x′ y* − *fy x′ y* (*y′* − *y*)) +
norm (*fy x y* (*y′* − *y*) − *fy x′ y* (*y′* − *y*)) +
norm (*f x′ y* − *f x y* − *fx* (*x′* − *x*))
by *norm*
also
assume *nz*: *norm* ((*x′*, *y′*) − (*x*, *y*)) ≠ *0*
and *nfy*: *norm* (*fy x′ y* − *fy x y*) < *e*
assume *norm* (*f x′ y′* − *f x′ y* − *blinfun_apply* (*fy x′ y*) (*y′* − *y*)) ≤ *norm* (*y′* − *y*) ∗ (*e* + *e*)
also assume *norm* (*f x′ y* − *f x y* − (*fx*) (*x′* − *x*)) ≤ *norm* (*x′* − *x*) ∗ *e*
also
have *norm* ((*fy x y*) (*y′* − *y*) − (*fy x′ y*) (*y′* − *y*)) ≤ *norm* ((*fy x y*) − (*fy x′ y*)) ∗ *norm* (*y′* − *y*)
by (*auto simp*: *blinfun.bilinear_simps*[*symmetric*] *intro*!: *norm_blinfun*)
also have . . . ≤ (*e* + *e*) ∗ *norm* (*y′* − *y*)
using ‹*e* > *0*› *nfy*
by (*auto simp*: *norm_minus_commute intro*!: *mult_right_mono*)
also have *norm* (*x′* − *x*) ∗ *e* ≤ *norm* (*x′* − *x*) ∗ (*e* + *e*)
using ‹*0* < *e*› **by** *simp*
also have *norm* (*y′* − *y*) ∗ (*e* + *e*) + (*e* + *e*) ∗ *norm* (*y′* − *y*) + *norm* (*x′* − *x*) ∗ (*e* + *e*) ≤

(*norm* (*y'* − *y*) + *norm* (*x'* − *x*)) ∗ (*4* ∗ *e*)
using ‹*e* > *0*›
by (*simp add*: *algebra_simps*)
also have ... ≤ *2* ∗ *norm* ((*x'*, *y'*) − (*x*, *y*)) ∗ (*4* ∗ *e*)
using ‹*0* < *e*› *real_sqrt_sum_squares_ge1*[*of norm* (*x'* − *x*) *norm* (*y'* − *y*)]
real_sqrt_sum_squares_ge2[*of norm* (*y'* − *y*) *norm* (*x'* − *x*)]
by (*auto intro*!: *mult_right_mono simp*: *norm_prod_def*
simp del: *real_sqrt_sum_squares_ge1 real_sqrt_sum_squares_ge2*)
also have ... ≤ *norm* ((*x'*, *y'*) − (*x*, *y*)) ∗ (*8* ∗ *e*)
by *simp*
also have ... < *norm* ((*x'*, *y'*) − (*x*, *y*)) ∗ *e'*
using ‹*0* < *e'*› *nz*
by (*auto simp*: *e_def*)
finally show *norm* ((*f x' y'* − *f x y* − (*fx* (*x'* − *x*) + *fy x y* (*y'* − *y*))) /$_R$ *norm* ((*x'*, *y'*) − (*x*, *y*))) < *e'*
by (*simp add*: *dist_norm*) (*auto simp add*: *field_split_simps*)
qed
then show *?case*
by *eventually_elim* (*auto simp*: *dist_norm field_simps*)
next
from *has_derivative_bounded_linear*[*OF fx*]
obtain *fxb* where *fx* = *blinfun_apply fxb*
by (*metis bounded_linear_Blinfun_apply*)
then show *bounded_linear* (λ(*tx*, *ty*). *fx tx* + *blinfun_apply* (*fy x y*) *ty*)
by (*auto intro*!: *bounded_linear_intros simp*: *split_beta'*)
qed

4.10.20 Differentiable case distinction

lemma *has_derivative_within_If_eq*:
((λ*x*. *if P x then f x else g x*) *has_derivative f'*) (*at x within S*) =
(*bounded_linear f'* ∧
((λ*y*.(*if P y then* (*f y* − ((*if P x then f x else g x*) + *f'* (*y* − *x*)))/$_R$ *norm* (*y* − *x*)
else (*g y* − ((*if P x then f x else g x*) + *f'* (*y* − *x*)))/$_R$ *norm* (*y* − *x*)))
⟶ *0*) (*at x within S*))
(is _ = (_ ∧ (*?if* ⟶ *0*) _))
proof −
have (λ*y*. (*1* / *norm* (*y* − *x*)) ∗$_R$
((*if P y then f y else g y*) −
((*if P x then f x else g x*) + *f'* (*y* − *x*)))) = *?if*
by (*auto simp*: *inverse_eq_divide*)
thus *?thesis* by (*auto simp*: *has_derivative_within*)
qed

lemma *has_derivative_If_within_closures*:
assumes *f'*: *x* ∈ *S* ∪ (*closure S* ∩ *closure T*) ⟹
(*f has_derivative f' x*) (*at x within S* ∪ (*closure S* ∩ *closure T*))
assumes *g'*: *x* ∈ *T* ∪ (*closure S* ∩ *closure T*) ⟹

(*g has_derivative g' x*) (*at x within T* ∪ (*closure S* ∩ *closure T*))
assumes *connect*: $x \in closure\ S \Longrightarrow x \in closure\ T \Longrightarrow f\ x = g\ x$
assumes *connect'*: $x \in closure\ S \Longrightarrow x \in closure\ T \Longrightarrow f'\ x = g'\ x$
assumes *x_in*: $x \in S \cup T$
shows ((λ*x*. *if x* ∈ *S then f x else g x*) *has_derivative*
(*if x* ∈ *S then f' x else g' x*)) (*at x within* (*S* ∪ *T*))
proof −
from *f' x_in* **interpret** *f'*: *bounded_linear if x* ∈ *S then f' x else* (λ*x*. *0*)
by (*auto simp add*: *has_derivative_within*)
from *g'* **interpret** *g'*: *bounded_linear if x* ∈ *T then g' x else* (λ*x*. *0*)
by (*auto simp add*: *has_derivative_within*)
have *bl*: *bounded_linear* (*if x* ∈ *S then f' x else g' x*)
using *f'.scaleR f'.bounded f'.add g'.scaleR g'.bounded g'.add x_in*
by (*unfold_locales*; *force*)
show *?thesis*
using *f' g' closure_subset*[*of T*] *closure_subset*[*of S*]
unfolding *has_derivative_within_If_eq*
by (*intro conjI bl tendsto_If_within_closures x_in*)
(*auto simp*: *has_derivative_within inverse_eq_divide connect connect' subsetD*)
qed

lemma *has_vector_derivative_If_within_closures*:
assumes *x_in*: $x \in S \cup T$
assumes $u = S \cup T$
assumes *f'*: *x* ∈ *S* ∪ (*closure S* ∩ *closure T*) ⟹
(*f has_vector_derivative f' x*) (*at x within S* ∪ (*closure S* ∩ *closure T*))
assumes *g'*: *x* ∈ *T* ∪ (*closure S* ∩ *closure T*) ⟹
(*g has_vector_derivative g' x*) (*at x within T* ∪ (*closure S* ∩ *closure T*))
assumes *connect*: $x \in closure\ S \Longrightarrow x \in closure\ T \Longrightarrow f\ x = g\ x$
assumes *connect'*: $x \in closure\ S \Longrightarrow x \in closure\ T \Longrightarrow f'\ x = g'\ x$
shows ((λ*x*. *if x* ∈ *S then f x else g x*) *has_vector_derivative*
(*if x* ∈ *S then f' x else g' x*)) (*at x within u*)
unfolding *has_vector_derivative_def assms*
using *x_in*
apply (*intro has_derivative_If_within_closures*[**where** *?f'* = λ*x a*. $a *_R f'\ x$ **and** *?g'* = λ*x a*. $a *_R g'\ x$,
THEN has_derivative_eq_rhs])
subgoal by (*rule f'*[*unfolded has_vector_derivative_def*]; *assumption*)
subgoal by (*rule g'*[*unfolded has_vector_derivative_def*]; *assumption*)
by (*auto simp*: *assms*)

4.10.21 The Inverse Function Theorem

lemma *linear_injective_contraction*:
assumes *linear f c* < *1* **and** *le*: ⋀*x*. *norm* (*f x* − *x*) ≤ *c* ∗ *norm x*
shows *inj f*
unfolding *linear_injective_0*[*OF* ‹*linear f*›]
proof *safe*
fix *x*

```
  assume f x = 0
  with le [of x] have norm x ≤ c ∗ norm x
    by simp
  then show x = 0
    using ‹c < 1› by (simp add: mult_le_cancel_right1)
qed
```

From an online proof by J. Michael Boardman, Department of Mathematics, Johns Hopkins University

```
lemma inverse_function_theorem_scaled:
  fixes f::'a::euclidean_space ⇒ 'a
    and f'::'a ⇒ ('a ⇒L 'a)
  assumes open U
    and derf: ⋀x. x ∈ U ⟹ (f has_derivative blinfun_apply (f' x)) (at x)
    and contf: continuous_on U f'
    and 0 ∈ U and [simp]: f 0 = 0
    and id: f' 0 = id_blinfun
  obtains U' V g g' where open U' U' ⊆ U 0 ∈ U' open V 0 ∈ V homeomorphism
U' V f g
               ⋀y. y ∈ V ⟹ (g has_derivative (g' y)) (at y)
               ⋀y. y ∈ V ⟹ g' y = inv (blinfun_apply (f'(g y)))
               ⋀y. y ∈ V ⟹ bij (blinfun_apply (f'(g y)))
proof –
  obtain d1 where cball 0 d1 ⊆ U d1 > 0
    using ‹open U› ‹0 ∈ U› open_contains_cball by blast
  obtain d2 where d2: ⋀x. ⟦x ∈ U; dist x 0 ≤ d2⟧ ⟹ dist (f' x) (f' 0) < 1/2
0 < d2
    using continuous_onE [OF contf, of 0 1/2] by (metis ‹0 ∈ U› half_gt_zero_iff
zero_less_one)
  obtain δ where le: ⋀x. norm x ≤ δ ⟹ dist (f' x) id_blinfun ≤ 1/2 and 0 <
δ
    and subU: cball 0 δ ⊆ U
  proof
    show min d1 d2 > 0
      by (simp add: ‹0 < d1› ‹0 < d2›)
    show cball 0 (min d1 d2) ⊆ U
      using ‹cball 0 d1 ⊆ U› by auto
    show dist (f' x) id_blinfun ≤ 1/2 if norm x ≤ min d1 d2 for x
      using ‹cball 0 d1 ⊆ U› d2 that id by fastforce
  qed
  let ?D = cball 0 δ
  define V:: 'a set where V ≡ ball 0 (δ/2)
  have 4: norm (f (x + h) − f x − h) ≤ 1/2 ∗ norm h
    if x ∈ ?D x+h ∈ ?D for x h
  proof –
    let ?w = λx. f x − x
    have B: ⋀x. x ∈ ?D ⟹ onorm (blinfun_apply (f' x − id_blinfun)) ≤ 1/2
      by (metis dist_norm le mem_cball_0 norm_blinfun.rep_eq)
    have ⋀x. x ∈ ?D ⟹ (?w has_derivative (blinfun_apply (f' x − id_blinfun)))
```

(*at x*)
  by (*rule derivative_eq_intros derf subsetD* [*OF subU*] | *force simp*: *blinfun.diff_left*)+
  then have *Dw*: ⋀*x*. *x* ∈ *?D* ⟹ (*?w has_derivative* (*blinfun_apply* (*f′ x* − *id_blinfun*))) (*at x within ?D*)
    using *has_derivative_at_withinI* **by** *blast*
  have *norm* (*?w* (*x*+*h*) − *?w x*) ≤ (*1/2*) ∗ *norm h*
    using *differentiable_bound* [*OF convex_cball Dw B*] *that* **by** *fastforce*
  then show *?thesis*
    by (*auto simp*: *algebra_simps*)
 qed
 have *for_g*: ∃!*x*. *norm x* < δ ∧ *f x* = *y* **if** *y*: *norm y* < δ/*2* **for** *y*
 proof −
  let *?u* = λ*x*. *x* + (*y* − *f x*)
  have ∗: *norm* (*?u x*) < δ **if** *x* ∈ *?D* **for** *x*
  proof −
   have *fxx*: *norm* (*f x* − *x*) ≤ δ/*2*
    using *4* [*of 0 x*] ‹*0* < δ› ‹*f 0* = *0*› *that* **by** *auto*
   have *norm* (*?u x*) ≤ *norm y* + *norm* (*f x* − *x*)
   by (*metis add.commute add_diff_eq norm_minus_commute norm_triangle_ineq*)
   also have . . . < δ/*2* + δ/*2*
    using *fxx y* **by** *auto*
   finally show *?thesis*
    by *simp*
  qed
  have ∃!*x* ∈ *?D*. *?u x* = *x*
  proof (*rule banach_fix*)
   show *cball 0* δ ≠ {}
    using ‹*0* < δ› **by** *auto*
   show (λ*x*. *x* + (*y* − *f x*)) ‘ *cball 0* δ ⊆ *cball 0* δ
    using ∗ **by** *force*
   have *dist* (*x* + (*y* − *f x*)) (*xh* + (*y* − *f xh*)) ∗ *2* ≤ *dist x xh*
    if *norm x* ≤ δ **and** *norm xh* ≤ δ **for** *x xh*
    using *that 4* [*of x xh*−*x*] **by** (*auto simp*: *dist_norm norm_minus_commute algebra_simps*)
   then show ∀*x*∈*cball 0* δ. ∀*ya*∈*cball 0* δ. *dist* (*x* + (*y* − *f x*)) (*ya* + (*y* − *f ya*)) ≤ (*1/2*) ∗ *dist x ya*
    by *auto*
  qed (*auto simp*: *complete_eq_closed*)
  then show *?thesis*
   by (*metis* ∗ *add_cancel_right_right eq_iff_diff_eq_0 le_less mem_cball_0*)
 qed
 define *g* **where** *g* ≡ λ*y*. *THE x*. *norm x* < δ ∧ *f x* = *y*
 have *g*: *norm* (*g y*) < δ ∧ *f* (*g y*) = *y* **if** *norm y* < δ/*2* **for** *y*
  unfolding *g_def* **using** *that theI′* [*OF for_g*] **by** *meson*
 then have *fg*[*simp*]: *f* (*g y*) = *y* **if** *y* ∈ *V* **for** *y*
  using *that* **by** (*auto simp*: *V_def*)
 have *5*: *norm* (*g y′* − *g y*) ≤ *2* ∗ *norm* (*y′* − *y*) **if** *y* ∈ *V y′* ∈ *V* **for** *y y′*
 proof −

```
    have no: norm (g y) ≤ δ norm (g y') ≤ δ and [simp]: f (g y) = y
      using that g unfolding V_def by force+
    have norm (g y' − g y) ≤ norm (g y' − g y − (y' − y)) + norm (y' − y)
      by (simp add: add.commute norm_triangle_sub)
    also have ... ≤ (1/2) ∗ norm (g y' − g y) + norm (y' − y)
       using 4 [of g y g y' − g y] that no by (simp add: g norm_minus_commute
V_def)
    finally show ?thesis
      by auto
  qed
  have contg: continuous_on V g
  proof
    fix y::'a and e::real
    assume 0 < e and y: y ∈ V
    show ∃ d>0. ∀ x'∈V. dist x' y < d ⟶ dist (g x') (g y) ≤ e
    proof (intro exI conjI ballI impI)
      show 0 < e/2
        by (simp add: ‹0 < e›)
    qed (use 5 y in ‹force simp: dist_norm›)
  qed
  show thesis
  proof
    define U' where U' ≡ (f −‘ V) ∩ ball 0 δ
    have contf: continuous_on U f
     using derf has_derivative_at_withinI by (fast intro: has_derivative_continuous_on)
    then have continuous_on (ball 0 δ) f
      by (meson ball_subset_cball continuous_on_subset subU)
    then show open U'
      by (simp add: U'_def V_def Int_commute continuous_open_preimage)
    show 0 ∈ U' U' ⊆ U open V 0 ∈ V
      using ‹0 < δ› subU by (auto simp: U'_def V_def)
    show hom: homeomorphism U' V f g
    proof
      show continuous_on U' f
        using ‹U' ⊆ U› contf continuous_on_subset by blast
      show continuous_on V g
        using contg by blast
      show f ‘ U' ⊆ V
        using U'_def by blast
      show g ‘ V ⊆ U'
        by (simp add: U'_def V_def g image_subset_iff)
      show g (f x) = x if x ∈ U' for x
        by (metis that fg Int_iff U'_def V_def for_g g mem_ball_0 vimage_eq)
      show f (g y) = y if y ∈ V for y
        using that by (simp add: g V_def)
    qed
    show bij: bij (blinfun_apply (f'(g y))) if y ∈ V for y
    proof −
      have inj: inj (blinfun_apply (f' (g y)))
```

proof (*rule linear_injective_contraction*)
show *linear* (*blinfun_apply* (f' (*g y*)))
using *blinfun.bounded_linear_right bounded_linear_def* **by** *blast*
next
fix *x*
have *norm* (*blinfun_apply* (f' (*g y*)) *x* − *x*) = *norm* (*blinfun_apply* (f' (*g y*) − *id_blinfun*) *x*)
by (*simp add*: *blinfun.diff_left*)
also have ... ≤ *norm* (f' (*g y*) − *id_blinfun*) ∗ *norm x*
by (*rule norm_blinfun*)
also have ... ≤ (*1/2*) ∗ *norm x*
proof (*rule mult_right_mono*)
show *norm* (f' (*g y*) − *id_blinfun*) ≤ *1/2*
using *that g* [*of y*] *le* **by** (*auto simp*: *V_def dist_norm*)
qed *auto*
finally show *norm* (*blinfun_apply* (f' (*g y*)) *x* − *x*) ≤ (*1/2*) ∗ *norm x* **.**
qed *auto*
moreover
have *surj* (*blinfun_apply* (f' (*g y*)))
using *blinfun.bounded_linear_right bounded_linear_def*
by (*blast intro*!: *linear_inj_imp_surj* [*OF _ inj*])
ultimately show *?thesis*
using *bijI* **by** *blast*
qed
define g' **where** g' ≡ λ*y*. *inv* (*blinfun_apply* (f'(*g y*)))
show (*g has_derivative* g' *y*) (*at y*) **if** *y* ∈ *V* **for** *y*
proof −
have *gy*: *g y* ∈ *U*
using *g subU that* **unfolding** *V_def* **by** *fastforce*
obtain *e* **where** *e*: ⋀*h*. *f* (*g y* + *h*) = *y* + *blinfun_apply* (f' (*g y*)) *h* + *e h*
and *e0*: (λ*h*. *norm* (*e h*) / *norm h*) −*0*→ *0*
using *iffD1* [*OF has_derivative_iff_Ex derf* [*OF gy*]] ‹*y* ∈ *V*› **by** *auto*
have [*simp*]: *e 0* = *0*
using *e* [*of 0*] *that* **by** *simp*
let *?INV* = *inv* (*blinfun_apply* (f' (*g y*)))
have *inj*: *inj* (*blinfun_apply* (f' (*g y*)))
using *bij bij_betw_def that* **by** *blast*
have (*g has_derivative* g' *y*) (*at y within V*)
unfolding *has_derivative_at_within_iff_Ex* [*OF* ‹*y* ∈ *V*› ‹*open V*›]
proof
show *blinv*: *bounded_linear* (g' *y*)
unfolding g'*_def* **using** *derf gy inj inj_linear_imp_inv_bounded_linear* **by** *blast*
define *eg* **where** *eg* ≡ λ*k*. − *?INV* (*e* (*g* (*y*+*k*) − *g y*))
have *g* (*y*+*k*) = *g y* + g' *y k* + *eg k* **if** *y* + *k* ∈ *V* **for** *k*
proof −
have *?INV k* = *?INV* (*blinfun_apply* (f' (*g y*)) (*g* (*y*+*k*) − *g y*) + *e* (*g* (*y*+*k*) − *g y*))
using *e* [*of g*(*y*+*k*) − *g y*] *that* **by** *simp*

then have *g* (*y*+*k*) = *g y* + *?INV k* − *?INV* (*e* (*g* (*y*+*k*) − *g y*))
using *inj blinv* **by** (*simp add*: *linear_simps g′_def*)
then show *?thesis*
by (*auto simp*: *eg_def g′_def*)
qed
moreover have (λ*k*. *norm* (*eg k*) / *norm k*) −*0*→ *0*
proof (*rule Lim_null_comparison*)
let *?g* = λ*k*. *2* ∗ *onorm ?INV* ∗ *norm* (*e* (*g* (*y*+*k*) − *g y*)) / *norm* (*g* (*y*+*k*) − *g y*)
show ∀$_F$ *k in at 0*. *norm* (*norm* (*eg k*) / *norm k*) ≤ *?g k*
unfolding *eventually_at_topological*
proof (*intro exI conjI ballI impI*)
show *open* ((+)(−*y*) ‘ *V*)
using ‹*open V*› *open_translation* **by** *blast*
show *0* ∈ (+)(−*y*) ‘ *V*
by (*simp add*: *that*)
show *norm* (*norm* (*eg k*) / *norm k*) ≤ *2* ∗ *onorm* (*inv* (*blinfun_apply* (*f′* (*g y*)))) ∗ *norm* (*e* (*g* (*y*+*k*) − *g y*)) / *norm* (*g* (*y*+*k*) − *g y*)
if *k* ∈ (+)(−*y*) ‘ *V k* ≠ *0* **for** *k*
proof −
have *y*+*k* ∈ *V*
using *that* **by** *auto*
have *norm* (*norm* (*eg k*) / *norm k*) ≤ *onorm ?INV* ∗ *norm* (*e* (*g* (*y*+*k*) − *g y*)) / *norm k*
using *blinv g′_def onorm* **by** (*force simp*: *eg_def divide_simps*)
also have . . . = (*norm* (*g* (*y*+*k*) − *g y*) / *norm k*) ∗ (*onorm ?INV* ∗ (*norm* (*e* (*g* (*y*+*k*) − *g y*)) / *norm* (*g* (*y*+*k*) − *g y*)))
by (*simp add*: *divide_simps*)
also have . . . ≤ *2* ∗ (*onorm ?INV* ∗ (*norm* (*e* (*g* (*y*+*k*) − *g y*)) / *norm* (*g* (*y*+*k*) − *g y*)))
apply (*rule mult_right_mono*)
using *5* [*of y y*+*k*] ‹*y* ∈ *V*› ‹*y* + *k* ∈ *V*› *onorm_pos_le* [*OF blinv*]
apply (*auto simp*: *divide_simps zero_le_mult_iff zero_le_divide_iff g′_def*)
done
finally show *norm* (*norm* (*eg k*) / *norm k*) ≤ *2* ∗ *onorm ?INV* ∗ *norm* (*e* (*g* (*y*+*k*) − *g y*)) / *norm* (*g* (*y*+*k*) − *g y*)
by *simp*
qed
qed
have *1*: (λ*h*. *norm* (*e h*) / *norm h*) −*0*→ (*norm* (*e 0*) / *norm 0*)
using *e0* **by** *auto*
have *2*: (λ*k*. *g* (*y*+*k*) − *g y*) −*0*→ *0*
using *contg* ‹*open V*› ‹*y* ∈ *V*› *LIM_offset_zero_iff LIM_zero_iff at_within_open continuous_on_def* **by** *fastforce*
from *tendsto_compose* [*OF 1 2*, *simplified*]
have (λ*k*. *norm* (*e* (*g* (*y*+*k*) − *g y*)) / *norm* (*g* (*y*+*k*) − *g y*)) −*0*→ *0* **.**
from *tendsto_mult_left* [*OF this*] **show** *?g* −*0*→ *0* **by** *auto*
qed

ultimately show $\exists e. (\forall k.\ y + k \in V \longrightarrow g\ (y{+}k) = g\ y + g'\ y\ k + e\ k) \wedge (\lambda k.\ norm\ (e\ k)\ /\ norm\ k) -0\rightarrow 0$
by *blast*
qed
then show *?thesis*
by (*metis* ‹*open V*› *at_within_open that*)
qed
show $g'\ y = inv\ (blinfun\_apply\ (f'\ (g\ y)))$
if $y \in V$ **for** y
by (*simp add*: *g'_def*)
qed
qed

We need all this to justify the scaling and translations.

theorem *inverse_function_theorem*:
fixes f::*'a*::*euclidean_space* ⇒ *'a*
and f'::*'a* ⇒ (*'a* $\Rightarrow_L$ *'a*)
assumes *open U*
and *derf*: $\bigwedge x.\ x \in U \Longrightarrow (f\ has\_derivative\ (blinfun\_apply\ (f'\ x)))\ (at\ x)$
and *contf*: *continuous_on U f'*
and $x0 \in U$
and *invf*: $invf\ o_L\ f'\ x0 = id\_blinfun$
obtains $U'\ V\ g\ g'$ **where** $open\ U'\ U' \subseteq U\ x0 \in U'\ open\ V\ f\ x0 \in V$ *homeomorphism* $U'\ V\ f\ g$
$\bigwedge y.\ y \in V \Longrightarrow (g\ has\_derivative\ (g'\ y))\ (at\ y)$
$\bigwedge y.\ y \in V \Longrightarrow g'\ y = inv\ (blinfun\_apply\ (f'(g\ y)))$
$\bigwedge y.\ y \in V \Longrightarrow bij\ (blinfun\_apply\ (f'(g\ y)))$
proof −
have *apply1* [*simp*]: $\bigwedge i.\ blinfun\_apply\ invf\ (blinfun\_apply\ (f'\ x0)\ i) = i$
by (*metis blinfun_apply_blinfun_compose blinfun_apply_id_blinfun invf*)
have *apply2* [*simp*]: $\bigwedge i.\ blinfun\_apply\ (f'\ x0)\ (blinfun\_apply\ invf\ i) = i$
by (*metis apply1 bij_inv_eq_iff blinfun_bij1 invf*)
have [*simp*]: (*range* (*blinfun_apply invf*)) = *UNIV*
using *apply1 surjI* **by** *blast*
let $?f = invf \circ (\lambda x.\ (f \circ (+)x0)x - f\ x0)$
let $?f' = \lambda x.\ invf\ o_L\ (f'\ (x + x0))$
obtain $U'\ V\ g\ g'$ **where** *open* U' **and** U': $U' \subseteq (+)(-x0)$ ‘ $U\ 0 \in U'$
and *open* $V\ 0 \in V$ **and** *hom*: *homeomorphism* $U'\ V\ ?f\ g$
and *derg*: $\bigwedge y.\ y \in V \Longrightarrow (g\ has\_derivative\ (g'\ y))\ (at\ y)$
and g': $\bigwedge y.\ y \in V \Longrightarrow g'\ y = inv\ (?f'(g\ y))$
and *bij*: $\bigwedge y.\ y \in V \Longrightarrow bij\ (?f'(g\ y))$
proof (*rule inverse_function_theorem_scaled* [*of* (+)(−*x0*) ‘ *U ?f ?f'*])
show *ope*: *open* ((+) (− *x0*) ‘ *U*)
using ‹*open U*› *open_translation* **by** *blast*
show (*?f has_derivative blinfun_apply* (*?f' x*)) (*at x*)
if $x \in (+)\ (-\ x0)$ ‘ U **for** x
using *that*
apply *clarify*
apply (*rule derf derivative_eq_intros* | *simp add*: *blinfun_compose.rep_eq*)+

```
      done
    have YY: (λx. f' (x + x0)) −u−x0→ f' u
      if f' −u→ f' u u ∈ U for u
      using that LIM_offset [where k = x0] by (auto simp: algebra_simps)
    then have continuous_on ((+) (− x0) ‘ U) (λx. f' (x + x0))
      using contf ‹open U› Lim_at_imp_Lim_at_within
      by (fastforce simp: continuous_on_def at_within_open_NO_MATCH ope)
    then show continuous_on ((+) (− x0) ‘ U) ?f'
      by (intro continuous_intros) simp
  qed (auto simp: invf ‹x0 ∈ U›)
  show thesis
  proof
    let ?U' = (+)x0 ‘ U'
    let ?V = ((+)(f x0) ∘ f' x0) ‘ V
    let ?g = (+)x0 ∘ g ∘ invf ∘ (+)(− f x0)
    let ?g' = λy. inv (blinfun_apply (f' (?g y)))
    show oU': open ?U'
      by (simp add: ‹open U'› open_translation)
    show subU: ?U' ⊆ U
      using ComplI ‹U' ⊆ (+) (− x0) ‘ U› by auto
    show x0 ∈ ?U'
      by (simp add: ‹0 ∈ U'›)
    show open ?V
      using blinfun_bij2 [OF invf]
      by (metis ‹open V› bij_is_surj blinfun.bounded_linear_right bounded_linear_def
image_comp open_surjective_linear_image open_translation)
    show f x0 ∈ ?V
      using ‹0 ∈ V› image_iff by fastforce
    show homeomorphism ?U' ?V f ?g
    proof
      show continuous_on ?U' f
      by (meson subU continuous_on_eq_continuous_at derf has_derivative_continuous
oU' subsetD)
      have ?f ‘ U' ⊆ V
        using hom homeomorphism_image1 by blast
      then show f ‘ ?U' ⊆ ?V
        unfolding image_subset_iff
        by (clarsimp simp: image_def) (metis apply2 add.commute diff_add_cancel)
      show ?g ‘ ?V ⊆ ?U'
        using hom invf by (auto simp: image_def homeomorphism_def)
      show ?g (f x) = x
        if x ∈ ?U' for x
        using that hom homeomorphism_apply1 by fastforce
      have continuous_on V g
        using hom homeomorphism_def by blast
      then show continuous_on ?V ?g
        by (intro continuous_intros) (auto elim!: continuous_on_subset)
      have fg: ?f (g x) = x if x ∈ V for x
        using hom homeomorphism_apply2 that by blast
```

```
    show f (?g y) = y
      if y ∈ ?V for y
    using that fg by (simp add: image_iff) (metis apply2 add.commute diff_add_cancel)
  qed
  show (?g has_derivative ?g' y) (at y) bij (blinfun_apply (f' (?g y)))
    if y ∈ ?V for y
  proof -
    have 1: bij (blinfun_apply invf)
      using blinfun_bij1 invf by blast
    then have 2: bij (blinfun_apply (f' (x0 + g x))) if x ∈ V for x
       by (metis add.commute bij bij_betw_comp_iff2 blinfun_compose.rep_eq that
top_greatest)
    then show bij (blinfun_apply (f' (?g y)))
      using that by auto
    have g' x ∘ blinfun_apply invf = inv (blinfun_apply (f' (x0 + g x)))
      if x ∈ V for x
      using that
        by (simp add: g' o_inv_distrib blinfun_compose.rep_eq 1 2 add.commute
bij_is_inj flip: o_assoc)
    then show (?g has_derivative ?g' y) (at y)
      using that invf
      by clarsimp (rule derg derivative_eq_intros | simp flip: id_def)+
  qed
 qed auto
qed
```

4.10.22 Piecewise differentiable functions

```
definition piecewise_differentiable_on
          (infixr piecewise'_differentiable'_on 50)
  where f piecewise_differentiable_on i ≡
          continuous_on i f ∧
          (∃ S. finite S ∧ (∀ x ∈ i − S. f differentiable (at x within i)))

lemma piecewise_differentiable_on_imp_continuous_on:
    f piecewise_differentiable_on S ⟹ continuous_on S f
by (simp add: piecewise_differentiable_on_def)

lemma piecewise_differentiable_on_subset:
    f piecewise_differentiable_on S ⟹ T ≤ S ⟹ f piecewise_differentiable_on T
  using continuous_on_subset
  unfolding piecewise_differentiable_on_def
  apply safe
  apply (blast elim: continuous_on_subset)
  by (meson Diff_iff differentiable_within_subset subsetCE)

lemma differentiable_on_imp_piecewise_differentiable:
  fixes a:: 'a::{linorder_topology,real_normed_vector}
  shows f differentiable_on {a..b} ⟹ f piecewise_differentiable_on {a..b}
```

apply (*simp add*: *piecewise_differentiable_on_def differentiable_imp_continuous_on*)
apply (*rule_tac x*={*a*,*b*} **in** *exI*, *simp add*: *differentiable_on_def*)
done

lemma *differentiable_imp_piecewise_differentiable*:
($\bigwedge x.\ x \in S \Longrightarrow$ *f differentiable* (*at x within S*))
$\Longrightarrow$ *f piecewise_differentiable_on S*
by (*auto simp*: *piecewise_differentiable_on_def differentiable_imp_continuous_on differentiable_on_def*
intro: *differentiable_within_subset*)

lemma *piecewise_differentiable_const* [*iff*]: ($\lambda x.\ z$) *piecewise_differentiable_on S*
by (*simp add*: *differentiable_imp_piecewise_differentiable*)

lemma *piecewise_differentiable_compose*:
⟦*f piecewise_differentiable_on S*; *g piecewise_differentiable_on* (*f* ‘ *S*);
$\bigwedge x$. *finite* ($S \cap f-$‘{x})⟧
$\Longrightarrow$ ($g \circ f$) *piecewise_differentiable_on S*
apply (*simp add*: *piecewise_differentiable_on_def*, *safe*)
apply (*blast intro*: *continuous_on_compose2*)
apply (*rename_tac A B*)
apply (*rule_tac x*=$A \cup (\bigcup x \in B.\ S \cap f-$‘{$x$}) **in** *exI*)
apply (*blast intro*!: *differentiable_chain_within*)
done

lemma *piecewise_differentiable_affine*:
fixes *m*::*real*
assumes *f piecewise_differentiable_on* (($\lambda x.\ m *_R x + c$) ‘ *S*)
shows ($f \circ (\lambda x.\ m *_R x + c)$) *piecewise_differentiable_on S*
proof (*cases m* = *0*)
case *True*
then show *?thesis*
unfolding *o_def*
by (*force intro*: *differentiable_imp_piecewise_differentiable differentiable_const*)
next
case *False*
show *?thesis*
apply (*rule piecewise_differentiable_compose* [*OF differentiable_imp_piecewise_differentiable*])
apply (*rule assms derivative_intros* | *simp add*: *False vimage_def real_vector_affinity_eq*)+
done
qed

lemma *piecewise_differentiable_cases*:
fixes *c*::*real*
assumes *f piecewise_differentiable_on* {*a*..*c*}
g piecewise_differentiable_on {*c*..*b*}
$a \le c$ $c \le b$ $f\ c = g\ c$
shows (λx. *if* $x \le c$ *then f x else g x*) *piecewise_differentiable_on* {*a*..*b*}
proof −

obtain *S T* **where** *st*: *finite S finite T*
  and *fd*: $\bigwedge x.\ x \in \{a..c\} - S \Longrightarrow$ *f differentiable at x within* $\{a..c\}$
  and *gd*: $\bigwedge x.\ x \in \{c..b\} - T \Longrightarrow$ *g differentiable at x within* $\{c..b\}$
 using *assms*
 by (*auto simp*: *piecewise_differentiable_on_def*)
have *finabc*: *finite* $(\{a,b,c\} \cup (S \cup T))$
 by (*metis* ‹*finite S*› ‹*finite T*› *finite_Un finite_insert finite.emptyI*)
have *continuous_on* $\{a..c\}$ *f continuous_on* $\{c..b\}$ *g*
 using *assms piecewise_differentiable_on_def* **by** *auto*
then have *continuous_on* $\{a..b\}$ ($\lambda x.$ *if* $x \le c$ *then f x else g x*)
 using *continuous_on_cases* [*OF closed_real_atLeastAtMost* [*of a c*],
  OF closed_real_atLeastAtMost [*of c b*],
  of f g $\lambda x.\ x \le c$] *assms*
 by (*force simp*: *ivl_disj_un_two_touch*)
moreover
{ fix *x*
 assume *x*: $x \in \{a..b\} - (\{a,b,c\} \cup (S \cup T))$
 have ($\lambda x.$ *if* $x \le c$ *then f x else g x*) *differentiable at x within* $\{a..b\}$ (**is** *?diff_fg*)
 proof (*cases x c rule*: *le_cases*)
  case *le* **show** *?diff_fg*
  proof (*rule differentiable_transform_within* [**where** $d = dist\ x\ c$])
   have *f differentiable at x*
    using *x le fd* [*of x*] *at_within_interior* [*of x* $\{a..c\}$] **by** *simp*
   then show *f differentiable at x within* $\{a..b\}$
    by (*simp add*: *differentiable_at_withinI*)
  qed (*use x le st dist_real_def* **in** *auto*)
 next
  case *ge* **show** *?diff_fg*
  proof (*rule differentiable_transform_within* [**where** $d = dist\ x\ c$])
   have *g differentiable at x*
    using *x ge gd* [*of x*] *at_within_interior* [*of x* $\{c..b\}$] **by** *simp*
   then show *g differentiable at x within* $\{a..b\}$
    by (*simp add*: *differentiable_at_withinI*)
  qed (*use x ge st dist_real_def* **in** *auto*)
 qed
}
then have $\exists S.$ *finite S* $\wedge$
  ($\forall x \in \{a..b\} - S.$ ($\lambda x.$ *if* $x \le c$ *then f x else g x*) *differentiable at x within* $\{a..b\}$)
 by (*meson finabc*)
ultimately show *?thesis*
 by (*simp add*: *piecewise_differentiable_on_def*)
qed

lemma *piecewise_differentiable_neg*:
 f piecewise_differentiable_on S $\Longrightarrow$ ($\lambda x.\ -(f\ x)$) *piecewise_differentiable_on S*
 by (*auto simp*: *piecewise_differentiable_on_def continuous_on_minus*)

lemma *piecewise_differentiable_add*:

assumes *f piecewise_differentiable_on i*
g piecewise_differentiable_on i
shows $(\lambda x.\ f\ x + g\ x)$ *piecewise_differentiable_on i*
proof –
obtain *S T* **where** *st*: *finite S finite T*
$\forall x \in i - S.$ *f differentiable at x within i*
$\forall x \in i - T.$ *g differentiable at x within i*
using *assms* **by** (*auto simp*: *piecewise_differentiable_on_def*)
then have *finite* $(S \cup T) \wedge (\forall x \in i - (S \cup T).\ (\lambda x.\ f\ x + g\ x)$ *differentiable at x within i*)
by *auto*
moreover have *continuous_on i f continuous_on i g*
using *assms piecewise_differentiable_on_def* **by** *auto*
ultimately show *?thesis*
by (*auto simp*: *piecewise_differentiable_on_def continuous_on_add*)
qed

lemma *piecewise_differentiable_diff*:
⟦*f piecewise_differentiable_on S*; *g piecewise_differentiable_on S*⟧
$\Longrightarrow (\lambda x.\ f\ x - g\ x)$ *piecewise_differentiable_on S*
unfolding *diff_conv_add_uminus*
by (*metis piecewise_differentiable_add piecewise_differentiable_neg*)

4.10.23 The concept of continuously differentiable

John Harrison writes as follows:

"The usual assumption in complex analysis texts is that a path γ should be piecewise continuously differentiable, which ensures that the path integral exists at least for any continuous f, since all piecewise continuous functions are integrable. However, our notion of validity is weaker, just piecewise differentiability. . . [namely] continuity plus differentiability except on a finite set. . . [Our] underlying theory of integration is the Kurzweil-Henstock theory. In contrast to the Riemann or Lebesgue theory (but in common with a simple notion based on antiderivatives), this can integrate all derivatives."

"Formalizing basic complex analysis." From Insight to Proof: Festschrift in Honour of Andrzej Trybulec. Studies in Logic, Grammar and Rhetoric 10.23 (2007): 151-165.

And indeed he does not assume that his derivatives are continuous, but the penalty is unreasonably difficult proofs concerning winding numbers. We need a self-contained and straightforward theorem asserting that all derivatives can be integrated before we can adopt Harrison's choice.

definition *C1_differentiable_on* :: (*real* ⇒ *'a*::*real_normed_vector*) ⇒ *real set* ⇒ *bool*
(**infix** *C1'_differentiable'_on 50*)
where
f C1_differentiable_on S ⟷

(∃ *D*. (∀ *x* ∈ *S*. (*f has_vector_derivative* (*D x*)) (*at x*)) ∧ *continuous_on S D*)

lemma *C1_differentiable_on_eq*:
f C1_differentiable_on S ⟷
(∀ *x* ∈ *S*. *f differentiable at x*) ∧ *continuous_on S* (λ*x*. *vector_derivative f* (*at x*))
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
unfolding *C1_differentiable_on_def*
by (*metis* (*no_types*, *lifting*) *continuous_on_eq differentiableI_vector vector_derivative_at*)
next
assume *?rhs*
then show *?lhs*
using *C1_differentiable_on_def vector_derivative_works* **by** *fastforce*
qed

lemma *C1_differentiable_on_subset*:
f C1_differentiable_on T ⟹ *S* ⊆ *T* ⟹ *f C1_differentiable_on S*
unfolding *C1_differentiable_on_def continuous_on_eq_continuous_within*
by (*blast intro*: *continuous_within_subset*)

lemma *C1_differentiable_compose*:
assumes *fg*: *f C1_differentiable_on S g C1_differentiable_on* (*f* ' *S*) **and** *fin*: ⋀*x*. *finite* (*S* ∩ *f*−'{*x*})
shows (*g* ∘ *f*) *C1_differentiable_on S*
proof −
have ⋀*x*. *x* ∈ *S* ⟹ *g* ∘ *f differentiable at x*
by (*meson C1_differentiable_on_eq assms differentiable_chain_at imageI*)
moreover have *continuous_on S* (λ*x*. *vector_derivative* (*g* ∘ *f*) (*at x*))
proof (*rule continuous_on_eq* [*of* _ λ*x*. *vector_derivative f* (*at x*) $*_R$ *vector_derivative g* (*at* (*f x*))])
show *continuous_on S* (λ*x*. *vector_derivative f* (*at x*) $*_R$ *vector_derivative g* (*at* (*f x*)))
using *fg*
apply (*clarsimp simp add*: *C1_differentiable_on_eq*)
apply (*rule Limits.continuous_on_scaleR*, *assumption*)
by (*metis* (*mono_tags*, *lifting*) *continuous_at_imp_continuous_on continuous_on_compose continuous_on_cong differentiable_imp_continuous_within o_def*)
show ⋀*x*. *x* ∈ *S* ⟹ *vector_derivative f* (*at x*) $*_R$ *vector_derivative g* (*at* (*f x*)) = *vector_derivative* (*g* ∘ *f*) (*at x*)
by (*metis* (*mono_tags*, *hide_lams*) *C1_differentiable_on_eq fg imageI vector_derivative_chain_at*)
qed
ultimately show *?thesis*
by (*simp add*: *C1_differentiable_on_eq*)
qed

lemma *C1_diff_imp_diff*: *f C1_differentiable_on S* ⟹ *f differentiable_on S*
by (*simp add*: *C1_differentiable_on_eq differentiable_at_imp_differentiable_on*)

lemma *C1_differentiable_on_ident* [*simp*, *derivative_intros*]: (λ*x*. *x*) *C1_differentiable_on S*
by (*auto simp*: *C1_differentiable_on_eq*)

lemma *C1_differentiable_on_const* [*simp*, *derivative_intros*]: (λ*z*. *a*) *C1_differentiable_on S*
by (*auto simp*: *C1_differentiable_on_eq*)

lemma *C1_differentiable_on_add* [*simp*, *derivative_intros*]:
f C1_differentiable_on S ⟹ *g C1_differentiable_on S* ⟹ (λ*x*. *f x* + *g x*) *C1_differentiable_on S*
unfolding *C1_differentiable_on_eq* **by** (*auto intro*: *continuous_intros*)

lemma *C1_differentiable_on_minus* [*simp*, *derivative_intros*]:
f C1_differentiable_on S ⟹ (λ*x*. − *f x*) *C1_differentiable_on S*
unfolding *C1_differentiable_on_eq* **by** (*auto intro*: *continuous_intros*)

lemma *C1_differentiable_on_diff* [*simp*, *derivative_intros*]:
f C1_differentiable_on S ⟹ *g C1_differentiable_on S* ⟹ (λ*x*. *f x* − *g x*) *C1_differentiable_on S*
unfolding *C1_differentiable_on_eq* **by** (*auto intro*: *continuous_intros*)

lemma *C1_differentiable_on_mult* [*simp*, *derivative_intros*]:
fixes *f g* :: *real* ⇒ ′*a* :: *real_normed_algebra*
shows *f C1_differentiable_on S* ⟹ *g C1_differentiable_on S* ⟹ (λ*x*. *f x* ∗ *g x*) *C1_differentiable_on S*
unfolding *C1_differentiable_on_eq*
by (*auto simp*: *continuous_on_add continuous_on_mult continuous_at_imp_continuous_on differentiable_imp_continuous_within*)

lemma *C1_differentiable_on_scaleR* [*simp*, *derivative_intros*]:
f C1_differentiable_on S ⟹ *g C1_differentiable_on S* ⟹ (λ*x*. *f x* $*_R$ *g x*) *C1_differentiable_on S*
unfolding *C1_differentiable_on_eq*
by (*rule continuous_intros* | *simp add*: *continuous_at_imp_continuous_on differentiable_imp_continuous_within*)+

definition *piecewise_C1_differentiable_on*
(**infixr** *piecewise′_C1′_differentiable′_on 50*)
where *f piecewise_C1_differentiable_on i* ≡
continuous_on i f ∧
(∃ *S*. *finite S* ∧ (*f C1_differentiable_on* (*i* − *S*)))

lemma *C1_differentiable_imp_piecewise*:
f C1_differentiable_on S ⟹ *f piecewise_C1_differentiable_on S*

by (*auto simp*: *piecewise_C1_differentiable_on_def C1_differentiable_on_eq continuous_at_imp_continuous_on differentiable_imp_continuous_within*)

lemma *piecewise_C1_imp_differentiable*:
f piecewise_C1_differentiable_on i $\Longrightarrow$ *f piecewise_differentiable_on i*
by (*auto simp*: *piecewise_C1_differentiable_on_def piecewise_differentiable_on_def C1_differentiable_on_def differentiable_def has_vector_derivative_def intro*: *has_derivative_at_withinI*)

lemma *piecewise_C1_differentiable_compose*:
assumes *fg*: *f piecewise_C1_differentiable_on S g piecewise_C1_differentiable_on* (*f* ' *S*) **and** *fin*: $\bigwedge x$. *finite* ($S \cap f-$'$\{x\}$)
shows ($g \circ f$) *piecewise_C1_differentiable_on S*
proof −
have *continuous_on S* (λx. *g* (*f x*))
by (*metis continuous_on_compose2 fg order_refl piecewise_C1_differentiable_on_def*)
moreover have $\exists$ *T*. *finite T* $\wedge$ $g \circ f$ *C1_differentiable_on S* − *T*
proof −
obtain *F* **where** *finite F* **and** *F*: *f C1_differentiable_on S* − *F* **and** *f*: *f piecewise_C1_differentiable_on S*
using *fg* **by** (*auto simp*: *piecewise_C1_differentiable_on_def*)
obtain *G* **where** *finite G* **and** *G*: *g C1_differentiable_on f* ' *S* − *G* **and** *g*: *g piecewise_C1_differentiable_on f* ' *S*
using *fg* **by** (*auto simp*: *piecewise_C1_differentiable_on_def*)
show *?thesis*
proof (*intro exI conjI*)
show *finite* ($F \cup (\bigcup x \in G.\ S \cap f-$'$\{x\}$))
using *fin* **by** (*auto simp only*: *Int_Union* ‹*finite F*› ‹*finite G*› *finite_UN finite_imageI*)
show $g \circ f$ *C1_differentiable_on S* − ($F \cup (\bigcup x \in G.\ S \cap f$ −' $\{x\}$))
apply (*rule C1_differentiable_compose*)
apply (*blast intro*: *C1_differentiable_on_subset* [*OF F*])
apply (*blast intro*: *C1_differentiable_on_subset* [*OF G*])
by (*simp add*: *C1_differentiable_on_subset G Diff_Int_distrib2 fin*)
qed
qed
ultimately show *?thesis*
by (*simp add*: *piecewise_C1_differentiable_on_def*)
qed

lemma *piecewise_C1_differentiable_on_subset*:
f piecewise_C1_differentiable_on S $\Longrightarrow$ $T \le S$ $\Longrightarrow$ *f piecewise_C1_differentiable_on T*
by (*auto simp*: *piecewise_C1_differentiable_on_def elim*!: *continuous_on_subset C1_differentiable_on_subse*

lemma *C1_differentiable_imp_continuous_on*:
f C1_differentiable_on S $\Longrightarrow$ *continuous_on S f*
unfolding *C1_differentiable_on_eq continuous_on_eq_continuous_within*
using *differentiable_at_withinI differentiable_imp_continuous_within* **by** *blast*

```
lemma C1_differentiable_on_empty [iff]: f C1_differentiable_on {}
  unfolding C1_differentiable_on_def
  by auto

lemma piecewise_C1_differentiable_affine:
  fixes m::real
  assumes f piecewise_C1_differentiable_on ((λx. m * x + c) ` S)
  shows (f ∘ (λx. m *R x + c)) piecewise_C1_differentiable_on S
proof (cases m = 0)
  case True
  then show ?thesis
    unfolding o_def by (auto simp: piecewise_C1_differentiable_on_def)
next
  case False
  have *: ⋀x. finite (S ∩ {y. m * y + c = x})
    using False not_finite_existsD by fastforce
  show ?thesis
   apply (rule piecewise_C1_differentiable_compose [OF C1_differentiable_imp_piecewise])
    apply (rule * assms derivative_intros | simp add: False vimage_def)+
    done
qed

lemma piecewise_C1_differentiable_cases:
  fixes c::real
  assumes f piecewise_C1_differentiable_on {a..c}
          g piecewise_C1_differentiable_on {c..b}
           a ≤ c c ≤ b f c = g c
  shows (λx. if x ≤ c then f x else g x) piecewise_C1_differentiable_on {a..b}
proof −
  obtain S T where st: f C1_differentiable_on ({a..c} − S)
                       g C1_differentiable_on ({c..b} − T)
                       finite S finite T
    using assms
    by (force simp: piecewise_C1_differentiable_on_def)
  then have f_diff: f differentiable_on {a..<c} − S
      and g_diff: g differentiable_on {c<..b} − T
   by (simp_all add: C1_differentiable_on_eq differentiable_at_withinI differentiable_on_def)
  have continuous_on {a..c} f continuous_on {c..b} g
    using assms piecewise_C1_differentiable_on_def by auto
  then have cab: continuous_on {a..b} (λx. if x ≤ c then f x else g x)
    using continuous_on_cases [OF closed_real_atLeastAtMost [of a c],
                               OF closed_real_atLeastAtMost [of c b],
                               of f g λx. x≤c]  assms
    by (force simp: ivl_disj_un_two_touch)
  { fix x
    assume x: x ∈ {a..b} − insert c (S ∪ T)
    have (λx. if x ≤ c then f x else g x) differentiable at x (is ?diff_fg)
    proof (cases x c rule: le_cases)
```

case *le* **show** *?diff_fg*
apply (*rule differentiable_transform_within* [**where** *f*=*f* **and** *d* = *dist x c*])
using *x dist_real_def le st* **by** (*auto simp*: *C1_differentiable_on_eq*)
next
case *ge* **show** *?diff_fg*
apply (*rule differentiable_transform_within* [**where** *f*=*g* **and** *d* = *dist x c*])
using *dist_nz x dist_real_def ge st x* **by** (*auto simp*: *C1_differentiable_on_eq*)
qed
}
then have (∀ *x* ∈ {*a*..*b*} − *insert c* (*S* ∪ *T*). (λ*x*. *if x* ≤ *c then f x else g x*) *differentiable at x*)
by *auto*
moreover
{ assume *fcon*: *continuous_on* ({*a*<..<*c*} − *S*) (λ*x*. *vector_derivative f* (*at x*))
and *gcon*: *continuous_on* ({*c*<..<*b*} − *T*) (λ*x*. *vector_derivative g* (*at x*))
have *open* ({*a*<..<*c*} − *S*) *open* ({*c*<..<*b*} − *T*)
using *st* **by** (*simp_all add*: *open_Diff finite_imp_closed*)
moreover have *continuous_on* ({*a*<..<*c*} − *S*) (λ*x*. *vector_derivative* (λ*x*. *if x* ≤ *c then f x else g x*) (*at x*))
proof −
have ((λ*x*. *if x* ≤ *c then f x else g x*) *has_vector_derivative vector_derivative f* (*at x*)) (*at x*)
if *a* < *x x* < *c x* ∉ *S* **for** *x*
proof −
have *f*: *f differentiable at x*
by (*meson C1_differentiable_on_eq Diff_iff atLeastAtMost_iff less_eq_real_def st*(*1*) *that*)
show *?thesis*
using *that*
apply (*rule_tac f*=*f* **and** *d*=*dist x c* **in** *has_vector_derivative_transform_within*)
apply (*auto simp*: *dist_norm vector_derivative_works* [*symmetric*] *f*)
done
qed
then show *?thesis*
by (*metis* (*no_types*, *lifting*) *continuous_on_eq* [*OF fcon*] *DiffE greaterThanLessThan_iff vector_derivative_at*)
qed
moreover have *continuous_on* ({*c*<..<*b*} − *T*) (λ*x*. *vector_derivative* (λ*x*. *if x* ≤ *c then f x else g x*) (*at x*))
proof −
have ((λ*x*. *if x* ≤ *c then f x else g x*) *has_vector_derivative vector_derivative g* (*at x*)) (*at x*)
if *c* < *x x* < *b x* ∉ *T* **for** *x*
proof −
have *g*: *g differentiable at x*
by (*metis C1_differentiable_on_eq DiffD1 DiffI atLeastAtMost_diff_ends greaterThanLessThan_iff st*(*2*) *that*)
show *?thesis*
using *that*

apply (*rule_tac f=g* **and** *d=dist x c* **in** *has_vector_derivative_transform_within*)
apply (*auto simp*: *dist_norm vector_derivative_works* [*symmetric*] *g*)
done
qed
then show *?thesis*
by (*metis* (*no_types*, *lifting*) *continuous_on_eq* [*OF gcon*] *DiffE greaterThanLessThan_iff vector_derivative_at*)
qed
ultimately have *continuous_on* ({*a*<..<*b*} − *insert c* (*S* ∪ *T*))
(λ*x*. *vector_derivative* (λ*x*. *if x* ≤ *c then f x else g x*) (*at x*))
by (*rule continuous_on_subset* [*OF continuous_on_open_Un*], *auto*)
} **note** ∗ = *this*
have *continuous_on* ({*a*<..<*b*} − *insert c* (*S* ∪ *T*)) (λ*x*. *vector_derivative* (λ*x*. *if x* ≤ *c then f x else g x*) (*at x*))
using *st*
by (*auto simp*: *C1_differentiable_on_eq elim*!: *continuous_on_subset intro*: ∗)
ultimately have ∃ *S*. *finite S* ∧ ((λ*x*. *if x* ≤ *c then f x else g x*) *C1_differentiable_on* {*a*..*b*} − *S*)
apply (*rule_tac x*={*a*,*b*,*c*} ∪ *S* ∪ *T* **in** *exI*)
using *st* **by** (*auto simp*: *C1_differentiable_on_eq elim*!: *continuous_on_subset*)
with *cab* **show** *?thesis*
by (*simp add*: *piecewise_C1_differentiable_on_def*)
qed

lemma *piecewise_C1_differentiable_neg*:
f piecewise_C1_differentiable_on S ⟹ (λ*x*. −(*f x*)) *piecewise_C1_differentiable_on S*
unfolding *piecewise_C1_differentiable_on_def*
by (*auto intro*!: *continuous_on_minus C1_differentiable_on_minus*)

lemma *piecewise_C1_differentiable_add*:
assumes *f piecewise_C1_differentiable_on i*
g piecewise_C1_differentiable_on i
shows (λ*x*. *f x* + *g x*) *piecewise_C1_differentiable_on i*
proof −
obtain *S t* **where** *st*: *finite S finite t*
f C1_differentiable_on (*i*−*S*)
g C1_differentiable_on (*i*−*t*)
using *assms* **by** (*auto simp*: *piecewise_C1_differentiable_on_def*)
then have *finite* (*S* ∪ *t*) ∧ (λ*x*. *f x* + *g x*) *C1_differentiable_on i* − (*S* ∪ *t*)
by (*auto intro*: *C1_differentiable_on_add elim*!: *C1_differentiable_on_subset*)
moreover have *continuous_on i f continuous_on i g*
using *assms piecewise_C1_differentiable_on_def* **by** *auto*
ultimately show *?thesis*
by (*auto simp*: *piecewise_C1_differentiable_on_def continuous_on_add*)
qed

lemma *piecewise_C1_differentiable_diff*:
⟦*f piecewise_C1_differentiable_on S*; *g piecewise_C1_differentiable_on S*⟧

```
    ⟹ (λx. f x − g x) piecewise_C1_differentiable_on S
  unfolding diff_conv_add_uminus
  by (metis piecewise_C1_differentiable_add piecewise_C1_differentiable_neg)

end
```

4.11 Finite Cartesian Products of Euclidean Spaces

```
theory Cartesian_Euclidean_Space
imports Derivative
begin

lemma subspace_special_hyperplane: subspace {x. x $ k = 0}
  by (simp add: subspace_def)

lemma sum_mult_product:
  sum h {..<A ∗ B :: nat} = (∑ i∈{..<A}. ∑ j∈{..<B}. h (j + i ∗ B))
  unfolding sum.nat_group[of h B A, unfolded atLeast0LessThan, symmetric]
proof (rule sum.cong, simp, rule sum.reindex_cong)
  fix i
  show inj_on (λj. j + i ∗ B) {..<B} by (auto intro!: inj_onI)
  show {i ∗ B..<i ∗ B + B} = (λj. j + i ∗ B) ‘ {..<B}
  proof safe
    fix j assume j ∈ {i ∗ B..<i ∗ B + B}
    then show j ∈ (λj. j + i ∗ B) ‘ {..<B}
      by (auto intro!: image_eqI[of _ _ j − i ∗ B])
  qed simp
qed simp

lemma interval_cbox_cart: {a::real^'n..b} = cbox a b
  by (auto simp add: less_eq_vec_def mem_box Basis_vec_def inner_axis)

lemma differentiable_vec:
  fixes S :: 'a::euclidean_space set
  shows vec differentiable_on S
  by (simp add: linear_linear bounded_linear_imp_differentiable_on)

lemma continuous_vec [continuous_intros]:
  fixes x :: 'a::euclidean_space
  shows isCont vec x
  apply (clarsimp simp add: continuous_def LIM_def dist_vec_def L2_set_def)
  apply (rule_tac x=r / sqrt (real CARD('b)) in exI)
  by (simp add: mult.commute pos_less_divide_eq real_sqrt_mult)

lemma box_vec_eq_empty [simp]:
  shows cbox (vec a) (vec b) = {} ⟷ cbox a b = {}
        box (vec a) (vec b) = {} ⟷ box a b = {}
  by (auto simp: Basis_vec_def mem_box box_eq_empty inner_axis)
```

4.11.1 Closures and interiors of halfspaces

```
lemma interior_halfspace_component_le [simp]:
    interior {x. x$k ≤ a} = {x :: (real^'n). x$k < a} (is ?LE)
  and interior_halfspace_component_ge [simp]:
    interior {x. x$k ≥ a} = {x :: (real^'n). x$k > a} (is ?GE)
proof -
  have axis k (1::real) ≠ 0
    by (simp add: axis_def vec_eq_iff)
  moreover have axis k (1::real) · x = x$k for x
    by (simp add: cart_eq_inner_axis inner_commute)
  ultimately show ?LE ?GE
    using interior_halfspace_le [of axis k (1::real) a]
          interior_halfspace_ge [of axis k (1::real) a] by auto
qed

lemma closure_halfspace_component_lt [simp]:
    closure {x. x$k < a} = {x :: (real^'n). x$k ≤ a} (is ?LE)
  and closure_halfspace_component_gt [simp]:
    closure {x. x$k > a} = {x :: (real^'n). x$k ≥ a} (is ?GE)
proof -
  have axis k (1::real) ≠ 0
    by (simp add: axis_def vec_eq_iff)
  moreover have axis k (1::real) · x = x$k for x
    by (simp add: cart_eq_inner_axis inner_commute)
  ultimately show ?LE ?GE
    using closure_halfspace_lt [of axis k (1::real) a]
          closure_halfspace_gt [of axis k (1::real) a] by auto
qed

lemma interior_standard_hyperplane:
   interior {x :: (real^'n). x$k = a} = {}
proof -
  have axis k (1::real) ≠ 0
    by (simp add: axis_def vec_eq_iff)
  moreover have axis k (1::real) · x = x$k for x
    by (simp add: cart_eq_inner_axis inner_commute)
  ultimately show ?thesis
    using interior_hyperplane [of axis k (1::real) a]
    by force
qed

lemma matrix_vector_mul_bounded_linear[intro, simp]: bounded_linear ((*v) A) for
A :: 'a::{euclidean_space,real_algebra_1}^'n^'m
  using matrix_vector_mul_linear[of A]
  by (simp add: linear_conv_bounded_linear linear_matrix_vector_mul_eq)

lemma
  fixes A :: 'a::{euclidean_space,real_algebra_1}^'n^'m
  shows matrix_vector_mult_linear_continuous_at [continuous_intros]: isCont ((*v)
```

A) *z*
and *matrix_vector_mult_linear_continuous_on* [*continuous_intros*]: *continuous_on S* ((*v) *A*)
by (*simp_all add*: *linear_continuous_at linear_continuous_on*)

4.11.2 Bounds on components etc. relative to operator norm

lemma *norm_column_le_onorm*:
fixes *A* :: *real^'n^'m*
shows *norm*(*column i A*) ≤ *onorm*((*v) *A*)
proof −
have *norm* (χ *j*. *A* $ *j* $ *i*) ≤ *norm* (*A* *v *axis i 1*)
by (*simp add*: *matrix_mult_dot cart_eq_inner_axis*)
also have ... ≤ *onorm* ((*v) *A*)
using *onorm* [*OF matrix_vector_mul_bounded_linear*, *of A axis i 1*] **by** *auto*
finally have *norm* (χ *j*. *A* $ *j* $ *i*) ≤ *onorm* ((*v) *A*) **.**
then show *?thesis*
unfolding *column_def* **.**
qed

lemma *matrix_component_le_onorm*:
fixes *A* :: *real^'n^'m*
shows |*A* $ *i* $ *j*| ≤ *onorm*((*v) *A*)
proof −
have |*A* $ *i* $ *j*| ≤ *norm* (χ *n*. (*A* $ *n* $ *j*))
by (*metis* (*full_types*, *lifting*) *component_le_norm_cart vec_lambda_beta*)
also have ... ≤ *onorm* ((*v) *A*)
by (*metis* (*no_types*) *column_def norm_column_le_onorm*)
finally show *?thesis* **.**
qed

lemma *component_le_onorm*:
fixes *f* :: *real^'m* ⇒ *real^'n*
shows *linear f* ⟹ |*matrix f* $ *i* $ *j*| ≤ *onorm f*
by (*metis matrix_component_le_onorm matrix_vector_mul*(*2*))

lemma *onorm_le_matrix_component_sum*:
fixes *A* :: *real^'n^'m*
shows *onorm*((*v) *A*) ≤ (∑*i*∈*UNIV*. ∑*j*∈*UNIV*. |*A* $ *i* $ *j*|)
proof (*rule onorm_le*)
fix *x*
have *norm* (*A* *v *x*) ≤ (∑*i*∈*UNIV*. |(*A* *v *x*) $ *i*|)
by (*rule norm_le_l1_cart*)
also have ... ≤ (∑*i*∈*UNIV*. ∑*j*∈*UNIV*. |*A* $ *i* $ *j*| * *norm x*)
proof (*rule sum_mono*)
fix *i*
have |(*A* *v *x*) $ *i*| ≤ |∑*j*∈*UNIV*. *A* $ *i* $ *j* * *x* $ *j*|
by (*simp add*: *matrix_vector_mult_def*)
also have ... ≤ (∑*j*∈*UNIV*. |*A* $ *i* $ *j* * *x* $ *j*|)

```
      by (rule sum_abs)
    also have ... ≤ (∑j∈UNIV. |A $ i $ j| * norm x)
     by (rule sum_mono) (simp add: abs_mult component_le_norm_cart mult_left_mono)
    finally show |(A *v x) $ i| ≤ (∑j∈UNIV. |A $ i $ j| * norm x) .
  qed
  finally show norm (A *v x) ≤ (∑i∈UNIV. ∑j∈UNIV. |A $ i $ j|) * norm x
    by (simp add: sum_distrib_right)
qed

lemma onorm_le_matrix_component:
  fixes A :: real^'n^'m
  assumes ⋀i j. abs(A$i$j) ≤ B
  shows onorm((*v) A) ≤ real (CARD('m)) * real (CARD('n)) * B
proof (rule onorm_le)
  fix x :: real^'n::_
  have norm (A *v x) ≤ (∑i∈UNIV. |(A *v x) $ i|)
    by (rule norm_le_l1_cart)
  also have ... ≤ (∑i::'m ∈UNIV. real (CARD('n)) * B * norm x)
  proof (rule sum_mono)
    fix i
    have |(A *v x) $ i| ≤ norm(A $ i) * norm x
      by (simp add: matrix_mult_dot Cauchy_Schwarz_ineq2)
    also have ... ≤ (∑j∈UNIV. |A $ i $ j|) * norm x
      by (simp add: mult_right_mono norm_le_l1_cart)
    also have ... ≤ real (CARD('n)) * B * norm x
      by (simp add: assms sum_bounded_above mult_right_mono)
    finally show |(A *v x) $ i| ≤ real (CARD('n)) * B * norm x .
  qed
  also have ... ≤ CARD('m) * real (CARD('n)) * B * norm x
    by simp
  finally show norm (A *v x) ≤ CARD('m) * real (CARD('n)) * B * norm x .
qed


lemma rational_approximation:
  assumes e > 0
  obtains r::real where r ∈ ℚ |r - x| < e
  using Rats_dense_in_real [of x - e/2 x + e/2] assms by auto

proposition matrix_rational_approximation:
  fixes A :: real^'n^'m
  assumes e > 0
  obtains B where ⋀i j. B$i$j ∈ ℚ onorm(λx. (A - B) *v x) < e
proof -
  have ∀i j. ∃q ∈ ℚ. |q - A $ i $ j| < e / (2 * CARD('m) * CARD('n))
    using assms by (force intro: rational_approximation [of e / (2 * CARD('m) *
CARD('n))])
  then obtain B where B: ⋀i j. B$i$j ∈ ℚ and Bclo: ⋀i j. |B$i$j - A $ i $
j| < e / (2 * CARD('m) * CARD('n))
```

by (*auto simp*: *lambda_skolem Bex_def*)
show *?thesis*
proof
have *onorm* ((∗*v*) (*A* − *B*)) ≤ *real CARD*(′*m*) ∗ *real CARD*(′*n*) ∗
(*e* / (*2* ∗ *real CARD*(′*m*) ∗ *real CARD*(′*n*)))
apply (*rule onorm_le_matrix_component*)
using *Bclo* **by** (*simp add*: *abs_minus_commute less_imp_le*)
also have ... < *e*
using ‹*0* < *e*› **by** (*simp add*: *field_split_simps*)
finally show *onorm* ((∗*v*) (*A* − *B*)) < *e* **.**
qed (*use B* **in** *auto*)
qed

lemma *vector_sub_project_orthogonal_cart*: (*b*::*real*ˆ′*n*) • (*x* − ((*b* • *x*) / (*b* • *b*)) ∗*s* *b*) = *0*
unfolding *inner_simps scalar_mult_eq_scaleR* **by** *auto*

lemma *infnorm_cart*:*infnorm* (*x*::*real*ˆ′*n*) = *Sup* {|*x*$*i*| |*i*. *i*∈*UNIV*}
by (*simp add*: *infnorm_def inner_axis Basis_vec_def*) (*metis* (*lifting*) *inner_axis real_inner_1_right*)

lemma *component_le_infnorm_cart*: |*x*$*i*| ≤ *infnorm* (*x*::*real*ˆ′*n*)
using *Basis_le_infnorm*[*of axis i 1 x*]
by (*simp add*: *Basis_vec_def axis_eq_axis inner_axis*)

lemma *continuous_component*[*continuous_intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *f x* $ *i*)
unfolding *continuous_def* **by** (*rule tendsto_vec_nth*)

lemma *continuous_on_component*[*continuous_intros*]: *continuous_on s f* ⟹ *continuous_on s* (λ*x*. *f x* $ *i*)
unfolding *continuous_on_def* **by** (*fast intro*: *tendsto_vec_nth*)

lemma *continuous_on_vec_lambda*[*continuous_intros*]:
(⋀*i*. *continuous_on S* (*f i*)) ⟹ *continuous_on S* (λ*x*. χ *i*. *f i x*)
unfolding *continuous_on_def* **by** (*auto intro*: *tendsto_vec_lambda*)

lemma *closed_positive_orthant*: *closed* {*x*::*real*ˆ′*n*. ∀ *i*. *0* ≤*x*$*i*}
by (*simp add*: *Collect_all_eq closed_INT closed_Collect_le continuous_on_component*)

lemma *bounded_component_cart*: *bounded s* ⟹ *bounded* ((λ*x*. *x* $ *i*) ‘ *s*)
unfolding *bounded_def*
apply *clarify*
apply (*rule_tac x*=*x* $ *i* **in** *exI*)
apply (*rule_tac x*=*e* **in** *exI*)
apply *clarify*
apply (*rule order_trans* [*OF dist_vec_nth_le*], *simp*)
done

```
lemma compact_lemma_cart:
  fixes f :: nat ⇒ 'a::heine_borel ^ 'n
  assumes f: bounded (range f)
  shows ∃ l r. strict_mono r ∧
        (∀ e>0. eventually (λn. ∀ i∈d. dist (f (r n) $ i) (l $ i) < e) sequentially)
    (is ?th d)
proof -
  have ∀ d' ⊆ d. ?th d'
    by (rule compact_lemma_general[where unproj=vec_lambda])
      (auto intro!: f bounded_component_cart)
  then show ?th d by simp
qed

instance vec :: (heine_borel, finite) heine_borel
proof
  fix f :: nat ⇒ 'a ^ 'b
  assume f: bounded (range f)
  then obtain l r where r: strict_mono r
      and l: ∀ e>0. eventually (λn. ∀ i∈UNIV. dist (f (r n) $ i) (l $ i) < e)
sequentially
    using compact_lemma_cart [OF f] by blast
  let ?d = UNIV::'b set
  { fix e::real assume e>0
    hence 0 < e / (real_of_nat (card ?d))
      using zero_less_card_finite divide_pos_pos[of e, of real_of_nat (card ?d)] by
auto
    with l have eventually (λn. ∀ i. dist (f (r n) $ i) (l $ i) < e / (real_of_nat
(card ?d))) sequentially
      by simp
    moreover
    { fix n
      assume n: ∀ i. dist (f (r n) $ i) (l $ i) < e / (real_of_nat (card ?d))
      have dist (f (r n)) l ≤ (∑ i∈?d. dist (f (r n) $ i) (l $ i))
        unfolding dist_vec_def using zero_le_dist by (rule L2_set_le_sum)
      also have ... < (∑ i∈?d. e / (real_of_nat (card ?d)))
        by (rule sum_strict_mono) (simp_all add: n)
      finally have dist (f (r n)) l < e by simp
    }
    ultimately have eventually (λn. dist (f (r n)) l < e) sequentially
      by (rule eventually_mono)
  }
  hence ((f ∘ r) ⟶ l) sequentially unfolding o_def tendsto_iff by simp
  with r show ∃ l r. strict_mono r ∧ ((f ∘ r) ⟶ l) sequentially by auto
qed

lemma interval_cart:
  fixes a :: real^'n
  shows box a b = {x::real^'n. ∀ i. a$i < x$i ∧ x$i < b$i}
    and cbox a b = {x::real^'n. ∀ i. a$i ≤ x$i ∧ x$i ≤ b$i}
```

by (*auto simp add*: *set_eq_iff less_vec_def less_eq_vec_def mem_box Basis_vec_def inner_axis*)

lemma *mem_box_cart*:
fixes a :: *real^'n*
shows $x \in box\ a\ b \longleftrightarrow (\forall i.\ a\$i < x\$i \wedge x\$i < b\$i)$
and $x \in cbox\ a\ b \longleftrightarrow (\forall i.\ a\$i \leq x\$i \wedge x\$i \leq b\$i)$
using *interval_cart*[*of a b*] **by** (*auto simp add*: *set_eq_iff less_vec_def less_eq_vec_def*)

lemma *interval_eq_empty_cart*:
fixes a :: *real^'n*
shows ($box\ a\ b = \{\} \longleftrightarrow (\exists i.\ b\$i \leq a\$i)$) (**is** *?th1*)
and ($cbox\ a\ b = \{\} \longleftrightarrow (\exists i.\ b\$i < a\$i)$) (**is** *?th2*)
proof −
{ **fix** $i\ x$ **assume** *as*:$b\$i \leq a\$i$ **and** *x*:$x \in box\ a\ b$
hence $a\ \$\ i < x\ \$\ i \wedge x\ \$\ i < b\ \$\ i$ **unfolding** *mem_box_cart* **by** *auto*
hence $a\$i < b\$i$ **by** *auto*
hence *False* **using** *as* **by** *auto* }
moreover
{ **assume** *as*:$\forall i.\ \neg\ (b\$i \leq a\$i)$
let $?x = (1/2) *_R (a + b)$
{ **fix** i
have $a\$i < b\$i$ **using** *as*[*THEN spec*[**where** $x=i$]] **by** *auto*
hence $a\$i < ((1/2) *_R (a+b))\ \$\ i\ ((1/2) *_R (a+b))\ \$\ i < b\$i$
unfolding *vector_smult_component* **and** *vector_add_component*
by *auto* }
hence $box\ a\ b \neq \{\}$ **using** *mem_box_cart*(*1*)[*of ?x a b*] **by** *auto* }
ultimately show *?th1* **by** *blast*

{ **fix** $i\ x$ **assume** *as*:$b\$i < a\$i$ **and** *x*:$x \in cbox\ a\ b$
hence $a\ \$\ i \leq x\ \$\ i \wedge x\ \$\ i \leq b\ \$\ i$ **unfolding** *mem_box_cart* **by** *auto*
hence $a\$i \leq b\$i$ **by** *auto*
hence *False* **using** *as* **by** *auto* }
moreover
{ **assume** *as*:$\forall i.\ \neg\ (b\$i < a\$i)$
let $?x = (1/2) *_R (a + b)$
{ **fix** i
have $a\$i \leq b\$i$ **using** *as*[*THEN spec*[**where** $x=i$]] **by** *auto*
hence $a\$i \leq ((1/2) *_R (a+b))\ \$\ i\ ((1/2) *_R (a+b))\ \$\ i \leq b\$i$
unfolding *vector_smult_component* **and** *vector_add_component*
by *auto* }
hence $cbox\ a\ b \neq \{\}$ **using** *mem_box_cart*(*2*)[*of ?x a b*] **by** *auto* }
ultimately show *?th2* **by** *blast*
qed

lemma *interval_ne_empty_cart*:
fixes a :: *real^'n*
shows $cbox\ a\ b \neq \{\} \longleftrightarrow (\forall i.\ a\$i \leq b\$i)$
and $box\ a\ b \neq \{\} \longleftrightarrow (\forall i.\ a\$i < b\$i)$

unfolding *interval_eq_empty_cart[of a b]* **by** (*auto simp add*: *not_less not_le*)

lemma *subset_interval_imp_cart*:
fixes *a* :: *real^'n*
shows (∀ *i*. *a$i* ≤ *c$i* ∧ *d$i* ≤ *b$i*) ⟹ *cbox c d* ⊆ *cbox a b*
and (∀ *i*. *a$i* < *c$i* ∧ *d$i* < *b$i*) ⟹ *cbox c d* ⊆ *box a b*
and (∀ *i*. *a$i* ≤ *c$i* ∧ *d$i* ≤ *b$i*) ⟹ *box c d* ⊆ *cbox a b*
and (∀ *i*. *a$i* ≤ *c$i* ∧ *d$i* ≤ *b$i*) ⟹ *box c d* ⊆ *box a b*
unfolding *subset_eq[unfolded Ball_def]* **unfolding** *mem_box_cart*
by (*auto intro*: *order_trans less_le_trans le_less_trans less_imp_le*)

lemma *interval_sing*:
fixes *a* :: *'a::linorder^'n*
shows {*a* .. *a*} = {*a*} ∧ {*a*<..<*a*} = {}
apply (*auto simp add*: *set_eq_iff less_vec_def less_eq_vec_def vec_eq_iff*)
done

lemma *subset_interval_cart*:
fixes *a* :: *real^'n*
shows *cbox c d* ⊆ *cbox a b* ⟷ (∀ *i*. *c$i* ≤ *d$i*) --> (∀ *i*. *a$i* ≤ *c$i* ∧ *d$i* ≤ *b$i*) (**is** *?th1*)
and *cbox c d* ⊆ *box a b* ⟷ (∀ *i*. *c$i* ≤ *d$i*) --> (∀ *i*. *a$i* < *c$i* ∧ *d$i* < *b$i*) (**is** *?th2*)
and *box c d* ⊆ *cbox a b* ⟷ (∀ *i*. *c$i* < *d$i*) --> (∀ *i*. *a$i* ≤ *c$i* ∧ *d$i* ≤ *b$i*) (**is** *?th3*)
and *box c d* ⊆ *box a b* ⟷ (∀ *i*. *c$i* < *d$i*) --> (∀ *i*. *a$i* ≤ *c$i* ∧ *d$i* ≤ *b$i*) (**is** *?th4*)
using *subset_box[of c d a b]* **by** (*simp_all add*: *Basis_vec_def inner_axis*)

lemma *disjoint_interval_cart*:
fixes *a::real^'n*
shows *cbox a b* ∩ *cbox c d* = {} ⟷ (∃ *i*. (*b$i* < *a$i* ∨ *d$i* < *c$i* ∨ *b$i* < *c$i* ∨ *d$i* < *a$i*)) (**is** *?th1*)
and *cbox a b* ∩ *box c d* = {} ⟷ (∃ *i*. (*b$i* < *a$i* ∨ *d$i* ≤ *c$i* ∨ *b$i* ≤ *c$i* ∨ *d$i* ≤ *a$i*)) (**is** *?th2*)
and *box a b* ∩ *cbox c d* = {} ⟷ (∃ *i*. (*b$i* ≤ *a$i* ∨ *d$i* < *c$i* ∨ *b$i* ≤ *c$i* ∨ *d$i* ≤ *a$i*)) (**is** *?th3*)
and *box a b* ∩ *box c d* = {} ⟷ (∃ *i*. (*b$i* ≤ *a$i* ∨ *d$i* ≤ *c$i* ∨ *b$i* ≤ *c$i* ∨ *d$i* ≤ *a$i*)) (**is** *?th4*)
using *disjoint_interval[of a b c d]* **by** (*simp_all add*: *Basis_vec_def inner_axis*)

lemma *Int_interval_cart*:
fixes *a* :: *real^'n*
shows *cbox a b* ∩ *cbox c d* = {(χ *i*. *max* (*a$i*) (*c$i*)) .. (χ *i*. *min* (*b$i*) (*d$i*))}
unfolding *Int_interval*
by (*auto simp*: *mem_box less_eq_vec_def*)
(*auto simp*: *Basis_vec_def inner_axis*)

lemma *closed_interval_left_cart*:
  fixes *b* :: *real^'n*
  shows *closed* {*x*::*real^'n*. ∀ *i*. *x$i* ≤ *b$i*}
 by (*simp add*: *Collect_all_eq closed_INT closed_Collect_le continuous_on_component*)

lemma *closed_interval_right_cart*:
  fixes *a*::*real^'n*
  shows *closed* {*x*::*real^'n*. ∀ *i*. *a$i* ≤ *x$i*}
 by (*simp add*: *Collect_all_eq closed_INT closed_Collect_le continuous_on_component*)

lemma *is_interval_cart*:
  is_interval (*s*::(*real^'n*) *set*) ⟷
    (∀ *a*∈*s*. ∀ *b*∈*s*. ∀ *x*. (∀ *i*. ((*a$i* ≤ *x$i* ∧ *x$i* ≤ *b$i*) ∨ (*b$i* ≤ *x$i* ∧ *x$i* ≤ *a$i*)))
⟶ *x* ∈ *s*)
  by (*simp add*: *is_interval_def Ball_def Basis_vec_def inner_axis imp_ex*)

lemma *closed_halfspace_component_le_cart*: *closed* {*x*::*real^'n*. *x$i* ≤ *a*}
  by (*simp add*: *closed_Collect_le continuous_on_component*)

lemma *closed_halfspace_component_ge_cart*: *closed* {*x*::*real^'n*. *x$i* ≥ *a*}
  by (*simp add*: *closed_Collect_le continuous_on_component*)

lemma *open_halfspace_component_lt_cart*: *open* {*x*::*real^'n*. *x$i* < *a*}
  by (*simp add*: *open_Collect_less continuous_on_component*)

lemma *open_halfspace_component_gt_cart*: *open* {*x*::*real^'n*. *x$i* > *a*}
  by (*simp add*: *open_Collect_less continuous_on_component*)

lemma *Lim_component_le_cart*:
  fixes *f* :: *'a* ⇒ *real^'n*
  assumes (*f* ⟶ *l*) *net* ¬ (*trivial_limit net*) *eventually* (λ*x*. *f x $i* ≤ *b*) *net*
  shows *l$i* ≤ *b*
 by (*rule tendsto_le*[*OF assms*(*2*) *tendsto_const tendsto_vec_nth*, *OF assms*(*1*, *3*)])

lemma *Lim_component_ge_cart*:
  fixes *f* :: *'a* ⇒ *real^'n*
  assumes (*f* ⟶ *l*) *net* ¬ (*trivial_limit net*) *eventually* (λ*x*. *b* ≤ (*f x*)*$i*) *net*
  shows *b* ≤ *l$i*
 by (*rule tendsto_le*[*OF assms*(*2*) *tendsto_vec_nth tendsto_const*, *OF assms*(*1*, *3*)])

lemma *Lim_component_eq_cart*:
  fixes *f* :: *'a* ⇒ *real^'n*
  assumes *net*: (*f* ⟶ *l*) *net* ¬ *trivial_limit net* **and** *ev*:*eventually* (λ*x*. *f*(*x*)*$i*
= *b*) *net*
  shows *l$i* = *b*
  using *ev*[*unfolded order_eq_iff eventually_conj_iff*] **and**
    Lim_component_ge_cart[*OF net*, *of b i*] **and**
    Lim_component_le_cart[*OF net*, *of i b*] **by** *auto*

lemma *connected_ivt_component_cart*:
  fixes x :: *real^'n*
  shows *connected s* $\Longrightarrow x \in s \Longrightarrow y \in s \Longrightarrow x\$k \leq a \Longrightarrow a \leq y\$k \Longrightarrow (\exists z{\in}s.\ z\$k = a)$
  using *connected_ivt_hyperplane*[*of s x y axis k 1 a*]
  by (*auto simp add*: *inner_axis inner_commute*)

lemma *subspace_substandard_cart*: *vec.subspace* $\{x.\ (\forall i.\ P\ i \longrightarrow x\$i = 0)\}$
  unfolding *vec.subspace_def* **by** *auto*

lemma *closed_substandard_cart*:
  closed $\{x$::*'a*::*real_normed_vector ^ 'n*. $\forall i.\ P\ i \longrightarrow x\$i = 0\}$
proof –
  { **fix** *i*::*'n*
    have *closed* $\{x$::*'a ^ 'n*. $P\ i \longrightarrow x\$i = 0\}$
      by (*cases P i*) (*simp_all add*: *closed_Collect_eq continuous_on_component*) }
  thus *?thesis*
    unfolding *Collect_all_eq* **by** (*simp add*: *closed_INT*)
qed

4.11.3 Convex Euclidean Space

lemma *Cart_1*:(*1*::*real^'n*) = $\sum$ *Basis*
  using *const_vector_cart*[*of 1*] **by** (*simp add*: *one_vec_def*)

declare *vector_add_ldistrib*[*simp*] *vector_ssub_ldistrib*[*simp*] *vector_smult_assoc*[*simp*] *vector_smult_rneg*[*simp*]
declare *vector_sadd_rdistrib*[*simp*] *vector_sub_rdistrib*[*simp*]

lemmas *vector_component_simps* = *vector_minus_component vector_smult_component vector_add_component less_eq_vec_def vec_lambda_beta vector_uminus_component*

lemma *convex_box_cart*:
  assumes $\bigwedge i$. *convex* $\{x.\ P\ i\ x\}$
  shows *convex* $\{x.\ \forall i.\ P\ i\ (x\$i)\}$
  using *assms* **unfolding** *convex_def* **by** *auto*

4.11.4 Derivative

definition *jacobian f net* = *matrix*(*frechet_derivative f net*)

proposition *jacobian_works*:
  (*f*::(*real^'a*) ⇒ (*real^'b*)) *differentiable net* ⟷
    (*f has_derivative* (λ*h*. (*jacobian f net*) ∗*v h*)) *net* (**is** *?lhs* = *?rhs*)
proof
  assume *?lhs* **then show** *?rhs*
    by (*simp add*: *frechet_derivative_works has_derivative_linear jacobian_def*)
next
  assume *?rhs* **then show** *?lhs*
    by (*rule differentiableI*)

qed

Component of the differential must be zero if it exists at a local maximum or minimum for that corresponding component

proposition *differential_zero_maxmin_cart*:
  fixes *f*::*real^'a ⇒ real^'b*
  assumes *0 < e* ((∀ *y* ∈ *ball x e*. (*f y*)$*k* ≤ (*f x*)$*k*) ∨ (∀ *y*∈*ball x e*. (*f x*)$*k* ≤ (*f y*)$*k*))
    f differentiable (*at x*)
  shows *jacobian f* (*at x*) $ *k* = *0*
  using *differential_zero_maxmin_component*[*of axis k 1 e x f*] *assms*
    vector_cart[*of* λ*j*. *frechet_derivative f* (*at x*) *j* $ *k*]
  by (*simp add*: *Basis_vec_def axis_eq_axis inner_axis jacobian_def matrix_def*)

4.11.5 Routine results connecting the types (*real*, *1*) *vec* and *real*

lemma *vec_cbox_1_eq* [*simp*]:
  shows *vec* ‘ *cbox u v* = *cbox* (*vec u*) (*vec v* ::*real^1*)
  by (*force simp*: *Basis_vec_def cart_eq_inner_axis* [*symmetric*] *mem_box*)

lemma *vec_nth_cbox_1_eq* [*simp*]:
  fixes *u v* :: *'a*::*euclidean_space^1*
  shows (λ*x*. *x* $ *1*) ‘ *cbox u v* = *cbox* (*u*$*1*) (*v*$*1*)
    by (*auto simp*: *Basis_vec_def cart_eq_inner_axis* [*symmetric*] *mem_box image_iff Bex_def inner_axis*) (*metis vec_component*)

lemma *vec_nth_1_iff_cbox* [*simp*]:
  fixes *a b* :: *'a*::*euclidean_space*
  shows (λ*x*::*'a^1*. *x* $ *1*) ‘ *S* = *cbox a b* ⟷ *S* = *cbox* (*vec a*) (*vec b*)
    (**is** *?lhs* = *?rhs*)
proof
  assume *L*: *?lhs* **show** *?rhs*
  proof (*intro equalityI subsetI*)
    fix *x*
    assume *x* ∈ *S*
    then have *x* $ *1* ∈ (λ*v*. *v* $ (*1*::*1*)) ‘ *cbox* (*vec a*) (*vec b*)
      using *L* **by** *auto*
    then show *x* ∈ *cbox* (*vec a*) (*vec b*)
      by (*metis* (*no_types*, *lifting*) *imageE vector_one_nth*)
  next
    fix *x* :: *'a^1*
    assume *x* ∈ *cbox* (*vec a*) (*vec b*)
    then show *x* ∈ *S*
     by (*metis* (*no_types*, *lifting*) *L imageE imageI vec_component vec_nth_cbox_1_eq vector_one_nth*)
  qed
qed *simp*

lemma *vec_nth_real_1_iff_cbox* [*simp*]:
  fixes *a b* :: *real*
  shows $(\lambda x{::}real\hat{}1.\ x\ \$\ 1)$ ` $S = \{a..b\} \longleftrightarrow S = cbox\ (vec\ a)\ (vec\ b)$
  using *vec_nth_1_iff_cbox*[*of S a b*]
  by *simp*

lemma *interval_split_cart*:
  $\{a..b{::}real\hat{}'n\} \cap \{x.\ x\$k \leq c\} = \{a\ ..\ (\chi\ i.\ if\ i = k\ then\ min\ (b\$k)\ c\ else\ b\$i)\}$
  $cbox\ a\ b \cap \{x.\ x\$k \geq c\} = \{(\chi\ i.\ if\ i = k\ then\ max\ (a\$k)\ c\ else\ a\$i)\ ..\ b\}$
  apply (*rule_tac*[!] *set_eqI*)
  unfolding *Int_iff mem_box_cart mem_Collect_eq interval_cbox_cart*
  unfolding *vec_lambda_beta*
  by *auto*

lemmas *cartesian_euclidean_space_uniform_limit_intros*[*uniform_limit_intros*] =
  bounded_linear.uniform_limit[*OF blinfun.bounded_linear_right*]
  bounded_linear.uniform_limit[*OF bounded_linear_vec_nth*]

end

Chapter 5

Unsorted

```
theory Starlike
  imports
    Convex_Euclidean_Space
    Line_Segment
begin

lemma affine_hull_closed_segment [simp]:
     affine hull (closed_segment a b) = affine hull {a,b}
  by (simp add: segment_convex_hull)

lemma affine_hull_open_segment [simp]:
    fixes a :: 'a::euclidean_space
    shows affine hull (open_segment a b) = (if a = b then {} else affine hull {a,b})
by (metis affine_hull_convex_hull affine_hull_empty closure_open_segment closure_same_affine_hull
segment_convex_hull)

lemma rel_interior_closure_convex_segment:
  fixes S :: _::euclidean_space set
  assumes convex S a ∈ rel_interior S b ∈ closure S
    shows open_segment a b ⊆ rel_interior S
proof
  fix x
  have [simp]: (1 − u) *R a + u *R b = b − (1 − u) *R (b − a) for u
    by (simp add: algebra_simps)
  assume x ∈ open_segment a b
  then show x ∈ rel_interior S
    unfolding closed_segment_def open_segment_def using assms
    by (auto intro: rel_interior_closure_convex_shrink)
qed

lemma convex_hull_insert_segments:
   convex hull (insert a S) =
   (if S = {} then {a} else ⋃x ∈ convex hull S. closed_segment a x)
  by (force simp add: convex_hull_insert_alt in_segment)
```

lemma *Int_convex_hull_insert_rel_exterior*:
fixes *z* :: *'a::euclidean_space*
assumes *convex C T* ⊆ *C* **and** *z*: *z* ∈ *rel_interior C* **and** *dis*: *disjnt S* (*rel_interior C*)
shows *S* ∩ (*convex hull* (*insert z T*)) = *S* ∩ (*convex hull T*) (**is** *?lhs* = *?rhs*)
proof
have *T* = {} ⟹ *z* ∉ *S*
using *dis z* **by** (*auto simp add*: *disjnt_def*)
then show *?lhs* ⊆ *?rhs*
proof (*clarsimp simp add*: *convex_hull_insert_segments*)
fix *x y*
assume *x* ∈ *S* **and** *y*: *y* ∈ *convex hull T* **and** *x* ∈ *closed_segment z y*
have *y* ∈ *closure C*
by (*metis y* ‹*convex C*› ‹*T* ⊆ *C*› *closure_subset contra_subsetD convex_hull_eq hull_mono*)
moreover have *x* ∉ *rel_interior C*
by (*meson* ‹*x* ∈ *S*› *dis disjnt_iff*)
moreover have *x* ∈ *open_segment z y* ∪ {*z*, *y*}
using ‹*x* ∈ *closed_segment z y*› *closed_segment_eq_open* **by** *blast*
ultimately show *x* ∈ *convex hull T*
using *rel_interior_closure_convex_segment* [*OF* ‹*convex C*› *z*]
using *y z* **by** *blast*
qed
show *?rhs* ⊆ *?lhs*
by (*meson hull_mono inf_mono subset_insertI subset_refl*)
qed

5.0.1 Shrinking towards the interior of a convex set

lemma *mem_interior_convex_shrink*:
fixes *S* :: *'a::euclidean_space set*
assumes *convex S*
and *c* ∈ *interior S*
and *x* ∈ *S*
and *0* < *e*
and *e* ≤ *1*
shows $x - e *_R (x - c)$ ∈ *interior S*
proof −
obtain *d* **where** *d* > *0* **and** *d*: *ball c d* ⊆ *S*
using *assms*(*2*) **unfolding** *mem_interior* **by** *auto*
show *?thesis*
unfolding *mem_interior*
proof (*intro exI subsetI conjI*)
fix *y*
assume *y* ∈ *ball* $(x - e *_R (x - c))$ (*e*∗*d*)
then have *as*: *dist* $(x - e *_R (x - c))$ *y* < *e* ∗ *d*
by *simp*
have ∗: $y = (1 - (1 - e)) *_R ((1 / e) *_R y - ((1 - e) / e) *_R x) + (1 -$

```
e) *R x
    using ‹e > 0› by (auto simp add: scaleR_left_diff_distrib scaleR_right_diff_distrib)
    have c − ((1 / e) *R y − ((1 − e) / e) *R x) = (1 / e) *R (e *R c − y +
(1 − e) *R x)
      using ‹e > 0›
      by (auto simp add: euclidean_eq_iff[where 'a='a] field_simps inner_simps)
    then have dist c ((1 / e) *R y − ((1 − e) / e) *R x) = |1/e| * norm (e *R
c − y + (1 − e) *R x)
      by (simp add: dist_norm)
    also have ... = |1/e| * norm (x − e *R (x − c) − y)
      by (auto intro!:arg_cong[where f=norm] simp add: algebra_simps)
    also have ... < d
      using as[unfolded dist_norm] and ‹e > 0›
      by (auto simp add:pos_divide_less_eq[OF ‹e > 0›] mult.commute)
    finally have (1 − (1 − e)) *R ((1 / e) *R y − ((1 − e) / e) *R x) + (1 −
e) *R x ∈ S
      using assms(3−5) d
      by (intro convexD_alt [OF ‹convex S›]) (auto intro: convexD_alt [OF ‹convex
S›])
    with ‹e > 0› show y ∈ S
      by (auto simp add: scaleR_left_diff_distrib scaleR_right_diff_distrib)
  qed (use ‹e>0› ‹d>0› in auto)
qed

lemma mem_interior_closure_convex_shrink:
  fixes S :: 'a::euclidean_space set
  assumes convex S
    and c ∈ interior S
    and x ∈ closure S
    and 0 < e
    and e ≤ 1
  shows x − e *R (x − c) ∈ interior S
proof −
  obtain d where d > 0 and d: ball c d ⊆ S
    using assms(2) unfolding mem_interior by auto
  have ∃y∈S. norm (y − x) * (1 − e) < e * d
  proof (cases x ∈ S)
    case True
    then show ?thesis
      using ‹e > 0› ‹d > 0› by force
  next
    case False
    then have x: x islimpt S
      using assms(3)[unfolded closure_def] by auto
    show ?thesis
    proof (cases e = 1)
      case True
      obtain y where y ∈ S y ≠ x dist y x < 1
        using x[unfolded islimpt_approachable,THEN spec[where x=1]] by auto
```

```
      then show ?thesis
        using True ‹0 < d› by auto
    next
      case False
      then have 0 < e * d / (1 − e) and *: 1 − e > 0
        using ‹e ≤ 1› ‹e > 0› ‹d > 0› by auto
      then obtain y where y ∈ S y ≠ x dist y x < e * d / (1 − e)
        using islimpt_approachable x by blast
      then have norm (y − x) * (1 − e) < e * d
        by (metis * dist_norm mult_imp_div_pos_le not_less)
      then show ?thesis
        using ‹y ∈ S› by blast
    qed
  qed
  then obtain y where y ∈ S and y: norm (y − x) * (1 − e) < e * d
    by auto
  define z where z = c + ((1 − e) / e) *R (x − y)
  have *: x − e *R (x − c) = y − e *R (y − z)
    unfolding z_def using ‹e > 0›
   by (auto simp add: scaleR_right_diff_distrib scaleR_right_distrib scaleR_left_diff_distrib)
  have (1 − e) * norm (x − y) / e < d
    using y ‹0 < e› by (simp add: field_simps norm_minus_commute)
  then have z ∈ interior (ball c d)
   using ‹0 < e› ‹e ≤ 1› by (simp add: interior_open[OF open_ball] z_def dist_norm)
  then have z ∈ interior S
    using d interiorI interior_ball by blast
  then show ?thesis
    unfolding * using mem_interior_convex_shrink ‹y ∈ S› assms by blast
qed

lemma in_interior_closure_convex_segment:
  fixes S :: 'a::euclidean_space set
  assumes convex S and a: a ∈ interior S and b: b ∈ closure S
    shows open_segment a b ⊆ interior S
proof (clarsimp simp: in_segment)
  fix u::real
  assume u: 0 < u u < 1
  have (1 − u) *R a + u *R b = b − (1 − u) *R (b − a)
    by (simp add: algebra_simps)
  also have ... ∈ interior S using mem_interior_closure_convex_shrink [OF assms]
u
    by simp
  finally show (1 − u) *R a + u *R b ∈ interior S .
qed

lemma convex_closure_interior:
  fixes S :: 'a::euclidean_space set
  assumes convex S and int: interior S ≠ {}
  shows closure(interior S) = closure S
```

```
proof -
  obtain a where a: a ∈ interior S
    using int by auto
  have closure S ⊆ closure(interior S)
  proof
    fix x
    assume x: x ∈ closure S
    show x ∈ closure (interior S)
    proof (cases x=a)
      case True
      then show ?thesis
        using ‹a ∈ interior S› closure_subset by blast
    next
      case False
      show ?thesis
      proof (clarsimp simp add: closure_def islimpt_approachable)
        fix e::real
        assume xnotS: x ∉ interior S and 0 < e
        show ∃x'∈interior S. x' ≠ x ∧ dist x' x < e
        proof (intro bexI conjI)
          show x - min (e/2 / norm (x - a)) 1 *R (x - a) ≠ x
            using False ‹0 < e› by (auto simp: algebra_simps min_def)
          show dist (x - min (e/2 / norm (x - a)) 1 *R (x - a)) x < e
            using ‹0 < e› by (auto simp: dist_norm min_def)
          show x - min (e/2 / norm (x - a)) 1 *R (x - a) ∈ interior S
            using ‹0 < e› False
            by (auto simp add: min_def a intro: mem_interior_closure_convex_shrink
[OF ‹convex S› a x])
        qed
      qed
    qed
  qed
  then show ?thesis
    by (simp add: closure_mono interior_subset subset_antisym)
qed

lemma closure_convex_Int_superset:
  fixes S :: 'a::euclidean_space set
  assumes convex S interior S ≠ {} interior S ⊆ closure T
  shows closure(S ∩ T) = closure S
proof -
  have closure S ⊆ closure(interior S)
    by (simp add: convex_closure_interior assms)
  also have ... ⊆ closure (S ∩ T)
    using interior_subset [of S] assms
   by (metis (no_types, lifting) Int_assoc Int_lower2 closure_mono closure_open_Int_superset
inf.orderE open_interior)
  finally show ?thesis
    by (simp add: closure_mono dual_order.antisym)
```

qed

5.0.2 Some obvious but surprisingly hard simplex lemmas

lemma *simplex*:
assumes *finite S*
and $0 \notin S$
shows *convex hull* (*insert 0 S*) = $\{y.\ \exists u.\ (\forall x{\in}S.\ 0 \le u\ x) \wedge sum\ u\ S \le 1 \wedge sum\ (\lambda x.\ u\ x *_R x)\ S = y\}$
proof (*simp add*: *convex_hull_finite set_eq_iff assms*, *safe*)
fix *x* **and** *u* :: *'a* ⇒ *real*
assume $0 \le u\ 0\ \forall x{\in}S.\ 0 \le u\ x\ u\ 0 + sum\ u\ S = 1$
then show $\exists v.\ (\forall x{\in}S.\ 0 \le v\ x) \wedge sum\ v\ S \le 1 \wedge (\sum x{\in}S.\ v\ x *_R x) = (\sum x{\in}S.\ u\ x *_R x)$
by *force*
next
fix *x* **and** *u* :: *'a* ⇒ *real*
assume $\forall x{\in}S.\ 0 \le u\ x\ sum\ u\ S \le 1$
then show $\exists v.\ 0 \le v\ 0 \wedge (\forall x{\in}S.\ 0 \le v\ x) \wedge v\ 0 + sum\ v\ S = 1 \wedge (\sum x{\in}S.\ v\ x *_R x) = (\sum x{\in}S.\ u\ x *_R x)$
by (*rule_tac x*=$\lambda x.$ *if* $x = 0$ *then* $1 - sum\ u\ S$ *else* *u x* **in** *exI*) (*auto simp*: *sum_delta_notmem assms if_smult*)
qed

lemma *substd_simplex*:
assumes *d*: $d \subseteq Basis$
shows *convex hull* (*insert 0 d*) =
$\{x.\ (\forall i{\in}Basis.\ 0 \le x \cdot i) \wedge (\sum i{\in}d.\ x \cdot i) \le 1 \wedge (\forall i{\in}Basis.\ i \notin d \longrightarrow x \cdot i = 0)\}$
(**is** *convex hull* (*insert 0 ?p*) = *?s*)
proof –
let *?D* = *d*
have $0 \notin ?p$
using *assms* **by** (*auto simp*: *image_def*)
from *d* **have** *finite d*
by (*blast intro*: *finite_subset finite_Basis*)
show *?thesis*
unfolding *simplex*[*OF* ‹*finite d*› ‹$0 \notin ?p$›]
proof (*intro set_eqI*; *safe*)
fix *u* :: *'a* ⇒ *real*
assume *as*: $\forall x{\in}?D.\ 0 \le u\ x\ sum\ u\ ?D \le 1$
let $?x = (\sum x{\in}?D.\ u\ x *_R x)$
have *ind*: $\forall i{\in}Basis.\ i \in d \longrightarrow u\ i = ?x \cdot i$
and *notind*: $(\forall i{\in}Basis.\ i \notin d \longrightarrow ?x \cdot i = 0)$
using *substdbasis_expansion_unique*[*OF assms*] **by** *blast+*
then have $**$: *sum u ?D* = *sum* ((·) *?x*) *?D*
using *assms* **by** (*auto intro*!: *sum.cong*)
show $0 \le ?x \cdot i$ **if** $i \in Basis$ **for** *i*
using *as*(*1*) *ind notind that* **by** *fastforce*

```
    show sum ((·) ?x) ?D ≤ 1
      using ** as(2) by linarith
    show ?x · i = 0 if i ∈ Basis i ∉ d for i
      using notind that by blast
  next
    fix x
    assume ∀i∈Basis. 0 ≤ x · i sum ((·) x) ?D ≤ 1 (∀i∈Basis. i ∉ d ⟶ x · i
= 0)
    with d show ∃u. (∀x∈?D. 0 ≤ u x) ∧ sum u ?D ≤ 1 ∧ (∑x∈?D. u x *R
x) = x
      unfolding substdbasis_expansion_unique[OF assms]
      by (rule_tac x=inner x in exI) auto
  qed
qed

lemma std_simplex:
  convex hull (insert 0 Basis) =
    {x::'a::euclidean_space. (∀i∈Basis. 0 ≤ x·i) ∧ sum (λi. x·i) Basis ≤ 1}
  using substd_simplex[of Basis] by auto

lemma interior_std_simplex:
  interior (convex hull (insert 0 Basis)) =
    {x::'a::euclidean_space. (∀i∈Basis. 0 < x·i) ∧ sum (λi. x·i) Basis < 1}
  unfolding set_eq_iff mem_interior std_simplex
proof (intro allI iffI CollectI; clarify)
  fix x :: 'a
  fix e
  assume e > 0 and as: ball x e ⊆ {x. (∀i∈Basis. 0 ≤ x · i) ∧ sum ((·) x) Basis
≤ 1}
  show (∀i∈Basis. 0 < x · i) ∧ sum ((·) x) Basis < 1
  proof safe
    fix i :: 'a
    assume i: i ∈ Basis
    then show 0 < x · i
      using as[THEN subsetD[where c=x − (e/2) *R i]] and ‹e > 0›
      by (force simp add: inner_simps)
  next
    have **: dist x (x + (e/2) *R (SOME i. i∈Basis)) < e using ‹e > 0›
      unfolding dist_norm
      by (auto intro!: mult_strict_left_mono simp: SOME_Basis)
    have ⋀i. i ∈ Basis ⟹ (x + (e/2) *R (SOME i. i∈Basis)) · i =
      x·i + (if i = (SOME i. i∈Basis) then e/2 else 0)
      by (auto simp: SOME_Basis inner_Basis inner_simps)
    then have *: sum ((·) (x + (e/2) *R (SOME i. i∈Basis))) Basis =
      sum (λi. x·i + (if (SOME i. i∈Basis) = i then e/2 else 0)) Basis
      by (auto simp: intro!: sum.cong)
    have sum ((·) x) Basis < sum ((·) (x + (e/2) *R (SOME i. i∈Basis))) Basis
      using ‹e > 0› DIM_positive by (auto simp: SOME_Basis sum.distrib *)
    also have ... ≤ 1
```

using ∗∗ *as* **by** *force*
finally show *sum* ((·) *x*) *Basis* < *1* **by** *auto*
qed
next
fix *x* :: *'a*
assume *as*: ∀ *i*∈*Basis*. *0* < *x* · *i sum* ((·) *x*) *Basis* < *1*
obtain *a* :: *'b* **where** *a* ∈ *UNIV* **using** *UNIV_witness* **..**
let *?d* = (*1* − *sum* ((·) *x*) *Basis*) / *real* (*DIM*(*'a*))
show ∃ *e*>*0*. *ball x e* ⊆ {*x*. (∀ *i*∈*Basis*. *0* ≤ *x* · *i*) ∧ *sum* ((·) *x*) *Basis* ≤ *1*}
proof (*rule_tac x*=*min* (*Min* (((·) *x*) ' *Basis*)) *D* **for** *D* **in** *exI*, *intro conjI subsetI CollectI*)
fix *y*
assume *y*: *y* ∈ *ball x* (*min* (*Min* ((·) *x* ' *Basis*)) *?d*)
have *sum* ((·) *y*) *Basis* ≤ *sum* (λ*i*. *x*·*i* + *?d*) *Basis*
proof (*rule sum_mono*)
fix *i* :: *'a*
assume *i*: *i* ∈ *Basis*
have |*y*·*i* − *x*·*i*| ≤ *norm* (*y* − *x*)
by (*metis Basis_le_norm i inner_commute inner_diff_right*)
also have ... < *?d*
using *y* **by** (*simp add*: *dist_norm norm_minus_commute*)
finally have |*y*·*i* − *x*·*i*| < *?d* **.**
then show *y* · *i* ≤ *x* · *i* + *?d* **by** *auto*
qed
also have . . . ≤ *1*
unfolding *sum.distrib sum_constant*
by (*auto simp add*: *Suc_le_eq*)
finally show *sum* ((·) *y*) *Basis* ≤ *1* **.**
show (∀ *i*∈*Basis*. *0* ≤ *y* · *i*)
proof *safe*
fix *i* :: *'a*
assume *i*: *i* ∈ *Basis*
have *norm* (*x* − *y*) < *Min* (((·) *x*) ' *Basis*)
using *y* **by** (*auto simp*: *dist_norm less_eq_real_def*)
also have ... ≤ *x*·*i*
using *i* **by** *auto*
finally have *norm* (*x* − *y*) < *x*·*i* **.**
then show *0* ≤ *y*·*i*
using *Basis_le_norm*[*OF i*, *of x* − *y*] **and** *as*(*1*)[*rule_format*, *OF i*]
by (*auto simp*: *inner_simps*)
qed
next
have *Min* (((·) *x*) ' *Basis*) > *0*
using *as* **by** *simp*
moreover have *?d* > *0*
using *as* **by** (*auto simp*: *Suc_le_eq*)
ultimately show *0* < *min* (*Min* ((·) *x* ' *Basis*)) ((*1* − *sum* ((·) *x*) *Basis*) / *real DIM*(*'a*))
by *linarith*

```
  qed
qed

lemma interior_std_simplex_nonempty:
  obtains a :: 'a::euclidean_space where
    a ∈ interior(convex hull (insert 0 Basis))
proof -
  let ?D = Basis :: 'a set
  let ?a = sum (λb::'a. inverse (2 * real DIM('a)) *R b) Basis
  {
    fix i :: 'a
    assume i: i ∈ Basis
    have ?a • i = inverse (2 * real DIM('a))
      by (rule trans[of _ sum (λj. if i = j then inverse (2 * real DIM('a)) else 0)
?D])
        (simp_all add: sum.If_cases i) }
  note ** = this
  show ?thesis
  proof
    show ?a ∈ interior(convex hull (insert 0 Basis))
      unfolding interior_std_simplex mem_Collect_eq
    proof safe
      fix i :: 'a
      assume i: i ∈ Basis
      show 0 < ?a • i
        unfolding **[OF i] by (auto simp add: Suc_le_eq)
    next
      have sum ((•) ?a) ?D = sum (λi. inverse (2 * real DIM('a))) ?D
        by (auto intro: sum.cong)
      also have ... < 1
        unfolding sum_constant divide_inverse[symmetric]
        by (auto simp add: field_simps)
      finally show sum ((•) ?a) ?D < 1 by auto
    qed
  qed
qed

lemma rel_interior_substd_simplex:
  assumes D: D ⊆ Basis
  shows rel_interior (convex hull (insert 0 D)) =
      {x::'a::euclidean_space. (∀ i∈D. 0 < x•i) ∧ (∑ i∈D. x•i) < 1 ∧ (∀ i∈Basis.
i ∉ D ⟶ x•i = 0)}
    (is _ = ?s)
proof -
  have finite D
    using D finite_Basis finite_subset by blast
  show ?thesis
  proof (cases D = {})
    case True
```

```
    then show ?thesis
      using rel_interior_sing using euclidean_eq_iff[of _ 0] by auto
  next
    case False
    have h0: affine hull (convex hull (insert 0 D)) =
             {x::'a::euclidean_space. (∀i∈Basis. i ∉ D ⟶ x·i = 0)}
      using affine_hull_convex_hull affine_hull_substd_basis assms by auto
    have aux: ⋀x::'a. ∀i∈Basis. (∀i∈D. 0 ≤ x·i) ∧ (∀i∈Basis. i ∉ D ⟶ x·i =
0) ⟶ 0 ≤ x·i
      by auto
    {
      fix x :: 'a::euclidean_space
      assume x: x ∈ rel_interior (convex hull (insert 0 D))
      then obtain e where e > 0 and
        ball x e ∩ {xa. (∀i∈Basis. i ∉ D ⟶ xa·i = 0)} ⊆ convex hull (insert 0 D)
        using mem_rel_interior_ball[of x convex hull (insert 0 D)] h0 by auto
      then have as: ⋀y. ⟦dist x y < e ∧ (∀i∈Basis. i ∉ D ⟶ y·i = 0)⟧ ⟹
                    (∀i∈D. 0 ≤ y · i) ∧ sum ((·) y) D ≤ 1
        using assms by (force simp: substd_simplex)
      have x0: (∀i∈Basis. i ∉ D ⟶ x·i = 0)
        using x rel_interior_subset  substd_simplex[OF assms] by auto
      have (∀i∈D. 0 < x · i) ∧ sum ((·) x) D < 1 ∧ (∀i∈Basis. i ∉ D ⟶ x·i
= 0)
      proof (intro conjI ballI)
        fix i :: 'a
        assume i ∈ D
        then have ∀j∈D. 0 ≤ (x − (e/2) *R i) · j
          using D ‹e > 0› x0
            by (intro as[THEN conjunct1]) (force simp: dist_norm inner_simps in-
ner_Basis)
        then show 0 < x · i
          using ‹e > 0› ‹i ∈ D› D  by (force simp: inner_simps inner_Basis)
      next
        obtain a where a: a ∈ D
          using ‹D ≠ {}› by auto
        then have **: dist x (x + (e/2) *R a) < e
          using ‹e > 0› norm_Basis[of a] D by (auto simp: dist_norm)
        have ⋀i. i ∈ Basis ⟹ (x + (e/2) *R a) · i = x·i + (if i = a then e/2
else 0)
          using a D by (auto simp: inner_simps inner_Basis)
        then have *: sum ((·) (x + (e/2) *R a)) D = sum (λi. x·i + (if a = i
then e/2 else 0)) D
          using D by (intro sum.cong) auto
        have a ∈ Basis
          using ‹a ∈ D› D by auto
        then have h1: (∀i∈Basis. i ∉ D ⟶ (x + (e/2) *R a) · i = 0)
          using x0 D ‹a∈D› by (auto simp add: inner_add_left inner_Basis)
        have sum ((·) x) D < sum ((·) (x + (e/2) *R a)) D
          using ‹e > 0› ‹a ∈ D› ‹finite D› by (auto simp add: * sum.distrib)
```

```
      also have ... ≤ 1
        using ** h1 as[rule_format, of x + (e/2) *R a]
        by auto
      finally show sum ((·) x) D < 1 ⋀i. i∈Basis ⟹ i ∉ D ⟶ x·i = 0
        using x0 by auto
    qed
  }
  moreover
  {
    fix x :: 'a::euclidean_space
    assume as: x ∈ ?s
    have ∀ i. 0 < x·i ∨ 0 = x·i ⟶ 0 ≤ x·i
      by auto
    moreover have ∀ i. i ∈ D ∨ i ∉ D by auto
    ultimately
    have ∀ i. (∀ i∈D. 0 < x·i) ∧ (∀ i. i ∉ D ⟶ x·i = 0) ⟶ 0 ≤ x·i
      by metis
    then have h2: x ∈ convex hull (insert 0 D)
      using as assms by (force simp add: substd_simplex)
    obtain a where a: a ∈ D
      using ‹D ≠ {}› by auto
    define d where d ≡ (1 − sum ((·) x) D) / real (card D)
     have ∃ e>0. ball x e ∩ {x. ∀ i∈Basis. i ∉ D ⟶ x · i = 0} ⊆ convex hull
insert 0 D
      unfolding substd_simplex[OF assms]
    proof (intro exI; safe)
      have 0 < card D using ‹D ≠ {}› ‹finite D›
        by (simp add: card_gt_0_iff)
      have Min (((·) x) ‘ D) > 0
        using as ‹D ≠ {}› ‹finite D› by (simp)
      moreover have d > 0
        using as ‹0 < card D› by (auto simp: d_def)
      ultimately show min (Min (((·) x) ‘ D)) d > 0
        by auto
      fix y :: 'a
      assume y2: ∀ i∈Basis. i ∉ D ⟶ y·i = 0
      assume y ∈ ball x (min (Min ((·) x ‘ D)) d)
      then have y: dist x y < min (Min ((·) x ‘ D)) d
        by auto
      have sum ((·) y) D ≤ sum (λi. x·i + d) D
      proof (rule sum_mono)
        fix i
        assume i ∈ D
        with D have i: i ∈ Basis
          by auto
        have |y·i − x·i| ≤ norm (y − x)
        by (metis i inner_commute inner_diff_right norm_bound_Basis_le order_refl)
        also have ... < d
          by (metis dist_norm min_less_iff_conj norm_minus_commute y)
```

```
          finally have |y·i − x·i| < d .
          then show y · i ≤ x · i + d by auto
        qed
        also have ... ≤ 1
          unfolding sum.distrib sum_constant d_def using ‹0 < card D›
          by auto
        finally show sum ((·) y) D ≤ 1 .

        fix i :: 'a
        assume i: i ∈ Basis
        then show 0 ≤ y·i
        proof (cases i∈D)
          case True
          have norm (x − y) < x·i
            using y Min_gr_iff[of (·) x ‘ D norm (x − y)] ‹0 < card D› ‹i ∈ D›
            by (simp add: dist_norm card_gt_0_iff)
          then show 0 ≤ y·i
            using Basis_le_norm[OF i, of x − y] and as(1)[rule_format]
            by (auto simp: inner_simps)
        qed (use y2 in auto)
      qed
      then have x ∈ rel_interior (convex hull (insert 0 D))
        using h0 h2 rel_interior_ball by force
    }
    ultimately have
      ⋀x. x ∈ rel_interior (convex hull insert 0 D) ⟷
        x ∈ {x. (∀i∈D. 0 < x · i) ∧ sum ((·) x) D < 1 ∧ (∀i∈Basis. i ∉ D ⟶
x · i = 0)}
      by blast
    then show ?thesis by (rule set_eqI)
  qed
qed

lemma rel_interior_substd_simplex_nonempty:
  assumes D ≠ {}
    and D ⊆ Basis
  obtains a :: 'a::euclidean_space
    where a ∈ rel_interior (convex hull (insert 0 D))
proof −
  let ?a = sum (λb::'a::euclidean_space. inverse (2 ∗ real (card D)) ∗R b) D
  have finite D
    using assms finite_Basis infinite_super by blast
  then have d1: 0 < real (card D)
    using ‹D ≠ {}› by auto
  {
    fix i
    assume i ∈ D
    have ?a · i = sum (λj. if i = j then inverse (2 ∗ real (card D)) else 0) D
      unfolding inner_sum_left
```

```
      using ‹i ∈ D› by (auto simp: inner_Basis subsetD[OF assms(2)] intro:
sum.cong)
    also have ... = inverse (2 * real (card D))
      using ‹i ∈ D› ‹finite D› by auto
    finally have ?a · i = inverse (2 * real (card D)) .
  }
  note ** = this
  show ?thesis
  proof
    show ?a ∈ rel_interior (convex hull (insert 0 D))
      unfolding rel_interior_substd_simplex[OF assms(2)]
    proof safe
      fix i
      assume i ∈ D
      have 0 < inverse (2 * real (card D))
        using d1 by auto
      also have . . . = ?a · i using **[of i] ‹i ∈ D›
        by auto
      finally show 0 < ?a · i by auto
    next
      have sum ((·) ?a) D = sum (λi. inverse (2 * real (card D))) D
        by (rule sum.cong) (rule refl, rule **)
      also have . . . < 1
        unfolding sum_constant divide_real_def[symmetric]
        by (auto simp add: field_simps)
      finally show sum ((·) ?a) D < 1 by auto
    next
      fix i
      assume i ∈ Basis and i ∉ D
      have ?a ∈ span D
      proof (rule span_sum[of D (λb. b /R (2 * real (card D))) D])
        {
          fix x :: 'a::euclidean_space
          assume x ∈ D
          then have x ∈ span D
            using span_base[of _ D] by auto
          then have x /R (2 * real (card D)) ∈ span D
            using span_mul[of x D (inverse (real (card D)) / 2)] by auto
        }
        then show ⋀x. x∈D ⟹ x /R (2 * real (card D)) ∈ span D
          by auto
      qed
      then show ?a · i = 0
       using ‹i ∉ D› unfolding span_substd_basis[OF assms(2)] using ‹i ∈ Basis›
by auto
    qed
  qed
qed
```

5.0.3 Relative interior of convex set

```
lemma rel_interior_convex_nonempty_aux:
  fixes S :: 'n::euclidean_space set
  assumes convex S
    and 0 ∈ S
  shows rel_interior S ≠ {}
proof (cases S = {0})
  case True
  then show ?thesis using rel_interior_sing by auto
next
  case False
  obtain B where B: independent B ∧ B ≤ S ∧ S ≤ span B ∧ card B = dim S
    using basis_exists[of S] by metis
  then have B ≠ {}
    using B assms ‹S ≠ {0}› span_empty by auto
  have insert 0 B ≤ span B
    using subspace_span[of B] subspace_0[of span B]
      span_superset by auto
  then have span (insert 0 B) ≤ span B
    using span_span[of B] span_mono[of insert 0 B span B] by blast
  then have convex hull insert 0 B ≤ span B
    using convex_hull_subset_span[of insert 0 B] by auto
  then have span (convex hull insert 0 B) ≤ span B
    using span_span[of B]
      span_mono[of convex hull insert 0 B span B] by blast
  then have *: span (convex hull insert 0 B) = span B
   using span_mono[of B convex hull insert 0 B] hull_subset[of insert 0 B] by auto
  then have span (convex hull insert 0 B) = span S
    using B span_mono[of B S] span_mono[of S span B]
      span_span[of B] by auto
  moreover have 0 ∈ affine hull (convex hull insert 0 B)
    using hull_subset[of convex hull insert 0 B] hull_subset[of insert 0 B] by auto
  ultimately have **: affine hull (convex hull insert 0 B) = affine hull S
    using affine_hull_span_0[of convex hull insert 0 B] affine_hull_span_0[of S]
      assms hull_subset[of S]
    by auto
  obtain d and f :: 'n ⇒ 'n where
    fd: card d = card B linear f f ` B = d
      f ` span B = {x. ∀ i∈Basis. i ∉ d ⟶ x · i = (0::real)} ∧ inj_on f (span B)
    and d: d ⊆ Basis
    using basis_to_substdbasis_subspace_isomorphism[of B,OF _ ] B by auto
  then have bounded_linear f
    using linear_conv_bounded_linear by auto
  have d ≠ {}
    using fd B ‹B ≠ {}› by auto
  have insert 0 d = f ` (insert 0 B)
    using fd linear_0 by auto
  then have (convex hull (insert 0 d)) = f ` (convex hull (insert 0 B))
    using convex_hull_linear_image[of f (insert 0 d)]
```

```
      convex_hull_linear_image[of f (insert 0 B)] ‹linear f›
    by auto
  moreover have rel_interior (f ‘ (convex hull insert 0 B)) = f ‘ rel_interior (convex
hull insert 0 B)
  proof (rule rel_interior_injective_on_span_linear_image[OF ‹bounded_linear f›])
    show inj_on f (span (convex hull insert 0 B))
      using fd * by auto
  qed
  ultimately have rel_interior (convex hull insert 0 B) ≠ {}
    using rel_interior_substd_simplex_nonempty[OF ‹d ≠ {}› d] by fastforce
  moreover have convex hull (insert 0 B) ⊆ S
    using B assms hull_mono[of insert 0 B S convex] convex_hull_eq by auto
  ultimately show ?thesis
    using subset_rel_interior[of convex hull insert 0 B S] ** by auto
qed

lemma rel_interior_eq_empty:
  fixes S :: 'n::euclidean_space set
  assumes convex S
  shows rel_interior S = {} ⟷ S = {}
proof -
  {
    assume S ≠ {}
    then obtain a where a ∈ S by auto
    then have 0 ∈ (+) (-a) ‘ S
      using assms exI[of (λx. x ∈ S ∧ - a + x = 0) a] by auto
    then have rel_interior ((+) (-a) ‘ S) ≠ {}
      using rel_interior_convex_nonempty_aux[of (+) (-a) ‘ S]
        convex_translation[of S -a] assms
      by auto
    then have rel_interior S ≠ {}
      using rel_interior_translation [of - a] by simp
  }
  then show ?thesis by auto
qed

lemma interior_simplex_nonempty:
  fixes S :: 'N :: euclidean_space set
  assumes independent S finite S card S = DIM('N)
  obtains a where a ∈ interior (convex hull (insert 0 S))
proof -
  have affine hull (insert 0 S) = UNIV
    by (simp add: hull_inc affine_hull_span_0 dim_eq_full[symmetric]
        assms(1) assms(3) dim_eq_card_independent)
  moreover have rel_interior (convex hull insert 0 S) ≠ {}
    using rel_interior_eq_empty [of convex hull (insert 0 S)] by auto
  ultimately have interior (convex hull insert 0 S) ≠ {}
    by (simp add: rel_interior_interior)
  with that show ?thesis
```

```
    by auto
qed

lemma convex_rel_interior:
  fixes S :: 'n::euclidean_space set
  assumes convex S
  shows convex (rel_interior S)
proof -
  {
    fix x y and u :: real
    assume assm: x ∈ rel_interior S y ∈ rel_interior S 0 ≤ u u ≤ 1
    then have x ∈ S
      using rel_interior_subset by auto
    have x − u *R (x−y) ∈ rel_interior S
    proof (cases 0 = u)
      case False
      then have 0 < u using assm by auto
      then show ?thesis
        using assm rel_interior_convex_shrink[of S y x u] assms ‹x ∈ S› by auto
    next
      case True
      then show ?thesis using assm by auto
    qed
    then have (1 − u) *R x + u *R y ∈ rel_interior S
      by (simp add: algebra_simps)
  }
  then show ?thesis
    unfolding convex_alt by auto
qed

lemma convex_closure_rel_interior:
  fixes S :: 'n::euclidean_space set
  assumes convex S
  shows closure (rel_interior S) = closure S
proof -
  have h1: closure (rel_interior S) ≤ closure S
    using closure_mono[of rel_interior S S] rel_interior_subset[of S] by auto
  show ?thesis
  proof (cases S = {})
    case False
    then obtain a where a: a ∈ rel_interior S
      using rel_interior_eq_empty assms by auto
    { fix x
      assume x: x ∈ closure S
      {
        assume x = a
        then have x ∈ closure (rel_interior S)
          using a unfolding closure_def by auto
      }
```

```
    moreover
    {
      assume x ≠ a
      {
        fix e :: real
        assume e > 0
        define e1 where e1 = min 1 (e/norm (x − a))
        then have e1: e1 > 0 e1 ≤ 1 e1 ∗ norm (x − a) ≤ e
          using ‹x ≠ a› ‹e > 0› le_divide_eq[of e1 e norm (x − a)]
          by simp_all
        then have ∗: x − e1 ∗R (x − a) ∈ rel_interior S
          using rel_interior_closure_convex_shrink[of S a x e1] assms x a e1_def
          by auto
        have ∃ y. y ∈ rel_interior S ∧ y ≠ x ∧ dist y x ≤ e
          using ∗ ‹x ≠ a› e1 by force
      }
      then have x islimpt rel_interior S
        unfolding islimpt_approachable_le by auto
      then have x ∈ closure(rel_interior S)
        unfolding closure_def by auto
    }
    ultimately have x ∈ closure(rel_interior S) by auto
  }
  then show ?thesis using h1 by auto
 qed auto
qed

lemma rel_interior_same_affine_hull:
  fixes S :: 'n::euclidean_space set
  assumes convex S
  shows affine hull (rel_interior S) = affine hull S
  by (metis assms closure_same_affine_hull convex_closure_rel_interior)

lemma rel_interior_aff_dim:
  fixes S :: 'n::euclidean_space set
  assumes convex S
  shows aff_dim (rel_interior S) = aff_dim S
  by (metis aff_dim_affine_hull2 assms rel_interior_same_affine_hull)

lemma rel_interior_rel_interior:
  fixes S :: 'n::euclidean_space set
  assumes convex S
  shows rel_interior (rel_interior S) = rel_interior S
proof −
  have openin (top_of_set (affine hull (rel_interior S))) (rel_interior S)
   using openin_rel_interior[of S] rel_interior_same_affine_hull[of S] assms by auto
  then show ?thesis
    using rel_interior_def by auto
qed
```

lemma *rel_interior_rel_open*:
  fixes *S* :: *'n::euclidean_space set*
  assumes *convex S*
  shows *rel_open* (*rel_interior S*)
  unfolding *rel_open_def* **using** *rel_interior_rel_interior assms* **by** *auto*

lemma *convex_rel_interior_closure_aux*:
  fixes *x y z* :: *'n::euclidean_space*
  assumes $0 < a$ $0 < b$ $(a + b) *_R z = a *_R x + b *_R y$
  obtains *e* **where** $0 < e$ $e < 1$ $z = y - e *_R (y - x)$
proof −
  define *e* **where** $e = a / (a + b)$
  have $z = (1 / (a + b)) *_R ((a + b) *_R z)$
    using *assms* **by** (*simp add*: *eq_vector_fraction_iff*)
  also have $\ldots = (1 / (a + b)) *_R (a *_R x + b *_R y)$
    using *assms scaleR_cancel_left*[*of* $1/(a+b)$ $(a + b) *_R z$ $a *_R x + b *_R y$]
    by *auto*
  also have $\ldots = y - e *_R (y-x)$
    using *e_def assms*
    by (*simp add*: *divide_simps vector_fraction_eq_iff*) (*simp add*: *algebra_simps*)
  finally have $z = y - e *_R (y-x)$
    by *auto*
  moreover have $e > 0$ $e < 1$ **using** *e_def assms* **by** *auto*
  ultimately show *?thesis* **using** *that*[*of e*] **by** *auto*
qed

lemma *convex_rel_interior_closure*:
  fixes *S* :: *'n::euclidean_space set*
  assumes *convex S*
  shows *rel_interior* (*closure S*) = *rel_interior S*
proof (*cases* $S = \{\}$)
  case *True*
  then show *?thesis*
    using *assms rel_interior_eq_empty* **by** *auto*
next
  case *False*
  have *rel_interior* (*closure S*) ⊇ *rel_interior S*
    using *subset_rel_interior*[*of S closure S*] *closure_same_affine_hull closure_subset*
    by *auto*
  moreover
  {
    fix *z*
    assume *z*: *z* ∈ *rel_interior* (*closure S*)
    obtain *x* **where** *x*: *x* ∈ *rel_interior S*
      using ‹$S \neq \{\}$› *assms rel_interior_eq_empty* **by** *auto*
    have *z* ∈ *rel_interior S*
    proof (*cases* $x = z$)
      case *True*

```
    then show ?thesis using x by auto
  next
    case False
    obtain e where e: e > 0 cball z e ∩ affine hull closure S ≤ closure S
      using z rel_interior_cball[of closure S] by auto
    hence *: 0 < e/norm(z−x) using e False by auto
    define y where y = z + (e/norm(z−x)) *R (z−x)
    have yball: y ∈ cball z e
      using y_def dist_norm[of z y] e by auto
    have x ∈ affine hull closure S
      using x rel_interior_subset_closure hull_inc[of x closure S] by blast
    moreover have z ∈ affine hull closure S
      using z rel_interior_subset hull_subset[of closure S] by blast
    ultimately have y ∈ affine hull closure S
      using y_def affine_affine_hull[of closure S]
        mem_affine_3_minus [of affine hull closure S z z x e/norm(z−x)] by auto
    then have y ∈ closure S using e yball by auto
    have (1 + (e/norm(z−x))) *R z = (e/norm(z−x)) *R x + y
      using y_def by (simp add: algebra_simps)
    then obtain e1 where 0 < e1 e1 < 1 z = y − e1 *R (y − x)
      using * convex_rel_interior_closure_aux[of e / norm (z − x) 1 z x y]
      by (auto simp add: algebra_simps)
    then show ?thesis
      using rel_interior_closure_convex_shrink assms x ‹y ∈ closure S›
      by fastforce
  qed
 }
 ultimately show ?thesis by auto
qed

lemma convex_interior_closure:
 fixes S :: 'n::euclidean_space set
 assumes convex S
 shows interior (closure S) = interior S
 using closure_aff_dim[of S] interior_rel_interior_gen[of S]
   interior_rel_interior_gen[of closure S]
   convex_rel_interior_closure[of S] assms
 by auto

lemma closure_eq_rel_interior_eq:
 fixes S1 S2 :: 'n::euclidean_space set
 assumes convex S1
   and convex S2
 shows closure S1 = closure S2 ⟷ rel_interior S1 = rel_interior S2
 by (metis convex_rel_interior_closure convex_closure_rel_interior assms)

lemma closure_eq_between:
 fixes S1 S2 :: 'n::euclidean_space set
 assumes convex S1
```

```
    and convex S2
  shows closure S1 = closure S2 ⟷ rel_interior S1 ≤ S2 ∧ S2 ⊆ closure S1
  (is ?A ⟷ ?B)
proof
  assume ?A
  then show ?B
    by (metis assms closure_subset convex_rel_interior_closure rel_interior_subset)
next
  assume ?B
  then have closure S1 ⊆ closure S2
    by (metis assms(1) convex_closure_rel_interior closure_mono)
  moreover from ‹?B› have closure S1 ⊇ closure S2
    by (metis closed_closure closure_minimal)
  ultimately show ?A ..
qed

lemma open_inter_closure_rel_interior:
  fixes S A :: 'n::euclidean_space set
  assumes convex S
    and open A
  shows A ∩ closure S = {} ⟷ A ∩ rel_interior S = {}
  by (metis assms convex_closure_rel_interior open_Int_closure_eq_empty)

lemma rel_interior_open_segment:
  fixes a :: 'a :: euclidean_space
  shows rel_interior(open_segment a b) = open_segment a b
proof (cases a = b)
  case True then show ?thesis by auto
next
  case False then
  have open_segment a b = affine hull {a, b} ∩ ball ((a + b) /R 2) (norm (b −
a) / 2)
    by (simp add: open_segment_as_ball)
  then show ?thesis
    unfolding rel_interior_eq openin_open
    by (metis Elementary_Metric_Spaces.open_ball False affine_hull_open_segment)
qed

lemma rel_interior_closed_segment:
  fixes a :: 'a :: euclidean_space
  shows rel_interior(closed_segment a b) =
         (if a = b then {a} else open_segment a b)
proof (cases a = b)
  case True then show ?thesis by auto
next
  case False then show ?thesis
    by simp
       (metis closure_open_segment convex_open_segment convex_rel_interior_closure
              rel_interior_open_segment)
```

qed

lemmas *rel_interior_segment* = *rel_interior_closed_segment rel_interior_open_segment*

5.0.4 The relative frontier of a set

definition *rel_frontier S* = *closure S* − *rel_interior S*

lemma *rel_frontier_empty* [*simp*]: *rel_frontier* {} = {}
by (*simp add*: *rel_frontier_def*)

lemma *rel_frontier_eq_empty*:
fixes *S* :: *'n::euclidean_space set*
shows *rel_frontier S* = {} ⟷ *affine S*
unfolding *rel_frontier_def*
using *rel_interior_subset_closure* **by** (*auto simp add*: *rel_interior_eq_closure* [*symmetric*])

lemma *rel_frontier_sing* [*simp*]:
fixes *a* :: *'n::euclidean_space*
shows *rel_frontier* {*a*} = {}
by (*simp add*: *rel_frontier_def*)

lemma *rel_frontier_affine_hull*:
fixes *S* :: *'a::euclidean_space set*
shows *rel_frontier S* ⊆ *affine hull S*
using *closure_affine_hull rel_frontier_def* **by** *fastforce*

lemma *rel_frontier_cball* [*simp*]:
fixes *a* :: *'n::euclidean_space*
shows *rel_frontier*(*cball a r*) = (*if r* = *0 then* {} *else sphere a r*)
proof (*cases rule*: *linorder_cases* [*of r 0*])
case *less* **then show** *?thesis*
by (*force simp*: *sphere_def*)
next
case *equal* **then show** *?thesis* **by** *simp*
next
case *greater* **then show** *?thesis*
by *simp* (*metis centre_in_ball empty_iff frontier_cball frontier_def interior_cball interior_rel_interior_gen rel_frontier_def*)
qed

lemma *rel_frontier_translation*:
fixes *a* :: *'a::euclidean_space*
shows *rel_frontier*((λ*x*. *a* + *x*) ‘ *S*) = (λ*x*. *a* + *x*) ‘ (*rel_frontier S*)
by (*simp add*: *rel_frontier_def translation_diff rel_interior_translation closure_translation*)

lemma *rel_frontier_nonempty_interior*:
fixes *S* :: *'n::euclidean_space set*
shows *interior S* ≠ {} ⟹ *rel_frontier S* = *frontier S*

```
  by (metis frontier_def interior_rel_interior_gen rel_frontier_def)

lemma rel_frontier_frontier:
  fixes S :: 'n::euclidean_space set
  shows affine hull S = UNIV ⟹ rel_frontier S = frontier S
  by (simp add: frontier_def rel_frontier_def rel_interior_interior)

lemma closest_point_in_rel_frontier:
   ⟦closed S; S ≠ {}; x ∈ affine hull S − rel_interior S⟧
   ⟹ closest_point S x ∈ rel_frontier S
  by (simp add: closest_point_in_rel_interior closest_point_in_set rel_frontier_def)

lemma closed_rel_frontier [iff]:
  fixes S :: 'n::euclidean_space set
  shows closed (rel_frontier S)
proof −
  have *: closedin (top_of_set (affine hull S)) (closure S − rel_interior S)
    by (simp add: closed_subset closedin_diff closure_affine_hull openin_rel_interior)
  show ?thesis
  proof (rule closedin_closed_trans[of affine hull S rel_frontier S])
    show closedin (top_of_set (affine hull S)) (rel_frontier S)
      by (simp add: ∗ rel_frontier_def)
  qed simp
qed

lemma closed_rel_boundary:
  fixes S :: 'n::euclidean_space set
  shows closed S ⟹ closed(S − rel_interior S)
  by (metis closed_rel_frontier closure_closed rel_frontier_def)

lemma compact_rel_boundary:
  fixes S :: 'n::euclidean_space set
  shows compact S ⟹ compact(S − rel_interior S)
 by (metis bounded_diff closed_rel_boundary closure_eq compact_closure compact_imp_closed)

lemma bounded_rel_frontier:
  fixes S :: 'n::euclidean_space set
  shows bounded S ⟹ bounded(rel_frontier S)
by (simp add: bounded_closure bounded_diff rel_frontier_def)

lemma compact_rel_frontier_bounded:
  fixes S :: 'n::euclidean_space set
  shows bounded S ⟹ compact(rel_frontier S)
using bounded_rel_frontier closed_rel_frontier compact_eq_bounded_closed by blast

lemma compact_rel_frontier:
  fixes S :: 'n::euclidean_space set
  shows compact S ⟹ compact(rel_frontier S)
by (meson compact_eq_bounded_closed compact_rel_frontier_bounded)
```

lemma *convex_same_rel_interior_closure*:
  fixes *S* :: *'n::euclidean_space set*
  shows ⟦*convex S*; *convex T*⟧
    ⟹ *rel_interior S* = *rel_interior T* ⟷ *closure S* = *closure T*
by (*simp add*: *closure_eq_rel_interior_eq*)

lemma *convex_same_rel_interior_closure_straddle*:
  fixes *S* :: *'n::euclidean_space set*
  shows ⟦*convex S*; *convex T*⟧
    ⟹ *rel_interior S* = *rel_interior T* ⟷
      rel_interior S ⊆ *T* ∧ *T* ⊆ *closure S*
by (*simp add*: *closure_eq_between convex_same_rel_interior_closure*)

lemma *convex_rel_frontier_aff_dim*:
  fixes *S1 S2* :: *'n::euclidean_space set*
  assumes *convex S1*
    and *convex S2*
    and *S2* ≠ {}
    and *S1* ≤ *rel_frontier S2*
  shows *aff_dim S1* < *aff_dim S2*
proof −
  have *S1* ⊆ *closure S2*
    using *assms* **unfolding** *rel_frontier_def* **by** *auto*
  then have ∗: *affine hull S1* ⊆ *affine hull S2*
    using *hull_mono*[*of S1 closure S2*] *closure_same_affine_hull*[*of S2*] **by** *blast*
  then have *aff_dim S1* ≤ *aff_dim S2*
    using ∗ *aff_dim_affine_hull*[*of S1*] *aff_dim_affine_hull*[*of S2*]
      aff_dim_subset[*of affine hull S1 affine hull S2*]
    by *auto*
  moreover
  {
    assume *eq*: *aff_dim S1* = *aff_dim S2*
    then have *S1* ≠ {}
      using *aff_dim_empty*[*of S1*] *aff_dim_empty*[*of S2*] ‹*S2* ≠ {}› **by** *auto*
    have ∗∗: *affine hull S1* = *affine hull S2*
      by (*simp_all add*: ∗ *eq* ‹*S1* ≠ {}› *affine_dim_equal*)
    obtain *a* **where** *a*: *a* ∈ *rel_interior S1*
      using ‹*S1* ≠ {}› *rel_interior_eq_empty assms* **by** *auto*
    obtain *T* **where** *T*: *open T a* ∈ *T* ∩ *S1 T* ∩ *affine hull S1* ⊆ *S1*
      using *mem_rel_interior*[*of a S1*] *a* **by** *auto*
    then have *a* ∈ *T* ∩ *closure S2*
      using *a assms* **unfolding** *rel_frontier_def* **by** *auto*
    then obtain *b* **where** *b*: *b* ∈ *T* ∩ *rel_interior S2*
      using *open_inter_closure_rel_interior*[*of S2 T*] *assms T* **by** *auto*
    then have *b* ∈ *affine hull S1*
      using *rel_interior_subset hull_subset*[*of S2*] ∗∗ **by** *auto*
    then have *b* ∈ *S1*
      using *T b* **by** *auto*

```
    then have False
      using b assms unfolding rel_frontier_def by auto
  }
  ultimately show ?thesis
    using less_le by auto
qed

lemma convex_rel_interior_if:
  fixes S :: 'n::euclidean_space set
  assumes convex S
    and z ∈ rel_interior S
  shows ∀x∈affine hull S. ∃m. m > 1 ∧ (∀e. e > 1 ∧ e ≤ m ⟶ (1 − e) *R x
+ e *R z ∈ S)
proof −
  obtain e1 where e1: e1 > 0 ∧ cball z e1 ∩ affine hull S ⊆ S
    using mem_rel_interior_cball[of z S] assms by auto
  {
    fix x
    assume x: x ∈ affine hull S
    {
      assume x ≠ z
      define m where m = 1 + e1/norm(x−z)
      hence m > 1 using e1 ‹x ≠ z› by auto
      {
        fix e
        assume e: e > 1 ∧ e ≤ m
        have z ∈ affine hull S
          using assms rel_interior_subset hull_subset[of S] by auto
        then have *: (1 − e)*R x + e *R z ∈ affine hull S
          using mem_affine[of affine hull S x z (1−e) e] affine_affine_hull[of S] x
          by auto
        have norm (z + e *R x − (x + e *R z)) = norm ((e − 1) *R (x − z))
          by (simp add: algebra_simps)
        also have ... = (e − 1) * norm (x−z)
          using norm_scaleR e by auto
        also have ... ≤ (m − 1) * norm (x − z)
          using e mult_right_mono[of _ _ norm(x−z)] by auto
        also have ... = (e1 / norm (x − z)) * norm (x − z)
          using m_def by auto
        also have ... = e1
          using ‹x ≠ z› e1 by simp
        finally have **: norm (z + e *R x − (x + e *R z)) ≤ e1
          by auto
        have (1 − e)*R x+ e *R z ∈ cball z e1
          using m_def **
          unfolding cball_def dist_norm
          by (auto simp add: algebra_simps)
        then have (1 − e) *R x+ e *R z ∈ S
          using e * e1 by auto
```

```
      }
      then have ∃m. m > 1 ∧ (∀e. e > 1 ∧ e ≤ m ⟶ (1 − e) *R x + e *R z
∈ S )
        using ‹m> 1 › by auto
    }
    moreover
    {
      assume x = z
      define m where m = 1 + e1
      then have m > 1
        using e1 by auto
      {
        fix e
        assume e: e > 1 ∧ e ≤ m
        then have (1 − e) *R x + e *R z ∈ S
          using e1 x ‹x = z› by (auto simp add: algebra_simps)
        then have (1 − e) *R x + e *R z ∈ S
          using e by auto
      }
      then have ∃m. m > 1 ∧ (∀e. e > 1 ∧ e ≤ m ⟶ (1 − e) *R x + e *R z
∈ S)
        using ‹m > 1› by auto
    }
    ultimately have ∃m. m > 1 ∧ (∀e. e > 1 ∧ e ≤ m ⟶ (1 − e) *R x + e
*R z ∈ S )
      by blast
  }
  then show ?thesis by auto
qed

lemma convex_rel_interior_if2:
  fixes S :: 'n::euclidean_space set
  assumes convex S
  assumes z ∈ rel_interior S
  shows ∀x∈affine hull S. ∃e. e > 1 ∧ (1 − e)*R x + e *R z ∈ S
  using convex_rel_interior_if[of S z] assms by auto

lemma convex_rel_interior_only_if:
  fixes S :: 'n::euclidean_space set
  assumes convex S
    and S ≠ {}
  assumes ∀x∈S. ∃e. e > 1 ∧ (1 − e) *R x + e *R z ∈ S
  shows z ∈ rel_interior S
proof −
  obtain x where x: x ∈ rel_interior S
    using rel_interior_eq_empty assms by auto
  then have x ∈ S
    using rel_interior_subset by auto
  then obtain e where e: e > 1 ∧ (1 − e) *R x + e *R z ∈ S
```

```
    using assms by auto
  define y where [abs_def]: y = (1 − e) *R x + e *R z
  then have y ∈ S using e by auto
  define e1 where e1 = 1/e
  then have 0 < e1 ∧ e1 < 1 using e by auto
  then have z =y − (1 − e1) *R (y − x)
    using e1_def y_def by (auto simp add: algebra_simps)
  then show ?thesis
    using rel_interior_convex_shrink[of S x y 1−e1] ‹0 < e1 ∧ e1 < 1› ‹y ∈ S› x
assms
    by auto
qed

lemma convex_rel_interior_iff:
  fixes S :: 'n::euclidean_space set
  assumes convex S
    and S ≠ {}
  shows z ∈ rel_interior S ⟷ (∀x∈S. ∃e. e > 1 ∧ (1 − e) *R x + e *R z ∈ S)
  using assms hull_subset[of S affine]
    convex_rel_interior_if[of S z] convex_rel_interior_only_if[of S z]
  by auto

lemma convex_rel_interior_iff2:
  fixes S :: 'n::euclidean_space set
  assumes convex S
    and S ≠ {}
  shows z ∈ rel_interior S ⟷ (∀x∈affine hull S. ∃e. e > 1 ∧ (1 − e) *R x +
e *R z ∈ S)
 using assms hull_subset[of S] convex_rel_interior_if2[of S z] convex_rel_interior_only_if[of
S z]
  by auto

lemma convex_interior_iff:
  fixes S :: 'n::euclidean_space set
  assumes convex S
  shows z ∈ interior S ⟷ (∀x. ∃e. e > 0 ∧ z + e *R x ∈ S)
proof (cases aff_dim S = int DIM('n))
  case False
  { assume z ∈ interior S
    then have False
      using False interior_rel_interior_gen[of S] by auto }
  moreover
  { assume r: ∀x. ∃e. e > 0 ∧ z + e *R x ∈ S
    { fix x
      obtain e1 where e1: e1 > 0 ∧ z + e1 *R (x − z) ∈ S
        using r by auto
      obtain e2 where e2: e2 > 0 ∧ z + e2 *R (z − x) ∈ S
        using r by auto
      define x1 where [abs_def]: x1 = z + e1 *R (x − z)
```

```
      then have x1: x1 ∈ affine hull S
        using e1 hull_subset[of S] by auto
      define x2 where [abs_def]: x2 = z + e2 *R (z − x)
      then have x2: x2 ∈ affine hull S
        using e2 hull_subset[of S] by auto
      have *: e1/(e1+e2) + e2/(e1+e2) = 1
        using add_divide_distrib[of e1 e2 e1+e2] e1 e2 by simp
      then have z = (e2/(e1+e2)) *R x1 + (e1/(e1+e2)) *R x2
        by (simp add: x1_def x2_def algebra_simps) (simp add: * pth_8)
      then have z: z ∈ affine hull S
        using mem_affine[of affine hull S x1 x2 e2/(e1+e2) e1/(e1+e2)]
          x1 x2 affine_affine_hull[of S] *
        by auto
      have x1 − x2 = (e1 + e2) *R (x − z)
        using x1_def x2_def by (auto simp add: algebra_simps)
      then have x = z+(1/(e1+e2)) *R (x1−x2)
        using e1 e2 by simp
      then have x ∈ affine hull S
        using mem_affine_3_minus[of affine hull S z x1 x2 1/(e1+e2)]
          x1 x2 z affine_affine_hull[of S]
        by auto
    }
    then have affine hull S = UNIV
      by auto
    then have aff_dim S = int DIM('n)
      using aff_dim_affine_hull[of S] by (simp)
    then have False
      using False by auto
  }
  ultimately show ?thesis by auto
next
  case True
  then have S ≠ {}
    using aff_dim_empty[of S] by auto
  have *: affine hull S = UNIV
    using True affine_hull_UNIV by auto
  {
    assume z ∈ interior S
    then have z ∈ rel_interior S
      using True interior_rel_interior_gen[of S] by auto
    then have **: ∀ x. ∃ e. e > 1 ∧ (1 − e) *R x + e *R z ∈ S
      using convex_rel_interior_iff2[of S z] assms ‹S ≠ {}› * by auto
    fix x
    obtain e1 where e1: e1 > 1 (1 − e1) *R (z − x) + e1 *R z ∈ S
      using **[rule_format, of z−x] by auto
    define e where [abs_def]: e = e1 − 1
    then have (1 − e1) *R (z − x) + e1 *R z = z + e *R x
      by (simp add: algebra_simps)
    then have e > 0 z + e *R x ∈ S
```

```
    using e1 e_def by auto
  then have ∃ e. e > 0 ∧ z + e *R x ∈ S
    by auto
}
moreover
{
  assume r: ∀ x. ∃ e. e > 0 ∧ z + e *R x ∈ S
  {
    fix x
    obtain e1 where e1: e1 > 0 z + e1 *R (z − x) ∈ S
      using r[rule_format, of z−x] by auto
    define e where e = e1 + 1
    then have z + e1 *R (z − x) = (1 − e) *R x + e *R z
      by (simp add: algebra_simps)
    then have e > 1 (1 − e)*R x + e *R z ∈ S
      using e1 e_def by auto
    then have ∃ e. e > 1 ∧ (1 − e) *R x + e *R z ∈ S by auto
  }
  then have z ∈ rel_interior S
    using convex_rel_interior_iff2[of S z] assms ‹S ≠ {}› by auto
  then have z ∈ interior S
    using True interior_rel_interior_gen[of S] by auto
}
ultimately show ?thesis by auto
qed
```

Relative interior and closure under common operations

```
lemma rel_interior_inter_aux: ⋂{rel_interior S |S. S ∈ I} ⊆ ⋂I
proof −
  {
    fix y
    assume y ∈ ⋂{rel_interior S |S. S ∈ I}
    then have y: ∀ S ∈ I. y ∈ rel_interior S
      by auto
    {
      fix S
      assume S ∈ I
      then have y ∈ S
        using rel_interior_subset y by auto
    }
    then have y ∈ ⋂I by auto
  }
  then show ?thesis by auto
qed

lemma convex_closure_rel_interior_inter:
  assumes ∀ S∈I. convex (S :: 'n::euclidean_space set)
    and ⋂{rel_interior S |S. S ∈ I} ≠ {}
```

```
  shows ⋂{closure S |S. S ∈ I} ≤ closure (⋂{rel_interior S |S. S ∈ I})
proof -
  obtain x where x: ∀S∈I. x ∈ rel_interior S
    using assms by auto
  {
    fix y
    assume y ∈ ⋂{closure S |S. S ∈ I}
    then have y: ∀S ∈ I. y ∈ closure S
      by auto
    {
      assume y = x
      then have y ∈ closure (⋂{rel_interior S |S. S ∈ I})
        using x closure_subset[of ⋂{rel_interior S |S. S ∈ I}] by auto
    }
    moreover
    {
      assume y ≠ x
      { fix e :: real
        assume e: e > 0
        define e1 where e1 = min 1 (e/norm (y - x))
        then have e1: e1 > 0 e1 ≤ 1 e1 * norm (y - x) ≤ e
          using ‹y ≠ x› ‹e > 0› le_divide_eq[of e1 e norm (y - x)]
          by simp_all
        define z where z = y - e1 *R (y - x)
        {
          fix S
          assume S ∈ I
          then have z ∈ rel_interior S
            using rel_interior_closure_convex_shrink[of S x y e1] assms x y e1 z_def
            by auto
        }
        then have *: z ∈ ⋂{rel_interior S |S. S ∈ I}
          by auto
        have ∃z. z ∈ ⋂{rel_interior S |S. S ∈ I} ∧ z ≠ y ∧ dist z y ≤ e
          using ‹y ≠ x› z_def * e1 e dist_norm[of z y]
          by (rule_tac x=z in exI) auto
      }
      then have y islimpt ⋂{rel_interior S |S. S ∈ I}
        unfolding islimpt_approachable_le by blast
      then have y ∈ closure (⋂{rel_interior S |S. S ∈ I})
        unfolding closure_def by auto
    }
    ultimately have y ∈ closure (⋂{rel_interior S |S. S ∈ I})
      by auto
  }
  then show ?thesis by auto
qed

lemma convex_closure_inter:
```

assumes $\forall S{\in}I.$ *convex* $(S :: 'n::euclidean\_space\ set)$
and $\bigcap\{rel\_interior\ S\ |S.\ S \in I\} \neq \{\}$
shows *closure* $(\bigcap I) = \bigcap\{closure\ S\ |S.\ S \in I\}$
proof –
have $\bigcap\{closure\ S\ |S.\ S \in I\} \leq closure\ (\bigcap\{rel\_interior\ S\ |S.\ S \in I\})$
using *convex_closure_rel_interior_inter assms* **by** *auto*
moreover
have *closure* $(\bigcap\{rel\_interior\ S\ |S.\ S \in I\}) \leq closure\ (\bigcap I)$
using *rel_interior_inter_aux closure_mono*[*of* $\bigcap\{rel\_interior\ S\ |S.\ S \in I\}\ \bigcap I$]
by *auto*
ultimately show *?thesis*
using *closure_Int*[*of I*] **by** *auto*
qed

lemma *convex_inter_rel_interior_same_closure*:
assumes $\forall S{\in}I.$ *convex* $(S :: 'n::euclidean\_space\ set)$
and $\bigcap\{rel\_interior\ S\ |S.\ S \in I\} \neq \{\}$
shows *closure* $(\bigcap\{rel\_interior\ S\ |S.\ S \in I\}) = closure\ (\bigcap I)$
proof –
have $\bigcap\{closure\ S\ |S.\ S \in I\} \leq closure\ (\bigcap\{rel\_interior\ S\ |S.\ S \in I\})$
using *convex_closure_rel_interior_inter assms* **by** *auto*
moreover
have *closure* $(\bigcap\{rel\_interior\ S\ |S.\ S \in I\}) \leq closure\ (\bigcap I)$
using *rel_interior_inter_aux closure_mono*[*of* $\bigcap\{rel\_interior\ S\ |S.\ S \in I\}\ \bigcap I$]
by *auto*
ultimately show *?thesis*
using *closure_Int*[*of I*] **by** *auto*
qed

lemma *convex_rel_interior_inter*:
assumes $\forall S{\in}I.$ *convex* $(S :: 'n::euclidean\_space\ set)$
and $\bigcap\{rel\_interior\ S\ |S.\ S \in I\} \neq \{\}$
shows *rel_interior* $(\bigcap I) \subseteq \bigcap\{rel\_interior\ S\ |S.\ S \in I\}$
proof –
have *convex* $(\bigcap I)$
using *assms convex_Inter* **by** *auto*
moreover
have *convex* $(\bigcap\{rel\_interior\ S\ |S.\ S \in I\})$
using *assms convex_rel_interior* **by** (*force intro*: *convex_Inter*)
ultimately
have *rel_interior* $(\bigcap\{rel\_interior\ S\ |S.\ S \in I\}) = rel\_interior\ (\bigcap I)$
using *convex_inter_rel_interior_same_closure assms*
closure_eq_rel_interior_eq[*of* $\bigcap\{rel\_interior\ S\ |S.\ S \in I\}\ \bigcap I$]
by *blast*
then show *?thesis*
using *rel_interior_subset*[*of* $\bigcap\{rel\_interior\ S\ |S.\ S \in I\}$] **by** *auto*
qed

lemma *convex_rel_interior_finite_inter*:

```
  assumes ∀ S∈I. convex (S :: 'n::euclidean_space set)
    and ⋂{rel_interior S |S. S ∈ I} ≠ {}
    and finite I
  shows rel_interior (⋂I) = ⋂{rel_interior S |S. S ∈ I}
proof −
  have ⋂I ≠ {}
    using assms rel_interior_inter_aux[of I] by auto
  have convex (⋂I)
    using convex_Inter assms by auto
  show ?thesis
  proof (cases I = {})
    case True
    then show ?thesis
      using Inter_empty rel_interior_UNIV by auto
  next
    case False
    {
      fix z
      assume z: z ∈ ⋂{rel_interior S |S. S ∈ I}
      {
        fix x
        assume x: x ∈ ⋂I
        {
          fix S
          assume S: S ∈ I
          then have z ∈ rel_interior S x ∈ S
            using z x by auto
          then have ∃ m. m > 1 ∧ (∀ e. e > 1 ∧ e ≤ m ⟶ (1 − e)*R x + e *R
z ∈ S)
            using convex_rel_interior_if[of S z] S assms hull_subset[of S] by auto
        }
        then obtain mS where
          mS: ∀ S∈I. mS S > 1 ∧ (∀ e. e > 1 ∧ e ≤ mS S ⟶ (1 − e) *R x + e
*R z ∈ S) by metis
        define e where e = Min (mS ` I)
        then have e ∈ mS ` I using assms ‹I ≠ {}› by simp
        then have e > 1 using mS by auto
        moreover have ∀ S∈I. e ≤ mS S
          using e_def assms by auto
        ultimately have ∃ e > 1. (1 − e) *R x + e *R z ∈ ⋂I
          using mS by auto
      }
      then have z ∈ rel_interior (⋂I)
        using convex_rel_interior_iff[of ⋂I z] ‹⋂I ≠ {}› ‹convex (⋂I)› by auto
    }
    then show ?thesis
      using convex_rel_interior_inter[of I] assms by auto
  qed
qed
```

lemma *convex_closure_inter_two*:
  fixes *S T* :: *'n::euclidean_space set*
  assumes *convex S*
    and *convex T*
  assumes *rel_interior S ∩ rel_interior T ≠ {}*
  shows *closure (S ∩ T) = closure S ∩ closure T*
  using *convex_closure_inter[of {S,T}] assms* **by** *auto*

lemma *convex_rel_interior_inter_two*:
  fixes *S T* :: *'n::euclidean_space set*
  assumes *convex S*
    and *convex T*
    and *rel_interior S ∩ rel_interior T ≠ {}*
  shows *rel_interior (S ∩ T) = rel_interior S ∩ rel_interior T*
  using *convex_rel_interior_finite_inter[of {S,T}] assms* **by** *auto*

lemma *convex_affine_closure_Int*:
  fixes *S T* :: *'n::euclidean_space set*
  assumes *convex S*
    and *affine T*
    and *rel_interior S ∩ T ≠ {}*
  shows *closure (S ∩ T) = closure S ∩ T*
proof –
  have *affine hull T = T*
    using *assms* **by** *auto*
  then have *rel_interior T = T*
    using *rel_interior_affine_hull[of T]* **by** *metis*
  moreover have *closure T = T*
    using *assms affine_closed[of T]* **by** *auto*
  ultimately show *?thesis*
    using *convex_closure_inter_two[of S T] assms affine_imp_convex* **by** *auto*
qed

lemma *connected_component_1_gen*:
  fixes *S* :: *'a :: euclidean_space set*
  assumes *DIM('a) = 1*
  shows *connected_component S a b ⟷ closed_segment a b ⊆ S*
unfolding *connected_component_def*
by (*metis* (*no_types*, *lifting*) *assms subsetD subsetI convex_contains_segment convex_segment(1)*
      ends_in_segment connected_convex_1_gen)

lemma *connected_component_1*:
  fixes *S* :: *real set*
  shows *connected_component S a b ⟷ closed_segment a b ⊆ S*
by (*simp add*: *connected_component_1_gen*)

lemma *convex_affine_rel_interior_Int*:

```
  fixes S T :: 'n::euclidean_space set
  assumes convex S
    and affine T
    and rel_interior S ∩ T ≠ {}
  shows rel_interior (S ∩ T) = rel_interior S ∩ T
proof -
  have affine hull T = T
    using assms by auto
  then have rel_interior T = T
    using rel_interior_affine_hull[of T] by metis
  moreover have closure T = T
    using assms affine_closed[of T] by auto
  ultimately show ?thesis
    using convex_rel_interior_inter_two[of S T] assms affine_imp_convex by auto
qed

lemma convex_affine_rel_frontier_Int:
   fixes S T :: 'n::euclidean_space set
  assumes convex S
    and affine T
    and interior S ∩ T ≠ {}
  shows rel_frontier(S ∩ T) = frontier S ∩ T
using assms
  unfolding rel_frontier_def  frontier_def
 using convex_affine_closure_Int convex_affine_rel_interior_Int rel_interior_nonempty_interior
by fastforce

lemma rel_interior_convex_Int_affine:
  fixes S :: 'a::euclidean_space set
  assumes convex S affine T interior S ∩ T ≠ {}
    shows rel_interior(S ∩ T) = interior S ∩ T
proof -
  obtain a where aS: a ∈ interior S and aT:a ∈ T
    using assms by force
  have rel_interior S = interior S
   by (metis (no_types) aS affine_hull_nonempty_interior equals0D rel_interior_interior)
  then show ?thesis
   by (metis (no_types) affine_imp_convex assms convex_rel_interior_inter_two hull_same
rel_interior_affine_hull)
qed

lemma closure_convex_Int_affine:
  fixes S :: 'a::euclidean_space set
  assumes convex S affine T rel_interior S ∩ T ≠ {}
  shows closure(S ∩ T) = closure S ∩ T
proof
  have closure (S ∩ T) ⊆ closure T
    by (simp add: closure_mono)
  also have ... ⊆ T
```

by (*simp add*: *affine_closed assms*)
finally show *closure*(*S* ∩ *T*) ⊆ *closure S* ∩ *T*
by (*simp add*: *closure_mono*)
next
obtain *a* **where** *a* ∈ *rel_interior S a* ∈ *T*
using *assms* **by** *auto*
then have *ssT*: *subspace* ((λ*x*. (−*a*)+*x*) ‘ *T*) **and** *a* ∈ *S*
using *affine_diffs_subspace rel_interior_subset assms* **by** *blast+*
show *closure S* ∩ *T* ⊆ *closure* (*S* ∩ *T*)
proof
fix *x* **assume** *x* ∈ *closure S* ∩ *T*
show *x* ∈ *closure* (*S* ∩ *T*)
proof (*cases x* = *a*)
case *True*
then show *?thesis*
using ‹*a* ∈ *S*› ‹*a* ∈ *T*› *closure_subset* **by** *fastforce*
next
case *False*
then have *x* ∈ *closure*(*open_segment a x*)
by *auto*
then show *?thesis*
using ‹*x* ∈ *closure S* ∩ *T*› *assms convex_affine_closure_Int* **by** *blast*
qed
qed
qed

lemma *subset_rel_interior_convex*:
fixes *S T* :: ′*n*::*euclidean_space set*
assumes *convex S*
and *convex T*
and *S* ≤ *closure T*
and ¬ *S* ⊆ *rel_frontier T*
shows *rel_interior S* ⊆ *rel_interior T*
proof −
have ∗: *S* ∩ *closure T* = *S*
using *assms* **by** *auto*
have ¬ *rel_interior S* ⊆ *rel_frontier T*
using *closure_mono*[*of rel_interior S rel_frontier T*] *closed_rel_frontier*[*of T*]
closure_closed[*of S*] *convex_closure_rel_interior*[*of S*] *closure_subset*[*of S*] *assms*
by *auto*
then have *rel_interior S* ∩ *rel_interior* (*closure T*) ≠ {}
using *assms rel_frontier_def*[*of T*] *rel_interior_subset convex_rel_interior_closure*[*of T*]
by *auto*
then have *rel_interior S* ∩ *rel_interior T* = *rel_interior* (*S* ∩ *closure T*)
using *assms convex_closure convex_rel_interior_inter_two*[*of S closure T*]
convex_rel_interior_closure[*of T*]
by *auto*
also have . . . = *rel_interior S*

```
    using * by auto
  finally show ?thesis
    by auto
qed

lemma rel_interior_convex_linear_image:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes linear f
    and convex S
  shows f ` (rel_interior S) = rel_interior (f ` S)
proof (cases S = {})
  case True
  then show ?thesis
    using assms by auto
next
  case False
  interpret linear f by fact
  have *: f ` (rel_interior S) ⊆ f ` S
    unfolding image_mono using rel_interior_subset by auto
  have f ` S ⊆ f ` (closure S)
    unfolding image_mono using closure_subset by auto
  also have ... = f ` (closure (rel_interior S))
    using convex_closure_rel_interior assms by auto
  also have ... ⊆ closure (f ` (rel_interior S))
    using closure_linear_image_subset assms by auto
  finally have closure (f ` S) = closure (f ` rel_interior S)
    using closure_mono[of f ` S closure (f ` rel_interior S)] closure_closure
      closure_mono[of f ` rel_interior S f ` S] *
    by auto
  then have rel_interior (f ` S) = rel_interior (f ` rel_interior S)
    using assms convex_rel_interior
      linear_conv_bounded_linear[of f] convex_linear_image[of _ S]
      convex_linear_image[of _ rel_interior S]
      closure_eq_rel_interior_eq[of f ` S f ` rel_interior S]
    by auto
  then have rel_interior (f ` S) ⊆ f ` rel_interior S
    using rel_interior_subset by auto
  moreover
  {
    fix z
    assume z ∈ f ` rel_interior S
    then obtain z1 where z1: z1 ∈ rel_interior S f z1 = z by auto
    {
      fix x
      assume x ∈ f ` S
      then obtain x1 where x1: x1 ∈ S f x1 = x by auto
      then obtain e where e: e > 1 (1 − e) *R x1 + e *R z1 ∈ S
        using convex_rel_interior_iff[of S z1] ‹convex S› x1 z1 by auto
      moreover have f ((1 − e) *R x1 + e *R z1) = (1 − e) *R x + e *R z
```

```
      using x1 z1 by (simp add: linear_add linear_scale ‹linear f›)
    ultimately have (1 − e) *R x + e *R z ∈ f ‘ S
      using imageI[of (1 − e) *R x1 + e *R z1 S f] by auto
    then have ∃ e. e > 1 ∧ (1 − e) *R x + e *R z ∈ f ‘ S
      using e by auto
  }
  then have z ∈ rel_interior (f ‘ S)
    using convex_rel_interior_iff[of f ‘ S z] ‹convex S› ‹linear f›
      ‹S ≠ {}› convex_linear_image[of f S]  linear_conv_bounded_linear[of f]
    by auto
}
ultimately show ?thesis by auto
qed

lemma rel_interior_convex_linear_preimage:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes linear f
    and convex S
    and f −‘ (rel_interior S) ≠ {}
  shows rel_interior (f −‘ S) = f −‘ (rel_interior S)
proof −
  interpret linear f by fact
  have S ≠ {}
    using assms by auto
  have nonemp: f −‘ S ≠ {}
    by (metis assms(3) rel_interior_subset subset_empty vimage_mono)
  then have S ∩ (range f) ≠ {}
    by auto
  have conv: convex (f −‘ S)
    using convex_linear_vimage assms by auto
  then have convex (S ∩ range f)
    by (simp add: assms(2) convex_Int convex_linear_image linear_axioms)
  {
    fix z
    assume z ∈ f −‘ (rel_interior S)
    then have z: f z ∈ rel_interior S
      by auto
    {
      fix x
      assume x ∈ f −‘ S
      then have f x ∈ S by auto
      then obtain e where e: e > 1 (1 − e) *R f x + e *R f z ∈ S
        using convex_rel_interior_iff[of S f z] z assms ‹S ≠ {}› by auto
      moreover have (1 − e) *R f x + e *R f z = f ((1 − e) *R x + e *R z)
        using ‹linear f› by (simp add: linear_iff)
      ultimately have ∃ e. e > 1 ∧ (1 − e) *R x + e *R z ∈ f −‘ S
        using e by auto
    }
    then have z ∈ rel_interior (f −‘ S)
```

```
      using convex_rel_interior_iff[of f -` S z] conv nonemp by auto
  }
  moreover
  {
    fix z
    assume z: z ∈ rel_interior (f -` S)
    {
      fix x
      assume x ∈ S ∩ range f
      then obtain y where y: f y = x y ∈ f -` S by auto
      then obtain e where e: e > 1 (1 - e) *R y + e *R z ∈ f -` S
        using convex_rel_interior_iff[of f -` S z] z conv by auto
      moreover have (1 - e) *R x + e *R f z = f ((1 - e) *R y + e *R z)
        using ‹linear f› y by (simp add: linear_iff)
      ultimately have ∃e. e > 1 ∧ (1 - e) *R x + e *R f z ∈ S ∩ range f
        using e by auto
    }
    then have f z ∈ rel_interior (S ∩ range f)
      using ‹convex (S ∩ (range f))› ‹S ∩ range f ≠ {}›
        convex_rel_interior_iff[of S ∩ (range f) f z]
      by auto
    moreover have affine (range f)
      by (simp add: linear_axioms linear_subspace_image subspace_imp_affine)
    ultimately have f z ∈ rel_interior S
      using convex_affine_rel_interior_Int[of S range f] assms by auto
    then have z ∈ f -` (rel_interior S)
      by auto
  }
  ultimately show ?thesis by auto
qed

lemma rel_interior_Times:
  fixes S :: 'n::euclidean_space set
    and T :: 'm::euclidean_space set
  assumes convex S
    and convex T
  shows rel_interior (S × T) = rel_interior S × rel_interior T
proof (cases S = {} ∨ T = {})
  case True
  then show ?thesis
    by auto
next
  case False
  then have S ≠ {} T ≠ {}
    by auto
  then have ri: rel_interior S ≠ {} rel_interior T ≠ {}
    using rel_interior_eq_empty assms by auto
  then have fst -` rel_interior S ≠ {}
    using fst_vimage_eq_Times[of rel_interior S] by auto
```

then have *rel_interior* ((*fst* :: ′*n* ∗ ′*m* ⇒ ′*n*) −‘ *S*) = *fst* −‘ *rel_interior S*
using *linear_fst* ‹*convex S*› *rel_interior_convex_linear_preimage*[*of fst S*] **by** *auto*
then have *s*: *rel_interior* (*S* × (*UNIV* :: ′*m set*)) = *rel_interior S* × *UNIV*
by (*simp add*: *fst_vimage_eq_Times*)
from *ri* **have** *snd* −‘ *rel_interior T* ≠ {}
using *snd_vimage_eq_Times*[*of rel_interior T*] **by** *auto*
then have *rel_interior* ((*snd* :: ′*n* ∗ ′*m* ⇒ ′*m*) −‘ *T*) = *snd* −‘ *rel_interior T*
using *linear_snd* ‹*convex T*› *rel_interior_convex_linear_preimage*[*of snd T*] **by** *auto*
then have *t*: *rel_interior* ((*UNIV* :: ′*n set*) × *T*) = *UNIV* × *rel_interior T*
by (*simp add*: *snd_vimage_eq_Times*)
from *s t* **have** ∗: *rel_interior* (*S* × (*UNIV* :: ′*m set*)) ∩ *rel_interior* ((*UNIV* :: ′*n set*) × *T*) =
rel_interior S × *rel_interior T* **by** *auto*
have *S* × *T* = *S* × (*UNIV* :: ′*m set*) ∩ (*UNIV* :: ′*n set*) × *T*
by *auto*
then have *rel_interior* (*S* × *T*) = *rel_interior* ((*S* × (*UNIV* :: ′*m set*)) ∩ ((*UNIV* :: ′*n set*) × *T*))
by *auto*
also have ... = *rel_interior* (*S* × (*UNIV* :: ′*m set*)) ∩ *rel_interior* ((*UNIV* :: ′*n set*) × *T*)
using ∗ *ri assms convex_Times*
by (*subst convex_rel_interior_inter_two*) *auto*
finally show *?thesis* **using** ∗ **by** *auto*
qed

lemma *rel_interior_scaleR*:
fixes *S* :: ′*n*::*euclidean_space set*
assumes *c* ≠ *0*
shows ((∗$_R$) *c*) ‘ (*rel_interior S*) = *rel_interior* (((∗$_R$) *c*) ‘ *S*)
using *rel_interior_injective_linear_image*[*of* ((∗$_R$) *c*) *S*]
linear_conv_bounded_linear[*of* (∗$_R$) *c*] *linear_scaleR injective_scaleR*[*of c*] *assms*
by *auto*

lemma *rel_interior_convex_scaleR*:
fixes *S* :: ′*n*::*euclidean_space set*
assumes *convex S*
shows ((∗$_R$) *c*) ‘ (*rel_interior S*) = *rel_interior* (((∗$_R$) *c*) ‘ *S*)
by (*metis assms linear_scaleR rel_interior_convex_linear_image*)

lemma *convex_rel_open_scaleR*:
fixes *S* :: ′*n*::*euclidean_space set*
assumes *convex S*
and *rel_open S*
shows *convex* (((∗$_R$) *c*) ‘ *S*) ∧ *rel_open* (((∗$_R$) *c*) ‘ *S*)
by (*metis assms convex_scaling rel_interior_convex_scaleR rel_open_def*)

lemma *convex_rel_open_finite_inter*:
assumes ∀ *S*∈*I*. *convex* (*S* :: ′*n*::*euclidean_space set*) ∧ *rel_open S*

```
    and finite I
  shows convex (⋂ I) ∧ rel_open (⋂ I)
proof (cases ⋂{rel_interior S |S. S ∈ I} = {})
  case True
  then have ⋂ I = {}
    using assms unfolding rel_open_def by auto
  then show ?thesis
    unfolding rel_open_def by auto
next
  case False
  then have rel_open (⋂ I)
    using assms unfolding rel_open_def
    using convex_rel_interior_finite_inter[of I]
    by auto
  then show ?thesis
    using convex_Inter assms by auto
qed

lemma convex_rel_open_linear_image:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes linear f
    and convex S
    and rel_open S
  shows convex (f ` S) ∧ rel_open (f ` S)
 by (metis assms convex_linear_image rel_interior_convex_linear_image rel_open_def)

lemma convex_rel_open_linear_preimage:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes linear f
    and convex S
    and rel_open S
  shows convex (f -` S) ∧ rel_open (f -` S)
proof (cases f -` (rel_interior S) = {})
  case True
  then have f -` S = {}
    using assms unfolding rel_open_def by auto
  then show ?thesis
    unfolding rel_open_def by auto
next
  case False
  then have rel_open (f -` S)
    using assms unfolding rel_open_def
    using rel_interior_convex_linear_preimage[of f S]
    by auto
  then show ?thesis
    using convex_linear_vimage assms
    by auto
qed
```

```
lemma rel_interior_projection:
  fixes S :: ('m::euclidean_space × 'n::euclidean_space) set
    and f :: 'm::euclidean_space ⇒ 'n::euclidean_space set
  assumes convex S
    and f = (λy. {z. (y, z) ∈ S})
  shows (y, z) ∈ rel_interior S ⟷ (y ∈ rel_interior {y. (f y ≠ {})} ∧ z ∈
rel_interior (f y))
proof −
  {
    fix y
    assume y ∈ {y. f y ≠ {}}
    then obtain z where (y, z) ∈ S
      using assms by auto
    then have ∃x. x ∈ S ∧ y = fst x
      by auto
    then obtain x where x ∈ S y = fst x
      by blast
    then have y ∈ fst ` S
      unfolding image_def by auto
  }
  then have fst ` S = {y. f y ≠ {}}
    unfolding fst_def using assms by auto
  then have h1: fst ` rel_interior S = rel_interior {y. f y ≠ {}}
    using rel_interior_convex_linear_image[of fst S] assms linear_fst by auto
  {
    fix y
    assume y ∈ rel_interior {y. f y ≠ {}}
    then have y ∈ fst ` rel_interior S
      using h1 by auto
    then have *: rel_interior S ∩ fst −` {y} ≠ {}
      by auto
    moreover have aff: affine (fst −` {y})
      unfolding affine_alt by (simp add: algebra_simps)
    ultimately have **: rel_interior (S ∩ fst −` {y}) = rel_interior S ∩ fst −`
{y}
      using convex_affine_rel_interior_Int[of S fst −` {y}] assms by auto
    have conv: convex (S ∩ fst −` {y})
      using convex_Int assms aff affine_imp_convex by auto
    {
      fix x
      assume x ∈ f y
      then have (y, x) ∈ S ∩ (fst −` {y})
        using assms by auto
      moreover have x = snd (y, x) by auto
      ultimately have x ∈ snd ` (S ∩ fst −` {y})
        by blast
    }
    then have snd ` (S ∩ fst −` {y}) = f y
      using assms by auto
```

```
    then have ***: rel_interior (f y) = snd ` rel_interior (S ∩ fst -` {y})
      using rel_interior_convex_linear_image[of snd S ∩ fst -` {y}] linear_snd conv
      by auto
    {
      fix z
      assume z ∈ rel_interior (f y)
      then have z ∈ snd ` rel_interior (S ∩ fst -` {y})
        using *** by auto
      moreover have {y} = fst ` rel_interior (S ∩ fst -` {y})
        using * ** rel_interior_subset by auto
      ultimately have (y, z) ∈ rel_interior (S ∩ fst -` {y})
        by force
      then have (y,z) ∈ rel_interior S
        using ** by auto
    }
    moreover
    {
      fix z
      assume (y, z) ∈ rel_interior S
      then have (y, z) ∈ rel_interior (S ∩ fst -` {y})
        using ** by auto
      then have z ∈ snd ` rel_interior (S ∩ fst -` {y})
        by (metis Range_iff snd_eq_Range)
      then have z ∈ rel_interior (f y)
        using *** by auto
    }
    ultimately have ⋀z. (y, z) ∈ rel_interior S ⟷ z ∈ rel_interior (f y)
      by auto
  }
  then have h2: ⋀y z. y ∈ rel_interior {t. f t ≠ {}} ⟹
    (y, z) ∈ rel_interior S ⟷ z ∈ rel_interior (f y)
    by auto
  {
    fix y z
    assume asm: (y, z) ∈ rel_interior S
    then have y ∈ fst ` rel_interior S
      by (metis Domain_iff fst_eq_Domain)
    then have y ∈ rel_interior {t. f t ≠ {}}
      using h1 by auto
    then have y ∈ rel_interior {t. f t ≠ {}} and (z ∈ rel_interior (f y))
      using h2 asm by auto
  }
  then show ?thesis using h2 by blast
qed

lemma rel_frontier_Times:
  fixes S :: 'n::euclidean_space set
    and T :: 'm::euclidean_space set
  assumes convex S
```

and *convex T*
shows *rel_frontier S × rel_frontier T ⊆ rel_frontier (S × T)*
by (*force simp*: *rel_frontier_def rel_interior_Times assms closure_Times*)

Relative interior of convex cone

lemma *cone_rel_interior*:
fixes *S* :: *'m::euclidean_space set*
assumes *cone S*
shows *cone* ({*0*} ∪ *rel_interior S*)
proof (*cases S* = {})
case *True*
then show *?thesis*
by (*simp add*: *cone_0*)
next
case *False*
then have ∗: *0* ∈ *S* ∧ (∀ *c*. *c* > *0* ⟶ ($*_R$) *c* ‘ *S* = *S*)
using *cone_iff* [*of S*] *assms* **by** *auto*
then have ∗: *0* ∈ ({*0*} ∪ *rel_interior S*)
and ∀ *c*. *c* > *0* ⟶ ($*_R$) *c* ‘ ({*0*} ∪ *rel_interior S*) = ({*0*} ∪ *rel_interior S*)
by (*auto simp add*: *rel_interior_scaleR*)
then show *?thesis*
using *cone_iff* [*of* {*0*} ∪ *rel_interior S*] **by** *auto*
qed

lemma *rel_interior_convex_cone_aux*:
fixes *S* :: *'m::euclidean_space set*
assumes *convex S*
shows (*c*, *x*) ∈ *rel_interior* (*cone hull* ({(*1* :: *real*)} × *S*)) ⟷
c > *0* ∧ *x* ∈ ((($*_R$) *c*) ‘ (*rel_interior S*))
proof (*cases S* = {})
case *True*
then show *?thesis*
by (*simp add*: *cone_hull_empty*)
next
case *False*
then obtain *s* **where** *s* ∈ *S* **by** *auto*
have *conv*: *convex* ({(*1* :: *real*)} × *S*)
using *convex_Times*[*of* {(*1* :: *real*)} *S*] *assms convex_singleton*[*of 1* :: *real*]
by *auto*
define *f* **where** *f y* = {*z*. (*y*, *z*) ∈ *cone hull* ({*1* :: *real*} × *S*)} **for** *y*
then have ∗: (*c*, *x*) ∈ *rel_interior* (*cone hull* ({(*1* :: *real*)} × *S*)) =
(*c* ∈ *rel_interior* {*y*. *f y* ≠ {}} ∧ *x* ∈ *rel_interior* (*f c*))
using *convex_cone_hull*[*of* {(*1* :: *real*)} × *S*] *conv*
by (*subst rel_interior_projection*) *auto*
{
fix *y* :: *real*
assume *y* ≥ *0*
then have *y* $*_R$ (*1*,*s*) ∈ *cone hull* ({*1* :: *real*} × *S*)

```
      using cone_hull_expl[of {(1 :: real)} × S] ‹s ∈ S› by auto
    then have f y ≠ {}
      using f_def by auto
  }
  then have {y. f y ≠ {}} = {0..}
    using f_def cone_hull_expl[of {1 :: real} × S] by auto
  then have **: rel_interior {y. f y ≠ {}} = {0<..}
    using rel_interior_real_semiline by auto
  {
    fix c :: real
    assume c > 0
    then have f c = ((*R) c ` S)
      using f_def cone_hull_expl[of {1 :: real} × S] by auto
    then have rel_interior (f c) = (*R) c ` rel_interior S
      using rel_interior_convex_scaleR[of S c] assms by auto
  }
  then show ?thesis using * ** by auto
qed

lemma rel_interior_convex_cone:
  fixes S :: 'm::euclidean_space set
  assumes convex S
  shows rel_interior (cone hull ({1 :: real} × S)) =
    {(c, c *R x) | c x. c > 0 ∧ x ∈ rel_interior S}
  (is ?lhs = ?rhs)
proof −
  {
    fix z
    assume z ∈ ?lhs
    have *: z = (fst z, snd z)
      by auto
    then have z ∈ ?rhs
        using rel_interior_convex_cone_aux[of S fst z snd z] assms ‹z ∈ ?lhs› by
fastforce
  }
  moreover
  {
    fix z
    assume z ∈ ?rhs
    then have z ∈ ?lhs
      using rel_interior_convex_cone_aux[of S fst z snd z] assms
      by auto
  }
  ultimately show ?thesis by blast
qed

lemma convex_hull_finite_union:
  assumes finite I
  assumes ∀ i∈I. convex (S i) ∧ (S i) ≠ {}
```

```
  shows convex hull (⋃(S ‘ I)) =
    {sum (λi. c i *R s i) I | c s. (∀ i∈I. c i ≥ 0) ∧ sum c I = 1 ∧ (∀ i∈I. s i ∈
S i)}
  (is ?lhs = ?rhs)
proof −
  have ?lhs ⊇ ?rhs
  proof
    fix x
    assume x ∈ ?rhs
    then obtain c s where *: sum (λi. c i *R s i) I = x sum c I = 1
      (∀ i∈I. c i ≥ 0) ∧ (∀ i∈I. s i ∈ S i) by auto
    then have ∀ i∈I. s i ∈ convex hull (⋃(S ‘ I))
      using hull_subset[of ⋃(S ‘ I) convex] by auto
    then show x ∈ ?lhs
      unfolding *(1)[symmetric]
      using * assms convex_convex_hull
      by (subst convex_sum) auto
  qed
  {
    fix i
    assume i ∈ I
    with assms have ∃ p. p ∈ S i by auto
  }
  then obtain p where p: ∀ i∈I. p i ∈ S i by metis
  {
    fix i
    assume i ∈ I
    {
      fix x
      assume x ∈ S i
      define c where c j = (if j = i then 1::real else 0) for j
      then have *: sum c I = 1
        using ‹finite I› ‹i ∈ I› sum.delta[of I i λj::'a. 1::real]
        by auto
      define s where s j = (if j = i then x else p j) for j
      then have ∀ j. c j *R s j = (if j = i then x else 0)
        using c_def by (auto simp add: algebra_simps)
      then have x = sum (λi. c i *R s i) I
        using s_def c_def ‹finite I› ‹i ∈ I› sum.delta[of I i λj::'a. x]
        by auto
      moreover have (∀ i∈I. 0 ≤ c i) ∧ sum c I = 1 ∧ (∀ i∈I. s i ∈ S i)
        using * c_def s_def p ‹x ∈ S i› by auto
      ultimately have x ∈ ?rhs
        by force
    }
    then have ?rhs ⊇ S i by auto
  }
  then have *: ?rhs ⊇ ⋃(S ‘ I) by auto
```

```
{
  fix u v :: real
  assume uv: u ≥ 0 ∧ v ≥ 0 ∧ u + v = 1
  fix x y
  assume xy: x ∈ ?rhs ∧ y ∈ ?rhs
  from xy obtain c s where
    xc: x = sum (λi. c i *R s i) I ∧ (∀ i∈I. c i ≥ 0) ∧ sum c I = 1 ∧ (∀ i∈I. s
i ∈ S i)
    by auto
  from xy obtain d t where
    yc: y = sum (λi. d i *R t i) I ∧ (∀ i∈I. d i ≥ 0) ∧ sum d I = 1 ∧ (∀ i∈I. t
i ∈ S i)
    by auto
  define e where e i = u * c i + v * d i for i
  have ge0: ∀ i∈I. e i ≥ 0
    using e_def xc yc uv by simp
  have sum (λi. u * c i) I = u * sum c I
    by (simp add: sum_distrib_left)
  moreover have sum (λi. v * d i) I = v * sum d I
    by (simp add: sum_distrib_left)
  ultimately have sum1: sum e I = 1
    using e_def xc yc uv by (simp add: sum.distrib)
  define q where q i = (if e i = 0 then p i else (u * c i / e i) *R s i + (v * d
i / e i) *R t i)
    for i
  {
    fix i
    assume i: i ∈ I
    have q i ∈ S i
    proof (cases e i = 0)
      case True
      then show ?thesis using i p q_def by auto
    next
      case False
      then show ?thesis
        using mem_convex_alt[of S i s i t i u * (c i) v * (d i)]
          mult_nonneg_nonneg[of u c i] mult_nonneg_nonneg[of v d i]
          assms q_def e_def i False xc yc uv
        by (auto simp del: mult_nonneg_nonneg)
    qed
  }
  then have qs: ∀ i∈I. q i ∈ S i by auto
  {
    fix i
    assume i: i ∈ I
    have (u * c i) *R s i + (v * d i) *R t i = e i *R q i
    proof (cases e i = 0)
      case True
      have ge: u * (c i) ≥ 0 ∧ v * d i ≥ 0
```

using *xc yc uv i* **by** *simp*
moreover from *ge* **have** $u * c\ i \leq 0 \wedge v * d\ i \leq 0$
using *True e_def i* **by** *simp*
ultimately have $u * c\ i = 0 \wedge v * d\ i = 0$ **by** *auto*
with *True* **show** *?thesis* **by** *auto*
next
case *False*
then have $(u * (c\ i)/(e\ i)) *_R (s\ i) + (v * (d\ i)/(e\ i)) *_R (t\ i) = q\ i$
using *q_def* **by** *auto*
then have $e\ i *_R ((u * (c\ i)/(e\ i)) *_R (s\ i) + (v * (d\ i)/(e\ i)) *_R (t\ i))$
$= (e\ i) *_R (q\ i)$ **by** *auto*
with *False* **show** *?thesis* **by** (*simp add*: *algebra_simps*)
qed
}
then have $*$: $\forall i \in I.\ (u * c\ i) *_R s\ i + (v * d\ i) *_R t\ i = e\ i *_R q\ i$
by *auto*
have $u *_R x + v *_R y = sum\ (\lambda i.\ (u * c\ i) *_R s\ i + (v * d\ i) *_R t\ i)\ I$
using *xc yc* **by** (*simp add*: *algebra_simps scaleR_right.sum sum.distrib*)
also have $\ldots = sum\ (\lambda i.\ e\ i *_R q\ i)\ I$
using $*$ **by** *auto*
finally have $u *_R x + v *_R y = sum\ (\lambda i.\ (e\ i) *_R (q\ i))\ I$
by *auto*
then have $u *_R x + v *_R y \in ?rhs$
using *ge0 sum1 qs* **by** *auto*
}
then have *convex ?rhs* **unfolding** *convex_def* **by** *auto*
then show *?thesis*
using ‹$?lhs \supseteq ?rhs$› $*$ *hull_minimal*[*of* $\bigcup(S\ `\ I)$ *?rhs convex*]
by *blast*
qed

lemma *convex_hull_union_two*:
fixes $S\ T$ :: $'m$::*euclidean_space set*
assumes *convex S*
and $S \neq \{\}$
and *convex T*
and $T \neq \{\}$
shows *convex hull* $(S \cup T) =$
$\{u *_R s + v *_R t \mid u\ v\ s\ t.\ u \geq 0 \wedge v \geq 0 \wedge u + v = 1 \wedge s \in S \wedge t \in T\}$
(**is** $?lhs = ?rhs$)
proof
define I :: *nat set* **where** $I = \{1,\ 2\}$
define s **where** $s\ i = ($*if* $i = (1::nat)$ *then S else T*$)$ **for** i
have $\bigcup(s\ `\ I) = S \cup T$
using *s_def I_def* **by** *auto*
then have *convex hull* $(\bigcup(s\ `\ I)) =$ *convex hull* $(S \cup T)$
by *auto*
moreover have *convex hull* $\bigcup(s\ `\ I) =$
$\{\sum\ i \in I.\ c\ i *_R sa\ i \mid c\ sa.\ (\forall i \in I.\ 0 \leq c\ i) \wedge sum\ c\ I = 1 \wedge (\forall i \in I.\ sa\ i \in s$

i)}
using assms s_def I_def
by (subst convex_hull_finite_union) auto
moreover have
{∑i∈I. c i *R sa i | c sa. (∀i∈I. 0 ≤ c i) ∧ sum c I = 1 ∧ (∀i∈I. sa i ∈ s i)} ≤ ?rhs
using s_def I_def **by** auto
ultimately show ?lhs ⊆ ?rhs **by** auto
{
fix x
assume x ∈ ?rhs
then obtain u v s t **where** *: x = u *R s + v *R t ∧ u ≥ 0 ∧ v ≥ 0 ∧ u + v = 1 ∧ s ∈ S ∧ t ∈ T
by auto
then have x ∈ convex hull {s, t}
using convex_hull_2[of s t] **by** auto
then have x ∈ convex hull (S ∪ T)
using * hull_mono[of {s, t} S ∪ T] **by** auto
}
then show ?lhs ⊇ ?rhs **by** blast
qed

proposition ray_to_rel_frontier:
fixes a :: 'a::real_inner
assumes bounded S
and a: a ∈ rel_interior S
and aff: (a + l) ∈ affine hull S
and l ≠ 0
obtains d **where** 0 < d (a + d *R l) ∈ rel_frontier S
⋀e. ⟦0 ≤ e; e < d⟧ ⟹ (a + e *R l) ∈ rel_interior S
proof −
have aaff: a ∈ affine hull S
by (meson a hull_subset rel_interior_subset rev_subsetD)
let ?D = {d. 0 < d ∧ a + d *R l ∉ rel_interior S}
obtain B **where** B > 0 **and** B: S ⊆ ball a B
using bounded_subset_ballD [OF ‹bounded S›] **by** blast
have a + (B / norm l) *R l ∉ ball a B
by (simp add: dist_norm ‹l ≠ 0›)
with B **have** a + (B / norm l) *R l ∉ rel_interior S
using rel_interior_subset subsetCE **by** blast
with ‹B > 0› ‹l ≠ 0› **have** nonMT: ?D ≠ {}
using divide_pos_pos zero_less_norm_iff **by** fastforce
have bdd: bdd_below ?D
by (metis (no_types, lifting) bdd_belowI le_less mem_Collect_eq)
have relin_Ex: ⋀x. x ∈ rel_interior S ⟹
∃e>0. ∀x'∈affine hull S. dist x' x < e ⟶ x' ∈ rel_interior S
using openin_rel_interior [of S] **by** (simp add: openin_euclidean_subtopology_iff)
define d **where** d = Inf ?D
obtain ε **where** 0 < ε **and** ε: ⋀η. ⟦0 ≤ η; η < ε⟧ ⟹ (a + η *R l) ∈ rel_interior

S
 proof −
  obtain *e* **where** *e>0*
   and *e*: $\bigwedge x'$. $x' \in$ *affine hull S* $\Longrightarrow$ *dist* x' *a* < *e* $\Longrightarrow$ $x' \in$ *rel_interior S*
   using *relin_Ex a* **by** *blast*
  show *thesis*
  proof (*rule_tac* ε = *e* / *norm l* **in** *that*)
   show *0* < *e* / *norm l* **by** (*simp add*: ‹*0* < *e*› ‹*l* ≠ *0*›)
  next
   show *a* + η $*_R$ *l* ∈ *rel_interior S* **if** *0* ≤ η η < *e* / *norm l* **for** η
   proof (*rule e*)
    show *a* + η $*_R$ *l* ∈ *affine hull S*
    by (*metis* (*no_types*) *add_diff_cancel_left' aff affine_affine_hull mem_affine_3_minus aaff*)
    show *dist* (*a* + η $*_R$ *l*) *a* < *e*
     using *that* **by** (*simp add*: ‹*l* ≠ *0*› *dist_norm pos_less_divide_eq*)
   qed
  qed
 qed
 have *inint*: $\bigwedge e$. ⟦*0* ≤ *e*; *e* < *d*⟧ $\Longrightarrow$ *a* + *e* $*_R$ *l* ∈ *rel_interior S*
  unfolding *d_def* **using** *cInf_lower* [*OF* _ *bdd*]
  by (*metis* (*no_types*, *lifting*) *a add.right_neutral le_less mem_Collect_eq not_less real_vector.scale_zero_left*)
 have ε ≤ *d*
  unfolding *d_def*
  using ε *dual_order.strict_implies_order le_less_linear*
  by (*blast intro*: *cInf_greatest* [*OF nonMT*])
 with ‹*0* < ε› **have** *0* < *d* **by** *simp*
 have *a* + *d* $*_R$ *l* ∉ *rel_interior S*
 proof
  assume *adl*: *a* + *d* $*_R$ *l* ∈ *rel_interior S*
  obtain *e* **where** *e* > *0*
    and *e*: $\bigwedge x'$. $x' \in$ *affine hull S* $\Longrightarrow$ *dist* x' (*a* + *d* $*_R$ *l*) < *e* $\Longrightarrow$ $x' \in$ *rel_interior S*
   using *relin_Ex adl* **by** *blast*
  have *d* + *e* / *norm l* ≤ *Inf* {*d*. *0* < *d* ∧ *a* + *d* $*_R$ *l* ∉ *rel_interior S*}
  proof (*rule cInf_greatest* [*OF nonMT*], *clarsimp*)
   fix *x*::*real*
   assume *0* < *x* **and** *nonrel*: *a* + *x* $*_R$ *l* ∉ *rel_interior S*
   show *d* + *e* / *norm l* ≤ *x*
   proof (*cases x* < *d*)
    case *True* **with** *inint nonrel* ‹*0* < *x*›
     show *?thesis* **by** *auto*
   next
    case *False*
     then have *dle*: *x* < *d* + *e* / *norm l* $\Longrightarrow$ *dist* (*a* + *x* $*_R$ *l*) (*a* + *d* $*_R$ *l*) < *e*
      by (*simp add*: *field_simps* ‹*l* ≠ *0*›)
     have *ain*: *a* + *x* $*_R$ *l* ∈ *affine hull S*

```
          by (metis add_diff_cancel_left' aff affine_affine_hull mem_affine_3_minus
aaff)
        show ?thesis
          using e [OF ain] nonrel dle by force
      qed
    qed
    then show False
      using ‹0 < e› ‹l ≠ 0› by (simp add: d_def [symmetric] field_simps)
  qed
  moreover have a + d *R l ∈ closure S
  proof (clarsimp simp: closure_approachable)
    fix η::real assume 0 < η
    have 1: a + (d - min d (η / 2 / norm l)) *R l ∈ S
    proof (rule subsetD [OF rel_interior_subset inint])
      show d - min d (η / 2 / norm l) < d
        using ‹l ≠ 0› ‹0 < d› ‹0 < η› by auto
    qed auto
    have norm l * min d (η / (norm l * 2)) ≤ norm l * (η / (norm l * 2))
      by (metis min_def mult_left_mono norm_ge_zero order_refl)
    also have ... < η
      using ‹l ≠ 0› ‹0 < η› by (simp add: field_simps)
    finally have 2: norm l * min d (η / (norm l * 2)) < η .
    show ∃y∈S. dist y (a + d *R l) < η
      using 1 2 ‹0 < d› ‹0 < η›
      by (rule_tac x=a + (d - min d (η / 2 / norm l)) *R l in bexI) (auto simp:
algebra_simps)
  qed
  ultimately have infront: a + d *R l ∈ rel_frontier S
    by (simp add: rel_frontier_def)
  show ?thesis
    by (rule that [OF ‹0 < d› infront inint])
qed

corollary ray_to_frontier:
  fixes a :: 'a::euclidean_space
  assumes bounded S
      and a: a ∈ interior S
      and l ≠ 0
  obtains d where 0 < d (a + d *R l) ∈ frontier S
          ⋀e. ⟦0 ≤ e; e < d⟧ ⟹ (a + e *R l) ∈ interior S
proof -
  have §: interior S = rel_interior S
    using a rel_interior_nonempty_interior by auto
  then have a ∈ rel_interior S
    using a by simp
  moreover have a + l ∈ affine hull S
    using a affine_hull_nonempty_interior by blast
  ultimately show thesis
    by (metis § ‹bounded S› ‹l ≠ 0› frontier_def ray_to_rel_frontier rel_frontier_def
```

that)
qed

lemma *segment_to_rel_frontier_aux*:
fixes *x* :: *'a::euclidean_space*
assumes *convex S bounded S* **and** *x*: $x \in$ *rel_interior S* **and** *y*: $y \in S$ **and** *xy*: $x \neq y$
obtains *z* **where** $z \in$ *rel_frontier S* $y \in$ *closed_segment x z*
open_segment x z $\subseteq$ *rel_interior S*
proof −
have $x + (y - x) \in$ *affine hull S*
using *hull_inc* [*OF y*] **by** *auto*
then obtain *d* **where** $0 < d$ **and** *df*: $(x + d *_R (y-x)) \in$ *rel_frontier S*
and *di*: $\bigwedge e.\ [\![0 \leq e;\ e < d]\!] \Longrightarrow (x + e *_R (y-x)) \in$ *rel_interior S*
by (*rule ray_to_rel_frontier* [*OF* ‹*bounded S*› *x*]) (*use xy* **in** *auto*)
show *?thesis*
proof
show $x + d *_R (y - x) \in$ *rel_frontier S*
by (*simp add*: *df*)
next
have *open_segment x y* $\subseteq$ *rel_interior S*
using *rel_interior_closure_convex_segment* [*OF* ‹*convex S*› *x*] *closure_subset y* **by** *blast*
moreover have $x + d *_R (y - x) \in$ *open_segment x y* **if** $d < 1$
using *xy* ‹$0 < d$› *that* **by** (*force simp*: *in_segment algebra_simps*)
ultimately have $1 \leq d$
using *df rel_frontier_def* **by** *fastforce*
moreover have $x = (1 / d) *_R x + ((d - 1) / d) *_R x$
by (*metis* ‹$0 < d$› *add.commute add_divide_distrib diff_add_cancel divide_self_if less_irrefl scaleR_add_left scaleR_one*)
ultimately show $y \in$ *closed_segment x* $(x + d *_R (y - x))$
unfolding *in_segment*
by (*rule_tac x=1/d* **in** *exI*) (*auto simp*: *algebra_simps*)
next
show *open_segment x* $(x + d *_R (y - x)) \subseteq$ *rel_interior S*
proof (*rule rel_interior_closure_convex_segment* [*OF* ‹*convex S*› *x*])
show $x + d *_R (y - x) \in$ *closure S*
using *df rel_frontier_def* **by** *auto*
qed
qed
qed

lemma *segment_to_rel_frontier*:
fixes *x* :: *'a::euclidean_space*
assumes *S*: *convex S bounded S* **and** *x*: $x \in$ *rel_interior S*
and *y*: $y \in S$ **and** *xy*: $\neg(x = y \wedge S = \{x\})$
obtains *z* **where** $z \in$ *rel_frontier S* $y \in$ *closed_segment x z*
open_segment x z $\subseteq$ *rel_interior S*

proof (*cases x=y*)
  case *True*
  with *xy* **have** $S \neq \{x\}$
    by *blast*
  with *True* **show** *?thesis*
   by (*metis Set.set_insert all_not_in_conv ends_in_segment*(*1*) *insert_iff segment_to_rel_frontier_aux*[*OF S x*] *that y*)
next
  case *False*
  then show *?thesis*
    using *segment_to_rel_frontier_aux* [*OF S x y*] *that* **by** *blast*
qed

proposition *rel_frontier_not_sing*:
  fixes *a* :: *'a::euclidean_space*
  assumes *bounded S*
    shows *rel_frontier S* $\neq \{a\}$
proof (*cases* $S = \{\}$)
  case *True* **then show** *?thesis* **by** *simp*
next
  case *False*
  then obtain *z* **where** $z \in S$
    by *blast*
  then show *?thesis*
  proof (*cases* $S = \{z\}$)
    case *True* **then show** *?thesis* **by** *simp*
  next
    case *False*
    then obtain *w* **where** $w \in S$ $w \neq z$
      using ‹$z \in S$› **by** *blast*
    show *?thesis*
    proof
      assume *rel_frontier S* $= \{a\}$
      then consider $w \notin$ *rel_frontier S* | $z \notin$ *rel_frontier S*
        using ‹$w \neq z$› **by** *auto*
      then show *False*
      proof *cases*
        case *1*
        then have *w*: $w \in$ *rel_interior S*
          using ‹$w \in S$› *closure_subset rel_frontier_def* **by** *fastforce*
        have $w + (w - z) \in$ *affine hull S*
          by (*metis* ‹$w \in S$› ‹$z \in S$› *affine_affine_hull hull_inc mem_affine_3_minus scaleR_one*)
        then obtain *e* **where** $0 < e$ $(w + e *_R (w - z)) \in$ *rel_frontier S*
          using ‹$w \neq z$› ‹$z \in S$› **by** (*metis assms ray_to_rel_frontier right_minus_eq w*)
        moreover obtain *d* **where** $0 < d$ $(w + d *_R (z - w)) \in$ *rel_frontier S*
          using *ray_to_rel_frontier* [*OF* ‹*bounded S*› *w*, *of 1* $*_R$ $(z - w)$] ‹$w \neq z$› ‹$z \in S$›

by (*metis add.commute add.right_neutral diff_add_cancel hull_inc scaleR_one*)
ultimately have $d *_R (z - w) = e *_R (w - z)$
using ‹*rel_frontier S* = {*a*}› **by** *force*
moreover have $e \neq -d$
using ‹$0 < e$› ‹$0 < d$› **by** *force*
ultimately show *False*
by (*metis* (*no_types*, *lifting*) ‹$w \neq z$› *eq_iff_diff_eq_0 minus_diff_eq real_vector.scale_cancel_right real_vector.scale_minus_right scaleR_left.minus*)
next
case *2*
then have *z*: *z* ∈ *rel_interior S*
using ‹*z* ∈ *S*› *closure_subset rel_frontier_def* **by** *fastforce*
have *z* + (*z* − *w*) ∈ *affine hull S*
by (*metis* ‹*z* ∈ *S*› ‹*w* ∈ *S*› *affine_affine_hull hull_inc mem_affine_3_minus scaleR_one*)
then obtain *e* **where** *0* < *e* (*z* + $e *_R$ (*z* − *w*)) ∈ *rel_frontier S*
using ‹$w \neq z$› ‹*w* ∈ *S*› **by** (*metis assms ray_to_rel_frontier right_minus_eq z*)
moreover obtain *d* **where** *0* < *d* (*z* + $d *_R$ (*w* − *z*)) ∈ *rel_frontier S*
using *ray_to_rel_frontier* [*OF* ‹*bounded S*› *z*, *of 1* $*_R$ (*w* − *z*)] ‹$w \neq z$› ‹*w* ∈ *S*›
by (*metis add.commute add.right_neutral diff_add_cancel hull_inc scaleR_one*)
ultimately have $d *_R (w - z) = e *_R (z - w)$
using ‹*rel_frontier S* = {*a*}› **by** *force*
moreover have $e \neq -d$
using ‹$0 < e$› ‹$0 < d$› **by** *force*
ultimately show *False*
by (*metis* (*no_types*, *lifting*) ‹$w \neq z$› *eq_iff_diff_eq_0 minus_diff_eq real_vector.scale_cancel_right real_vector.scale_minus_right scaleR_left.minus*)
qed
qed
qed
qed

5.0.5 Convexity on direct sums

lemma *closure_sum*:
fixes *S T* :: ′*a*::*real_normed_vector set*
shows *closure S* + *closure T* ⊆ *closure* (*S* + *T*)
unfolding *set_plus_image closure_Times* [*symmetric*] *split_def*
by (*intro closure_bounded_linear_image_subset bounded_linear_add bounded_linear_fst bounded_linear_snd*)

lemma *rel_interior_sum*:
fixes *S T* :: ′*n*::*euclidean_space set*
assumes *convex S*
and *convex T*
shows *rel_interior* (*S* + *T*) = *rel_interior S* + *rel_interior T*
proof −

```
  have rel_interior S + rel_interior T = (λ(x,y). x + y) ‘ (rel_interior S ×
rel_interior T)
    by (simp add: set_plus_image)
  also have ... = (λ(x,y). x + y) ‘ rel_interior (S × T)
    using rel_interior_Times assms by auto
  also have ... = rel_interior (S + T)
    using fst_snd_linear convex_Times assms
      rel_interior_convex_linear_image[of (λ(x,y). x + y) S × T]
    by (auto simp add: set_plus_image)
  finally show ?thesis ..
qed

lemma rel_interior_sum_gen:
  fixes S :: 'a ⇒ 'n::euclidean_space set
  assumes ⋀i. i∈I ⟹ convex (S i)
  shows rel_interior (sum S I) = sum (λi. rel_interior (S i)) I
  using rel_interior_sum rel_interior_sing[of 0] assms
  by (subst sum_set_cond_linear[of convex], auto simp add: convex_set_plus)

lemma convex_rel_open_direct_sum:
  fixes S T :: 'n::euclidean_space set
  assumes convex S
    and rel_open S
    and convex T
    and rel_open T
  shows convex (S × T) ∧ rel_open (S × T)
  by (metis assms convex_Times rel_interior_Times rel_open_def)

lemma convex_rel_open_sum:
  fixes S T :: 'n::euclidean_space set
  assumes convex S
    and rel_open S
    and convex T
    and rel_open T
  shows convex (S + T) ∧ rel_open (S + T)
  by (metis assms convex_set_plus rel_interior_sum rel_open_def)

lemma convex_hull_finite_union_cones:
  assumes finite I
    and I ≠ {}
  assumes ⋀i. i∈I ⟹ convex (S i) ∧ cone (S i) ∧ S i ≠ {}
  shows convex hull (⋃(S ‘ I)) = sum S I
  (is ?lhs = ?rhs)
proof -
  {
    fix x
    assume x ∈ ?lhs
    then obtain c xs where
      x: x = sum (λi. c i *R xs i) I ∧ (∀i∈I. c i ≥ 0) ∧ sum c I = 1 ∧ (∀i∈I.
```

```
xs i ∈ S i)
      using convex_hull_finite_union[of I S] assms by auto
    define s where s i = c i *R xs i for i
    have ∀ i∈I. s i ∈ S i
        using s_def x assms by (simp add: mem_cone)
    moreover have x = sum s I using x s_def by auto
    ultimately have x ∈ ?rhs
      using set_sum_alt[of I S] assms by auto
  }
  moreover
  {
    fix x
    assume x ∈ ?rhs
    then obtain s where x: x = sum s I ∧ (∀ i∈I. s i ∈ S i)
      using set_sum_alt[of I S] assms by auto
    define xs where xs i = of_nat(card I) *R s i for i
    then have x = sum (λi. ((1 :: real) / of_nat(card I)) *R xs i) I
      using x assms by auto
    moreover have ∀ i∈I. xs i ∈ S i
      using x xs_def assms by (simp add: cone_def)
    moreover have ∀ i∈I. (1 :: real) / of_nat (card I) ≥ 0
      by auto
    moreover have sum (λi. (1 :: real) / of_nat (card I)) I = 1
      using assms by auto
    ultimately have x ∈ ?lhs
      using assms
      apply (simp add: convex_hull_finite_union[of I S])
      by (rule_tac x = (λi. 1 / (card I)) in exI) auto
  }
  ultimately show ?thesis by auto
qed

lemma convex_hull_union_cones_two:
  fixes S T :: 'm::euclidean_space set
  assumes convex S
    and cone S
    and S ≠ {}
  assumes convex T
    and cone T
    and T ≠ {}
  shows convex hull (S ∪ T) = S + T
proof −
  define I :: nat set where I = {1, 2}
  define A where A i = (if i = (1::nat) then S else T) for i
  have ⋃(A ‘ I) = S ∪ T
    using A_def I_def by auto
  then have convex hull (⋃(A ‘ I)) = convex hull (S ∪ T)
    by auto
  moreover have convex hull ⋃(A ‘ I) = sum A I
```

using *A_def I_def*
by (*metis assms convex_hull_finite_union_cones empty_iff finite.emptyI finite.insertI insertI1*)
moreover have *sum A I = S + T*
using *A_def I_def* **by** (*force simp add: set_plus_def*)
ultimately show *?thesis* **by** *auto*
qed

lemma *rel_interior_convex_hull_union*:
fixes $S :: 'a \Rightarrow 'n::euclidean\_space\ set$
assumes *finite I*
and $\forall i\in I.\ convex\ (S\ i) \wedge S\ i \neq \{\}$
shows $rel\_interior\ (convex\ hull\ (\bigcup(S\ `\ I))) =$
$\{sum\ (\lambda i.\ c\ i *_R s\ i)\ I \mid c\ s.\ (\forall i\in I.\ c\ i > 0) \wedge sum\ c\ I = 1 \wedge$
$(\forall i\in I.\ s\ i \in rel\_interior(S\ i))\}$
(**is** *?lhs = ?rhs*)
proof (*cases* $I = \{\}$)
case *True*
then show *?thesis*
using *convex_hull_empty* **by** *auto*
next
case *False*
define *C0* **where** $C0 = convex\ hull\ (\bigcup(S\ `\ I))$
have $\forall i\in I.\ C0 \geq S\ i$
unfolding *C0_def* **using** $hull\_subset[of\ \bigcup(S\ `\ I)]$ **by** *auto*
define *K0* **where** $K0 = cone\ hull\ (\{1 :: real\} \times C0)$
define *K* **where** $K\ i = cone\ hull\ (\{1 :: real\} \times S\ i)$ **for** *i*
have $\forall i\in I.\ K\ i \neq \{\}$
unfolding *K_def* **using** *assms*
by (*simp add: cone_hull_empty_iff*[*symmetric*])
have *convK*: $\forall i\in I.\ convex\ (K\ i)$
unfolding *K_def*
by (*simp add: assms(2) convex_Times convex_cone_hull*)
have $K0 \supseteq K\ i$ **if** $i \in I$ **for** *i*
unfolding *K0_def K_def*
by (*simp add: Sigma_mono* ‹$\forall i\in I.\ S\ i \subseteq C0$› *hull_mono that*)
then have $K0 \supseteq \bigcup(K\ `\ I)$ **by** *auto*
moreover have *convex K0*
unfolding *K0_def* **by** (*simp add: C0_def convex_Times convex_cone_hull*)
ultimately have *geq*: $K0 \supseteq convex\ hull\ (\bigcup(K\ `\ I))$
using *hull_minimal*[*of _ K0 convex*] **by** *blast*
have $\forall i\in I.\ K\ i \supseteq \{1 :: real\} \times S\ i$
using *K_def* **by** (*simp add: hull_subset*)
then have $\bigcup(K\ `\ I) \supseteq \{1 :: real\} \times \bigcup(S\ `\ I)$
by *auto*
then have $convex\ hull\ \bigcup(K\ `\ I) \supseteq convex\ hull\ (\{1 :: real\} \times \bigcup(S\ `\ I))$
by (*simp add: hull_mono*)
then have $convex\ hull\ \bigcup(K\ `\ I) \supseteq \{1 :: real\} \times C0$
unfolding *C0_def*

using *convex_hull_Times*[*of* {(*1* :: *real*)} ⋃(*S* ' *I*)] *convex_hull_singleton*
by *auto*
moreover have *cone* (*convex hull* (⋃(*K* ' *I*)))
by (*simp add*: *K_def cone_Union cone_cone_hull cone_convex_hull*)
ultimately have *convex hull* (⋃(*K* ' *I*)) ⊇ *K0*
unfolding *K0_def*
using *hull_minimal*[*of* _ *convex hull* (⋃(*K* ' *I*)) *cone*]
by *blast*
then have *K0* = *convex hull* (⋃(*K* ' *I*))
using *geq* **by** *auto*
also have ... = *sum K I*
using *assms False* ‹∀ *i*∈*I*. *K i* ≠ {}› *cone_hull_eq convK*
by (*intro convex_hull_finite_union_cones*; *fastforce simp*: *K_def*)
finally have *K0* = *sum K I* **by** *auto*
then have ∗: *rel_interior K0* = *sum* (λ*i*. (*rel_interior* (*K i*))) *I*
using *rel_interior_sum_gen*[*of I K*] *convK* **by** *auto*
{
fix *x*
assume *x* ∈ *?lhs*
then have (*1*::*real*, *x*) ∈ *rel_interior K0*
using *K0_def C0_def rel_interior_convex_cone_aux*[*of C0 1*::*real x*] *convex_convex_hull*
by *auto*
then obtain *k* **where** *k*: (*1*::*real*, *x*) = *sum k I* ∧ (∀ *i*∈*I*. *k i* ∈ *rel_interior* (*K*
i))
using ‹*finite I*› ∗ *set_sum_alt*[*of I* λ*i*. *rel_interior* (*K i*)] **by** *auto*
{
fix *i*
assume *i* ∈ *I*
then have *convex* (*S i*) ∧ *k i* ∈ *rel_interior* (*cone hull* {*1*} × *S i*)
using *k K_def assms* **by** *auto*
then have ∃ *ci si*. *k i* = (*ci*, *ci* $*_R$ *si*) ∧ *0* < *ci* ∧ *si* ∈ *rel_interior* (*S i*)
using *rel_interior_convex_cone*[*of S i*] **by** *auto*
}
then obtain *c s* **where** *cs*: ∀ *i*∈*I*. *k i* = (*c i*, *c i* $*_R$ *s i*) ∧ *0* < *c i* ∧ *s i* ∈
rel_interior (*S i*)
by *metis*
then have *x* = (∑ *i*∈*I*. *c i* $*_R$ *s i*) ∧ *sum c I* = *1*
using *k* **by** (*simp add*: *sum_prod*)
then have *x* ∈ *?rhs*
using *k cs* **by** *auto*
}
moreover
{
fix *x*
assume *x* ∈ *?rhs*
then obtain *c s* **where** *cs*: *x* = *sum* (λ*i*. *c i* $*_R$ *s i*) *I* ∧
(∀ *i*∈*I*. *c i* > *0*) ∧ *sum c I* = *1* ∧ (∀ *i*∈*I*. *s i* ∈ *rel_interior* (*S i*))
by *auto*
define *k* **where** *k i* = (*c i*, *c i* $*_R$ *s i*) **for** *i*

```
  {
    fix i assume i ∈ I
    then have k i ∈ rel_interior (K i)
      using k_def K_def assms cs rel_interior_convex_cone[of S i]
      by auto
  }
  then have (1, x) ∈ rel_interior K0
    using * set_sum_alt[of I (λi. rel_interior (K i))] assms cs
    by (simp add: k_def) (metis (mono_tags, lifting) sum_prod)
  then have x ∈ ?lhs
    using K0_def C0_def rel_interior_convex_cone_aux[of C0 1 x]
    by auto
 }
 ultimately show ?thesis by blast
qed


lemma convex_le_Inf_differential:
 fixes f :: real ⇒ real
 assumes convex_on I f
  and x ∈ interior I
  and y ∈ I
 shows f y ≥ f x + Inf ((λt. (f x − f t) / (x − t)) ‘ ({x<..} ∩ I)) ∗ (y − x)
 (is _ ≥ _ + Inf (?F x) ∗ (y − x))
proof (cases rule: linorder_cases)
 assume x < y
 moreover
 have open (interior I) by auto
 from openE[OF this ‹x ∈ interior I›]
 obtain e where e: 0 < e ball x e ⊆ interior I .
 moreover define t where t = min (x + e / 2) ((x + y) / 2)
 ultimately have x < t t < y t ∈ ball x e
  by (auto simp: dist_real_def field_simps split: split_min)
 with ‹x ∈ interior I› e interior_subset[of I] have t ∈ I x ∈ I by auto

 define K where K = x − e / 2
 with ‹0 < e› have K ∈ ball x e K < x
  by (auto simp: dist_real_def)
 then have K ∈ I
  using ‹interior I ⊆ I› e(2) by blast

 have Inf (?F x) ≤ (f x − f y) / (x − y)
 proof (intro bdd_belowI cInf_lower2)
  show (f x − f t) / (x − t) ∈ ?F x
    using ‹t ∈ I› ‹x < t› by auto
  show (f x − f t) / (x − t) ≤ (f x − f y) / (x − y)
    using ‹convex_on I f› ‹x ∈ I› ‹y ∈ I› ‹x < t› ‹t < y›
    by (rule convex_on_diff)
 next
```

```
    fix y
    assume y ∈ ?F x
    with order_trans[OF convex_on_diff[OF ‹convex_on I f› ‹K ∈ I› _ ‹K < x› _]]
    show (f K − f x) / (K − x) ≤ y by auto
  qed
  then show ?thesis
    using ‹x < y› by (simp add: field_simps)
next
  assume y < x
  moreover
  have open (interior I) by auto
  from openE[OF this ‹x ∈ interior I›]
  obtain e where e: 0 < e ball x e ⊆ interior I .
  moreover define t where t = x + e / 2
  ultimately have x < t t ∈ ball x e
    by (auto simp: dist_real_def field_simps)
  with ‹x ∈ interior I› e interior_subset[of I] have t ∈ I x ∈ I by auto

  have (f x − f y) / (x − y) ≤ Inf (?F x)
  proof (rule cInf_greatest)
    have (f x − f y) / (x − y) = (f y − f x) / (y − x)
      using ‹y < x› by (auto simp: field_simps)
    also
    fix z
    assume z ∈ ?F x
    with order_trans[OF convex_on_diff[OF ‹convex_on I f› ‹y ∈ I› _ ‹y < x›]]
    have (f y − f x) / (y − x) ≤ z
      by auto
    finally show (f x − f y) / (x − y) ≤ z .
  next
    have x + e / 2 ∈ ball x e
      using e by (auto simp: dist_real_def)
    with e interior_subset[of I] have x + e / 2 ∈ {x<..} ∩ I
      by auto
    then show ?F x ≠ {}
      by blast
  qed
  then show ?thesis
    using ‹y < x› by (simp add: field_simps)
qed simp
```

5.0.6 Explicit formulas for interior and relative interior of convex hull

```
lemma at_within_cbox_finite:
  assumes x ∈ box a b x ∉ S finite S
  shows (at x within cbox a b − S) = at x
proof −
  have interior (cbox a b − S) = box a b − S
```

```
    using ‹finite S› by (simp add: interior_diff finite_imp_closed)
  then show ?thesis
    using at_within_interior assms by fastforce
qed

lemma affine_independent_convex_affine_hull:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent S T ⊆ S
    shows convex hull T = affine hull T ∩ convex hull S
proof -
  have fin: finite S finite T using assms aff_independent_finite finite_subset by
auto
  have convex hull T ⊆ affine hull T
    using convex_hull_subset_affine_hull by blast
  moreover have convex hull T ⊆ convex hull S
    using assms hull_mono by blast
  moreover have affine hull T ∩ convex hull S ⊆ convex hull T
  proof -
    have 0: ⋀u. sum u S = 0 ⟹ (∀v∈S. u v = 0) ∨ (∑v∈S. u v *R v) ≠ 0
      using affine_dependent_explicit_finite assms(1) fin(1) by auto
    show ?thesis
    proof (clarsimp simp add: affine_hull_finite fin)
      fix u
      assume S: (∑v∈T. u v *R v) ∈ convex hull S
        and T1: sum u T = 1
      then obtain v where v: ∀x∈S. 0 ≤ v x sum v S = 1 (∑x∈S. v x *R x)
= (∑v∈T. u v *R v)
        by (auto simp add: convex_hull_finite fin)
      { fix x
        assumex ∈ T
        then have S: S = (S − T) ∪ T — split into separate cases
          using assms by auto
        have [simp]: (∑x∈T. v x *R x) + (∑x∈S − T. v x *R x) = (∑x∈T. u
x *R x)
          sum v T + sum v (S − T) = 1
          using v fin S
          by (auto simp: sum.union_disjoint [symmetric] Un_commute)
        have (∑x∈S. if x ∈ T then v x − u x else v x) = 0
             (∑x∈S. (if x ∈ T then v x − u x else v x) *R x) = 0
          using v fin T1
        by (subst S, subst sum.union_disjoint, auto simp: algebra_simps sum_subtractf)+
      } note [simp] = this
      have (∀x∈T. 0 ≤ u x)
        using 0 [of λx. if x ∈ T then v x − u x else v x] ‹T ⊆ S› v(1) by fastforce
      then show (∑v∈T. u v *R v) ∈ convex hull T
        using 0 [of λx. if x ∈ T then v x − u x else v x] ‹T ⊆ S› T1
        by (fastforce simp add: convex_hull_finite fin)
    qed
  qed
```

```
  ultimately show ?thesis
    by blast
qed

lemma affine_independent_span_eq:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent S card S = Suc (DIM ('a))
    shows affine hull S = UNIV
proof (cases S = {})
  case True then show ?thesis
    using assms by simp
next
  case False
    then obtain a T where T: a ∉ T S = insert a T
      by blast
    then have fin: finite T using assms
      by (metis finite_insert aff_independent_finite)
    have UNIV ⊆ (+) a ` span ((λx. x − a) ` T)
    proof (intro card_ge_dim_independent Fun.vimage_subsetD)
      show independent ((λx. x − a) ` T)
        using T affine_dependent_iff_dependent assms(1) by auto
      show dim ((+) a −` UNIV) ≤ card ((λx. x − a) ` T)
        using assms T fin by (auto simp: card_image inj_on_def)
    qed (use surj_plus in auto)
    then show ?thesis
      using T(2) affine_hull_insert_span_gen equalityI by fastforce
qed

lemma affine_independent_span_gt:
  fixes S :: 'a::euclidean_space set
  assumes ind: ¬ affine_dependent S and dim: DIM ('a) < card S
    shows affine hull S = UNIV
proof (intro affine_independent_span_eq [OF ind] antisym)
  show card S ≤ Suc DIM('a)
    using aff_independent_finite affine_dependent_biggerset ind by fastforce
  show Suc DIM('a) ≤ card S
    using Suc_leI dim by blast
qed

lemma empty_interior_affine_hull:
  fixes S :: 'a::euclidean_space set
  assumes finite S and dim: card S ≤ DIM ('a)
    shows interior(affine hull S) = {}
  using assms
proof (induct S rule: finite_induct)
  case (insert x S)
  then have dim (span ((λy. y − x) ` S)) < DIM('a)
   by (auto simp: Suc_le_lessD card_image_le dual_order.trans intro!: dim_le_card'[THEN
le_less_trans])
```

then show *?case*
by (*simp add*: *empty_interior_lowdim affine_hull_insert_span_gen interior_translation*)
qed *auto*

lemma *empty_interior_convex_hull*:
fixes S :: $'a$::*euclidean_space set*
assumes *finite S* **and** *dim*: *card S* ≤ *DIM* ('*a*)
shows *interior*(*convex hull S*) = {}
by (*metis Diff_empty Diff_eq_empty_iff convex_hull_subset_affine_hull interior_mono empty_interior_affine_hull* [*OF assms*])

lemma *explicit_subset_rel_interior_convex_hull*:
fixes S :: $'a$::*euclidean_space set*
shows *finite S*
⟹ {*y*. ∃ *u*. (∀ *x* ∈ *S*. *0* < *u x* ∧ *u x* < *1*) ∧ *sum u S* = *1* ∧ *sum* (λ*x*. *u x* $*_R$ *x*) *S* = *y*}
⊆ *rel_interior* (*convex hull S*)
by (*force simp add*: *rel_interior_convex_hull_union* [**where** *S*=λ*x*. {*x*} **and** *I*=*S*, *simplified*])

lemma *explicit_subset_rel_interior_convex_hull_minimal*:
fixes S :: $'a$::*euclidean_space set*
shows *finite S*
⟹ {*y*. ∃ *u*. (∀ *x* ∈ *S*. *0* < *u x*) ∧ *sum u S* = *1* ∧ *sum* (λ*x*. *u x* $*_R$ *x*) *S* = *y*}
⊆ *rel_interior* (*convex hull S*)
by (*force simp add*: *rel_interior_convex_hull_union* [**where** *S*=λ*x*. {*x*} **and** *I*=*S*, *simplified*])

lemma *rel_interior_convex_hull_explicit*:
fixes S :: $'a$::*euclidean_space set*
assumes ¬ *affine_dependent S*
shows *rel_interior*(*convex hull S*) =
{*y*. ∃ *u*. (∀ *x* ∈ *S*. *0* < *u x*) ∧ *sum u S* = *1* ∧ *sum* (λ*x*. *u x* $*_R$ *x*) *S* = *y*}
(**is** *?lhs* = *?rhs*)
proof
show *?rhs* ≤ *?lhs*
by (*simp add*: *aff_independent_finite explicit_subset_rel_interior_convex_hull_minimal assms*)
next
show *?lhs* ≤ *?rhs*
proof (*cases* ∃ *a*. *S* = {*a*})
case *True* **then show** *?lhs* ≤ *?rhs*
by *force*
next
case *False*
have *fs*: *finite S*
using *assms* **by** (*simp add*: *aff_independent_finite*)
{ **fix** *a b* **and** *d*::*real*

assume *ab*: $a \in S$ $b \in S$ $a \neq b$
then have *S*: $S = (S - \{a,b\}) \cup \{a,b\}$ — split into separate cases
by *auto*
have $(\sum x\in S.\ \mathit{if}\ x = a\ \mathit{then} - d\ \mathit{else\ if}\ x = b\ \mathit{then}\ d\ \mathit{else}\ 0) = 0$
$(\sum x\in S.\ (\mathit{if}\ x = a\ \mathit{then} - d\ \mathit{else\ if}\ x = b\ \mathit{then}\ d\ \mathit{else}\ 0) *_R x) = d *_R b - d *_R a$
using *ab fs*
by (*subst S*, *subst sum.union_disjoint*, *auto*)+
} **note** [*simp*] = *this*
{ **fix** *y*
assume *y*: $y \in$ *convex hull S* $y \notin$ *?rhs*
have ∗: *False* **if**
ua: $\forall x\in S.\ 0 \leq u\ x$ *sum u S* = *1* ¬ $0 < u\ a$ $a \in S$
and *yT*: $y = (\sum x\in S.\ u\ x *_R x)$ $y \in T$ *open T*
and *sb*: $T \cap$ *affine hull S* $\subseteq \{w.\ \exists u.\ (\forall x\in S.\ 0 \leq u\ x) \wedge$ *sum u S* = *1* $\wedge (\sum x\in S.\ u\ x *_R x) = w\}$
for *u T a*
proof −
have *ua0*: $u\ a = 0$
using *ua* **by** *auto*
obtain *b* **where** *b*: $b\in S$ $a \neq b$
using *ua False* **by** *auto*
obtain *e* **where** *e*: $0 < e$ *ball* $(\sum x\in S.\ u\ x *_R x)$ $e \subseteq T$
using *yT* **by** (*auto elim*: *openE*)
with *b* **obtain** *d* **where** *d*: $0 < d$ $norm(d *_R (a-b)) < e$
by (*auto intro*: *that* [*of e* / *2* / *norm*(*a*−*b*)])
have $(\sum x\in S.\ u\ x *_R x) \in$ *affine hull S*
using *yT y* **by** (*metis affine_hull_convex_hull hull_redundant_eq*)
then have $(\sum x\in S.\ u\ x *_R x) - d *_R (a - b) \in$ *affine hull S*
using *ua b* **by** (*auto simp*: *hull_inc intro*: *mem_affine_3_minus2*)
then have $y - d *_R (a - b) \in T \cap$ *affine hull S*
using *d e yT* **by** *auto*
then obtain *v* **where** *v*: $\forall x\in S.\ 0 \leq v\ x$
sum v S = *1*
$(\sum x\in S.\ v\ x *_R x) = (\sum x\in S.\ u\ x *_R x) - d *_R (a - b)$
using *subsetD* [*OF sb*] *yT*
by *auto*
have *aff*: $\bigwedge u.$ *sum u S* = *0* $\Longrightarrow (\forall v\in S.\ u\ v = 0) \vee (\sum v\in S.\ u\ v *_R v) \neq 0$
using *assms* **by** (*simp add*: *affine_dependent_explicit_finite fs*)
show *False*
using *ua b d v aff* [*of* $\lambda x.\ (v\ x - u\ x) - (\mathit{if}\ x = a\ \mathit{then} -d\ \mathit{else\ if}\ x = b\ \mathit{then}\ d\ \mathit{else}\ 0)$]
by (*auto simp*: *algebra_simps sum_subtractf sum.distrib*)
qed
have $y \notin$ *rel_interior* (*convex hull S*)
using *y*
apply (*simp add*: *mem_rel_interior*)
apply (*auto simp*: *convex_hull_finite* [*OF fs*])

```
      apply (drule_tac x=u in spec)
      apply (auto intro: *)
      done
  } with rel_interior_subset show ?lhs ≤ ?rhs
    by blast
 qed
qed

lemma interior_convex_hull_explicit_minimal:
 fixes S :: 'a::euclidean_space set
 assumes ¬ affine_dependent S
 shows
  interior(convex hull S) =
         (if card(S) ≤ DIM('a) then {}
          else {y. ∃u. (∀x ∈ S. 0 < u x) ∧ sum u S = 1 ∧ (∑x∈S. u x *R x)
= y})
  (is _ = (if _ then _ else ?rhs))
proof (clarsimp simp: aff_independent_finite empty_interior_convex_hull assms)
 assume S: ¬ card S ≤ DIM('a)
 have interior (convex hull S) = rel_interior(convex hull S)
   using assms S by (simp add: affine_independent_span_gt rel_interior_interior)
 then show interior(convex hull S) = ?rhs
   by (simp add: assms S rel_interior_convex_hull_explicit)
qed

lemma interior_convex_hull_explicit:
 fixes S :: 'a::euclidean_space set
 assumes ¬ affine_dependent S
 shows
  interior(convex hull S) =
         (if card(S) ≤ DIM('a) then {}
          else {y. ∃u. (∀x ∈ S. 0 < u x ∧ u x < 1) ∧ sum u S = 1 ∧ (∑x∈S.
u x *R x) = y})
proof –
 { fix u :: 'a ⇒ real and a
   assume card Basis < card S and u: ⋀x. x∈S ⟹ 0 < u x sum u S = 1 and
a: a ∈ S
   then have cs: Suc 0 < card S
     by (metis DIM_positive less_trans_Suc)
   obtain b where b: b ∈ S a ≠ b
   proof (cases S ≤ {a})
     case True
     then show thesis
       using cs subset_singletonD by fastforce
   qed blast
   have u a + u b ≤ sum u {a,b}
     using a b by simp
   also have ... ≤ sum u S
     using a b u
```

by (*intro Groups_Big.sum_mono2*) (*auto simp*: *less_imp_le aff_independent_finite assms*)
finally have *u a* < *1*
using ‹*b* ∈ *S*› *u* **by** *fastforce*
} **note** [*simp*] = *this*
show *?thesis*
using *assms* **by** (*force simp add*: *not_le interior_convex_hull_explicit_minimal*)
qed

lemma *interior_closed_segment_ge2*:
fixes *a* :: ′*a*::*euclidean_space*
assumes *2* ≤ *DIM*(′*a*)
shows *interior*(*closed_segment a b*) = {}
using *assms* **unfolding** *segment_convex_hull*
proof −
have *card* {*a*, *b*} ≤ *DIM*(′*a*)
using *assms*
by (*simp add*: *card_insert_if linear not_less_eq_eq numeral_2_eq_2*)
then show *interior* (*convex hull* {*a*, *b*}) = {}
by (*metis empty_interior_convex_hull finite.insertI finite.emptyI*)
qed

lemma *interior_open_segment*:
fixes *a* :: ′*a*::*euclidean_space*
shows *interior*(*open_segment a b*) =
(*if 2* ≤ *DIM*(′*a*) *then* {} *else open_segment a b*)
proof (*simp add*: *not_le*, *intro conjI impI*)
assume *2* ≤ *DIM*(′*a*)
then show *interior* (*open_segment a b*) = {}
using *interior_closed_segment_ge2 interior_mono segment_open_subset_closed* **by** *blast*
next
assume *le2*: *DIM*(′*a*) < *2*
show *interior* (*open_segment a b*) = *open_segment a b*
proof (*cases a* = *b*)
case *True* **then show** *?thesis* **by** *auto*
next
case *False*
with *le2* **have** *affine hull* (*open_segment a b*) = *UNIV*
by (*simp add*: *False affine_independent_span_gt*)
then show *interior* (*open_segment a b*) = *open_segment a b*
using *rel_interior_interior rel_interior_open_segment* **by** *blast*
qed
qed

lemma *interior_closed_segment*:
fixes *a* :: ′*a*::*euclidean_space*
shows *interior*(*closed_segment a b*) =
(*if 2* ≤ *DIM*(′*a*) *then* {} *else open_segment a b*)

```
proof (cases a = b)
  case True then show ?thesis by simp
next
  case False
  then have closure (open_segment a b) = closed_segment a b
    by simp
  then show ?thesis
   by (metis (no_types) convex_interior_closure convex_open_segment interior_open_segment)
qed

lemmas interior_segment = interior_closed_segment interior_open_segment

lemma closed_segment_eq [simp]:
  fixes a :: 'a::euclidean_space
  shows closed_segment a b = closed_segment c d ⟷ {a,b} = {c,d}
proof
  assume abcd: closed_segment a b = closed_segment c d
  show {a,b} = {c,d}
  proof (cases a=b ∨ c=d)
    case True with abcd show ?thesis by force
  next
    case False
    then have neq: a ≠ b ∧ c ≠ d by force
    have *: closed_segment c d − {a, b} = rel_interior (closed_segment c d)
     using neq abcd by (metis (no_types) open_segment_def rel_interior_closed_segment)
    have b ∈ {c, d}
    proof −
      have insert b (closed_segment c d) = closed_segment c d
        using abcd by blast
      then show ?thesis
       by (metis DiffD2 Diff_insert2 False ∗ insertI1 insert_Diff_if open_segment_def
rel_interior_closed_segment)
    qed
    moreover have a ∈ {c, d}
      by (metis Diff_iff False ∗ abcd ends_in_segment(1) insertI1 open_segment_def
rel_interior_closed_segment)
    ultimately show {a, b} = {c, d}
      using neq by fastforce
  qed
next
  assume {a,b} = {c,d}
  then show closed_segment a b = closed_segment c d
    by (simp add: segment_convex_hull)
qed

lemma closed_open_segment_eq [simp]:
  fixes a :: 'a::euclidean_space
  shows closed_segment a b ≠ open_segment c d
by (metis DiffE closed_segment_neq_empty closure_closed_segment closure_open_segment
```

ends_in_segment(*1*) *insertI1 open_segment_def*)

lemma *open_closed_segment_eq* [*simp*]:
  fixes *a* :: *'a*::*euclidean_space*
  shows *open_segment a b* ≠ *closed_segment c d*
using *closed_open_segment_eq* **by** *blast*

lemma *open_segment_eq* [*simp*]:
  fixes *a* :: *'a*::*euclidean_space*
  shows *open_segment a b* = *open_segment c d* ⟷ *a* = *b* ∧ *c* = *d* ∨ {*a*,*b*} = {*c*,*d*}
    (**is** *?lhs* = *?rhs*)
proof
  assume *abcd*: *?lhs*
  show *?rhs*
  proof (*cases a*=*b* ∨ *c*=*d*)
    case *True* **with** *abcd* **show** *?thesis*
      using *finite_open_segment* **by** *fastforce*
  next
    case *False*
    then have *a2*: *a* ≠ *b* ∧ *c* ≠ *d* **by** *force*
    with *abcd* **show** *?rhs*
      unfolding *open_segment_def*
      by (*metis* (*no_types*) *abcd closed_segment_eq closure_open_segment*)
  qed
next
  assume *?rhs*
  then show *?lhs*
   by (*metis Diff_cancel convex_hull_singleton insert_absorb2 open_segment_def segment_convex_hull*)
qed

5.0.7 Similar results for closure and (relative or absolute) frontier

lemma *closure_convex_hull* [*simp*]:
  fixes *S* :: *'a*::*euclidean_space set*
  shows *compact S* ==> *closure*(*convex hull S*) = *convex hull S*
  by (*simp add*: *compact_imp_closed compact_convex_hull*)

lemma *rel_frontier_convex_hull_explicit*:
  fixes *S* :: *'a*::*euclidean_space set*
  assumes ¬ *affine_dependent S*
  shows *rel_frontier*(*convex hull S*) =
    {*y*. ∃ *u*. (∀ *x* ∈ *S*. *0* ≤ *u x*) ∧ (∃ *x* ∈ *S*. *u x* = *0*) ∧ *sum u S* = *1* ∧ *sum* (λ*x*. *u x* $*_R$ *x*) *S* = *y*}
proof −
  have *fs*: *finite S*
    using *assms* **by** (*simp add*: *aff_independent_finite*)

```
  have ⋀u y v.
      ⟦y ∈ S; u y = 0; sum u S = 1; ∀x∈S. 0 < v x;
       sum v S = 1; (∑x∈S. v x *R x) = (∑x∈S. u x *R x)⟧
      ⟹ ∃u. sum u S = 0 ∧ (∃v∈S. u v ≠ 0) ∧ (∑v∈S. u v *R v) = 0
    apply (rule_tac x = λx. u x − v x in exI)
    apply (force simp: sum_subtractf scaleR_diff_left)
    done
  then show ?thesis
    using fs assms
   apply (simp add: rel_frontier_def finite_imp_compact rel_interior_convex_hull_explicit)
    apply (auto simp: convex_hull_finite)
    apply (metis less_eq_real_def)
    by (simp add: affine_dependent_explicit_finite)
qed

lemma frontier_convex_hull_explicit:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent S
  shows frontier(convex hull S) =
        {y. ∃u. (∀x ∈ S. 0 ≤ u x) ∧ (DIM ('a) < card S ⟶ (∃x ∈ S. u x = 0))
∧
             sum u S = 1 ∧ sum (λx. u x *R x) S = y}
proof −
  have fs: finite S
    using assms by (simp add: aff_independent_finite)
  show ?thesis
  proof (cases DIM ('a) < card S)
    case True
    with assms fs show ?thesis
        by (simp add: rel_frontier_def frontier_def rel_frontier_convex_hull_explicit
[symmetric]
                interior_convex_hull_explicit_minimal rel_interior_convex_hull_explicit)
  next
    case False
    then have card S ≤ DIM ('a)
      by linarith
    then show ?thesis
      using assms fs
     apply (simp add: frontier_def interior_convex_hull_explicit finite_imp_compact)
      apply (simp add: convex_hull_finite)
      done
  qed
qed

lemma rel_frontier_convex_hull_cases:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent S
  shows rel_frontier(convex hull S) = ⋃{convex hull (S − {x}) |x. x ∈ S}
proof −
```

have *fs*: *finite S*
  using *assms* **by** (*simp add*: *aff_independent_finite*)
{ fix *u a*
have ∀ *x*∈*S*. *0* ≤ *u x* ⟹ *a* ∈ *S* ⟹ *u a* = *0* ⟹ *sum u S* = *1* ⟹
    ∃ *x v*. *x* ∈ *S* ∧
      (∀ *x*∈*S* − {*x*}. *0* ≤ *v x*) ∧
        sum v (*S* − {*x*}) = *1* ∧ (∑ *x*∈*S* − {*x*}. *v x* $*_R$ *x*) = (∑ *x*∈*S*. *u x* $*_R$ *x*)
  apply (*rule_tac x*=*a* **in** *exI*)
  apply (*rule_tac x*=*u* **in** *exI*)
  apply (*simp add*: *Groups_Big.sum_diff1 fs*)
  done }
moreover
{ fix *a u*
  have *a* ∈ *S* ⟹ ∀ *x*∈*S* − {*a*}. *0* ≤ *u x* ⟹ *sum u* (*S* − {*a*}) = *1* ⟹
    ∃ *v*. (∀ *x*∈*S*. *0* ≤ *v x*) ∧
      (∃ *x*∈*S*. *v x* = *0*) ∧ *sum v S* = *1* ∧ (∑ *x*∈*S*. *v x* $*_R$ *x*) = (∑ *x*∈*S* − {*a*}. *u x* $*_R$ *x*)
  apply (*rule_tac x*=λ*x*. *if x* = *a then 0 else u x* **in** *exI*)
  apply (*auto simp*: *sum.If_cases Diff_eq if_smult fs*)
  done }
ultimately show *?thesis*
  using *assms*
  apply (*simp add*: *rel_frontier_convex_hull_explicit*)
  apply (*auto simp add*: *convex_hull_finite fs Union_SetCompr_eq*)
  done
qed

lemma *frontier_convex_hull_eq_rel_frontier*:
  fixes *S* :: ′*a*::*euclidean_space set*
  assumes ¬ *affine_dependent S*
  shows *frontier*(*convex hull S*) =
    (*if card S* ≤ *DIM* (′*a*) *then convex hull S else rel_frontier*(*convex hull S*))
  using *assms*
  unfolding *rel_frontier_def frontier_def*
  by (*simp add*: *affine_independent_span_gt rel_interior_interior*
      finite_imp_compact empty_interior_convex_hull aff_independent_finite)

lemma *frontier_convex_hull_cases*:
  fixes *S* :: ′*a*::*euclidean_space set*
  assumes ¬ *affine_dependent S*
  shows *frontier*(*convex hull S*) =
    (*if card S* ≤ *DIM* (′*a*) *then convex hull S else* ⋃{*convex hull* (*S* − {*x*}) |*x*. *x* ∈ *S*})
by (*simp add*: *assms frontier_convex_hull_eq_rel_frontier rel_frontier_convex_hull_cases*)

lemma *in_frontier_convex_hull*:
  fixes *S* :: ′*a*::*euclidean_space set*
  assumes *finite S card S* ≤ *Suc* (*DIM* (′*a*)) *x* ∈ *S*

shows $x \in$ *frontier*(*convex hull S*)
proof (*cases affine_dependent S*)
case *True*
with *assms* **obtain** *y* **where** $y \in S$ **and** *y*: $y \in$ *affine hull* ($S - \{y\}$)
by (*auto simp*: *affine_dependent_def*)
moreover have $x \in$ *closure* (*convex hull S*)
by (*meson closure_subset hull_inc subset_eq* ‹$x \in S$›)
moreover have $x \notin$ *interior* (*convex hull S*)
using *assms*
by (*metis Suc_mono affine_hull_convex_hull affine_hull_nonempty_interior* ‹*y* $\in S$› *y card.remove empty_iff empty_interior_affine_hull finite_Diff hull_redundant insert_Diff interior_UNIV not_less*)
ultimately show *?thesis*
unfolding *frontier_def* **by** *blast*
next
case *False*
{ **assume** *card S* = *Suc* (*card Basis*)
then have *cs*: *Suc 0* < *card S*
by (*simp*)
with *subset_singletonD* **have** $\exists y \in S.\ y \neq x$
by (*cases* $S \leq \{x\}$) *fastforce+*
} **note** [*dest*!] = *this*
show *?thesis* **using** *assms*
unfolding *frontier_convex_hull_cases* [*OF False*] *Union_SetCompr_eq*
by (*auto simp*: *le_Suc_eq hull_inc*)
qed

lemma *not_in_interior_convex_hull*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *finite S card S* ≤ *Suc* (*DIM* (*'a*)) $x \in S$
shows $x \notin$ *interior*(*convex hull S*)
using *in_frontier_convex_hull* [*OF assms*]
by (*metis Diff_iff frontier_def*)

lemma *interior_convex_hull_eq_empty*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *card S* = *Suc* (*DIM* (*'a*))
shows *interior*(*convex hull S*) = {} ⟷ *affine_dependent S*
proof
show *affine_dependent S* ⟹ *interior* (*convex hull S*) = {}
proof (*clarsimp simp*: *affine_dependent_def*)
fix *a b*
assume $b \in S$ $b \in$ *affine hull* ($S - \{b\}$)
then have *interior*(*affine hull S*) = {} **using** *assms*
by (*metis DIM_positive One_nat_def Suc_mono card.remove card.infinite empty_interior_affine_hull eq_iff hull_redundant insert_Diff not_less zero_le_one*)
then show *interior* (*convex hull S*) = {}
using *affine_hull_nonempty_interior* **by** *fastforce*
qed

next
  show *interior* (*convex hull S*) = {} $\Longrightarrow$ *affine_dependent S*
    by (*metis affine_hull_convex_hull affine_hull_empty affine_independent_span_eq assms convex_convex_hull empty_not_UNIV rel_interior_eq_empty rel_interior_interior*)
qed

5.0.8 Coplanarity, and collinearity in terms of affine hull

definition *coplanar* **where**
  coplanar S ≡ ∃ *u v w*. *S* ⊆ *affine hull* {*u*,*v*,*w*}

lemma *collinear_affine_hull*:
  collinear S ⟷ (∃ *u v*. *S* ⊆ *affine hull* {*u*,*v*})
proof (*cases S*={})
  case *True* **then show** *?thesis*
    by *simp*
next
  case *False*
  then obtain *x* **where** *x*: *x* ∈ *S* **by** *auto*
  { **fix** *u*
    assume ∗: ⋀*x y*. ⟦*x*∈*S*; *y*∈*S*⟧ $\Longrightarrow$ ∃ *c*. *x* − *y* = *c* $*_R$ *u*
    have ⋀*y c*. *x* − *y* = *c* $*_R$ *u* $\Longrightarrow$ ∃ *a b*. *y* = *a* $*_R$ *x* + *b* $*_R$ (*x* + *u*) ∧ *a* + *b* = *1*
      by (*rule_tac x=1+c* **in** *exI*, *rule_tac x=−c* **in** *exI*, *simp add*: *algebra_simps*)
    then have ∃ *u v*. *S* ⊆ {*a* $*_R$ *u* + *b* $*_R$ *v* |*a b*. *a* + *b* = *1*}
      using ∗ [*OF x*] **by** (*rule_tac x=x* **in** *exI*, *rule_tac x=x+u* **in** *exI*, *force*)
  } **moreover**
  { **fix** *u v x y*
    assume ∗: *S* ⊆ {*a* $*_R$ *u* + *b* $*_R$ *v* |*a b*. *a* + *b* = *1*}
    have ∃ *c*. *x* − *y* = *c* $*_R$ (*v*−*u*) **if** *x*∈*S y*∈*S*
    proof −
      obtain *a r* **where** *a* + *r* = *1 x* = *a* $*_R$ *u* + *r* $*_R$ *v*
        using ∗ ‹*x* ∈ *S*› **by** *blast*
      moreover
      obtain *b s* **where** *b* + *s* = *1 y* = *b* $*_R$ *u* + *s* $*_R$ *v*
        using ∗ ‹*y* ∈ *S*› **by** *blast*
      ultimately have *x* − *y* = (*r*−*s*) $*_R$ (*v*−*u*)
        by (*simp add*: *algebra_simps*) (*metis scaleR_left.add*)
      then show *?thesis*
        by *blast*
    qed
  } **ultimately**
  show *?thesis*
  unfolding *collinear_def affine_hull_2*
    by *blast*
qed

lemma *collinear_closed_segment* [*simp*]: *collinear* (*closed_segment a b*)
 by (*metis affine_hull_convex_hull collinear_affine_hull hull_subset segment_convex_hull*)

lemma *collinear_open_segment* [*simp*]: *collinear* (*open_segment a b*)
  unfolding *open_segment_def*
  by (*metis convex_hull_subset_affine_hull segment_convex_hull dual_order.trans*
    convex_hull_subset_affine_hull Diff_subset collinear_affine_hull)

lemma *collinear_between_cases*:
  fixes *c* :: *'a::euclidean_space*
  shows *collinear* {*a*,*b*,*c*} ⟷ *between* (*b*,*c*) *a* ∨ *between* (*c*,*a*) *b* ∨ *between* (*a*,*b*)
c
    (**is** *?lhs* = *?rhs*)
proof
  assume *?lhs*
  then obtain *u v* **where** *uv*: $\bigwedge x.\ x \in \{a,\ b,\ c\} \Longrightarrow \exists c.\ x = u + c *_R v$
    by (*auto simp*: *collinear_alt*)
  show *?rhs*
    using *uv* [*of a*] *uv* [*of b*] *uv* [*of c*] **by** (*auto simp*: *between_1*)
next
  assume *?rhs*
  then show *?lhs*
    unfolding *between_mem_convex_hull*
   by (*metis* (*no_types*, *hide_lams*) *collinear_closed_segment collinear_subset hull_redundant*
hull_subset insert_commute segment_convex_hull)
qed

lemma *subset_continuous_image_segment_1*:
  fixes *f* :: *'a::euclidean_space* ⇒ *real*
  assumes *continuous_on* (*closed_segment a b*) *f*
  shows *closed_segment* (*f a*) (*f b*) ⊆ *image f* (*closed_segment a b*)
by (*metis connected_segment convex_contains_segment ends_in_segment imageI*
    is_interval_connected_1 is_interval_convex connected_continuous_image [*OF*
assms])

lemma *continuous_injective_image_segment_1*:
  fixes *f* :: *'a::euclidean_space* ⇒ *real*
  assumes *contf*: *continuous_on* (*closed_segment a b*) *f*
    and *injf*: *inj_on f* (*closed_segment a b*)
  shows *f* ‘ (*closed_segment a b*) = *closed_segment* (*f a*) (*f b*)
proof
  show *closed_segment* (*f a*) (*f b*) ⊆ *f* ‘ *closed_segment a b*
    by (*metis subset_continuous_image_segment_1 contf*)
  show *f* ‘ *closed_segment a b* ⊆ *closed_segment* (*f a*) (*f b*)
  proof (*cases a* = *b*)
    case *True*
    then show *?thesis* **by** *auto*
  next
    case *False*
    then have *fnot*: *f a* ≠ *f b*

using *inj_onD injf* **by** *fastforce*
moreover
have *f a* ∉ *open_segment* (*f c*) (*f b*) **if** *c*: *c* ∈ *closed_segment a b* **for** *c*
proof (*clarsimp simp add*: *open_segment_def*)
assume *fa*: *f a* ∈ *closed_segment* (*f c*) (*f b*)
moreover have *closed_segment* (*f c*) (*f b*) ⊆ *f* ‘ *closed_segment c b*
by (*meson closed_segment_subset contf continuous_on_subset convex_closed_segment ends_in_segment*(*2*) *subset_continuous_image_segment_1 that*)
ultimately have *f a* ∈ *f* ‘ *closed_segment c b*
by *blast*
then have *a*: *a* ∈ *closed_segment c b*
by (*meson ends_in_segment inj_on_image_mem_iff injf subset_closed_segment that*)
have *cb*: *closed_segment c b* ⊆ *closed_segment a b*
by (*simp add*: *closed_segment_subset that*)
show *f a* = *f c*
proof (*rule between_antisym*)
show *between* (*f c*, *f b*) (*f a*)
by (*simp add*: *between_mem_segment fa*)
show *between* (*f a*, *f b*) (*f c*)
by (*metis a cb between_antisym between_mem_segment between_triv1 subset_iff*)
qed
qed
moreover
have *f b* ∉ *open_segment* (*f a*) (*f c*) **if** *c*: *c* ∈ *closed_segment a b* **for** *c*
proof (*clarsimp simp add*: *open_segment_def fnot eq_commute*)
assume *fb*: *f b* ∈ *closed_segment* (*f a*) (*f c*)
moreover have *closed_segment* (*f a*) (*f c*) ⊆ *f* ‘ *closed_segment a c*
by (*meson contf continuous_on_subset ends_in_segment*(*1*) *subset_closed_segment subset_continuous_image_segment_1 that*)
ultimately have *f b* ∈ *f* ‘ *closed_segment a c*
by *blast*
then have *b*: *b* ∈ *closed_segment a c*
by (*meson ends_in_segment inj_on_image_mem_iff injf subset_closed_segment that*)
have *ca*: *closed_segment a c* ⊆ *closed_segment a b*
by (*simp add*: *closed_segment_subset that*)
show *f b* = *f c*
proof (*rule between_antisym*)
show *between* (*f c*, *f a*) (*f b*)
by (*simp add*: *between_commute between_mem_segment fb*)
show *between* (*f b*, *f a*) (*f c*)
by (*metis b between_antisym between_commute between_mem_segment between_triv2 that*)
qed
qed
ultimately show *?thesis*
by (*force simp*: *closed_segment_eq_real_ivl open_segment_eq_real_ivl split*: *if_split_asm*)

```
  qed
qed

lemma continuous_injective_image_open_segment_1:
  fixes f :: 'a::euclidean_space ⇒ real
  assumes contf: continuous_on (closed_segment a b) f
      and injf: inj_on f (closed_segment a b)
    shows f ` (open_segment a b) = open_segment (f a) (f b)
proof -
  have f ` (open_segment a b) = f ` (closed_segment a b) - {f a, f b}
   by (metis (no_types, hide_lams) empty_subsetI ends_in_segment image_insert im-
age_is_empty inj_on_image_set_diff injf insert_subset open_segment_def segment_open_subset_closed)
  also have ... = open_segment (f a) (f b)
    using continuous_injective_image_segment_1 [OF assms]
    by (simp add: open_segment_def inj_on_image_set_diff [OF injf])
  finally show ?thesis .
qed

lemma collinear_imp_coplanar:
  collinear s ==> coplanar s
by (metis collinear_affine_hull coplanar_def insert_absorb2)

lemma collinear_small:
  assumes finite s card s ≤ 2
    shows collinear s
proof -
  have card s = 0 ∨ card s = 1 ∨ card s = 2
    using assms by linarith
  then show ?thesis using assms
    using card_eq_SucD numeral_2_eq_2 by (force simp: card_1_singleton_iff)
qed

lemma coplanar_small:
  assumes finite s card s ≤ 3
    shows coplanar s
proof -
  consider card s ≤ 2 | card s = Suc (Suc (Suc 0))
    using assms by linarith
  then show ?thesis
  proof cases
    case 1
    then show ?thesis
      by (simp add: ‹finite s› collinear_imp_coplanar collinear_small)
  next
    case 2
    then show ?thesis
      using hull_subset [of {_,_,_}]
      by (fastforce simp: coplanar_def dest!: card_eq_SucD)
  qed
```

qed

lemma *coplanar_empty*: *coplanar* {}
by (*simp add*: *coplanar_small*)

lemma *coplanar_sing*: *coplanar* {*a*}
by (*simp add*: *coplanar_small*)

lemma *coplanar_2*: *coplanar* {*a*,*b*}
by (*auto simp*: *card_insert_if coplanar_small*)

lemma *coplanar_3*: *coplanar* {*a*,*b*,*c*}
by (*auto simp*: *card_insert_if coplanar_small*)

lemma *collinear_affine_hull_collinear*: *collinear*(*affine hull s*) ⟷ *collinear s*
unfolding *collinear_affine_hull*
by (*metis affine_affine_hull subset_hull hull_hull hull_mono*)

lemma *coplanar_affine_hull_coplanar*: *coplanar*(*affine hull s*) ⟷ *coplanar s*
unfolding *coplanar_def*
by (*metis affine_affine_hull subset_hull hull_hull hull_mono*)

lemma *coplanar_linear_image*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*real_normed_vector*
assumes *coplanar S linear f* **shows** *coplanar*(*f* ‘ *S*)
proof –
{ fix *u v w*
assume *S* ⊆ *affine hull* {*u*, *v*, *w*}
then have *f* ‘ *S* ⊆ *f* ‘ (*affine hull* {*u*, *v*, *w*})
by (*simp add*: *image_mono*)
then have *f* ‘ *S* ⊆ *affine hull* (*f* ‘ {*u*, *v*, *w*})
by (*metis assms*(*2*) *linear_conv_bounded_linear affine_hull_linear_image*)
} then
show *?thesis*
by *auto* (*meson assms*(*1*) *coplanar_def*)
qed

lemma *coplanar_translation_imp*:
assumes *coplanar S* **shows** *coplanar* ((λ*x*. *a* + *x*) ‘ *S*)
proof –
obtain *u v w* **where** *S* ⊆ *affine hull* {*u*,*v*,*w*}
by (*meson assms coplanar_def*)
then have (+) *a* ‘ *S* ⊆ *affine hull* {*u* + *a*, *v* + *a*, *w* + *a*}
using *affine_hull_translation* [*of a* {*u*,*v*,*w*} **for** *u v w*]
by (*force simp*: *add.commute*)
then show *?thesis*
unfolding *coplanar_def* **by** *blast*
qed

lemma *coplanar_translation_eq*: *coplanar*((λ*x*. *a* + *x*) ‘ *S*) ⟷ *coplanar S*
  by (*metis* (*no_types*) *coplanar_translation_imp translation_galois*)

lemma *coplanar_linear_image_eq*:
  fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
  assumes *linear f inj f* **shows** *coplanar*(*f* ‘ *S*) = *coplanar S*
proof
  assume *coplanar S*
  then show *coplanar* (*f* ‘ *S*)
    using *assms*(*1*) *coplanar_linear_image* **by** *blast*
next
  obtain *g* **where** *g*: *linear g g* ∘ *f* = *id*
    using *linear_injective_left_inverse* [*OF assms*]
    by *blast*
  assume *coplanar* (*f* ‘ *S*)
  then show *coplanar S*
    by (*metis coplanar_linear_image g*(*1*) *g*(*2*) *id_apply image_comp image_id*)
qed

lemma *coplanar_subset*: ⟦*coplanar t*; *S* ⊆ *t*⟧ ⟹ *coplanar S*
  by (*meson coplanar_def order_trans*)

lemma *affine_hull_3_imp_collinear*: *c* ∈ *affine hull* {*a*,*b*} ⟹ *collinear* {*a*,*b*,*c*}
  by (*metis collinear_2 collinear_affine_hull_collinear hull_redundant insert_commute*)

lemma *collinear_3_imp_in_affine_hull*:
  assumes *collinear* {*a*,*b*,*c*} *a* ≠ *b* **shows** *c* ∈ *affine hull* {*a*,*b*}
proof −
  obtain *u x y* **where** *b* − *a* = *y* $*_R$ *u c* − *a* = *x* $*_R$ *u*
    using *assms* **unfolding** *collinear_def* **by** *auto*
  with ‹*a* ≠ *b*› **have** ∃*v*. *c* = (*1* − *x* / *y*) $*_R$ *a* + *v* $*_R$ *b* ∧ *1* − *x* / *y* + *v* = *1*
    by (*simp add*: *algebra_simps*)
  then show *?thesis*
    by (*simp add*: *hull_inc mem_affine*)
qed

lemma *collinear_3_affine_hull*:
  assumes *a* ≠ *b*
  shows *collinear* {*a*,*b*,*c*} ⟷ *c* ∈ *affine hull* {*a*,*b*}
  using *affine_hull_3_imp_collinear assms collinear_3_imp_in_affine_hull* **by** *blast*

lemma *collinear_3_eq_affine_dependent*:
  collinear{*a*,*b*,*c*} ⟷ *a* = *b* ∨ *a* = *c* ∨ *b* = *c* ∨ *affine_dependent* {*a*,*b*,*c*}
proof (*cases a* = *b* ∨ *a* = *c* ∨ *b* = *c*)
  case *True*
  then show *?thesis*
    by (*auto simp*: *insert_commute*)
next
  case *False*

then have *collinear*{*a*,*b*,*c*} **if** *affine_dependent* {*a*,*b*,*c*}
using *that* **unfolding** *affine_dependent_def*
by (*auto simp*: *insert_Diff_if*; *metis affine_hull_3_imp_collinear insert_commute*)
moreover
have *affine_dependent* {*a*,*b*,*c*} **if** *collinear*{*a*,*b*,*c*}
using *False that* **by** (*auto simp*: *affine_dependent_def collinear_3_affine_hull insert_Diff_if*)
ultimately
show *?thesis*
using *False* **by** *blast*
qed

lemma *affine_dependent_imp_collinear_3*:
affine_dependent {*a*,*b*,*c*} $\Longrightarrow$ *collinear*{*a*,*b*,*c*}
by (*simp add*: *collinear_3_eq_affine_dependent*)

lemma *collinear_3*: *NO_MATCH 0 x* $\Longrightarrow$ *collinear* {*x*,*y*,*z*} $\longleftrightarrow$ *collinear* {*0*, *x*−*y*, *z*−*y*}
by (*auto simp add*: *collinear_def*)

lemma *collinear_3_expand*:
collinear{*a*,*b*,*c*} $\longleftrightarrow$ $a = c \vee (\exists u.\ b = u *_R a + (1 - u) *_R c)$
proof −
have *collinear*{*a*,*b*,*c*} = *collinear*{*a*,*c*,*b*}
by (*simp add*: *insert_commute*)
also have ... = *collinear* {*0*, *a* − *c*, *b* − *c*}
by (*simp add*: *collinear_3*)
also have ... $\longleftrightarrow$ $(a = c \vee b = c \vee (\exists ca.\ b - c = ca *_R (a - c)))$
by (*simp add*: *collinear_lemma*)
also have ... $\longleftrightarrow$ $a = c \vee (\exists u.\ b = u *_R a + (1 - u) *_R c)$
by (*cases* $a = c \vee b = c$) (*auto simp*: *algebra_simps*)
finally show *?thesis* **.**
qed

lemma *collinear_aff_dim*: *collinear S* $\longleftrightarrow$ *aff_dim S* $\leq$ *1*
proof
assume *collinear S*
then obtain *u* **and** *v* :: *'a* **where** *aff_dim S* $\leq$ *aff_dim* {*u*,*v*}
by (*metis* ‹*collinear S*› *aff_dim_affine_hull aff_dim_subset collinear_affine_hull*)
then show *aff_dim S* $\leq$ *1*
using *order_trans* **by** *fastforce*
next
assume *aff_dim S* $\leq$ *1*
then have *le1*: *aff_dim* (*affine hull S*) $\leq$ *1*
by *simp*
obtain *B* **where** *B* $\subseteq$ *S* **and** *B*: ¬ *affine_dependent B affine hull S* = *affine hull B*
using *affine_basis_exists* [*of S*] **by** *auto*
then have *finite B card B* $\leq$ *2*

```
    using B le1 by (auto simp: affine_independent_iff_card)
  then have collinear B
    by (rule collinear_small)
  then show collinear S
    by (metis ‹affine hull S = affine hull B› collinear_affine_hull_collinear)
qed

lemma collinear_midpoint: collinear{a, midpoint a b, b}
proof –
  have §: ⟦a ≠ midpoint a b; b − midpoint a b ≠ − 1 *R (a − midpoint a b)⟧ ⟹
b = midpoint a b
    by (simp add: algebra_simps)
  show ?thesis
    by (auto simp: collinear_3 collinear_lemma intro: §)
qed

lemma midpoint_collinear:
  fixes a b c :: 'a::real_normed_vector
  assumes a ≠ c
    shows b = midpoint a c ⟷ collinear{a,b,c} ∧ dist a b = dist b c
proof –
  have *: a − (u *R a + (1 − u) *R c) = (1 − u) *R (a − c)
          u *R a + (1 − u) *R c − c = u *R (a − c)
          |1 − u| = |u| ⟷ u = 1/2 for u::real
    by (auto simp: algebra_simps)
  have b = midpoint a c ⟹ collinear{a,b,c}
    using collinear_midpoint by blast
  moreover have b = midpoint a c ⟷ dist a b = dist b c if collinear{a,b,c}
  proof –
    consider a = c | u where b = u *R a + (1 − u) *R c
      using ‹collinear {a,b,c}› unfolding collinear_3_expand by blast
    then show ?thesis
    proof cases
      case 2
      with assms have dist a b = dist b c ⟹ b = midpoint a c
      by (simp add: dist_norm * midpoint_def scaleR_add_right del: divide_const_simps)
      then show ?thesis
        by (auto simp: dist_midpoint)
    qed (use assms in auto)
  qed
  ultimately show ?thesis by blast
qed

lemma between_imp_collinear:
  fixes x :: 'a :: euclidean_space
  assumes between (a,b) x
    shows collinear {a,x,b}
proof (cases x = a ∨ x = b ∨ a = b)
  case True with assms show ?thesis
```

```
    by (auto simp: dist_commute)
next
  case False
  then have False if ⋀c. b − x ≠ c *R (a − x)
    using that [of −(norm(b − x) / norm(x − a))] assms
   by (simp add: between_norm vector_add_divide_simps flip: real_vector.scale_minus_right)
  then show ?thesis
    by (auto simp: collinear_3 collinear_lemma)
qed

lemma midpoint_between:
  fixes a b :: 'a::euclidean_space
  shows b = midpoint a c ⟷ between (a,c) b ∧ dist a b = dist b c
proof (cases a = c)
  case False
  show ?thesis
     using False between_imp_collinear between_midpoint(1) midpoint_collinear by
blast
qed (auto simp: dist_commute)

lemma collinear_triples:
  assumes a ≠ b
    shows collinear(insert a (insert b S)) ⟷ (∀x ∈ S. collinear{a,b,x})
          (is ?lhs = ?rhs)
proof safe
  fix x
  assume ?lhs and x ∈ S
  then show collinear {a, b, x}
    using collinear_subset by force
next
  assume ?rhs
  then have ∀x ∈ S. collinear{a,x,b}
    by (simp add: insert_commute)
  then have *: ∃u. x = u *R a + (1 − u) *R b if x ∈ insert a (insert b S) for x
    using that assms collinear_3_expand by fastforce+
  have ∃c. x − y = c *R (b − a)
    if x: x ∈ insert a (insert b S) and y: y ∈ insert a (insert b S) for x y
  proof −
    obtain u v where x = u *R a + (1 − u) *R b y = v *R a + (1 − v) *R b
      using * x y by presburger
    then have x − y = (v − u) *R (b − a)
      by (simp add: scale_left_diff_distrib scale_right_diff_distrib)
    then show ?thesis ..
  qed
  then show ?lhs
    unfolding collinear_def by metis
qed

lemma collinear_4_3:
```

assumes $a \neq b$
shows *collinear* $\{a,b,c,d\} \longleftrightarrow$ *collinear*$\{a,b,c\} \wedge$ *collinear*$\{a,b,d\}$
using *collinear_triples* [*OF assms*, *of* $\{c,d\}$] **by** (*force simp*:)

lemma *collinear_3_trans*:
assumes *collinear*$\{a,b,c\}$ *collinear*$\{b,c,d\}$ $b \neq c$
shows *collinear*$\{a,b,d\}$
proof −
have *collinear*$\{b,c,a,d\}$
by (*metis* (*full_types*) *assms collinear_4_3 insert_commute*)
then show *?thesis*
by (*simp add*: *collinear_subset*)
qed

lemma *affine_hull_2_alt*:
fixes *a b* :: *'a::real_vector*
shows *affine hull* $\{a,b\}$ = *range* ($\lambda u.\ a + u *_R (b - a)$)
proof −
have *1*: $u *_R a + v *_R b = a + v *_R (b - a)$ **if** $u + v = 1$ **for** *u v*
using *that*
by (*simp add*: *algebra_simps flip*: *scaleR_add_left*)
have *2*: $a + u *_R (b - a) = (1 - u) *_R a + u *_R b$ **for** *u*
by (*auto simp*: *algebra_simps*)
show *?thesis*
by (*force simp add*: *affine_hull_2 dest*: *1 intro*!: *2*)
qed

lemma *interior_convex_hull_3_minimal*:
fixes *a* :: *'a::euclidean_space*
assumes ¬ *collinear*$\{a,b,c\}$ **and** *2*: *DIM*(*'a*) = *2*
shows *interior*(*convex hull* $\{a,b,c\}$) =
$\{v.\ \exists x\ y\ z.\ 0 < x \wedge 0 < y \wedge 0 < z \wedge x + y + z = 1 \wedge x *_R a + y *_R b$
$+ z *_R c = v\}$
(**is** *?lhs* = *?rhs*)
proof
have *abc*: $a \neq b$ $a \neq c$ $b \neq c$ ¬ *affine_dependent* $\{a, b, c\}$
using *assms* **by** (*auto simp*: *collinear_3_eq_affine_dependent*)
with *2* **show** *?lhs* ⊆ *?rhs*
by (*fastforce simp add*: *interior_convex_hull_explicit_minimal*)
show *?rhs* ⊆ *?lhs*
using *abc 2*
apply (*clarsimp simp add*: *interior_convex_hull_explicit_minimal*)
subgoal for *x y z*
by (*rule_tac x*=λ*r*. (*if r*=*a then x else if r*=*b then y else if r*=*c then z else*
0) **in** *exI*) *auto*
done
qed

5.0.9 Basic lemmas about hyperplanes and halfspaces

lemma *halfspace_Int_eq*:
$\{x.\ a \cdot x \leq b\} \cap \{x.\ b \leq a \cdot x\} = \{x.\ a \cdot x = b\}$
$\{x.\ b \leq a \cdot x\} \cap \{x.\ a \cdot x \leq b\} = \{x.\ a \cdot x = b\}$
by *auto*

lemma *hyperplane_eq_Ex*:
assumes $a \neq 0$ **obtains** x **where** $a \cdot x = b$
by (*rule_tac* $x = (b\ /\ (a \cdot a)) *_R a$ **in** *that*) (*simp add*: *assms*)

lemma *hyperplane_eq_empty*:
$\{x.\ a \cdot x = b\} = \{\} \longleftrightarrow a = 0 \wedge b \neq 0$
using *hyperplane_eq_Ex*
by (*metis* (*mono_tags*, *lifting*) *empty_Collect_eq inner_zero_left*)

lemma *hyperplane_eq_UNIV*:
$\{x.\ a \cdot x = b\} = UNIV \longleftrightarrow a = 0 \wedge b = 0$
proof −
have $a = 0 \wedge b = 0$ **if** $UNIV \subseteq \{x.\ a \cdot x = b\}$
using *subsetD* [*OF that*, **where** $c = ((b+1)\ /\ (a \cdot a)) *_R a$]
by (*simp add*: *field_split_simps split*: *if_split_asm*)
then show *?thesis* **by** *force*
qed

lemma *halfspace_eq_empty_lt*:
$\{x.\ a \cdot x < b\} = \{\} \longleftrightarrow a = 0 \wedge b \leq 0$
proof −
have $a = 0 \wedge b \leq 0$ **if** $\{x.\ a \cdot x < b\} \subseteq \{\}$
using *subsetD* [*OF that*, **where** $c = ((b-1)\ /\ (a \cdot a)) *_R a$]
by (*force simp add*: *field_split_simps split*: *if_split_asm*)
then show *?thesis* **by** *force*
qed

lemma *halfspace_eq_empty_gt*:
$\{x.\ a \cdot x > b\} = \{\} \longleftrightarrow a = 0 \wedge b \geq 0$
using *halfspace_eq_empty_lt* [*of* $-a$ $-b$]
by *simp*

lemma *halfspace_eq_empty_le*:
$\{x.\ a \cdot x \leq b\} = \{\} \longleftrightarrow a = 0 \wedge b < 0$
proof −
have $a = 0 \wedge b < 0$ **if** $\{x.\ a \cdot x \leq b\} \subseteq \{\}$
using *subsetD* [*OF that*, **where** $c = ((b-1)\ /\ (a \cdot a)) *_R a$]
by (*force simp add*: *field_split_simps split*: *if_split_asm*)
then show *?thesis* **by** *force*
qed

lemma *halfspace_eq_empty_ge*:
$\{x.\ a \cdot x \geq b\} = \{\} \longleftrightarrow a = 0 \wedge b > 0$

using *halfspace_eq_empty_le* [*of* $-a$ $-b$] **by** *simp*

5.0.10 Use set distance for an easy proof of separation properties

proposition *separation_closures*:
  fixes S :: $'a$::*euclidean_space set*
  assumes $S \cap$ *closure* $T = \{\}$ $T \cap$ *closure* $S = \{\}$
  obtains U V **where** $U \cap V = \{\}$ *open* U *open* V $S \subseteq U$ $T \subseteq V$
proof (*cases* $S = \{\} \vee T = \{\}$)
  case *True* **with** *that* **show** *?thesis* **by** *auto*
next
  case *False*
  define f **where** $f \equiv \lambda x.$ *setdist* $\{x\}$ T $-$ *setdist* $\{x\}$ S
  have *contf*: *continuous_on UNIV f*
    unfolding *f_def* **by** (*intro continuous_intros continuous_on_setdist*)
  show *?thesis*
  proof (*rule_tac* $U = \{x.\ f\ x > 0\}$ **and** $V = \{x.\ f\ x < 0\}$ **in** *that*)
    show $\{x.\ 0 < f\ x\} \cap \{x.\ f\ x < 0\} = \{\}$
      by *auto*
    show *open* $\{x.\ 0 < f\ x\}$
      by (*simp add*: *open_Collect_less contf*)
    show *open* $\{x.\ f\ x < 0\}$
      by (*simp add*: *open_Collect_less contf*)
    have $\bigwedge x.\ x \in S \Longrightarrow$ *setdist* $\{x\}$ $T \neq 0$ $\bigwedge x.\ x \in T \Longrightarrow$ *setdist* $\{x\}$ $S \neq 0$
      by (*meson False assms disjoint_iff setdist_eq_0_sing_1*)+
    then show $S \subseteq \{x.\ 0 < f\ x\}$ $T \subseteq \{x.\ f\ x < 0\}$
      using *less_eq_real_def* **by** (*fastforce simp add*: *f_def setdist_sing_in_set*)+
  qed
qed

lemma *separation_normal*:
  fixes S :: $'a$::*euclidean_space set*
  assumes *closed* S *closed* T $S \cap T = \{\}$
  obtains U V **where** *open* U *open* V $S \subseteq U$ $T \subseteq V$ $U \cap V = \{\}$
using *separation_closures* [*of* S T]
by (*metis assms closure_closed disjnt_def inf_commute*)

lemma *separation_normal_local*:
  fixes S :: $'a$::*euclidean_space set*
  assumes *US*: *closedin* (*top_of_set* U) S
    and *UT*: *closedin* (*top_of_set* U) T
    and $S \cap T = \{\}$
  obtains S' T' **where** *openin* (*top_of_set* U) S'
        openin (*top_of_set* U) T'
        $S \subseteq S'$ $T \subseteq T'$ $S' \cap T' = \{\}$
proof (*cases* $S = \{\} \vee T = \{\}$)
  case *True* **with** *that* **show** *?thesis*
    using *UT US* **by** (*blast dest*: *closedin_subset*)

next
  case *False*
  define *f* **where** $f \equiv \lambda x.\ setdist\ \{x\}\ T\ -\ setdist\ \{x\}\ S$
  have *contf*: *continuous_on U f*
    unfolding *f_def* **by** (*intro continuous_intros*)
  show *?thesis*
  proof (*rule_tac* $S' = (U \cap f\ -`\ \{0<..\})$ **and** $T' = (U \cap f\ -`\ \{..<0\})$ **in** *that*)
    show $(U \cap f\ -`\ \{0<..\}) \cap (U \cap f\ -`\ \{..<0\}) = \{\}$
      by *auto*
    show *openin* (*top_of_set U*) $(U \cap f\ -`\ \{0<..\})$
     by (*rule continuous_openin_preimage* [**where** *T=UNIV*]) (*simp_all add*: *contf*)
  next
    show *openin* (*top_of_set U*) $(U \cap f\ -`\ \{..<0\})$
     by (*rule continuous_openin_preimage* [**where** *T=UNIV*]) (*simp_all add*: *contf*)
  next
    have $S \subseteq U\ T \subseteq U$
      using *closedin_imp_subset assms* **by** *blast+*
    then show $S \subseteq U \cap f\ -`\ \{0<..\}\ T \subseteq U \cap f\ -`\ \{..<0\}$
       using *assms False* **by** (*force simp add*: *f_def setdist_sing_in_set intro*!: *setdist_gt_0_closedin*)+
  qed
qed

lemma *separation_normal_compact*:
  fixes *S* :: *'a::euclidean_space set*
  assumes *compact S closed T* $S \cap T = \{\}$
  obtains *U V* **where** *open U compact*(*closure U*) *open V* $S \subseteq U\ T \subseteq V\ U \cap V = \{\}$
proof −
  have *closed S bounded S*
    using *assms* **by** (*auto simp*: *compact_eq_bounded_closed*)
  then obtain *r* **where** *r>0* **and** *r*: $S \subseteq ball\ 0\ r$
    by (*auto dest*!: *bounded_subset_ballD*)
  have **: *closed* $(T \cup -\ ball\ 0\ r)$ $S \cap (T \cup -\ ball\ 0\ r) = \{\}$
    using *assms r* **by** *blast+*
  then obtain *U V* **where** *UV*: *open U open V* $S \subseteq U\ T \cup -\ ball\ 0\ r \subseteq V\ U \cap V = \{\}$
    by (*meson* ‹*closed S*› *separation_normal*)
  then have *compact*(*closure U*)
     by (*meson bounded_ball bounded_subset compact_closure compl_le_swap2 disjoint_eq_subset_Compl le_sup_iff*)
  with *UV* **show** *thesis*
    using *that* **by** *auto*
qed

5.0.11 Connectedness of the intersection of a chain

proposition *connected_chain*:
  fixes $\mathcal{F}$ :: *'a* :: *euclidean_space set set*

```
  assumes cc: ⋀S. S ∈ ℱ ⟹ compact S ∧ connected S
    and linear: ⋀S T. S ∈ ℱ ∧ T ∈ ℱ ⟹ S ⊆ T ∨ T ⊆ S
  shows connected(⋂ℱ)
proof (cases ℱ = {})
  case True then show ?thesis
    by auto
next
  case False
  then have cf: compact(⋂ℱ)
    by (simp add: cc compact_Inter)
  have False if AB: closed A closed B A ∩ B = {}
              and ABeq: A ∪ B = ⋂ℱ and A ≠ {} B ≠ {} for A B
  proof –
    obtain U V where open U open V A ⊆ U B ⊆ V U ∩ V = {}
      using separation_normal [OF AB] by metis
    obtain K where K ∈ ℱ compact K
      using cc False by blast
    then obtain N where open N and K ⊆ N
      by blast
    let ?C = insert (U ∪ V) ((λS. N – S) ‘ ℱ)
    obtain 𝒟 where 𝒟 ⊆ ?C finite 𝒟 K ⊆ ⋃𝒟
    proof (rule compactE [OF ‹compact K›])
      show K ⊆ ⋃(insert (U ∪ V) ((–) N ‘ ℱ))
        using ‹K ⊆ N› ABeq ‹A ⊆ U› ‹B ⊆ V› by auto
      show ⋀B. B ∈ insert (U ∪ V) ((–) N ‘ ℱ) ⟹ open B
       by (auto simp: ‹open U› ‹open V› open_Un ‹open N› cc compact_imp_closed
open_Diff)
    qed
    then have finite(𝒟 – {U ∪ V})
      by blast
    moreover have 𝒟 – {U ∪ V} ⊆ (λS. N – S) ‘ ℱ
      using ‹𝒟 ⊆ ?C› by blast
    ultimately obtain 𝒢 where 𝒢 ⊆ ℱ finite 𝒢 and Deq: 𝒟 – {U ∪ V} = (λS.
N–S) ‘ 𝒢
      using finite_subset_image by metis
    obtain J where J ∈ ℱ and J: (⋃S∈𝒢. N – S) ⊆ N – J
    proof (cases 𝒢 = {})
      case True
      with ‹ℱ ≠ {}› that show ?thesis
        by auto
    next
      case False
      have ⋀S T. ⟦S ∈ 𝒢; T ∈ 𝒢⟧ ⟹ S ⊆ T ∨ T ⊆ S
        by (meson ‹𝒢 ⊆ ℱ› in_mono local.linear)
      with ‹finite 𝒢› ‹𝒢 ≠ {}›
      have ∃J ∈ 𝒢. (⋃S∈𝒢. N – S) ⊆ N – J
      proof induction
        case (insert X ℋ)
        show ?case
```

proof (*cases* $\mathcal{H} = \{\}$)
case *True* **then show** *?thesis* **by** *auto*
next
case *False*
then have $\bigwedge S\ T.\ [\![S \in \mathcal{H};\ T \in \mathcal{H}]\!] \Longrightarrow S \subseteq T \vee T \subseteq S$
by (*simp add*: *insert.prems*)
with *insert.IH False* **obtain** J **where** $J \in \mathcal{H}$ **and** J: $(\bigcup Y{\in}\mathcal{H}.\ N - Y)$ $\subseteq N - J$
by *metis*
have $N - J \subseteq N - X \vee N - X \subseteq N - J$
by (*meson Diff_mono* ‹$J \in \mathcal{H}$› *insert.prems*(*2*) *insert_iff order_refl*)
then show *?thesis*
proof
assume $N - J \subseteq N - X$ **with** J **show** *?thesis*
by *auto*
next
assume $N - X \subseteq N - J$
with J **have** $N - X \cup \bigcup\ ((-)\ N\ `\ \mathcal{H}) \subseteq N - J$
by *auto*
with ‹$J \in \mathcal{H}$› **show** *?thesis*
by *blast*
qed
qed
qed *simp*
with ‹$\mathcal{G} \subseteq \mathcal{F}$› **show** *?thesis* **by** (*blast intro*: *that*)
qed
have $K \subseteq \bigcup(insert\ (U \cup V)\ (\mathcal{D} - \{U \cup V\}))$
using ‹$K \subseteq \bigcup\mathcal{D}$› **by** *auto*
also have ... $\subseteq (U \cup V) \cup (N - J)$
by (*metis* (*no_types*, *hide_lams*) *Deq Un_subset_iff Un_upper2 J Union_insert order_trans sup_ge1*)
finally have $J \cap K \subseteq U \cup V$
by *blast*
moreover have *connected*$(J \cap K)$
by (*metis Int_absorb1* ‹$J \in \mathcal{F}$› ‹$K \in \mathcal{F}$› *cc inf.orderE local.linear*)
moreover have $U \cap (J \cap K) \neq \{\}$
using *ABeq* ‹$J \in \mathcal{F}$› ‹$K \in \mathcal{F}$› ‹$A \neq \{\}$› ‹$A \subseteq U$› **by** *blast*
moreover have $V \cap (J \cap K) \neq \{\}$
using *ABeq* ‹$J \in \mathcal{F}$› ‹$K \in \mathcal{F}$› ‹$B \neq \{\}$› ‹$B \subseteq V$› **by** *blast*
ultimately show *False*
using *connectedD* [*of* $J \cap K\ U\ V$] ‹*open* U› ‹*open* V› ‹$U \cap V = \{\}$› **by** *auto*
qed
with *cf* **show** *?thesis*
by (*auto simp*: *connected_closed_set compact_imp_closed*)
qed

lemma *connected_chain_gen*:
fixes $\mathcal{F}$ :: *'a* :: *euclidean_space set set*

assumes *X*: $X \in \mathcal{F}$ *compact X*
  and *cc*: $\bigwedge T.\ T \in \mathcal{F} \Longrightarrow$ *closed* $T \wedge$ *connected* T
  and *linear*: $\bigwedge S\ T.\ S \in \mathcal{F} \wedge T \in \mathcal{F} \Longrightarrow S \subseteq T \vee T \subseteq S$
shows *connected*$(\bigcap \mathcal{F})$
proof –
  have $\bigcap \mathcal{F} = (\bigcap T{\in}\mathcal{F}.\ X \cap T)$
    using *X* **by** *blast*
  moreover have *connected* $(\bigcap T{\in}\mathcal{F}.\ X \cap T)$
  proof (*rule connected_chain*)
    show $\bigwedge T.\ T \in (\cap)\ X$ ‘ $\mathcal{F} \Longrightarrow$ *compact* $T \wedge$ *connected* T
      using *cc X* **by** *auto* (*metis inf.absorb2 inf.orderE local.linear*)
    show $\bigwedge S\ T.\ S \in (\cap)\ X$ ‘ $\mathcal{F} \wedge T \in (\cap)\ X$ ‘ $\mathcal{F} \Longrightarrow S \subseteq T \vee T \subseteq S$
      using *local.linear* **by** *blast*
  qed
  ultimately show *?thesis*
    by *metis*
qed

lemma *connected_nest*:
  fixes S :: *'a::linorder* $\Rightarrow$ *'b::euclidean_space set*
  assumes *S*: $\bigwedge n.$ *compact*$(S\ n)$ $\bigwedge n.$ *connected*$(S\ n)$
    and *nest*: $\bigwedge m\ n.\ m \le n \Longrightarrow S\ n \subseteq S\ m$
  shows *connected*$(\bigcap$ (*range S*))
proof (*rule connected_chain*)
  show $\bigwedge A\ T.\ A \in$ *range* $S \wedge T \in$ *range* $S \Longrightarrow A \subseteq T \vee T \subseteq A$
  by (*metis image_iff le_cases nest*)
qed (*use S* **in** *blast*)

lemma *connected_nest_gen*:
  fixes S :: *'a::linorder* $\Rightarrow$ *'b::euclidean_space set*
  assumes *S*: $\bigwedge n.$ *closed*$(S\ n)$ $\bigwedge n.$ *connected*$(S\ n)$ *compact*$(S\ k)$
    and *nest*: $\bigwedge m\ n.\ m \le n \Longrightarrow S\ n \subseteq S\ m$
  shows *connected*$(\bigcap$ (*range S*))
proof (*rule connected_chain_gen* [*of S k*])
  show $\bigwedge A\ T.\ A \in$ *range* $S \wedge T \in$ *range* $S \Longrightarrow A \subseteq T \vee T \subseteq A$
    by (*metis imageE le_cases nest*)
qed (*use S* **in** *auto*)

5.0.12 Proper maps, including projections out of compact sets

lemma *finite_indexed_bound*:
  assumes *A*: *finite A* $\bigwedge x.\ x \in A \Longrightarrow \exists n$::*'a::linorder*. $P\ x\ n$
    shows $\exists m.\ \forall x \in A.\ \exists k{\le}m.\ P\ x\ k$
using *A*
proof (*induction A*)
  case *empty* **then show** *?case* **by** *force*
next
  case (*insert a A*)

then obtain m n **where** $\forall x \in A.\ \exists k{\le}m.\ P\ x\ k\ P\ a\ n$
by *force*
then show *?case*
by (*metis dual_order.trans insert_iff le_cases*)
qed

proposition *proper_map*:
fixes f :: $'a$::*heine_borel* $\Rightarrow$ $'b$::*heine_borel*
assumes *closedin* (*top_of_set* S) K
and *com*: $\bigwedge U.\ [\![U \subseteq T;\ compact\ U]\!] \Longrightarrow compact\ (S \cap f -`\ U)$
and $f\ `\ S \subseteq T$
shows *closedin* (*top_of_set* T) ($f\ `\ K$)
proof −
have $K \subseteq S$
using *assms closedin_imp_subset* **by** *metis*
obtain C **where** *closed* C **and** *Keq*: $K = S \cap C$
using *assms* **by** (*auto simp*: *closedin_closed*)
have ∗: $y \in f\ `\ K$ **if** $y \in T$ **and** $y$: $y$ *islimpt* $f\ `\ K$ **for** y
proof −
obtain h **where** $\forall n.\ (\exists x{\in}K.\ h\ n = f\ x) \wedge h\ n \neq y$ *inj* h **and** *hlim*: ($h \longrightarrow$ y) *sequentially*
using ‹$y \in T$› y **by** (*force simp*: *limpt_sequential_inj*)
then obtain X **where** X: $\bigwedge n.\ X\ n \in K \wedge h\ n = f\ (X\ n) \wedge h\ n \neq y$
by *metis*
then have *fX*: $\bigwedge n.\ f\ (X\ n) = h\ n$
by *metis*
define Ψ **where** $\Psi \equiv \lambda n.\ \{a \in K.\ f\ a \in insert\ y\ (range\ (\lambda i.\ f\ (X\ (n + i))))\}$
have *compact* $(C \cap (S \cap f -`\ insert\ y\ (range\ (\lambda i.\ f(X(n + i))))))$ **for** n
proof (*intro closed_Int_compact* [*OF* ‹*closed C*› *com*] *compact_sequence_with_limit*)
show *insert* y (*range* ($\lambda i.\ f\ (X\ (n + i))$)) $\subseteq T$
using X ‹$K \subseteq S$› ‹$f\ `\ S \subseteq T$› ‹$y \in T$› **by** *blast*
show $(\lambda i.\ f\ (X\ (n + i))) \longrightarrow y$
by (*simp add*: *fX add.commute* [*of n*] *LIMSEQ_ignore_initial_segment* [*OF* *hlim*])
qed
then have *comf*: *compact* ($\Psi\ n$) **for** n
by (*simp add*: *Keq Int_def* Ψ*_def conj_commute*)
have *ne*: $\bigcap\mathcal{F} \neq \{\}$
if *finite* $\mathcal{F}$
and $\mathcal{F}$: $\bigwedge t.\ t \in \mathcal{F} \Longrightarrow (\exists n.\ t = \Psi\ n)$
for $\mathcal{F}$
proof −
obtain m **where** m: $\bigwedge t.\ t \in \mathcal{F} \Longrightarrow \exists k{\le}m.\ t = \Psi\ k$
by (*rule exE* [*OF finite_indexed_bound* [*OF* ‹*finite* $\mathcal{F}$› $\mathcal{F}$]], *force*+)
have $X\ m \in \bigcap\mathcal{F}$
using X *le_Suc_ex* **by** (*fastforce simp*: Ψ*_def dest*: m)
then show *?thesis* **by** *blast*
qed
have $(\bigcap n.\ \Psi\ n) \neq \{\}$

```
    proof (rule compact_fip_Heine_Borel)
      show ⋀ℱ'. ⟦finite ℱ'; ℱ' ⊆ range Ψ⟧ ⟹ ⋂ ℱ' ≠ {}
        by (meson ne rangeE subset_eq)
    qed (use comf in blast)
    then obtain x where x ∈ K ⋀n. (f x = y ∨ (∃u. f x = h (n + u)))
      by (force simp add: Ψ_def fX)
    then show ?thesis
     unfolding image_iff by (metis ‹inj h› le_add1 not_less_eq_eq rangeI range_ex1_eq)
  qed
  with assms closedin_subset show ?thesis
    by (force simp: closedin_limpt)
qed


lemma compact_continuous_image_eq:
  fixes f :: 'a::heine_borel ⇒ 'b::heine_borel
  assumes f: inj_on f S
  shows continuous_on S f ⟷ (∀T. compact T ∧ T ⊆ S ⟶ compact(f ` T))
          (is ?lhs = ?rhs)
proof
  assume ?lhs then show ?rhs
    by (metis continuous_on_subset compact_continuous_image)
next
  assume RHS: ?rhs
  obtain g where gf: ⋀x. x ∈ S ⟹ g (f x) = x
    by (metis inv_into_f_f f)
  then have *: (S ∩ f -` U) = g ` U if U ⊆ f ` S for U
    using that by fastforce
  have gfim: g ` f ` S ⊆ S using gf by auto
  have **: compact (f ` S ∩ g -` C) if C: C ⊆ S compact C for C
  proof -
    obtain h where h C ∈ C ∧ h C ∉ S ∨ compact (f ` C)
      by (force simp: C RHS)
    moreover have f ` C = (f ` S ∩ g -` C)
      using C gf by auto
    ultimately show ?thesis
      using C by auto
  qed
  show ?lhs
    using proper_map [OF _ _ gfim] **
    by (simp add: continuous_on_closed * closedin_imp_subset)
qed
```

5.0.13 Trivial fact: convexity equals connectedness for collinear sets

```
lemma convex_connected_collinear:
  fixes S :: 'a::euclidean_space set
  assumes collinear S
```

shows *convex S* ⟷ *connected S*
proof
assume *convex S*
then show *connected S*
using *convex_connected* **by** *blast*
next
assume *S*: *connected S*
show *convex S*
proof (*cases S* = {})
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then obtain *a* **where** *a* ∈ *S* **by** *auto*
have *collinear* (*affine hull S*)
by (*simp add*: *assms collinear_affine_hull_collinear*)
then obtain *z* **where** *z* ≠ *0* ⋀*x*. *x* ∈ *affine hull S* ⟹ ∃ *c*. *x* − *a* = *c* $*_R$ *z*
by (*meson* ‹*a* ∈ *S*› *collinear hull_inc*)
then obtain *f* **where** *f*: ⋀*x*. *x* ∈ *affine hull S* ⟹ *x* − *a* = *f x* $*_R$ *z*
by *metis*
then have *inj_f*: *inj_on f* (*affine hull S*)
by (*metis diff_add_cancel inj_onI*)
have *diff*: *x* − *y* = (*f x* − *f y*) $*_R$ *z* **if** *x*: *x* ∈ *affine hull S* **and** *y*: *y* ∈ *affine hull S* **for** *x y*
proof −
have *f x* $*_R$ *z* = *x* − *a*
by (*simp add*: *f hull_inc x*)
moreover have *f y* $*_R$ *z* = *y* − *a*
by (*simp add*: *f hull_inc y*)
ultimately show *?thesis*
by (*simp add*: *scaleR_left.diff*)
qed
have *cont_f*: *continuous_on* (*affine hull S*) *f*
proof (*clarsimp simp*: *dist_norm continuous_on_iff diff*)
show ⋀*x e*. *0* < *e* ⟹ ∃ *d*>*0*. ∀ *y* ∈ *affine hull S*. |*f y* − *f x*| ∗ *norm z* < *d* ⟶ |*f y* − *f x*| < *e*
by (*metis* ‹*z* ≠ *0*› *mult_pos_pos mult_less_iff1 zero_less_norm_iff*)
qed
then have *conn_fS*: *connected* (*f* ‘ *S*)
by (*meson S connected_continuous_image continuous_on_subset hull_subset*)
show *?thesis*
proof (*clarsimp simp*: *convex_contains_segment*)
fix *x y z*
assume *x* ∈ *S y* ∈ *S z* ∈ *closed_segment x y*
have *False* **if** *z* ∉ *S*
proof −
have *f* ‘ (*closed_segment x y*) = *closed_segment* (*f x*) (*f y*)
proof (*rule continuous_injective_image_segment_1*)
show *continuous_on* (*closed_segment x y*) *f*

by (*meson* ‹$x \in S$› ‹$y \in S$› *convex_affine_hull convex_contains_segment hull_inc continuous_on_subset* [*OF cont_f*])
show *inj_on f* (*closed_segment x y*)
by (*meson* ‹$x \in S$› ‹$y \in S$› *convex_affine_hull convex_contains_segment hull_inc inj_on_subset* [*OF inj_f*])
qed
then have *fz*: $f\ z \in$ *closed_segment* $(f\ x)\ (f\ y)$
using ‹$z \in$ *closed_segment* $x\ y$› **by** *blast*
have $z \in$ *affine hull* S
by (*meson* ‹$x \in S$› ‹$y \in S$› ‹$z \in$ *closed_segment* $x\ y$› *convex_affine_hull convex_contains_segment hull_inc subset_eq*)
then have *fz_notin*: $f\ z \notin f$ ‘ S
using *hull_subset inj_f inj_onD that* **by** *fastforce*
moreover have $\{..<f\ z\} \cap f$ ‘ $S \neq \{\}$ $\{f\ z<..\} \cap f$ ‘ $S \neq \{\}$
proof −
consider $f\ x \le f\ z \wedge f\ z \le f\ y \mid f\ y \le f\ z \wedge f\ z \le f\ x$
using *fz*
by (*auto simp add*: *closed_segment_eq_real_ivl split*: *if_split_asm*)
then have $\{..<f\ z\} \cap f$ ‘ $\{x,y\} \neq \{\} \wedge \{f\ z<..\} \cap f$ ‘ $\{x,y\} \neq \{\}$
by *cases* (*use fz_notin* ‹$x \in S$› ‹$y \in S$› **in** ‹*auto simp*: *image_iff*›)
then show $\{..<f\ z\} \cap f$ ‘ $S \neq \{\}$ $\{f\ z<..\} \cap f$ ‘ $S \neq \{\}$
using ‹$x \in S$› ‹$y \in S$› **by** *blast+*
qed
ultimately show *False*
using *connectedD* [*OF conn_fS*, *of* $\{..<f\ z\}$ $\{f\ z<..\}$] **by** *force*
qed
then show $z \in S$ **by** *meson*
qed
qed
qed

lemma *compact_convex_collinear_segment_alt*:
fixes S :: *'a::euclidean_space set*
assumes $S \neq \{\}$ *compact* S *connected* S *collinear* S
obtains $a\ b$ **where** $S =$ *closed_segment* $a\ b$
proof −
obtain ξ **where** $\xi \in S$ **using** ‹$S \neq \{\}$› **by** *auto*
have *collinear* (*affine hull* S)
by (*simp add*: *assms collinear_affine_hull_collinear*)
then obtain z **where** $z \neq 0$ $\bigwedge x.\ x \in$ *affine hull* $S \Longrightarrow \exists c.\ x - \xi = c *_R z$
by (*meson* ‹$\xi \in S$› *collinear hull_inc*)
then obtain f **where** f: $\bigwedge x.\ x \in$ *affine hull* $S \Longrightarrow x - \xi = f\ x *_R z$
by *metis*
let $?g = \lambda r.\ r *_R z + \xi$
have *gf*: $?g\ (f\ x) = x$ **if** $x \in$ *affine hull* S **for** x
by (*metis diff_add_cancel f that*)
then have *inj_f*: *inj_on f* (*affine hull* S)
by (*metis inj_onI*)
have *diff*: $x - y = (f\ x - f\ y) *_R z$ **if** x: $x \in$ *affine hull* S **and** y: $y \in$ *affine*

hull S **for** *x y*
  proof −
    have *f x* $*_R$ *z* = *x* − *ξ*
      by (*simp add*: *f hull_inc x*)
    moreover have *f y* $*_R$ *z* = *y* − *ξ*
      by (*simp add*: *f hull_inc y*)
    ultimately show *?thesis*
      by (*simp add*: *scaleR_left.diff*)
  qed
  have *cont_f*: *continuous_on* (*affine hull S*) *f*
  proof (*clarsimp simp*: *dist_norm continuous_on_iff diff*)
    show ⋀*x e*. *0* < *e* ⟹ ∃ *d*>*0*. ∀ *y* ∈ *affine hull S*. |*f y* − *f x*| ∗ *norm z* < *d* ⟶ |*f y* − *f x*| < *e*
      by (*metis* ‹*z* ≠ *0*› *mult_pos_pos mult_less_iff1 zero_less_norm_iff*)
  qed
  then have *connected* (*f* ‘ *S*)
      by (*meson* ‹*connected S*› *connected_continuous_image continuous_on_subset hull_subset*)
  moreover have *compact* (*f* ‘ *S*)
    by (*meson* ‹*compact S*› *compact_continuous_image_eq cont_f hull_subset inj_f*)
  ultimately obtain *x y* **where** *f* ‘ *S* = {*x*..*y*}
    by (*meson connected_compact_interval_1*)
  then have *fS_eq*: *f* ‘ *S* = *closed_segment x y*
    using ‹*S* ≠ {}› *closed_segment_eq_real_ivl* **by** *auto*
  obtain *a b* **where** *a* ∈ *S f a* = *x b* ∈ *S f b* = *y*
    by (*metis* (*full_types*) *ends_in_segment fS_eq imageE*)
  have *f* ‘ (*closed_segment a b*) = *closed_segment* (*f a*) (*f b*)
  proof (*rule continuous_injective_image_segment_1*)
    show *continuous_on* (*closed_segment a b*) *f*
      by (*meson* ‹*a* ∈ *S*› ‹*b* ∈ *S*› *convex_affine_hull convex_contains_segment hull_inc continuous_on_subset* [*OF cont_f*])
    show *inj_on f* (*closed_segment a b*)
      by (*meson* ‹*a* ∈ *S*› ‹*b* ∈ *S*› *convex_affine_hull convex_contains_segment hull_inc inj_on_subset* [*OF inj_f*])
  qed
  then have *f* ‘ (*closed_segment a b*) = *f* ‘ *S*
    by (*simp add*: ‹*f a* = *x*› ‹*f b* = *y*› *fS_eq*)
  then have *?g* ‘ *f* ‘ (*closed_segment a b*) = *?g* ‘ *f* ‘ *S*
    by *simp*
  moreover have (*λx*. *f x* $*_R$ *z* + *ξ*) ‘ *closed_segment a b* = *closed_segment a b*
    unfolding *image_def* **using** ‹*a* ∈ *S*› ‹*b* ∈ *S*›
    by (*safe*; *metis* (*mono_tags*, *lifting*) *convex_affine_hull convex_contains_segment gf hull_subset subsetCE*)
  ultimately have *closed_segment a b* = *S*
    using *gf* **by** (*simp add*: *image_comp o_def hull_inc cong*: *image_cong*)
  then show *?thesis*
    using *that* **by** *blast*
qed

lemma *compact_convex_collinear_segment*:
  fixes *S* :: *'a::euclidean_space set*
  assumes *S* ≠ {} *compact S convex S collinear S*
  obtains *a b* **where** *S* = *closed_segment a b*
  using *assms convex_connected_collinear compact_convex_collinear_segment_alt* **by** *blast*

lemma *proper_map_from_compact*:
  fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
  assumes *contf*: *continuous_on S f* **and** *imf*: *f* ' *S* ⊆ *T* **and** *compact S*
    closedin (*top_of_set T*) *K*
  shows *compact* (*S* ∩ *f* −' *K*)
by (*rule closedin_compact* [*OF* ‹*compact S*›] *continuous_closedin_preimage_gen assms*)+

lemma *proper_map_fst*:
  assumes *compact T K* ⊆ *S compact K*
    shows *compact* (*S* × *T* ∩ *fst* −' *K*)
proof −
  have (*S* × *T* ∩ *fst* −' *K*) = *K* × *T*
    using *assms* **by** *auto*
  then show *?thesis* **by** (*simp add*: *assms compact_Times*)
qed

lemma *closed_map_fst*:
  fixes *S* :: *'a::euclidean_space set* **and** *T* :: *'b::euclidean_space set*
  assumes *compact T closedin* (*top_of_set* (*S* × *T*)) *c*
   shows *closedin* (*top_of_set S*) (*fst* ' *c*)
proof −
  have ∗: *fst* ' (*S* × *T*) ⊆ *S*
    by *auto*
  show *?thesis*
    using *proper_map* [*OF* _ _ ∗] **by** (*simp add*: *proper_map_fst assms*)
qed

lemma *proper_map_snd*:
  assumes *compact S K* ⊆ *T compact K*
    shows *compact* (*S* × *T* ∩ *snd* −' *K*)
proof −
  have (*S* × *T* ∩ *snd* −' *K*) = *S* × *K*
    using *assms* **by** *auto*
  then show *?thesis* **by** (*simp add*: *assms compact_Times*)
qed

lemma *closed_map_snd*:
  fixes *S* :: *'a::euclidean_space set* **and** *T* :: *'b::euclidean_space set*
  assumes *compact S closedin* (*top_of_set* (*S* × *T*)) *c*
   shows *closedin* (*top_of_set T*) (*snd* ' *c*)
proof −

have ∗: *snd* ‘ (*S* × *T*) ⊆ *T*
by *auto*
show *?thesis*
using *proper_map* [*OF* _ _ ∗] **by** (*simp add*: *proper_map_snd assms*)
qed

lemma *closedin_compact_projection*:
fixes *S* :: ′*a*::*euclidean_space set* **and** *T* :: ′*b*::*euclidean_space set*
assumes *compact S* **and** *clo*: *closedin* (*top_of_set* (*S* × *T*)) *U*
shows *closedin* (*top_of_set T*) {*y*. ∃ *x*. *x* ∈ *S* ∧ (*x*, *y*) ∈ *U*}
proof −
have *U* ⊆ *S* × *T*
by (*metis clo closedin_imp_subset*)
then have {*y*. ∃ *x*. *x* ∈ *S* ∧ (*x*, *y*) ∈ *U*} = *snd* ‘ *U*
by *force*
moreover have *closedin* (*top_of_set T*) (*snd* ‘ *U*)
by (*rule closed_map_snd* [*OF assms*])
ultimately show *?thesis*
by *simp*
qed

lemma *closed_compact_projection*:
fixes *S* :: ′*a*::*euclidean_space set*
and *T* :: (′*a* ∗ ′*b*::*euclidean_space*) *set*
assumes *compact S* **and** *clo*: *closed T*
shows *closed* {*y*. ∃ *x*. *x* ∈ *S* ∧ (*x*, *y*) ∈ *T*}
proof −
have ∗: {*y*. ∃ *x*. *x* ∈ *S* ∧ *Pair x y* ∈ *T*} = {*y*. ∃ *x*. *x* ∈ *S* ∧ *Pair x y* ∈ ((*S* × *UNIV*) ∩ *T*)}
by *auto*
show *?thesis*
unfolding ∗
by (*intro clo closedin_closed_Int closedin_closed_trans* [*OF* _ *closed_UNIV*] *closedin_compact_projection* [*OF* ‹*compact S*›])
qed

Representing affine hull as a finite intersection of hyperplanes

proposition *affine_hull_convex_Int_nonempty_interior*:
fixes *S* :: ′*a*::*real_normed_vector set*
assumes *convex S S* ∩ *interior T* ≠ {}
shows *affine hull* (*S* ∩ *T*) = *affine hull S*
proof
show *affine hull* (*S* ∩ *T*) ⊆ *affine hull S*
by (*simp add*: *hull_mono*)
next
obtain *a* **where** *a* ∈ *S a* ∈ *T* **and** *at*: *a* ∈ *interior T*
using *assms interior_subset* **by** *blast*

```
  then obtain e where e > 0 and e: cball a e ⊆ T
    using mem_interior_cball by blast
  have *: x ∈ (+) a ‘ span ((λx. x − a) ‘ (S ∩ T)) if x ∈ S for x
  proof (cases x = a)
    case True with that span_0 eq_add_iff image_def mem_Collect_eq show ?thesis
      by blast
  next
    case False
    define k where k = min (1/2) (e / norm (x−a))
    have k: 0 < k k < 1
      using ‹e > 0› False by (auto simp: k_def)
    then have xa: (x−a) = inverse k *R k *R (x−a)
      by simp
    have e / norm (x − a) ≥ k
      using k_def by linarith
    then have a + k *R (x − a) ∈ cball a e
      using ‹0 < k› False
      by (simp add: dist_norm) (simp add: field_simps)
    then have T: a + k *R (x − a) ∈ T
      using e by blast
    have S: a + k *R (x − a) ∈ S
      using k ‹a ∈ S› convexD [OF ‹convex S› ‹a ∈ S› ‹x ∈ S›, of 1−k k]
      by (simp add: algebra_simps)
    have inverse k *R k *R (x−a) ∈ span ((λx. x − a) ‘ (S ∩ T))
       by (intro span_mul [OF span_base] image_eqI [where x = a + k *R (x −
a)]) (auto simp: S T)
    with xa image_iff show ?thesis  by fastforce
  qed
  have S ⊆ affine hull (S ∩ T)
    by (force simp: * ‹a ∈ S› ‹a ∈ T› hull_inc affine_hull_span_gen [of a])
  then show affine hull S ⊆ affine hull (S ∩ T)
    by (simp add: subset_hull)
qed

corollary affine_hull_convex_Int_open:
  fixes S :: 'a::real_normed_vector set
  assumes convex S open T S ∩ T ≠ {}
  shows affine hull (S ∩ T) = affine hull S
  using affine_hull_convex_Int_nonempty_interior assms interior_eq by blast

corollary affine_hull_affine_Int_nonempty_interior:
  fixes S :: 'a::real_normed_vector set
  assumes affine S S ∩ interior T ≠ {}
  shows affine hull (S ∩ T) = affine hull S
 by (simp add: affine_hull_convex_Int_nonempty_interior affine_imp_convex assms)

corollary affine_hull_affine_Int_open:
  fixes S :: 'a::real_normed_vector set
  assumes affine S open T S ∩ T ≠ {}
```

shows *affine hull* $(S \cap T)$ = *affine hull* S
by (*simp add*: *affine_hull_convex_Int_open affine_imp_convex assms*)

corollary *affine_hull_convex_Int_openin*:
fixes S :: *'a::real_normed_vector set*
assumes *convex* S *openin* (*top_of_set* (*affine hull* S)) T $S \cap T \neq \{\}$
shows *affine hull* $(S \cap T)$ = *affine hull* S
using *assms* **unfolding** *openin_open*
by (*metis affine_hull_convex_Int_open hull_subset inf.orderE inf_assoc*)

corollary *affine_hull_openin*:
fixes S :: *'a::real_normed_vector set*
assumes *openin* (*top_of_set* (*affine hull* T)) S $S \neq \{\}$
shows *affine hull* S = *affine hull* T
using *assms* **unfolding** *openin_open*
by (*metis affine_affine_hull affine_hull_affine_Int_open hull_hull*)

corollary *affine_hull_open*:
fixes S :: *'a::real_normed_vector set*
assumes *open* S $S \neq \{\}$
shows *affine hull* S = *UNIV*
by (*metis affine_hull_convex_Int_nonempty_interior assms convex_UNIV hull_UNIV inf_top.left_neutral interior_open*)

lemma *aff_dim_convex_Int_nonempty_interior*:
fixes S :: *'a::euclidean_space set*
shows ⟦*convex* S; $S \cap$ *interior* $T \neq \{\}$⟧ $\Longrightarrow$ *aff_dim*$(S \cap T)$ = *aff_dim* S
using *aff_dim_affine_hull2 affine_hull_convex_Int_nonempty_interior* **by** *blast*

lemma *aff_dim_convex_Int_open*:
fixes S :: *'a::euclidean_space set*
shows ⟦*convex* S; *open* T; $S \cap T \neq \{\}$⟧ $\Longrightarrow$ *aff_dim*$(S \cap T)$ = *aff_dim* S
using *aff_dim_convex_Int_nonempty_interior interior_eq* **by** *blast*

lemma *affine_hull_Diff*:
fixes S:: *'a::real_normed_vector set*
assumes *ope*: *openin* (*top_of_set* (*affine hull* S)) S **and** *finite* F $F \subset S$
shows *affine hull* $(S - F)$ = *affine hull* S
proof −
have *clo*: *closedin* (*top_of_set* S) F
using *assms finite_imp_closedin* **by** *auto*
moreover have $S - F \neq \{\}$
using *assms* **by** *auto*
ultimately show *?thesis*
by (*metis ope closedin_def topspace_euclidean_subtopology affine_hull_openin openin_trans*)
qed

lemma *affine_hull_halfspace_lt*:

fixes *a* :: *'a::euclidean_space*
shows *affine hull* {*x. a* · *x* < *r*} = (*if a* = *0* ∧ *r* ≤ *0 then* {} *else UNIV*)
using *halfspace_eq_empty_lt* [*of a r*]
by (*simp add*: *open_halfspace_lt affine_hull_open*)

lemma *affine_hull_halfspace_le*:
fixes *a* :: *'a::euclidean_space*
shows *affine hull* {*x. a* · *x* ≤ *r*} = (*if a* = *0* ∧ *r* < *0 then* {} *else UNIV*)
proof (*cases a* = *0*)
case *True* **then show** *?thesis* **by** *simp*
next
case *False*
then have *affine hull closure* {*x. a* · *x* < *r*} = *UNIV*
using *affine_hull_halfspace_lt closure_same_affine_hull* **by** *fastforce*
moreover have {*x. a* · *x* < *r*} ⊆ {*x. a* · *x* ≤ *r*}
by (*simp add*: *Collect_mono*)
ultimately show *?thesis* **using** *False antisym_conv hull_mono top_greatest*
by (*metis affine_hull_halfspace_lt*)
qed

lemma *affine_hull_halfspace_gt*:
fixes *a* :: *'a::euclidean_space*
shows *affine hull* {*x. a* · *x* > *r*} = (*if a* = *0* ∧ *r* ≥ *0 then* {} *else UNIV*)
using *halfspace_eq_empty_gt* [*of r a*]
by (*simp add*: *open_halfspace_gt affine_hull_open*)

lemma *affine_hull_halfspace_ge*:
fixes *a* :: *'a::euclidean_space*
shows *affine hull* {*x. a* · *x* ≥ *r*} = (*if a* = *0* ∧ *r* > *0 then* {} *else UNIV*)
using *affine_hull_halfspace_le* [*of* −*a* −*r*] **by** *simp*

lemma *aff_dim_halfspace_lt*:
fixes *a* :: *'a::euclidean_space*
shows *aff_dim* {*x. a* · *x* < *r*} =
(*if a* = *0* ∧ *r* ≤ *0 then* −*1 else DIM*(*'a*))
by *simp* (*metis aff_dim_open halfspace_eq_empty_lt open_halfspace_lt*)

lemma *aff_dim_halfspace_le*:
fixes *a* :: *'a::euclidean_space*
shows *aff_dim* {*x. a* · *x* ≤ *r*} =
(*if a* = *0* ∧ *r* < *0 then* −*1 else DIM*(*'a*))
proof −
have *int* (*DIM*(*'a*)) = *aff_dim* (*UNIV*::*'a set*)
by (*simp*)
then have *aff_dim* (*affine hull* {*x. a* · *x* ≤ *r*}) = *DIM*(*'a*) **if** (*a* = *0* ⟶ *r* ≥ *0*)
using *that* **by** (*simp add*: *affine_hull_halfspace_le not_less*)
then show *?thesis*
by (*force*)

qed

lemma *aff_dim_halfspace_gt*:
  fixes *a* :: *'a::euclidean_space*
  shows *aff_dim* {*x*. *a* · *x* > *r*} =
        (*if a* = *0* ∧ *r* ≥ *0 then* −*1 else DIM*(*'a*))
by *simp* (*metis aff_dim_open halfspace_eq_empty_gt open_halfspace_gt*)

lemma *aff_dim_halfspace_ge*:
  fixes *a* :: *'a::euclidean_space*
  shows *aff_dim* {*x*. *a* · *x* ≥ *r*} =
        (*if a* = *0* ∧ *r* > *0 then* −*1 else DIM*(*'a*))
using *aff_dim_halfspace_le* [*of* −*a* −*r*] **by** *simp*

proposition *aff_dim_eq_hyperplane*:
  fixes *S* :: *'a::euclidean_space set*
  shows *aff_dim S* = *DIM*(*'a*) − *1* ⟷ (∃ *a b*. *a* ≠ *0* ∧ *affine hull S* = {*x*. *a* · *x*
= *b*})
  (**is** *?lhs* = *?rhs*)
proof (*cases S* = {})
  case *True* **then show** *?thesis*
    by (*auto simp*: *dest*: *hyperplane_eq_Ex*)
next
  case *False*
  then obtain *c* **where** *c* ∈ *S* **by** *blast*
  show *?thesis*
  proof (*cases c* = *0*)
    case *True*
    have *?lhs* ⟷ (∃ *a*. *a* ≠ *0* ∧ *span* ((λ*x*. *x* − *c*) ‘ *S*) = {*x*. *a* · *x* = *0*})
      by (*simp add*: *aff_dim_eq_dim* [*of c*] ‹*c* ∈ *S*› *hull_inc dim_eq_hyperplane del*:
One_nat_def)
    also have ... ⟷ *?rhs*
      using *span_zero* [*of S*] *True* ‹*c* ∈ *S*› *affine_hull_span_0 hull_inc*
      by (*fastforce simp add*: *affine_hull_span_gen* [*of c*] ‹*c* = *0*›)
    finally show *?thesis* **.**
  next
    case *False*
    have *xc_im*: *x* ∈ (+) *c* ‘ {*y*. *a* · *y* = *0*} **if** *a* · *x* = *a* · *c* **for** *a x*
    proof −
      have ∃ *y*. *a* · *y* = *0* ∧ *c* + *y* = *x*
        by (*metis that add.commute diff_add_cancel inner_commute inner_diff_left
right_minus_eq*)
      then show *x* ∈ (+) *c* ‘ {*y*. *a* · *y* = *0*}
        by *blast*
    qed
    have *2*: *span* ((λ*x*. *x* − *c*) ‘ *S*) = {*x*. *a* · *x* = *0*}
        if (+) *c* ‘ *span* ((λ*x*. *x* − *c*) ‘ *S*) = {*x*. *a* · *x* = *b*} **for** *a b*
    proof −
      have *b* = *a* · *c*

using *span_0 that* **by** *fastforce*
with *that* **have** $(+)$ *c* ‘ *span* $((\lambda x.\ x - c)$ ‘ $S) = \{x.\ a \cdot x = a \cdot c\}$
by *simp*
then have *span* $((\lambda x.\ x - c)$ ‘ $S) = (\lambda x.\ x - c)$ ‘ $\{x.\ a \cdot x = a \cdot c\}$
by (*metis* (*no_types*) *image_cong translation_galois uminus_add_conv_diff*)
also have ... $= \{x.\ a \cdot x = 0\}$
by (*force simp*: *inner_distrib inner_diff_right*
intro: *image_eqI* [**where** $x{=}x{+}c$ **for** *x*])
finally show *?thesis* **.**
qed
have *?lhs* $= (\exists\, a.\ a \neq 0 \wedge$ *span* $((\lambda x.\ x - c)$ ‘ $S) = \{x.\ a \cdot x = 0\})$
by (*simp add*: *aff_dim_eq_dim* [*of c*] ‹$c \in S$› *hull_inc dim_eq_hyperplane del*: *One_nat_def*)
also have ... = *?rhs*
by (*fastforce simp add*: *affine_hull_span_gen* [*of c*] ‹$c \in S$› *hull_inc inner_distrib intro*: *xc_im intro*!: *2*)
finally show *?thesis* **.**
qed
qed

corollary *aff_dim_hyperplane* [*simp*]:
fixes *a* :: *'a*::*euclidean_space*
shows $a \neq 0 \Longrightarrow$ *aff_dim* $\{x.\ a \cdot x = r\} = DIM('a) - 1$
by (*metis aff_dim_eq_hyperplane affine_hull_eq affine_hyperplane*)

5.0.14 Some stepping theorems

lemma *aff_dim_insert*:
fixes *a* :: *'a*::*euclidean_space*
shows *aff_dim* (*insert a S*) = (*if* $a \in$ *affine hull S then aff_dim S else aff_dim S* + *1*)
proof (*cases* $S = \{\}$)
case *True* **then show** *?thesis*
by *simp*
next
case *False*
then obtain *x s'* **where** *S*: *S* = *insert x s'* $x \notin s'$
by (*meson Set.set_insert all_not_in_conv*)
show *?thesis* **using** *S*
by (*force simp add*: *affine_hull_insert_span_gen span_zero insert_commute* [*of a*] *aff_dim_eq_dim* [*of x*] *dim_insert*)
qed

lemma *affine_dependent_choose*:
fixes *a* :: *'a* :: *euclidean_space*
assumes ¬(*affine_dependent S*)
shows *affine_dependent*(*insert a S*) $\longleftrightarrow a \notin S \wedge a \in$ *affine hull S*
(**is** *?lhs* = *?rhs*)
proof *safe*

assume *affine_dependent* (*insert a S*) **and** $a \in S$
then show *False*
using ‹$a \in S$› *assms insert_absorb* **by** *fastforce*
next
assume *lhs*: *affine_dependent* (*insert a S*)
then have $a \notin S$
by (*metis* (*no_types*) *assms insert_absorb*)
moreover have *finite S*
using *affine_independent_iff_card assms* **by** *blast*
moreover have *aff_dim* (*insert a S*) $\neq$ *int* (*card S*)
using ‹*finite S*› *affine_independent_iff_card* ‹$a \notin S$› *lhs* **by** *fastforce*
ultimately show $a \in$ *affine hull S*
by (*metis aff_dim_affine_independent aff_dim_insert assms*)
next
assume $a \notin S$ **and** $a \in$ *affine hull S*
show *affine_dependent* (*insert a S*)
by (*simp add*: ‹$a \in$ *affine hull S*› ‹$a \notin S$› *affine_dependent_def*)
qed

lemma *affine_independent_insert*:
fixes *a* :: *'a* :: *euclidean_space*
shows ⟦¬ *affine_dependent S*; $a \notin$ *affine hull S*⟧ ⟹ ¬ *affine_dependent*(*insert a S*)
by (*simp add*: *affine_dependent_choose*)

lemma *subspace_bounded_eq_trivial*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *subspace S*
shows *bounded S* ⟷ *S* = {*0*}
proof −
have *False* **if** *bounded S* $x \in S$ $x \neq 0$ **for** *x*
proof −
obtain *B* **where** *B*: ⋀*y*. $y \in S$ ⟹ *norm y* < *B B* > *0*
using ‹*bounded S*› **by** (*force simp*: *bounded_pos_less*)
have (*B* / *norm x*) $*_R$ $x \in S$
using *assms subspace_mul* ‹$x \in S$› **by** *auto*
moreover have *norm* ((*B* / *norm x*) $*_R$ *x*) = *B*
using *that B* **by** (*simp add*: *algebra_simps*)
ultimately show *False* **using** *B* **by** *force*
qed
then have *bounded S* ⟹ *S* = {*0*}
using *assms subspace_0* **by** *fastforce*
then show *?thesis*
by *blast*
qed

lemma *affine_bounded_eq_trivial*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *affine S*

```
  shows bounded S ⟷ S = {} ∨ (∃ a. S = {a})
proof (cases S = {})
  case True then show ?thesis
    by simp
next
  case False
  then obtain b where b ∈ S by blast
  with False assms
  have bounded S ⟹ S = {b}
    using affine_diffs_subspace [OF assms ‹b ∈ S›]
   by (metis (no_types, lifting) ab_group_add_class.ab_left_minus bounded_translation
image_empty image_insert subspace_bounded_eq_trivial translation_invert)
  then show ?thesis by auto
qed

lemma affine_bounded_eq_lowdim:
  fixes S :: 'a::euclidean_space set
  assumes affine S
  shows bounded S ⟷ aff_dim S ≤ 0
proof
  show aff_dim S ≤ 0 ⟹ bounded S
  by (metis aff_dim_sing aff_dim_subset affine_dim_equal affine_sing all_not_in_conv
assms bounded_empty bounded_insert dual_order.antisym empty_subsetI insert_subset)
qed (use affine_bounded_eq_trivial assms in fastforce)


lemma bounded_hyperplane_eq_trivial_0:
  fixes a :: 'a::euclidean_space
  assumes a ≠ 0
  shows bounded {x. a · x = 0} ⟷ DIM('a) = 1
proof
  assume bounded {x. a · x = 0}
  then have aff_dim {x. a · x = 0} ≤ 0
    by (simp add: affine_bounded_eq_lowdim affine_hyperplane)
  with assms show DIM('a) = 1
    by (simp add: le_Suc_eq)
next
  assume DIM('a) = 1
  then show bounded {x. a · x = 0}
    by (simp add: affine_bounded_eq_lowdim affine_hyperplane assms)
qed

lemma bounded_hyperplane_eq_trivial:
  fixes a :: 'a::euclidean_space
  shows bounded {x. a · x = r} ⟷ (if a = 0 then r ≠ 0 else DIM('a) = 1)
proof (simp add: bounded_hyperplane_eq_trivial_0, clarify)
  assume r ≠ 0 a ≠ 0
  have aff_dim {x. y · x = 0} = aff_dim {x. a · x = r} if y ≠ 0 for y::'a
    by (metis that ‹a ≠ 0› aff_dim_hyperplane)
```

then show *bounded* {*x*. *a* · *x* = *r*} = (*DIM*(′*a*) = *Suc 0*)
by (*metis One_nat_def* ‹*a* ≠ *0*› *affine_bounded_eq_lowdim affine_hyperplane bounded_hyperplane_eq_trivial_0*)
qed

5.0.15 General case without assuming closure and getting non-strict separation

proposition *separating_hyperplane_closed_point_inset*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *convex S closed S S* ≠ {} *z* ∉ *S*
obtains *a b* **where** *a* ∈ *S* (*a* − *z*) · *z* < *b* ⋀*x*. *x* ∈ *S* ⟹ *b* < (*a* − *z*) · *x*
proof −
obtain *y* **where** *y* ∈ *S* **and** *y*: ⋀*u*. *u* ∈ *S* ⟹ *dist z y* ≤ *dist z u*
using *distance_attains_inf* [*of S z*] *assms* **by** *auto*
then have ∗: (*y* − *z*) · *z* < (*y* − *z*) · *z* + (*norm* (*y* − *z*))^2 / *2*
using ‹*y* ∈ *S*› ‹*z* ∉ *S*› **by** *auto*
show *?thesis*
proof (*rule that* [*OF* ‹*y* ∈ *S*› ∗])
fix *x*
assume *x* ∈ *S*
have *yz*: *0* < (*y* − *z*) · (*y* − *z*)
using ‹*y* ∈ *S*› ‹*z* ∉ *S*› **by** *auto*
{ assume *0*: *0* < ((*z* − *y*) · (*x* − *y*))
with *any_closest_point_dot* [*OF* ‹*convex S*› ‹*closed S*›]
have *False*
using *y* ‹*x* ∈ *S*› ‹*y* ∈ *S*› *not_less* **by** *blast*
}
then have *0* ≤ ((*y* − *z*) · (*x* − *y*))
by (*force simp*: *not_less inner_diff_left*)
with *yz* **have** *0* < *2* ∗ ((*y* − *z*) · (*x* − *y*)) + (*y* − *z*) · (*y* − *z*)
by (*simp add*: *algebra_simps*)
then show (*y* − *z*) · *z* + (*norm* (*y* − *z*))^2 / *2* < (*y* − *z*) · *x*
by (*simp add*: *field_simps inner_diff_left inner_diff_right dot_square_norm* [*symmetric*])
qed
qed

lemma *separating_hyperplane_closed_0_inset*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *convex S closed S S* ≠ {} *0* ∉ *S*
obtains *a b* **where** *a* ∈ *S a* ≠ *0 0* < *b* ⋀*x*. *x* ∈ *S* ⟹ *a* · *x* > *b*
using *separating_hyperplane_closed_point_inset* [*OF assms*] **by** *simp* (*metis* ‹*0* ∉ *S*›)

proposition *separating_hyperplane_set_0_inspan*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *convex S S* ≠ {} *0* ∉ *S*

```
  obtains a where a ∈ span S a ≠ 0 ⋀x. x ∈ S ⟹ 0 ≤ a · x
proof −
  define k where [abs_def]: k c = {x. 0 ≤ c · x} for c :: 'a
  have span S ∩ frontier (cball 0 1) ∩ ⋂f' ≠ {}
        if f': finite f' f' ⊆ k ` S for f'
  proof −
    obtain C where C ⊆ S finite C and C: f' = k ` C
      using finite_subset_image [OF f'] by blast
    obtain a where a ∈ S a ≠ 0
      using ‹S ≠ {}› ‹0 ∉ S› ex_in_conv by blast
    then have norm (a /R (norm a)) = 1
      by simp
    moreover have a /R (norm a) ∈ span S
      by (simp add: ‹a ∈ S› span_scale span_base)
    ultimately have ass: a /R (norm a) ∈ span S ∩ sphere 0 1
      by simp
    show ?thesis
    proof (cases C = {})
      case True with C ass show ?thesis
        by auto
    next
      case False
      have closed (convex hull C)
         using ‹finite C› compact_eq_bounded_closed finite_imp_compact_convex_hull
by auto
      moreover have convex hull C ≠ {}
        by (simp add: False)
      moreover have 0 ∉ convex hull C
          by (metis ‹C ⊆ S› ‹convex S› ‹0 ∉ S› convex_hull_subset hull_same in-
sert_absorb insert_subset)
      ultimately obtain a b
            where a ∈ convex hull C a ≠ 0 0 < b
                  and ab: ⋀x. x ∈ convex hull C ⟹ a · x > b
        using separating_hyperplane_closed_0_inset by blast
      then have a ∈ S
        by (metis ‹C ⊆ S› assms(1) subsetCE subset_hull)
      moreover have norm (a /R (norm a)) = 1
        using ‹a ≠ 0› by simp
      moreover have a /R (norm a) ∈ span S
        by (simp add: ‹a ∈ S› span_scale span_base)
      ultimately have ass: a /R (norm a) ∈ span S ∩ sphere 0 1
        by simp
      have ⋀x. ⟦a ≠ 0; x ∈ C⟧ ⟹ 0 ≤ x · a
       using ab ‹0 < b› by (metis hull_inc inner_commute less_eq_real_def less_trans)
      then have aa: a /R (norm a) ∈ (⋂c∈C. {x. 0 ≤ c · x})
        by (auto simp add: field_split_simps)
      show ?thesis
        unfolding C k_def
        using ass aa Int_iff empty_iff by force
```

qed
qed
moreover have $\bigwedge T.\ T \in k\ `\ S \Longrightarrow closed\ T$
using *closed_halfspace_ge k_def* **by** *blast*
ultimately have $(span\ S \cap frontier(cball\ 0\ 1)) \cap (\bigcap\ (k\ `\ S)) \neq \{\}$
by (*metis compact_imp_fip closed_Int_compact closed_span compact_cball compact_frontier*)
then show *?thesis*
unfolding *set_eq_iff k_def*
by *simp* (*metis inner_commute norm_eq_zero that zero_neq_one*)
qed

lemma *separating_hyperplane_set_point_inaff*:
fixes $S :: 'a::euclidean\_space\ set$
assumes $convex\ S\ S \neq \{\}$ **and** *zno*: $z \notin S$
obtains $a\ b$ **where** $(z + a) \in affine\ hull\ (insert\ z\ S)$
and $a \neq 0$ **and** $a \cdot z \leq b$
and $\bigwedge x.\ x \in S \Longrightarrow a \cdot x \geq b$
proof −
from *separating_hyperplane_set_0_inspan* [*of image* $(\lambda x.\ -z + x)\ S$]
have *convex* $((+)\ (-\ z)\ `\ S)$
using ‹*convex S*› **by** *simp*
moreover have $(+)\ (-\ z)\ `\ S \neq \{\}$
by (*simp add*: ‹$S \neq \{\}$›)
moreover have $0 \notin (+)\ (-\ z)\ `\ S$
using *zno* **by** *auto*
ultimately obtain a **where** $a \in span\ ((+)\ (-\ z)\ `\ S)\ a \neq 0$
and a: $\bigwedge x.\ x \in ((+)\ (-\ z)\ `\ S) \Longrightarrow 0 \leq a \cdot x$
using *separating_hyperplane_set_0_inspan* [*of image* $(\lambda x.\ -z + x)\ S$]
by *blast*
then have *szx*: $\bigwedge x.\ x \in S \Longrightarrow a \cdot z \leq a \cdot x$
by (*metis* (*no_types*, *lifting*) *imageI inner_minus_right inner_right_distrib minus_add neg_le_0_iff_le neg_le_iff_le real_add_le_0_iff*)
moreover
have $z + a \in affine\ hull\ insert\ z\ S$
using ‹$a \in span\ ((+)\ (-\ z)\ `\ S)$› *affine_hull_insert_span_gen* **by** *blast*
ultimately show *?thesis*
using ‹$a \neq 0$› *szx that* **by** *auto*
qed

proposition *supporting_hyperplane_rel_boundary*:
fixes $S :: 'a::euclidean\_space\ set$
assumes $convex\ S\ x \in S$ **and** *xno*: $x \notin rel\_interior\ S$
obtains a **where** $a \neq 0$
and $\bigwedge y.\ y \in S \Longrightarrow a \cdot x \leq a \cdot y$
and $\bigwedge y.\ y \in rel\_interior\ S \Longrightarrow a \cdot x < a \cdot y$
proof −
obtain $a\ b$ **where** *aff*: $(x + a) \in affine\ hull\ (insert\ x\ (rel\_interior\ S))$

```
                  and a ≠ 0 and a · x ≤ b
                  and ageb: ⋀u. u ∈ (rel_interior S) ⟹ a · u ≥ b
    using separating_hyperplane_set_point_inaff [of rel_interior S x] assms
    by (auto simp: rel_interior_eq_empty convex_rel_interior)
  have le_ay: a · x ≤ a · y if y ∈ S for y
  proof −
    have con: continuous_on (closure (rel_interior S)) ((·) a)
      by (rule continuous_intros continuous_on_subset | blast)+
    have y: y ∈ closure (rel_interior S)
      using ‹convex S› closure_def convex_closure_rel_interior ‹y ∈ S›
      by fastforce
    show ?thesis
      using continuous_ge_on_closure [OF con y] ageb ‹a · x ≤ b›
      by fastforce
  qed
  have 3: a · x < a · y if y ∈ rel_interior S for y
  proof −
    obtain e where 0 < e y ∈ S and e: cball y e ∩ affine hull S ⊆ S
      using ‹y ∈ rel_interior S› by (force simp: rel_interior_cball)
    define y' where y' = y − (e / norm a) *R ((x + a) − x)
    have y' ∈ cball y e
      unfolding y'_def using ‹0 < e› by force
    moreover have y' ∈ affine hull S
      unfolding y'_def
      by (metis ‹x ∈ S› ‹y ∈ S› ‹convex S› aff affine_affine_hull hull_redundant
               rel_interior_same_affine_hull hull_inc mem_affine_3_minus2)
    ultimately have y' ∈ S
      using e by auto
    have a · x ≤ a · y
      using le_ay ‹a ≠ 0› ‹y ∈ S› by blast
    moreover have a · x ≠ a · y
      using le_ay [OF ‹y' ∈ S›] ‹a ≠ 0› ‹0 < e› not_le
      by (fastforce simp add: y'_def inner_diff dot_square_norm power2_eq_square)
    ultimately show ?thesis by force
  qed
  show ?thesis
    by (rule that [OF ‹a ≠ 0› le_ay 3])
qed

lemma supporting_hyperplane_relative_frontier:
  fixes S :: 'a::euclidean_space set
  assumes convex S x ∈ closure S x ∉ rel_interior S
  obtains a where a ≠ 0
              and ⋀y. y ∈ closure S ⟹ a · x ≤ a · y
              and ⋀y. y ∈ rel_interior S ⟹ a · x < a · y
using supporting_hyperplane_rel_boundary [of closure S x]
by (metis assms convex_closure convex_rel_interior_closure)
```

5.0.16 Some results on decomposing convex hulls: intersections, simplicial subdivision

lemma
fixes *S* :: *'a::euclidean_space set*
assumes ¬ *affine_dependent*(*S* ∪ *T*)
shows *convex_hull_Int_subset*: *convex hull S* ∩ *convex hull T* ⊆ *convex hull* (*S* ∩ *T*) (**is** *?C*)
and *affine_hull_Int_subset*: *affine hull S* ∩ *affine hull T* ⊆ *affine hull* (*S* ∩ *T*) (**is** *?A*)
proof −
have [*simp*]: *finite S finite T*
using *aff_independent_finite assms* **by** *blast+*
have *sum u* (*S* ∩ *T*) = *1* ∧
($\sum$ *v*∈*S* ∩ *T*. *u v* $*_R$ *v*) = ($\sum$ *v*∈*S*. *u v* $*_R$ *v*)
if [*simp*]: *sum u S* = *1*
sum v T = *1*
and *eq*: ($\sum$ *x*∈*T*. *v x* $*_R$ *x*) = ($\sum$ *x*∈*S*. *u x* $*_R$ *x*) **for** *u v*
proof −
define *f* **where** *f x* = (*if x* ∈ *S then u x else 0*) − (*if x* ∈ *T then v x else 0*) **for** *x*
have *sum f* (*S* ∪ *T*) = *0*
by (*simp add*: *f_def sum_Un sum_subtractf flip*: *sum.inter_restrict*)
moreover have ($\sum$ *x*∈(*S* ∪ *T*). *f x* $*_R$ *x*) = *0*
by (*simp add*: *eq f_def sum_Un scaleR_left_diff_distrib sum_subtractf if_smult flip*: *sum.inter_restrict cong*: *if_cong*)
ultimately have $\bigwedge$*v*. *v* ∈ *S* ∪ *T* ⟹ *f v* = *0*
using *aff_independent_finite assms* **unfolding** *affine_dependent_explicit*
by *blast*
then have *u* [*simp*]: $\bigwedge$*x*. *x* ∈ *S* ⟹ *u x* = (*if x* ∈ *T then v x else 0*)
by (*simp add*: *f_def*) *presburger*
have *sum u* (*S* ∩ *T*) = *sum u S*
by (*simp add*: *sum.inter_restrict*)
then have *sum u* (*S* ∩ *T*) = *1*
using *that* **by** *linarith*
moreover have ($\sum$ *v*∈*S* ∩ *T*. *u v* $*_R$ *v*) = ($\sum$ *v*∈*S*. *u v* $*_R$ *v*)
by (*auto simp*: *if_smult sum.inter_restrict intro*: *sum.cong*)
ultimately show *?thesis*
by *force*
qed
then show *?A ?C*
by (*auto simp*: *convex_hull_finite affine_hull_finite*)
qed

proposition *affine_hull_Int*:
fixes *S* :: *'a::euclidean_space set*
assumes ¬ *affine_dependent*(*S* ∪ *T*)
shows *affine hull* (*S* ∩ *T*) = *affine hull S* ∩ *affine hull T*
by (*simp add*: *affine_hull_Int_subset assms hull_mono subset_antisym*)

```
proposition convex_hull_Int:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent(S ∪ T)
    shows convex hull (S ∩ T) = convex hull S ∩ convex hull T
  by (simp add: convex_hull_Int_subset assms hull_mono subset_antisym)

proposition
  fixes S :: 'a::euclidean_space set set
  assumes ¬ affine_dependent (⋃ S)
    shows affine_hull_Inter: affine hull (⋂ S) = (⋂ T∈S. affine hull T) (is ?A)
      and convex_hull_Inter: convex hull (⋂ S) = (⋂ T∈S. convex hull T) (is ?C)
proof -
  have finite S
    using aff_independent_finite assms finite_UnionD by blast
  then have ?A ∧ ?C using assms
  proof (induction S rule: finite_induct)
    case empty then show ?case by auto
  next
    case (insert T F)
    then show ?case
    proof (cases F={})
      case True then show ?thesis by simp
    next
      case False
      with insert.prems have [simp]: ¬ affine_dependent (T ∪ ⋂ F)
        by (auto intro: affine_dependent_subset)
      have [simp]: ¬ affine_dependent (⋃ F)
        using affine_independent_subset insert.prems by fastforce
      show ?thesis
        by (simp add: affine_hull_Int convex_hull_Int insert.IH)
    qed
  qed
  then show ?A ?C
    by auto
qed

proposition in_convex_hull_exchange_unique:
  fixes S :: 'a::euclidean_space set
  assumes naff: ¬ affine_dependent S and a: a ∈ convex hull S
      and S: T ⊆ S T' ⊆ S
      and x:  x ∈ convex hull (insert a T)
      and x': x ∈ convex hull (insert a T')
    shows x ∈ convex hull (insert a (T ∩ T'))
proof (cases a ∈ S)
  case True
  then have ¬ affine_dependent (insert a T ∪ insert a T')
    using affine_dependent_subset assms by auto
  then have x ∈ convex hull (insert a T ∩ insert a T')
```

by (*metis IntI convex_hull_Int x x'*)
then show *?thesis*
by *simp*
next
case *False*
then have *anot*: $a \notin T$ $a \notin T'$
using *assms* **by** *auto*
have [*simp*]: *finite S*
by (*simp add*: *aff_independent_finite assms*)
then obtain *b* **where** *b0*: $\bigwedge s.\ s \in S \Longrightarrow 0 \le b\ s$
and *b1*: *sum b S* = *1* **and** *aeq*: $a = (\sum s{\in}S.\ b\ s *_R s)$
using *a* **by** (*auto simp*: *convex_hull_finite*)
have *fin* [*simp*]: *finite T finite T'*
using *assms infinite_super* ‹*finite S*› **by** *blast+*
then obtain *c c'* **where** *c0*: $\bigwedge t.\ t \in insert\ a\ T \Longrightarrow 0 \le c\ t$
and *c1*: *sum c* (*insert a T*) = *1*
and *xeq*: $x = (\sum t \in insert\ a\ T.\ c\ t *_R t)$
and *c'0*: $\bigwedge t.\ t \in insert\ a\ T' \Longrightarrow 0 \le c'\ t$
and *c'1*: *sum c'* (*insert a T'*) = *1*
and *x'eq*: $x = (\sum t \in insert\ a\ T'.\ c'\ t *_R t)$
using *x x'* **by** (*auto simp*: *convex_hull_finite*)
with *fin anot*
have *sumTT'*: *sum c T* = *1* − *c a sum c' T'* = *1* − *c' a*
and *wsumT*: $(\sum t \in T.\ c\ t *_R t) = x - c\ a *_R a$
by *simp_all*
have *wsumT'*: $(\sum t \in T'.\ c'\ t *_R t) = x - c'\ a *_R a$
using *x'eq fin anot* **by** *simp*
define *cc* **where** $cc \equiv \lambda x.$ *if* $x \in T$ *then c x else 0*
define *cc'* **where** $cc' \equiv \lambda x.$ *if* $x \in T'$ *then c' x else 0*
define *dd* **where** $dd \equiv \lambda x.\ cc\ x - cc'\ x + (c\ a - c'\ a) * b\ x$
have *sumSS'*: *sum cc S* = *1* − *c a sum cc' S* = *1* − *c' a*
unfolding *cc_def cc'_def* **using** *S*
by (*simp_all add*: *Int_absorb1 Int_absorb2 sum_subtractf sum.inter_restrict* [*symmetric*] *sumTT'*)
have *wsumSS*: $(\sum t \in S.\ cc\ t *_R t) = x - c\ a *_R a$ $(\sum t \in S.\ cc'\ t *_R t) = x - c'\ a *_R a$
unfolding *cc_def cc'_def* **using** *S*
by (*simp_all add*: *Int_absorb1 Int_absorb2 if_smult sum.inter_restrict* [*symmetric*] *wsumT wsumT' cong*: *if_cong*)
have *sum_dd0*: *sum dd S* = *0*
unfolding *dd_def* **using** *S*
by (*simp add*: *sumSS' comm_monoid_add_class.sum.distrib sum_subtractf*
algebra_simps sum_distrib_right [*symmetric*] *b1*)
have $(\sum v{\in}S.\ (b\ v * x) *_R v) = x *_R (\sum v{\in}S.\ b\ v *_R v)$ **for** *x*
by (*simp add*: *pth_5 real_vector.scale_sum_right mult.commute*)
then have $*$: $(\sum v{\in}S.\ (b\ v * x) *_R v) = x *_R a$ **for** *x*
using *aeq* **by** *blast*
have $(\sum v \in S.\ dd\ v *_R v) = 0$
unfolding *dd_def* **using** *S*

```
    by (simp add: * wsumSS sum.distrib sum_subtractf algebra_simps)
  then have dd0: dd v = 0 if v ∈ S for v
    using naff [unfolded affine_dependent_explicit not_ex, rule_format, of S dd]
    using that sum_dd0 by force
  consider c′ a ≤ c a | c a ≤ c′ a by linarith
  then show ?thesis
  proof cases
    case 1
    then have sum cc S ≤ sum cc′ S
      by (simp add: sumSS′)
    then have le: cc x ≤ cc′ x if x ∈ S for x
      using dd0 [OF that] 1 b0 mult_left_mono that
      by (fastforce simp add: dd_def algebra_simps)
    have cc0: cc x = 0 if x ∈ S x ∉ T ∩ T′ for x
      using le [OF ‹x ∈ S›] that c0
      by (force simp: cc_def cc′_def split: if_split_asm)
    show ?thesis
    proof (simp add: convex_hull_finite, intro exI conjI)
      show ∀x∈T ∩ T′. 0 ≤ (cc(a := c a)) x
        by (simp add: c0 cc_def)
      show 0 ≤ (cc(a := c a)) a
        by (simp add: c0)
     have sum (cc(a := c a)) (insert a (T ∩ T′)) = c a + sum (cc(a := c a)) (T
∩ T′)
        by (simp add: anot)
      also have ... = c a + sum (cc(a := c a)) S
     using ‹T ⊆ S› False cc0 cc_def ‹a ∉ S› by (fastforce intro!: sum.mono_neutral_left
split: if_split_asm)
      also have ... = c a + (1 − c a)
        by (metis ‹a ∉ S› fun_upd_other sum.cong sumSS′(1))
      finally show sum (cc(a := c a)) (insert a (T ∩ T′)) = 1
        by simp
     have (∑x∈insert a (T ∩ T′). (cc(a := c a)) x *R x) = c a *R a + (∑x ∈
T ∩ T′. (cc(a := c a)) x *R x)
        by (simp add: anot)
      also have ... = c a *R a + (∑x ∈ S. (cc(a := c a)) x *R x)
         using ‹T ⊆ S› False cc0 by (fastforce intro!: sum.mono_neutral_left split:
if_split_asm)
      also have ... = c a *R a + x − c a *R a
        by (simp add: wsumSS ‹a ∉ S› if_smult sum_delta_notmem)
      finally show (∑x∈insert a (T ∩ T′). (cc(a := c a)) x *R x) = x
        by simp
    qed
  next
    case 2
    then have sum cc′ S ≤ sum cc S
      by (simp add: sumSS′)
    then have le: cc′ x ≤ cc x if x ∈ S for x
      using dd0 [OF that] 2 b0 mult_left_mono that
```

by (*fastforce simp add*: *dd_def algebra_simps*)
have *cc0*: $cc'\ x = 0$ **if** $x \in S\ x \notin T \cap T'$ **for** x
using *le* [*OF* ‹$x \in S$›] *that c'0*
by (*force simp*: *cc_def cc'_def split*: *if_split_asm*)
show *?thesis*
proof (*simp add*: *convex_hull_finite*, *intro exI conjI*)
show $\forall x \in T \cap T'.\ 0 \leq (cc'(a := c'\ a))\ x$
by (*simp add*: *c'0 cc'_def*)
show $0 \leq (cc'(a := c'\ a))\ a$
by (*simp add*: *c'0*)
have $sum\ (cc'(a := c'\ a))\ (insert\ a\ (T \cap T')) = c'\ a + sum\ (cc'(a := c'\ a))$ $(T \cap T')$
by (*simp add*: *anot*)
also have ... $= c'\ a + sum\ (cc'(a := c'\ a))\ S$
using ‹$T \subseteq S$› *False cc0* **by** (*fastforce intro*!: *sum.mono_neutral_left split*: *if_split_asm*)
also have ... $= c'\ a + (1 - c'\ a)$
by (*metis* ‹$a \notin S$› *fun_upd_other sum.cong sumSS'*)
finally show $sum\ (cc'(a := c'\ a))\ (insert\ a\ (T \cap T')) = 1$
by *simp*
have $(\sum x \in insert\ a\ (T \cap T').\ (cc'(a := c'\ a))\ x *_R x) = c'\ a *_R a + (\sum x \in T \cap T'.\ (cc'(a := c'\ a))\ x *_R x)$
by (*simp add*: *anot*)
also have ... $= c'\ a *_R a + (\sum x \in S.\ (cc'(a := c'\ a))\ x *_R x)$
using ‹$T \subseteq S$› *False cc0* **by** (*fastforce intro*!: *sum.mono_neutral_left split*: *if_split_asm*)
also have ... $= c\ a *_R a + x - c\ a *_R a$
by (*simp add*: *wsumSS* ‹$a \notin S$› *if_smult sum_delta_notmem*)
finally show $(\sum x \in insert\ a\ (T \cap T').\ (cc'(a := c'\ a))\ x *_R x) = x$
by *simp*
qed
qed
qed

corollary *convex_hull_exchange_Int*:
fixes a :: *'a::euclidean_space*
assumes $\neg$ *affine_dependent* $S\ a \in$ *convex hull* $S\ T \subseteq S\ T' \subseteq S$
shows (*convex hull* (*insert a T*)) $\cap$ (*convex hull* (*insert a T'*)) =
convex hull (*insert a* $(T \cap T')$) (**is** *?lhs* = *?rhs*)
proof (*rule subset_antisym*)
show *?lhs* $\subseteq$ *?rhs*
using *in_convex_hull_exchange_unique assms* **by** *blast*
show *?rhs* $\subseteq$ *?lhs*
by (*metis hull_mono inf_le1 inf_le2 insert_inter_insert le_inf_iff*)
qed

lemma *Int_closed_segment*:
fixes b :: *'a::euclidean_space*
assumes $b \in$ *closed_segment* $a\ c \lor \neg$ *collinear*$\{a,b,c\}$

```
  shows closed_segment a b ∩ closed_segment b c = {b}
proof (cases c = a)
  case True
  then show ?thesis
    using assms collinear_3_eq_affine_dependent by fastforce
next
  case False
  from assms show ?thesis
  proof
    assume b ∈ closed_segment a c
    moreover have ¬ affine_dependent {a, c}
      by (simp)
    ultimately show ?thesis
      using False convex_hull_exchange_Int [of {a,c} b {a} {c}]
      by (simp add: segment_convex_hull insert_commute)
  next
    assume ncoll: ¬ collinear {a, b, c}
    have False if closed_segment a b ∩ closed_segment b c ≠ {b}
    proof −
      have b ∈ closed_segment a b and b ∈ closed_segment b c
        by auto
      with that obtain d where b ≠ d d ∈ closed_segment a b d ∈ closed_segment
b c
        by force
      then have d: collinear {a, d, b}  collinear {b, d, c}
        by (auto simp:  between_mem_segment between_imp_collinear)
      have collinear {a, b, c}
        by (metis (full_types) ‹b ≠ d› collinear_3_trans d insert_commute)
      with ncoll show False ..
    qed
    then show ?thesis
      by blast
  qed
qed

lemma affine_hull_finite_intersection_hyperplanes:
  fixes S :: 'a::euclidean_space set
  obtains ℱ where
    finite ℱ
    of_nat (card ℱ) + aff_dim S = DIM('a)
    affine hull S = ⋂ℱ
    ⋀h. h ∈ ℱ ⟹ ∃ a b. a ≠ 0 ∧ h = {x. a · x = b}
proof −
  obtain b where b ⊆ S
           and indb: ¬ affine_dependent b
           and eq: affine hull S = affine hull b
    using affine_basis_exists by blast
  obtain c where indc: ¬ affine_dependent c and b ⊆ c
           and affc: affine hull c = UNIV
```

by (*metis extend_to_affine_basis affine_UNIV hull_same indb subset_UNIV*)
then have *finite c*
by (*simp add*: *aff_independent_finite*)
then have *fbc*: *finite b card b* ≤ *card c*
using ‹*b* ⊆ *c*› *infinite_super* **by** (*auto simp*: *card_mono*)
have *imeq*: (λ*x*. *affine hull x*) ‘ ((λ*a*. *c* − {*a*}) ‘ (*c* − *b*)) = ((λ*a*. *affine hull* (*c* − {*a*})) ‘ (*c* − *b*))
by *blast*
have *card_cb*: (*card* (*c* − *b*)) + *aff_dim S* = *DIM*(′*a*)
proof −
have *aff*: *aff_dim* (*UNIV*::′*a set*) = *aff_dim c*
by (*metis aff_dim_affine_hull affc*)
have *aff_dim b* = *aff_dim S*
by (*metis* (*no_types*) *aff_dim_affine_hull eq*)
then have *int* (*card b*) = *1* + *aff_dim S*
by (*simp add*: *aff_dim_affine_independent indb*)
then show *?thesis*
using *fbc aff*
by (*simp add*: ‹¬ *affine_dependent c*› ‹*b* ⊆ *c*› *aff_dim_affine_independent card_Diff_subset of_nat_diff*)
qed
show *?thesis*
proof (*cases c* = *b*)
case *True* **show** *?thesis*
proof
show *int* (*card* {}) + *aff_dim S* = *int DIM*(′*a*)
using *True card_cb* **by** *auto*
show *affine hull S* = ⋂ {}
using *True affc eq* **by** *blast*
qed *auto*
next
case *False*
have *ind*: ¬ *affine_dependent* (⋃ *a*∈*c* − *b*. *c* − {*a*})
by (*rule affine_independent_subset* [*OF indc*]) *auto*
have ∗: *1* + *aff_dim* (*c* − {*t*}) = *int* (*DIM*(′*a*)) **if** *t*: *t* ∈ *c* **for** *t*
proof −
have *insert t c* = *c*
using *t* **by** *blast*
then show *?thesis*
by (*metis* (*full_types*) *add.commute aff_dim_affine_hull aff_dim_insert aff_dim_UNIV affc affine_dependent_def indc insert_Diff_single t*)
qed
let *?F* = (λ*x*. *affine hull x*) ‘ ((λ*a*. *c* − {*a*}) ‘ (*c* − *b*))
show *?thesis*
proof
have *card* ((λ*a*. *affine hull* (*c* − {*a*})) ‘ (*c* − *b*)) = *card* (*c* − *b*)
proof (*rule card_image*)
show *inj_on* (λ*a*. *affine hull* (*c* − {*a*})) (*c* − *b*)
unfolding *inj_on_def*

```
          by (metis Diff_eq_empty_iff Diff_iff indc affine_dependent_def hull_subset
insert_iff)
      qed
      then show int (card ?ℱ) + aff_dim S = int DIM('a)
        by (simp add: imeq card_cb)
      show affine hull S = ⋂ ?ℱ
        by (metis Diff_eq_empty_iff INT_simps(4) UN_singleton order_refl ‹b ⊆ c›
False eq double_diff affine_hull_Inter [OF ind])
      have ⋀a. ⟦a ∈ c; a ∉ b⟧ ⟹ aff_dim (c − {a}) = int (DIM('a) − Suc 0)
          by (metis ∗ DIM_ge_Suc0 One_nat_def add_diff_cancel_left' int_ops(2)
of_nat_diff)
      then show ⋀h. h ∈ ?ℱ ⟹ ∃ a b. a ≠ 0 ∧ h = {x. a · x = b}
        by (auto simp only: One_nat_def aff_dim_eq_hyperplane [symmetric])
      qed (use ‹finite c› in auto)
  qed
qed

lemma affine_hyperplane_sums_eq_UNIV_0:
  fixes S :: 'a :: euclidean_space set
  assumes affine S
     and 0 ∈ S and w ∈ S
     and a · w ≠ 0
   shows {x + y| x y. x ∈ S ∧ a · y = 0} = UNIV
proof −
  have subspace S
    by (simp add: assms subspace_affine)
  have span1: span {y. a · y = 0} ⊆ span {x + y |x y. x ∈ S ∧ a · y = 0}
    using ‹0 ∈ S› add.left_neutral by (intro span_mono) force
  have w ∉ span {y. a · y = 0}
    using ‹a · w ≠ 0› span_induct subspace_hyperplane by auto
  moreover have w ∈ span {x + y |x y. x ∈ S ∧ a · y = 0}
    using ‹w ∈ S›
   by (metis (mono_tags, lifting) inner_zero_right mem_Collect_eq pth_d span_base)
  ultimately have span2: span {y. a · y = 0} ≠ span {x + y |x y. x ∈ S ∧ a ·
y = 0}
    by blast
  have a ≠ 0 using assms inner_zero_left by blast
  then have DIM('a) − 1 = dim {y. a · y = 0}
    by (simp add: dim_hyperplane)
  also have ... < dim {x + y |x y. x ∈ S ∧ a · y = 0}
    using span1 span2 by (blast intro: dim_psubset)
  finally have DIM('a) − 1 < dim {x + y |x y. x ∈ S ∧ a · y = 0} .
  then have DD: dim {x + y |x y. x ∈ S ∧ a · y = 0} = DIM('a)
    using antisym dim_subset_UNIV lowdim_subset_hyperplane not_le by fastforce
  have subs: subspace {x + y| x y. x ∈ S ∧ a · y = 0}
    using subspace_sums [OF ‹subspace S› subspace_hyperplane] by simp
  moreover have span {x + y| x y. x ∈ S ∧ a · y = 0} = UNIV
    using DD dim_eq_full by blast
  ultimately show ?thesis
```

```
  by (simp add: subs) (metis (lifting) span_eq_iff subs)
qed

proposition affine_hyperplane_sums_eq_UNIV:
  fixes S :: 'a :: euclidean_space set
  assumes affine S
      and S ∩ {v. a · v = b} ≠ {}
      and S − {v. a · v = b} ≠ {}
    shows {x + y| x y. x ∈ S ∧ a · y = b} = UNIV
proof (cases a = 0)
  case True with assms show ?thesis
    by (auto simp: if_splits)
next
  case False
  obtain c where c ∈ S and c: a · c = b
    using assms by force
  with affine_diffs_subspace [OF ‹affine S›]
  have subspace ((+) (− c) ‘ S) by blast
  then have aff: affine ((+) (− c) ‘ S)
    by (simp add: subspace_imp_affine)
  have 0: 0 ∈ (+) (− c) ‘ S
    by (simp add: ‹c ∈ S›)
  obtain d where d ∈ S and a · d ≠ b and dc: d−c ∈ (+) (− c) ‘ S
    using assms by auto
  then have adc: a · (d − c) ≠ 0
    by (simp add: c inner_diff_right)
  define U where U ≡ {x + y |x y. x ∈ (+) (− c) ‘ S ∧ a · y = 0}
  have u + v ∈ (+) (c+c) ‘ U
    if u ∈ S b = a · v for u v
  proof
    show u + v = c + c + (u + v − c − c)
      by (simp add: algebra_simps)
    have ∃x y. u + v − c − c = x + y ∧ (∃xa∈S. x = xa − c) ∧ a · y = 0
    proof (intro exI conjI)
      show u + v − c − c = (u−c) + (v−c) a · (v − c) = 0
        by (simp_all add: algebra_simps c that)
    qed (use that in auto)
    then show u + v − c − c ∈ U
      by (auto simp: U_def image_def)
  qed
  moreover have ⟦a · v = 0; u ∈ S⟧
      ⟹ ∃x ya. v + (u + c) = x + ya ∧ x ∈ S ∧ a · ya = b for v u
    by (metis add.left_commute c inner_right_distrib pth_d)
  ultimately have {x + y |x y. x ∈ S ∧ a · y = b} = (+) (c+c) ‘ U
    by (fastforce simp: algebra_simps U_def)
  also have ... = range ((+) (c + c))
    by (simp only: U_def affine_hyperplane_sums_eq_UNIV_0 [OF aff 0 dc adc])
  also have ... = UNIV
    by simp
```

finally show *?thesis* **.**
qed

lemma *aff_dim_sums_Int_0*:
assumes *affine S*
and *affine T*
and *0 ∈ S 0 ∈ T*
shows *aff_dim* $\{x + y|\ x\ y.\ x \in S \wedge y \in T\} = (aff\_dim\ S + aff\_dim\ T) - aff\_dim(S \cap T)$
proof −
have $0 \in \{x + y\ |x\ y.\ x \in S \wedge y \in T\}$
using *assms* **by** *force*
then have *0*: $0 \in$ *affine hull* $\{x + y\ |x\ y.\ x \in S \wedge y \in T\}$
by (*metis* (*lifting*) *hull_inc*)
have *sub*: *subspace S subspace T*
using *assms* **by** (*auto simp*: *subspace_affine*)
show *?thesis*
using *dim_sums_Int* [*OF sub*] **by** (*simp add*: *aff_dim_zero assms 0 hull_inc*)
qed

proposition *aff_dim_sums_Int*:
assumes *affine S*
and *affine T*
and $S \cap T \neq \{\}$
shows *aff_dim* $\{x + y|\ x\ y.\ x \in S \wedge y \in T\} = (aff\_dim\ S + aff\_dim\ T) - aff\_dim(S \cap T)$
proof −
obtain *a* **where** *a*: $a \in S\ a \in T$ **using** *assms* **by** *force*
have *aff*: *affine* $((+)\ (-a)\ `\ S)$ *affine* $((+)\ (-a)\ `\ T)$
using *affine_translation* [*symmetric*, *of* − *a*] *assms* **by** (*simp_all cong*: *image_cong_simp*)
have *zero*: $0 \in ((+)\ (-a)\ `\ S)\ \ 0 \in ((+)\ (-a)\ `\ T)$
using *a assms* **by** *auto*
have $\{x + y\ |x\ y.\ x \in (+)\ (-\ a)\ `\ S \wedge y \in (+)\ (-\ a)\ `\ T\} =$
$(+)\ (-\ 2 *_R a)\ `\ \{x + y|\ x\ y.\ x \in S \wedge y \in T\}$
by (*force simp*: *algebra_simps scaleR_2*)
moreover have $(+)\ (-\ a)\ `\ S \cap (+)\ (-\ a)\ `\ T = (+)\ (-\ a)\ `\ (S \cap T)$
by *auto*
ultimately show *?thesis*
using *aff_dim_sums_Int_0* [*OF aff zero*] *aff_dim_translation_eq*
by (*metis* (*lifting*))
qed

lemma *aff_dim_affine_Int_hyperplane*:
fixes *a* :: *'a::euclidean_space*
assumes *affine S*
shows $aff\_dim(S \cap \{x.\ a \cdot x = b\}) =$
$(if\ S \cap \{v.\ a \cdot v = b\} = \{\}\ then\ -\ 1$
$else\ if\ S \subseteq \{v.\ a \cdot v = b\}\ then\ aff\_dim\ S$

else aff_dim S − *1*)
proof (*cases a* = *0*)
 case *True* **with** *assms* **show** *?thesis*
  by *auto*
next
 case *False*
 then have *aff_dim* ($S \cap \{x.\ a \cdot x = b\}$) = *aff_dim S* − *1*
  if $x \in S$ $a \cdot x \neq b$ **and** *non*: $S \cap \{v.\ a \cdot v = b\} \neq \{\}$ **for** *x*
 proof −
  have [*simp*]: $\{x + y \mid x\ y.\ x \in S \wedge a \cdot y = b\}$ = *UNIV*
   using *affine_hyperplane_sums_eq_UNIV* [*OF assms non*] *that* **by** *blast*
  show *?thesis*
   using *aff_dim_sums_Int* [*OF assms affine_hyperplane non*]
   by (*simp add*: *of_nat_diff False*)
 qed
 then show *?thesis*
  by (*metis* (*mono_tags*, *lifting*) *inf.orderE aff_dim_empty_eq mem_Collect_eq subsetI*)
qed

lemma *aff_dim_lt_full*:
 fixes *S* :: *'a*::*euclidean_space set*
 shows *aff_dim S* < *DIM*(*'a*) ⟷ (*affine hull S* ≠ *UNIV*)
by (*metis* (*no_types*) *aff_dim_affine_hull aff_dim_le_DIM aff_dim_UNIV affine_hull_UNIV less_le*)

lemma *aff_dim_openin*:
 fixes *S* :: *'a*::*euclidean_space set*
 assumes *ope*: *openin* (*top_of_set T*) *S* **and** *affine T* $S \neq \{\}$
 shows *aff_dim S* = *aff_dim T*
proof −
 show *?thesis*
 proof (*rule order_antisym*)
  show *aff_dim S* ≤ *aff_dim T*
   by (*blast intro*: *aff_dim_subset* [*OF openin_imp_subset*] *ope*)
 next
  obtain *a* **where** $a \in S$
   using ‹$S \neq \{\}$› **by** *blast*
  have $S \subseteq T$
   using *ope openin_imp_subset* **by** *auto*
  then have $a \in T$
   using ‹$a \in S$› **by** *auto*
  then have *subT'*: *subspace* ((λ*x*. − *a* + *x*) ‘ *T*)
   using *affine_diffs_subspace* ‹*affine T*› **by** *auto*
  then obtain *B* **where** *Bsub*: *B* ⊆ ((λ*x*. − *a* + *x*) ‘ *T*) **and** *po*: *pairwise orthogonal B*
   and *eq1*: ⋀*x*. *x* ∈ *B* ⟹ *norm x* = *1* **and** *independent B*
   and *cardB*: *card B* = *dim* ((λ*x*. − *a* + *x*) ‘ *T*)
   and *spanB*: *span B* = ((λ*x*. − *a* + *x*) ‘ *T*)

```
      by (rule orthonormal_basis_subspace) auto
    obtain e where 0 < e and e: cball a e ∩ T ⊆ S
      by (meson ‹a ∈ S› openin_contains_cball ope)
    have aff_dim T = aff_dim ((λx. − a + x) ‘ T)
      by (metis aff_dim_translation_eq)
    also have ... = dim ((λx. − a + x) ‘ T)
      using aff_dim_subspace subT' by blast
    also have ... = card B
      by (simp add: cardB)
    also have ... = card ((λx. e *R x) ‘ B)
      using ‹0 < e› by (force simp: inj_on_def card_image)
    also have ... ≤ dim ((λx. − a + x) ‘ S)
    proof (simp, rule independent_card_le_dim)
      have e': cball 0 e ∩ (λx. x − a) ‘ T ⊆ (λx. x − a) ‘ S
        using e by (auto simp: dist_norm norm_minus_commute subset_eq)
      have (λx. e *R x) ‘ B ⊆ cball 0 e ∩ (λx. x − a) ‘ T
        using Bsub ‹0 < e› eq1 subT' ‹a ∈ T› by (auto simp: subspace_def)
      then show (λx. e *R x) ‘ B ⊆ (λx. x − a) ‘ S
        using e' by blast
      have inj_on ((*R) e) (span B)
        using ‹0 < e› inj_on_def by fastforce
      then show independent ((λx. e *R x) ‘ B)
       using linear_scale_self ‹independent B› linear_dependent_inj_imageD by blast
    qed
    also have ... = aff_dim S
      using ‹a ∈ S› aff_dim_eq_dim hull_inc by (force cong: image_cong_simp)
    finally show aff_dim T ≤ aff_dim S .
  qed
qed

lemma dim_openin:
  fixes S :: 'a::euclidean_space set
  assumes ope: openin (top_of_set T) S and subspace T S ≠ {}
  shows dim S = dim T
proof (rule order_antisym)
  show dim S ≤ dim T
    by (metis ope dim_subset openin_subset topspace_euclidean_subtopology)
next
  have dim T = aff_dim S
    using aff_dim_openin
    by (metis aff_dim_subspace ‹subspace T› ‹S ≠ {}› ope subspace_affine)
  also have ... ≤ dim S
    by (metis aff_dim_subset aff_dim_subspace dim_span span_superset
        subspace_span)
  finally show dim T ≤ dim S by simp
qed
```

5.0.17 Lower-dimensional affine subsets are nowhere dense

```
proposition dense_complement_subspace:
  fixes S :: 'a :: euclidean_space set
  assumes dim_less: dim T < dim S and subspace S shows closure(S − T) = S
proof −
  have closure(S − U) = S if dim U < dim S U ⊆ S for U
  proof −
    have span U ⊂ span S
      by (metis neq_iff psubsetI span_eq_dim span_mono that)
   then obtain a where a ≠ 0 a ∈ span S and a: ⋀y. y ∈ span U ⟹ orthogonal
a y
      using orthogonal_to_subspace_exists_gen by metis
    show ?thesis
    proof
      have closed S
        by (simp add: ‹subspace S› closed_subspace)
      then show closure (S − U) ⊆ S
        by (simp add: closure_minimal)
      show S ⊆ closure (S − U)
      proof (clarsimp simp: closure_approachable)
        fix x and e::real
        assume x ∈ S 0 < e
        show ∃ y∈S − U. dist y x < e
        proof (cases x ∈ U)
          case True
          let ?y = x + (e/2 / norm a) *R a
          show ?thesis
          proof
            show dist ?y x < e
              using ‹0 < e› by (simp add: dist_norm)
          next
            have ?y ∈ S
                by (metis ‹a ∈ span S› ‹x ∈ S› assms(2) span_eq_iff subspace_add
subspace_scale)
            moreover have ?y ∉ U
            proof −
              have e/2 / norm a ≠ 0
                using ‹0 < e› ‹a ≠ 0› by auto
              then show ?thesis
                    by (metis True ‹a ≠ 0› a orthogonal_scaleR orthogonal_self
real_vector.scale_eq_0_iff span_add_eq span_base)
            qed
            ultimately show ?y ∈ S − U by blast
          qed
        next
          case False
          with ‹0 < e› ‹x ∈ S› show ?thesis by force
        qed
      qed
```

qed
qed
moreover have *S* − *S* ∩ *T* = *S*−*T*
by *blast*
moreover have *dim* (*S* ∩ *T*) < *dim S*
by (*metis dim_less dim_subset inf.cobounded2 inf.orderE inf.strict_boundedE not_le*)
ultimately show *?thesis*
by *force*
qed

corollary *dense_complement_affine*:
fixes *S* :: ′*a* :: *euclidean_space set*
assumes *less*: *aff_dim T* < *aff_dim S* **and** *affine S* **shows** *closure*(*S* − *T*) = *S*
proof (*cases S* ∩ *T* = {})
case *True*
then show *?thesis*
by (*metis Diff_triv affine_hull_eq* ‹*affine S*› *closure_same_affine_hull closure_subset hull_subset subset_antisym*)
next
case *False*
then obtain *z* **where** *z*: *z* ∈ *S* ∩ *T* **by** *blast*
then have *subspace* ((+) (− *z*) ‘ *S*)
by (*meson IntD1 affine_diffs_subspace* ‹*affine S*›)
moreover have *int* (*dim* ((+) (− *z*) ‘ *T*)) < *int* (*dim* ((+) (− *z*) ‘ *S*))
thm *aff_dim_eq_dim*
using *z less* **by** (*simp add*: *aff_dim_eq_dim_subtract* [*of z*] *hull_inc cong*: *image_cong_simp*)
ultimately have *closure*(((+) (− *z*) ‘ *S*) − ((+) (− *z*) ‘ *T*)) = ((+) (− *z*) ‘ *S*)
by (*simp add*: *dense_complement_subspace*)
then show *?thesis*
by (*metis closure_translation translation_diff translation_invert*)
qed

corollary *dense_complement_openin_affine_hull*:
fixes *S* :: ′*a* :: *euclidean_space set*
assumes *less*: *aff_dim T* < *aff_dim S*
and *ope*: *openin* (*top_of_set* (*affine hull S*)) *S*
shows *closure*(*S* − *T*) = *closure S*
proof −
have *affine hull S* − *T* ⊆ *affine hull S*
by *blast*
then have *closure* (*S* ∩ *closure* (*affine hull S* − *T*)) = *closure* (*S* ∩ (*affine hull S* − *T*))
by (*rule closure_openin_Int_closure* [*OF ope*])
then show *?thesis*
by (*metis Int_Diff aff_dim_affine_hull affine_affine_hull dense_complement_affine hull_subset inf.orderE less*)
qed

corollary *dense_complement_convex*:
fixes S :: $'a$ :: *euclidean_space set*
assumes *aff_dim* T < *aff_dim* S *convex* S
shows *closure*(S − T) = *closure* S
proof
show *closure* (S − T) ⊆ *closure* S
by (*simp add*: *closure_mono*)
have *closure* (*rel_interior* S − T) = *closure* (*rel_interior* S)
by (*simp add*: *assms dense_complement_openin_affine_hull openin_rel_interior rel_interior_aff_dim rel_interior_same_affine_hull*)
then show *closure* S ⊆ *closure* (S − T)
by (*metis Diff_mono* ‹*convex* S› *closure_mono convex_closure_rel_interior order_refl rel_interior_subset*)
qed

corollary *dense_complement_convex_closed*:
fixes S :: $'a$ :: *euclidean_space set*
assumes *aff_dim* T < *aff_dim* S *convex* S *closed* S
shows *closure*(S − T) = S
by (*simp add*: *assms dense_complement_convex*)

5.0.18 Parallel slices, etc

If we take a slice out of a set, we can do it perpendicularly, with the normal vector to the slice parallel to the affine hull.

proposition *affine_parallel_slice*:
fixes S :: $'a$ :: *euclidean_space set*
assumes *affine* S
and $S \cap \{x.\ a \cdot x \le b\} \ne \{\}$
and $\neg (S \subseteq \{x.\ a \cdot x \le b\})$
obtains a' b' **where** $a' \ne 0$
$S \cap \{x.\ a' \cdot x \le b'\} = S \cap \{x.\ a \cdot x \le b\}$
$S \cap \{x.\ a' \cdot x = b'\} = S \cap \{x.\ a \cdot x = b\}$
$\bigwedge w.\ w \in S \Longrightarrow (w + a') \in S$
proof (*cases* $S \cap \{x.\ a \cdot x = b\} = \{\}$)
case *True*
then obtain u v **where** $u \in S$ $v \in S$ $a \cdot u \le b$ $a \cdot v > b$
using *assms* **by** (*auto simp*: *not_le*)
define η **where** $\eta = u + ((b - a \cdot u) / (a \cdot v - a \cdot u)) *_R (v - u)$
have $\eta \in S$
by (*simp add*: *η_def* ‹$u \in S$› ‹$v \in S$› ‹*affine* S› *mem_affine_3_minus*)
moreover have $a \cdot \eta = b$
using ‹$a \cdot u \le b$› ‹$b < a \cdot v$›
by (*simp add*: *η_def algebra_simps*) (*simp add*: *field_simps*)
ultimately have *False*
using *True* **by** *force*
then show *?thesis* **..**
next

```
  case False
  then obtain z where z ∈ S and z: a · z = b
    using assms by auto
  with affine_diffs_subspace [OF ‹affine S›]
  have sub: subspace ((+) (− z) ` S) by blast
  then have aff: affine ((+) (− z) ` S) and span: span ((+) (− z) ` S) = ((+)
(− z) ` S)
    by (auto simp: subspace_imp_affine)
  obtain a′ a′′ where a′: a′ ∈ span ((+) (− z) ` S) and a: a = a′ + a′′
                 and ⋀w. w ∈ span ((+) (− z) ` S) ⟹ orthogonal a′′ w
    using orthogonal_subspace_decomp_exists [of (+) (− z) ` S a] by metis
  then have ⋀w. w ∈ S ⟹ a′′ · (w−z) = 0
    by (simp add: span_base orthogonal_def)
  then have a′′: ⋀w. w ∈ S ⟹ a′′ · w = (a − a′) · z
    by (simp add: a inner_diff_right)
  then have ba′′: ⋀w. w ∈ S ⟹ a′′ · w = b − a′ · z
    by (simp add: inner_diff_left z)
  show ?thesis
  proof (cases a′ = 0)
    case True
    with a assms True a′′ diff_zero less_irrefl show ?thesis
      by auto
  next
    case False
    show ?thesis
    proof
      show S ∩ {x. a′ · x ≤ a′ · z} = S ∩ {x. a · x ≤ b}
        S ∩ {x. a′ · x = a′ · z} = S ∩ {x. a · x = b}
        by (auto simp: a ba′′ inner_left_distrib)
      have ⋀w. w ∈ (+) (− z) ` S ⟹ (w + a′) ∈ (+) (− z) ` S
        by (metis subspace_add a′ span_eq_iff sub)
      then show ⋀w. w ∈ S ⟹ (w + a′) ∈ S
        by fastforce
    qed (use False in auto)
  qed
qed

lemma diffs_affine_hull_span:
  assumes a ∈ S
    shows (λx. x − a) ` (affine hull S) = span ((λx. x − a) ` S)
proof −
  have ∗: ((λx. x − a) ` (S − {a})) = ((λx. x − a) ` S) − {0}
    by (auto simp: algebra_simps)
  show ?thesis
    by (auto simp add: algebra_simps affine_hull_span2 [OF assms] ∗)
qed

lemma aff_dim_dim_affine_diffs:
  fixes S :: ′a :: euclidean_space set
```

assumes *affine S a* ∈ *S*
shows *aff_dim S* = *dim* ((λ*x*. *x* − *a*) ‘ *S*)
proof −
obtain *B* **where** *aff*: *affine hull B* = *affine hull S*
and *ind*: ¬ *affine_dependent B*
and *card*: *of_nat* (*card B*) = *aff_dim S* + *1*
using *aff_dim_basis_exists* **by** *blast*
then have *B* ≠ {} **using** *assms*
by (*metis affine_hull_eq_empty ex_in_conv*)
then obtain *c* **where** *c* ∈ *B* **by** *auto*
then have *c* ∈ *S*
by (*metis aff affine_hull_eq* ‹*affine S*› *hull_inc*)
have *xy*: *x* − *c* = *y* − *a* ⟷ *y* = *x* + *1* $*_R$ (*a* − *c*) **for** *x y c* **and** *a*::′*a*
by (*auto simp*: *algebra_simps*)
have ∗: (λ*x*. *x* − *c*) ‘ *S* = (λ*x*. *x* − *a*) ‘ *S*
using *assms* ‹*c* ∈ *S*›
by (*auto simp*: *image_iff xy*; *metis mem_affine_3_minus pth_1*)
have *affS*: *affine hull S* = *S*
by (*simp add*: ‹*affine S*›)
have *aff_dim S* = *of_nat* (*card B*) − *1*
using *card* **by** *simp*
also have ... = *dim* ((λ*x*. *x* − *c*) ‘ *B*)
using *affine_independent_card_dim_diffs* [*OF ind* ‹*c* ∈ *B*›]
by (*simp add*: *affine_independent_card_dim_diffs* [*OF ind* ‹*c* ∈ *B*›])
also have ... = *dim* ((λ*x*. *x* − *c*) ‘ (*affine hull B*))
by (*simp add*: *diffs_affine_hull_span* ‹*c* ∈ *B*›)
also have ... = *dim* ((λ*x*. *x* − *a*) ‘ *S*)
by (*simp add*: *affS aff* ∗)
finally show *?thesis* **.**
qed

lemma *aff_dim_linear_image_le*:
assumes *linear f*
shows *aff_dim*(*f* ‘ *S*) ≤ *aff_dim S*
proof −
have *aff_dim* (*f* ‘ *T*) ≤ *aff_dim T* **if** *affine T* **for** *T*
proof (*cases T* = {})
case *True* **then show** *?thesis* **by** (*simp add*: *aff_dim_geq*)
next
case *False*
then obtain *a* **where** *a* ∈ *T* **by** *auto*
have *1*: ((λ*x*. *x* − *f a*) ‘ *f* ‘ *T*) = {*x* − *f a* |*x*. *x* ∈ *f* ‘ *T*}
by *auto*
have *2*: {*x* − *f a*| *x*. *x* ∈ *f* ‘ *T*} = *f* ‘ ((λ*x*. *x* − *a*) ‘ *T*)
by (*force simp*: *linear_diff* [*OF assms*])
have *aff_dim* (*f* ‘ *T*) = *int* (*dim* {*x* − *f a* |*x*. *x* ∈ *f* ‘ *T*})
by (*simp add*: ‹*a* ∈ *T*› *hull_inc aff_dim_eq_dim* [*of f a*] *1 cong*: *image_cong_simp*)
also have ... = *int* (*dim* (*f* ‘ ((λ*x*. *x* − *a*) ‘ *T*)))
by (*force simp*: *linear_diff* [*OF assms*] *2*)

```
    also have ... ≤ int (dim ((λx. x − a) ‘ T))
      by (simp add: dim_image_le [OF assms])
    also have ... ≤ aff_dim T
      by (simp add: aff_dim_dim_affine_diffs [symmetric] ‹a ∈ T› ‹affine T›)
    finally show ?thesis .
  qed
  then
  have aff_dim (f ‘ (affine hull S)) ≤ aff_dim (affine hull S)
    using affine_affine_hull [of S] by blast
  then show ?thesis
    using affine_hull_linear_image assms linear_conv_bounded_linear by fastforce
qed

lemma aff_dim_injective_linear_image [simp]:
  assumes linear f inj f
    shows aff_dim (f ‘ S) = aff_dim S
proof (rule antisym)
  show aff_dim (f ‘ S) ≤ aff_dim S
    by (simp add: aff_dim_linear_image_le assms(1))
next
  obtain g where linear g g ∘ f = id
    using assms(1) assms(2) linear_injective_left_inverse by blast
  then have aff_dim S ≤ aff_dim(g ‘ f ‘ S)
    by (simp add: image_comp)
  also have ... ≤ aff_dim (f ‘ S)
    by (simp add: ‹linear g› aff_dim_linear_image_le)
  finally show aff_dim S ≤ aff_dim (f ‘ S) .
qed


lemma choose_affine_subset:
  assumes affine S −1 ≤ d and dle: d ≤ aff_dim S
  obtains T where affine T T ⊆ S aff_dim T = d
proof (cases d = −1 ∨ S={})
  case True with assms show ?thesis
    by (metis aff_dim_empty affine_empty bot.extremum that eq_iff)
next
  case False
  with assms obtain a where a ∈ S 0 ≤ d by auto
  with assms have ss: subspace ((+) (− a) ‘ S)
    by (simp add: affine_diffs_subspace_subtract cong: image_cong_simp)
  have nat d ≤ dim ((+) (− a) ‘ S)
    by (metis aff_dim_subspace aff_dim_translation_eq dle nat_int nat_mono ss)
  then obtain T where subspace T and Tsb: T ⊆ span ((+) (− a) ‘ S)
                and Tdim: dim T = nat d
    using choose_subspace_of_subspace [of nat d (+) (− a) ‘ S] by blast
  then have affine T
    using subspace_affine by blast
  then have affine ((+) a ‘ T)
```

by (*metis affine_hull_eq affine_hull_translation*)
moreover have $(+)\ a\ `\ T \subseteq S$
proof −
have $T \subseteq (+)\ (-\ a)\ `\ S$
by (*metis* (*no_types*) *span_eq_iff Tsb ss*)
then show $(+)\ a\ `\ T \subseteq S$
using *add_ac* **by** *auto*
qed
moreover have *aff_dim* $((+)\ a\ `\ T) = d$
by (*simp add*: *aff_dim_subspace Tdim* ‹$0 \le d$› ‹*subspace T*› *aff_dim_translation_eq*)
ultimately show *?thesis*
by (*rule that*)
qed

5.0.19 Paracompactness

proposition *paracompact*:
fixes S :: $'a$ :: {*metric_space*,*second_countable_topology*} *set*
assumes $S \subseteq \bigcup\mathcal{C}$ **and** *opC*: $\bigwedge T.\ T \in \mathcal{C} \Longrightarrow$ *open* T
obtains $\mathcal{C}'$ **where** $S \subseteq \bigcup\ \mathcal{C}'$
and $\bigwedge U.\ U \in \mathcal{C}' \Longrightarrow$ *open* $U \wedge (\exists\, T.\ T \in \mathcal{C} \wedge U \subseteq T)$
and $\bigwedge x.\ x \in S$
$\Longrightarrow \exists\, V.$ *open* $V \wedge x \in V \wedge$ *finite* $\{U.\ U \in \mathcal{C}' \wedge (U \cap V \neq$
$\{\})\}$
proof (*cases* $S = \{\}$)
case *True* **with** *that* **show** *?thesis* **by** *blast*
next
case *False*
have $\exists\, T\ U.\ x \in U \wedge$ *open* $U \wedge$ *closure* $U \subseteq T \wedge T \in \mathcal{C}$ **if** $x \in S$ **for** x
proof −
obtain T **where** $x \in T$ $T \in \mathcal{C}$ *open* T
using *assms* ‹$x \in S$› **by** *blast*
then obtain e **where** $e > 0$ *cball* x $e \subseteq T$
by (*force simp*: *open_contains_cball*)
then show *?thesis*
by (*meson open_ball* ‹$T \in \mathcal{C}$› *ball_subset_cball centre_in_ball closed_cball closure_minimal dual_order.trans*)
qed
then obtain F G **where** *Gin*: $x \in G\ x$ **and** *oG*: *open* $(G\ x)$
and *clos*: *closure* $(G\ x) \subseteq F\ x$ **and** *Fin*: $F\ x \in \mathcal{C}$
if $x \in S$ **for** x
by *metis*
then obtain $\mathcal{F}$ **where** $\mathcal{F} \subseteq G\ `\ S$ *countable* $\mathcal{F}$ $\bigcup\mathcal{F} = \bigcup(G\ `\ S)$
using *Lindelof* [*of* $G\ `\ S$] **by** (*metis image_iff*)
then obtain K **where** K: $K \subseteq S$ *countable* K **and** *eq*: $\bigcup(G\ `\ K) = \bigcup(G\ `\ S)$
by (*metis countable_subset_image*)
with *False Gin* **have** $K \neq \{\}$ **by** *force*
then obtain a :: *nat* $\Rightarrow$ $'a$ **where** *range* $a = K$
by (*metis range_from_nat_into* ‹*countable* K›)

then have *odif*: $\bigwedge n.$ *open* $(F\ (a\ n) - \bigcup\{closure\ (G\ (a\ m))\ |m.\ m < n\})$
using ‹$K \subseteq S$› *Fin opC* **by** (*fastforce simp add:*)
let $?C = range\ (\lambda n.\ F(a\ n) - \bigcup\{closure(G(a\ m))\ |m.\ m < n\})$
have *enum_S*: $\exists\, n.\ x \in F(a\ n) \wedge x \in G(a\ n)$ **if** $x \in S$ **for** x
proof −
have $\exists\, y \in K.\ x \in G\ y$ **using** *eq that Gin* **by** *fastforce*
then show *?thesis*
using *clos K* ‹*range a* = *K*› *closure_subset* **by** *blast*
qed
show *?thesis*
proof
show $S \subseteq$ *Union ?C*
proof
fix x **assume** $x \in S$
define n **where** $n \equiv LEAST\ n.\ x \in F(a\ n)$
have n: $x \in F(a\ n)$
using *enum_S* [*OF* ‹$x \in S$›] **by** (*force simp*: *n_def intro*: *LeastI*)
have *notn*: $x \notin F(a\ m)$ **if** $m < n$ **for** m
using *that not_less_Least* **by** (*force simp*: *n_def*)
then have $x \notin \bigcup\{closure\ (G\ (a\ m))\ |m.\ m < n\}$
using n ‹$K \subseteq S$› ‹*range a* = *K*› *clos notn* **by** *fastforce*
with n **show** $x \in$ *Union ?C*
by *blast*
qed
show $\bigwedge U.\ U \in\ ?C \Longrightarrow open\ U \wedge (\exists\, T.\ T \in \mathcal{C} \wedge U \subseteq T)$
using *Fin* ‹$K \subseteq S$› ‹*range a* = *K*› **by** (*auto simp*: *odif*)
show $\exists\, V.\ open\ V \wedge x \in V \wedge finite\ \{U.\ U \in\ ?C \wedge (U \cap V \neq \{\})\}$ **if** $x \in S$ **for** x
proof −
obtain n **where** n: $x \in F(a\ n)\ x \in G(a\ n)$
using ‹$x \in S$› *enum_S* **by** *auto*
have $\{U \in\ ?C.\ U \cap G\ (a\ n) \neq \{\}\} \subseteq (\lambda n.\ F(a\ n) - \bigcup\{closure(G(a\ m))\ |m.\ m < n\})$ ‘ *atMost n*
proof *clarsimp*
fix k **assume** $(F\ (a\ k) - \bigcup\{closure\ (G\ (a\ m))\ |m.\ m < k\}) \cap G\ (a\ n) \neq \{\}$
then have $k \le n$
by *auto* (*metis closure_subset not_le subsetCE*)
then show $F\ (a\ k) - \bigcup\{closure\ (G\ (a\ m))\ |m.\ m < k\}$
$\in (\lambda n.\ F\ (a\ n) - \bigcup\{closure\ (G\ (a\ m))\ |m.\ m < n\})$ ‘ $\{..n\}$
by *force*
qed
moreover have *finite* $((\lambda n.\ F(a\ n) - \bigcup\{closure(G(a\ m))\ |m.\ m < n\})$ ‘ *atMost n*)
by *force*
ultimately have $*$: *finite* $\{U \in\ ?C.\ U \cap G\ (a\ n) \neq \{\}\}$
using *finite_subset* **by** *blast*
have $a\ n \in S$
using ‹$K \subseteq S$› ‹*range a* = *K*› **by** *blast*

then show *?thesis*
by (*blast intro*: *oG n* ∗)
qed
qed
qed

corollary *paracompact_closedin*:
fixes S :: $'a$ :: {*metric_space*,*second_countable_topology*} *set*
assumes *cin*: *closedin* (*top_of_set* U) S
and *oin*: $\bigwedge T.\ T \in \mathcal{C} \Longrightarrow$ *openin* (*top_of_set* U) T
and $S \subseteq \bigcup\mathcal{C}$
obtains $\mathcal{C}'$ **where** $S \subseteq \bigcup\ \mathcal{C}'$
and $\bigwedge V.\ V \in \mathcal{C}' \Longrightarrow$ *openin* (*top_of_set* U) $V \wedge (\exists\, T.\ T \in \mathcal{C} \wedge V \subseteq$
$T)$
and $\bigwedge x.\ x \in U$
$\Longrightarrow \exists\, V.$ *openin* (*top_of_set* U) $V \wedge x \in V \wedge$
finite $\{X.\ X \in \mathcal{C}' \wedge (X \cap V \neq \{\})\}$
proof −
have $\exists\, Z.$ *open* $Z \wedge (T = U \cap Z)$ **if** $T \in \mathcal{C}$ **for** T
using *oin* [*OF that*] **by** (*auto simp*: *openin_open*)
then obtain F **where** *opF*: *open* (F T) **and** *intF*: $U \cap F\ T = T$ **if** $T \in \mathcal{C}$ **for**
T
by *metis*
obtain K **where** K: *closed* K $U \cap K = S$
using *cin* **by** (*auto simp*: *closedin_closed*)
have *1*: $U \subseteq \bigcup$(*insert* $(-\ K)$ (F ‘ $\mathcal{C}$))
by *clarsimp* (*metis Int_iff Union_iff* ‹$U \cap K = S$› ‹$S \subseteq \bigcup\mathcal{C}$› *subsetD intF*)
have *2*: $\bigwedge T.\ T \in$ *insert* $(-\ K)$ (F ‘ $\mathcal{C}$) $\Longrightarrow$ *open* T
using ‹*closed* K› **by** (*auto simp*: *opF*)
obtain $\mathcal{D}$ **where** $U \subseteq \bigcup\mathcal{D}$
and *D1*: $\bigwedge U.\ U \in \mathcal{D} \Longrightarrow$ *open* $U \wedge (\exists\, T.\ T \in$ *insert* $(-\ K)$ (F ‘ $\mathcal{C}$) $\wedge$
$U \subseteq T)$
and *D2*: $\bigwedge x.\ x \in U \Longrightarrow \exists\, V.$ *open* $V \wedge x \in V \wedge$ *finite* $\{U \in \mathcal{D}.\ U \cap$
$V \neq \{\}\}$
by (*blast intro*: *paracompact* [*OF 1 2*])
let $?C = \{U \cap V\ |V.\ V \in \mathcal{D} \wedge (V \cap K \neq \{\})\}$
show *?thesis*
proof (*rule_tac* $\mathcal{C}' = \{U \cap V\ |V.\ V \in \mathcal{D} \wedge (V \cap K \neq \{\})\}$ **in** *that*)
show $S \subseteq \bigcup ?C$
using ‹$U \cap K = S$› ‹$U \subseteq \bigcup\mathcal{D}$› K **by** (*blast dest*!: *subsetD*)
show $\bigwedge V.\ V \in ?C \Longrightarrow$ *openin* (*top_of_set* U) $V \wedge (\exists\, T.\ T \in \mathcal{C} \wedge V \subseteq T)$
using *D1 intF* **by** *fastforce*
have ∗: $\{X.\ (\exists\, V.\ X = U \cap V \wedge V \in \mathcal{D} \wedge V \cap K \neq \{\}) \wedge X \cap (U \cap V) \neq$
$\{\}\} \subseteq$
$(\lambda x.\ U \cap x)$ ‘ $\{U \in \mathcal{D}.\ U \cap V \neq \{\}\}$ **for** V
by *blast*
show $\exists\, V.$ *openin* (*top_of_set* U) $V \wedge x \in V \wedge$ *finite* $\{X \in ?C.\ X \cap V \neq \{\}\}$
if $x \in U$ **for** x
proof −

from *D2* [*OF that*] **obtain** *V* **where** *open V x* ∈ *V finite* {*U* ∈ 𝒟. *U* ∩ *V* ≠ {}}
  by *auto*
 with ∗ **show** *?thesis*
  by (*rule_tac x*=*U* ∩ *V* **in** *exI*) (*auto intro*: *that finite_subset* [*OF* ∗])
 qed
 qed
qed

corollary *paracompact_closed*:
 fixes *S* :: *'a* :: {*metric_space*,*second_countable_topology*} *set*
 assumes *closed S*
  and *opC*: ⋀*T*. *T* ∈ 𝒞 ⟹ *open T*
  and *S* ⊆ ⋃𝒞
 obtains 𝒞*'* **where** *S* ⊆ ⋃𝒞*'*
   and ⋀*U*. *U* ∈ 𝒞*'* ⟹ *open U* ∧ (∃ *T*. *T* ∈ 𝒞 ∧ *U* ⊆ *T*)
   and ⋀*x*. ∃ *V*. *open V* ∧ *x* ∈ *V* ∧
     finite {*U*. *U* ∈ 𝒞*'* ∧ (*U* ∩ *V* ≠ {})}
 by (*rule paracompact_closedin* [*of UNIV S* 𝒞]) (*auto simp*: *assms*)

5.0.20 Closed-graph characterization of continuity

lemma *continuous_closed_graph_gen*:
 fixes *T* :: *'b*::*real_normed_vector set*
 assumes *contf*: *continuous_on S f* **and** *fim*: *f* ‘ *S* ⊆ *T*
  shows *closedin* (*top_of_set* (*S* × *T*)) ((λ*x*. *Pair x* (*f x*)) ‘ *S*)
proof −
 have *eq*: ((λ*x*. *Pair x* (*f x*)) ‘ *S*) = (*S* × *T* ∩ (λ*z*. (*f* ∘ *fst*)*z* − *snd z*) −‘ {*0*})
  using *fim* **by** *auto*
 show *?thesis*
  unfolding *eq*
  by (*intro continuous_intros continuous_closedin_preimage continuous_on_subset* [*OF contf*]) *auto*
qed

lemma *continuous_closed_graph_eq*:
 fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
 assumes *compact T* **and** *fim*: *f* ‘ *S* ⊆ *T*
 shows *continuous_on S f* ⟷
   closedin (*top_of_set* (*S* × *T*)) ((λ*x*. *Pair x* (*f x*)) ‘ *S*)
   (**is** *?lhs* = *?rhs*)
proof −
 have *?lhs* **if** *?rhs*
 proof (*clarsimp simp add*: *continuous_on_closed_gen* [*OF fim*])
  fix *U*
  assume *U*: *closedin* (*top_of_set T*) *U*
  have *eq*: (*S* ∩ *f* −‘ *U*) = *fst* ‘ (((λ*x*. *Pair x* (*f x*)) ‘ *S*) ∩ (*S* × *U*))
   by (*force simp*: *image_iff*)
  show *closedin* (*top_of_set S*) (*S* ∩ *f* −‘ *U*)

by (*simp add*: *U closedin_Int closedin_Times closed_map_fst* [*OF* ‹*compact T*›] *that eq*)
qed
with *continuous_closed_graph_gen assms* **show** *?thesis* **by** *blast*
qed

lemma *continuous_closed_graph*:
fixes *f* :: *'a::topological_space* ⇒ *'b::real_normed_vector*
assumes *closed S* **and** *contf*: *continuous_on S f*
shows *closed* ((λ*x*. *Pair x* (*f x*)) ‘ *S*)
proof (*rule closedin_closed_trans*)
show *closedin* (*top_of_set* (*S* × *UNIV*)) ((λ*x*. (*x*, *f x*)) ‘ *S*)
by (*rule continuous_closed_graph_gen* [*OF contf subset_UNIV*])
qed (*simp add*: ‹*closed S*› *closed_Times*)

lemma *continuous_from_closed_graph*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *compact T* **and** *fim*: *f* ‘ *S* ⊆ *T* **and** *clo*: *closed* ((λ*x*. *Pair x* (*f x*)) ‘ *S*)
shows *continuous_on S f*
using *fim clo*
by (*auto intro*: *closed_subset simp*: *continuous_closed_graph_eq* [*OF* ‹*compact T*› *fim*])

lemma *continuous_on_Un_local_open*:
assumes *opS*: *openin* (*top_of_set* (*S* ∪ *T*)) *S*
and *opT*: *openin* (*top_of_set* (*S* ∪ *T*)) *T*
and *contf*: *continuous_on S f* **and** *contg*: *continuous_on T f*
shows *continuous_on* (*S* ∪ *T*) *f*
using *pasting_lemma* [*of* {*S*,*T*} *top_of_set* (*S* ∪ *T*) *id euclidean* λ*i*. *f f*] *contf contg opS opT*
by (*simp add*: *subtopology_subtopology*) (*metis inf.absorb2 openin_imp_subset*)

lemma *continuous_on_cases_local_open*:
assumes *opS*: *openin* (*top_of_set* (*S* ∪ *T*)) *S*
and *opT*: *openin* (*top_of_set* (*S* ∪ *T*)) *T*
and *contf*: *continuous_on S f* **and** *contg*: *continuous_on T g*
and *fg*: ⋀*x*. *x* ∈ *S* ∧ ¬*P x* ∨ *x* ∈ *T* ∧ *P x* ⟹ *f x* = *g x*
shows *continuous_on* (*S* ∪ *T*) (λ*x*. *if P x then f x else g x*)
proof −
have ⋀*x*. *x* ∈ *S* ⟹ (*if P x then f x else g x*) = *f x* ⋀*x*. *x* ∈ *T* ⟹ (*if P x then f x else g x*) = *g x*
by (*simp_all add*: *fg*)
then have *continuous_on S* (λ*x*. *if P x then f x else g x*) *continuous_on T* (λ*x*. *if P x then f x else g x*)
by (*simp_all add*: *contf contg cong*: *continuous_on_cong*)
then show *?thesis*
by (*rule continuous_on_Un_local_open* [*OF opS opT*])
qed

5.0.21 The union of two collinear segments is another segment

proposition *in_convex_hull_exchange*:
  fixes *a* :: *'a::euclidean_space*
  assumes *a*: *a* ∈ *convex hull S* **and** *xS*: *x* ∈ *convex hull S*
  obtains *b* **where** *b* ∈ *S x* ∈ *convex hull* (*insert a* (*S* − {*b*}))
proof (*cases a* ∈ *S*)
  case *True*
  with *xS insert_Diff that* **show** *?thesis* **by** *fastforce*
next
  case *False*
  show *?thesis*
  proof (*cases finite S* ∧ *card S* ≤ *Suc* (*DIM*(*'a*)))
    case *True*
    then obtain *u* **where** *u0*: ⋀*i*. *i* ∈ *S* ⟹ *0* ≤ *u i* **and** *u1*: *sum u S* = *1*
                and *ua*: (∑*i*∈*S*. *u i* *R *i*) = *a*
      using *a* **by** (*auto simp*: *convex_hull_finite*)
    obtain *v* **where** *v0*: ⋀*i*. *i* ∈ *S* ⟹ *0* ≤ *v i* **and** *v1*: *sum v S* = *1*
            and *vx*: (∑*i*∈*S*. *v i* *R *i*) = *x*
      using *True xS* **by** (*auto simp*: *convex_hull_finite*)
    show *?thesis*
    proof (*cases* ∃*b*. *b* ∈ *S* ∧ *v b* = *0*)
      case *True*
      then obtain *b* **where** *b*: *b* ∈ *S v b* = *0*
        by *blast*
      show *?thesis*
      proof
        have *fin*: *finite* (*insert a* (*S* − {*b*}))
          using *sum.infinite v1* **by** *fastforce*
        show *x* ∈ *convex hull insert a* (*S* − {*b*})
          unfolding *convex_hull_finite* [*OF fin*] *mem_Collect_eq*
        proof (*intro conjI exI ballI*)
          have (∑*x* ∈ *insert a* (*S* − {*b*}). *if x* = *a then 0 else v x*) =
                (∑*x* ∈ *S* − {*b*}. *if x* = *a then 0 else v x*)
            using *fin* **by** (*force intro*: *sum.mono_neutral_right*)
          also have ... = (∑*x* ∈ *S* − {*b*}. *v x*)
            using *b False* **by** (*auto intro*!: *sum.cong split*: *if_split_asm*)
          also have ... = (∑*x*∈*S*. *v x*)
            by (*metis* ‹*v b* = *0*› *diff_zero sum.infinite sum_diff1 u1 zero_neq_one*)
          finally show (∑*x*∈*insert a* (*S* − {*b*}). *if x* = *a then 0 else v x*) = *1*
            by (*simp add*: *v1*)
          show ⋀*x*. *x* ∈ *insert a* (*S* − {*b*}) ⟹ *0* ≤ (*if x* = *a then 0 else v x*)
            by (*auto simp*: *v0*)
          have (∑*x* ∈ *insert a* (*S* − {*b*}). (*if x* = *a then 0 else v x*) *R *x*) =
                (∑*x* ∈ *S* − {*b*}. (*if x* = *a then 0 else v x*) *R *x*)
            using *fin* **by** (*force intro*: *sum.mono_neutral_right*)
          also have ... = (∑*x* ∈ *S* − {*b*}. *v x* *R *x*)
            using *b False* **by** (*auto intro*!: *sum.cong split*: *if_split_asm*)
          also have ... = (∑*x*∈*S*. *v x* *R *x*)

by (*metis* (*no_types*, *lifting*) *b*(*2*) *diff_zero fin finite.emptyI finite_Diff2 finite_insert scale_eq_0_iff sum_diff1*)
finally show ($\sum$*x*∈*insert a* (*S* − {*b*}). (*if x* = *a then 0 else v x*) $*_R$ *x*) = *x*
by (*simp add*: *vx*)
qed
qed (*rule* ‹*b* ∈ *S*›)
next
case *False*
have *le_Max*: *u i* / *v i* ≤ *Max* ((λ*i*. *u i* / *v i*) ' *S*) **if** *i* ∈ *S* **for** *i*
by (*simp add*: *True that*)
have *Max* ((λ*i*. *u i* / *v i*) ' *S*) ∈ (λ*i*. *u i* / *v i*) ' *S*
using *True v1* **by** (*auto intro*: *Max_in*)
then obtain *b* **where** *b* ∈ *S* **and** *beq*: *Max* ((λ*b*. *u b* / *v b*) ' *S*) = *u b* / *v b*
by *blast*
then have *0* ≠ *u b* / *v b*
using *le_Max beq divide_le_0_iff le_numeral_extra*(*2*) *sum_nonpos u1*
by (*metis False eq_iff v0*)
then have *0* < *u b 0* < *v b*
using *False* ‹*b* ∈ *S*› *u0 v0* **by** *force+*
have *fin*: *finite* (*insert a* (*S* − {*b*}))
using *sum.infinite v1* **by** *fastforce*
show *?thesis*
proof
show *x* ∈ *convex hull insert a* (*S* − {*b*})
unfolding *convex_hull_finite* [*OF fin*] *mem_Collect_eq*
proof (*intro conjI exI ballI*)
have ($\sum$*x* ∈ *insert a* (*S* − {*b*}). *if x*=*a then v b* / *u b else v x* − (*v b* / *u b*) ∗ *u x*) =
v b / *u b* + ($\sum$*x* ∈ *S* − {*b*}. *v x* − (*v b* / *u b*) ∗ *u x*)
using ‹*a* ∉ *S*› ‹*b* ∈ *S*› *True*
by (*auto intro*!: *sum.cong split*: *if_split_asm*)
also have ... = *v b* / *u b* + ($\sum$*x* ∈ *S* − {*b*}. *v x*) − (*v b* / *u b*) ∗ ($\sum$*x* ∈ *S* − {*b*}. *u x*)
by (*simp add*: *Groups_Big.sum_subtractf sum_distrib_left*)
also have ... = ($\sum$*x*∈*S*. *v x*)
using ‹*0* < *u b*› *True* **by** (*simp add*: *Groups_Big.sum_diff1 u1 field_simps*)
finally show *sum* (λ*x*. *if x*=*a then v b* / *u b else v x* − (*v b* / *u b*) ∗ *u x*) (*insert a* (*S* − {*b*})) = *1*
by (*simp add*: *v1*)
show *0* ≤ (*if i* = *a then v b* / *u b else v i* − *v b* / *u b* ∗ *u i*)
if *i* ∈ *insert a* (*S* − {*b*}) **for** *i*
using ‹*0* < *u b*› ‹*0* < *v b*› *v0* [*of i*] *le_Max* [*of i*] *beq that False*
by (*auto simp*: *field_simps split*: *if_split_asm*)
have ($\sum$*x*∈*insert a* (*S* − {*b*}). (*if x*=*a then v b* / *u b else v x* − *v b* / *u b* ∗ *u x*) $*_R$ *x*) =
(*v b* / *u b*) $*_R$ *a* + ($\sum$*x*∈*S* − {*b*}. (*v x* − *v b* / *u b* ∗ *u x*) $*_R$ *x*)
using ‹*a* ∉ *S*› ‹*b* ∈ *S*› *True* **by** (*auto intro*!: *sum.cong split*: *if_split_asm*)
also have ... = (*v b* / *u b*) $*_R$ *a* + ($\sum$*x* ∈ *S* − {*b*}. *v x* $*_R$ *x*) − (*v b* /

u b) $*_R$ ($\sum x \in S - \{b\}$. *u x* $*_R$ *x*)
by (*simp add*: *Groups_Big.sum_subtractf scaleR_left_diff_distrib sum_distrib_left scale_sum_right*)
also have ... = ($\sum x{\in}S$. *v x* $*_R$ *x*)
using ‹*0* < *u b*› *True* **by** (*simp add*: *ua vx Groups_Big.sum_diff1 algebra_simps*)
finally
show ($\sum x{\in}$*insert a* ($S - \{b\}$). (*if x=a then v b / u b else v x − v b / u b ∗ u x*) $*_R$ *x*) = *x*
by (*simp add*: *vx*)
qed
qed (*rule* ‹*b* ∈ *S*›)
qed
next
case *False*
obtain *T* **where** *finite T T* ⊆ *S* **and** *caT*: *card T* ≤ *Suc* (*DIM*(′*a*)) **and** *xT*: *x* ∈ *convex hull T*
using *xS* **by** (*auto simp*: *caratheodory* [*of S*])
with *False* **obtain** *b* **where** *b*: *b* ∈ *S b* ∉ *T*
by (*metis antisym subsetI*)
show *?thesis*
proof
show *x* ∈ *convex hull insert a* (*S* − {*b*})
using ‹*T* ⊆ *S*› *b* **by** (*blast intro*: *subsetD* [*OF hull_mono xT*])
qed (*rule* ‹*b* ∈ *S*›)
qed
qed

lemma *convex_hull_exchange_Union*:
fixes *a* :: ′*a*::*euclidean_space*
assumes *a* ∈ *convex hull S*
shows *convex hull S* = ($\bigcup b \in S$. *convex hull* (*insert a* (*S* − {*b*}))) (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
by (*blast intro*: *in_convex_hull_exchange* [*OF assms*])
show *?rhs* ⊆ *?lhs*
proof *clarify*
fix *x b*
assume*b* ∈ *S x* ∈ *convex hull insert a* (*S* − {*b*})
then show *x* ∈ *convex hull S* **if** *b* ∈ *S*
by (*metis* (*no_types*) *that assms order_refl hull_mono hull_redundant insert_Diff_single insert_subset subsetCE*)
qed
qed

lemma *Un_closed_segment*:
fixes *a* :: ′*a*::*euclidean_space*
assumes *b* ∈ *closed_segment a c*

shows *closed_segment a b* ∪ *closed_segment b c* = *closed_segment a c*
proof (*cases c* = *a*)
case *True*
with *assms* **show** *?thesis* **by** *simp*
next
case *False*
with *assms* **have** *convex hull* {*a*, *b*} ∪ *convex hull* {*b*, *c*} = (⋃ *ba*∈{*a*, *c*}. *convex hull insert b* ({*a*, *c*} − {*ba*}))
by (*auto simp*: *insert_Diff_if insert_commute*)
then show *?thesis*
using *convex_hull_exchange_Union*
by (*metis assms segment_convex_hull*)
qed

lemma *Un_open_segment*:
fixes *a* :: *'a*::*euclidean_space*
assumes *b* ∈ *open_segment a c*
shows *open_segment a b* ∪ {*b*} ∪ *open_segment b c* = *open_segment a c* (**is** *?lhs* = *?rhs*)
proof −
have *b*: *b* ∈ *closed_segment a c*
by (*simp add*: *assms open_closed_segment*)
have ∗: *?rhs* ⊆ *insert b* (*open_segment a b* ∪ *open_segment b c*)
if {*b*,*c*,*a*} ∪ *open_segment a b* ∪ *open_segment b c* = {*c*,*a*} ∪ *?rhs*
proof −
have *insert a* (*insert c* (*insert b* (*open_segment a b* ∪ *open_segment b c*))) = *insert a* (*insert c* (*?rhs*))
using *that* **by** (*simp add*: *insert_commute*)
then show *?thesis*
by (*metis* (*no_types*) *Diff_cancel Diff_eq_empty_iff Diff_insert2 open_segment_def*)
qed
show *?thesis*
proof
show *?lhs* ⊆ *?rhs*
by (*simp add*: *assms b subset_open_segment*)
show *?rhs* ⊆ *?lhs*
using *Un_closed_segment* [*OF b*] ∗
by (*simp add*: *closed_segment_eq_open insert_commute*)
qed
qed

5.0.22 Covering an open set by a countable chain of compact sets

proposition *open_Union_compact_subsets*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *open S*
obtains *C* **where** ⋀*n*. *compact*(*C n*) ⋀*n*. *C n* ⊆ *S*
⋀*n*. *C n* ⊆ *interior*(*C*(*Suc n*))

```
                 ⋃(range C) = S
                 ⋀K. ⟦compact K; K ⊆ S⟧ ⟹ ∃N. ∀n≥N. K ⊆ (C n)
proof (cases S = {})
  case True
  then show ?thesis
    by (rule_tac C = λn. {} in that) auto
next
  case False
  then obtain a where a ∈ S
    by auto
  let ?C = λn. cball a (real n) − (⋃x ∈ −S. ⋃e ∈ ball 0 (1 / real(Suc n)). {x
+ e})
  have ∃N. ∀n≥N. K ⊆ (f n)
        if ⋀n. compact(f n) and sub_int: ⋀n. f n ⊆ interior (f(Suc n))
            and eq: ⋃(range f) = S and compact K K ⊆ S for f K
  proof −
    have *: ∀n. f n ⊆ (⋃n. interior (f n))
      by (meson Sup_upper2 UNIV_I ‹⋀n. f n ⊆ interior (f (Suc n))› image_iff)
    have mono: ⋀m n. m ≤ n ⟹f m ⊆ f n
      by (meson dual_order.trans interior_subset lift_Suc_mono_le sub_int)
    obtain I where finite I and I: K ⊆ (⋃i∈I. interior (f i))
    proof (rule compactE_image [OF ‹compact K›])
      show K ⊆ (⋃n. interior (f n))
        using ‹K ⊆ S› ‹⋃(f ‘ UNIV) = S› * by blast
    qed auto
    { fix n
      assume n: Max I ≤ n
      have (⋃i∈I. interior (f i)) ⊆ f n
        by (rule UN_least) (meson dual_order.trans interior_subset mono I Max_ge
[OF ‹finite I›] n)
      then have K ⊆ f n
        using I by auto
    }
    then show ?thesis
      by blast
  qed
   moreover have ∃f. (∀n. compact(f n)) ∧ (∀n. (f n) ⊆ S) ∧ (∀n. (f n) ⊆
interior(f(Suc n))) ∧
                        ((⋃(range f) = S))
  proof (intro exI conjI allI)
    show ⋀n. compact (?C n)
      by (auto simp: compact_diff open_sums)
    show ⋀n. ?C n ⊆ S
      by auto
    show ?C n ⊆ interior (?C (Suc n)) for n
    proof (simp add: interior_diff, rule Diff_mono)
      show cball a (real n) ⊆ ball a (1 + real n)
        by (simp add: cball_subset_ball_iff)
      have cl: closed (⋃x∈− S. ⋃e∈cball 0 (1 / (2 + real n)). {x + e})
```

```
      using assms by (auto intro: closed_compact_sums)
    have closure (⋃x∈− S. ⋃y∈ball 0 (1 / (2 + real n)). {x + y})
         ⊆ (⋃x ∈ −S. ⋃e ∈ cball 0 (1 / (2 + real n)). {x + e})
      by (intro closure_minimal UN_mono ball_subset_cball order_refl cl)
    also have ... ⊆ (⋃x ∈ −S. ⋃y∈ball 0 (1 / (1 + real n)). {x + y})
      by (simp add: cball_subset_ball_iff field_split_simps UN_mono)
    finally show closure (⋃x∈− S. ⋃y∈ball 0 (1 / (2 + real n)). {x + y})
                 ⊆ (⋃x ∈ −S. ⋃y∈ball 0 (1 / (1 + real n)). {x + y}) .
  qed
  have S ⊆ ⋃ (range ?C)
  proof
    fix x
    assume x: x ∈ S
    then obtain e where e > 0 and e: ball x e ⊆ S
      using assms open_contains_ball by blast
    then obtain N1 where N1 > 0 and N1: real N1 > 1/e
      using reals_Archimedean2
      by (metis divide_less_0_iff less_eq_real_def neq0_conv not_le of_nat_0 of_nat_1
of_nat_less_0_iff)
    obtain N2 where N2: norm(x − a) ≤ real N2
      by (meson real_arch_simple)
    have N12: inverse((N1 + N2) + 1) ≤ inverse(N1)
      using ‹N1 > 0› by (auto simp: field_split_simps)
    have x ≠ y + z if y ∉ S norm z < 1 / (1 + (real N1 + real N2)) for y z
    proof −
      have e ∗ real N1 < e ∗ (1 + (real N1 + real N2))
        by (simp add: ‹0 < e›)
      then have 1 / (1 + (real N1 + real N2)) < e
        using N1 ‹e > 0›
        by (metis divide_less_eq less_trans mult.commute of_nat_add of_nat_less_0_iff
of_nat_Suc)
      then have x − z ∈ ball x e
        using that by simp
      then have x − z ∈ S
        using e by blast
      with that show ?thesis
        by auto
    qed
    with N2 show x ∈ ⋃ (range ?C)
    by (rule_tac a = N1+N2 in UN_I) (auto simp: dist_norm norm_minus_commute)
  qed
  then show ⋃ (range ?C) = S by auto
 qed
 ultimately show ?thesis
   using that by metis
qed
```

5.0.23 Orthogonal complement

definition *orthogonal_comp* ($\_^{\perp}$ [80] 80)
  where *orthogonal_comp* $W \equiv \{x.\ \forall y \in W.$ *orthogonal* $y\ x\}$

proposition *subspace_orthogonal_comp*: *subspace* $(W^{\perp})$
  unfolding *subspace_def orthogonal_comp_def orthogonal_def*
  by (*auto simp*: *inner_right_distrib*)

lemma *orthogonal_comp_anti_mono*:
  assumes $A \subseteq B$
  shows $B^{\perp} \subseteq A^{\perp}$
proof
  fix x **assume** x: $x \in B^{\perp}$
  show $x \in$ *orthogonal_comp* A **using** x **unfolding** *orthogonal_comp_def*
    by (*simp add*: *orthogonal_def*, *metis assms in_mono*)
qed

lemma *orthogonal_comp_null* [*simp*]: $\{0\}^{\perp} =$ *UNIV*
  by (*auto simp*: *orthogonal_comp_def orthogonal_def*)

lemma *orthogonal_comp_UNIV* [*simp*]: $UNIV^{\perp} = \{0\}$
  unfolding *orthogonal_comp_def orthogonal_def*
  by *auto* (*use inner_eq_zero_iff* **in** *blast*)

lemma *orthogonal_comp_subset*: $U \subseteq U^{\perp\perp}$
  by (*auto simp*: *orthogonal_comp_def orthogonal_def inner_commute*)

lemma *subspace_sum_minimal*:
  assumes $S \subseteq U$ $T \subseteq U$ *subspace* U
  shows $S + T \subseteq U$
proof
  fix x
  assume $x \in S + T$
  then obtain *xs xt* **where** $xs \in S$ $xt \in T$ $x = xs+xt$
    by (*meson set_plus_elim*)
  then show $x \in U$
    by (*meson assms subsetCE subspace_add*)
qed

proposition *subspace_sum_orthogonal_comp*:
  fixes U :: $'a$ :: *euclidean_space set*
  assumes *subspace* U
  shows $U + U^{\perp} =$ *UNIV*
proof −
  obtain B **where** $B \subseteq U$
    and *ortho*: *pairwise orthogonal* B $\bigwedge x.\ x \in B \Longrightarrow$ *norm* $x = 1$
    and *independent* B *card* $B =$ *dim* U *span* $B = U$
    using *orthonormal_basis_subspace* [*OF assms*] **by** *metis*
  then have *finite* B

by (*simp add*: *indep_card_eq_dim_span*)
have ∗: ∀ *x*∈*B*. ∀ *y*∈*B*. *x* · *y* = (*if x*=*y then 1 else 0*)
using *ortho norm_eq_1* **by** (*auto simp*: *orthogonal_def pairwise_def*)
{ fix *v*
let *?u* = ∑ *b*∈*B*. (*v* · *b*) ∗$_R$ *b*
have *v* = *?u* + (*v* − *?u*)
by *simp*
moreover have *?u* ∈ *U*
by (*metis* (*no_types*, *lifting*) ‹*span B* = *U*› *assms subspace_sum span_base span_mul*)
moreover have (*v* − *?u*) ∈ *U*$^\perp$
proof (*clarsimp simp*: *orthogonal_comp_def orthogonal_def*)
fix *y*
assume *y* ∈ *U*
with ‹*span B* = *U*› *span_finite* [*OF* ‹*finite B*›]
obtain *u* **where** *u*: *y* = (∑ *b*∈*B*. *u b* ∗$_R$ *b*)
by *auto*
have *b* · (*v* − *?u*) = *0* **if** *b* ∈ *B* **for** *b*
using *that* ‹*finite B*›
by (*simp add*: ∗ *algebra_simps inner_sum_right if_distrib* [*of* (∗)*v* **for** *v*] *inner_commute cong*: *if_cong*)
then show *y* · (*v* − *?u*) = *0*
by (*simp add*: *u inner_sum_left*)
qed
ultimately have *v* ∈ *U* + *U*$^\perp$
using *set_plus_intro* **by** *fastforce*
} then show *?thesis*
by *auto*
qed

lemma *orthogonal_Int_0*:
assumes *subspace U*
shows *U* ∩ *U*$^\perp$ = {*0*}
using *orthogonal_comp_def orthogonal_self*
by (*force simp*: *assms subspace_0 subspace_orthogonal_comp*)

lemma *orthogonal_comp_self*:
fixes *U* :: *'a* :: *euclidean_space set*
assumes *subspace U*
shows *U*$^{\perp\perp}$ = *U*
proof
have *ssU'*: *subspace* (*U*$^\perp$)
by (*simp add*: *subspace_orthogonal_comp*)
have *u* ∈ *U* **if** *u* ∈ *U*$^{\perp\perp}$ **for** *u*
proof −
obtain *v w* **where** *u* = *v*+*w v* ∈ *U w* ∈ *U*$^\perp$
using *subspace_sum_orthogonal_comp* [*OF assms*] *set_plus_elim* **by** *blast*
then have *u*−*v* ∈ *U*$^\perp$
by *simp*

```
    moreover have v ∈ U⊥⊥
      using ‹v ∈ U› orthogonal_comp_subset by blast
    then have u−v ∈ U⊥⊥
      by (simp add: subspace_diff subspace_orthogonal_comp that)
    ultimately have u−v = 0
      using orthogonal_Int_0 ssU' by blast
    with ‹v ∈ U› show ?thesis
      by auto
  qed
  then show U⊥⊥ ⊆ U
    by auto
qed (use orthogonal_comp_subset in auto)

lemma ker_orthogonal_comp_adjoint:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes linear f
  shows f −` {0} = (range (adjoint f))⊥
proof −
  have ⋀x. ⟦∀ y. y · f x = 0⟧ ⟹ f x = 0
    using assms inner_commute all_zero_iff by metis
  then show ?thesis
    using assms
   by (auto simp: orthogonal_comp_def orthogonal_def adjoint_works inner_commute)
qed
```

5.0.24 A non-injective linear function maps into a hyperplane.

```
lemma linear_surj_adj_imp_inj:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes linear f surj (adjoint f)
  shows inj f
proof −
  have ∃ x. y = adjoint f x for y
    using assms by (simp add: surjD)
  then show inj f
    using assms unfolding inj_on_def image_def
    by (metis (no_types) adjoint_works euclidean_eqI)
qed
```

— https://mathonline.wikidot.com/injectivity-and-surjectivity-of-the-adjoint-of-a-linear-map

```
lemma surj_adjoint_iff_inj [simp]:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes linear f
  shows  surj (adjoint f) ⟷ inj f
proof
  assume surj (adjoint f)
  then show inj f
    by (simp add: assms linear_surj_adj_imp_inj)
```

next
  assume *inj f*
  have $f -` \{0\} = \{0\}$
    using *assms* ‹*inj f*› *linear_0 linear_injective_0* **by** *fastforce*
  moreover have $f -` \{0\} = range\ (adjoint\ f)^{\perp}$
    by (*intro ker_orthogonal_comp_adjoint assms*)
  ultimately have $range\ (adjoint\ f)^{\perp\perp} = UNIV$
    by (*metis orthogonal_comp_null*)
  then show *surj* (*adjoint f*)
    using *adjoint_linear* ‹*linear f*›
    by (*subst* (*asm*) *orthogonal_comp_self*)
      (*simp add*: *adjoint_linear linear_subspace_image*)
qed

lemma *inj_adjoint_iff_surj* [*simp*]:
  fixes *f* :: *'m*::*euclidean_space* ⇒ *'n*::*euclidean_space*
  assumes *linear f*
  shows *inj* (*adjoint f*) ⟷ *surj f*
proof
  assume *inj* (*adjoint f*)
  have $(adjoint\ f) -` \{0\} = \{0\}$
   by (*metis* ‹*inj* (*adjoint f*)› *adjoint_linear assms surj_adjoint_iff_inj ker_orthogonal_comp_adjoint orthogonal_comp_UNIV*)
  then have $(range(f))^{\perp} = \{0\}$
   by (*metis* (*no_types*, *hide_lams*) *adjoint_adjoint adjoint_linear assms ker_orthogonal_comp_adjoint set_zero*)
  then show *surj f*
   by (*metis* ‹*inj* (*adjoint f*)› *adjoint_adjoint adjoint_linear assms surj_adjoint_iff_inj*)
next
  assume *surj f*
  then have $range\ f = (adjoint\ f -` \{0\})^{\perp}$
   by (*simp add*: *adjoint_adjoint adjoint_linear assms ker_orthogonal_comp_adjoint*)
  then have $\{0\} = adjoint\ f -` \{0\}$
    using ‹*surj f*› *adjoint_adjoint adjoint_linear assms ker_orthogonal_comp_adjoint* **by** *force*
  then show *inj* (*adjoint f*)
   by (*simp add*: ‹*surj f*› *adjoint_adjoint adjoint_linear assms linear_surj_adj_imp_inj*)
qed

lemma *linear_singular_into_hyperplane*:
  fixes *f* :: *'n*::*euclidean_space* ⇒ *'n*
  assumes *linear f*
  shows ¬ *inj f* ⟷ $(\exists a.\ a \neq 0 \wedge (\forall x.\ a \cdot f\ x = 0))$ (**is** _ = *?rhs*)
proof
  assume ¬*inj f*
  then show *?rhs*
    using *all_zero_iff*
    by (*metis* (*no_types*, *hide_lams*) *adjoint_clauses*(*2*) *adjoint_linear assms*
      linear_injective_0 linear_injective_imp_surjective linear_surj_adj_imp_inj)

next
assume *?rhs*
then show $\neg$*inj f*
by (*metis assms linear_injective_isomorphism all_zero_iff*)
qed

lemma *linear_singular_image_hyperplane*:
fixes *f* :: *'n::euclidean_space* $\Rightarrow$ *'n*
assumes *linear f* $\neg$*inj f*
obtains *a* **where** $a \neq 0$ $\bigwedge S.\ f\ `\ S \subseteq \{x.\ a \cdot x = 0\}$
using *assms* **by** (*fastforce simp add*: *linear_singular_into_hyperplane*)

end

5.1 The binary product topology

theory *Product_Topology*
imports *Function_Topology*
begin

5.2 Product Topology

5.2.1 Definition

definition *prod_topology* :: *'a topology* $\Rightarrow$ *'b topology* $\Rightarrow$ (*'a* $\times$ *'b*) *topology* **where**
prod_topology X Y $\equiv$ *topology* (*arbitrary union_of* ($\lambda U.\ U \in \{S \times T\ |S\ T.$ *openin* *X S* $\wedge$ *openin Y T*$\}$))

lemma *open_product_open*:
assumes *open A*
shows $\exists \mathcal{U}.\ \mathcal{U} \subseteq \{S \times T\ |S\ T.$ *open S* $\wedge$ *open T*$\}$ $\wedge \bigcup \mathcal{U} = A$
proof −
obtain *f g* **where** $*$: $\bigwedge u.\ u \in A \Longrightarrow$ *open* (*f u*) $\wedge$ *open* (*g u*) $\wedge$ $u \in (f\ u) \times (g\ u) \wedge (f\ u) \times (g\ u) \subseteq A$
using *open_prod_def* [*of A*] *assms* **by** *metis*
let $?\mathcal{U} = (\lambda u.\ f\ u \times g\ u)\ `\ A$
show *?thesis*
by (*rule_tac x=?*$\mathcal{U}$ **in** *exI*) (*auto simp*: *dest*: $*$)
qed

lemma *open_product_open_eq*: (*arbitrary union_of* ($\lambda U.\ \exists S\ T.\ U = S \times T$ $\wedge$ *open S* $\wedge$ *open T*)) = *open*
by (*force simp*: *union_of_def arbitrary_def intro*: *open_product_open open_Times*)

lemma *openin_prod_topology*:
openin (*prod_topology X Y*) = *arbitrary union_of* ($\lambda U.\ U \in \{S \times T\ |S\ T.$ *openin X S* $\wedge$ *openin Y T*$\}$)
unfolding *prod_topology_def*

proof (*rule topology_inverse'*)
  show *istopology* (*arbitrary union_of* (λU. $U \in \{S \times T$ |*S T*. *openin X S* $\wedge$ *openin Y T*$\}$))
    apply (*rule istopology_base*, *simp*)
    by (*metis openin_Int Times_Int_Times*)
qed

lemma *topspace_prod_topology* [*simp*]:
  topspace (*prod_topology X Y*) = *topspace X* $\times$ *topspace Y*
proof −
  have *topspace*(*prod_topology X Y*) = $\bigcup$ (*Collect* (*openin* (*prod_topology X Y*))) (**is** _ = *?Z*)
    unfolding *topspace_def* **..**
  also have ... = *topspace X* $\times$ *topspace Y*
  proof
    show *?Z* $\subseteq$ *topspace X* $\times$ *topspace Y*
      apply (*auto simp*: *openin_prod_topology union_of_def arbitrary_def*)
      using *openin_subset* **by** *force+*
  next
    have $*$: $\exists$ *A B*. *topspace X* $\times$ *topspace Y* = *A* $\times$ *B* $\wedge$ *openin X A* $\wedge$ *openin Y B*
      by *blast*
    show *topspace X* $\times$ *topspace Y* $\subseteq$ *?Z*
      apply (*rule Union_upper*)
      using $*$ **by** (*simp add*: *openin_prod_topology arbitrary_union_of_inc*)
  qed
  finally show *?thesis* **.**
qed

lemma *subtopology_Times*:
  shows *subtopology* (*prod_topology X Y*) (*S* $\times$ *T*) = *prod_topology* (*subtopology X S*) (*subtopology Y T*)
proof −
  have ((λU. $\exists$ *S T*. *U* = *S* $\times$ *T* $\wedge$ *openin X S* $\wedge$ *openin Y T*) *relative_to S* $\times$ *T*) =
        (λU. $\exists$ *S' T'*. *U* = *S'* $\times$ *T'* $\wedge$ (*openin X relative_to S*) *S'* $\wedge$ (*openin Y relative_to T*) *T'*)
    by (*auto simp*: *relative_to_def Times_Int_Times fun_eq_iff*) *metis*
  then show *?thesis*
    by (*simp add*: *topology_eq openin_prod_topology arbitrary_union_of_relative_to flip*: *openin_relative_to*)
qed

lemma *prod_topology_subtopology*:
    prod_topology (*subtopology X S*) *Y* = *subtopology* (*prod_topology X Y*) (*S* $\times$ *topspace Y*)
    prod_topology X (*subtopology Y T*) = *subtopology* (*prod_topology X Y*) (*topspace X* $\times$ *T*)
by (*auto simp*: *subtopology_Times*)

lemma *prod_topology_discrete_topology*:
discrete_topology (*S* × *T*) = *prod_topology* (*discrete_topology S*) (*discrete_topology T*)
by (*auto simp*: *discrete_topology_unique openin_prod_topology intro*: *arbitrary_union_of_inc*)

lemma *prod_topology_euclidean* [*simp*]: *prod_topology euclidean euclidean* = *euclidean*
by (*simp add*: *prod_topology_def open_product_open_eq*)

lemma *prod_topology_subtopology_eu* [*simp*]:
prod_topology (*subtopology euclidean S*) (*subtopology euclidean T*) = *subtopology euclidean* (*S* × *T*)
by (*simp add*: *prod_topology_subtopology subtopology_subtopology Times_Int_Times*)

lemma *openin_prod_topology_alt*:
openin (*prod_topology X Y*) *S* ⟷
(∀ *x y*. (*x*,*y*) ∈ *S* ⟶ (∃ *U V*. *openin X U* ∧ *openin Y V* ∧ *x* ∈ *U* ∧ *y* ∈ *V* ∧ *U* × *V* ⊆ *S*))
apply (*auto simp*: *openin_prod_topology arbitrary_union_of_alt*, *fastforce*)
by (*metis mem_Sigma_iff*)

lemma *open_map_fst*: *open_map* (*prod_topology X Y*) *X fst*
unfolding *open_map_def openin_prod_topology_alt*
by (*force simp*: *openin_subopen* [*of X fst* ' _] *intro*: *subset_fst_imageI*)

lemma *open_map_snd*: *open_map* (*prod_topology X Y*) *Y snd*
unfolding *open_map_def openin_prod_topology_alt*
by (*force simp*: *openin_subopen* [*of Y snd* ' _] *intro*: *subset_snd_imageI*)

lemma *openin_prod_Times_iff*:
openin (*prod_topology X Y*) (*S* × *T*) ⟷ *S* = {} ∨ *T* = {} ∨ *openin X S* ∧ *openin Y T*
proof (*cases S* = {} ∨ *T* = {})
case *False*
then show *?thesis*
apply (*simp add*: *openin_prod_topology_alt openin_subopen* [*of X S*] *openin_subopen* [*of Y T*] *times_subset_iff*, *safe*)
apply (*meson*|*force*)+
done
qed *force*

lemma *closure_of_Times*:
(*prod_topology X Y*) *closure_of* (*S* × *T*) = (*X closure_of S*) × (*Y closure_of T*)
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
by (*clarsimp simp*: *closure_of_def openin_prod_topology_alt*) *blast*
show *?rhs* ⊆ *?lhs*

by (*clarsimp simp*: *closure_of_def openin_prod_topology_alt*) (*meson SigmaI subsetD*)
qed

lemma *closedin_prod_Times_iff*:
closedin (*prod_topology* X Y) ($S \times T$) $\longleftrightarrow$ $S = \{\} \vee T = \{\} \vee$ *closedin* X S $\wedge$ *closedin* Y T
by (*auto simp*: *closure_of_Times times_eq_iff simp flip*: *closure_of_eq*)

lemma *interior_of_Times*: (*prod_topology* X Y) *interior_of* ($S \times T$) = (X *interior_of* S) $\times$ (Y *interior_of* T)
proof (*rule interior_of_unique*)
show (X *interior_of* S) $\times$ Y *interior_of* $T \subseteq S \times T$
by (*simp add*: *Sigma_mono interior_of_subset*)
show *openin* (*prod_topology* X Y) ((X *interior_of* S) $\times$ Y *interior_of* T)
by (*simp add*: *openin_prod_Times_iff*)
next
show $T' \subseteq$ (X *interior_of* S) $\times$ Y *interior_of* T **if** $T' \subseteq S \times T$ *openin* (*prod_topology* X Y) T' **for** T'
proof (*clarsimp*; *intro conjI*)
fix a :: $'a$ **and** b :: $'b$
assume $(a, b) \in T'$
with *that* **obtain** U V **where** UV: *openin* X U *openin* Y V $a \in U$ $b \in V$ $U \times V \subseteq T'$
by (*metis openin_prod_topology_alt*)
then show $a \in X$ *interior_of* S
using *interior_of_maximal_eq that*(*1*) **by** *fastforce*
show $b \in Y$ *interior_of* T
using UV *interior_of_maximal_eq that*(*1*)
by (*metis SigmaI mem_Sigma_iff subset_eq*)
qed
qed

5.2.2 Continuity

lemma *continuous_map_pairwise*:
continuous_map Z (*prod_topology* X Y) f $\longleftrightarrow$ *continuous_map* Z X ($fst \circ f$) $\wedge$ *continuous_map* Z Y ($snd \circ f$)
(**is** *?lhs* = *?rhs*)
proof −
let $?g = fst \circ f$ **and** $?h = snd \circ f$
have f: $f\ x = (?g\ x,\ ?h\ x)$ **for** x
by *auto*
show *?thesis*
proof (*cases* ($\forall x \in$ *topspace* Z. $?g\ x \in$ *topspace* X) $\wedge$ ($\forall x \in$ *topspace* Z. $?h\ x \in$ *topspace* Y))
case *True*
show *?thesis*
proof *safe*

assume *continuous_map Z (prod_topology X Y) f*
then have *openin Z {x ∈ topspace Z. fst (f x) ∈ U}* **if** *openin X U* **for** *U*
unfolding *continuous_map_def* **using** *True that*
apply *clarify*
apply (*drule_tac x=U × topspace Y* **in** *spec*)
by (*simp add: openin_prod_Times_iff mem_Times_iff cong: conj_cong*)
with *True* **show** *continuous_map Z X (fst ∘ f)*
by (*auto simp: continuous_map_def*)
next
assume *continuous_map Z (prod_topology X Y) f*
then have *openin Z {x ∈ topspace Z. snd (f x) ∈ V}* **if** *openin Y V* **for** *V*
unfolding *continuous_map_def* **using** *True that*
apply *clarify*
apply (*drule_tac x=topspace X × V* **in** *spec*)
by (*simp add: openin_prod_Times_iff mem_Times_iff cong: conj_cong*)
with *True* **show** *continuous_map Z Y (snd ∘ f)*
by (*auto simp: continuous_map_def*)
next
assume *Z: continuous_map Z X (fst ∘ f) continuous_map Z Y (snd ∘ f)*
have *∗: openin Z {x ∈ topspace Z. f x ∈ W}*
if *⋀w. w ∈ W ⟹ ∃ U V. openin X U ∧ openin Y V ∧ w ∈ U × V ∧ U × V ⊆ W* **for** *W*
proof (*subst openin_subopen, clarify*)
fix *x :: 'a*
assume *x ∈ topspace Z* **and** *f x ∈ W*
with *that* [*OF ‹f x ∈ W›*]
obtain *U V* **where** *UV: openin X U openin Y V f x ∈ U × V U × V ⊆ W*
by *auto*
with *Z UV* **show** *∃ T. openin Z T ∧ x ∈ T ∧ T ⊆ {x ∈ topspace Z. f x ∈ W}*
apply (*rule_tac x={x ∈ topspace Z. ?g x ∈ U} ∩ {x ∈ topspace Z. ?h x ∈ V}* **in** *exI*)
apply (*auto simp: ‹x ∈ topspace Z› continuous_map_def*)
done
qed
show *continuous_map Z (prod_topology X Y) f*
using *True* **by** (*simp add: continuous_map_def openin_prod_topology_alt mem_Times_iff ∗*)
qed
qed (*auto simp: continuous_map_def*)
qed

lemma *continuous_map_paired*:
continuous_map Z (prod_topology X Y) (λx. (f x,g x)) ⟷ continuous_map Z X f ∧ continuous_map Z Y g
by (*simp add: continuous_map_pairwise o_def*)

lemma *continuous_map_pairedI* [*continuous_intros*]:

⟦*continuous_map Z X f*; *continuous_map Z Y g*⟧ ⟹ *continuous_map Z* (*prod_topology X Y*) (λ*x*. (*f x*,*g x*))
by (*simp add*: *continuous_map_pairwise o_def*)

lemma *continuous_map_fst* [*continuous_intros*]: *continuous_map* (*prod_topology X Y*) *X fst*
using *continuous_map_pairwise* [*of prod_topology X Y X Y id*]
by (*simp add*: *continuous_map_pairwise*)

lemma *continuous_map_snd* [*continuous_intros*]: *continuous_map* (*prod_topology X Y*) *Y snd*
using *continuous_map_pairwise* [*of prod_topology X Y X Y id*]
by (*simp add*: *continuous_map_pairwise*)

lemma *continuous_map_fst_of* [*continuous_intros*]:
continuous_map Z (*prod_topology X Y*) *f* ⟹ *continuous_map Z X* (*fst* ∘ *f*)
by (*simp add*: *continuous_map_pairwise*)

lemma *continuous_map_snd_of* [*continuous_intros*]:
continuous_map Z (*prod_topology X Y*) *f* ⟹ *continuous_map Z Y* (*snd* ∘ *f*)
by (*simp add*: *continuous_map_pairwise*)

lemma *continuous_map_prod_fst*:
i ∈ *I* ⟹ *continuous_map* (*prod_topology* (*product_topology* (λ*i*. *Y*) *I*) *X*) *Y* (λ*x*. *fst x i*)
using *continuous_map_componentwise_UNIV continuous_map_fst* **by** *fastforce*

lemma *continuous_map_prod_snd*:
i ∈ *I* ⟹ *continuous_map* (*prod_topology X* (*product_topology* (λ*i*. *Y*) *I*)) *Y* (λ*x*. *snd x i*)
using *continuous_map_componentwise_UNIV continuous_map_snd* **by** *fastforce*

lemma *continuous_map_if_iff* [*simp*]: *continuous_map X Y* (λ*x*. *if P then f x else g x*) ⟷ *continuous_map X Y* (*if P then f else g*)
by *simp*

lemma *continuous_map_if* [*continuous_intros*]: ⟦*P* ⟹ *continuous_map X Y f*; ~*P* ⟹ *continuous_map X Y g*⟧
⟹ *continuous_map X Y* (λ*x*. *if P then f x else g x*)
by *simp*

lemma *continuous_map_subtopology_fst* [*continuous_intros*]: *continuous_map* (*subtopology* (*prod_topology X Y*) *Z*) *X fst*
using *continuous_map_from_subtopology continuous_map_fst* **by** *force*

lemma *continuous_map_subtopology_snd* [*continuous_intros*]: *continuous_map* (*subtopology* (*prod_topology X Y*) *Z*) *Y snd*
using *continuous_map_from_subtopology continuous_map_snd* **by** *force*

lemma *quotient_map_fst* [*simp*]:
quotient_map(*prod_topology* *X* *Y*) *X* *fst* ⟷ (*topspace* *Y* = {} ⟶ *topspace* *X* = {})
by (*auto simp*: *continuous_open_quotient_map open_map_fst continuous_map_fst*)

lemma *quotient_map_snd* [*simp*]:
quotient_map(*prod_topology* *X* *Y*) *Y* *snd* ⟷ (*topspace* *X* = {} ⟶ *topspace* *Y* = {})
by (*auto simp*: *continuous_open_quotient_map open_map_snd continuous_map_snd*)

lemma *retraction_map_fst*:
retraction_map (*prod_topology* *X* *Y*) *X* *fst* ⟷ (*topspace* *Y* = {} ⟶ *topspace* *X* = {})
proof (*cases topspace* *Y* = {})
case *True*
then show *?thesis*
using *continuous_map_image_subset_topspace*
by (*fastforce simp*: *retraction_map_def retraction_maps_def continuous_map_fst continuous_map_on_empty*)
next
case *False*
have ∃ *g*. *continuous_map* *X* (*prod_topology* *X* *Y*) *g* ∧ (∀ *x*∈*topspace* *X*. *fst* (*g* *x*) = *x*)
if *y*: *y* ∈ *topspace* *Y* **for** *y*
by (*rule_tac x*=λ*x*. (*x*,*y*) **in** *exI*) (*auto simp*: *y continuous_map_paired*)
with *False* **have** *retraction_map* (*prod_topology* *X* *Y*) *X* *fst*
by (*fastforce simp*: *retraction_map_def retraction_maps_def continuous_map_fst*)
with *False* **show** *?thesis*
by *simp*
qed

lemma *retraction_map_snd*:
retraction_map (*prod_topology* *X* *Y*) *Y* *snd* ⟷ (*topspace* *X* = {} ⟶ *topspace* *Y* = {})
proof (*cases topspace* *X* = {})
case *True*
then show *?thesis*
using *continuous_map_image_subset_topspace*
by (*fastforce simp*: *retraction_map_def retraction_maps_def continuous_map_fst continuous_map_on_empty*)
next
case *False*
have ∃ *g*. *continuous_map* *Y* (*prod_topology* *X* *Y*) *g* ∧ (∀ *y*∈*topspace* *Y*. *snd* (*g* *y*) = *y*)
if *x*: *x* ∈ *topspace* *X* **for** *x*
by (*rule_tac x*=λ*y*. (*x*,*y*) **in** *exI*) (*auto simp*: *x continuous_map_paired*)
with *False* **have** *retraction_map* (*prod_topology* *X* *Y*) *Y* *snd*
by (*fastforce simp*: *retraction_map_def retraction_maps_def continuous_map_snd*)
with *False* **show** *?thesis*

by *simp*
qed

lemma *continuous_map_of_fst*:
continuous_map (*prod_topology* X Y) Z ($f \circ$ *fst*) $\longleftrightarrow$ *topspace* Y = {} $\vee$ *continuous_map* X Z f
proof (*cases topspace* Y = {})
case *True*
then show *?thesis*
by (*simp add*: *continuous_map_on_empty*)
next
case *False*
then show *?thesis*
by (*simp add*: *continuous_compose_quotient_map_eq*)
qed

lemma *continuous_map_of_snd*:
continuous_map (*prod_topology* X Y) Z ($f \circ$ *snd*) $\longleftrightarrow$ *topspace* X = {} $\vee$ *continuous_map* Y Z f
proof (*cases topspace* X = {})
case *True*
then show *?thesis*
by (*simp add*: *continuous_map_on_empty*)
next
case *False*
then show *?thesis*
by (*simp add*: *continuous_compose_quotient_map_eq*)
qed

lemma *continuous_map_prod_top*:
continuous_map (*prod_topology* X Y) (*prod_topology* X' Y') ($\lambda(x,y)$. (f x, g y)) $\longleftrightarrow$
topspace (*prod_topology* X Y) = {} $\vee$ *continuous_map* X X' f $\wedge$ *continuous_map* Y Y' g
proof (*cases topspace* (*prod_topology* X Y) = {})
case *True*
then show *?thesis*
by (*simp add*: *continuous_map_on_empty*)
next
case *False*
then show *?thesis*
by (*simp add*: *continuous_map_paired case_prod_unfold continuous_map_of_fst* [*unfolded o_def*] *continuous_map_of_snd* [*unfolded o_def*])
qed

lemma *in_prod_topology_closure_of*:
assumes $z \in$ (*prod_topology* X Y) *closure_of* S
shows *fst* $z \in X$ *closure_of* (*fst* ‘ S) *snd* $z \in Y$ *closure_of* (*snd* ‘ S)

using *assms continuous_map_eq_image_closure_subset continuous_map_fst* **apply** *fastforce*
using *assms continuous_map_eq_image_closure_subset continuous_map_snd* **apply** *fastforce*
done

proposition *compact_space_prod_topology*:
$compact\_space(prod\_topology\ X\ Y) \longleftrightarrow topspace(prod\_topology\ X\ Y) = \{\} \vee compact\_space\ X \wedge compact\_space\ Y$
proof $(cases\ topspace(prod\_topology\ X\ Y) = \{\})$
case *True*
then show *?thesis*
using *compact_space_topspace_empty* **by** *blast*
next
case *False*
then have *non_mt*: $topspace\ X \neq \{\}\ topspace\ Y \neq \{\}$
by *auto*
have *compact_space X compact_space Y* **if** *compact_space(prod_topology X Y)*
proof −
have $compactin\ X\ (fst\ `\ (topspace\ X \times topspace\ Y))$
by (*metis compact_space_def continuous_map_fst image_compactin that topspace_prod_topology*)
moreover
have $compactin\ Y\ (snd\ `\ (topspace\ X \times topspace\ Y))$
by (*metis compact_space_def continuous_map_snd image_compactin that topspace_prod_topology*)
ultimately show *compact_space X compact_space Y*
by (*simp_all add*: *non_mt compact_space_def*)
qed
moreover
define $\mathcal{X}$ **where** $\mathcal{X} \equiv (\lambda V.\ topspace\ X \times V)\ `\ Collect\ (openin\ Y)$
define $\mathcal{Y}$ **where** $\mathcal{Y} \equiv (\lambda U.\ U \times topspace\ Y)\ `\ Collect\ (openin\ X)$
have *compact_space(prod_topology X Y)* **if** *compact_space X compact_space Y*
proof (*rule Alexander_subbase_alt*)
show *toptop*: $topspace\ X \times topspace\ Y \subseteq \bigcup(\mathcal{X} \cup \mathcal{Y})$
unfolding $\mathcal{X}$*_def* $\mathcal{Y}$*_def* **by** *auto*
fix $\mathcal{C}$ :: $('a \times 'b)\ set\ set$
assume $\mathcal{C}$: $\mathcal{C} \subseteq \mathcal{X} \cup \mathcal{Y}\ topspace\ X \times topspace\ Y \subseteq \bigcup\mathcal{C}$
then obtain $\mathcal{X}'\ \mathcal{Y}'$ **where** *XY*: $\mathcal{X}' \subseteq \mathcal{X}\ \mathcal{Y}' \subseteq \mathcal{Y}$ **and** $\mathcal{C}eq$: $\mathcal{C} = \mathcal{X}' \cup \mathcal{Y}'$
using *subset_UnE* **by** *metis*
then have *sub*: $topspace\ X \times topspace\ Y \subseteq \bigcup(\mathcal{X}' \cup \mathcal{Y}')$
using $\mathcal{C}$ **by** *simp*
obtain $\mathcal{U}\ \mathcal{V}$ **where** $\mathcal{U}$: $\bigwedge U.\ U \in \mathcal{U} \Longrightarrow openin\ X\ U\ \mathcal{Y}' = (\lambda U.\ U \times topspace\ Y)\ `\ \mathcal{U}$
and $\mathcal{V}$: $\bigwedge V.\ V \in \mathcal{V} \Longrightarrow openin\ Y\ V\ \mathcal{X}' = (\lambda V.\ topspace\ X \times V)\ `\ \mathcal{V}$
using *XY* **by** (*clarsimp simp add*: $\mathcal{X}$*_def* $\mathcal{Y}$*_def subset_image_iff*) (*force simp add*: *subset_iff*)
have $\exists\mathcal{D}.\ finite\ \mathcal{D} \wedge \mathcal{D} \subseteq \mathcal{X}' \cup \mathcal{Y}' \wedge topspace\ X \times topspace\ Y \subseteq \bigcup\ \mathcal{D}$
proof −
have $topspace\ X \subseteq \bigcup\mathcal{U} \vee topspace\ Y \subseteq \bigcup\mathcal{V}$

using $\mathcal{U}$ $\mathcal{V}$ $\mathcal{C}$ $\mathcal{C}$*eq* **by** *auto*
then have ∗: ∃ $\mathcal{D}$. *finite* $\mathcal{D}$ ∧
(∀ *x* ∈ $\mathcal{D}$. *x* ∈ (λ*V*. *topspace X* × *V*) ' $\mathcal{V}$ ∨ *x* ∈ (λ*U*. *U* × *topspace Y*) ' $\mathcal{U}$) ∧
(*topspace X* × *topspace Y* ⊆ ⋃$\mathcal{D}$)
proof
assume *topspace X* ⊆ ⋃$\mathcal{U}$
with ‹*compact_space X*› $\mathcal{U}$ **obtain** $\mathcal{F}$ **where** *finite* $\mathcal{F}$ $\mathcal{F}$ ⊆ $\mathcal{U}$ *topspace X* ⊆ ⋃$\mathcal{F}$
by (*meson compact_space_alt*)
with *that* **show** *?thesis*
by (*rule_tac x*=(λ*D*. *D* × *topspace Y*) ' $\mathcal{F}$ **in** *exI*) *auto*
next
assume *topspace Y* ⊆ ⋃$\mathcal{V}$
with ‹*compact_space Y*› $\mathcal{V}$ **obtain** $\mathcal{F}$ **where** *finite* $\mathcal{F}$ $\mathcal{F}$ ⊆ $\mathcal{V}$ *topspace Y* ⊆ ⋃$\mathcal{F}$
by (*meson compact_space_alt*)
with *that* **show** *?thesis*
by (*rule_tac x*=(λ*C*. *topspace X* × *C*) ' $\mathcal{F}$ **in** *exI*) *auto*
qed
then show *?thesis*
using *that* $\mathcal{U}$ $\mathcal{V}$ **by** *blast*
qed
then show ∃ $\mathcal{D}$. *finite* $\mathcal{D}$ ∧ $\mathcal{D}$ ⊆ $\mathcal{C}$ ∧ *topspace X* × *topspace Y* ⊆ ⋃ $\mathcal{D}$
using $\mathcal{C}$ $\mathcal{C}$*eq* **by** *blast*
next
have (*finite intersection_of* (λ*x*. *x* ∈ $\mathcal{X}$ ∨ *x* ∈ $\mathcal{Y}$) *relative_to topspace X* × *topspace Y*)
= (λ*U*. ∃ *S T*. *U* = *S* × *T* ∧ *openin X S* ∧ *openin Y T*)
(**is** *?lhs* = *?rhs*)
proof −
have *?rhs U* **if** *?lhs U* **for** *U*
proof −
have *topspace X* × *topspace Y* ∩ ⋂ *T* ∈ {*A* × *B* |*A B*. *A* ∈ *Collect* (*openin X*) ∧ *B* ∈ *Collect* (*openin Y*)}
if *finite T T* ⊆ $\mathcal{X}$ ∪ $\mathcal{Y}$ **for** *T*
using *that*
proof *induction*
case (*insert B* $\mathcal{B}$)
then show *?case*
unfolding $\mathcal{X}$*_def* $\mathcal{Y}$*_def*
apply (*simp add*: *Int_ac subset_eq image_def*)
apply (*metis* (*no_types*) *openin_Int openin_topspace Times_Int_Times*)
done
qed *auto*
then show *?thesis*
using *that*
by (*auto simp*: *subset_eq elim*!: *relative_toE intersection_ofE*)
qed

```
      moreover
      have ?lhs Z if Z: ?rhs Z for Z
      proof -
        obtain U V where Z = U × V openin X U openin Y V
          using Z by blast
        then have UV: U × V = (topspace X × topspace Y) ∩ (U × V)
          by (simp add: Sigma_mono inf_absorb2 openin_subset)
        moreover
        have ?lhs ((topspace X × topspace Y) ∩ (U × V))
        proof (rule relative_to_inc)
          show (finite intersection_of (λx. x ∈ 𝒳 ∨ x ∈ 𝒴)) (U × V)
            apply (simp add: intersection_of_def 𝒳_def 𝒴_def)
            apply (rule_tac x={(U × topspace Y),(topspace X × V)} in exI)
              using ‹openin X U› ‹openin Y V› openin_subset UV apply (fastforce
simp add:)
            done
        qed
        ultimately show ?thesis
          using ‹Z = U × V› by auto
      qed
      ultimately show ?thesis
        by meson
    qed
    then show topology (arbitrary union_of (finite intersection_of (λx. x ∈ 𝒳 ∪
𝒴)
            relative_to (topspace X × topspace Y))) =
        prod_topology X Y
      by (simp add: prod_topology_def)
  qed
  ultimately show ?thesis
    using False by blast
qed

lemma compactin_Times:
  compactin (prod_topology X Y) (S × T) ⟷ S = {} ∨ T = {} ∨ compactin X
S ∧ compactin Y T
  by (auto simp: compactin_subspace subtopology_Times compact_space_prod_topology)
```

5.2.3 Homeomorphic maps

```
lemma homeomorphic_maps_prod:
  homeomorphic_maps (prod_topology X Y) (prod_topology X' Y') (λ(x,y). (f x, g
y)) (λ(x,y). (f' x, g' y)) ⟷
        topspace(prod_topology X Y) = {} ∧
        topspace(prod_topology X' Y') = {} ∨
        homeomorphic_maps X X' f f' ∧
        homeomorphic_maps Y Y' g g'
  unfolding homeomorphic_maps_def continuous_map_prod_top
  by (auto simp: continuous_map_def homeomorphic_maps_def continuous_map_prod_top)
```

lemma *homeomorphic_maps_swap*:
homeomorphic_maps (*prod_topology X Y*) (*prod_topology Y X*)
(λ(*x*,*y*). (*y*,*x*)) (λ(*y*,*x*). (*x*,*y*))
by (*auto simp*: *homeomorphic_maps_def case_prod_unfold continuous_map_fst continuous_map_pairedI continuous_map_snd*)

lemma *homeomorphic_map_swap*:
homeomorphic_map (*prod_topology X Y*) (*prod_topology Y X*) (λ(*x*,*y*). (*y*,*x*))
using *homeomorphic_map_maps homeomorphic_maps_swap* **by** *metis*

lemma *embedding_map_graph*:
embedding_map X (*prod_topology X Y*) (λ*x*. (*x*, *f x*)) ⟷ *continuous_map X Y f*
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
have *snd* ∘ (λ*x*. (*x*, *f x*)) = *f*
by *force*
moreover have *continuous_map X Y* (*snd* ∘ (λ*x*. (*x*, *f x*)))
using *L*
unfolding *embedding_map_def*
by (*meson continuous_map_in_subtopology continuous_map_snd_of homeomorphic_imp_continuous_map*)
ultimately show *?rhs*
by *simp*
next
assume *R*: *?rhs*
then show *?lhs*
unfolding *homeomorphic_map_maps embedding_map_def homeomorphic_maps_def*
by (*rule_tac x*=*fst* **in** *exI*)
(*auto simp*: *continuous_map_in_subtopology continuous_map_paired continuous_map_from_subtopology*
continuous_map_fst)
qed

lemma *homeomorphic_space_prod_topology*:
⟦*X homeomorphic_space X''*; *Y homeomorphic_space Y'*⟧
⟹ *prod_topology X Y homeomorphic_space prod_topology X'' Y'*
using *homeomorphic_maps_prod* **unfolding** *homeomorphic_space_def* **by** *blast*

lemma *prod_topology_homeomorphic_space_left*:
topspace Y = {*b*} ⟹ *prod_topology X Y homeomorphic_space X*
unfolding *homeomorphic_space_def*
by (*rule_tac x*=*fst* **in** *exI*) (*simp add*: *homeomorphic_map_def inj_on_def flip*: *homeomorphic_map_maps*)

lemma *prod_topology_homeomorphic_space_right*:
topspace X = {*a*} ⟹ *prod_topology X Y homeomorphic_space Y*

unfolding *homeomorphic_space_def*
by (*rule_tac x=snd* **in** *exI*) (*simp add*: *homeomorphic_map_def inj_on_def flip*: *homeomorphic_map_maps*)

lemma *homeomorphic_space_prod_topology_sing1*:
$b \in$ *topspace* $Y \Longrightarrow X$ *homeomorphic_space* (*prod_topology* X (*subtopology* Y $\{b\}$))
by (*metis empty_subsetI homeomorphic_space_sym inf.absorb_iff2 insert_subset prod_topology_homeomorphic_space_left topspace_subtopology*)

lemma *homeomorphic_space_prod_topology_sing2*:
$a \in$ *topspace* $X \Longrightarrow Y$ *homeomorphic_space* (*prod_topology* (*subtopology* X $\{a\}$) Y)
by (*metis empty_subsetI homeomorphic_space_sym inf.absorb_iff2 insert_subset prod_topology_homeomorphic_space_right topspace_subtopology*)

lemma *topological_property_of_prod_component*:
assumes *major*: P(*prod_topology* X Y)
and X: $\bigwedge x$. $[\![x \in$ *topspace* X; P(*prod_topology* X Y)$]\!] \Longrightarrow P$(*subtopology* (*prod_topology* X Y) ($\{x\} \times$ *topspace* Y))
and Y: $\bigwedge y$. $[\![y \in$ *topspace* Y; P(*prod_topology* X Y)$]\!] \Longrightarrow P$(*subtopology* (*prod_topology* X Y) (*topspace* $X \times \{y\}$))
and *PQ*: $\bigwedge X$ X'. X *homeomorphic_space* $X' \Longrightarrow (P\ X \longleftrightarrow Q\ X')$
and *PR*: $\bigwedge X$ X'. X *homeomorphic_space* $X' \Longrightarrow (P\ X \longleftrightarrow R\ X')$
shows *topspace*(*prod_topology* X Y) $= \{\} \vee Q\ X \wedge R\ Y$
proof −
have $Q\ X \wedge R\ Y$ **if** *topspace*(*prod_topology* X Y) $\neq \{\}$
proof −
from *that* **obtain** *a b* **where** *a*: $a \in$ *topspace* X **and** *b*: $b \in$ *topspace* Y
by *force*
show *?thesis*
using X [*OF a major*] **and** Y [*OF b major*] *homeomorphic_space_prod_topology_sing1* [*OF b*, *of X*] *homeomorphic_space_prod_topology_sing2* [*OF a*, *of Y*]
by (*simp add*: *subtopology_Times*) (*meson PQ PR homeomorphic_space_prod_topology_sing2 homeomorphic_space_sym*)
qed
then show *?thesis* **by** *metis*
qed

lemma *limitin_pairwise*:
limitin (*prod_topology* X Y) f l $F \longleftrightarrow$ *limitin* X (*fst* $\circ f$) (*fst* l) $F \wedge$ *limitin* Y (*snd* $\circ f$) (*snd* l) F
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then obtain *a b* **where** *ev*: $\bigwedge U$. $[\![(a,b) \in U$; *openin* (*prod_topology* X Y) $U]\!] \Longrightarrow \forall_F\ x$ *in* F. $f\ x \in U$
and *a*: $a \in$ *topspace* X **and** *b*: $b \in$ *topspace* Y **and** *l*: $l = (a,b)$

```
    by (auto simp: limitin_def)
  moreover have ∀F x in F. fst (f x) ∈ U if openin X U a ∈ U for U
  proof −
    have ∀F c in F. f c ∈ U × topspace Y
      using b that ev [of U × topspace Y] by (auto simp: openin_prod_topology_alt)
    then show ?thesis
      by (rule eventually_mono) (metis (mono_tags, lifting) SigmaE2 prod.collapse)
  qed
  moreover have ∀F x in F. snd (f x) ∈ U if openin Y U b ∈ U for U
  proof −
    have ∀F c in F. f c ∈ topspace X × U
      using a that ev [of topspace X × U] by (auto simp: openin_prod_topology_alt)
    then show ?thesis
      by (rule eventually_mono) (metis (mono_tags, lifting) SigmaE2 prod.collapse)
  qed
  ultimately show ?rhs
    by (simp add: limitin_def)
next
  have limitin (prod_topology X Y) f (a,b) F
    if limitin X (fst ∘ f) a F limitin Y (snd ∘ f) b F for a b
    using that
  proof (clarsimp simp: limitin_def)
    fix Z :: ('a × 'b) set
    assume a: a ∈ topspace X ∀U. openin X U ∧ a ∈ U ⟶ (∀F x in F. fst (f
x) ∈ U)
      and b: b ∈ topspace Y ∀U. openin Y U ∧ b ∈ U ⟶ (∀F x in F. snd (f x)
∈ U)
      and Z: openin (prod_topology X Y) Z (a, b) ∈ Z
    then obtain U V where openin X U openin Y V a ∈ U b ∈ V U × V ⊆ Z
      using Z by (force simp: openin_prod_topology_alt)
    then have ∀F x in F. fst (f x) ∈ U ∀F x in F. snd (f x) ∈ V
      by (simp_all add: a b)
    then show ∀F x in F. f x ∈ Z
      by (rule eventually_elim2) (use ‹U × V ⊆ Z› subsetD in auto)
  qed
  then show ?rhs ⟹ ?lhs
    by (metis prod.collapse)
qed

end
```

5.3 T1 and Hausdorff spaces

```
theory T1_Spaces
imports Product_Topology
begin
```

5.4 T1 spaces with equivalences to many naturally "nice" properties.

definition *t1_space* **where**
t1_space $X \equiv \forall x \in topspace\ X.\ \forall y \in topspace\ X.\ x{\neq}y \longrightarrow (\exists U.\ openin\ X\ U \wedge x \in U \wedge y \notin U)$

lemma *t1_space_expansive*:
$[\![topspace\ Y = topspace\ X; \bigwedge U.\ openin\ X\ U \Longrightarrow openin\ Y\ U]\!] \Longrightarrow t1\_space\ X \Longrightarrow t1\_space\ Y$
by (*metis t1_space_def*)

lemma *t1_space_alt*:
$t1\_space\ X \longleftrightarrow (\forall x \in topspace\ X.\ \forall y \in topspace\ X.\ x{\neq}y \longrightarrow (\exists U.\ closedin\ X\ U \wedge x \in U \wedge y \notin U))$
by (*metis DiffE DiffI closedin_def openin_closedin_eq t1_space_def*)

lemma *t1_space_empty*: $topspace\ X = \{\} \Longrightarrow t1\_space\ X$
by (*simp add*: *t1_space_def*)

lemma *t1_space_derived_set_of_singleton*:
$t1\_space\ X \longleftrightarrow (\forall x \in topspace\ X.\ X\ derived\_set\_of\ \{x\} = \{\})$
apply (*simp add*: *t1_space_def derived_set_of_def*, *safe*)
apply (*metis openin_topspace*)
by *force*

lemma *t1_space_derived_set_of_finite*:
$t1\_space\ X \longleftrightarrow (\forall S.\ finite\ S \longrightarrow X\ derived\_set\_of\ S = \{\})$
proof (*intro iffI allI impI*)
fix $S :: 'a\ set$
assume *finite S*
then have *fin*: $finite\ ((\lambda x.\ \{x\})\ `\ (topspace\ X \cap S))$
by *blast*
assume *t1_space X*
then have $X\ derived\_set\_of\ (\bigcup x \in topspace\ X \cap S.\ \{x\}) = \{\}$
unfolding *derived_set_of_Union* [*OF fin*]
by (*auto simp*: *t1_space_derived_set_of_singleton*)
then have $X\ derived\_set\_of\ (topspace\ X \cap S) = \{\}$
by *simp*
then show $X\ derived\_set\_of\ S = \{\}$
by *simp*
qed (*auto simp*: *t1_space_derived_set_of_singleton*)

lemma *t1_space_closedin_singleton*:
$t1\_space\ X \longleftrightarrow (\forall x \in topspace\ X.\ closedin\ X\ \{x\})$
apply (*rule iffI*)
apply (*simp add*: *closedin_contains_derived_set t1_space_derived_set_of_singleton*)
using *t1_space_alt* **by** *auto*

lemma *closedin_t1_singleton*:
  $[\![t1\_space\ X;\ a \in topspace\ X]\!] \Longrightarrow closedin\ X\ \{a\}$
  by (*simp add*: *t1_space_closedin_singleton*)

lemma *t1_space_closedin_finite*:
  $t1\_space\ X \longleftrightarrow (\forall S.\ finite\ S \wedge S \subseteq topspace\ X \longrightarrow closedin\ X\ S)$
  apply (*rule iffI*)
  apply (*simp add*: *closedin_contains_derived_set t1_space_derived_set_of_finite*)
  by (*simp add*: *t1_space_closedin_singleton*)

lemma *closure_of_singleton*:
  $t1\_space\ X \Longrightarrow X\ closure\_of\ \{a\} = (if\ a \in topspace\ X\ then\ \{a\}\ else\ \{\})$
  by (*simp add*: *closure_of_eq t1_space_closedin_singleton closure_of_eq_empty_gen*)

lemma *separated_in_singleton*:
  assumes *t1_space X*
  shows $separatedin\ X\ \{a\}\ S \longleftrightarrow a \in topspace\ X \wedge S \subseteq topspace\ X \wedge (a \notin X\ closure\_of\ S)$
    $separatedin\ X\ S\ \{a\} \longleftrightarrow a \in topspace\ X \wedge S \subseteq topspace\ X \wedge (a \notin X\ closure\_of\ S)$
  unfolding *separatedin_def*
  using *assms closure_of closure_of_singleton* **by** *fastforce*+

lemma *t1_space_openin_delete*:
  $t1\_space\ X \longleftrightarrow (\forall U\ x.\ openin\ X\ U \wedge x \in U \longrightarrow openin\ X\ (U - \{x\}))$
  apply (*rule iffI*)
  apply (*meson closedin_t1_singleton in_mono openin_diff openin_subset*)
  by (*simp add*: *closedin_def t1_space_closedin_singleton*)

lemma *t1_space_openin_delete_alt*:
  $t1\_space\ X \longleftrightarrow (\forall U\ x.\ openin\ X\ U \longrightarrow openin\ X\ (U - \{x\}))$
  by (*metis Diff_empty Diff_insert0 t1_space_openin_delete*)

lemma *t1_space_singleton_Inter_open*:
  $t1\_space\ X \longleftrightarrow (\forall x \in topspace\ X.\ \bigcap\{U.\ openin\ X\ U \wedge x \in U\} = \{x\})$ (**is** *?P=?Q*)
  and *t1_space_Inter_open_supersets*:
  $t1\_space\ X \longleftrightarrow (\forall S.\ S \subseteq topspace\ X \longrightarrow \bigcap\{U.\ openin\ X\ U \wedge S \subseteq U\} = S)$ (**is** *?P=?R*)
proof −
  have *?R* ⟹ *?Q*
    apply *clarify*
    apply (*drule_tac x*={*x*} **in** *spec*, *simp*)
    done
  moreover have *?Q* ⟹ *?P*
    apply (*clarsimp simp add*: *t1_space_def*)
    apply (*drule_tac x*=*x* **in** *bspec*)
     apply (*simp_all add*: *set_eq_iff*)

by (*metis* (*no_types*, *lifting*))
moreover have *?P* ⟹ *?R*
proof (*clarsimp simp add*: *t1_space_closedin_singleton*, *rule subset_antisym*)
fix *S*
assume *S*: ∀ *x*∈*topspace X*. *closedin X* {*x*} *S* ⊆ *topspace X*
then show ⋂ {*U*. *openin X U* ∧ *S* ⊆ *U*} ⊆ *S*
apply *clarsimp*
by (*metis Diff_insert_absorb Set.set_insert closedin_def openin_topspace subset_insert*)
qed *force*
ultimately show *?P*=*?Q* *?P*=*?R*
by *auto*
qed

lemma *t1_space_derived_set_of_infinite_openin*:
t1_space X ⟷
(∀ *S*. *X derived_set_of S* =
{*x* ∈ *topspace X*. ∀ *U*. *x* ∈ *U* ∧ *openin X U* ⟶ *infinite*(*S* ∩ *U*)})
(**is** _ = *?rhs*)
proof
assume *t1_space X*
show *?rhs*
proof *safe*
fix *S x U*
assume *x* ∈ *X derived_set_of S x* ∈ *U openin X U finite* (*S* ∩ *U*)
with ‹*t1_space X*› **show** *False*
apply (*simp add*: *t1_space_derived_set_of_finite*)
by (*metis IntI empty_iff empty_subsetI inf_commute openin_Int_derived_set_of_subset subset_antisym*)
next
fix *S x*
have *eq*: (∃ *y*. (*y* ≠ *x*) ∧ *y* ∈ *S* ∧ *y* ∈ *T*) ⟷ ~((*S* ∩ *T*) ⊆ {*x*}) **for** *x S T*
by *blast*
assume *x* ∈ *topspace X* ∀ *U*. *x* ∈ *U* ∧ *openin X U* ⟶ *infinite* (*S* ∩ *U*)
then show *x* ∈ *X derived_set_of S*
apply (*clarsimp simp add*: *derived_set_of_def eq*)
by (*meson finite.emptyI finite.insertI finite_subset*)
qed (*auto simp*: *in_derived_set_of*)
qed (*auto simp*: *t1_space_derived_set_of_singleton*)

lemma *finite_t1_space_imp_discrete_topology*:
⟦*topspace X* = *U*; *finite U*; *t1_space X*⟧ ⟹ *X* = *discrete_topology U*
by (*metis discrete_topology_unique_derived_set t1_space_derived_set_of_finite*)

lemma *t1_space_subtopology*: *t1_space X* ⟹ *t1_space*(*subtopology X U*)
by (*simp add*: *derived_set_of_subtopology t1_space_derived_set_of_finite*)

lemma *closedin_derived_set_of_gen*:
t1_space X ⟹ *closedin X* (*X derived_set_of S*)

apply (*clarsimp simp add*: *in_derived_set_of closedin_contains_derived_set derived_set_of_subset_topspace*)
by (*metis DiffD2 insert_Diff insert_iff t1_space_openin_delete*)

lemma *derived_set_of_derived_set_subset_gen*:
t1_space X ⟹ *X derived_set_of* (*X derived_set_of S*) ⊆ *X derived_set_of S*
by (*meson closedin_contains_derived_set closedin_derived_set_of_gen*)

lemma *subtopology_eq_discrete_topology_gen_finite*:
⟦*t1_space X*; *finite S*⟧ ⟹ *subtopology X S* = *discrete_topology*(*topspace X* ∩ *S*)
by (*simp add*: *subtopology_eq_discrete_topology_gen t1_space_derived_set_of_finite*)

lemma *subtopology_eq_discrete_topology_finite*:
⟦*t1_space X*; *S* ⊆ *topspace X*; *finite S*⟧
⟹ *subtopology X S* = *discrete_topology S*
by (*simp add*: *subtopology_eq_discrete_topology_eq t1_space_derived_set_of_finite*)

lemma *t1_space_closed_map_image*:
⟦*closed_map X Y f*; *f* ' (*topspace X*) = *topspace Y*; *t1_space X*⟧ ⟹ *t1_space Y*
by (*metis closed_map_def finite_subset_image t1_space_closedin_finite*)

lemma *homeomorphic_t1_space*: *X homeomorphic_space Y* ⟹ (*t1_space X* ⟷ *t1_space Y*)
apply (*clarsimp simp add*: *homeomorphic_space_def*)
by (*meson homeomorphic_eq_everything_map homeomorphic_maps_map t1_space_closed_map_image*)

proposition *t1_space_product_topology*:
t1_space (*product_topology X I*)
⟷ *topspace*(*product_topology X I*) = {} ∨ (∀ *i* ∈ *I*. *t1_space* (*X i*))
proof (*cases topspace*(*product_topology X I*) = {})
case *True*
then show *?thesis*
using *True t1_space_empty* **by** *blast*
next
case *False*
then obtain *f* **where** *f*: *f* ∈ (Π_E *i*∈*I*. *topspace*(*X i*))
by *fastforce*
have *t1_space* (*product_topology X I*) ⟷ (∀ *i*∈*I*. *t1_space* (*X i*))
proof (*intro iffI ballI*)
show *t1_space* (*X i*) **if** *t1_space* (*product_topology X I*) **and** *i* ∈ *I* **for** *i*
proof −
have *clo*: ⋀*h*. *h* ∈ (Π_E *i*∈*I*. *topspace* (*X i*)) ⟹ *closedin* (*product_topology X I*) {*h*}
using *that* **by** (*simp add*: *t1_space_closedin_singleton*)
show *?thesis*
unfolding *t1_space_closedin_singleton*
proof *clarify*
show *closedin* (*X i*) {*xi*} **if** *xi* ∈ *topspace* (*X i*) **for** *xi*
using *clo* [*of* λ*j* ∈ *I*. *if i*=*j then xi else f j*] *f that* ‹*i* ∈ *I*›

by (*fastforce simp add*: *closedin_product_topology_singleton*)
qed
qed
next
next
show *t1_space* (*product_topology X I*) **if** ∀ *i*∈*I*. *t1_space* (*X i*)
using *that*
by (*simp add*: *t1_space_closedin_singleton Ball_def PiE_iff closedin_product_topology_singleton*)
qed
then show *?thesis*
using *False* **by** *blast*
qed

lemma *t1_space_prod_topology*:
t1_space(*prod_topology X Y*) ⟷ *topspace*(*prod_topology X Y*) = {} ∨ *t1_space X* ∧ *t1_space Y*
proof (*cases topspace* (*prod_topology X Y*) = {})
case *True* **then show** *?thesis*
by (*auto simp*: *t1_space_empty*)
next
case *False*
have *eq*: {(*x*,*y*)} = {*x*} × {*y*} **for** *x y*
by *simp*
have *t1_space* (*prod_topology X Y*) ⟷ (*t1_space X* ∧ *t1_space Y*)
using *False*
by (*force simp*: *t1_space_closedin_singleton closedin_prod_Times_iff eq simp del*: *insert_Times_insert*)
with *False* **show** *?thesis*
by *simp*
qed

5.4.1 Hausdorff Spaces

definition *Hausdorff_space*
where
Hausdorff_space X ≡
∀ *x y*. *x* ∈ *topspace X* ∧ *y* ∈ *topspace X* ∧ (*x* ≠ *y*)
⟶ (∃ *U V*. *openin X U* ∧ *openin X V* ∧ *x* ∈ *U* ∧ *y* ∈ *V* ∧ *disjnt U V*)

lemma *Hausdorff_space_expansive*:
⟦*Hausdorff_space X*; *topspace X* = *topspace Y*; ⋀*U*. *openin X U* ⟹ *openin Y U*⟧ ⟹ *Hausdorff_space Y*
by (*metis Hausdorff_space_def*)

lemma *Hausdorff_space_topspace_empty*:
topspace X = {} ⟹ *Hausdorff_space X*
by (*simp add*: *Hausdorff_space_def*)

lemma *Hausdorff_imp_t1_space*:
  Hausdorff_space X $\Longrightarrow$ *t1_space X*
  by (*metis Hausdorff_space_def disjnt_iff t1_space_def*)

lemma *closedin_derived_set_of*:
  Hausdorff_space X $\Longrightarrow$ *closedin X* (*X derived_set_of S*)
  by (*simp add*: *Hausdorff_imp_t1_space closedin_derived_set_of_gen*)

lemma *t1_or_Hausdorff_space*:
  t1_space X $\vee$ *Hausdorff_space X* $\longleftrightarrow$ *t1_space X*
  using *Hausdorff_imp_t1_space* **by** *blast*

lemma *Hausdorff_space_sing_Inter_opens*:
  $[\![$*Hausdorff_space X*; $a \in$ *topspace X*$]\!]$ $\Longrightarrow$ $\bigcap\{u.$ *openin X u* $\wedge\ a \in u\} = \{a\}$
  using *Hausdorff_imp_t1_space t1_space_singleton_Inter_open* **by** *force*

lemma *Hausdorff_space_subtopology*:
  assumes *Hausdorff_space X* **shows** *Hausdorff_space*(*subtopology X S*)
proof −
  have ∗: *disjnt U V* $\Longrightarrow$ *disjnt* ($S \cap U$) ($S \cap V$) **for** *U V*
    by (*simp add*: *disjnt_iff*)
  from *assms* **show** *?thesis*
    apply (*simp add*: *Hausdorff_space_def openin_subtopology_alt*)
    apply (*fast intro*: ∗ *elim*!: *all_forward*)
    done
qed

lemma *Hausdorff_space_compact_separation*:
  assumes *X*: *Hausdorff_space X* **and** *S*: *compactin X S* **and** *T*: *compactin X T*
and *disjnt S T*
  obtains *U V* **where** *openin X U openin X V* $S \subseteq U$ $T \subseteq V$ *disjnt U V*
proof (*cases* $S = \{\}$)
  case *True*
  then show *thesis*
   by (*metis* ‹*compactin X T*› *compactin_subset_topspace disjnt_empty1 empty_subsetI*
openin_empty openin_topspace that)
next
  case *False*
  have $\forall x \in S.\ \exists U\ V.$ *openin X U* $\wedge$ *openin X V* $\wedge\ x \in U \wedge T \subseteq V \wedge$ *disjnt U*
V
  proof
    fix *a*
    assume $a \in S$
    then have $a \notin T$
      by (*meson assms*(*4*) *disjnt_iff*)
    have *a*: $a \in$ *topspace X*
      using *S* ‹$a \in S$› *compactin_subset_topspace* **by** *blast*
    show $\exists U\ V.$ *openin X U* $\wedge$ *openin X V* $\wedge\ a \in U \wedge T \subseteq V \wedge$ *disjnt U V*
    proof (*cases* $T = \{\}$)

```
      case True
      then show ?thesis
        using a disjnt_empty2 openin_empty by blast
    next
      case False
      have ∀x ∈ topspace X − {a}. ∃U V. openin X U ∧ openin X V ∧ x ∈ U
∧ a ∈ V ∧ disjnt U V
        using X a by (simp add: Hausdorff_space_def)
      then obtain U V where UV: ∀x ∈ topspace X − {a}. openin X (U x) ∧
openin X (V x) ∧ x ∈ U x ∧ a ∈ V x ∧ disjnt (U x) (V x)
        by metis
      with ‹a ∉ T› compactin_subset_topspace [OF T]
      have Topen: ∀W ∈ U ` T. openin X W and Tsub: T ⊆ ⋃ (U ` T)
        by (auto simp: )
      then obtain ℱ where ℱ: finite ℱ ℱ ⊆ U ` T and T ⊆ ⋃ℱ
        using T unfolding compactin_def by meson
      then obtain F where F: finite F F ⊆ T ℱ = U ` F and SUF: T ⊆ ⋃(U
` F) and a ∉ F
        using finite_subset_image [OF ℱ] ‹a ∉ T› by (metis subsetD)
      have U: ⋀x. ⟦x ∈ topspace X; x ≠ a⟧ ⟹ openin X (U x)
        and V: ⋀x. ⟦x ∈ topspace X; x ≠ a⟧ ⟹ openin X (V x)
        and disj: ⋀x. ⟦x ∈ topspace X; x ≠ a⟧ ⟹ disjnt (U x) (V x)
        using UV by blast+
      show ?thesis
      proof (intro exI conjI)
        have F ≠ {}
          using False SUF by blast
        with ‹a ∉ F› show openin X (⋂(V ` F))
          using F compactin_subset_topspace [OF T] by (force intro: V)
        show openin X (⋃(U ` F))
          using F Topen Tsub by (force intro: U)
        show disjnt (⋂(V ` F)) (⋃(U ` F))
          using disj
          apply (auto simp: disjnt_def)
          using ‹F ⊆ T› ‹a ∉ F› compactin_subset_topspace [OF T] by blast
        show a ∈ (⋂(V ` F))
          using ‹F ⊆ T› T UV ‹a ∉ T› compactin_subset_topspace by blast
      qed (auto simp: SUF)
    qed
  qed
  then obtain U V where UV: ∀x ∈ S. openin X (U x) ∧ openin X (V x) ∧ x
∈ U x ∧ T ⊆ V x ∧ disjnt (U x) (V x)
    by metis
  then have S ⊆ ⋃ (U ` S)
    by auto
  moreover have ∀W ∈ U ` S. openin X W
    using UV by blast
  ultimately obtain I where I: S ⊆ ⋃ (U ` I) I ⊆ S finite I
    by (metis S compactin_def finite_subset_image)
```

show *thesis*
proof
show *openin X* ($\bigcup$(*U* ‘ *I*))
using ‹*I* ⊆ *S*› *UV* **by** *blast*
show *openin X* ($\bigcap$ (*V* ‘ *I*))
using *False UV* ‹*I* ⊆ *S*› ‹*S* ⊆ $\bigcup$ (*U* ‘ *I*)› ‹*finite I*› **by** *blast*
show *disjnt* ($\bigcup$(*U* ‘ *I*)) ($\bigcap$ (*V* ‘ *I*))
by *simp* (*meson UV* ‹*I* ⊆ *S*› *disjnt_subset2 in_mono le_INF_iff order_refl*)
qed (*use UV I* **in** *auto*)
qed

lemma *Hausdorff_space_compact_sets*:
Hausdorff_space X ⟷
(∀ *S T*. *compactin X S* ∧ *compactin X T* ∧ *disjnt S T*
⟶ (∃ *U V*. *openin X U* ∧ *openin X V* ∧ *S* ⊆ *U* ∧ *T* ⊆ *V* ∧ *disjnt U V*))
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*meson Hausdorff_space_compact_separation*)
next
assume *R* [*rule_format*]: *?rhs*
show *?lhs*
proof (*clarsimp simp add*: *Hausdorff_space_def*)
fix *x y*
assume *x* ∈ *topspace X y* ∈ *topspace X x* ≠ *y*
then show ∃ *U*. *openin X U* ∧ (∃ *V*. *openin X V* ∧ *x* ∈ *U* ∧ *y* ∈ *V* ∧ *disjnt U V*)
using *R* [*of* {*x*} {*y*}] **by** *auto*
qed
qed

lemma *compactin_imp_closedin*:
assumes *X*: *Hausdorff_space X* **and** *S*: *compactin X S* **shows** *closedin X S*
proof −
have *S* ⊆ *topspace X*
by (*simp add*: *assms compactin_subset_topspace*)
moreover
have ∃ *T*. *openin X T* ∧ *x* ∈ *T* ∧ *T* ⊆ *topspace X* − *S* **if** *x* ∈ *topspace X x* ∉ *S* **for** *x*
using *Hausdorff_space_compact_separation* [*OF X _ S*, *of* {*x*}] *that*
apply (*simp add*: *disjnt_def*)
by (*metis Diff_mono Diff_triv openin_subset*)
ultimately show *?thesis*
using *closedin_def openin_subopen* **by** *force*
qed

lemma *closedin_Hausdorff_singleton*:
⟦*Hausdorff_space X*; $x \in$ *topspace X*⟧ $\Longrightarrow$ *closedin X* $\{x\}$
by (*simp add*: *Hausdorff_imp_t1_space closedin_t1_singleton*)

lemma *closedin_Hausdorff_sing_eq*:
Hausdorff_space X $\Longrightarrow$ *closedin X* $\{x\} \longleftrightarrow x \in$ *topspace X*
by (*meson closedin_Hausdorff_singleton closedin_subset insert_subset*)

lemma *Hausdorff_space_discrete_topology* [*simp*]:
Hausdorff_space (*discrete_topology U*)
unfolding *Hausdorff_space_def*
apply *safe*
by (*metis discrete_topology_unique_alt disjnt_empty2 disjnt_insert2 insert_iff mk_disjoint_insert topspace_discrete_topology*)

lemma *compactin_Int*:
⟦*Hausdorff_space X*; *compactin X S*; *compactin X T*⟧ $\Longrightarrow$ *compactin X* $(S \cap T)$
by (*simp add*: *closed_Int_compactin compactin_imp_closedin*)

lemma *finite_topspace_imp_discrete_topology*:
⟦*topspace X* $= U$; *finite U*; *Hausdorff_space X*⟧ $\Longrightarrow X =$ *discrete_topology U*
using *Hausdorff_imp_t1_space finite_t1_space_imp_discrete_topology* **by** *blast*

lemma *derived_set_of_finite*:
⟦*Hausdorff_space X*; *finite S*⟧ $\Longrightarrow$ *X derived_set_of S* $= \{\}$
using *Hausdorff_imp_t1_space t1_space_derived_set_of_finite* **by** *auto*

lemma *derived_set_of_singleton*:
Hausdorff_space X $\Longrightarrow$ *X derived_set_of* $\{x\} = \{\}$
by (*simp add*: *derived_set_of_finite*)

lemma *closedin_Hausdorff_finite*:
⟦*Hausdorff_space X*; $S \subseteq$ *topspace X*; *finite S*⟧ $\Longrightarrow$ *closedin X S*
by (*simp add*: *compactin_imp_closedin finite_imp_compactin_eq*)

lemma *open_in_Hausdorff_delete*:
⟦*Hausdorff_space X*; *openin X S*⟧ $\Longrightarrow$ *openin X* $(S - \{x\})$
using *Hausdorff_imp_t1_space t1_space_openin_delete_alt* **by** *auto*

lemma *closedin_Hausdorff_finite_eq*:
⟦*Hausdorff_space X*; *finite S*⟧ $\Longrightarrow$ *closedin X S* $\longleftrightarrow S \subseteq$ *topspace X*
by (*meson closedin_Hausdorff_finite closedin_def*)

lemma *derived_set_of_infinite_openin*:
Hausdorff_space X
$\Longrightarrow$ *X derived_set_of S* $=$
$\{x \in$ *topspace X*. $\forall U.\ x \in U \land$ *openin X U* $\longrightarrow$ *infinite*$(S \cap U)\}$
using *Hausdorff_imp_t1_space t1_space_derived_set_of_infinite_openin* **by** *fastforce*

lemma *Hausdorff_space_discrete_compactin*:
Hausdorff_space X
$\Longrightarrow$ *S* ∩ *X derived_set_of S* = {} ∧ *compactin X S* ⟷ *S* ⊆ *topspace X* ∧ *finite S*
using *derived_set_of_finite discrete_compactin_eq_finite* **by** *fastforce*

lemma *Hausdorff_space_finite_topspace*:
Hausdorff_space X $\Longrightarrow$ *X derived_set_of* (*topspace X*) = {} ∧ *compact_space X* ⟷ *finite*(*topspace X*)
using *derived_set_of_finite discrete_compact_space_eq_finite* **by** *auto*

lemma *derived_set_of_derived_set_subset*:
Hausdorff_space X $\Longrightarrow$ *X derived_set_of* (*X derived_set_of S*) ⊆ *X derived_set_of S*
by (*simp add*: *Hausdorff_imp_t1_space derived_set_of_derived_set_subset_gen*)

lemma *Hausdorff_space_injective_preimage*:
assumes *Hausdorff_space Y* **and** *cmf*: *continuous_map X Y f* **and** *inj_on f* (*topspace X*)
shows *Hausdorff_space X*
unfolding *Hausdorff_space_def*
proof *clarify*
fix *x y*
assume *x*: *x* ∈ *topspace X* **and** *y*: *y* ∈ *topspace X* **and** *x* ≠ *y*
then obtain *U V* **where** *openin Y U openin Y V f x* ∈ *U f y* ∈ *V disjnt U V*
using *assms* **unfolding** *Hausdorff_space_def continuous_map_def* **by** (*meson inj_onD*)
show ∃ *U V*. *openin X U* ∧ *openin X V* ∧ *x* ∈ *U* ∧ *y* ∈ *V* ∧ *disjnt U V*
proof (*intro exI conjI*)
show *openin X* {*x* ∈ *topspace X*. *f x* ∈ *U*}
using ‹*openin Y U*› *cmf continuous_map* **by** *fastforce*
show *openin X* {*x* ∈ *topspace X*. *f x* ∈ *V*}
using ‹*openin Y V*› *cmf openin_continuous_map_preimage* **by** *blast*
show *disjnt* {*x* ∈ *topspace X*. *f x* ∈ *U*} {*x* ∈ *topspace X*. *f x* ∈ *V*}
using ‹*disjnt U V*› **by** (*auto simp add*: *disjnt_def*)
qed (*use x* ‹*f x* ∈ *U*› *y* ‹*f y* ∈ *V*› **in** *auto*)
qed

lemma *homeomorphic_Hausdorff_space*:
X homeomorphic_space Y $\Longrightarrow$ *Hausdorff_space X* ⟷ *Hausdorff_space Y*
unfolding *homeomorphic_space_def homeomorphic_maps_map*
by (*auto simp*: *homeomorphic_eq_everything_map Hausdorff_space_injective_preimage*)

lemma *Hausdorff_space_retraction_map_image*:
⟦*retraction_map X Y r*; *Hausdorff_space X*⟧ $\Longrightarrow$ *Hausdorff_space Y*
unfolding *retraction_map_def*
using *Hausdorff_space_subtopology homeomorphic_Hausdorff_space retraction_maps_section_image2*
by *blast*

lemma *compact_Hausdorff_space_optimal*:
assumes *eq*: *topspace Y* = *topspace X* **and** *XY*: ⋀*U*. *openin X U* ⟹ *openin Y U*
and *Hausdorff_space X compact_space Y*
shows *Y* = *X*
proof −
have ⋀*U*. *closedin X U* ⟹ *closedin Y U*
using *XY* **using** *topology_finer_closedin* [*OF eq*]
by *metis*
have *openin Y S* = *openin X S* **for** *S*
by (*metis XY assms*(*3*) *assms*(*4*) *closedin_compact_space compactin_contractive compactin_imp_closedin eq openin_closedin_eq*)
then show *?thesis*
by (*simp add*: *topology_eq*)
qed

lemma *continuous_map_imp_closed_graph*:
assumes *f*: *continuous_map X Y f* **and** *Y*: *Hausdorff_space Y*
shows *closedin* (*prod_topology X Y*) ((λ*x*. (*x*,*f x*)) ' *topspace X*)
unfolding *closedin_def*
proof
show (λ*x*. (*x*, *f x*)) ' *topspace X* ⊆ *topspace* (*prod_topology X Y*)
using *continuous_map_def f* **by** *fastforce*
show *openin* (*prod_topology X Y*) (*topspace* (*prod_topology X Y*) − (λ*x*. (*x*, *f x*)) ' *topspace X*)
unfolding *openin_prod_topology_alt*
proof (*intro allI impI*)
show ∃ *U V*. *openin X U* ∧ *openin Y V* ∧ *x* ∈ *U* ∧ *y* ∈ *V* ∧ *U* × *V* ⊆ *topspace* (*prod_topology X Y*) − (λ*x*. (*x*, *f x*)) ' *topspace X*
if (*x*,*y*) ∈ *topspace* (*prod_topology X Y*) − (λ*x*. (*x*, *f x*)) ' *topspace X*
for *x y*
proof −
have *x* ∈ *topspace X y* ∈ *topspace Y y* ≠ *f x*
using *that* **by** *auto*
moreover have *f x* ∈ *topspace Y*
by (*meson* ‹*x* ∈ *topspace X*› *continuous_map_def f*)
ultimately obtain *U V* **where** *UV*: *openin Y U openin Y V f x* ∈ *U y* ∈ *V disjnt U V*
using *Y Hausdorff_space_def* **by** *metis*
show *?thesis*
proof (*intro exI conjI*)
show *openin X* {*x* ∈ *topspace X*. *f x* ∈ *U*}
using ‹*openin Y U*› *f openin_continuous_map_preimage* **by** *blast*
show {*x* ∈ *topspace X*. *f x* ∈ *U*} × *V* ⊆ *topspace* (*prod_topology X Y*) − (λ*x*. (*x*, *f x*)) ' *topspace X*
using *UV* **by** (*auto simp*: *disjnt_iff dest*: *openin_subset*)
qed (*use UV* ‹*x* ∈ *topspace X*› **in** *auto*)
qed

qed
qed

lemma *continuous_imp_closed_map*:
⟦*continuous_map X Y f*; *compact_space X*; *Hausdorff_space Y*⟧ ⟹ *closed_map X Y f*
by (*meson closed_map_def closedin_compact_space compactin_imp_closedin image_compactin*)

lemma *continuous_imp_quotient_map*:
⟦*continuous_map X Y f*; *compact_space X*; *Hausdorff_space Y*; *f* ‘ (*topspace X*) = *topspace Y*⟧
⟹ *quotient_map X Y f*
by (*simp add*: *continuous_imp_closed_map continuous_closed_imp_quotient_map*)

lemma *continuous_imp_homeomorphic_map*:
⟦*continuous_map X Y f*; *compact_space X*; *Hausdorff_space Y*;
f ‘ (*topspace X*) = *topspace Y*; *inj_on f* (*topspace X*)⟧
⟹ *homeomorphic_map X Y f*
by (*simp add*: *continuous_imp_closed_map bijective_closed_imp_homeomorphic_map*)

lemma *continuous_imp_embedding_map*:
⟦*continuous_map X Y f*; *compact_space X*; *Hausdorff_space Y*; *inj_on f* (*topspace X*)⟧
⟹ *embedding_map X Y f*
by (*simp add*: *continuous_imp_closed_map injective_closed_imp_embedding_map*)

lemma *continuous_inverse_map*:
assumes *compact_space X Hausdorff_space Y*
and *cmf*: *continuous_map X Y f* **and** *gf*: $\bigwedge$*x*. *x* ∈ *topspace X* ⟹ *g*(*f x*) = *x*
and *Sf*: *S* ⊆ *f* ‘ (*topspace X*)
shows *continuous_map* (*subtopology Y S*) *X g*
proof (*rule continuous_map_from_subtopology_mono* [*OF* _ ‹*S* ⊆ *f* ‘ (*topspace X*)›])
show *continuous_map* (*subtopology Y* (*f* ‘ (*topspace X*))) *X g*
unfolding *continuous_map_closedin*
proof (*intro conjI ballI allI impI*)
fix *x*
assume *x* ∈ *topspace* (*subtopology Y* (*f* ‘ *topspace X*))
then show *g x* ∈ *topspace X*
by (*auto simp*: *gf*)
next
fix *C*
assume *C*: *closedin X C*
show *closedin* (*subtopology Y* (*f* ‘ *topspace X*))
{*x* ∈ *topspace* (*subtopology Y* (*f* ‘ *topspace X*)). *g x* ∈ *C*}
proof (*rule compactin_imp_closedin*)
show *Hausdorff_space* (*subtopology Y* (*f* ‘ *topspace X*))
using *Hausdorff_space_subtopology* [*OF* ‹*Hausdorff_space Y*›] **by** *blast*
have *compactin Y* (*f* ‘ *C*)

using *C cmf image_compactin closedin_compact_space* [*OF* ‹*compact_space X*›] **by** *blast*
moreover have {*x* ∈ *topspace Y*. *x* ∈ *f* ‘ *topspace X* ∧ *g x* ∈ *C*} = *f* ‘ *C*
using *closedin_subset* [*OF C*] *cmf* **by** (*auto simp*: *gf continuous_map_def*)
ultimately have *compactin Y* {*x* ∈ *topspace Y*. *x* ∈ *f* ‘ *topspace X* ∧ *g x* ∈ *C*}
by *simp*
then show *compactin* (*subtopology Y* (*f* ‘ *topspace X*))
{*x* ∈ *topspace* (*subtopology Y* (*f* ‘ *topspace X*)). *g x* ∈ *C*}
by (*auto simp add*: *compactin_subtopology*)
qed
qed
qed

lemma *closed_map_paired_continuous_map_right*:
⟦*continuous_map X Y f*; *Hausdorff_space Y*⟧ ⟹ *closed_map X* (*prod_topology X Y*) (λ*x*. (*x*,*f x*))
by (*simp add*: *continuous_map_imp_closed_graph embedding_map_graph embedding_imp_closed_map*)

lemma *closed_map_paired_continuous_map_left*:
assumes *f*: *continuous_map X Y f* **and** *Y*: *Hausdorff_space Y*
shows *closed_map X* (*prod_topology Y X*) (λ*x*. (*f x*,*x*))
proof −
have *eq*: (λ*x*. (*f x*,*x*)) = (λ(*a*,*b*). (*b*,*a*)) ∘ (λ*x*. (*x*,*f x*))
by *auto*
show *?thesis*
unfolding *eq*
proof (*rule closed_map_compose*)
show *closed_map X* (*prod_topology X Y*) (λ*x*. (*x*, *f x*))
using *Y closed_map_paired_continuous_map_right f* **by** *blast*
show *closed_map* (*prod_topology X Y*) (*prod_topology Y X*) (λ(*a*, *b*). (*b*, *a*))
by (*metis homeomorphic_map_swap homeomorphic_imp_closed_map*)
qed
qed

lemma *proper_map_paired_continuous_map_right*:
⟦*continuous_map X Y f*; *Hausdorff_space Y*⟧
⟹ *proper_map X* (*prod_topology X Y*) (λ*x*. (*x*,*f x*))
using *closed_injective_imp_proper_map closed_map_paired_continuous_map_right*
by (*metis* (*mono_tags*, *lifting*) *Pair_inject inj_onI*)

lemma *proper_map_paired_continuous_map_left*:
⟦*continuous_map X Y f*; *Hausdorff_space Y*⟧
⟹ *proper_map X* (*prod_topology Y X*) (λ*x*. (*f x*,*x*))
using *closed_injective_imp_proper_map closed_map_paired_continuous_map_left*
by (*metis* (*mono_tags*, *lifting*) *Pair_inject inj_onI*)

lemma *Hausdorff_space_prod_topology*:

Hausdorff_space(*prod_topology X Y*) ⟷ *topspace*(*prod_topology X Y*) = {} ∨ *Hausdorff_space X* ∧ *Hausdorff_space Y*
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*rule topological_property_of_prod_component*) (*auto simp*: *Hausdorff_space_subtopology homeomorphic_Hausdorff_space*)
next
assume *R*: *?rhs*
show *?lhs*
proof (*cases* (*topspace X* × *topspace Y*) = {})
case *False*
with *R* **have** *ne*: *topspace X* ≠ {} *topspace Y* ≠ {} **and** *X*: *Hausdorff_space X* **and** *Y*: *Hausdorff_space Y*
by *auto*
show *?thesis*
unfolding *Hausdorff_space_def*
proof *clarify*
fix *x y x′ y′*
assume *xy*: (*x*, *y*) ∈ *topspace* (*prod_topology X Y*)
and *xy′*: (*x′*,*y′*) ∈ *topspace* (*prod_topology X Y*)
and ∗: ∄ *U V*. *openin* (*prod_topology X Y*) *U* ∧ *openin* (*prod_topology X Y*) *V*
∧ (*x*, *y*) ∈ *U* ∧ (*x′*, *y′*) ∈ *V* ∧ *disjnt U V*
have *False* **if** *x* ≠ *x′* ∨ *y* ≠ *y′*
using *that*
proof
assume *x* ≠ *x′*
then obtain *U V* **where** *openin X U openin X V x* ∈ *U x′* ∈ *V disjnt U V*
by (*metis Hausdorff_space_def X mem_Sigma_iff topspace_prod_topology xy xy′*)
let *?U* = *U* × *topspace Y*
let *?V* = *V* × *topspace Y*
have *openin* (*prod_topology X Y*) *?U openin* (*prod_topology X Y*) *?V*
by (*simp_all add*: *openin_prod_Times_iff* ‹*openin X U*› ‹*openin X V*›)
moreover have *disjnt ?U ?V*
by (*simp add*: ‹*disjnt U V*›)
ultimately show *False*
using ∗ ‹*x* ∈ *U*› ‹*x′* ∈ *V*› *xy xy′* **by** (*metis SigmaD2 SigmaI topspace_prod_topology*)
next
assume *y* ≠ *y′*
then obtain *U V* **where** *openin Y U openin Y V y* ∈ *U y′* ∈ *V disjnt U V*
by (*metis Hausdorff_space_def Y mem_Sigma_iff topspace_prod_topology xy xy′*)
let *?U* = *topspace X* × *U*
let *?V* = *topspace X* × *V*
have *openin* (*prod_topology X Y*) *?U openin* (*prod_topology X Y*) *?V*
by (*simp_all add*: *openin_prod_Times_iff* ‹*openin Y U*› ‹*openin Y V*›)

```
        moreover have disjnt ?U ?V
          by (simp add: ‹disjnt U V›)
        ultimately show False
        using * ‹y ∈ U› ‹y' ∈ V› xy xy' by (metis SigmaD1 SigmaI topspace_prod_topology)
      qed
      then show x = x' ∧ y = y'
        by blast
    qed
  qed (simp add: Hausdorff_space_topspace_empty)
qed


lemma Hausdorff_space_product_topology:
    Hausdorff_space (product_topology X I) ⟷ (Π_E i∈I. topspace(X i)) = {} ∨
(∀ i ∈ I. Hausdorff_space (X i))
  (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    apply (rule topological_property_of_product_component)
    apply (blast dest: Hausdorff_space_subtopology homeomorphic_Hausdorff_space)+
    done
next
  assume R: ?rhs
  show ?lhs
  proof (cases (Π_E i∈I. topspace(X i)) = {})
    case True
    then show ?thesis
      by (simp add: Hausdorff_space_topspace_empty)
  next
    case False
    have ∃ U V. openin (product_topology X I) U ∧ openin (product_topology X I)
V ∧ f ∈ U ∧ g ∈ V ∧ disjnt U V
      if f: f ∈ (Π_E i∈I. topspace (X i)) and g: g ∈ (Π_E i∈I. topspace (X i)) and
f ≠ g
      for f g :: 'a ⇒ 'b
    proof −
      obtain m where f m ≠ g m
        using ‹f ≠ g› by blast
      then have m ∈ I
        using f g by fastforce
      then have Hausdorff_space (X m)
        using False that R by blast
      then obtain U V where U: openin (X m) U and V: openin (X m) V and
f m ∈ U g m ∈ V disjnt U V
        by (metis Hausdorff_space_def PiE_mem ‹f m ≠ g m› ‹m ∈ I› f g)
      show ?thesis
      proof (intro exI conjI)
        let ?U = (Π_E i∈I. topspace(X i)) ∩ {x. x m ∈ U}
```

```
      let ?V = (Π_E i∈I. topspace(X i)) ∩ {x. x m ∈ V}
      show openin (product_topology X I) ?U openin (product_topology X I) ?V
        using ‹m ∈ I› U V
         by (force simp add: openin_product_topology intro: arbitrary_union_of_inc
relative_to_inc finite_intersection_of_inc)+
      show f ∈ ?U
        using ‹f m ∈ U› f by blast
      show g ∈ ?V
        using ‹g m ∈ V› g by blast
      show disjnt ?U ?V
        using ‹disjnt U V› by (auto simp: PiE_def Pi_def disjnt_def)
      qed
    qed
    then show ?thesis
      by (simp add: Hausdorff_space_def)
  qed
qed

end
```

5.5 Path-Connectedness

```
theory Path_Connected
imports
  Starlike
  T1_Spaces
begin
```

5.5.1 Paths and Arcs

```
definition path :: (real ⇒ 'a::topological_space) ⇒ bool
  where path g ⟷ continuous_on {0..1} g

definition pathstart :: (real ⇒ 'a::topological_space) ⇒ 'a
  where pathstart g = g 0

definition pathfinish :: (real ⇒ 'a::topological_space) ⇒ 'a
  where pathfinish g = g 1

definition path_image :: (real ⇒ 'a::topological_space) ⇒ 'a set
  where path_image g = g ` {0 .. 1}

definition reversepath :: (real ⇒ 'a::topological_space) ⇒ real ⇒ 'a
  where reversepath g = (λx. g(1 − x))

definition joinpaths :: (real ⇒ 'a::topological_space) ⇒ (real ⇒ 'a) ⇒ real ⇒ 'a
    (infixr +++ 75)
  where g1 +++ g2 = (λx. if x ≤ 1/2 then g1 (2 ∗ x) else g2 (2 ∗ x − 1))
```

definition *simple_path* :: (*real* ⇒ ′*a*::*topological_space*) ⇒ *bool*
  where *simple_path g* ⟷
    path g ∧ (∀*x*∈{*0..1*}. ∀*y*∈{*0..1*}. *g x* = *g y* ⟶ *x* = *y* ∨ *x* = *0* ∧ *y* = *1* ∨ *x* = *1* ∧ *y* = *0*)

definition *arc* :: (*real* ⇒ ′*a* :: *topological_space*) ⇒ *bool*
  where *arc g* ⟷ *path g* ∧ *inj_on g* {*0..1*}

5.5.2 Invariance theorems

lemma *path_eq*: *path p* ⟹ (⋀*t*. *t* ∈ {*0..1*} ⟹ *p t* = *q t*) ⟹ *path q*
  using *continuous_on_eq path_def* **by** *blast*

lemma *path_continuous_image*: *path g* ⟹ *continuous_on* (*path_image g*) *f* ⟹ *path*(*f* ∘ *g*)
  unfolding *path_def path_image_def*
  using *continuous_on_compose* **by** *blast*

lemma *continuous_on_translation_eq*:
  fixes *g* :: ′*a* :: *real_normed_vector* ⇒ ′*b* :: *real_normed_vector*
  shows *continuous_on A* ((+) *a* ∘ *g*) = *continuous_on A g*
proof −
  have *g*: *g* = (λ*x*. −*a* + *x*) ∘ ((λ*x*. *a* + *x*) ∘ *g*)
    by (*rule ext*) *simp*
  show *?thesis*
    by (*metis* (*no_types*, *hide_lams*) *g continuous_on_compose homeomorphism_def homeomorphism_translation*)
qed

lemma *path_translation_eq*:
  fixes *g* :: *real* ⇒ ′*a* :: *real_normed_vector*
  shows *path*((λ*x*. *a* + *x*) ∘ *g*) = *path g*
  using *continuous_on_translation_eq path_def* **by** *blast*

lemma *path_linear_image_eq*:
  fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
   assumes *linear f inj f*
    shows *path*(*f* ∘ *g*) = *path g*
proof −
  from *linear_injective_left_inverse* [*OF assms*]
  obtain *h* **where** *h*: *linear h h* ∘ *f* = *id*
    by *blast*
  then have *g*: *g* = *h* ∘ (*f* ∘ *g*)
    by (*metis comp_assoc id_comp*)
  show *?thesis*
    unfolding *path_def*
    using *h assms*
   by (*metis g continuous_on_compose linear_continuous_on linear_conv_bounded_linear*)
qed

lemma *pathstart_translation*: *pathstart*((λ*x*. *a* + *x*) ∘ *g*) = *a* + *pathstart g*
  by (*simp add*: *pathstart_def*)

lemma *pathstart_linear_image_eq*: *linear f* ⟹ *pathstart*(*f* ∘ *g*) = *f*(*pathstart g*)
  by (*simp add*: *pathstart_def*)

lemma *pathfinish_translation*: *pathfinish*((λ*x*. *a* + *x*) ∘ *g*) = *a* + *pathfinish g*
  by (*simp add*: *pathfinish_def*)

lemma *pathfinish_linear_image*: *linear f* ⟹ *pathfinish*(*f* ∘ *g*) = *f*(*pathfinish g*)
  by (*simp add*: *pathfinish_def*)

lemma *path_image_translation*: *path_image*((λ*x*. *a* + *x*) ∘ *g*) = (λ*x*. *a* + *x*) ‘ (*path_image g*)
  by (*simp add*: *image_comp path_image_def*)

lemma *path_image_linear_image*: *linear f* ⟹ *path_image*(*f* ∘ *g*) = *f* ‘ (*path_image g*)
  by (*simp add*: *image_comp path_image_def*)

lemma *reversepath_translation*: *reversepath*((λ*x*. *a* + *x*) ∘ *g*) = (λ*x*. *a* + *x*) ∘ *reversepath g*
  by (*rule ext*) (*simp add*: *reversepath_def*)

lemma *reversepath_linear_image*: *linear f* ⟹ *reversepath*(*f* ∘ *g*) = *f* ∘ *reversepath g*
  by (*rule ext*) (*simp add*: *reversepath_def*)

lemma *joinpaths_translation*:
    ((λ*x*. *a* + *x*) ∘ *g1*) +++ ((λ*x*. *a* + *x*) ∘ *g2*) = (λ*x*. *a* + *x*) ∘ (*g1* +++ *g2*)
  by (*rule ext*) (*simp add*: *joinpaths_def*)

lemma *joinpaths_linear_image*: *linear f* ⟹ (*f* ∘ *g1*) +++ (*f* ∘ *g2*) = *f* ∘ (*g1* +++ *g2*)
  by (*rule ext*) (*simp add*: *joinpaths_def*)

lemma *simple_path_translation_eq*:
  fixes *g* :: *real* ⇒ ′*a*::*euclidean_space*
  shows *simple_path*((λ*x*. *a* + *x*) ∘ *g*) = *simple_path g*
  by (*simp add*: *simple_path_def path_translation_eq*)

lemma *simple_path_linear_image_eq*:
  fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
  assumes *linear f inj f*
    shows *simple_path*(*f* ∘ *g*) = *simple_path g*
  using *assms inj_on_eq_iff* [*of f*]
  by (*auto simp*: *path_linear_image_eq simple_path_def path_translation_eq*)

lemma *arc_translation_eq*:
  fixes g :: *real* ⇒ *'a::euclidean_space*
  shows *arc*((λ*x*. *a* + *x*) ∘ *g*) = *arc g*
  by (*auto simp*: *arc_def inj_on_def path_translation_eq*)

lemma *arc_linear_image_eq*:
  fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
  assumes *linear f inj f*
  shows *arc*(*f* ∘ *g*) = *arc g*
  using *assms inj_on_eq_iff* [*of f*]
  by (*auto simp*: *arc_def inj_on_def path_linear_image_eq*)

5.5.3 Basic lemmas about paths

lemma *pathin_iff_path_real* [*simp*]: *pathin euclideanreal g* ⟷ *path g*
  by (*simp add*: *pathin_def path_def*)

lemma *continuous_on_path*: *path f* ⟹ *t* ⊆ {*0..1*} ⟹ *continuous_on t f*
  using *continuous_on_subset path_def* **by** *blast*

lemma *arc_imp_simple_path*: *arc g* ⟹ *simple_path g*
  by (*simp add*: *arc_def inj_on_def simple_path_def*)

lemma *arc_imp_path*: *arc g* ⟹ *path g*
  using *arc_def* **by** *blast*

lemma *arc_imp_inj_on*: *arc g* ⟹ *inj_on g* {*0..1*}
  by (*auto simp*: *arc_def*)

lemma *simple_path_imp_path*: *simple_path g* ⟹ *path g*
  using *simple_path_def* **by** *blast*

lemma *simple_path_cases*: *simple_path g* ⟹ *arc g* ∨ *pathfinish g* = *pathstart g*
  unfolding *simple_path_def arc_def inj_on_def pathfinish_def pathstart_def*
  by *force*

lemma *simple_path_imp_arc*: *simple_path g* ⟹ *pathfinish g* ≠ *pathstart g* ⟹ *arc g*
  using *simple_path_cases* **by** *auto*

lemma *arc_distinct_ends*: *arc g* ⟹ *pathfinish g* ≠ *pathstart g*
  unfolding *arc_def inj_on_def pathfinish_def pathstart_def*
  by *fastforce*

lemma *arc_simple_path*: *arc g* ⟷ *simple_path g* ∧ *pathfinish g* ≠ *pathstart g*
  using *arc_distinct_ends arc_imp_simple_path simple_path_cases* **by** *blast*

lemma *simple_path_eq_arc*: *pathfinish g* ≠ *pathstart g* ⟹ (*simple_path g* = *arc g*)
  by (*simp add*: *arc_simple_path*)

lemma *path_image_const* [*simp*]: *path_image* ($\lambda t.$ *a*) = {*a*}
by (*force simp*: *path_image_def*)

lemma *path_image_nonempty* [*simp*]: *path_image g* ≠ {}
unfolding *path_image_def image_is_empty box_eq_empty*
by *auto*

lemma *pathstart_in_path_image*[*intro*]: *pathstart g* ∈ *path_image g*
unfolding *pathstart_def path_image_def*
by *auto*

lemma *pathfinish_in_path_image*[*intro*]: *pathfinish g* ∈ *path_image g*
unfolding *pathfinish_def path_image_def*
by *auto*

lemma *connected_path_image*[*intro*]: *path g* ⟹ *connected* (*path_image g*)
unfolding *path_def path_image_def*
using *connected_continuous_image connected_Icc* **by** *blast*

lemma *compact_path_image*[*intro*]: *path g* ⟹ *compact* (*path_image g*)
unfolding *path_def path_image_def*
using *compact_continuous_image connected_Icc* **by** *blast*

lemma *reversepath_reversepath*[*simp*]: *reversepath* (*reversepath g*) = *g*
unfolding *reversepath_def*
by *auto*

lemma *pathstart_reversepath*[*simp*]: *pathstart* (*reversepath g*) = *pathfinish g*
unfolding *pathstart_def reversepath_def pathfinish_def*
by *auto*

lemma *pathfinish_reversepath*[*simp*]: *pathfinish* (*reversepath g*) = *pathstart g*
unfolding *pathstart_def reversepath_def pathfinish_def*
by *auto*

lemma *pathstart_join*[*simp*]: *pathstart* (*g1* +++ *g2*) = *pathstart g1*
unfolding *pathstart_def joinpaths_def pathfinish_def*
by *auto*

lemma *pathfinish_join*[*simp*]: *pathfinish* (*g1* +++ *g2*) = *pathfinish g2*
unfolding *pathstart_def joinpaths_def pathfinish_def*
by *auto*

lemma *path_image_reversepath*[*simp*]: *path_image* (*reversepath g*) = *path_image g*
proof −
have ∗: ⋀*g*. *path_image* (*reversepath g*) ⊆ *path_image g*
unfolding *path_image_def subset_eq reversepath_def Ball_def image_iff*
by *force*

```
  show ?thesis
    using *[of g] *[of reversepath g]
    unfolding reversepath_reversepath
    by auto
qed

lemma path_reversepath [simp]: path (reversepath g) ⟷ path g
proof -
  have *: ⋀g. path g ⟹ path (reversepath g)
    unfolding path_def reversepath_def
    apply (rule continuous_on_compose[unfolded o_def, of _ λx. 1 - x])
    apply (auto intro: continuous_intros continuous_on_subset[of {0..1}])
    done
  show ?thesis
    using * by force
qed

lemma arc_reversepath:
  assumes arc g shows arc(reversepath g)
proof -
  have injg: inj_on g {0..1}
    using assms
    by (simp add: arc_def)
  have **: ⋀x y::real. 1-x = 1-y ⟹ x = y
    by simp
  show ?thesis
    using assms by (clarsimp simp: arc_def intro!: inj_onI) (simp add: inj_onD
reversepath_def **)
qed

lemma simple_path_reversepath: simple_path g ⟹ simple_path (reversepath g)
  apply (simp add: simple_path_def)
  apply (force simp: reversepath_def)
  done

lemmas reversepath_simps =
  path_reversepath path_image_reversepath pathstart_reversepath pathfinish_reversepath

lemma path_join[simp]:
  assumes pathfinish g1 = pathstart g2
  shows path (g1 +++ g2) ⟷ path g1 ∧ path g2
  unfolding path_def pathfinish_def pathstart_def
proof safe
  assume cont: continuous_on {0..1} (g1 +++ g2)
  have g1: continuous_on {0..1} g1 ⟷ continuous_on {0..1} ((g1 +++ g2) ∘
(λx. x / 2))
    by (intro continuous_on_cong refl) (auto simp: joinpaths_def)
  have g2: continuous_on {0..1} g2 ⟷ continuous_on {0..1} ((g1 +++ g2) ∘
(λx. x / 2 + 1/2))
```

using *assms*
by (*intro continuous_on_cong refl*) (*auto simp*: *joinpaths_def pathfinish_def pathstart_def*)
show *continuous_on* {*0..1*} *g1* **and** *continuous_on* {*0..1*} *g2*
unfolding *g1 g2*
by (*auto intro*!: *continuous_intros continuous_on_subset*[*OF cont*] *simp del*: *o_apply*)
next
assume *g1g2*: *continuous_on* {*0..1*} *g1 continuous_on* {*0..1*} *g2*
have *01*: {*0* .. *1*} = {*0..1/2*} ∪ {*1/2* .. *1*::*real*}
by *auto*
{
fix *x* :: *real*
assume *0* ≤ *x* **and** *x* ≤ *1*
then have *x* ∈ (λ*x*. *x* ∗ *2*) ‘ {*0..1* / *2*}
by (*intro image_eqI*[**where** *x*=*x*/*2*]) *auto*
}
note *1* = *this*
{
fix *x* :: *real*
assume *0* ≤ *x* **and** *x* ≤ *1*
then have *x* ∈ (λ*x*. *x* ∗ *2* − *1*) ‘ {*1* / *2..1*}
by (*intro image_eqI*[**where** *x*=*x*/*2* + *1*/*2*]) *auto*
}
note *2* = *this*
show *continuous_on* {*0..1*} (*g1* +++ *g2*)
using *assms*
unfolding *joinpaths_def 01*
apply (*intro continuous_on_cases closed_atLeastAtMost g1g2*[*THEN continuous_on_compose2*] *continuous_intros*)
apply (*auto simp*: *field_simps pathfinish_def pathstart_def intro*!: *1 2*)
done
qed

5.5.4 Path Images

lemma *bounded_path_image*: *path g* ⟹ *bounded*(*path_image g*)
by (*simp add*: *compact_imp_bounded compact_path_image*)

lemma *closed_path_image*:
fixes *g* :: *real* ⇒ *'a*::*t2_space*
shows *path g* ⟹ *closed*(*path_image g*)
by (*metis compact_path_image compact_imp_closed*)

lemma *connected_simple_path_image*: *simple_path g* ⟹ *connected*(*path_image g*)
by (*metis connected_path_image simple_path_imp_path*)

lemma *compact_simple_path_image*: *simple_path g* ⟹ *compact*(*path_image g*)
by (*metis compact_path_image simple_path_imp_path*)

lemma *bounded_simple_path_image*: *simple_path g* ⟹ *bounded*(*path_image g*)
  by (*metis bounded_path_image simple_path_imp_path*)

lemma *closed_simple_path_image*:
  fixes *g* :: *real* ⇒ *'a*::*t2_space*
  shows *simple_path g* ⟹ *closed*(*path_image g*)
  by (*metis closed_path_image simple_path_imp_path*)

lemma *connected_arc_image*: *arc g* ⟹ *connected*(*path_image g*)
  by (*metis connected_path_image arc_imp_path*)

lemma *compact_arc_image*: *arc g* ⟹ *compact*(*path_image g*)
  by (*metis compact_path_image arc_imp_path*)

lemma *bounded_arc_image*: *arc g* ⟹ *bounded*(*path_image g*)
  by (*metis bounded_path_image arc_imp_path*)

lemma *closed_arc_image*:
  fixes *g* :: *real* ⇒ *'a*::*t2_space*
  shows *arc g* ⟹ *closed*(*path_image g*)
  by (*metis closed_path_image arc_imp_path*)

lemma *path_image_join_subset*: *path_image* (*g1* +++ *g2*) ⊆ *path_image g1* ∪ *path_image g2*
  unfolding *path_image_def joinpaths_def*
  by *auto*

lemma *subset_path_image_join*:
  assumes *path_image g1* ⊆ *s*
    and *path_image g2* ⊆ *s*
  shows *path_image* (*g1* +++ *g2*) ⊆ *s*
  using *path_image_join_subset*[*of g1 g2*] **and** *assms*
  by *auto*

lemma *path_image_join*:
  assumes *pathfinish g1* = *pathstart g2*
  shows *path_image*(*g1* +++ *g2*) = *path_image g1* ∪ *path_image g2*
proof −
  have *path_image g1* ⊆ *path_image* (*g1* +++ *g2*)
  proof (*clarsimp simp*: *path_image_def joinpaths_def*)
    fix *u*::*real*
    assume *0* ≤ *u u* ≤ *1*
    then show *g1 u* ∈ (λ*x*. *g1* (*2* ∗ *x*)) ‘ ({*0*..*1*} ∩ {*x*. *x* ∗ *2* ≤ *1*})
      by (*rule_tac x*=*u*/*2* **in** *image_eqI*) *auto*
  qed
  moreover
  have §: *g2 u* ∈ (λ*x*. *g2* (*2* ∗ *x* − *1*)) ‘ ({*0*..*1*} ∩ {*x*. ¬ *x* ∗ *2* ≤ *1*})
    if *0* < *u u* ≤ *1* **for** *u*

using *that assms*
by (*rule_tac x*=(*u*+*1*)/*2* **in** *image_eqI*) (*auto simp*: *field_simps pathfinish_def pathstart_def*)
have *g2 0* ∈ (λ*x*. *g1* (*2* ∗ *x*)) ' ({*0*..*1*} ∩ {*x*. *x* ∗ *2* ≤ *1*})
using *assms*
by (*rule_tac x*=*1*/*2* **in** *image_eqI*) (*auto simp*: *pathfinish_def pathstart_def*)
then have *path_image g2* ⊆ *path_image* (*g1* +++ *g2*)
by (*auto simp*: *path_image_def joinpaths_def intro*!: §)
ultimately show *?thesis*
using *path_image_join_subset* **by** *blast*
qed

lemma *not_in_path_image_join*:
assumes *x* ∉ *path_image g1*
and *x* ∉ *path_image g2*
shows *x* ∉ *path_image* (*g1* +++ *g2*)
using *assms* **and** *path_image_join_subset*[*of g1 g2*]
by *auto*

lemma *pathstart_compose*: *pathstart*(*f* ∘ *p*) = *f*(*pathstart p*)
by (*simp add*: *pathstart_def*)

lemma *pathfinish_compose*: *pathfinish*(*f* ∘ *p*) = *f*(*pathfinish p*)
by (*simp add*: *pathfinish_def*)

lemma *path_image_compose*: *path_image* (*f* ∘ *p*) = *f* ' (*path_image p*)
by (*simp add*: *image_comp path_image_def*)

lemma *path_compose_join*: *f* ∘ (*p* +++ *q*) = (*f* ∘ *p*) +++ (*f* ∘ *q*)
by (*rule ext*) (*simp add*: *joinpaths_def*)

lemma *path_compose_reversepath*: *f* ∘ *reversepath p* = *reversepath*(*f* ∘ *p*)
by (*rule ext*) (*simp add*: *reversepath_def*)

lemma *joinpaths_eq*:
(⋀*t*. *t* ∈ {*0*..*1*} ⟹ *p t* = *p*′ *t*) ⟹
(⋀*t*. *t* ∈ {*0*..*1*} ⟹ *q t* = *q*′ *t*)
⟹ *t* ∈ {*0*..*1*} ⟹ (*p* +++ *q*) *t* = (*p*′ +++ *q*′) *t*
by (*auto simp*: *joinpaths_def*)

lemma *simple_path_inj_on*: *simple_path g* ⟹ *inj_on g* {*0*<..<*1*}
by (*auto simp*: *simple_path_def path_image_def inj_on_def less_eq_real_def Ball_def*)

5.5.5 Simple paths with the endpoints removed

lemma *simple_path_endless*:
assumes *simple_path c*
shows *path_image c* − {*pathstart c*,*pathfinish c*} = *c* ' {*0*<..<*1*} (**is** *?lhs* = *?rhs*)
proof

show *?lhs* ⊆ *?rhs*
using *less_eq_real_def* **by** (*auto simp*: *path_image_def pathstart_def pathfinish_def*)
show *?rhs* ⊆ *?lhs*
using *assms*
apply (*auto simp*: *simple_path_def path_image_def pathstart_def pathfinish_def Ball_def*)
using *less_eq_real_def zero_le_one* **by** *blast+*
qed

lemma *connected_simple_path_endless*:
assumes *simple_path c*
shows *connected*(*path_image c* − {*pathstart c*,*pathfinish c*})
proof −
have *continuous_on* {*0*<..<*1*} *c*
using *assms* **by** (*simp add*: *simple_path_def continuous_on_path path_def subset_iff*)
then have *connected* (*c* ' {*0*<..<*1*})
using *connected_Ioo connected_continuous_image* **by** *blast*
then show *?thesis*
using *assms* **by** (*simp add*: *simple_path_endless*)
qed

lemma *nonempty_simple_path_endless*:
simple_path c ⟹ *path_image c* − {*pathstart c*,*pathfinish c*} ≠ {}
by (*simp add*: *simple_path_endless*)

5.5.6 The operations on paths

lemma *path_image_subset_reversepath*: *path_image*(*reversepath g*) ≤ *path_image g*
by *simp*

lemma *path_imp_reversepath*: *path g* ⟹ *path*(*reversepath g*)
by *simp*

lemma *half_bounded_equal*: *1* ≤ *x* ∗ *2* ⟹ *x* ∗ *2* ≤ *1* ⟷ *x* = (*1*/*2*::*real*)
by *simp*

lemma *continuous_on_joinpaths*:
assumes *continuous_on* {*0*..*1*} *g1 continuous_on* {*0*..*1*} *g2 pathfinish g1* = *pathstart g2*
shows *continuous_on* {*0*..*1*} (*g1* +++ *g2*)
proof −
have {*0*..*1*::*real*} = {*0*..*1*/*2*} ∪ {*1*/*2*..*1*}
by *auto*
then show *?thesis*
using *assms* **by** (*metis path_def path_join*)
qed

lemma *path_join_imp*: ⟦*path g1*; *path g2*; *pathfinish g1* = *pathstart g2*⟧ ⟹ *path*(*g1* +++ *g2*)
  by *simp*

lemma *simple_path_join_loop*:
  assumes *arc g1 arc g2*
    pathfinish g1 = *pathstart g2* *pathfinish g2* = *pathstart g1*
    path_image g1 ∩ *path_image g2* ⊆ {*pathstart g1*, *pathstart g2*}
  shows *simple_path*(*g1* +++ *g2*)
proof −
  have *injg1*: *inj_on g1* {*0*..*1*}
    using *assms*
    by (*simp add*: *arc_def*)
  have *injg2*: *inj_on g2* {*0*..*1*}
    using *assms*
    by (*simp add*: *arc_def*)
  have *g12*: *g1 1* = *g2 0*
   and *g21*: *g2 1* = *g1 0*
   and *sb*: *g1* ‘ {*0*..*1*} ∩ *g2* ‘ {*0*..*1*} ⊆ {*g1 0*, *g2 0*}
    using *assms*
    by (*simp_all add*: *arc_def pathfinish_def pathstart_def path_image_def*)
  { **fix** *x* **and** *y*::*real*
    assume *g2_eq*: *g2* (*2* ∗ *x* − *1*) = *g1* (*2* ∗ *y*)
      and *xyI*: *x* ≠ *1* ∨ *y* ≠ *0*
      and *xy*: *x* ≤ *1 0* ≤ *y* *y* ∗ *2* ≤ *1* ¬ *x* ∗ *2* ≤ *1*
    then consider *g1* (*2* ∗ *y*) = *g1 0* | *g1* (*2* ∗ *y*) = *g2 0*
      using *sb* **by** *force*
    then have *False*
    proof *cases*
      case *1*
      then have *y* = *0*
        using *xy g2_eq* **by** (*auto dest*!: *inj_onD* [*OF injg1*])
      then show *?thesis*
        using *xy g2_eq xyI* **by** (*auto dest*: *inj_onD* [*OF injg2*] *simp flip*: *g21*)
    next
      case *2*
      then have *2*∗*x* = *1*
          using *g2_eq g12 inj_onD* [*OF injg2*] *atLeastAtMost_iff xy*(*1*) *xy*(*4*) **by** *fastforce*
      with *xy* **show** *False* **by** *auto*
    qed
  } **note** ∗ = *this*
  { **fix** *x* **and** *y*::*real*
    assume *xy*: *g1* (*2* ∗ *x*) = *g2* (*2* ∗ *y* − *1*) *y* ≤ *1 0* ≤ *x* ¬ *y* ∗ *2* ≤ *1 x* ∗ *2* ≤ *1*
    then have *x* = *0* ∧ *y* = *1*
      using ∗ *xy* **by** *force*
  } **note** ∗∗ = *this*
  show *?thesis*
    using *assms*

apply (*simp add*: *arc_def simple_path_def*)
apply (*auto simp*: *joinpaths_def split*: *if_split_asm*
dest!: ∗ ∗∗ *dest*: *inj_onD* [*OF injg1*] *inj_onD* [*OF injg2*])
done
qed

lemma *arc_join*:
assumes *arc g1 arc g2*
pathfinish g1 = *pathstart g2*
path_image g1 ∩ *path_image g2* ⊆ {*pathstart g2*}
shows *arc*(*g1* +++ *g2*)
proof −
have *injg1*: *inj_on g1* {*0*..*1*}
using *assms*
by (*simp add*: *arc_def*)
have *injg2*: *inj_on g2* {*0*..*1*}
using *assms*
by (*simp add*: *arc_def*)
have *g11*: *g1 1* = *g2 0*
and *sb*: *g1* ‘ {*0*..*1*} ∩ *g2* ‘ {*0*..*1*} ⊆ {*g2 0*}
using *assms*
by (*simp_all add*: *arc_def pathfinish_def pathstart_def path_image_def*)
{ **fix** *x* **and** *y*::*real*
assume *xy*: *g2* (*2* ∗ *x* − *1*) = *g1* (*2* ∗ *y*) *x* ≤ *1 0* ≤ *y* *y* ∗ *2* ≤ *1* ¬ *x* ∗ *2* ≤ *1*
then have *g1* (*2* ∗ *y*) = *g2 0*
using *sb* **by** *force*
then have *False*
using *xy inj_onD injg2* **by** *fastforce*
} **note** ∗ = *this*
show *?thesis*
using *assms*
apply (*simp add*: *arc_def inj_on_def*)
apply (*auto simp*: *joinpaths_def arc_imp_path split*: *if_split_asm*
dest: ∗ ∗[*OF sym*] *inj_onD* [*OF injg1*] *inj_onD* [*OF injg2*])
done
qed

lemma *reversepath_joinpaths*:
pathfinish g1 = *pathstart g2* ⟹ *reversepath*(*g1* +++ *g2*) = *reversepath g2* +++ *reversepath g1*
unfolding *reversepath_def pathfinish_def pathstart_def joinpaths_def*
by (*rule ext*) (*auto simp*: *mult.commute*)

5.5.7 Some reversed and "if and only if" versions of joining theorems

lemma *path_join_path_ends*:
fixes *g1* :: *real* ⇒ *'a*::*metric_space*
assumes *path*(*g1* +++ *g2*) *path g2*

shows *pathfinish g1 = pathstart g2*
proof (*rule ccontr*)
define *e* **where** *e = dist* (*g1 1*) (*g2 0*)
assume *Neg*: *pathfinish g1* ≠ *pathstart g2*
then have *0 < dist* (*pathfinish g1*) (*pathstart g2*)
by *auto*
then have *e > 0*
by (*metis e_def pathfinish_def pathstart_def*)
then have ∀ *e>0.* ∃ *d>0.* ∀ *x'*∈{*0..1*}. *dist x' 0 < d* ⟶ *dist* (*g2 x'*) (*g2 0*) < *e*
using ‹*path g2*› *atLeastAtMost_iff zero_le_one* **unfolding** *path_def continuous_on_iff*
by *blast*
then obtain *d1* **where** *d1 > 0*
and *d1*: ⋀*x'*. ⟦*x'*∈{*0..1*}; *norm x' < d1*⟧ ⟹ *dist* (*g2 x'*) (*g2 0*) < *e/2*
by (*metis* ‹*0 < e*› *half_gt_zero_iff norm_conv_dist*)
obtain *d2* **where** *d2 > 0*
and *d2*: ⋀*x'*. ⟦*x'*∈{*0..1*}; *dist x'* (*1/2*) < *d2*⟧
⟹ *dist* ((*g1 +++ g2*) *x'*) (*g1 1*) < *e/2*
using *assms*(*1*) ‹*e > 0*› **unfolding** *path_def continuous_on_iff*
apply (*drule_tac x=1/2* **in** *bspec*, *simp*)
apply (*drule_tac x=e/2* **in** *spec*, *force simp*: *joinpaths_def*)
done
have *int01_1*: *min* (*1/2*) (*min d1 d2*) / *2* ∈ {*0..1*}
using ‹*d1 > 0*› ‹*d2 > 0*› **by** (*simp add*: *min_def*)
have *dist1*: *norm* (*min* (*1 / 2*) (*min d1 d2*) / *2*) < *d1*
using ‹*d1 > 0*› ‹*d2 > 0*› **by** (*simp add*: *min_def dist_norm*)
have *int01_2*: *1/2 + min* (*1/2*) (*min d1 d2*) / *4* ∈ {*0..1*}
using ‹*d1 > 0*› ‹*d2 > 0*› **by** (*simp add*: *min_def*)
have *dist2*: *dist* (*1 / 2 + min* (*1 / 2*) (*min d1 d2*) / *4*) (*1 / 2*) < *d2*
using ‹*d1 > 0*› ‹*d2 > 0*› **by** (*simp add*: *min_def dist_norm*)
have [*simp*]: ¬ *min* (*1 / 2*) (*min d1 d2*) ≤ *0*
using ‹*d1 > 0*› ‹*d2 > 0*› **by** (*simp add*: *min_def*)
have *dist* (*g2* (*min* (*1 / 2*) (*min d1 d2*) / *2*)) (*g1 1*) < *e/2*
dist (*g2* (*min* (*1 / 2*) (*min d1 d2*) / *2*)) (*g2 0*) < *e/2*
using *d1* [*OF int01_1 dist1*] *d2* [*OF int01_2 dist2*] **by** (*simp_all add*: *joinpaths_def*)
then have *dist* (*g1 1*) (*g2 0*) < *e/2 + e/2*
using *dist_triangle_half_r e_def* **by** *blast*
then show *False*
by (*simp add*: *e_def* [*symmetric*])
qed

lemma *path_join_eq* [*simp*]:
fixes *g1* :: *real* ⇒ *'a*::*metric_space*
assumes *path g1 path g2*
shows *path*(*g1 +++ g2*) ⟷ *pathfinish g1 = pathstart g2*
using *assms* **by** (*metis path_join_path_ends path_join_imp*)

lemma *simple_path_joinE*:

assumes *simple_path*(*g1* +++ *g2*) **and** *pathfinish g1* = *pathstart g2*
obtains *arc g1 arc g2*
path_image g1 ∩ *path_image g2* ⊆ {*pathstart g1*, *pathstart g2*}
proof −
have ∗: ⋀*x y*. ⟦*0* ≤ *x*; *x* ≤ *1*; *0* ≤ *y*; *y* ≤ *1*; (*g1* +++ *g2*) *x* = (*g1* +++ *g2*) *y*⟧
⟹ *x* = *y* ∨ *x* = *0* ∧ *y* = *1* ∨ *x* = *1* ∧ *y* = *0*
using *assms* **by** (*simp add*: *simple_path_def*)
have *path g1*
using *assms path_join simple_path_imp_path* **by** *blast*
moreover have *inj_on g1* {*0*..*1*}
proof (*clarsimp simp*: *inj_on_def*)
fix *x y*
assume *g1 x* = *g1 y 0* ≤ *x x* ≤ *1 0* ≤ *y y* ≤ *1*
then show *x* = *y*
using ∗ [*of x*/*2 y*/*2*] **by** (*simp add*: *joinpaths_def split_ifs*)
qed
ultimately have *arc g1*
using *assms* **by** (*simp add*: *arc_def*)
have [*simp*]: *g2 0* = *g1 1*
using *assms* **by** (*metis pathfinish_def pathstart_def*)
have *path g2*
using *assms path_join simple_path_imp_path* **by** *blast*
moreover have *inj_on g2* {*0*..*1*}
proof (*clarsimp simp*: *inj_on_def*)
fix *x y*
assume *g2 x* = *g2 y 0* ≤ *x x* ≤ *1 0* ≤ *y y* ≤ *1*
then show *x* = *y*
using ∗ [*of* (*x* + *1*) / *2* (*y* + *1*) / *2*]
by (*force simp*: *joinpaths_def split_ifs field_split_simps*)
qed
ultimately have *arc g2*
using *assms* **by** (*simp add*: *arc_def*)
have *g2 y* = *g1 0* ∨ *g2 y* = *g1 1*
if *g1 x* = *g2 y 0* ≤ *x x* ≤ *1 0* ≤ *y y* ≤ *1* **for** *x y*
using ∗ [*of x* / *2* (*y* + *1*) / *2*] *that*
by (*auto simp*: *joinpaths_def split_ifs field_split_simps*)
then have *path_image g1* ∩ *path_image g2* ⊆ {*pathstart g1*, *pathstart g2*}
by (*fastforce simp*: *pathstart_def pathfinish_def path_image_def*)
with ‹*arc g1*› ‹*arc g2*› **show** *?thesis* **using** *that* **by** *blast*
qed

lemma *simple_path_join_loop_eq*:
assumes *pathfinish g2* = *pathstart g1 pathfinish g1* = *pathstart g2*
shows *simple_path*(*g1* +++ *g2*) ⟷
arc g1 ∧ *arc g2* ∧ *path_image g1* ∩ *path_image g2* ⊆ {*pathstart g1*, *pathstart g2*}
by (*metis assms simple_path_joinE simple_path_join_loop*)

lemma *arc_join_eq*:
  assumes *pathfinish g1 = pathstart g2*
    shows *arc*(*g1 +++ g2*) $\longleftrightarrow$
          arc g1 $\wedge$ *arc g2* $\wedge$ *path_image g1* $\cap$ *path_image g2* $\subseteq$ {*pathstart g2*}
          (**is** *?lhs = ?rhs*)
proof
  assume *?lhs*
  then have *simple_path*(*g1 +++ g2*) **by** (*rule arc_imp_simple_path*)
  then have ∗: $\bigwedge$*x y*. ⟦*0* ≤ *x*; *x* ≤ *1*; *0* ≤ *y*; *y* ≤ *1*; (*g1 +++ g2*) *x* = (*g1 +++ g2*) *y*⟧
      $\Longrightarrow$ *x = y* ∨ *x = 0* ∧ *y = 1* ∨ *x = 1* ∧ *y = 0*
    using *assms* **by** (*simp add*: *simple_path_def*)
  have *False* **if** *g1 0 = g2 u 0* ≤ *u u* ≤ *1* **for** *u*
    using ∗ [*of 0* (*u* + *1*) / *2*] *that assms arc_distinct_ends* [*OF* ‹*?lhs*›]
   by (*auto simp*: *joinpaths_def pathstart_def pathfinish_def split_ifs field_split_simps*)
  then have *n1*: *pathstart g1* ∉ *path_image g2*
    unfolding *pathstart_def path_image_def*
    using *atLeastAtMost_iff* **by** *blast*
  show *?rhs* **using** ‹*?lhs*›
    using ‹*simple_path* (*g1 +++ g2*)› *assms n1 simple_path_joinE* **by** *auto*
next
  assume *?rhs* **then show** *?lhs*
    using *assms*
    by (*fastforce simp*: *pathfinish_def pathstart_def intro*!: *arc_join*)
qed

lemma *arc_join_eq_alt*:
      pathfinish g1 = pathstart g2
      $\Longrightarrow$ (*arc*(*g1 +++ g2*) $\longleftrightarrow$
          arc g1 ∧ *arc g2* ∧
          path_image g1 ∩ *path_image g2* = {*pathstart g2*})
using *pathfinish_in_path_image* **by** (*fastforce simp*: *arc_join_eq*)

5.5.8 The joining of paths is associative

lemma *path_assoc*:
    ⟦*pathfinish p = pathstart q*; *pathfinish q = pathstart r*⟧
     $\Longrightarrow$ *path*(*p +++* (*q +++ r*)) $\longleftrightarrow$ *path*((*p +++ q*) *+++ r*)
by *simp*

lemma *simple_path_assoc*:
  assumes *pathfinish p = pathstart q pathfinish q = pathstart r*
   shows *simple_path* (*p +++* (*q +++ r*)) $\longleftrightarrow$ *simple_path* ((*p +++ q*) *+++ r*)
proof (*cases pathstart p = pathfinish r*)
  case *True* **show** *?thesis*
  proof
    assume *simple_path* (*p +++ q +++ r*)
    with *assms True* **show** *simple_path* ((*p +++ q*) *+++ r*)
      by (*fastforce simp add*: *simple_path_join_loop_eq arc_join_eq path_image_join*

```
                dest: arc_distinct_ends [of r])
  next
    assume 0: simple_path ((p +++ q) +++ r)
    with assms True have q: pathfinish r ∉ path_image q
      using arc_distinct_ends
      by (fastforce simp add: simple_path_join_loop_eq arc_join_eq path_image_join)
    have pathstart r ∉ path_image p
      using assms
      by (metis 0 IntI arc_distinct_ends arc_join_eq_alt empty_iff insert_iff
              pathfinish_in_path_image pathfinish_join simple_path_joinE)
    with assms 0 q True show simple_path (p +++ q +++ r)
      by (auto simp: simple_path_join_loop_eq arc_join_eq path_image_join
               dest!: subsetD [OF _ IntI])
  qed
next
  case False
  { fix x :: 'a
    assume a: path_image p ∩ path_image q ⊆ {pathstart q}
              (path_image p ∪ path_image q) ∩ path_image r ⊆ {pathstart r}
              x ∈ path_image p x ∈ path_image r
    have pathstart r ∈ path_image q
      by (metis assms(2) pathfinish_in_path_image)
    with a have x = pathstart q
      by blast
  }
  with False assms show ?thesis
   by (auto simp: simple_path_eq_arc simple_path_join_loop_eq arc_join_eq path_image_join)
qed

lemma arc_assoc:
     ⟦pathfinish p = pathstart q; pathfinish q = pathstart r⟧
      ⟹ arc(p +++ (q +++ r)) ⟷ arc((p +++ q) +++ r)
by (simp add: arc_simple_path simple_path_assoc)
```

Symmetry and loops

```
lemma path_sym:
    ⟦pathfinish p = pathstart q; pathfinish q = pathstart p⟧ ⟹ path(p +++ q) ⟷
path(q +++ p)
  by auto

lemma simple_path_sym:
    ⟦pathfinish p = pathstart q; pathfinish q = pathstart p⟧
     ⟹ simple_path(p +++ q) ⟷ simple_path(q +++ p)
by (metis (full_types) inf_commute insert_commute simple_path_joinE simple_path_join_loop)

lemma path_image_sym:
    ⟦pathfinish p = pathstart q; pathfinish q = pathstart p⟧
     ⟹ path_image(p +++ q) = path_image(q +++ p)
```

by (*simp add*: *path_image_join sup_commute*)

5.5.9 Subpath

definition *subpath* :: *real* ⇒ *real* ⇒ (*real* ⇒ *'a*) ⇒ *real* ⇒ *'a*::*real_normed_vector*
where *subpath a b g* ≡ λ*x*. *g*((*b* − *a*) ∗ *x* + *a*)

lemma *path_image_subpath_gen*:
fixes *g* :: _ ⇒ *'a*::*real_normed_vector*
shows *path_image*(*subpath u v g*) = *g* ‘ (*closed_segment u v*)
by (*auto simp add*: *closed_segment_real_eq path_image_def subpath_def*)

lemma *path_image_subpath*:
fixes *g* :: *real* ⇒ *'a*::*real_normed_vector*
shows *path_image*(*subpath u v g*) = (*if u* ≤ *v then g* ‘ {*u*..*v*} *else g* ‘ {*v*..*u*})
by (*simp add*: *path_image_subpath_gen closed_segment_eq_real_ivl*)

lemma *path_image_subpath_commute*:
fixes *g* :: *real* ⇒ *'a*::*real_normed_vector*
shows *path_image*(*subpath u v g*) = *path_image*(*subpath v u g*)
by (*simp add*: *path_image_subpath_gen closed_segment_eq_real_ivl*)

lemma *path_subpath* [*simp*]:
fixes *g* :: *real* ⇒ *'a*::*real_normed_vector*
assumes *path g u* ∈ {*0*..*1*} *v* ∈ {*0*..*1*}
shows *path*(*subpath u v g*)
proof −
have *continuous_on* {*0*..*1*} (*g* ∘ (λ*x*. ((*v*−*u*) ∗ *x*+ *u*)))
using *assms*
apply (*intro continuous_intros*; *simp add*: *image_affinity_atLeastAtMost* [**where** *c*=*u*])
apply (*auto simp*: *path_def continuous_on_subset*)
done
then show *?thesis*
by (*simp add*: *path_def subpath_def*)
qed

lemma *pathstart_subpath* [*simp*]: *pathstart*(*subpath u v g*) = *g*(*u*)
by (*simp add*: *pathstart_def subpath_def*)

lemma *pathfinish_subpath* [*simp*]: *pathfinish*(*subpath u v g*) = *g*(*v*)
by (*simp add*: *pathfinish_def subpath_def*)

lemma *subpath_trivial* [*simp*]: *subpath 0 1 g* = *g*
by (*simp add*: *subpath_def*)

lemma *subpath_reversepath*: *subpath 1 0 g* = *reversepath g*
by (*simp add*: *reversepath_def subpath_def*)

lemma *reversepath_subpath*: *reversepath*(*subpath u v g*) = *subpath v u g*
  by (*simp add*: *reversepath_def subpath_def algebra_simps*)

lemma *subpath_translation*: *subpath u v* ((λx. *a* + *x*) ∘ *g*) = (λx. *a* + *x*) ∘ *subpath u v g*
  by (*rule ext*) (*simp add*: *subpath_def*)

lemma *subpath_image*: *subpath u v* (*f* ∘ *g*) = *f* ∘ *subpath u v g*
  by (*rule ext*) (*simp add*: *subpath_def*)

lemma *affine_ineq*:
  fixes *x* :: *'a*::*linordered_idom*
  assumes *x* ≤ *1* *v* ≤ *u*
    shows *v* + *x* ∗ *u* ≤ *u* + *x* ∗ *v*
proof −
  have (*1*−*x*)∗(*u*−*v*) ≥ *0*
    using *assms* **by** *auto*
  then show *?thesis*
    by (*simp add*: *algebra_simps*)
qed

lemma *sum_le_prod1*:
  fixes *a*::*real* **shows** ⟦*a* ≤ *1*; *b* ≤ *1*⟧ ⟹ *a* + *b* ≤ *1* + *a* ∗ *b*
by (*metis add.commute affine_ineq mult.right_neutral*)

lemma *simple_path_subpath_eq*:
  simple_path(*subpath u v g*) ⟷
    path(*subpath u v g*) ∧ *u*≠*v* ∧
    (∀ *x y*. *x* ∈ *closed_segment u v* ∧ *y* ∈ *closed_segment u v* ∧ *g x* = *g y*
          ⟶ *x* = *y* ∨ *x* = *u* ∧ *y* = *v* ∨ *x* = *v* ∧ *y* = *u*)
    (**is** *?lhs* = *?rhs*)
proof
  assume *?lhs*
  then have *p*: *path* (λx. *g* ((*v* − *u*) ∗ *x* + *u*))
      and *sim*: (⋀*x y*. ⟦*x*∈{*0*..*1*}; *y*∈{*0*..*1*}; *g* ((*v* − *u*) ∗ *x* + *u*) = *g* ((*v* − *u*) ∗ *y* + *u*)⟧
          ⟹ *x* = *y* ∨ *x* = *0* ∧ *y* = *1* ∨ *x* = *1* ∧ *y* = *0*)
    by (*auto simp*: *simple_path_def subpath_def*)
  { **fix** *x y*
    assume *x* ∈ *closed_segment u v* *y* ∈ *closed_segment u v* *g x* = *g y*
    then have *x* = *y* ∨ *x* = *u* ∧ *y* = *v* ∨ *x* = *v* ∧ *y* = *u*
      using *sim* [*of* (*x*−*u*)/(*v*−*u*) (*y*−*u*)/(*v*−*u*)] *p*
    by (*auto split*: *if_split_asm simp add*: *closed_segment_real_eq image_affinity_atLeastAtMost*)
      (*simp_all add*: *field_split_simps*)
  } **moreover**
  have *path*(*subpath u v g*) ∧ *u*≠*v*
    using *sim* [*of 1*/*3 2*/*3*] *p*
    by (*auto simp*: *subpath_def*)
  ultimately show *?rhs*

by *metis*
next
assume *?rhs*
then
have *d1*: $\bigwedge x\ y.\ [\![g\ x = g\ y;\ u \leq x;\ x \leq v;\ u \leq y;\ y \leq v]\!] \Longrightarrow x = y \vee x = u \wedge y = v \vee x = v \wedge y = u$
and *d2*: $\bigwedge x\ y.\ [\![g\ x = g\ y;\ v \leq x;\ x \leq u;\ v \leq y;\ y \leq u]\!] \Longrightarrow x = y \vee x = u \wedge y = v \vee x = v \wedge y = u$
and *ne*: $u < v \vee v < u$
and *psp*: *path* (*subpath u v g*)
by (*auto simp*: *closed_segment_real_eq image_affinity_atLeastAtMost*)
have [*simp*]: $\bigwedge x.\ u + x * v = v + x * u \longleftrightarrow u{=}v \vee x{=}1$
by *algebra*
show *?lhs* **using** *psp ne*
unfolding *simple_path_def subpath_def*
by (*fastforce simp add*: *algebra_simps affine_ineq mult_left_mono crossproduct_eq dest*: *d1 d2*)
qed

lemma *arc_subpath_eq*:
arc(*subpath u v g*) $\longleftrightarrow$ *path*(*subpath u v g*) $\wedge$ $u{\neq}v$ $\wedge$ *inj_on g* (*closed_segment u v*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then have *p*: *path* ($\lambda x.\ g\ ((v - u) * x + u)$)
and *sim*: ($\bigwedge x\ y.\ [\![x{\in}\{0..1\};\ y{\in}\{0..1\};\ g\ ((v - u) * x + u) = g\ ((v - u) * y + u)]\!]$
$\Longrightarrow x = y$)
by (*auto simp*: *arc_def inj_on_def subpath_def*)
{ fix *x y*
assume $x \in$ *closed_segment u v* $y \in$ *closed_segment u v* $g\ x = g\ y$
then have $x = y$
using *sim* [*of* $(x{-}u)/(v{-}u)$ $(y{-}u)/(v{-}u)$] *p*
by (*cases* $v = u$)
(*simp_all split*: *if_split_asm add*: *inj_on_def closed_segment_real_eq image_affinity_atLeastAtMost*, *simp add*: *field_simps*)
} moreover
have *path*(*subpath u v g*) $\wedge$ $u{\neq}v$
using *sim* [*of 1/3 2/3*] *p*
by (*auto simp*: *subpath_def*)
ultimately show *?rhs*
unfolding *inj_on_def*
by *metis*
next
assume *?rhs*
then
have *d1*: $\bigwedge x\ y.\ [\![g\ x = g\ y;\ u \leq x;\ x \leq v;\ u \leq y;\ y \leq v]\!] \Longrightarrow x = y$
and *d2*: $\bigwedge x\ y.\ [\![g\ x = g\ y;\ v \leq x;\ x \leq u;\ v \leq y;\ y \leq u]\!] \Longrightarrow x = y$

and *ne*: *u* < *v* ∨ *v* < *u*
and *psp*: *path* (*subpath u v g*)
by (*auto simp*: *inj_on_def closed_segment_real_eq image_affinity_atLeastAtMost*)
show *?lhs* **using** *psp ne*
unfolding *arc_def subpath_def inj_on_def*
by (*auto simp*: *algebra_simps affine_ineq mult_left_mono crossproduct_eq dest*: *d1 d2*)
qed

lemma *simple_path_subpath*:
assumes *simple_path g u* ∈ {*0..1*} *v* ∈ {*0..1*} *u* ≠ *v*
shows *simple_path*(*subpath u v g*)
using *assms*
apply (*simp add*: *simple_path_subpath_eq simple_path_imp_path*)
apply (*simp add*: *simple_path_def closed_segment_real_eq image_affinity_atLeastAtMost*, *fastforce*)
done

lemma *arc_simple_path_subpath*:
⟦*simple_path g*; *u* ∈ {*0..1*}; *v* ∈ {*0..1*}; *g u* ≠ *g v*⟧ ⟹ *arc*(*subpath u v g*)
by (*force intro*: *simple_path_subpath simple_path_imp_arc*)

lemma *arc_subpath_arc*:
⟦*arc g*; *u* ∈ {*0..1*}; *v* ∈ {*0..1*}; *u* ≠ *v*⟧ ⟹ *arc*(*subpath u v g*)
by (*meson arc_def arc_imp_simple_path arc_simple_path_subpath inj_onD*)

lemma *arc_simple_path_subpath_interior*:
⟦*simple_path g*; *u* ∈ {*0..1*}; *v* ∈ {*0..1*}; *u* ≠ *v*; |*u*−*v*| < *1*⟧ ⟹ *arc*(*subpath u v g*)
by (*force simp*: *simple_path_def intro*: *arc_simple_path_subpath*)

lemma *path_image_subpath_subset*:
⟦*u* ∈ {*0..1*}; *v* ∈ {*0..1*}⟧ ⟹ *path_image*(*subpath u v g*) ⊆ *path_image g*
by (*metis atLeastAtMost_iff atLeastatMost_subset_iff path_image_def path_image_subpath subset_image_iff*)

lemma *join_subpaths_middle*: *subpath* (*0*) ((*1* / *2*)) *p* +++ *subpath* ((*1* / *2*)) *1 p* = *p*
by (*rule ext*) (*simp add*: *joinpaths_def subpath_def field_split_simps*)

5.5.10 There is a subpath to the frontier

lemma *subpath_to_frontier_explicit*:
fixes *S* :: *'a::metric_space set*
assumes *g*: *path g* **and** *pathfinish g* ∉ *S*
obtains *u* **where** *0* ≤ *u u* ≤ *1*
⋀*x*. *0* ≤ *x* ∧ *x* < *u* ⟹ *g x* ∈ *interior S*
(*g u* ∉ *interior S*) (*u* = *0* ∨ *g u* ∈ *closure S*)

proof −
  have *gcon*: *continuous_on* {*0..1*} *g*
    using *g* **by** (*simp add*: *path_def*)
  moreover have *bounded* ({*u*. *g u* ∈ *closure* (− *S*)} ∩ {*0..1*})
    using *compact_eq_bounded_closed* **by** *fastforce*
  ultimately have *com*: *compact* ({*0..1*} ∩ {*u*. *g u* ∈ *closure* (− *S*)})
    using *closed_vimage_Int*
     by (*metis* (*full_types*) *Int_commute closed_atLeastAtMost closed_closure compact_eq_bounded_closed vimage_def*)
  have *1* ∈ {*u*. *g u* ∈ *closure* (− *S*)}
    using *assms* **by** (*simp add*: *pathfinish_def closure_def*)
  then have *dis*: {*0..1*} ∩ {*u*. *g u* ∈ *closure* (− *S*)} ≠ {}
    using *atLeastAtMost_iff zero_le_one* **by** *blast*
  then obtain *u* **where** *0* ≤ *u u* ≤ *1* **and** *gu*: *g u* ∈ *closure* (− *S*)
                and *umin*: ⋀*t*. ⟦*0* ≤ *t*; *t* ≤ *1*; *g t* ∈ *closure* (− *S*)⟧ ⟹ *u* ≤ *t*
    using *compact_attains_inf* [*OF com dis*] **by** *fastforce*
  then have *umin'*: ⋀*t*. ⟦*0* ≤ *t*; *t* ≤ *1*; *t* < *u*⟧ ⟹ *g t* ∈ *S*
    using *closure_def* **by** *fastforce*
  have §: *g u* ∈ *closure S* **if** *u* ≠ *0*
  proof −
    have *u* > *0* **using** *that* ‹*0* ≤ *u*› **by** *auto*
    { fix *e*::*real* **assume** *e* > *0*
      obtain *d* **where** *d*>*0* **and** *d*: ⋀*x'*. ⟦*x'* ∈ {*0..1*}; *dist x' u* ≤ *d*⟧ ⟹ *dist* (*g x'*) (*g u*) < *e*
        using *continuous_onE* [*OF gcon* _ ‹*e* > *0*›] ‹*0* ≤ _› ‹_ ≤ *1*› *atLeastAtMost_iff* **by** *auto*
      have ∗: *dist* (*max 0* (*u* − *d* / *2*)) *u* ≤ *d*
        using ‹*0* ≤ *u*› ‹*u* ≤ *1*› ‹*d* > *0*› **by** (*simp add*: *dist_real_def*)
      have ∃ *y*∈*S*. *dist y* (*g u*) < *e*
        using ‹*0* < *u*› ‹*u* ≤ *1*› ‹*d* > *0*›
        by (*force intro*: *d* [*OF* _ ∗] *umin'*)
    }
    then show *?thesis*
      by (*simp add*: *frontier_def closure_approachable*)
  qed
  show *?thesis*
  proof
    show ⋀*x*. *0* ≤ *x* ∧ *x* < *u* ⟹ *g x* ∈ *interior S*
      using ‹*u* ≤ *1*› *interior_closure umin* **by** *fastforce*
    show *g u* ∉ *interior S*
      by (*simp add*: *gu interior_closure*)
  qed (*use* ‹*0* ≤ *u*› ‹*u* ≤ *1*› § **in** *auto*)
qed

lemma *subpath_to_frontier_strong*:
    assumes *g*: *path g* **and** *pathfinish g* ∉ *S*
    obtains *u* **where** *0* ≤ *u u* ≤ *1 g u* ∉ *interior S*
                u = *0* ∨ (∀ *x*. *0* ≤ *x* ∧ *x* < *1* ⟶ *subpath 0 u g x* ∈ *interior S*) ∧ *g u* ∈ *closure S*

```
proof -
  obtain u where 0 ≤ u u ≤ 1
           and gxin: ⋀x. 0 ≤ x ∧ x < u ⟹ g x ∈ interior S
           and gunot: (g u ∉ interior S) and u0: (u = 0 ∨ g u ∈ closure S)
    using subpath_to_frontier_explicit [OF assms] by blast
  show ?thesis
  proof
    show g u ∉ interior S
      using gunot by blast
  qed (use ‹0 ≤ u› ‹u ≤ 1› u0 in ‹(force simp: subpath_def gxin)+›)
qed

lemma subpath_to_frontier:
    assumes g: path g and g0: pathstart g ∈ closure S and g1: pathfinish g ∉ S
    obtains u where 0 ≤ u u ≤ 1 g u ∈ frontier S path_image(subpath 0 u g) −
{g u} ⊆ interior S
proof -
  obtain u where 0 ≤ u u ≤ 1
           and notin: g u ∉ interior S
           and disj: u = 0 ∨
                      (∀x. 0 ≤ x ∧ x < 1 ⟶ subpath 0 u g x ∈ interior S) ∧ g u
∈ closure S
                      (is _ ∨ ?P)
    using subpath_to_frontier_strong [OF g g1] by blast
  show ?thesis
  proof
    show g u ∈ frontier S
      by (metis DiffI disj frontier_def g0 notin pathstart_def)
    show path_image (subpath 0 u g) − {g u} ⊆ interior S
      using disj
    proof
      assume u = 0
      then show ?thesis
        by (simp add: path_image_subpath)
    next
      assume P: ?P
      show ?thesis
      proof (clarsimp simp add: path_image_subpath_gen)
        fix y
        assume y: y ∈ closed_segment 0 u g y ∉ interior S
        with ‹0 ≤ u› have 0 ≤ y y ≤ u
          by (auto simp: closed_segment_eq_real_ivl split: if_split_asm)
        then have y=u ∨ subpath 0 u g (y/u) ∈ interior S
          using P less_eq_real_def by force
        then show g y = g u
          using y by (auto simp: subpath_def split: if_split_asm)
      qed
    qed
  qed (use ‹0 ≤ u› ‹u ≤ 1› in auto)
```

qed

lemma *exists_path_subpath_to_frontier*:
fixes *S* :: *'a::real_normed_vector set*
assumes *path g pathstart g* ∈ *closure S pathfinish g* ∉ *S*
obtains *h* **where** *path h pathstart h* = *pathstart g path_image h* ⊆ *path_image g*
path_image h − {*pathfinish h*} ⊆ *interior S*
pathfinish h ∈ *frontier S*
proof −
obtain *u* **where** *u*: *0* ≤ *u u* ≤ *1 g u* ∈ *frontier S* (*path_image*(*subpath 0 u g*) − {*g u*}) ⊆ *interior S*
using *subpath_to_frontier* [*OF assms*] **by** *blast*
show *?thesis*
proof
show *path_image* (*subpath 0 u g*) ⊆ *path_image g*
by (*simp add*: *path_image_subpath_subset u*)
show *pathstart* (*subpath 0 u g*) = *pathstart g*
by (*metis pathstart_def pathstart_subpath*)
qed (*use assms u* **in** ‹*auto simp*: *path_image_subpath*›)
qed

lemma *exists_path_subpath_to_frontier_closed*:
fixes *S* :: *'a::real_normed_vector set*
assumes *S*: *closed S* **and** *g*: *path g* **and** *g0*: *pathstart g* ∈ *S* **and** *g1*: *pathfinish g* ∉ *S*
obtains *h* **where** *path h pathstart h* = *pathstart g path_image h* ⊆ *path_image g* ∩ *S*
pathfinish h ∈ *frontier S*
proof −
obtain *h* **where** *h*: *path h pathstart h* = *pathstart g path_image h* ⊆ *path_image g*
path_image h − {*pathfinish h*} ⊆ *interior S*
pathfinish h ∈ *frontier S*
using *exists_path_subpath_to_frontier* [*OF g _ g1*] *closure_closed* [*OF S*] *g0* **by** *auto*
show *?thesis*
proof
show *path_image h* ⊆ *path_image g* ∩ *S*
using *assms h interior_subset* [*of S*] **by** (*auto simp*: *frontier_def*)
qed (*use h* **in** *auto*)
qed

5.5.11 Shift Path to Start at Some Given Point

definition *shiftpath* :: *real* ⇒ (*real* ⇒ *'a::topological_space*) ⇒ *real* ⇒ *'a*
where *shiftpath a f* = (λ*x*. *if* (*a* + *x*) ≤ *1 then f* (*a* + *x*) *else f* (*a* + *x* − *1*))

lemma *shiftpath_alt_def*: *shiftpath a f* = (λ*x*. *if x* ≤ *1*−*a then f* (*a* + *x*) *else f* (*a*

+ x − 1))
  by (*auto simp*: *shiftpath_def*)

lemma *pathstart_shiftpath*: *a* ≤ *1* ⟹ *pathstart* (*shiftpath a g*) = *g a*
  unfolding *pathstart_def shiftpath_def* **by** *auto*

lemma *pathfinish_shiftpath*:
  assumes *0* ≤ *a*
    and *pathfinish g* = *pathstart g*
  shows *pathfinish* (*shiftpath a g*) = *g a*
  using *assms*
  unfolding *pathstart_def pathfinish_def shiftpath_def*
  by *auto*

lemma *endpoints_shiftpath*:
  assumes *pathfinish g* = *pathstart g*
    and *a* ∈ {*0* .. *1*}
  shows *pathfinish* (*shiftpath a g*) = *g a*
    and *pathstart* (*shiftpath a g*) = *g a*
  using *assms*
  by (*auto intro*!: *pathfinish_shiftpath pathstart_shiftpath*)

lemma *closed_shiftpath*:
  assumes *pathfinish g* = *pathstart g*
    and *a* ∈ {*0*..*1*}
  shows *pathfinish* (*shiftpath a g*) = *pathstart* (*shiftpath a g*)
  using *endpoints_shiftpath*[*OF assms*]
  by *auto*

lemma *path_shiftpath*:
  assumes *path g*
    and *pathfinish g* = *pathstart g*
    and *a* ∈ {*0*..*1*}
  shows *path* (*shiftpath a g*)
proof −
  have ∗: {*0* .. *1*} = {*0* .. *1*−*a*} ∪ {*1*−*a* .. *1*}
    using *assms*(*3*) **by** *auto*
  have ∗∗: ⋀*x*. *x* + *a* = *1* ⟹ *g* (*x* + *a* − *1*) = *g* (*x* + *a*)
    using *assms*(*2*)[*unfolded pathfinish_def pathstart_def*]
    by *auto*
  show *?thesis*
    unfolding *path_def shiftpath_def* ∗
  proof (*rule continuous_on_closed_Un*)
    have *contg*: *continuous_on* {*0*..*1*} *g*
      using ‹*path g*› *path_def* **by** *blast*
    show *continuous_on* {*0*..*1*−*a*} (λ*x*. *if a* + *x* ≤ *1 then g* (*a* + *x*) *else g* (*a* + *x*
− *1*))
    proof (*rule continuous_on_eq*)
      show *continuous_on* {*0*..*1*−*a*} (*g* ∘ (+) *a*)

by (*intro continuous_intros continuous_on_subset* [*OF contg*]) (*use* ‹$a \in \{0..1\}$› **in** *auto*)
qed *auto*
show *continuous_on* $\{1-a..1\}$ (λx. *if* $a + x \leq 1$ *then* g ($a + x$) *else* g ($a + x - 1$))
proof (*rule continuous_on_eq*)
show *continuous_on* $\{1-a..1\}$ ($g \circ (+)$ ($a - 1$))
by (*intro continuous_intros continuous_on_subset* [*OF contg*]) (*use* ‹$a \in \{0..1\}$› **in** *auto*)
qed (*auto simp*: ∗∗ *add.commute add_diff_eq*)
qed *auto*
qed

lemma *shiftpath_shiftpath*:
assumes *pathfinish g* = *pathstart g*
and $a \in \{0..1\}$
and $x \in \{0..1\}$
shows *shiftpath* ($1 - a$) (*shiftpath a g*) $x = g\ x$
using *assms*
unfolding *pathfinish_def pathstart_def shiftpath_def*
by *auto*

lemma *path_image_shiftpath*:
assumes *a*: $a \in \{0..1\}$
and *pathfinish g* = *pathstart g*
shows *path_image* (*shiftpath a g*) = *path_image g*
proof −
{ fix *x*
assume *g*: $g\ 1 = g\ 0\ x \in \{0..1::real\}$ **and** *gne*: $\bigwedge y.\ y \in \{0..1\} \cap \{x.\ \neg\ a + x \leq 1\} \Longrightarrow g\ x \neq g\ (a + y - 1)$
then have $\exists y \in \{0..1\} \cap \{x.\ a + x \leq 1\}.\ g\ x = g\ (a + y)$
proof (*cases* $a \leq x$)
case *False*
then show *?thesis*
apply (*rule_tac* $x=1 + x - a$ **in** *bexI*)
using *g gne*[*of* $1 + x - a$] *a* **by** (*force simp*: *field_simps*)+
next
case *True*
then show *?thesis*
using *g a* **by** (*rule_tac* $x=x - a$ **in** *bexI*) (*auto simp*: *field_simps*)
qed
}
then show *?thesis*
using *assms*
unfolding *shiftpath_def path_image_def pathfinish_def pathstart_def*
by (*auto simp*: *image_iff*)
qed

lemma *simple_path_shiftpath*:

assumes *simple_path g pathfinish g* = *pathstart g* **and** *a*: *0* ≤ *a a* ≤ *1*
shows *simple_path* (*shiftpath a g*)
unfolding *simple_path_def*
proof (*intro conjI impI ballI*)
show *path* (*shiftpath a g*)
by (*simp add*: *assms path_shiftpath simple_path_imp_path*)
have ∗: ⋀*x y*. ⟦*g x* = *g y*; *x* ∈ {*0..1*}; *y* ∈ {*0..1*}⟧ ⟹ *x* = *y* ∨ *x* = *0* ∧ *y* = *1* ∨ *x* = *1* ∧ *y* = *0*
using *assms* **by** (*simp add*: *simple_path_def*)
show *x* = *y* ∨ *x* = *0* ∧ *y* = *1* ∨ *x* = *1* ∧ *y* = *0*
if *x* ∈ {*0..1*} *y* ∈ {*0..1*} *shiftpath a g x* = *shiftpath a g y* **for** *x y*
using *that a* **unfolding** *shiftpath_def*
by (*force split*: *if_split_asm dest*!: ∗)
qed

5.5.12 Straight-Line Paths

definition *linepath* :: *'a*::*real_normed_vector* ⇒ *'a* ⇒ *real* ⇒ *'a*
where *linepath a b* = (λ*x*. (*1* − *x*) $*_R$ *a* + *x* $*_R$ *b*)

lemma *pathstart_linepath*[*simp*]: *pathstart* (*linepath a b*) = *a*
unfolding *pathstart_def linepath_def*
by *auto*

lemma *pathfinish_linepath*[*simp*]: *pathfinish* (*linepath a b*) = *b*
unfolding *pathfinish_def linepath_def*
by *auto*

lemma *linepath_inner*: *linepath a b x* · *v* = *linepath* (*a* · *v*) (*b* · *v*) *x*
by (*simp add*: *linepath_def algebra_simps*)

lemma *Re_linepath'*: *Re* (*linepath a b x*) = *linepath* (*Re a*) (*Re b*) *x*
by (*simp add*: *linepath_def*)

lemma *Im_linepath'*: *Im* (*linepath a b x*) = *linepath* (*Im a*) (*Im b*) *x*
by (*simp add*: *linepath_def*)

lemma *linepath_0'*: *linepath a b 0* = *a*
by (*simp add*: *linepath_def*)

lemma *linepath_1'*: *linepath a b 1* = *b*
by (*simp add*: *linepath_def*)

lemma *continuous_linepath_at*[*intro*]: *continuous* (*at x*) (*linepath a b*)
unfolding *linepath_def*
by (*intro continuous_intros*)

lemma *continuous_on_linepath* [*intro*,*continuous_intros*]: *continuous_on s* (*linepath a b*)

using *continuous_linepath_at*
by (*auto intro*!: *continuous_at_imp_continuous_on*)

lemma *path_linepath*[*iff*]: *path* (*linepath a b*)
unfolding *path_def*
by (*rule continuous_on_linepath*)

lemma *path_image_linepath*[*simp*]: *path_image* (*linepath a b*) = *closed_segment a b*
unfolding *path_image_def segment linepath_def*
by *auto*

lemma *reversepath_linepath*[*simp*]: *reversepath* (*linepath a b*) = *linepath b a*
unfolding *reversepath_def linepath_def*
by *auto*

lemma *linepath_0* [*simp*]: *linepath 0 b x* = $x *_R b$
by (*simp add*: *linepath_def*)

lemma *linepath_cnj*: *cnj* (*linepath a b x*) = *linepath* (*cnj a*) (*cnj b*) *x*
by (*simp add*: *linepath_def*)

lemma *arc_linepath*:
assumes $a \neq b$ **shows** [*simp*]: *arc* (*linepath a b*)
proof –
{
fix *x y* :: *real*
assume $x *_R b + y *_R a = x *_R a + y *_R b$
then have $(x - y) *_R a = (x - y) *_R b$
by (*simp add*: *algebra_simps*)
with *assms* **have** $x = y$
by *simp*
}
then show *?thesis*
unfolding *arc_def inj_on_def*
by (*fastforce simp*: *algebra_simps linepath_def*)
qed

lemma *simple_path_linepath*[*intro*]: $a \neq b \Longrightarrow$ *simple_path* (*linepath a b*)
by (*simp add*: *arc_imp_simple_path*)

lemma *linepath_trivial* [*simp*]: *linepath a a x* = *a*
by (*simp add*: *linepath_def real_vector.scale_left_diff_distrib*)

lemma *linepath_refl*: *linepath a a* = ($\lambda x.$ *a*)
by *auto*

lemma *subpath_refl*: *subpath a a g* = *linepath* (*g a*) (*g a*)
by (*simp add*: *subpath_def linepath_def algebra_simps*)

lemma *linepath_of_real*: (*linepath* (*of_real a*) (*of_real b*) *x*) = *of_real* ((*1* − *x*)∗*a* + *x*∗*b*)
  by (*simp add*: *scaleR_conv_of_real linepath_def*)

lemma *of_real_linepath*: *of_real* (*linepath a b x*) = *linepath* (*of_real a*) (*of_real b*) *x*
  by (*metis linepath_of_real mult.right_neutral of_real_def real_scaleR_def*)

lemma *inj_on_linepath*:
  assumes *a* ≠ *b* **shows** *inj_on* (*linepath a b*) {*0..1*}
proof (*clarsimp simp*: *inj_on_def linepath_def*)
  fix *x y*
  assume (*1* − *x*) $*_R$ *a* + *x* $*_R$ *b* = (*1* − *y*) $*_R$ *a* + *y* $*_R$ *b* *0* ≤ *x* *x* ≤ *1* *0* ≤ *y* *y* ≤ *1*
  then have *x* $*_R$ (*a* − *b*) = *y* $*_R$ (*a* − *b*)
    by (*auto simp*: *algebra_simps*)
  then show *x*=*y*
    using *assms* **by** *auto*
qed

lemma *linepath_le_1*:
  fixes *a*::′*a*::*linordered_idom* **shows** ⟦*a* ≤ *1*; *b* ≤ *1*; *0* ≤ *u*; *u* ≤ *1*⟧ ⟹ (*1* − *u*) ∗ *a* + *u* ∗ *b* ≤ *1*
  using *mult_left_le* [*of a 1*−*u*] *mult_left_le* [*of b u*] **by** *auto*

lemma *linepath_in_path*:
  shows *x* ∈ {*0..1*} ⟹ *linepath a b x* ∈ *closed_segment a b*
  by (*auto simp*: *segment linepath_def*)

lemma *linepath_image_01*: *linepath a b* ‘ {*0..1*} = *closed_segment a b*
  by (*auto simp*: *segment linepath_def*)

lemma *linepath_in_convex_hull*:
  fixes *x*::*real*
  assumes *a*: *a* ∈ *convex hull S*
    and *b*: *b* ∈ *convex hull S*
    and *x*: *0*≤*x* *x*≤*1*
  shows *linepath a b x* ∈ *convex hull S*
proof −
  have *linepath a b x* ∈ *closed_segment a b*
    using *x* **by** (*auto simp flip*: *linepath_image_01*)
  then show *?thesis*
    using *a b convex_contains_segment* **by** *blast*
qed

lemma *Re_linepath*: *Re*(*linepath* (*of_real a*) (*of_real b*) *x*) = (*1* − *x*)∗*a* + *x*∗*b*
  by (*simp add*: *linepath_def*)

lemma *Im_linepath*: *Im*(*linepath* (*of_real a*) (*of_real b*) *x*) = *0*
  by (*simp add*: *linepath_def*)

lemma *bounded_linear_linepath*:
assumes *bounded_linear f*
shows *f* (*linepath a b x*) = *linepath* (*f a*) (*f b*) *x*
proof −
interpret *f*: *bounded_linear f* **by** *fact*
show *?thesis* **by** (*simp add*: *linepath_def f.add f.scale*)
qed

lemma *bounded_linear_linepath'*:
assumes *bounded_linear f*
shows *f* ∘ *linepath a b* = *linepath* (*f a*) (*f b*)
using *bounded_linear_linepath*[*OF assms*] **by** (*simp add*: *fun_eq_iff*)

lemma *linepath_cnj'*: *cnj* ∘ *linepath a b* = *linepath* (*cnj a*) (*cnj b*)
by (*simp add*: *linepath_def fun_eq_iff*)

lemma *differentiable_linepath* [*intro*]: *linepath a b differentiable at x within A*
by (*auto simp*: *linepath_def*)

lemma *has_vector_derivative_linepath_within*:
(*linepath a b has_vector_derivative* (*b* − *a*)) (*at x within S*)
by (*force intro*: *derivative_eq_intros simp add*: *linepath_def has_vector_derivative_def algebra_simps*)

5.5.13 Segments via convex hulls

lemma *segments_subset_convex_hull*:
closed_segment a b ⊆ (*convex hull* {*a*,*b*,*c*})
closed_segment a c ⊆ (*convex hull* {*a*,*b*,*c*})
closed_segment b c ⊆ (*convex hull* {*a*,*b*,*c*})
closed_segment b a ⊆ (*convex hull* {*a*,*b*,*c*})
closed_segment c a ⊆ (*convex hull* {*a*,*b*,*c*})
closed_segment c b ⊆ (*convex hull* {*a*,*b*,*c*})
by (*auto simp*: *segment_convex_hull linepath_of_real elim*!: *rev_subsetD* [*OF _ hull_mono*])

lemma *midpoints_in_convex_hull*:
assumes *x* ∈ *convex hull s y* ∈ *convex hull s*
shows *midpoint x y* ∈ *convex hull s*
proof −
have (*1* − *inverse*(*2*)) $*_R$ *x* + *inverse*(*2*) $*_R$ *y* ∈ *convex hull s*
by (*rule convexD_alt*) (*use assms* **in** *auto*)
then show *?thesis*
by (*simp add*: *midpoint_def algebra_simps*)
qed

lemma *not_in_interior_convex_hull_3*:
fixes *a* :: *complex*
shows *a* ∉ *interior*(*convex hull* {*a*,*b*,*c*})

```
    b ∉ interior(convex hull {a,b,c})
    c ∉ interior(convex hull {a,b,c})
  by (auto simp: card_insert_le_m1 not_in_interior_convex_hull)

lemma midpoint_in_closed_segment [simp]: midpoint a b ∈ closed_segment a b
  using midpoints_in_convex_hull segment_convex_hull by blast

lemma midpoint_in_open_segment [simp]: midpoint a b ∈ open_segment a b ⟷ a
≠ b
  by (simp add: open_segment_def)

lemma continuous_IVT_local_extremum:
  fixes f :: 'a::euclidean_space ⇒ real
  assumes contf: continuous_on (closed_segment a b) f
      and a ≠ b f a = f b
  obtains z where z ∈ open_segment a b
               (∀ w ∈ closed_segment a b. (f w) ≤ (f z)) ∨
                (∀ w ∈ closed_segment a b. (f z) ≤ (f w))
proof −
  obtain c where c ∈ closed_segment a b and c: ⋀y. y ∈ closed_segment a b ⟹
f y ≤ f c
    using continuous_attains_sup [of closed_segment a b f] contf by auto
  obtain d where d ∈ closed_segment a b and d: ⋀y. y ∈ closed_segment a b ⟹
f d ≤ f y
    using continuous_attains_inf [of closed_segment a b f] contf by auto
  show ?thesis
  proof (cases c ∈ open_segment a b ∨ d ∈ open_segment a b)
    case True
    then show ?thesis
      using c d that by blast
  next
    case False
    then have (c = a ∨ c = b) ∧ (d = a ∨ d = b)
     by (simp add: ‹c ∈ closed_segment a b› ‹d ∈ closed_segment a b› open_segment_def)
    with ‹a ≠ b› ‹f a = f b› c d show ?thesis
      by (rule_tac z = midpoint a b in that) (fastforce+)
  qed
qed
```

An injective map into R is also an open map w.r.T. the universe, and conversely.

```
proposition injective_eq_1d_open_map_UNIV:
  fixes f :: real ⇒ real
  assumes contf: continuous_on S f and S: is_interval S
    shows inj_on f S ⟷ (∀ T. open T ∧ T ⊆ S ⟶ open(f ` T))
        (is ?lhs = ?rhs)
proof safe
  fix T
  assume injf: ?lhs and open T and T ⊆ S
```

have ∃ *U*. *open U* ∧ *f x* ∈ *U* ∧ *U* ⊆ *f* ' *T* **if** *x* ∈ *T* **for** *x*
proof −
obtain δ **where** δ > *0* **and** δ: *cball x* δ ⊆ *T*
using ‹*open T*› ‹*x* ∈ *T*› *open_contains_cball_eq* **by** *blast*
show *?thesis*
proof (*intro exI conjI*)
have *closed_segment* (*x*−δ) (*x*+δ) = {*x*−δ..*x*+δ}
using ‹*0* < δ› **by** (*auto simp*: *closed_segment_eq_real_ivl*)
also have ... ⊆ *S*
using δ ‹*T* ⊆ *S*› **by** (*auto simp*: *dist_norm subset_eq*)
finally have *f* ' (*open_segment* (*x*−δ) (*x*+δ)) = *open_segment* (*f* (*x*−δ)) (*f* (*x*+δ))
using *continuous_injective_image_open_segment_1*
by (*metis continuous_on_subset* [*OF contf*] *inj_on_subset* [*OF injf*])
then show *open* (*f* ' {*x*−δ<..<*x*+δ})
using ‹*0* < δ› **by** (*simp add*: *open_segment_eq_real_ivl*)
show *f x* ∈ *f* ' {*x* − δ<..<*x* + δ}
by (*auto simp*: ‹δ > *0*›)
show *f* ' {*x* − δ<..<*x* + δ} ⊆ *f* ' *T*
using δ **by** (*auto simp*: *dist_norm subset_iff*)
qed
qed
with *open_subopen* **show** *open* (*f* ' *T*)
by *blast*
next
assume *R*: *?rhs*
have *False* **if** *xy*: *x* ∈ *S y* ∈ *S* **and** *f x* = *f y x* ≠ *y* **for** *x y*
proof −
have *open* (*f* ' *open_segment x y*)
using *R*
by (*metis S convex_contains_open_segment is_interval_convex open_greaterThanLessThan open_segment_eq_real_ivl xy*)
moreover
have *continuous_on* (*closed_segment x y*) *f*
by (*meson S closed_segment_subset contf continuous_on_subset is_interval_convex that*)
then obtain ξ **where** ξ ∈ *open_segment x y*
and ξ: (∀ *w* ∈ *closed_segment x y*. (*f w*) ≤ (*f* ξ)) ∨
(∀ *w* ∈ *closed_segment x y*. (*f* ξ) ≤ (*f w*))
using *continuous_IVT_local_extremum* [*of x y f*] ‹*f x* = *f y*› ‹*x* ≠ *y*› **by** *blast*
ultimately obtain *e* **where** *e*>*0* **and** *e*: ⋀*u*. *dist u* (*f* ξ) < *e* ⟹ *u* ∈ *f* ' *open_segment x y*
using *open_dist* **by** (*metis image_eqI*)
have *fin*: *f* ξ + (*e*/*2*) ∈ *f* ' *open_segment x y f* ξ − (*e*/*2*) ∈ *f* ' *open_segment x y*
using *e* [*of f* ξ + (*e*/*2*)] *e* [*of f* ξ − (*e*/*2*)] ‹*e* > *0*› **by** (*auto simp*: *dist_norm*)
show *?thesis*
using ξ ‹*0* < *e*› *fin open_closed_segment* **by** *fastforce*
qed

then show *?lhs*
  by (*force simp*: *inj_on_def*)
qed

5.5.14 Bounding a point away from a path

lemma *not_on_path_ball*:
  fixes *g* :: *real* ⇒ *'a*::*heine_borel*
  assumes *path g*
    and *z*: *z* ∉ *path_image g*
  shows ∃ *e* > *0*. *ball z e* ∩ *path_image g* = {}
proof −
  have *closed* (*path_image g*)
    by (*simp add*: ‹*path g*› *closed_path_image*)
  then obtain *a* **where** *a* ∈ *path_image g* ∀ *y* ∈ *path_image g*. *dist z a* ≤ *dist z y*
    by (*auto intro*: *distance_attains_inf*[*OF* _ *path_image_nonempty*, *of g z*])
  then show *?thesis*
    by (*rule_tac x*=*dist z a* **in** *exI*) (*use dist_commute z* **in** *auto*)
qed

lemma *not_on_path_cball*:
  fixes *g* :: *real* ⇒ *'a*::*heine_borel*
  assumes *path g*
    and *z* ∉ *path_image g*
  shows ∃ *e*>*0*. *cball z e* ∩ (*path_image g*) = {}
proof −
  obtain *e* **where** *ball z e* ∩ *path_image g* = {} *e* > *0*
    using *not_on_path_ball*[*OF assms*] **by** *auto*
  moreover have *cball z* (*e*/*2*) ⊆ *ball z e*
    using ‹*e* > *0*› **by** *auto*
  ultimately show *?thesis*
    by (*rule_tac x*=*e*/*2* **in** *exI*) *auto*
qed

5.5.15 Path component

Original formalization by Tom Hales

definition *path_component S x y* ≡
  (∃ *g*. *path g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*)

abbreviation
  path_component_set S x ≡ *Collect* (*path_component S x*)

lemmas *path_defs* = *path_def pathstart_def pathfinish_def path_image_def path_component_def*

lemma *path_component_mem*:
  assumes *path_component S x y*
  shows *x* ∈ *S* **and** *y* ∈ *S*
  using *assms*

unfolding *path_defs*
by *auto*

lemma *path_component_refl*:
assumes $x \in S$
shows *path_component* $S\ x\ x$
using *assms*
unfolding *path_defs*
by (*metis* (*full_types*) *assms continuous_on_const image_subset_iff path_image_def*)

lemma *path_component_refl_eq*: *path_component* $S\ x\ x \longleftrightarrow x \in S$
by (*auto intro*!: *path_component_mem path_component_refl*)

lemma *path_component_sym*: *path_component* $S\ x\ y \Longrightarrow$ *path_component* $S\ y\ x$
unfolding *path_component_def*
by (*metis* (*no_types*) *path_image_reversepath path_reversepath pathfinish_reversepath pathstart_reversepath*)

lemma *path_component_trans*:
assumes *path_component* $S\ x\ y$ **and** *path_component* $S\ y\ z$
shows *path_component* $S\ x\ z$
using *assms*
unfolding *path_component_def*
by (*metis path_join pathfinish_join pathstart_join subset_path_image_join*)

lemma *path_component_of_subset*: $S \subseteq T \Longrightarrow$ *path_component* $S\ x\ y \Longrightarrow$ *path_component* $T\ x\ y$
unfolding *path_component_def* **by** *auto*

lemma *path_component_linepath*:
fixes S :: *'a::real_normed_vector set*
shows *closed_segment* $a\ b \subseteq S \Longrightarrow$ *path_component* $S\ a\ b$
unfolding *path_component_def*
by (*rule_tac x=linepath a b* **in** *exI*, *auto*)

Path components as sets

lemma *path_component_set*:
path_component_set $S\ x =$
$\{y.\ (\exists\, g.$ *path* $g \wedge$ *path_image* $g \subseteq S \wedge$ *pathstart* $g = x \wedge$ *pathfinish* $g = y)\}$
by (*auto simp*: *path_component_def*)

lemma *path_component_subset*: *path_component_set* $S\ x \subseteq S$
by (*auto simp*: *path_component_mem*(*2*))

lemma *path_component_eq_empty*: *path_component_set* $S\ x = \{\} \longleftrightarrow x \notin S$
using *path_component_mem path_component_refl_eq*
by *fastforce*

lemma *path_component_mono*:
$S \subseteq T \Longrightarrow$ (*path_component_set S x*) $\subseteq$ (*path_component_set T x*)
by (*simp add*: *Collect_mono path_component_of_subset*)

lemma *path_component_eq*:
$y \in$ *path_component_set S x* $\Longrightarrow$ *path_component_set S y* = *path_component_set S x*
by (*metis* (*no_types*, *lifting*) *Collect_cong mem_Collect_eq path_component_sym path_component_trans*)

5.5.16 Path connectedness of a space

definition *path_connected S* $\longleftrightarrow$
($\forall x \in S$. $\forall y \in S$. $\exists g$. *path g* $\wedge$ *path_image g* $\subseteq S$ $\wedge$ *pathstart g* = x $\wedge$ *pathfinish g* = y)

lemma *path_connectedin_iff_path_connected_real* [*simp*]:
path_connectedin euclideanreal S $\longleftrightarrow$ *path_connected S*
by (*simp add*: *path_connectedin path_connected_def path_defs*)

lemma *path_connected_component*: *path_connected S* $\longleftrightarrow$ ($\forall x \in S$. $\forall y \in S$. *path_component S x y*)
unfolding *path_connected_def path_component_def* **by** *auto*

lemma *path_connected_component_set*: *path_connected S* $\longleftrightarrow$ ($\forall x \in S$. *path_component_set S x* = S)
unfolding *path_connected_component path_component_subset*
using *path_component_mem* **by** *blast*

lemma *path_component_maximal*:
$[\![x \in T$; *path_connected T*; $T \subseteq S]\!] \Longrightarrow T \subseteq$ (*path_component_set S x*)
by (*metis path_component_mono path_connected_component_set*)

lemma *convex_imp_path_connected*:
fixes *S* :: *'a::real_normed_vector set*
assumes *convex S*
shows *path_connected S*
unfolding *path_connected_def*
using *assms convex_contains_segment* **by** *fastforce*

lemma *path_connected_UNIV* [*iff*]: *path_connected* (*UNIV* :: *'a::real_normed_vector set*)
by (*simp add*: *convex_imp_path_connected*)

lemma *path_component_UNIV*: *path_component_set UNIV x* = (*UNIV* :: *'a::real_normed_vector set*)
using *path_connected_component_set* **by** *auto*

lemma *path_connected_imp_connected*:
assumes *path_connected S*

```
  shows connected S
proof (rule connectedI)
  fix e1 e2
  assume as: open e1 open e2 S ⊆ e1 ∪ e2 e1 ∩ e2 ∩ S = {} e1 ∩ S ≠ {} e2 ∩
S ≠ {}
  then obtain x1 x2 where obt:x1 ∈ e1 ∩ S x2 ∈ e2 ∩ S
    by auto
  then obtain g where g: path g path_image g ⊆ S pathstart g = x1 pathfinish g
= x2
    using assms[unfolded path_connected_def,rule_format,of x1 x2] by auto
  have *: connected {0..1::real}
    by (auto intro!: convex_connected)
  have {0..1} ⊆ {x ∈ {0..1}. g x ∈ e1} ∪ {x ∈ {0..1}. g x ∈ e2}
    using as(3) g(2)[unfolded path_defs] by blast
  moreover have {x ∈ {0..1}. g x ∈ e1} ∩ {x ∈ {0..1}. g x ∈ e2} = {}
    using as(4) g(2)[unfolded path_defs]
    unfolding subset_eq
    by auto
  moreover have {x ∈ {0..1}. g x ∈ e1} ≠ {} ∧ {x ∈ {0..1}. g x ∈ e2} ≠ {}
    using g(3,4)[unfolded path_defs]
    using obt
    by (simp add: ex_in_conv [symmetric], metis zero_le_one order_refl)
  ultimately show False
    using *[unfolded connected_local not_ex, rule_format,
      of {0..1} ∩ g −‘ e1 {0..1} ∩ g −‘ e2]
    using continuous_openin_preimage_gen[OF g(1)[unfolded path_def] as(1)]
    using continuous_openin_preimage_gen[OF g(1)[unfolded path_def] as(2)]
    by auto
qed

lemma open_path_component:
  fixes S :: 'a::real_normed_vector set
  assumes open S
  shows open (path_component_set S x)
  unfolding open_contains_ball
proof
  fix y
  assume as: y ∈ path_component_set S x
  then have y ∈ S
    by (simp add: path_component_mem(2))
  then obtain e where e: e > 0 ball y e ⊆ S
    using assms openE by blast
have ⋀u. dist y u < e ⟹ path_component S x u
     by (metis (full_types) as centre_in_ball convex_ball convex_imp_path_connected e
mem_Collect_eq mem_ball path_component_eq path_component_of_subset path_connected_component)
  then show ∃ e > 0. ball y e ⊆ path_component_set S x
    using ‹e>0› by auto
qed
```

lemma *open_non_path_component*:
  fixes *S* :: *'a::real_normed_vector set*
  assumes *open S*
  shows *open* (*S* − *path_component_set S x*)
  unfolding *open_contains_ball*
proof
  fix *y*
  assume *y*: *y* ∈ *S* − *path_component_set S x*
  then obtain *e* **where** *e*: *e* > *0 ball y e* ⊆ *S*
    using *assms openE* **by** *auto*
  show ∃ *e*>*0*. *ball y e* ⊆ *S* − *path_component_set S x*
  proof (*intro exI conjI subsetI DiffI notI*)
    show ⋀*x*. *x* ∈ *ball y e* ⟹ *x* ∈ *S*
      using *e* **by** *blast*
    show *False* **if** *z* ∈ *ball y e z* ∈ *path_component_set S x* **for** *z*
    proof −
      have *y* ∈ *path_component_set S z*
      by (*meson assms convex_ball convex_imp_path_connected e open_contains_ball_eq open_path_component path_component_maximal that*(*1*))
      then have *y* ∈ *path_component_set S x*
        using *path_component_eq that*(*2*) **by** *blast*
      then show *False*
        using *y* **by** *blast*
    qed
  qed (*use e* **in** *auto*)
qed

lemma *connected_open_path_connected*:
  fixes *S* :: *'a::real_normed_vector set*
  assumes *open S*
    and *connected S*
  shows *path_connected S*
  unfolding *path_connected_component_set*
proof (*rule*, *rule*, *rule path_component_subset*, *rule*)
  fix *x y*
  assume *x* ∈ *S* **and** *y* ∈ *S*
  show *y* ∈ *path_component_set S x*
  proof (*rule ccontr*)
    assume ¬ *?thesis*
    moreover have *path_component_set S x* ∩ *S* ≠ {}
      using ‹*x* ∈ *S*› *path_component_eq_empty path_component_subset*[*of S x*]
      by *auto*
    ultimately
    show *False*
    using ‹*y* ∈ *S*› *open_non_path_component*[*OF assms*(*1*)] *open_path_component*[*OF assms*(*1*)]
      using *assms*(*2*)[*unfolded connected_def not_ex*, *rule_format*,
        of path_component_set S x S − *path_component_set S x*]
      by *auto*

qed
qed

lemma *path_connected_continuous_image*:
assumes *contf*: *continuous_on S f*
and *path_connected S*
shows *path_connected* (*f* ‘ *S*)
unfolding *path_connected_def*
proof (*rule*, *rule*)
fix x' y'
assume $x' \in f$ ‘ S $y' \in f$ ‘ S
then obtain *x y* **where** *x*: $x \in S$ **and** *y*: $y \in S$ **and** x': $x' = f\ x$ **and** y': $y' = f\ y$
by *auto*
from *x y* **obtain** *g* **where** *path g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*
using *assms*(*2*)[*unfolded path_connected_def*] **by** *fast*
then show ∃ *g*. *path g* ∧ *path_image g* ⊆ *f* ‘ *S* ∧ *pathstart g* = x' ∧ *pathfinish g* = y'
unfolding x' y' *path_defs*
by (*fastforce intro*: *continuous_on_compose continuous_on_subset*[*OF contf*])
qed

lemma *path_connected_translationI*:
fixes *a* :: ′*a* :: *topological_group_add*
assumes *path_connected S* **shows** *path_connected* ((λ*x*. *a* + *x*) ‘ *S*)
by (*intro path_connected_continuous_image assms continuous_intros*)

lemma *path_connected_translation*:
fixes *a* :: ′*a* :: *topological_group_add*
shows *path_connected* ((λ*x*. *a* + *x*) ‘ *S*) = *path_connected S*
proof −
have ∀ *x y*. (+) (*x*::′*a*) ‘ (+) (*0* − *x*) ‘ *y* = *y*
by (*simp add*: *image_image*)
then show *?thesis*
by (*metis* (*no_types*) *path_connected_translationI*)
qed

lemma *path_connected_segment* [*simp*]:
fixes *a* :: ′*a*::*real_normed_vector*
shows *path_connected* (*closed_segment a b*)
by (*simp add*: *convex_imp_path_connected*)

lemma *path_connected_open_segment* [*simp*]:
fixes *a* :: ′*a*::*real_normed_vector*
shows *path_connected* (*open_segment a b*)
by (*simp add*: *convex_imp_path_connected*)

lemma *homeomorphic_path_connectedness*:

```
  S homeomorphic T ⟹ path_connected S ⟷ path_connected T
 unfolding homeomorphic_def homeomorphism_def by (metis path_connected_continuous_image)

lemma path_connected_empty [simp]: path_connected {}
  unfolding path_connected_def by auto

lemma path_connected_singleton [simp]: path_connected {a}
  unfolding path_connected_def pathstart_def pathfinish_def path_image_def
  using path_def by fastforce

lemma path_connected_Un:
  assumes path_connected S
    and path_connected T
    and S ∩ T ≠ {}
  shows path_connected (S ∪ T)
  unfolding path_connected_component
proof (intro ballI)
  fix x y
  assume x: x ∈ S ∪ T and y: y ∈ S ∪ T
  from assms obtain z where z: z ∈ S z ∈ T
    by auto
  show path_component (S ∪ T) x y
    using x y
  proof safe
    assume x ∈ S y ∈ S
    then show path_component (S ∪ T) x y
     by (meson Un_upper1 ‹path_connected S› path_component_of_subset path_connected_component)
  next
    assume x ∈ S y ∈ T
    then show path_component (S ∪ T) x y
     by (metis z assms(1−2) le_sup_iff order_refl path_component_of_subset path_component_trans
path_connected_component)
  next
  assume x ∈ T y ∈ S
    then show path_component (S ∪ T) x y
     by (metis z assms(1−2) le_sup_iff order_refl path_component_of_subset path_component_trans
path_connected_component)
  next
    assume x ∈ T y ∈ T
    then show path_component (S ∪ T) x y
     by (metis Un_upper1 assms(2) path_component_of_subset path_connected_component
sup_commute)
  qed
qed

lemma path_connected_UNION:
  assumes ⋀i. i ∈ A ⟹ path_connected (S i)
    and ⋀i. i ∈ A ⟹ z ∈ S i
  shows path_connected (⋃i∈A. S i)
```

```
  unfolding path_connected_component
proof clarify
  fix x i y j
  assume *: i ∈ A x ∈ S i j ∈ A y ∈ S j
  then have path_component (S i) x z and path_component (S j) z y
    using assms by (simp_all add: path_connected_component)
  then have path_component (⋃ i∈A. S i) x z and path_component (⋃ i∈A. S i)
z y
    using *(1,3) by (auto elim!: path_component_of_subset [rotated])
  then show path_component (⋃ i∈A. S i) x y
    by (rule path_component_trans)
qed

lemma path_component_path_image_pathstart:
  assumes p: path p and x: x ∈ path_image p
  shows path_component (path_image p) (pathstart p) x
proof -
  obtain y where x: x = p y and y: 0 ≤ y y ≤ 1
    using x by (auto simp: path_image_def)
  show ?thesis
    unfolding path_component_def
  proof (intro exI conjI)
    have continuous_on ((*) y ` {0..1}) p
      by (simp add: continuous_on_path image_mult_atLeastAtMost_if p y)
    then have continuous_on {0..1} (p ∘ ((*) y))
      using continuous_on_compose continuous_on_mult_const by blast
    then show path (λu. p (y * u))
      by (simp add: path_def)
    show path_image (λu. p (y * u)) ⊆ path_image p
      using y mult_le_one by (fastforce simp: path_image_def image_iff)
  qed (auto simp: pathstart_def pathfinish_def x)
qed

lemma path_connected_path_image: path p ⟹ path_connected(path_image p)
  unfolding path_connected_component
 by (meson path_component_path_image_pathstart path_component_sym path_component_trans)

lemma path_connected_path_component [simp]:
   path_connected (path_component_set s x)
proof -
  { fix y z
    assume pa: path_component s x y path_component s x z
    then have pae: path_component_set s x = path_component_set s y
      using path_component_eq by auto
    have yz: path_component s y z
      using pa path_component_sym path_component_trans by blast
    then have ∃ g. path g ∧ path_image g ⊆ path_component_set s x ∧ pathstart g
= y ∧ pathfinish g = z
      apply (simp add: path_component_def)
```

```
    by (metis pae path_component_maximal path_connected_path_image pathstart_in_path_image)
  }
  then show ?thesis
    by (simp add: path_connected_def)
qed

lemma path_component: path_component S x y ⟷ (∃ t. path_connected t ∧ t ⊆
S ∧ x ∈ t ∧ y ∈ t)
  apply (intro iffI)
   apply (metis path_connected_path_image path_defs(5) pathfinish_in_path_image
pathstart_in_path_image)
  using path_component_of_subset path_connected_component by blast

lemma path_component_path_component [simp]:
   path_component_set (path_component_set S x) x = path_component_set S x
proof (cases x ∈ S)
  case True show ?thesis
   by (metis True mem_Collect_eq path_component_refl path_connected_component_set
path_connected_path_component)
next
  case False then show ?thesis
    by (metis False empty_iff path_component_eq_empty)
qed

lemma path_component_subset_connected_component:
   (path_component_set S x) ⊆ (connected_component_set S x)
proof (cases x ∈ S)
  case True show ?thesis
   by (simp add: True connected_component_maximal path_component_refl path_component_subset
path_connected_imp_connected)
next
  case False then show ?thesis
    using path_component_eq_empty by auto
qed
```

5.5.17 Lemmas about path-connectedness

```
lemma path_connected_linear_image:
  fixes f :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
  assumes path_connected S bounded_linear f
    shows path_connected(f ` S)
by (auto simp: linear_continuous_on assms path_connected_continuous_image)

lemma is_interval_path_connected: is_interval S ⟹ path_connected S
  by (simp add: convex_imp_path_connected is_interval_convex)

lemma path_connected_Ioi[simp]: path_connected {a<..} for a :: real
  by (simp add: convex_imp_path_connected)
```

lemma path_connected_Ici[simp]: path_connected {a..} **for** a :: real
  by (simp add: convex_imp_path_connected)

lemma path_connected_Iio[simp]: path_connected {..<a} **for** a :: real
  by (simp add: convex_imp_path_connected)

lemma path_connected_Iic[simp]: path_connected {..a} **for** a :: real
  by (simp add: convex_imp_path_connected)

lemma path_connected_Ioo[simp]: path_connected {a<..<b} **for** a b :: real
  by (simp add: convex_imp_path_connected)

lemma path_connected_Ioc[simp]: path_connected {a<..b} **for** a b :: real
  by (simp add: convex_imp_path_connected)

lemma path_connected_Ico[simp]: path_connected {a..<b} **for** a b :: real
  by (simp add: convex_imp_path_connected)

lemma path_connectedin_path_image:
  assumes pathin X g **shows** path_connectedin X (g ‘ ({0..1}))
  unfolding pathin_def
proof (rule path_connectedin_continuous_map_image)
  show continuous_map (subtopology euclideanreal {0..1}) X g
    using assms pathin_def **by** blast
qed (auto simp: is_interval_1 is_interval_path_connected)

lemma path_connected_space_subconnected:
    path_connected_space X ⟷
     (∀ x ∈ topspace X. ∀ y ∈ topspace X. ∃ S. path_connectedin X S ∧ x ∈ S ∧ y ∈ S)
  by (metis path_connectedin path_connectedin_topspace path_connected_space_def)

lemma connectedin_path_image: pathin X g ⟹ connectedin X (g ‘ ({0..1}))
  by (simp add: path_connectedin_imp_connectedin path_connectedin_path_image)

lemma compactin_path_image: pathin X g ⟹ compactin X (g ‘ ({0..1}))
  unfolding pathin_def
  by (rule image_compactin [of top_of_set {0..1}]) auto

lemma linear_homeomorphism_image:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes linear f inj f
  obtains g **where** homeomorphism (f ‘ S) S g f
proof −
  obtain g **where** linear g g ∘ f = id
    using assms linear_injective_left_inverse **by** blast
  then have homeomorphism (f ‘ S) S g f
    using assms **unfolding** homeomorphism_def

by (*auto simp*: *eq_id_iff* [*symmetric*] *image_comp linear_conv_bounded_linear linear_continuous_on*)
then show *thesis* **..**
qed

lemma *linear_homeomorphic_image*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *linear f inj f*
shows *S homeomorphic f ‘ S*
by (*meson homeomorphic_def homeomorphic_sym linear_homeomorphism_image* [*OF assms*])

lemma *path_connected_Times*:
assumes *path_connected s path_connected t*
shows *path_connected* (*s* × *t*)
proof (*simp add*: *path_connected_def Sigma_def*, *clarify*)
fix *x1 y1 x2 y2*
assume *x1* ∈ *s y1* ∈ *t x2* ∈ *s y2* ∈ *t*
obtain *g* **where** *path g* **and** *g*: *path_image g* ⊆ *s* **and** *gs*: *pathstart g* = *x1* **and** *gf*: *pathfinish g* = *x2*
using ‹*x1* ∈ *s*› ‹*x2* ∈ *s*› *assms* **by** (*force simp*: *path_connected_def*)
obtain *h* **where** *path h* **and** *h*: *path_image h* ⊆ *t* **and** *hs*: *pathstart h* = *y1* **and** *hf*: *pathfinish h* = *y2*
using ‹*y1* ∈ *t*› ‹*y2* ∈ *t*› *assms* **by** (*force simp*: *path_connected_def*)
have *path* (λ*z*. (*x1*, *h z*))
using ‹*path h*›
unfolding *path_def*
by (*intro continuous_intros continuous_on_compose2* [**where** *g* = *Pair _*]; *force*)
moreover have *path* (λ*z*. (*g z*, *y2*))
using ‹*path g*›
unfolding *path_def*
by (*intro continuous_intros continuous_on_compose2* [**where** *g* = *Pair _*]; *force*)
ultimately have *1*: *path* ((λ*z*. (*x1*, *h z*)) +++ (λ*z*. (*g z*, *y2*)))
by (*metis hf gs path_join_imp pathstart_def pathfinish_def*)
have *path_image* ((λ*z*. (*x1*, *h z*)) +++ (λ*z*. (*g z*, *y2*))) ⊆ *path_image* (λ*z*. (*x1*, *h z*)) ∪ *path_image* (λ*z*. (*g z*, *y2*))
by (*rule Path_Connected.path_image_join_subset*)
also have ... ⊆ (⋃*x*∈*s*. ⋃*x1*∈*t*. {(*x*, *x1*)})
using *g h* ‹*x1* ∈ *s*› ‹*y2* ∈ *t*› **by** (*force simp*: *path_image_def*)
finally have *2*: *path_image* ((λ*z*. (*x1*, *h z*)) +++ (λ*z*. (*g z*, *y2*))) ⊆ (⋃*x*∈*s*. ⋃*x1*∈*t*. {(*x*, *x1*)}) **.**
show ∃ *g*. *path g* ∧ *path_image g* ⊆ (⋃*x*∈*s*. ⋃*x1*∈*t*. {(*x*, *x1*)}) ∧
pathstart g = (*x1*, *y1*) ∧ *pathfinish g* = (*x2*, *y2*)
using *1 2 gf hs*
by (*metis* (*no_types*, *lifting*) *pathfinish_def pathfinish_join pathstart_def pathstart_join*)
qed

lemma *is_interval_path_connected_1*:

fixes *s* :: *real set*
shows *is_interval s* ⟷ *path_connected s*
using *is_interval_connected_1 is_interval_path_connected path_connected_imp_connected*
by *blast*

5.5.18 Path components

lemma *Union_path_component* [*simp*]:
Union {*path_component_set S x* |*x*. *x* ∈ *S*} = *S*
apply (*rule subset_antisym*)
using *path_component_subset* **apply** *force*
using *path_component_refl* **by** *auto*

lemma *path_component_disjoint*:
disjnt (*path_component_set S a*) (*path_component_set S b*) ⟷
(*a* ∉ *path_component_set S b*)
unfolding *disjnt_iff*
using *path_component_sym path_component_trans* **by** *blast*

lemma *path_component_eq_eq*:
path_component S x = *path_component S y* ⟷
(*x* ∉ *S*) ∧ (*y* ∉ *S*) ∨ *x* ∈ *S* ∧ *y* ∈ *S* ∧ *path_component S x y*
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*metis* (*no_types*) *path_component_mem*(*1*) *path_component_refl*)
next
assume *?rhs* **then show** *?lhs*
proof
assume *x* ∉ *S* ∧ *y* ∉ *S* **then show** *?lhs*
by (*metis Collect_empty_eq_bot path_component_eq_empty*)
next
assume *S*: *x* ∈ *S* ∧ *y* ∈ *S* ∧ *path_component S x y* **show** *?lhs*
by (*rule ext*) (*metis S path_component_trans path_component_sym*)
qed
qed

lemma *path_component_unique*:
assumes *x* ∈ *c c* ⊆ *S path_connected c*
⋀*c'*. ⟦*x* ∈ *c'*; *c'* ⊆ *S*; *path_connected c'*⟧ ⟹ *c'* ⊆ *c*
shows *path_component_set S x* = *c*
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
using *assms*
by (*metis mem_Collect_eq path_component_refl path_component_subset path_connected_path_component subsetD*)
qed (*simp add*: *assms path_component_maximal*)

lemma *path_component_intermediate_subset*:
path_component_set u a ⊆ *t* ∧ *t* ⊆ *u*
⟹ *path_component_set t a* = *path_component_set u a*
by (*metis* (*no_types*) *path_component_mono path_component_path_component subset_antisym*)

lemma *complement_path_component_Union*:
fixes *x* :: *'a* :: *topological_space*
shows *S* − *path_component_set S x* =
⋃({*path_component_set S y*| *y*. *y* ∈ *S*} − {*path_component_set S x*})
proof −
have ∗: (⋀*x*. *x* ∈ *S* − {*a*} ⟹ *disjnt a x*) ⟹ ⋃*S* − *a* = ⋃(*S* − {*a*})
for *a*::*'a set* **and** *S*
by (*auto simp*: *disjnt_def*)
have ⋀*y*. *y* ∈ {*path_component_set S x* |*x*. *x* ∈ *S*} − {*path_component_set S x*}
⟹ *disjnt* (*path_component_set S x*) *y*
using *path_component_disjoint path_component_eq* **by** *fastforce*
then have ⋃{*path_component_set S x* |*x*. *x* ∈ *S*} − *path_component_set S x* =
⋃({*path_component_set S y* |*y*. *y* ∈ *S*} − {*path_component_set S x*})
by (*meson* ∗)
then show *?thesis* **by** *simp*
qed

5.5.19 Path components

definition *path_component_of*
where *path_component_of X x y* ≡ ∃ *g*. *pathin X g* ∧ *g 0* = *x* ∧ *g 1* = *y*

abbreviation *path_component_of_set*
where *path_component_of_set X x* ≡ *Collect* (*path_component_of X x*)

definition *path_components_of* :: *'a topology* ⇒ *'a set set*
where *path_components_of X* ≡ *path_component_of_set X* ‘ *topspace X*

lemma *pathin_canon_iff*: *pathin* (*top_of_set T*) *g* ⟷ *path g* ∧ *g* ‘ {*0..1*} ⊆ *T*
by (*simp add*: *path_def pathin_def*)

lemma *path_component_of_canon_iff* [*simp*]:
path_component_of (*top_of_set T*) *a b* ⟷ *path_component T a b*
by (*simp add*: *path_component_of_def pathin_canon_iff path_defs*)

lemma *path_component_in_topspace*:
path_component_of X x y ⟹ *x* ∈ *topspace X* ∧ *y* ∈ *topspace X*
by (*auto simp*: *path_component_of_def pathin_def continuous_map_def*)

lemma *path_component_of_refl*:
path_component_of X x x ⟷ *x* ∈ *topspace X*
by (*metis path_component_in_topspace path_component_of_def pathin_const*)

lemma *path_component_of_sym*:
assumes *path_component_of X x y*
shows *path_component_of X y x*
using *assms*
apply (*clarsimp simp*: *path_component_of_def pathin_def*)
apply (*rule_tac x*=*g* ∘ (λ*t*. *1* − *t*) **in** *exI*)
apply (*auto intro*!: *continuous_map_compose simp*: *continuous_map_in_subtopology continuous_on_op_minus*)
done

lemma *path_component_of_sym_iff*:
path_component_of X x y ⟷ *path_component_of X y x*
by (*metis path_component_of_sym*)

lemma *continuous_map_cases_le*:
assumes *contp*: *continuous_map X euclideanreal p*
and *contq*: *continuous_map X euclideanreal q*
and *contf*: *continuous_map* (*subtopology X* {*x*. *x* ∈ *topspace X* ∧ *p x* ≤ *q x*}) *Y f*
and *contg*: *continuous_map* (*subtopology X* {*x*. *x* ∈ *topspace X* ∧ *q x* ≤ *p x*}) *Y g*
and *fg*: ⋀*x*. ⟦*x* ∈ *topspace X*; *p x* = *q x*⟧ ⟹ *f x* = *g x*
shows *continuous_map X Y* (λ*x*. *if p x* ≤ *q x then f x else g x*)
proof −
have *continuous_map X Y* (λ*x*. *if q x* − *p x* ∈ {*0*..} *then f x else g x*)
proof (*rule continuous_map_cases_function*)
show *continuous_map X euclideanreal* (λ*x*. *q x* − *p x*)
by (*intro contp contq continuous_intros*)
show *continuous_map* (*subtopology X* {*x* ∈ *topspace X*. *q x* − *p x* ∈ *euclideanreal closure_of* {*0*..}}) *Y f*
by (*simp add*: *contf*)
show *continuous_map* (*subtopology X* {*x* ∈ *topspace X*. *q x* − *p x* ∈ *euclideanreal closure_of* (*topspace euclideanreal* − {*0*..})}) *Y g*
by (*simp add*: *contg flip*: *Compl_eq_Diff_UNIV*)
qed (*auto simp*: *fg*)
then show *?thesis*
by *simp*
qed

lemma *continuous_map_cases_lt*:
assumes *contp*: *continuous_map X euclideanreal p*
and *contq*: *continuous_map X euclideanreal q*
and *contf*: *continuous_map* (*subtopology X* {*x*. *x* ∈ *topspace X* ∧ *p x* ≤ *q x*}) *Y f*
and *contg*: *continuous_map* (*subtopology X* {*x*. *x* ∈ *topspace X* ∧ *q x* ≤ *p x*}) *Y g*
and *fg*: ⋀*x*. ⟦*x* ∈ *topspace X*; *p x* = *q x*⟧ ⟹ *f x* = *g x*
shows *continuous_map X Y* (λ*x*. *if p x* < *q x then f x else g x*)
proof −

have *continuous_map X Y* (λ*x*. *if q x* − *p x* ∈ {*0*<..} *then f x else g x*)
proof (*rule continuous_map_cases_function*)
show *continuous_map X euclideanreal* (λ*x*. *q x* − *p x*)
by (*intro contp contq continuous_intros*)
show *continuous_map* (*subtopology X* {*x* ∈ *topspace X*. *q x* − *p x* ∈ *euclideanreal closure_of* {*0*<..}}) *Y f*
by (*simp add*: *contf*)
show *continuous_map* (*subtopology X* {*x* ∈ *topspace X*. *q x* − *p x* ∈ *euclideanreal closure_of* (*topspace euclideanreal* − {*0*<..})}) *Y g*
by (*simp add*: *contg flip*: *Compl_eq_Diff_UNIV*)
qed (*auto simp*: *fg*)
then show *?thesis*
by *simp*
qed

lemma *path_component_of_trans*:
assumes *path_component_of X x y* **and** *path_component_of X y z*
shows *path_component_of X x z*
unfolding *path_component_of_def pathin_def*
proof −
let *?T01* = *top_of_set* {*0*..*1*::*real*}
obtain *g1 g2* **where** *g1*: *continuous_map ?T01 X g1 x* = *g1 0 y* = *g1 1*
and *g2*: *continuous_map ?T01 X g2 g2 0* = *g1 1 z* = *g2 1*
using *assms* **unfolding** *path_component_of_def pathin_def* **by** *blast*
let *?g* = λ*x*. *if x* ≤ *1*/*2 then* (*g1* ∘ (λ*t*. *2* ∗ *t*)) *x else* (*g2* ∘ (λ*t*. *2* ∗ *t* −*1*)) *x*
show ∃ *g*. *continuous_map ?T01 X g* ∧ *g 0* = *x* ∧ *g 1* = *z*
proof (*intro exI conjI*)
show *continuous_map* (*subtopology euclideanreal* {*0*..*1*}) *X ?g*
proof (*intro continuous_map_cases_le continuous_map_compose*, *force*, *force*)
show *continuous_map* (*subtopology ?T01* {*x* ∈ *topspace ?T01*. *x* ≤ *1*/*2*}) *?T01* ((∗) *2*)
by (*auto simp*: *continuous_map_in_subtopology continuous_map_from_subtopology*)
have *continuous_map*
(*subtopology* (*top_of_set* {*0*..*1*}) {*x*. *0* ≤ *x* ∧ *x* ≤ *1* ∧ *1* ≤ *x* ∗ *2*})
euclideanreal (λ*t*. *2* ∗ *t* − *1*)
by (*intro continuous_intros*) (*force intro*: *continuous_map_from_subtopology*)
then show *continuous_map* (*subtopology ?T01* {*x* ∈ *topspace ?T01*. *1*/*2* ≤ *x*}) *?T01* (λ*t*. *2* ∗ *t* − *1*)
by (*force simp*: *continuous_map_in_subtopology*)
show (*g1* ∘ (∗) *2*) *x* = (*g2* ∘ (λ*t*. *2* ∗ *t* − *1*)) *x* **if** *x* ∈ *topspace ?T01 x* = *1*/*2* **for** *x*
using *that* **by** (*simp add*: *g2*(*2*) *mult.commute continuous_map_from_subtopology*)
qed (*auto simp*: *g1 g2*)
qed (*auto simp*: *g1 g2*)
qed

lemma *path_component_of_mono*:
⟦*path_component_of* (*subtopology X S*) *x y*; *S* ⊆ *T*⟧ ⟹ *path_component_of* (*subtopology X T*) *x y*

unfolding *path_component_of_def*
by (*metis subsetD pathin_subtopology*)

lemma *path_component_of*:
path_component_of X x y ⟷ (∃ *T*. *path_connectedin X T* ∧ *x* ∈ *T* ∧ *y* ∈ *T*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*metis atLeastAtMost_iff image_eqI order_refl path_component_of_def path_connectedin_path_image zero_le_one*)
next
assume *?rhs* **then show** *?lhs*
by (*metis path_component_of_def path_connectedin*)
qed

lemma *path_component_of_set*:
path_component_of X x y ⟷ (∃ *g*. *pathin X g* ∧ *g 0* = *x* ∧ *g 1* = *y*)
by (*auto simp*: *path_component_of_def*)

lemma *path_component_of_subset_topspace*:
Collect(*path_component_of X x*) ⊆ *topspace X*
using *path_component_in_topspace* **by** *fastforce*

lemma *path_component_of_eq_empty*:
Collect(*path_component_of X x*) = {} ⟷ (*x* ∉ *topspace X*)
using *path_component_in_topspace path_component_of_refl* **by** *fastforce*

lemma *path_connected_space_iff_path_component*:
path_connected_space X ⟷ (∀ *x* ∈ *topspace X*. ∀ *y* ∈ *topspace X*. *path_component_of X x y*)
by (*simp add*: *path_component_of path_connected_space_subconnected*)

lemma *path_connected_space_imp_path_component_of*:
⟦*path_connected_space X*; *a* ∈ *topspace X*; *b* ∈ *topspace X*⟧
⟹ *path_component_of X a b*
by (*simp add*: *path_connected_space_iff_path_component*)

lemma *path_connected_space_path_component_set*:
path_connected_space X ⟷ (∀ *x* ∈ *topspace X*. *Collect*(*path_component_of X x*) = *topspace X*)
using *path_component_of_subset_topspace path_connected_space_iff_path_component*
by *fastforce*

lemma *path_component_of_maximal*:
⟦*path_connectedin X s*; *x* ∈ *s*⟧ ⟹ *s* ⊆ *Collect*(*path_component_of X x*)
using *path_component_of* **by** *fastforce*

lemma *path_component_of_equiv*:
path_component_of X x y ⟷ *x* ∈ *topspace X* ∧ *y* ∈ *topspace X* ∧ *path_component_of*

X x = *path_component_of X y*
  (**is** *?lhs* = *?rhs*)
proof
  assume *?lhs*
  then show *?rhs*
    apply (*simp add*: *fun_eq_iff path_component_in_topspace*)
    apply (*meson path_component_of_sym path_component_of_trans*)
    done
qed (*simp add*: *path_component_of_refl*)

lemma *path_component_of_disjoint*:
    disjnt (*Collect* (*path_component_of X x*)) (*Collect* (*path_component_of X y*))
⟷
    ~(*path_component_of X x y*)
  by (*force simp*: *disjnt_def path_component_of_eq_empty path_component_of_equiv*)

lemma *path_component_of_eq*:
  path_component_of X x = *path_component_of X y* ⟷
    (*x* ∉ *topspace X*) ∧ (*y* ∉ *topspace X*) ∨
    x ∈ *topspace X* ∧ *y* ∈ *topspace X* ∧ *path_component_of X x y*
 by (*metis Collect_empty_eq_bot path_component_of_eq_empty path_component_of_equiv*)

lemma *path_component_of_aux*:
  path_component_of X x y
    ⟹ *path_component_of* (*subtopology X* (*Collect* (*path_component_of X x*)))
x y
   by (*meson path_component_of path_component_of_maximal path_connectedin_subtopology*)

lemma *path_connectedin_path_component_of*:
  path_connectedin X (*Collect* (*path_component_of X x*))
proof −
 have *topspace* (*subtopology X* (*path_component_of_set X x*)) = *path_component_of_set*
X x
   by (*meson path_component_of_subset_topspace topspace_subtopology_subset*)
  then have *path_connected_space* (*subtopology X* (*path_component_of_set X x*))
   by (*metis* (*full_types*) *path_component_of_aux mem_Collect_eq path_component_of_equiv*
path_connected_space_iff_path_component)
  then show *?thesis*
   by (*simp add*: *path_component_of_subset_topspace path_connectedin_def*)
qed

lemma *path_connectedin_euclidean* [*simp*]:
  path_connectedin euclidean S ⟷ *path_connected S*
 by (*auto simp*: *path_connectedin_def path_connected_space_iff_path_component path_connected_component*)

lemma *path_connected_space_euclidean_subtopology* [*simp*]:
  path_connected_space(*subtopology euclidean S*) ⟷ *path_connected S*
  using *path_connectedin_topspace* **by** *force*

lemma *Union_path_components_of*:
$\bigcup$(*path_components_of* X) = *topspace* X
by (*auto simp*: *path_components_of_def path_component_of_equiv*)

lemma *path_components_of_maximal*:
$[\![C \in$ *path_components_of* X; *path_connectedin* X S; ~*disjnt* C $S]\!] \Longrightarrow S \subseteq C$
apply (*auto simp*: *path_components_of_def path_component_of_equiv*)
using *path_component_of_maximal path_connectedin_def* **apply** *fastforce*
by (*meson disjnt_subset2 path_component_of_disjoint path_component_of_equiv path_component_of_maxim*

lemma *pairwise_disjoint_path_components_of*:
pairwise disjnt (*path_components_of* X)
by (*auto simp*: *path_components_of_def pairwise_def path_component_of_disjoint path_component_of_equiv*)

lemma *complement_path_components_of_Union*:
$C \in$ *path_components_of* X
$\Longrightarrow$ *topspace* $X - C = \bigcup$(*path_components_of* $X - \{C\}$)
by (*metis Diff_cancel Diff_subset Union_path_components_of cSup_singleton diff_Union_pairwise_disjoint insert_subset pairwise_disjoint_path_components_of*)

lemma *nonempty_path_components_of*:
assumes $C \in$ *path_components_of* X **shows** $C \neq \{\}$
proof −
have $C \in$ *path_component_of_set* X ‘ *topspace* X
using *assms path_components_of_def* **by** *blast*
then show *?thesis*
using *path_component_of_refl* **by** *fastforce*
qed

lemma *path_components_of_subset*: $C \in$ *path_components_of* $X \Longrightarrow C \subseteq$ *topspace* X
by (*auto simp*: *path_components_of_def path_component_of_equiv*)

lemma *path_connectedin_path_components_of*:
$C \in$ *path_components_of* $X \Longrightarrow$ *path_connectedin* X C
by (*auto simp*: *path_components_of_def path_connectedin_path_component_of*)

lemma *path_component_in_path_components_of*:
Collect (*path_component_of* X a) $\in$ *path_components_of* $X \longleftrightarrow a \in$ *topspace* X
by (*metis imageI nonempty_path_components_of path_component_of_eq_empty path_components_of_def*)

lemma *path_connectedin_Union*:
assumes $\mathcal{A}$: $\bigwedge S.\ S \in \mathcal{A} \Longrightarrow$ *path_connectedin* X S $\bigcap\mathcal{A} \neq \{\}$
shows *path_connectedin* X $(\bigcup\mathcal{A})$
proof −
obtain a **where** $\bigwedge S.\ S \in \mathcal{A} \Longrightarrow a \in S$
using *assms* **by** *blast*
then have $\bigwedge x.\ x \in$ *topspace* (*subtopology* X $(\bigcup\mathcal{A})$) $\Longrightarrow$ *path_component_of*

(*subtopology X* (⋃𝒜)) *a x*
  by *simp* (*meson Union_upper* 𝒜 *path_component_of path_connectedin_subtopology*)
 then show *?thesis*
  using 𝒜 **unfolding** *path_connectedin_def*
  by (*metis Sup_le_iff path_component_of_equiv path_connected_space_iff_path_component*)
qed

lemma *path_connectedin_Un*:
  ⟦*path_connectedin X S*; *path_connectedin X T*; $S \cap T \neq \{\}$⟧
  ⟹ *path_connectedin X* $(S \cup T)$
 by (*blast intro*: *path_connectedin_Union* [*of* {*S*,*T*}, *simplified*])

lemma *path_connected_space_iff_components_eq*:
 path_connected_space X ⟷
  (∀ *C* ∈ *path_components_of X*. ∀ *C′* ∈ *path_components_of X*. *C* = *C′*)
 unfolding *path_components_of_def*
proof (*intro iffI ballI*)
 assume ∀ *C* ∈ *path_component_of_set X* ‘ *topspace X*.
      ∀ *C′* ∈ *path_component_of_set X* ‘ *topspace X*. *C* = *C′*
 then show *path_connected_space X*
  using *path_component_of_refl path_connected_space_iff_path_component* **by** *fastforce*
qed (*auto simp*: *path_connected_space_path_component_set*)

lemma *path_components_of_eq_empty*:
  path_components_of X = {} ⟷ *topspace X* = {}
 using *Union_path_components_of nonempty_path_components_of* **by** *fastforce*

lemma *path_components_of_empty_space*:
  topspace X = {} ⟹ *path_components_of X* = {}
 by (*simp add*: *path_components_of_eq_empty*)

lemma *path_components_of_subset_singleton*:
 path_components_of X ⊆ {*S*} ⟷
    path_connected_space X ∧ (*topspace X* = {} ∨ *topspace X* = *S*)
proof (*cases topspace X* = {})
 case *True*
 then show *?thesis*
  by (*auto simp*: *path_components_of_empty_space path_connected_space_topspace_empty*)
next
 case *False*
 have (*path_components_of X* = {*S*}) ⟷ (*path_connected_space X* ∧ *topspace X*
= *S*)
 proof (*intro iffI conjI*)
  assume *L*: *path_components_of X* = {*S*}
  then show *path_connected_space X*
   by (*simp add*: *path_connected_space_iff_components_eq*)
  show *topspace X* = *S*
   by (*metis L ccpo_Sup_singleton* [*of S*] *Union_path_components_of*)
 next

assume *R*: *path_connected_space X* ∧ *topspace X* = *S*
then show *path_components_of X* = {*S*}
using *ccpo_Sup_singleton* [*of S*]
by (*metis False all_not_in_conv insert_iff mk_disjoint_insert path_component_in_path_components_of path_connected_space_iff_components_eq path_connected_space_path_component_set*)
qed
with *False* **show** *?thesis*
by (*simp add*: *path_components_of_eq_empty subset_singleton_iff*)
qed

lemma *path_connected_space_iff_components_subset_singleton*:
path_connected_space X ⟷ (∃ *a*. *path_components_of X* ⊆ {*a*})
by (*simp add*: *path_components_of_subset_singleton*)

lemma *path_components_of_eq_singleton*:
path_components_of X = {*S*} ⟷ *path_connected_space X* ∧ *topspace X* ≠ {} ∧ *S* = *topspace X*
by (*metis cSup_singleton insert_not_empty path_components_of_subset_singleton subset_singleton_iff*)

lemma *path_components_of_path_connected_space*:
path_connected_space X ⟹ *path_components_of X* = (*if topspace X* = {} *then* {} *else* {*topspace X*})
by (*simp add*: *path_components_of_eq_empty path_components_of_eq_singleton*)

lemma *path_component_subset_connected_component_of*:
path_component_of_set X x ⊆ *connected_component_of_set X x*
proof (*cases x* ∈ *topspace X*)
case *True*
then show *?thesis*
by (*simp add*: *connected_component_of_maximal path_component_of_refl path_connectedin_imp_connected path_connectedin_path_component_of*)
next
case *False*
then show *?thesis*
using *path_component_of_eq_empty* **by** *fastforce*
qed

lemma *exists_path_component_of_superset*:
assumes *S*: *path_connectedin X S* **and** *ne*: *topspace X* ≠ {}
obtains *C* **where** *C* ∈ *path_components_of X S* ⊆ *C*
proof (*cases S* = {})
case *True*
then show *?thesis*
using *ne path_components_of_eq_empty that* **by** *fastforce*
next
case *False*
then obtain *a* **where** *a* ∈ *S*
by *blast*

show *?thesis*
proof
show *Collect* (*path_component_of* X a) $\in$ *path_components_of* X
by (*meson* ‹$a \in S$› S *subsetD path_component_in_path_components_of path_connectedin_subset_topspace*)
show $S \subseteq$ *Collect* (*path_component_of* X a)
by (*simp add*: S ‹$a \in S$› *path_component_of_maximal*)
qed
qed

lemma *path_component_of_eq_overlap*:
path_component_of X x = *path_component_of* X y $\longleftrightarrow$
($x \notin$ *topspace* X) $\wedge$ ($y \notin$ *topspace* X) $\vee$
Collect (*path_component_of* X x) $\cap$ *Collect* (*path_component_of* X y) $\neq$ {}
by (*metis disjnt_def empty_iff inf_bot_right mem_Collect_eq path_component_of_disjoint path_component_of_eq path_component_of_eq_empty*)

lemma *path_component_of_nonoverlap*:
Collect (*path_component_of* X x) $\cap$ *Collect* (*path_component_of* X y) = {} $\longleftrightarrow$
($x \notin$ *topspace* X) $\vee$ ($y \notin$ *topspace* X) $\vee$
path_component_of X x $\neq$ *path_component_of* X y
by (*metis inf.idem path_component_of_eq_empty path_component_of_eq_overlap*)

lemma *path_component_of_overlap*:
Collect (*path_component_of* X x) $\cap$ *Collect* (*path_component_of* X y) $\neq$ {} $\longleftrightarrow$
$x \in$ *topspace* X $\wedge$ $y \in$ *topspace* X $\wedge$ *path_component_of* X x = *path_component_of* X y
by (*meson path_component_of_nonoverlap*)

lemma *path_components_of_disjoint*:
$[\![C \in$ *path_components_of* X; $C' \in$ *path_components_of* $X]\!] \Longrightarrow$ *disjnt* C C' $\longleftrightarrow$ $C \neq C'$
by (*auto simp*: *path_components_of_def path_component_of_disjoint path_component_of_equiv*)

lemma *path_components_of_overlap*:
$[\![C \in$ *path_components_of* X; $C' \in$ *path_components_of* $X]\!] \Longrightarrow C \cap C' \neq \{\}$ $\longleftrightarrow C = C'$
by (*auto simp*: *path_components_of_def path_component_of_equiv*)

lemma *path_component_of_unique*:
$[\![x \in C$; *path_connectedin* X C; $\bigwedge C'$. $[\![x \in C'$; *path_connectedin* X $C']\!] \Longrightarrow C' \subseteq C]\!]$
$\Longrightarrow$ *Collect* (*path_component_of* X x) = C
by (*meson subsetD eq_iff path_component_of_maximal path_connectedin_path_component_of*)

lemma *path_component_of_discrete_topology* [*simp*]:
Collect (*path_component_of* (*discrete_topology* U) x) = (*if* $x \in U$ *then* $\{x\}$ *else* {})
proof −
have $\bigwedge C'$. $[\![x \in C'$; *path_connectedin* (*discrete_topology* U) $C']\!] \Longrightarrow C' \subseteq \{x\}$

by (*metis path_connectedin_discrete_topology subsetD singletonD*)
then have *x* ∈ *U* ⟹ *Collect* (*path_component_of* (*discrete_topology U*) *x*) = {*x*}
by (*simp add*: *path_component_of_unique*)
then show *?thesis*
using *path_component_in_topspace* **by** *fastforce*
qed

lemma *path_component_of_discrete_topology_iff* [*simp*]:
path_component_of (*discrete_topology U*) *x y* ⟷ *x* ∈ *U* ∧ *y*=*x*
by (*metis empty_iff insertI1 mem_Collect_eq path_component_of_discrete_topology singletonD*)

lemma *path_components_of_discrete_topology* [*simp*]:
path_components_of (*discrete_topology U*) = (λ*x*. {*x*}) ' *U*
by (*auto simp*: *path_components_of_def image_def fun_eq_iff*)

lemma *homeomorphic_map_path_component_of*:
assumes *f*: *homeomorphic_map X Y f* **and** *x*: *x* ∈ *topspace X*
shows *Collect* (*path_component_of Y* (*f x*)) = *f* ' *Collect*(*path_component_of X x*)
proof −
obtain *g* **where** *g*: *homeomorphic_maps X Y f g*
using *f homeomorphic_map_maps* **by** *blast*
show *?thesis*
proof
have *Collect* (*path_component_of Y* (*f x*)) ⊆ *topspace Y*
by (*simp add*: *path_component_of_subset_topspace*)
moreover have *g* ' *Collect*(*path_component_of Y* (*f x*)) ⊆ *Collect* (*path_component_of X* (*g* (*f x*)))
using *g x* **unfolding** *homeomorphic_maps_def*
by (*metis f homeomorphic_imp_surjective_map imageI mem_Collect_eq path_component_of_maximal path_component_of_refl path_connectedin_continuous_map_image path_connectedin_path_component_of*)
ultimately show *Collect* (*path_component_of Y* (*f x*)) ⊆ *f* ' *Collect* (*path_component_of X x*)
using *g x* **unfolding** *homeomorphic_maps_def continuous_map_def image_iff subset_iff*
by *metis*
show *f* ' *Collect* (*path_component_of X x*) ⊆ *Collect* (*path_component_of Y* (*f x*))
proof (*rule path_component_of_maximal*)
show *path_connectedin Y* (*f* ' *Collect* (*path_component_of X x*))
by (*meson f homeomorphic_map_path_connectedness_eq path_connectedin_path_component_of*)
qed (*simp add*: *path_component_of_refl x*)
qed
qed

lemma *homeomorphic_map_path_components_of*:
assumes *homeomorphic_map X Y f*

```
  shows path_components_of Y = (image f) ` (path_components_of X)
    (is ?lhs = ?rhs)
  unfolding path_components_of_def homeomorphic_imp_surjective_map [OF assms,
symmetric]
  using assms homeomorphic_map_path_component_of by fastforce
```

5.5.20 Sphere is path-connected

```
lemma path_connected_punctured_universe:
  assumes 2 ≤ DIM('a::euclidean_space)
  shows path_connected (− {a::'a})
proof −
  let ?A = {x::'a. ∃ i∈Basis. x · i < a · i}
  let ?B = {x::'a. ∃ i∈Basis. a · i < x · i}

  have A: path_connected ?A
    unfolding Collect_bex_eq
  proof (rule path_connected_UNION)
    fix i :: 'a
    assume i ∈ Basis
    then show (∑ i∈Basis. (a · i − 1)*R i) ∈ {x::'a. x · i < a · i}
      by simp
    show path_connected {x. x · i < a · i}
      using convex_imp_path_connected [OF convex_halfspace_lt, of i a · i]
      by (simp add: inner_commute)
  qed
  have B: path_connected ?B
    unfolding Collect_bex_eq
  proof (rule path_connected_UNION)
    fix i :: 'a
    assume i ∈ Basis
    then show (∑ i∈Basis. (a · i + 1) *R i) ∈ {x::'a. a · i < x · i}
      by simp
    show path_connected {x. a · i < x · i}
      using convex_imp_path_connected [OF convex_halfspace_gt, of a · i i]
      by (simp add: inner_commute)
  qed
  obtain S :: 'a set where S ⊆ Basis and card S = Suc (Suc 0)
    using ex_card[OF assms]
    by auto
  then obtain b0 b1 :: 'a where b0 ∈ Basis and b1 ∈ Basis and b0 ≠ b1
    unfolding card_Suc_eq by auto
  then have a + b0 − b1 ∈ ?A ∩ ?B
    by (auto simp: inner_simps inner_Basis)
  then have ?A ∩ ?B ≠ {}
    by fast
  with A B have path_connected (?A ∪ ?B)
    by (rule path_connected_Un)
  also have ?A ∪ ?B = {x. ∃ i∈Basis. x · i ≠ a · i}
```

unfolding *neq_iff bex_disj_distrib Collect_disj_eq* **..**
also have ... = $\{x.\ x \neq a\}$
unfolding *euclidean_eq_iff* [**where** *'a='a*]
by (*simp add*: *Bex_def*)
also have ... = − {*a*}
by *auto*
finally show *?thesis* **.**
qed

corollary *connected_punctured_universe*:
2 ≤ *DIM*(*'N*::*euclidean_space*) ⟹ *connected*(− {*a*::*'N*})
by (*simp add*: *path_connected_punctured_universe path_connected_imp_connected*)

proposition *path_connected_sphere*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *2* ≤ *DIM*(*'a*)
shows *path_connected*(*sphere a r*)
proof (*cases r 0*::*real rule*: *linorder_cases*)
case *less*
then show *?thesis*
by (*simp*)
next
case *equal*
then show *?thesis*
by (*simp*)
next
case *greater*
then have *eq*: (*sphere* (*0*::*'a*) *r*) = (λ*x*. (*r* / *norm x*) $*_R$ *x*) ‘ (− {*0*::*'a*})
by (*force simp*: *image_iff split*: *if_split_asm*)
have *continuous_on* (− {*0*::*'a*}) (λ*x*. (*r* / *norm x*) $*_R$ *x*)
by (*intro continuous_intros*) *auto*
then have *path_connected* ((λ*x*. (*r* / *norm x*) $*_R$ *x*) ‘ (− {*0*::*'a*}))
by (*intro path_connected_continuous_image path_connected_punctured_universe assms*)
with *eq* **have** *path_connected* (*sphere* (*0*::*'a*) *r*)
by *auto*
then have *path_connected*((+) *a* ‘ (*sphere* (*0*::*'a*) *r*))
by (*simp add*: *path_connected_translation*)
then show *?thesis*
by (*metis add.right_neutral sphere_translation*)
qed

lemma *connected_sphere*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *2* ≤ *DIM*(*'a*)
shows *connected*(*sphere a r*)
using *path_connected_sphere* [*OF assms*]
by (*simp add*: *path_connected_imp_connected*)

corollary *path_connected_complement_bounded_convex*:
  fixes S :: *'a* :: *euclidean_space set*
  assumes *bounded S convex S* **and** *2*: $2 \leq DIM('a)$
  shows *path_connected* $(- S)$
proof (*cases* $S = \{\}$)
  case *True* **then show** *?thesis*
    using *convex_imp_path_connected* **by** *auto*
next
  case *False*
  then obtain a **where** $a \in S$ **by** *auto*
  have § [*rule_format*]: $\forall y \in S.\ \forall u.\ 0 \leq u \wedge u \leq 1 \longrightarrow (1 - u) *_R a + u *_R y \in S$
    using ‹*convex S*› ‹$a \in S$› **by** (*simp add*: *convex_alt*)
  { **fix** x y **assume** $x \notin S$ $y \notin S$
    then have $x \neq a$ $y \neq a$ **using** ‹$a \in S$› **by** *auto*
    then have *bxy*: *bounded*(*insert x* (*insert y S*))
      by (*simp add*: ‹*bounded S*›)
    then obtain B::*real* **where** B: $0 < B$ **and** *Bx*: *norm* $(a - x) < B$ **and** *By*: *norm* $(a - y) < B$
                 and $S \subseteq$ *ball a B*
      using *bounded_subset_ballD* [*OF bxy*, *of a*] **by** (*auto simp*: *dist_norm*)
    define C **where** $C = B$ / *norm*$(x - a)$
    let $?Cxa = a + C *_R (x - a)$
    { **fix** u
      assume u: $(1 - u) *_R x + u *_R ?Cxa \in S$ **and** $0 \leq u$ $u \leq 1$
      have *CC*: $1 \leq 1 + (C - 1) * u$
        using ‹$x \neq a$› ‹$0 \leq u$› *Bx*
        by (*auto simp add*: *C_def norm_minus_commute*)
      have $*$: $\bigwedge v.\ (1 - u) *_R x + u *_R (a + v *_R (x - a)) = a + (1 + (v - 1) * u) *_R (x - a)$
        by (*simp add*: *algebra_simps*)
      have $a + ((1 / (1 + C * u - u)) *_R x + ((u / (1 + C * u - u)) *_R a + (C * u / (1 + C * u - u)) *_R x)) =$
          $(1 + (u / (1 + C * u - u))) *_R a + ((1 / (1 + C * u - u)) + (C * u / (1 + C * u - u))) *_R x$
        by (*simp add*: *algebra_simps*)
      also have ... $= (1 + (u / (1 + C * u - u))) *_R a + (1 + (u / (1 + C * u - u))) *_R x$
        using *CC* **by** (*simp add*: *field_simps*)
      also have ... $= x + (1 + (u / (1 + C * u - u))) *_R a + (u / (1 + C * u - u)) *_R x$
        by (*simp add*: *algebra_simps*)
      also have ... $= x + ((1 / (1 + C * u - u)) *_R a +$
            $((u / (1 + C * u - u)) *_R x + (C * u / (1 + C * u - u)) *_R a))$
      using *CC* **by** (*simp add*: *field_simps*) (*simp add*: *add_divide_distrib scaleR_add_left*)
      finally have *xeq*: $(1 - 1 / (1 + (C - 1) * u)) *_R a + (1 / (1 + (C - 1) * u)) *_R (a + (1 + (C - 1) * u) *_R (x - a)) = x$
        by (*simp add*: *algebra_simps*)

have *False*
using § [*of* $a + (1 + (C - 1) * u) *_R (x - a)$ *1* / $(1 + (C - 1) * u)$]
using *u* ‹$x \neq a$› ‹$x \notin S$› ‹$0 \leq u$› *CC*
by (*auto simp*: *xeq* ∗)
}
then have *pcx*: *path_component* (− *S*) *x ?Cxa*
by (*force simp*: *closed_segment_def intro*!: *path_component_linepath*)
define *D* **where** $D = B / norm(y - a)$ — massive duplication with the proof above
let *?Dya* = $a + D *_R (y - a)$
{ fix *u*
assume *u*: $(1 - u) *_R y + u *_R$ *?Dya* ∈ *S* **and** $0 \leq u$ $u \leq 1$
have *DD*: $1 \leq 1 + (D - 1) * u$
using ‹$y \neq a$› ‹$0 \leq u$› *By*
by (*auto simp add*: *D_def norm_minus_commute*)
have ∗: $\bigwedge v.\ (1 - u) *_R y + u *_R (a + v *_R (y - a)) = a + (1 + (v - 1) * u) *_R (y - a)$
by (*simp add*: *algebra_simps*)
have $a + ((1 / (1 + D * u - u)) *_R y + ((u / (1 + D * u - u)) *_R a + (D * u / (1 + D * u - u)) *_R y)) =$
$(1 + (u / (1 + D * u - u))) *_R a + ((1 / (1 + D * u - u)) + (D * u / (1 + D * u - u))) *_R y$
by (*simp add*: *algebra_simps*)
also have ... = $(1 + (u / (1 + D * u - u))) *_R a + (1 + (u / (1 + D * u - u))) *_R y$
using *DD* **by** (*simp add*: *field_simps*)
also have ... = $y + (1 + (u / (1 + D * u - u))) *_R a + (u / (1 + D * u - u)) *_R y$
by (*simp add*: *algebra_simps*)
also have ... = $y + ((1 / (1 + D * u - u)) *_R a +$
$((u / (1 + D * u - u)) *_R y + (D * u / (1 + D * u - u)) *_R a))$
using *DD* **by** (*simp add*: *field_simps*) (*simp add*: *add_divide_distrib scaleR_add_left*)
finally have *xeq*: $(1 - 1 / (1 + (D - 1) * u)) *_R a + (1 / (1 + (D - 1) * u)) *_R (a + (1 + (D - 1) * u) *_R (y - a)) = y$
by (*simp add*: *algebra_simps*)
have *False*
using § [*of* $a + (1 + (D - 1) * u) *_R (y - a)$ *1* / $(1 + (D - 1) * u)$]
using *u* ‹$y \neq a$› ‹$y \notin S$› ‹$0 \leq u$› *DD*
by (*auto simp*: *xeq* ∗)
}
then have *pdy*: *path_component* (− *S*) *y ?Dya*
by (*force simp*: *closed_segment_def intro*!: *path_component_linepath*)
have *pyx*: *path_component* (− *S*) *?Dya ?Cxa*
proof (*rule path_component_of_subset*)
show *sphere a B* ⊆ − *S*
using ‹*S* ⊆ *ball a B*› **by** (*force simp*: *ball_def dist_norm norm_minus_commute*)
have *aB*: *?Dya* ∈ *sphere a B* *?Cxa* ∈ *sphere a B*
using ‹$x \neq a$› **using** ‹$y \neq a$› *B* **by** (*auto simp*: *dist_norm C_def D_def*)
then show *path_component* (*sphere a B*) *?Dya ?Cxa*

```
      using path_connected_sphere [OF 2] path_connected_component by blast
    qed
    have path_component (− S) x y
      by (metis path_component_trans path_component_sym pcx pdy pyx)
  }
  then show ?thesis
    by (auto simp: path_connected_component)
qed

lemma connected_complement_bounded_convex:
    fixes S :: 'a :: euclidean_space set
    assumes bounded S convex S 2 ≤ DIM('a)
      shows  connected (− S)
  using path_connected_complement_bounded_convex [OF assms] path_connected_imp_connected
by blast

lemma connected_diff_ball:
    fixes S :: 'a :: euclidean_space set
    assumes connected S cball a r ⊆ S 2 ≤ DIM('a)
      shows connected (S − ball a r)
proof (rule connected_diff_open_from_closed [OF ball_subset_cball])
  show connected (cball a r − ball a r)
    using assms connected_sphere by (auto simp: cball_diff_eq_sphere)
qed (auto simp: assms dist_norm)

proposition connected_open_delete:
  assumes open S connected S and 2: 2 ≤ DIM('N::euclidean_space)
    shows connected(S − {a::'N})
proof (cases a ∈ S)
  case True
  with ‹open S› obtain ε where ε > 0 and ε: cball a ε ⊆ S
    using open_contains_cball_eq by blast
  define b where b ≡ a + ε *R (SOME i. i ∈ Basis)
  have dist a b = ε
    by (simp add: b_def dist_norm SOME_Basis ‹0 < ε› less_imp_le)
  with ε have b ∈ ⋂{S − ball a r |r. 0 < r ∧ r < ε}
    by auto
  then have nonemp: (⋂{S − ball a r |r. 0 < r ∧ r < ε}) = {} ⟹ False
    by auto
  have con: ⋀r. r < ε ⟹ connected (S − ball a r)
    using ε by (force intro: connected_diff_ball [OF ‹connected S› _ 2])
  have x ∈ ⋃{S − ball a r |r. 0 < r ∧ r < ε} if x ∈ S − {a} for x
     using that ‹0 < ε›
     by (intro UnionI [of S − ball a (min ε (dist a x) / 2)]) auto
  then have S − {a} = ⋃{S − ball a r | r. 0 < r ∧ r < ε}
    by auto
  then show ?thesis
    by (auto intro: connected_Union con dest!: nonemp)
next
```

case *False* **then show** *?thesis*
by (*simp add*: ‹*connected S*›)
qed

corollary *path_connected_open_delete*:
assumes *open S connected S* **and** *2*: $2 \le DIM('N::euclidean\_space)$
shows *path_connected*$(S - \{a::'N\})$
by (*simp add*: *assms connected_open_delete connected_open_path_connected open_delete*)

corollary *path_connected_punctured_ball*:
$2 \le DIM('N::euclidean\_space) \Longrightarrow$ *path_connected*(*ball a r* $- \{a::'N\})$
by (*simp add*: *path_connected_open_delete*)

corollary *connected_punctured_ball*:
$2 \le DIM('N::euclidean\_space) \Longrightarrow$ *connected*(*ball a r* $- \{a::'N\})$
by (*simp add*: *connected_open_delete*)

corollary *connected_open_delete_finite*:
fixes *S T*::*'a*::*euclidean_space set*
assumes *S*: *open S connected S* **and** *2*: $2 \le DIM('a)$ **and** *finite T*
shows *connected*$(S - T)$
using ‹*finite T*› *S*
proof (*induct T*)
case *empty*
show *?case* **using** ‹*connected S*› **by** *simp*
next
case (*insert x F*)
then have *connected* $(S-F)$ **by** *auto*
moreover have *open* $(S - F)$ **using** *finite_imp_closed*[*OF* ‹*finite F*›] ‹*open S*›
by *auto*
ultimately have *connected* $(S - F - \{x\})$ **using** *connected_open_delete*[*OF* _ _
2] **by** *auto*
thus *?case* **by** (*metis Diff_insert*)
qed

lemma *sphere_1D_doubleton_zero*:
assumes *1*: $DIM('a) = 1$ **and** $r > 0$
obtains *x y*::*'a*::*euclidean_space*
where *sphere 0 r* $= \{x,y\} \wedge$ *dist x y* $= 2*r$
proof −
obtain *b*::*'a* **where** *b*: *Basis* $= \{b\}$
using *1 card_1_singletonE* **by** *blast*
show *?thesis*
proof (*intro that conjI*)
have $x = norm\ x *_R b \vee x = -\ norm\ x *_R b$ **if** $r = norm\ x$ **for** *x*
proof −
have *xb*: $(x \cdot b) *_R b = x$
using *euclidean_representation* [*of x*, *unfolded b*] **by** *force*
then have *norm* $((x \cdot b) *_R b) = norm\ x$

```
      by simp
    with b have |x · b| = norm x
      using norm_Basis by (simp add: b)
    with xb show ?thesis
      by (metis (mono_tags, hide_lams) abs_eq_iff abs_norm_cancel)
  qed
  with ‹r > 0› b show sphere 0 r = {r *R b, − r *R b}
    by (force simp: sphere_def dist_norm)
  have dist (r *R b) (− r *R b) = norm (r *R b + r *R b)
    by (simp add: dist_norm)
  also have ... = norm ((2*r) *R b)
    by (metis mult_2 scaleR_add_left)
  also have ... = 2*r
    using ‹r > 0› b norm_Basis by fastforce
  finally show dist (r *R b) (− r *R b) = 2*r .
 qed
qed

lemma sphere_1D_doubleton:
  fixes a :: 'a :: euclidean_space
  assumes DIM('a) = 1 and r > 0
  obtains x y where sphere a r = {x,y} ∧ dist x y = 2*r
proof −
  have sphere a r = (+) a ` sphere 0 r
    by (metis add.right_neutral sphere_translation)
  then show ?thesis
    using sphere_1D_doubleton_zero [OF assms]
    by (metis (mono_tags, lifting) dist_add_cancel image_empty image_insert that)
qed

lemma psubset_sphere_Compl_connected:
  fixes S :: 'a::euclidean_space set
  assumes S: S ⊂ sphere a r and 0 < r and 2: 2 ≤ DIM('a)
  shows connected(− S)
proof −
  have S ⊆ sphere a r
    using S by blast
  obtain b where dist a b = r and b ∉ S
    using S mem_sphere by blast
  have CS: − S = {x. dist a x ≤ r ∧ (x ∉ S)} ∪ {x. r ≤ dist a x ∧ (x ∉ S)}
    by auto
  have {x. dist a x ≤ r ∧ x ∉ S} ∩ {x. r ≤ dist a x ∧ x ∉ S} ≠ {}
    using ‹b ∉ S› ‹dist a b = r› by blast
  moreover have connected {x. dist a x ≤ r ∧ x ∉ S}
    using assms
    by (force intro: connected_intermediate_closure [of ball a r])
  moreover
  have connected {x. r ≤ dist a x ∧ x ∉ S}
  proof (rule connected_intermediate_closure [of − cball a r])
```

```
    show {x. r ≤ dist a x ∧ x ∉ S} ⊆ closure (− cball a r)
     using interior_closure by (force intro: connected_complement_bounded_convex)
  qed (use assms connected_complement_bounded_convex in auto)
  ultimately show ?thesis
    by (simp add: CS connected_Un)
qed
```

5.5.21 Every annulus is a connected set

```
lemma path_connected_2DIM_I:
  fixes a :: 'N::euclidean_space
  assumes 2: 2 ≤ DIM('N) and pc: path_connected {r. 0 ≤ r ∧ P r}
  shows path_connected {x. P(norm(x − a))}
proof −
  have {x. P(norm(x − a))} = (+) a ` {x. P(norm x)}
    by force
  moreover have path_connected {x::'N. P(norm x)}
  proof −
    let ?D = {x. 0 ≤ x ∧ P x} × sphere (0::'N) 1
    have x ∈ (λz. fst z *R snd z) ` ?D
      if P (norm x) for x::'N
    proof (cases x=0)
      case True
      with that show ?thesis
        apply (simp add: image_iff)
       by (metis (no_types) mem_sphere_0 order_refl vector_choose_size zero_le_one)
    next
      case False
      with that show ?thesis
        by (rule_tac x=(norm x, x /R norm x) in image_eqI) auto
    qed
    then have *: {x::'N. P(norm x)} = (λz. fst z *R snd z) ` ?D
      by auto
    have continuous_on ?D (λz:: real×'N. fst z *R snd z)
      by (intro continuous_intros)
    moreover have path_connected ?D
      by (metis path_connected_Times [OF pc] path_connected_sphere 2)
    ultimately show ?thesis
      by (simp add: * path_connected_continuous_image)
  qed
  ultimately show ?thesis
    using path_connected_translation by metis
qed

proposition path_connected_annulus:
  fixes a :: 'N::euclidean_space
  assumes 2 ≤ DIM('N)
  shows path_connected {x. r1 < norm(x − a) ∧ norm(x − a) < r2}
        path_connected {x. r1 < norm(x − a) ∧ norm(x − a) ≤ r2}
```

path_connected $\{x.\ r1 \le norm(x - a) \wedge norm(x - a) < r2\}$
path_connected $\{x.\ r1 \le norm(x - a) \wedge norm(x - a) \le r2\}$
by (*auto simp*: *is_interval_def intro*!: *is_interval_convex convex_imp_path_connected path_connected_2DIM_I* [*OF assms*])

proposition *connected_annulus*:
fixes *a* :: *'N::euclidean_space*
assumes $2 \le DIM('N::euclidean\_space)$
shows *connected* $\{x.\ r1 < norm(x - a) \wedge norm(x - a) < r2\}$
connected $\{x.\ r1 < norm(x - a) \wedge norm(x - a) \le r2\}$
connected $\{x.\ r1 \le norm(x - a) \wedge norm(x - a) < r2\}$
connected $\{x.\ r1 \le norm(x - a) \wedge norm(x - a) \le r2\}$
by (*auto simp*: *path_connected_annulus* [*OF assms*] *path_connected_imp_connected*)

5.5.22 Relations between components and path components

lemma *open_connected_component*:
fixes *S* :: *'a::real_normed_vector set*
assumes *open S*
shows *open* (*connected_component_set S x*)
proof (*clarsimp simp*: *open_contains_ball*)
fix *y*
assume *xy*: *connected_component S x y*
then obtain *e* **where** *e>0 ball y e* ⊆ *S*
using *assms connected_component_in openE* **by** *blast*
then show ∃ *e>0. ball y e* ⊆ *connected_component_set S x*
by (*metis xy centre_in_ball connected_ball connected_component_eq_eq connected_component_in connected_component_maximal*)
qed

corollary *open_components*:
fixes *S* :: *'a::real_normed_vector set*
shows ⟦*open u*; *S* ∈ *components u*⟧ ⟹ *open S*
by (*simp add*: *components_iff*) (*metis open_connected_component*)

lemma *in_closure_connected_component*:
fixes *S* :: *'a::real_normed_vector set*
assumes *x*: *x* ∈ *S* **and** *S*: *open S*
shows *x* ∈ *closure* (*connected_component_set S y*) ⟷ *x* ∈ *connected_component_set S y*
proof −
{ assume *x* ∈ *closure* (*connected_component_set S y*)
moreover have *x* ∈ *connected_component_set S x*
using *x* **by** *simp*
ultimately have *x* ∈ *connected_component_set S y*
using *S* **by** (*meson Compl_disjoint closure_iff_nhds_not_empty connected_component_disjoint disjoint_eq_subset_Compl open_connected_component*)
}
then show *?thesis*

by (*auto simp*: *closure_def*)
qed

lemma *connected_disjoint_Union_open_pick*:
assumes *pairwise disjnt B*
$\bigwedge S.\ S \in A \Longrightarrow connected\ S \wedge S \neq \{\}$
$\bigwedge S.\ S \in B \Longrightarrow open\ S$
$\bigcup A \subseteq \bigcup B$
$S \in A$
obtains T **where** $T \in B$ $S \subseteq T$ $S \cap \bigcup(B - \{T\}) = \{\}$
proof −
have $S \subseteq \bigcup B$ *connected* S $S \neq \{\}$
using *assms* ‹$S \in A$› **by** *blast*+
then obtain T **where** $T \in B$ $S \cap T \neq \{\}$
by (*metis Sup_inf_eq_bot_iff inf.absorb_iff2 inf_commute*)
have *1*: *open* T **by** (*simp add*: ‹$T \in B$› *assms*)
have *2*: *open* $(\bigcup(B-\{T\}))$ **using** *assms* **by** *blast*
have *3*: $S \subseteq T \cup \bigcup(B - \{T\})$ **using** ‹$S \subseteq \bigcup B$› **by** *blast*
have $T \cap \bigcup(B - \{T\}) = \{\}$ **using** ‹$T \in B$› ‹*pairwise disjnt B*›
by (*auto simp*: *pairwise_def disjnt_def*)
then have *4*: $T \cap \bigcup(B - \{T\}) \cap S = \{\}$ **by** *auto*
from *connectedD* [*OF* ‹*connected S*› *1 2 4 3*]
have $S \cap \bigcup(B-\{T\}) = \{\}$
by (*auto simp*: *Int_commute* ‹$S \cap T \neq \{\}$›)
with ‹$T \in B$› **have** $S \subseteq T$
using *3* **by** *auto*
show *?thesis*
using ‹$S \cap \bigcup(B - \{T\}) = \{\}$› ‹$S \subseteq T$› ‹$T \in B$› *that* **by** *auto*
qed

lemma *connected_disjoint_Union_open_subset*:
assumes *A*: *pairwise disjnt A* **and** *B*: *pairwise disjnt B*
and *SA*: $\bigwedge S.\ S \in A \Longrightarrow open\ S \wedge connected\ S \wedge S \neq \{\}$
and *SB*: $\bigwedge S.\ S \in B \Longrightarrow open\ S \wedge connected\ S \wedge S \neq \{\}$
and *eq* [*simp*]: $\bigcup A = \bigcup B$
shows $A \subseteq B$
proof
fix S
assume $S \in A$
obtain T **where** $T \in B$ $S \subseteq T$ $S \cap \bigcup(B - \{T\}) = \{\}$
using *SA SB* ‹$S \in A$› *connected_disjoint_Union_open_pick* [*OF B*, *of A*] *eq order_refl* **by** *blast*
moreover obtain S' **where** $S' \in A$ $T \subseteq S'$ $T \cap \bigcup(A - \{S'\}) = \{\}$
using *SA SB* ‹$T \in B$› *connected_disjoint_Union_open_pick* [*OF A*, *of B*] *eq order_refl* **by** *blast*
ultimately have $S' = S$
by (*metis A Int_subset_iff SA* ‹$S \in A$› *disjnt_def inf.orderE pairwise_def*)
with ‹$T \subseteq S'$› **have** $T \subseteq S$ **by** *simp*
with ‹$S \subseteq T$› **have** $S = T$ **by** *blast*

```
  with ‹T ∈ B› show S ∈ B by simp
qed

lemma connected_disjoint_Union_open_unique:
  assumes A: pairwise disjnt A and B: pairwise disjnt B
    and SA: ⋀S. S ∈ A ⟹ open S ∧ connected S ∧ S ≠ {}
    and SB: ⋀S. S ∈ B ⟹ open S ∧ connected S ∧ S ≠ {}
    and eq [simp]: ⋃A = ⋃B
  shows A = B
by (rule subset_antisym; metis connected_disjoint_Union_open_subset assms)

proposition components_open_unique:
 fixes S :: 'a::real_normed_vector set
  assumes pairwise disjnt A ⋃A = S
        ⋀X. X ∈ A ⟹ open X ∧ connected X ∧ X ≠ {}
    shows components S = A
proof –
  have open S using assms by blast
  show ?thesis
  proof (rule connected_disjoint_Union_open_unique)
    show disjoint (components S)
      by (simp add: components_eq disjnt_def pairwise_def)
 qed (use ‹open S› in ‹simp_all add: assms open_components in_components_connected
in_components_nonempty›)
qed
```

5.5.23 Existence of unbounded components

```
lemma cobounded_unbounded_component:
    fixes S :: 'a :: euclidean_space set
    assumes bounded (−S)
      shows ∃x. x ∈ S ∧ ¬ bounded (connected_component_set S x)
proof –
  obtain i::'a where i: i ∈ Basis
    using nonempty_Basis by blast
  obtain B where B: B>0 −S ⊆ ball 0 B
    using bounded_subset_ballD [OF assms, of 0] by auto
  then have ∗: ⋀x. B ≤ norm x ⟹ x ∈ S
    by (force simp: ball_def dist_norm)
  have unbounded_inner: ¬ bounded {x. inner i x ≥ B}
  proof (clarsimp simp: bounded_def dist_norm)
    fix e x
    show ∃y. B ≤ i · y ∧ ¬ norm (x − y) ≤ e
      using i
        by (rule_tac x=x + (max B e + 1 + |i · x|) ∗R i in exI) (auto simp:
inner_right_distrib)
  qed
  have §: ⋀x. B ≤ i · x ⟹ x ∈ S
   using ∗ Basis_le_norm [OF i] by (metis abs_ge_self inner_commute order_trans)
```

have $\{x.\ B \le i \cdot x\} \subseteq$ *connected_component_set S* $(B *_R i)$
by (*intro connected_component_maximal*) (*auto simp*: *i intro*: *convex_connected convex_halfspace_ge* [*of B*] §)
then have ¬ *bounded* (*connected_component_set S* $(B *_R i)$)
using *bounded_subset unbounded_inner* **by** *blast*
moreover have $B *_R i \in S$
by (*rule* ∗) (*simp add*: *norm_Basis* [*OF i*])
ultimately show *?thesis*
by *blast*
qed

lemma *cobounded_unique_unbounded_component*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *bs*: *bounded* (−*S*) **and** *2* ≤ *DIM*(*'a*)
and *bo*: ¬ *bounded*(*connected_component_set S x*)
¬ *bounded*(*connected_component_set S y*)
shows *connected_component_set S x* = *connected_component_set S y*
proof −
obtain *i*::*'a* **where** *i*: $i \in$ *Basis*
using *nonempty_Basis* **by** *blast*
obtain *B* **where** *B*: *B*>*0* −*S* ⊆ *ball 0 B*
using *bounded_subset_ballD* [*OF bs*, *of 0*] **by** *auto*
then have ∗: $\bigwedge x.\ B \le$ *norm x* $\Longrightarrow x \in S$
by (*force simp*: *ball_def dist_norm*)
obtain *x'* **where** *x'*: *connected_component S x x' norm x'* > *B*
using *bo* [*unfolded bounded_def dist_norm*, *simplified*, *rule_format*]
by (*metis diff_zero norm_minus_commute not_less*)
obtain *y'* **where** *y'*: *connected_component S y y' norm y'* > *B*
using *bo* [*unfolded bounded_def dist_norm*, *simplified*, *rule_format*]
by (*metis diff_zero norm_minus_commute not_less*)
have *x'y'*: *connected_component S x' y'*
unfolding *connected_component_def*
proof (*intro exI conjI*)
show *connected* (− *ball 0 B* :: *'a set*)
using *assms* **by** (*auto intro*: *connected_complement_bounded_convex*)
qed (*use x' y' dist_norm* ∗ **in** *auto*)
show *?thesis*
proof (*rule connected_component_eq*)
show $x \in$ *connected_component_set S y*
using *x' y' x'y'*
by (*metis* (*no_types*) *connected_component_eq_eq connected_component_in mem_Collect_eq*)
qed
qed

lemma *cobounded_unbounded_components*:
fixes *S* :: *'a* :: *euclidean_space set*
shows *bounded* (−*S*) $\Longrightarrow \exists c.\ c \in$ *components S* ∧ ¬*bounded c*
by (*metis cobounded_unbounded_component components_def imageI*)

lemma *cobounded_unique_unbounded_components*:
  fixes S :: *'a* :: *euclidean_space set*
  shows ⟦*bounded* $(- S)$; $c \in$ *components* S; ¬ *bounded* c; $c' \in$ *components* S; ¬ *bounded* c'; $2 \leq$ *DIM*(*'a*)⟧ ⟹ $c' = c$
  unfolding *components_iff*
  by (*metis cobounded_unique_unbounded_component*)

lemma *cobounded_has_bounded_component*:
  fixes S :: *'a* :: *euclidean_space set*
  assumes *bounded* $(- S)$ ¬ *connected* S $2 \leq$ *DIM*(*'a*)
  obtains C **where** $C \in$ *components* S *bounded* C
  by (*meson cobounded_unique_unbounded_components connected_eq_connected_components_eq assms*)

5.5.24 The *inside* and *outside* of a Set

The inside comprises the points in a bounded connected component of the set's complement. The outside comprises the points in unbounded connected component of the complement.

definition *inside* **where**
  inside $S \equiv \{x.\ (x \notin S) \wedge$ *bounded*(*connected_component_set* $(- S)$ x)}

definition *outside* **where**
  outside $S \equiv -S \cap \{x.$ ¬ *bounded*(*connected_component_set* $(- S)$ x)}

lemma *outside*: *outside* $S = \{x.$ ¬ *bounded*(*connected_component_set* $(- S)$ x)}
  by (*auto simp*: *outside_def*) (*metis Compl_iff bounded_empty connected_component_eq_empty*)

lemma *inside_no_overlap* [*simp*]: *inside* $S \cap S = \{\}$
  by (*auto simp*: *inside_def*)

lemma *outside_no_overlap* [*simp*]:
  outside $S \cap S = \{\}$
  by (*auto simp*: *outside_def*)

lemma *inside_Int_outside* [*simp*]: *inside* $S \cap$ *outside* $S = \{\}$
  by (*auto simp*: *inside_def outside_def*)

lemma *inside_Un_outside* [*simp*]: *inside* $S \cup$ *outside* $S = (- S)$
  by (*auto simp*: *inside_def outside_def*)

lemma *inside_eq_outside*:
  inside $S =$ *outside* $S \longleftrightarrow S =$ *UNIV*
  by (*auto simp*: *inside_def outside_def*)

lemma *inside_outside*: *inside* $S = (- (S \cup$ *outside* $S))$
  by (*force simp*: *inside_def outside*)

lemma *outside_inside*: *outside S* = (− (*S* ∪ *inside S*))
  by (*auto simp*: *inside_outside*) (*metis IntI equals0D outside_no_overlap*)

lemma *union_with_inside*: *S* ∪ *inside S* = − *outside S*
  by (*auto simp*: *inside_outside*) (*simp add*: *outside_inside*)

lemma *union_with_outside*: *S* ∪ *outside S* = − *inside S*
  by (*simp add*: *inside_outside*)

lemma *outside_mono*: *S* ⊆ *T* ⟹ *outside T* ⊆ *outside S*
  by (*auto simp*: *outside bounded_subset connected_component_mono*)

lemma *inside_mono*: *S* ⊆ *T* ⟹ *inside S* − *T* ⊆ *inside T*
  by (*auto simp*: *inside_def bounded_subset connected_component_mono*)

lemma *segment_bound_lemma*:
  fixes *u*::*real*
  assumes *x* ≥ *B y* ≥ *B 0* ≤ *u u* ≤ *1*
  shows (*1* − *u*) ∗ *x* + *u* ∗ *y* ≥ *B*
proof −
  obtain *dx dy* **where** *dx* ≥ *0 dy* ≥ *0 x* = *B* + *dx y* = *B* + *dy*
    using *assms* **by** *auto* (*metis add.commute diff_add_cancel*)
  with ‹*0* ≤ *u*› ‹*u* ≤ *1*› **show** *?thesis*
    by (*simp add*: *add_increasing2 mult_left_le field_simps*)
qed

lemma *cobounded_outside*:
  fixes *S* :: *'a* :: *real_normed_vector set*
  assumes *bounded S* **shows** *bounded* (− *outside S*)
proof −
  obtain *B* **where** *B*: *B*>*0 S* ⊆ *ball 0 B*
    using *bounded_subset_ballD* [*OF assms*, *of 0*] **by** *auto*
  { fix *x*::*'a* **and** *C*::*real*
    assume *Bno*: *B* ≤ *norm x* **and** *C*: *0* < *C*
    have ∃ *y*. *connected_component* (− *S*) *x y* ∧ *norm y* > *C*
    proof (*cases x* = *0*)
      case *True* **with** *B Bno* **show** *?thesis* **by** *force*
    next
      case *False*
      have *closed_segment x* (((*B* + *C*) / *norm x*) $*_R$ *x*) ⊆ − *ball 0 B*
      proof
        fix *w*
        assume *w* ∈ *closed_segment x* (((*B* + *C*) / *norm x*) $*_R$ *x*)
        then obtain *u* **where**
          w: *w* = (*1* − *u* + *u* ∗ (*B* + *C*) / *norm x*) $*_R$ *x 0* ≤ *u u* ≤ *1*
            by (*auto simp add*: *closed_segment_def real_vector_class.scaleR_add_left*
[*symmetric*])
        with *False B C* **have** *B* ≤ (*1* − *u*) ∗ *norm x* + *u* ∗ (*B* + *C*)
          using *segment_bound_lemma* [*of B norm x B* + *C u*] *Bno*

by *simp*
with *False B C* **show** $w \in -$ *ball 0 B*
using *distrib_right* [*of _ _ norm x*]
by (*simp add*: *ball_def w not_less*)
qed
also have ... $\subseteq -S$
by (*simp add*: *B*)
finally have $\exists T$. *connected T* $\wedge$ $T \subseteq - S \wedge x \in T \wedge ((B + C) / norm\ x) *_R x \in T$
by (*rule_tac x=closed_segment x (((B+C)/norm x)* $*_R$ *x)* **in** *exI*) *simp*
with *False B*
show *?thesis*
by (*rule_tac x=((B+C)/norm x)* $*_R$ *x* **in** *exI*) (*simp add*: *connected_component_def*)
qed
}
then show *?thesis*
apply (*simp add*: *outside_def assms*)
apply (*rule bounded_subset* [*OF bounded_ball* [*of 0 B*]])
apply (*force simp*: *dist_norm not_less bounded_pos*)
done
qed

lemma *unbounded_outside*:
fixes *S* :: *'a*::{*real_normed_vector*, *perfect_space*} *set*
shows *bounded S* $\Longrightarrow$ $\neg$ *bounded*(*outside S*)
using *cobounded_imp_unbounded cobounded_outside* **by** *blast*

lemma *bounded_inside*:
fixes *S* :: *'a*::{*real_normed_vector*, *perfect_space*} *set*
shows *bounded S* $\Longrightarrow$ *bounded*(*inside S*)
by (*simp add*: *bounded_Int cobounded_outside inside_outside*)

lemma *connected_outside*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *bounded S* $2 \leq$ *DIM*(*'a*)
shows *connected*(*outside S*)
apply (*clarsimp simp add*: *connected_iff_connected_component outside*)
apply (*rule_tac S=connected_component_set* (− *S*) *x* **in** *connected_component_of_subset*)
apply (*metis* (*no_types*) *assms cobounded_unbounded_component cobounded_unique_unbounded_component connected_component_eq_eq connected_component_idemp double_complement mem_Collect_eq*)
by (*simp add*: *Collect_mono connected_component_eq*)

lemma *outside_connected_component_lt*:
outside S = $\{x. \forall B. \exists y. B < norm(y) \wedge$ *connected_component* (− *S*) *x y*}
apply (*auto simp*: *outside bounded_def dist_norm*)
apply (*metis diff_0 norm_minus_cancel not_less*)
by (*metis less_diff_eq norm_minus_commute norm_triangle_ineq2 order.trans pinf*(*6*))

lemma *outside_connected_component_le*:

outside S = {*x*. ∀ *B*. ∃ *y*. *B* ≤ *norm*(*y*) ∧ *connected_component* (− *S*) *x y*}
apply (*simp add*: *outside_connected_component_lt Set.set_eq_iff*)
by (*meson gt_ex leD le_less_linear less_imp_le order.trans*)

lemma *not_outside_connected_component_lt*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *S*: *bounded S* **and** *2* ≤ *DIM*(′*a*)
shows − (*outside S*) = {*x*. ∀ *B*. ∃ *y*. *B* < *norm*(*y*) ∧ ¬ *connected_component* (− *S*) *x y*}
proof −
obtain *B*::*real* **where** *B*: *0* < *B* **and** *Bno*: ⋀*x*. *x* ∈ *S* ⟹ *norm x* ≤ *B*
using *S* [*simplified bounded_pos*] **by** *auto*
{ **fix** *y*::′*a* **and** *z*::′*a*
assume *yz*: *B* < *norm z B* < *norm y*
have *connected_component* (− *cball 0 B*) *y z*
using *assms yz*
by (*force simp*: *dist_norm intro*: *connected_componentI* [*OF* _ *subset_refl*] *connected_complement_bounded_convex*)
then have *connected_component* (− *S*) *y z*
by (*metis connected_component_of_subset Bno Compl_anti_mono mem_cball_0 subset_iff*)
} **note** *cyz* = *this*
show *?thesis*
apply (*auto simp*: *outside bounded_pos*)
apply (*metis Compl_iff bounded_iff cobounded_imp_unbounded mem_Collect_eq not_le*)
by (*metis B connected_component_trans cyz not_le*)
qed

lemma *not_outside_connected_component_le*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *S*: *bounded S* *2* ≤ *DIM*(′*a*)
shows − (*outside S*) = {*x*. ∀ *B*. ∃ *y*. *B* ≤ *norm*(*y*) ∧ ¬ *connected_component* (− *S*) *x y*}
apply (*auto intro*: *less_imp_le simp*: *not_outside_connected_component_lt* [*OF assms*])
by (*meson gt_ex less_le_trans*)

lemma *inside_connected_component_lt*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *S*: *bounded S* *2* ≤ *DIM*(′*a*)
shows *inside S* = {*x*. (*x* ∉ *S*) ∧ (∀ *B*. ∃ *y*. *B* < *norm*(*y*) ∧ ¬ *connected_component* (− *S*) *x y*)}
by (*auto simp*: *inside_outside not_outside_connected_component_lt* [*OF assms*])

lemma *inside_connected_component_le*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *S*: *bounded S* *2* ≤ *DIM*(′*a*)
shows *inside S* = {*x*. (*x* ∉ *S*) ∧ (∀ *B*. ∃ *y*. *B* ≤ *norm*(*y*) ∧ ¬ *con-*

```
nected_component (− S) x y)}
  by (auto simp: inside_outside not_outside_connected_component_le [OF assms])

lemma inside_subset:
  assumes connected U and ¬ bounded U and T ∪ U = − S
  shows inside S ⊆ T
  apply (auto simp: inside_def)
 by (metis bounded_subset [of connected_component_set (− S) _] connected_component_maximal
     Compl_iff Un_iff assms subsetI)

lemma frontier_not_empty:
  fixes S :: 'a :: real_normed_vector set
  shows ⟦S ≠ {}; S ≠ UNIV⟧ ⟹ frontier S ≠ {}
    using connected_Int_frontier [of UNIV S] by auto

lemma frontier_eq_empty:
  fixes S :: 'a :: real_normed_vector set
  shows frontier S = {} ⟷ S = {} ∨ S = UNIV
using frontier_UNIV frontier_empty frontier_not_empty by blast

lemma frontier_of_connected_component_subset:
  fixes S :: 'a::real_normed_vector set
  shows frontier(connected_component_set S x) ⊆ frontier S
proof −
  { fix y
    assume y1: y ∈ closure (connected_component_set S x)
       and y2: y ∉ interior (connected_component_set S x)
    have y ∈ closure S
      using y1 closure_mono connected_component_subset by blast
    moreover have z ∈ interior (connected_component_set S x)
          if 0 < e ball y e ⊆ interior S dist y z < e for e z
    proof −
      have ball y e ⊆ connected_component_set S y
        using connected_component_maximal that interior_subset
        by (metis centre_in_ball connected_ball subset_trans)
      then show ?thesis
        using y1 apply (simp add: closure_approachable open_contains_ball_eq [OF
open_interior])
         by (metis connected_component_eq dist_commute mem_Collect_eq mem_ball
mem_interior subsetD ‹0 < e› y2)
    qed
    then have y ∉ interior S
      using y2 by (force simp: open_contains_ball_eq [OF open_interior])
    ultimately have y ∈ frontier S
      by (auto simp: frontier_def)
  }
  then show ?thesis by (auto simp: frontier_def)
qed
```

lemma *frontier_Union_subset_closure*:
  fixes *F* :: *'a::real_normed_vector set set*
  shows *frontier*($\bigcup$*F*) ⊆ *closure*($\bigcup$ *t* ∈ *F*. *frontier t*)
proof −
  have ∃ *y*∈*F*. ∃ *y*∈*frontier y*. *dist y x* < *e*
      if *T* ∈ *F y* ∈ *T dist y x* < *e*
         x ∉ *interior* ($\bigcup$*F*) *0* < *e* **for** *x y e T*
  proof (*cases x* ∈ *T*)
    case *True* **with** *that* **show** *?thesis*
     by (*metis Diff_iff Sup_upper closure_subset contra_subsetD dist_self frontier_def interior_mono*)
  next
    case *False*
    have *1*: *closed_segment x y* ∩ *T* ≠ {}
      using ‹*y* ∈ *T*› **by** *blast*
    have *2*: *closed_segment x y* − *T* ≠ {}
      using *False* **by** *blast*
    obtain *c* **where** *c* ∈ *closed_segment x y c* ∈ *frontier T*
       using *False connected_Int_frontier* [*OF connected_segment 1 2*] **by** *auto*
    then show *?thesis*
    proof −
      have *norm* (*y* − *x*) < *e*
        by (*metis dist_norm* ‹*dist y x* < *e*›)
      moreover have *norm* (*c* − *x*) ≤ *norm* (*y* − *x*)
        by (*simp add*: ‹*c* ∈ *closed_segment x y*› *segment_bound*(*1*))
      ultimately have *norm* (*c* − *x*) < *e*
        by *linarith*
      then show *?thesis*
        by (*metis* (*no_types*) ‹*c* ∈ *frontier T*› *dist_norm that*(*1*))
    qed
  qed
  then show *?thesis*
    by (*fastforce simp add*: *frontier_def closure_approachable*)
qed

lemma *frontier_Union_subset*:
  fixes *F* :: *'a::real_normed_vector set set*
  shows *finite F* ⟹ *frontier*($\bigcup$*F*) ⊆ ($\bigcup$ *t* ∈ *F*. *frontier t*)
by (*rule order_trans* [*OF frontier_Union_subset_closure*])
   (*auto simp*: *closure_subset_eq*)

lemma *frontier_of_components_subset*:
  fixes *S* :: *'a::real_normed_vector set*
  shows *C* ∈ *components S* ⟹ *frontier C* ⊆ *frontier S*
 by (*metis Path_Connected.frontier_of_connected_component_subset components_iff*)

lemma *frontier_of_components_closed_complement*:
  fixes *S* :: *'a::real_normed_vector set*
  shows ⟦*closed S*; *C* ∈ *components* (− *S*)⟧ ⟹ *frontier C* ⊆ *S*

using *frontier_complement frontier_of_components_subset frontier_subset_eq* **by** *blast*

lemma *frontier_minimal_separating_closed*:
fixes *S* :: *'a::real_normed_vector set*
assumes *closed S*
and *nconn*: ¬ *connected*(− *S*)
and *C*: *C* ∈ *components* (− *S*)
and *conn*: ⋀*T*. ⟦*closed T*; *T* ⊂ *S*⟧ ⟹ *connected*(− *T*)
shows *frontier C* = *S*
proof (*rule ccontr*)
assume *frontier C* ≠ *S*
then have *frontier C* ⊂ *S*
using *frontier_of_components_closed_complement* [*OF* ‹*closed S*› *C*] **by** *blast*
then have *connected*(− (*frontier C*))
by (*simp add*: *conn*)
have ¬ *connected*(− (*frontier C*))
unfolding *connected_def not_not*
proof (*intro exI conjI*)
show *open C*
using *C* ‹*closed S*› *open_components* **by** *blast*
show *open* (− *closure C*)
by *blast*
show *C* ∩ − *closure C* ∩ − *frontier C* = {}
using *closure_subset* **by** *blast*
show *C* ∩ − *frontier C* ≠ {}
using *C* ‹*open C*› *components_eq frontier_disjoint_eq* **by** *fastforce*
show − *frontier C* ⊆ *C* ∪ − *closure C*
by (*simp add*: ‹*open C*› *closed_Compl frontier_closures*)
then show − *closure C* ∩ − *frontier C* ≠ {}
by (*metis* (*no_types*, *lifting*) *C Compl_subset_Compl_iff* ‹*frontier C* ⊂ *S*› *compl_sup frontier_closures in_components_subset psubsetE sup.absorb_iff2 sup.boundedE sup_bot.right_neutral sup_inf_absorb*)
qed
then show *False*
using ‹*connected* (− *frontier C*)› **by** *blast*
qed

lemma *connected_component_UNIV* [*simp*]:
fixes *x* :: *'a::real_normed_vector*
shows *connected_component_set UNIV x* = *UNIV*
using *connected_iff_eq_connected_component_set* [*of UNIV*::*'a set*] *connected_UNIV* **by** *auto*

lemma *connected_component_eq_UNIV*:
fixes *x* :: *'a::real_normed_vector*
shows *connected_component_set s x* = *UNIV* ⟷ *s* = *UNIV*
using *connected_component_in connected_component_UNIV* **by** *blast*

lemma *components_UNIV* [*simp*]: *components UNIV* = {*UNIV* :: ′*a*::*real_normed_vector set*}
  by (*auto simp*: *components_eq_sing_iff*)

lemma *interior_inside_frontier*:
  fixes *S* :: ′*a*::*real_normed_vector set*
  assumes *bounded S*
    shows *interior S* ⊆ *inside* (*frontier S*)
proof −
  { **fix** *x y*
    assume *x*: *x* ∈ *interior S* **and** *y*: *y* ∉ *S*
      and *cc*: *connected_component* (− *frontier S*) *x y*
    have *connected_component_set* (− *frontier S*) *x* ∩ *frontier S* ≠ {}
    proof (*rule connected_Int_frontier*; *simp add*: *set_eq_iff*)
      show ∃ *u*. *connected_component* (− *frontier S*) *x u* ∧ *u* ∈ *S*
        by (*meson cc connected_component_in connected_component_refl_eq interior_subset subsetD x*)
      show ∃ *u*. *connected_component* (− *frontier S*) *x u* ∧ *u* ∉ *S*
        using *y cc* **by** *blast*
    qed
    then have *bounded* (*connected_component_set* (− *frontier S*) *x*)
      using *connected_component_in* **by** *auto*
  }
  then show *?thesis*
    apply (*auto simp*: *inside_def frontier_def*)
    apply (*rule classical*)
    apply (*rule bounded_subset* [*OF assms*], *blast*)
    done
qed

lemma *inside_empty* [*simp*]: *inside* {} = ({} :: ′*a* :: {*real_normed_vector*, *perfect_space*} *set*)
  by (*simp add*: *inside_def*)

lemma *outside_empty* [*simp*]: *outside* {} = (*UNIV* :: ′*a* :: {*real_normed_vector*, *perfect_space*} *set*)
  using *inside_empty inside_Un_outside* **by** *blast*

lemma *inside_same_component*:
  ⟦*connected_component* (− *S*) *x y*; *x* ∈ *inside S*⟧ ⟹ *y* ∈ *inside S*
  using *connected_component_eq connected_component_in*
  by (*fastforce simp add*: *inside_def*)

lemma *outside_same_component*:
  ⟦*connected_component* (− *S*) *x y*; *x* ∈ *outside S*⟧ ⟹ *y* ∈ *outside S*
  using *connected_component_eq connected_component_in*
  by (*fastforce simp add*: *outside_def*)

lemma *convex_in_outside*:

```
  fixes S :: 'a :: {real_normed_vector, perfect_space} set
  assumes S: convex S and z: z ∉ S
    shows z ∈ outside S
proof (cases S={})
  case True then show ?thesis by simp
next
  case False then obtain a where a ∈ S by blast
  with z have zna: z ≠ a by auto
  { assume bounded (connected_component_set (− S) z)
   with bounded_pos_less obtain B where B>0 and B: ⋀x. connected_component
(− S) z x ⟹ norm x < B
      by (metis mem_Collect_eq)
    define C where C = (B + 1 + norm z) / norm (z−a)
    have C > 0
     using ‹0 < B› zna by (simp add: C_def field_split_simps add_strict_increasing)
    have |norm (z + C *R (z−a)) − norm (C *R (z−a))| ≤ norm z
      by (metis add_diff_cancel norm_triangle_ineq3)
    moreover have norm (C *R (z−a)) > norm z + B
      using zna ‹B>0› by (simp add: C_def le_max_iff_disj)
    ultimately have C: norm (z + C *R (z−a)) > B by linarith
    { fix u::real
      assume u: 0≤u u≤1 and ins: (1 − u) *R z + u *R (z + C *R (z − a)) ∈
S
      then have Cpos: 1 + u * C > 0
      by (meson ‹0 < C› add_pos_nonneg less_eq_real_def zero_le_mult_iff zero_less_one)
      then have *: (1 / (1 + u * C)) *R z + (u * C / (1 + u * C)) *R z = z
        by (simp add: scaleR_add_left [symmetric] field_split_simps)
      then have False
       using convexD_alt [OF S ‹a ∈ S› ins, of 1/(u*C + 1)] ‹C>0› ‹z ∉ S› Cpos
u
        by (simp add: * field_split_simps)
    } note contra = this
    have connected_component (− S) z (z + C *R (z−a))
    proof (rule connected_componentI [OF connected_segment])
      show closed_segment z (z + C *R (z − a)) ⊆ − S
        using contra by (force simp add: closed_segment_def)
    qed auto
    then have False
      using zna B [of z + C *R (z−a)] C
      by (auto simp: field_split_simps max_mult_distrib_right)
  }
  then show ?thesis
    by (auto simp: outside_def z)
qed

lemma outside_convex:
  fixes S :: 'a :: {real_normed_vector, perfect_space} set
  assumes convex S
    shows outside S = − S
```

by (*metis ComplD assms convex_in_outside equalityI inside_Un_outside subsetI sup.cobounded2*)

lemma *outside_singleton* [*simp*]:
fixes x :: $'a$ :: {*real_normed_vector*, *perfect_space*}
shows *outside* $\{x\} = -\{x\}$
by (*auto simp*: *outside_convex*)

lemma *inside_convex*:
fixes S :: $'a$ :: {*real_normed_vector*, *perfect_space*} *set*
shows *convex* $S \Longrightarrow$ *inside* $S = \{\}$
by (*simp add*: *inside_outside outside_convex*)

lemma *inside_singleton* [*simp*]:
fixes x :: $'a$ :: {*real_normed_vector*, *perfect_space*}
shows *inside* $\{x\} = \{\}$
by (*auto simp*: *inside_convex*)

lemma *outside_subset_convex*:
fixes S :: $'a$ :: {*real_normed_vector*, *perfect_space*} *set*
shows $[\![$*convex* T; $S \subseteq T]\!] \Longrightarrow -\ T \subseteq$ *outside* S
using *outside_convex outside_mono* **by** *blast*

lemma *outside_Un_outside_Un*:
fixes S :: $'a$::*real_normed_vector set*
assumes $S \cap$ *outside*$(T \cup U) = \{\}$
shows *outside*$(T \cup U) \subseteq$ *outside*$(T \cup S)$
proof
fix x
assume x: $x \in$ *outside* $(T \cup U)$
have $Y \subseteq -\ S$ **if** *connected* Y $Y \subseteq -\ T$ $Y \subseteq -\ U$ $x \in Y$ $u \in Y$ **for** u Y
proof −
have $Y \subseteq$ *connected_component_set* $(-\ (T \cup U))$ x
by (*simp add*: *connected_component_maximal that*)
also have $\ldots \subseteq$ *outside*$(T \cup U)$
by (*metis* (*mono_tags*, *lifting*) *Collect_mono mem_Collect_eq outside outside_same_component x*)
finally have $Y \subseteq$ *outside*$(T \cup U)$ **.**
with *assms* **show** *?thesis* **by** *auto*
qed
with x **show** $x \in$ *outside* $(T \cup S)$
by (*simp add*: *outside_connected_component_lt connected_component_def*) *meson*
qed

lemma *outside_frontier_misses_closure*:
fixes S :: $'a$::*real_normed_vector set*
assumes *bounded* S
shows *outside*(*frontier* S) $\subseteq -\ $*closure* S
unfolding *outside_inside Lattices.boolean_algebra_class.compl_le_compl_iff*

proof −
  { **assume** *interior S* ⊆ *inside* (*frontier S*)
    hence *interior S* ∪ *inside* (*frontier S*) = *inside* (*frontier S*)
      by (*simp add*: *subset_Un_eq*)
    then have *closure S* ⊆ *frontier S* ∪ *inside* (*frontier S*)
      using *frontier_def* **by** *auto*
  }
  then show *closure S* ⊆ *frontier S* ∪ *inside* (*frontier S*)
    using *interior_inside_frontier* [*OF assms*] **by** *blast*
qed

lemma *outside_frontier_eq_complement_closure*:
  fixes *S* :: *'a* :: {*real_normed_vector*, *perfect_space*} *set*
    assumes *bounded S convex S*
      shows *outside*(*frontier S*) = − *closure S*
by (*metis Diff_subset assms convex_closure frontier_def outside_frontier_misses_closure outside_subset_convex subset_antisym*)

lemma *inside_frontier_eq_interior*:
     fixes *S* :: *'a* :: {*real_normed_vector*, *perfect_space*} *set*
     shows ⟦*bounded S*; *convex S*⟧ ⟹ *inside*(*frontier S*) = *interior S*
  apply (*simp add*: *inside_outside outside_frontier_eq_complement_closure*)
  using *closure_subset interior_subset*
  apply (*auto simp*: *frontier_def*)
  done

lemma *open_inside*:
    fixes *S* :: *'a*::*real_normed_vector set*
    assumes *closed S*
      shows *open* (*inside S*)
proof −
  { **fix** *x* **assume** *x*: *x* ∈ *inside S*
    have *open* (*connected_component_set* (− *S*) *x*)
      using *assms open_connected_component* **by** *blast*
   then obtain *e* **where** *e*: *e*>*0* **and** *e*: ⋀*y*. *dist y x* < *e* ⟶ *connected_component* (− *S*) *x y*
      using *dist_not_less_zero*
      apply (*simp add*: *open_dist*)
      by (*metis* (*no_types*, *lifting*) *Compl_iff connected_component_refl_eq inside_def mem_Collect_eq x*)
    then have ∃ *e*>*0*. *ball x e* ⊆ *inside S*
      by (*metis e dist_commute inside_same_component mem_ball subsetI x*)
  }
  then show *?thesis*
    by (*simp add*: *open_contains_ball*)
qed

lemma *open_outside*:
    fixes *S* :: *'a*::*real_normed_vector set*

assumes *closed S*
shows *open* (*outside S*)
proof −
{ **fix** *x* **assume** *x*: *x* ∈ *outside S*
have *open* (*connected_component_set* (− *S*) *x*)
using *assms open_connected_component* **by** *blast*
then obtain *e* **where** *e*: *e*>*0* **and** *e*: ⋀*y*. *dist y x* < *e* ⟶ *connected_component* (− *S*) *x y*
using *dist_not_less_zero x*
by (*auto simp add*: *open_dist outside_def intro*: *connected_component_refl*)
then have ∃ *e*>*0*. *ball x e* ⊆ *outside S*
by (*metis e dist_commute outside_same_component mem_ball subsetI x*)
}
then show *?thesis*
by (*simp add*: *open_contains_ball*)
qed

lemma *closure_inside_subset*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *closed S*
shows *closure*(*inside S*) ⊆ *S* ∪ *inside S*
by (*metis assms closure_minimal open_closed open_outside sup.cobounded2 union_with_inside*)

lemma *frontier_inside_subset*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *closed S*
shows *frontier*(*inside S*) ⊆ *S*
proof −
have *closure* (*inside S*) ∩ − *inside S* = *closure* (*inside S*) − *interior* (*inside S*)
by (*metis* (*no_types*) *Diff_Compl assms closure_closed interior_closure open_closed open_inside*)
moreover have − *inside S* ∩ − *outside S* = *S*
by (*metis* (*no_types*) *compl_sup double_compl inside_Un_outside*)
moreover have *closure* (*inside S*) ⊆ − *outside S*
by (*metis* (*no_types*) *assms closure_inside_subset union_with_inside*)
ultimately have *closure* (*inside S*) − *interior* (*inside S*) ⊆ *S*
by *blast*
then show *?thesis*
by (*simp add*: *frontier_def open_inside interior_open*)
qed

lemma *closure_outside_subset*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *closed S*
shows *closure*(*outside S*) ⊆ *S* ∪ *outside S*
by (*metis assms closed_open closure_minimal inside_outside open_inside sup_ge2*)

lemma *frontier_outside_subset*:
fixes *S* :: *'a*::*real_normed_vector set*

assumes *closed S*
shows *frontier(outside S) ⊆ S*
unfolding *frontier_def*
by (*metis Diff_subset_conv assms closure_outside_subset interior_eq open_outside sup_aci(1)*)

lemma *inside_complement_unbounded_connected_empty*:
⟦*connected* (− *S*); ¬ *bounded* (− *S*)⟧ ⟹ *inside S* = {}
using *inside_subset* **by** *blast*

lemma *inside_bounded_complement_connected_empty*:
fixes *S* :: *'a*::{*real_normed_vector*, *perfect_space*} *set*
shows ⟦*connected* (− *S*); *bounded S*⟧ ⟹ *inside S* = {}
by (*metis inside_complement_unbounded_connected_empty cobounded_imp_unbounded*)

lemma *inside_inside*:
assumes *S* ⊆ *inside T*
shows *inside S* − *T* ⊆ *inside T*
unfolding *inside_def*
proof *clarify*
fix *x*
assume *x*: *x* ∉ *T* *x* ∉ *S* **and** *bo*: *bounded* (*connected_component_set* (− *S*) *x*)
show *bounded* (*connected_component_set* (− *T*) *x*)
proof (*cases S* ∩ *connected_component_set* (− *T*) *x* = {})
case *True* **then show** *?thesis*
by (*metis bounded_subset* [*OF bo*] *compl_le_compl_iff connected_component_idemp connected_component_mono disjoint_eq_subset_Compl double_compl*)
next
case *False*
then obtain *y* **where** *y*: *y* ∈ *S* *y* ∈ *connected_component_set* (− *T*) *x*
by (*meson disjoint_iff*)
then have *bounded* (*connected_component_set* (− *T*) *y*)
using *assms* [*unfolded inside_def*] **by** *blast*
with *y* **show** *?thesis*
by (*metis connected_component_eq*)
qed
qed

lemma *inside_inside_subset*: *inside*(*inside S*) ⊆ *S*
using *inside_inside union_with_outside* **by** *fastforce*

lemma *inside_outside_intersect_connected*:
⟦*connected T*; *inside S* ∩ *T* ≠ {}; *outside S* ∩ *T* ≠ {}⟧ ⟹ *S* ∩ *T* ≠ {}
apply (*simp add*: *inside_def outside_def ex_in_conv* [*symmetric*] *disjoint_eq_subset_Compl*, *clarify*)
by (*metis* (*no_types*, *hide_lams*) *Compl_anti_mono connected_component_eq connected_component_maximal contra_subsetD double_compl*)

lemma *outside_bounded_nonempty*:

fixes *S* :: *'a* :: {*real_normed_vector*, *perfect_space*} *set*
assumes *bounded S* **shows** *outside S* ≠ {}
by (*metis* (*no_types*, *lifting*) *Collect_empty_eq Collect_mem_eq Compl_eq_Diff_UNIV Diff_cancel*
Diff_disjoint UNIV_I assms ball_eq_empty bounded_diff cobounded_outside convex_ball
double_complement order_refl outside_convex outside_def)

lemma *outside_compact_in_open*:
fixes *S* :: *'a* :: {*real_normed_vector*,*perfect_space*} *set*
assumes *S*: *compact S* **and** *T*: *open T* **and** *S* ⊆ *T T* ≠ {}
shows *outside S* ∩ *T* ≠ {}
proof −
have *outside S* ≠ {}
by (*simp add*: *compact_imp_bounded outside_bounded_nonempty S*)
with *assms* **obtain** *a b* **where** *a*: *a* ∈ *outside S* **and** *b*: *b* ∈ *T* **by** *auto*
show *?thesis*
proof (*cases a* ∈ *T*)
case *True* **with** *a* **show** *?thesis* **by** *blast*
next
case *False*
have *front*: *frontier T* ⊆ − *S*
using ‹*S* ⊆ *T*› *frontier_disjoint_eq T* **by** *auto*
{ **fix** γ
assume *path* γ **and** *pimg_sbs*: *path_image* γ − {*pathfinish* γ} ⊆ *interior* (−*T*)
and *pf*: *pathfinish* γ ∈ *frontier T* **and** *ps*: *pathstart* γ = *a*
define *c* **where** *c* = *pathfinish* γ
have *c* ∈ −*S* **unfolding** *c_def* **using** *front pf* **by** *blast*
moreover have *open* (−*S*) **using** *S compact_imp_closed* **by** *blast*
ultimately obtain ε::*real* **where** ε > *0* **and** ε: *cball c* ε ⊆ −*S*
using *open_contains_cball*[*of* −*S*] *S* **by** *blast*
then obtain *d* **where** *d* ∈ *T* **and** *d*: *dist d c* < ε
using *closure_approachable* [*of c T*] *pf* **unfolding** *c_def*
by (*metis Diff_iff frontier_def*)
then have *d* ∈ −*S* **using** ε
using *dist_commute* **by** (*metis contra_subsetD mem_cball not_le not_less_iff_gr_or_eq*)
have *pimg_sbs_cos*: *path_image* γ ⊆ −*S*
using ‹*c* ∈ − *S*› ‹*S* ⊆ *T*› *c_def interior_subset pimg_sbs* **by** *fastforce*
have *closed_segment c d* ≤ *cball c* ε
by (*metis* ‹*0* < ε› *centre_in_cball closed_segment_subset convex_cball d dist_commute less_eq_real_def mem_cball*)
with ε **have** *closed_segment c d* ⊆ −*S* **by** *blast*
moreover have *con_gcd*: *connected* (*path_image* γ ∪ *closed_segment c d*)
by (*rule connected_Un*) (*auto simp*: *c_def* ‹*path* γ› *connected_path_image*)
ultimately have *connected_component* (− *S*) *a d*
unfolding *connected_component_def* **using** *pimg_sbs_cos ps* **by** *blast*
then have *outside S* ∩ *T* ≠ {}
using *outside_same_component* [*OF* _ *a*] **by** (*metis IntI* ‹*d* ∈ *T*› *empty_iff*)

} **note** ∗ = *this*
have *pal*: *pathstart* (*linepath a b*) ∈ *closure* (− *T*)
by (*auto simp*: *False closure_def*)
show *?thesis*
by (*rule exists_path_subpath_to_frontier* [*OF path_linepath pal* _ ∗]) (*auto simp*: *b*)
qed
qed

lemma *inside_inside_compact_connected*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *S*: *closed S* **and** *T*: *compact T* **and** *connected T S* ⊆ *inside T*
shows *inside S* ⊆ *inside T*
proof (*cases inside T* = {})
case *True* **with** *assms* **show** *?thesis* **by** *auto*
next
case *False*
consider *DIM*(*'a*) = *1* | *DIM*(*'a*) ≥ *2*
using *antisym not_less_eq_eq* **by** *fastforce*
then show *?thesis*
proof *cases*
case *1* **then show** *?thesis*
using *connected_convex_1_gen assms False inside_convex* **by** *blast*
next
case *2*
have *bounded S*
using *assms* **by** (*meson bounded_inside bounded_subset compact_imp_bounded*)
then have *coms*: *compact S*
by (*simp add*: *S compact_eq_bounded_closed*)
then have *bst*: *bounded* (*S* ∪ *T*)
by (*simp add*: *compact_imp_bounded T*)
then obtain *r* **where** *0* < *r* **and** *r*: *S* ∪ *T* ⊆ *ball 0 r*
using *bounded_subset_ballD* **by** *blast*
have *outst*: *outside S* ∩ *outside T* ≠ {}
proof −
have − *ball 0 r* ⊆ *outside S*
by (*meson convex_ball le_supE outside_subset_convex r*)
moreover have − *ball 0 r* ⊆ *outside T*
by (*meson convex_ball le_supE outside_subset_convex r*)
ultimately show *?thesis*
by (*metis Compl_subset_Compl_iff Int_subset_iff bounded_ball inf.orderE outside_bounded_nonempty outside_no_overlap*)
qed
have *S* ∩ *T* = {} **using** *assms*
by (*metis disjoint_iff_not_equal inside_no_overlap subsetCE*)
moreover have *outside S* ∩ *inside T* ≠ {}
by (*meson False assms*(*4*) *compact_eq_bounded_closed coms open_inside outside_compact_in_open T*)
ultimately have *inside S* ∩ *T* = {}

using *inside_outside_intersect_connected* [*OF* ‹*connected T*›, *of S*]
by (*metis 2 compact_eq_bounded_closed coms connected_outside inf.commute inside_outside_intersect_connected outst*)
then show *?thesis*
using *inside_inside* [*OF* ‹$S \subseteq$ *inside T*›] **by** *blast*
qed
qed

lemma *connected_with_inside*:
fixes *S* :: *'a* :: *real_normed_vector set*
assumes *S*: *closed S* **and** *cons*: *connected S*
shows *connected*($S \cup$ *inside S*)
proof (*cases* $S \cup$ *inside S* = *UNIV*)
case *True* **with** *assms* **show** *?thesis* **by** *auto*
next
case *False*
then obtain *b* **where** *b*: $b \notin S$ $b \notin$ *inside S* **by** *blast*
have ∗: $\exists y\ T.\ y \in S \wedge$ *connected T* $\wedge a \in T \wedge y \in T \wedge T \subseteq (S \cup$ *inside S*)
if $a \in S \cup$ *inside S* **for** *a*
using *that*
proof
assume $a \in S$ **then show** *?thesis*
by (*rule_tac x=a* **in** *exI*, *rule_tac x*={*a*} **in** *exI*, *simp*)
next
assume *a*: $a \in$ *inside S*
then have *ain*: $a \in$ *closure* (*inside S*)
by (*simp add*: *closure_def*)
show *?thesis*
apply (*rule exists_path_subpath_to_frontier* [*OF path_linepath* [*of a b*], *of inside S*])
apply (*simp_all add*: *ain b*)
subgoal for *h*
apply (*rule_tac x=pathfinish h* **in** *exI*)
apply (*simp add*: *subsetD* [*OF frontier_inside_subset*[*OF S*]])
apply (*rule_tac x=path_image h* **in** *exI*)
apply (*simp add*: *pathfinish_in_path_image connected_path_image*, *auto*)
by (*metis Diff_single_insert S frontier_inside_subset insert_iff interior_subset subsetD*)
done
qed
show *?thesis*
apply (*simp add*: *connected_iff_connected_component*)
apply (*clarsimp simp add*: *connected_component_def dest*!: ∗)
subgoal for *x y u u' T t'*
by (*rule_tac x*=($S \cup T \cup t'$) **in** *exI*) (*auto intro*!: *connected_Un cons*)
done
qed

The proof is virtually the same as that above.

lemma *connected_with_outside*:
  fixes *S* :: *'a* :: *real_normed_vector set*
  assumes *S*: *closed S* **and** *cons*: *connected S*
    shows *connected*(*S* ∪ *outside S*)
proof (*cases S* ∪ *outside S* = *UNIV*)
  case *True* **with** *assms* **show** *?thesis* **by** *auto*
next
  case *False*
  then obtain *b* **where** *b*: *b* ∉ *S b* ∉ *outside S* **by** *blast*
  have ∗: ∃ *y T*. *y* ∈ *S* ∧ *connected T* ∧ *a* ∈ *T* ∧ *y* ∈ *T* ∧ *T* ⊆ (*S* ∪ *outside S*)
if *a* ∈ (*S* ∪ *outside S*) **for** *a*
  using *that* **proof**
    assume *a* ∈ *S* **then show** *?thesis*
      by (*rule_tac x*=*a* **in** *exI*, *rule_tac x*={*a*} **in** *exI*, *simp*)
  next
    assume *a*: *a* ∈ *outside S*
    then have *ain*: *a* ∈ *closure* (*outside S*)
      by (*simp add*: *closure_def*)
    show *?thesis*
    apply (*rule exists_path_subpath_to_frontier* [*OF path_linepath* [*of a b*], *of outside S*])
        apply (*simp_all add*: *ain b*)
      subgoal for *h*
      apply (*rule_tac x*=*pathfinish h* **in** *exI*)
        apply (*simp add*: *subsetD* [*OF frontier_outside_subset*[*OF S*]])
      apply (*rule_tac x*=*path_image h* **in** *exI*)
      apply (*simp add*: *pathfinish_in_path_image connected_path_image*, *auto*)
        by (*metis* (*no_types*, *lifting*) *frontier_outside_subset insertE insert_Diff interior_eq open_outside pathfinish_in_path_image S subsetCE*)
      done
  qed
  show *?thesis*
    apply (*simp add*: *connected_iff_connected_component*)
    apply (*clarsimp simp add*: *connected_component_def dest*!: ∗)
    subgoal for *x y u u' T t'*
      by (*rule_tac x*=(*S* ∪ *T* ∪ *t'*) **in** *exI*) (*auto intro*!: *connected_Un cons*)
    done
qed

lemma *inside_inside_eq_empty* [*simp*]:
  fixes *S* :: *'a* :: {*real_normed_vector*, *perfect_space*} *set*
  assumes *S*: *closed S* **and** *cons*: *connected S*
    shows *inside* (*inside S*) = {}
  by (*metis* (*no_types*) *unbounded_outside connected_with_outside* [*OF assms*] *bounded_Un inside_complement_unbounded_connected_empty unbounded_outside union_with_outside*)

lemma *inside_in_components*:
  inside S ∈ *components* (− *S*) ⟷ *connected*(*inside S*) ∧ *inside S* ≠ {} (**is** *?lhs* = *?rhs*)

proof
  assume *R*: *?rhs*
  then have $\bigwedge x.\ [\![x \in S;\ x \in inside\ S]\!] \Longrightarrow \neg\ connected\ (inside\ S)$
    by (*simp add*: *inside_outside*)
  with *R* **show** *?lhs*
    unfolding *in_components_maximal*
    by (*auto intro*: *inside_same_component connected_componentI*)
qed (*simp add*: *in_components_maximal*)

The proof is like that above.

lemma *outside_in_components*:
  $outside\ S \in components\ (-\ S) \longleftrightarrow connected(outside\ S) \wedge outside\ S \neq \{\}$ (**is** *?lhs* = *?rhs*)
proof
  assume *R*: *?rhs*
  then have $\bigwedge x.\ [\![x \in S;\ x \in outside\ S]\!] \Longrightarrow \neg\ connected\ (outside\ S)$
    by (*meson disjoint_iff outside_no_overlap*)
  with *R* **show** *?lhs*
    unfolding *in_components_maximal*
    by (*auto intro*: *outside_same_component connected_componentI*)
qed (*simp add*: *in_components_maximal*)

lemma *bounded_unique_outside*:
  fixes *S* :: *'a* :: *euclidean_space set*
  assumes *bounded S* $DIM('a) \geq 2$
  shows $(c \in components\ (-\ S) \wedge \neg\ bounded\ c \longleftrightarrow c = outside\ S)$
  using *assms*
  by (*metis cobounded_unique_unbounded_components connected_outside double_compl outside_bounded_nonempty outside_in_components unbounded_outside*)

5.5.25 Condition for an open map's image to contain a ball

proposition *ball_subset_open_map_image*:
  fixes *f* :: *'a::heine_borel* ⇒ *'b* :: {*real_normed_vector,heine_borel*}
  assumes *contf*: *continuous_on* (*closure S*) *f*
    and *oint*: *open* (*f* ‘ *interior S*)
    and *le_no*: $\bigwedge z.\ z \in frontier\ S \Longrightarrow r \leq norm(f\ z - f\ a)$
    and *bounded S* $a \in S$ $0 < r$
  shows *ball* (*f a*) $r \subseteq f$ ‘ *S*
proof (*cases f* ‘ *S* = *UNIV*)
  case *True* **then show** *?thesis* **by** *simp*
next
  case *False*
  then have *closed* (*frontier* (*f* ‘ *S*)) *frontier* (*f* ‘ *S*) ≠ {}
    using ‹$a \in S$› **by** (*auto simp*: *frontier_eq_empty*)
  then obtain *w* **where** *w*: *w* ∈ *frontier* (*f* ‘ *S*)
    and *dw_le*: $\bigwedge y.\ y \in frontier\ (f\ `\ S) \Longrightarrow norm\ (f\ a - w) \leq norm\ (f\ a - y)$
    by (*auto simp add*: *dist_norm intro*: *distance_attains_inf* [*of frontier*(*f* ‘ *S*) *f a*])
  then obtain ξ **where** ξ: $\bigwedge n.\ \xi\ n \in f$ ‘ *S* **and** *tendsw*: $\xi \longrightarrow w$

by (*metis Diff_iff frontier_def closure_sequential*)
then have $\bigwedge n. \exists x \in S. \xi\ n = f\ x$ **by** *force*
then obtain z **where** *zs*: $\bigwedge n. z\ n \in S$ **and** *fz*: $\bigwedge n. \xi\ n = f\ (z\ n)$
by *metis*
then obtain $y\ K$ **where** *y*: $y \in closure\ S$ **and** *strict_mono* $(K :: nat \Rightarrow nat)$
and *Klim*: $(z \circ K) \longrightarrow y$
using ‹*bounded S*›
unfolding *compact_closure* [*symmetric*] *compact_def* **by** (*meson closure_subset subset_iff*)
then have *ftendsw*: $((\lambda n. f\ (z\ n)) \circ K) \longrightarrow w$
by (*metis LIMSEQ_subseq_LIMSEQ fun.map_cong0 fz tendsw*)
have *zKs*: $\bigwedge n. (z \circ K)\ n \in S$ **by** (*simp add*: *zs*)
have *fz*: $f \circ z = \xi\ \ (\lambda n. f\ (z\ n)) = \xi$
using *fz* **by** *auto*
then have $(\xi \circ K) \longrightarrow f\ y$
by (*metis* (*no_types*) *Klim zKs y contf comp_assoc continuous_on_closure_sequentially*)
with *fz* **have** *wy*: $w = f\ y$ **using** *fz LIMSEQ_unique ftendsw* **by** *auto*
have *rle*: $r \le norm\ (f\ y - f\ a)$
proof (*rule le_no*)
show $y \in frontier\ S$
using *w wy oint* **by** (*force simp*: *imageI image_mono interiorI interior_subset frontier_def y*)
qed
have $**$: $(b \cap (-\ S) \neq \{\} \wedge b - (-\ S) \neq \{\} \Longrightarrow b \cap f \neq \{\})$
$\Longrightarrow (b \cap S \neq \{\}) \Longrightarrow b \cap f = \{\} \Longrightarrow b \subseteq S$
for $b\ f$ **and** $S :: 'b\ set$
by *blast*
have §: $\bigwedge y. [\![norm\ (f\ a - y) < r;\ y \in frontier\ (f\ `\ S)]\!] \Longrightarrow False$
by (*metis dw_le norm_minus_commute not_less order_trans rle wy*)
show *?thesis*
apply (*rule* $**$ [*OF connected_Int_frontier* [**where** $t = f`S$, *OF connected_ball*]])

using ‹$a \in S$› ‹$0 < r$› **by** (*auto simp*: *disjoint_iff_not_equal dist_norm dest*: §)
qed

Special characterizations of classes of functions into and out of R.

lemma *Hausdorff_space_euclidean* [*simp*]: *Hausdorff_space* (*euclidean* :: *'a::metric_space topology*)
proof –
have $\exists U\ V.\ open\ U \wedge open\ V \wedge x \in U \wedge y \in V \wedge disjnt\ U\ V$
if $x \neq y$
for $x\ y :: 'a$
proof (*intro exI conjI*)
let $?r = dist\ x\ y\ /\ 2$
have [*simp*]: $?r > 0$
by (*simp add*: *that*)
show *open* (*ball x ?r*) *open* (*ball y ?r*) $x \in$ (*ball x ?r*) $y \in$ (*ball y ?r*)
by (*auto simp add*: *that*)

```
    show disjnt (ball x ?r) (ball y ?r)
      unfolding disjnt_def by (simp add: disjoint_ballI)
  qed
  then show ?thesis
    by (simp add: Hausdorff_space_def)
qed

proposition embedding_map_into_euclideanreal:
  assumes path_connected_space X
  shows embedding_map X euclideanreal f ⟷
        continuous_map X euclideanreal f ∧ inj_on f (topspace X)
  proof safe
  show continuous_map X euclideanreal f
    if embedding_map X euclideanreal f
    using continuous_map_in_subtopology homeomorphic_imp_continuous_map that
    unfolding embedding_map_def by blast
  show inj_on f (topspace X)
    if embedding_map X euclideanreal f
    using that homeomorphic_imp_injective_map
    unfolding embedding_map_def by blast
  show embedding_map X euclideanreal f
    if cont: continuous_map X euclideanreal f and inj: inj_on f (topspace X)
  proof −
    obtain g where gf: ⋀x. x ∈ topspace X ⟹ g (f x) = x
      using inv_into_f_f [OF inj] by auto
    show ?thesis
     unfolding embedding_map_def homeomorphic_map_maps homeomorphic_maps_def
    proof (intro exI conjI)
      show continuous_map X (top_of_set (f ‘ topspace X)) f
        by (simp add: cont continuous_map_in_subtopology)
      let ?S = f ‘ topspace X
      have eq: {x ∈ ?S. g x ∈ U} = f ‘ U if openin X U for U
        using openin_subset [OF that] by (auto simp: gf)
      have 1: g ‘ ?S ⊆ topspace X
        using eq by blast
      have openin (top_of_set ?S) {x ∈ ?S. g x ∈ T}
        if openin X T for T
      proof −
        have T ⊆ topspace X
          by (simp add: openin_subset that)
        have RR: ∀x ∈ ?S ∩ g −‘ T. ∃d>0. ∀x′ ∈ ?S ∩ ball x d. g x′ ∈ T
        proof (clarsimp simp add: gf)
          have pcS: path_connectedin euclidean ?S
         using assms cont path_connectedin_continuous_map_image path_connectedin_topspace
by blast
          show ∃d>0. ∀x′∈f ‘ topspace X ∩ ball (f x) d. g x′ ∈ T
            if x ∈ T for x
          proof −
            have x: x ∈ topspace X
```

```
  using ‹T ⊆ topspace X› ‹x ∈ T› by blast
obtain u v d where 0 < d u ∈ topspace X v ∈ topspace X
       and sub_fuv: ?S ∩ {f x − d .. f x + d} ⊆ {f u..f v}
proof (cases ∃ u ∈ topspace X. f u < f x)
  case True
  then obtain u where u: u ∈ topspace X f u < f x ..
  show ?thesis
  proof (cases ∃ v ∈ topspace X. f x < f v)
    case True
    then obtain v where v: v ∈ topspace X f x < f v ..
    show ?thesis
    proof
      let ?d = min (f x − f u) (f v − f x)
      show 0 < ?d
        by (simp add: ‹f u < f x› ‹f x < f v›)
      show f ‘ topspace X ∩ {f x − ?d..f x + ?d} ⊆ {f u..f v}
        by fastforce
    qed (auto simp: u v)
  next
    case False
    show ?thesis
    proof
      let ?d = f x − f u
      show 0 < ?d
        by (simp add: u)
      show f ‘ topspace X ∩ {f x − ?d..f x + ?d} ⊆ {f u..f x}
        using x u False by auto
    qed (auto simp: x u)
  qed
next
  case False
  note no_u = False
  show ?thesis
  proof (cases ∃ v ∈ topspace X. f x < f v)
    case True
    then obtain v where v: v ∈ topspace X f x < f v ..
    show ?thesis
    proof
      let ?d = f v − f x
      show 0 < ?d
        by (simp add: v)
      show f ‘ topspace X ∩ {f x − ?d..f x + ?d} ⊆ {f x..f v}
        using False by auto
    qed (auto simp: x v)
  next
    case False
    show ?thesis
    proof
      show f ‘ topspace X ∩ {f x − 1..f x + 1} ⊆ {f x..f x}
```

using *False no_u* **by** *fastforce*
qed (*auto simp*: *x*)
qed
qed
then obtain *h* **where** *pathin X h h 0 = u h 1 = v*
using *assms* **unfolding** *path_connected_space_def* **by** *blast*
obtain *C* **where** *compactin X C connectedin X C u ∈ C v ∈ C*
proof
show *compactin X* (*h ‘* {*0..1*})
using *that* **by** (*simp add*: ‹*pathin X h*› *compactin_path_image*)
show *connectedin X* (*h ‘* {*0..1*})
using ‹*pathin X h*› *connectedin_path_image* **by** *blast*
qed (*use* ‹*h 0 = u*› ‹*h 1 = v*› **in** *auto*)
have *continuous_map* (*subtopology euclideanreal* (*?S* ∩ {*f x − d .. f x + d*})) (*subtopology X C*) *g*
proof (*rule continuous_inverse_map*)
show *compact_space* (*subtopology X C*)
using ‹*compactin X C*› *compactin_subspace* **by** *blast*
show *continuous_map* (*subtopology X C*) *euclideanreal f*
by (*simp add*: *cont continuous_map_from_subtopology*)
have {*f u .. f v*} ⊆ *f ‘ topspace* (*subtopology X C*)
proof (*rule connected_contains_Icc*)
show *connected* (*f ‘ topspace* (*subtopology X C*))
using *connectedin_continuous_map_image* [*OF cont*]
by (*simp add*: ‹*compactin X C*› ‹*connectedin X C*› *compactin_subset_topspace inf_absorb2*)
show *f u ∈ f ‘ topspace* (*subtopology X C*)
by (*simp add*: ‹*u ∈ C*› ‹*u ∈ topspace X*›)
show *f v ∈ f ‘ topspace* (*subtopology X C*)
by (*simp add*: ‹*v ∈ C*› ‹*v ∈ topspace X*›)
qed
then show *f ‘ topspace X* ∩ {*f x − d..f x + d*} ⊆ *f ‘ topspace* (*subtopology X C*)
using *sub_fuv* **by** *blast*
qed (*auto simp*: *gf*)
then have *contg*: *continuous_map* (*subtopology euclideanreal* (*?S* ∩ {*f x − d .. f x + d*})) *X g*
using *continuous_map_in_subtopology* **by** *blast*
have ∃ *e>0*. ∀ *x* ∈ *?S* ∩ {*f x − d .. f x + d*} ∩ *ball* (*f x*) *e*. *g x* ∈ *T*
using *openin_continuous_map_preimage* [*OF contg* ‹*openin X T*›] *x* ‹*x* ∈ *T*› ‹*0 < d*›
unfolding *openin_euclidean_subtopology_iff*
by (*force simp*: *gf dist_commute*)
then obtain *e* **where** *e > 0* ∧ (∀ *x*∈*f ‘ topspace X* ∩ {*f x − d..f x + d*} ∩ *ball* (*f x*) *e*. *g x* ∈ *T*)
by *metis*
with ‹*0 < d*› **have** *min d e > 0* ∀ *u*. *u ∈ topspace X* ⟶ |*f x − f u*| < *min d e* ⟶ *u ∈ T*
using *dist_real_def gf* **by** *force+*

```
          then show ?thesis
            by (metis (full_types) Int_iff dist_real_def image_iff mem_ball gf)
        qed
      qed
      then obtain d where d: ⋀r. r ∈ ?S ∩ g −‘ T ⟹
              d r > 0 ∧ (∀x ∈ ?S ∩ ball r (d r). g x ∈ T)
        by metis
      show ?thesis
        unfolding openin_subtopology
      proof (intro exI conjI)
          show {x ∈ ?S. g x ∈ T} = (⋃r ∈ ?S ∩ g −‘ T. ball r (d r)) ∩ f ‘
topspace X
          using d by (auto simp: gf)
      qed auto
    qed
    then show continuous_map (top_of_set ?S) X g
      by (simp add: continuous_map_def gf)
  qed (auto simp: gf)
 qed
qed
```

An injective function into R is a homeomorphism and so an open map.

```
lemma injective_into_1d_eq_homeomorphism:
  fixes f :: 'a::topological_space ⇒ real
  assumes f: continuous_on S f and S: path_connected S
  shows inj_on f S ⟷ (∃g. homeomorphism S (f ‘ S) f g)
proof
  show ∃g. homeomorphism S (f ‘ S) f g
    if inj_on f S
  proof −
    have embedding_map (top_of_set S) euclideanreal f
      using that embedding_map_into_euclideanreal [of top_of_set S f] assms by auto
    then show ?thesis
      by (simp add: embedding_map_def) (metis all_closedin_homeomorphic_image f
homeomorphism_injective_closed_map that)
  qed
qed (metis homeomorphism_def inj_onI)

lemma injective_into_1d_imp_open_map:
  fixes f :: 'a::topological_space ⇒ real
  assumes continuous_on S f path_connected S inj_on f S openin (subtopology eu-
clidean S) T
  shows openin (subtopology euclidean (f ‘ S)) (f ‘ T)
  using assms homeomorphism_imp_open_map injective_into_1d_eq_homeomorphism
by blast

lemma homeomorphism_into_1d:
```

fixes *f* :: *'a::topological_space* ⇒ *real*
assumes *path_connected S continuous_on S f f ' S* = *T inj_on f S*
shows ∃ *g. homeomorphism S T f g*
using *assms injective_into_1d_eq_homeomorphism* **by** *blast*

5.5.26 Rectangular paths

definition *rectpath* **where**
rectpath a1 a3 = (*let a2* = *Complex* (*Re a3*) (*Im a1*); *a4* = *Complex* (*Re a1*) (*Im a3*)
in linepath a1 a2 +++ *linepath a2 a3* +++ *linepath a3 a4* +++ *linepath a4 a1*)

lemma *path_rectpath* [*simp*, *intro*]: *path* (*rectpath a b*)
by (*simp add*: *Let_def rectpath_def*)

lemma *pathstart_rectpath* [*simp*]: *pathstart* (*rectpath a1 a3*) = *a1*
by (*simp add*: *rectpath_def Let_def*)

lemma *pathfinish_rectpath* [*simp*]: *pathfinish* (*rectpath a1 a3*) = *a1*
by (*simp add*: *rectpath_def Let_def*)

lemma *simple_path_rectpath* [*simp*, *intro*]:
assumes *Re a1* ≠ *Re a3 Im a1* ≠ *Im a3*
shows *simple_path* (*rectpath a1 a3*)
unfolding *rectpath_def Let_def* **using** *assms*
by (*intro simple_path_join_loop arc_join arc_linepath*)
(*auto simp*: *complex_eq_iff path_image_join closed_segment_same_Re closed_segment_same_Im*)

lemma *path_image_rectpath*:
assumes *Re a1* ≤ *Re a3 Im a1* ≤ *Im a3*
shows *path_image* (*rectpath a1 a3*) =
{*z. Re z* ∈ {*Re a1*, *Re a3*} ∧ *Im z* ∈ {*Im a1..Im a3*}} ∪
{*z. Im z* ∈ {*Im a1*, *Im a3*} ∧ *Re z* ∈ {*Re a1..Re a3*}} (**is** *?lhs* = *?rhs*)
proof −
define *a2 a4* **where** *a2* = *Complex* (*Re a3*) (*Im a1*) **and** *a4* = *Complex* (*Re a1*) (*Im a3*)
have *?lhs* = *closed_segment a1 a2* ∪ *closed_segment a2 a3* ∪
closed_segment a4 a3 ∪ *closed_segment a1 a4*
by (*simp_all add*: *rectpath_def Let_def path_image_join closed_segment_commute a2_def a4_def Un_assoc*)
also have ... = *?rhs* **using** *assms*
by (*auto simp*: *rectpath_def Let_def path_image_join a2_def a4_def closed_segment_same_Re closed_segment_same_Im closed_segment_eq_real_ivl*)
finally show *?thesis* **.**
qed

lemma *path_image_rectpath_subset_cbox*:
assumes *Re a* ≤ *Re b Im a* ≤ *Im b*

shows *path_image* (*rectpath a b*) ⊆ *cbox a b*
using *assms* **by** (*auto simp*: *path_image_rectpath in_cbox_complex_iff*)

lemma *path_image_rectpath_inter_box*:
assumes *Re a* ≤ *Re b Im a* ≤ *Im b*
shows *path_image* (*rectpath a b*) ∩ *box a b* = {}
using *assms* **by** (*auto simp*: *path_image_rectpath in_box_complex_iff*)

lemma *path_image_rectpath_cbox_minus_box*:
assumes *Re a* ≤ *Re b Im a* ≤ *Im b*
shows *path_image* (*rectpath a b*) = *cbox a b* − *box a b*
using *assms* **by** (*auto simp*: *path_image_rectpath in_cbox_complex_iff in_box_complex_iff*)

end

5.6 Bernstein-Weierstrass and Stone-Weierstrass

By L C Paulson (2015)

theory *Weierstrass_Theorems*
imports *Uniform_Limit Path_Connected Derivative*
begin

5.6.1 Bernstein polynomials

definition *Bernstein* :: [*nat*,*nat*,*real*] ⇒ *real* **where**
Bernstein n k x ≡ *of_nat* (*n choose k*) ∗ *x*ˆ*k* ∗ (*1* − *x*)ˆ(*n* − *k*)

lemma *Bernstein_nonneg*: ⟦*0* ≤ *x*; *x* ≤ *1*⟧ ⟹ *0* ≤ *Bernstein n k x*
by (*simp add*: *Bernstein_def*)

lemma *Bernstein_pos*: ⟦*0* < *x*; *x* < *1*; *k* ≤ *n*⟧ ⟹ *0* < *Bernstein n k x*
by (*simp add*: *Bernstein_def*)

lemma *sum_Bernstein* [*simp*]: (∑*k*≤*n*. *Bernstein n k x*) = *1*
using *binomial_ring* [*of x 1*−*x n*]
by (*simp add*: *Bernstein_def*)

lemma *binomial_deriv1*:
(∑*k*≤*n*. (*of_nat k* ∗ *of_nat* (*n choose k*)) ∗ *a*ˆ(*k*−*1*) ∗ *b*ˆ(*n*−*k*)) = *real_of_nat n* ∗ (*a*+*b*)ˆ(*n*−*1*)
apply (*rule DERIV_unique* [**where** *f* = λ*a*. (*a*+*b*)ˆ*n* **and** *x*=*a*])
apply (*subst binomial_ring*)
apply (*rule derivative_eq_intros sum.cong* | *simp add*: *atMost_atLeast0*)+
done

lemma *binomial_deriv2*:

($\sum k \le n.$ (*of_nat k* $*$ *of_nat* ($k-1$) $*$ *of_nat* (*n choose k*)) $*$ *a* ^($k-2$) $*$ *b* ^($n-k$))
=
of_nat n $*$ *of_nat* ($n-1$) $*$ (*a*+*b*::*real*) ^($n-2$)
apply (*rule DERIV_unique* [**where** *f* = $\lambda a.$ *of_nat n* $*$ (*a*+*b*::*real*) ^($n-1$) **and** *x*=*a*])
apply (*subst binomial_deriv1* [*symmetric*])
apply (*rule derivative_eq_intros sum.cong* | *simp add*: *Num.numeral_2_eq_2*)+
done

lemma *sum_k_Bernstein* [*simp*]: ($\sum k \le n.$ *real k* $*$ *Bernstein n k x*) = *of_nat n* $*$ *x*
apply (*subst binomial_deriv1* [*of n x* 1−*x*, *simplified*, *symmetric*])
apply (*simp add*: *sum_distrib_right*)
apply (*auto simp*: *Bernstein_def algebra_simps power_eq_if intro*!: *sum.cong*)
done

lemma *sum_kk_Bernstein* [*simp*]: ($\sum k \le n.$ *real k* $*$ (*real k* − *1*) $*$ *Bernstein n k x*) = *real n* $*$ (*real n* − *1*) $*$ x^2
proof −
have ($\sum k \le n.$ *real k* $*$ (*real k* − *1*) $*$ *Bernstein n k x*) =
($\sum k \le n.$ *real k* $*$ *real* (*k* − *Suc 0*) $*$ *real* (*n choose k*) $*$ *x* ^(*k* − *2*) $*$ (*1* − *x*) ^(*n* − *k*) $*$ x^2)
proof (*rule sum.cong* [*OF refl*], *simp*)
fix *k*
assume $k \le n$
then consider *k* = *0* | *k* = *1* | *k′* **where** *k* = *Suc* (*Suc k′*)
by (*metis One_nat_def not0_implies_Suc*)
then show *k* = *0* ∨
(*real k* − *1*) $*$ *Bernstein n k x* =
real (*k* − *Suc 0*) $*$
(*real* (*n choose k*) $*$ (*x* ^(*k* − *2*) $*$ ((*1* − *x*) ^(*n* − *k*) $*$ x^2)))
by *cases* (*auto simp add*: *Bernstein_def power2_eq_square algebra_simps*)
qed
also have ... = *real_of_nat n* $*$ *real_of_nat* (*n* − *Suc 0*) $*$ x^2
by (*subst binomial_deriv2* [*of n x* 1−*x*, *simplified*, *symmetric*]) (*simp add*: *sum_distrib_right*)
also have ... = *n* $*$ (*n* − *1*) $*$ x^2
by *auto*
finally show *?thesis*
by *auto*
qed

5.6.2 Explicit Bernstein version of the 1D Weierstrass approximation theorem

theorem *Bernstein_Weierstrass*:
fixes *f* :: *real* ⇒ *real*
assumes *contf*: *continuous_on* {*0..1*} *f* **and** *e*: *0* < *e*
shows ∃*N*. ∀*n x*. *N* ≤ *n* ∧ *x* ∈ {*0..1*}

```
                    ⟶ |f x − (∑k≤n. f(k/n) ∗ Bernstein n k x)| < e
proof −
  have bounded (f ‘ {0..1})
    using compact_continuous_image compact_imp_bounded contf by blast
  then obtain M where M: ⋀x. 0 ≤ x ⟹ x ≤ 1 ⟹ |f x| ≤ M
    by (force simp add: bounded_iff)
  then have 0 ≤ M by force
  have ucontf: uniformly_continuous_on {0..1} f
    using compact_uniformly_continuous contf by blast
  then obtain d where d: d>0 ⋀x x'. ⟦ x ∈ {0..1}; x' ∈ {0..1}; |x' − x| < d⟧
⟹ |f x' − f x| < e/2
     apply (rule uniformly_continuous_onE [where e = e/2])
     using e by (auto simp: dist_norm)
  { fix n::nat and x::real
    assume n: Suc (nat⌈4∗M/(e∗d²)⌉) ≤ n and x: 0 ≤ x x ≤ 1
    have 0 < n using n by simp
    have ed0: − (e ∗ d²) < 0
      using e ‹0<d› by simp
    also have ... ≤ M ∗ 4
      using ‹0≤M› by simp
    finally have [simp]: real_of_int (nat ⌈4 ∗ M / (e ∗ d²)⌉) = real_of_int ⌈4 ∗ M
/ (e ∗ d²)⌉
      using ‹0≤M› e ‹0<d›
      by (simp add: field_simps)
    have 4∗M/(e∗d²) + 1 ≤ real (Suc (nat⌈4∗M/(e∗d²)⌉))
      by (simp add: real_nat_ceiling_ge)
    also have ... ≤ real n
      using n by (simp add: field_simps)
    finally have nbig: 4∗M/(e∗d²) + 1 ≤ real n .
    have sum_bern: (∑k≤n. (x − k/n)² ∗ Bernstein n k x) = x ∗ (1 − x) / n
    proof −
      have ∗: ⋀a b x::real. (a − b)² ∗ x = a ∗ (a − 1) ∗ x + (1 − 2 ∗ b) ∗ a ∗ x
+ b ∗ b ∗ x
        by (simp add: algebra_simps power2_eq_square)
      have (∑k≤n. (k − n ∗ x)² ∗ Bernstein n k x) = n ∗ x ∗ (1 − x)
        apply (simp add: ∗ sum.distrib)
        apply (simp flip: sum_distrib_left add: mult.assoc)
        apply (simp add: algebra_simps power2_eq_square)
        done
      then have (∑k≤n. (k − n ∗ x)² ∗ Bernstein n k x)/nˆ2 = x ∗ (1 − x) / n
        by (simp add: power2_eq_square)
      then show ?thesis
       using n by (simp add: sum_divide_distrib field_split_simps power2_commute)
    qed
    { fix k
      assume k: k ≤ n
      then have kn: 0 ≤ k / n k / n ≤ 1
        by (auto simp: field_split_simps)
      consider (lessd) |x − k / n| < d | (ged) d ≤ |x − k / n|
```

```
        by linarith
      then have |(f x − f (k/n))| ≤ e/2 + 2 ∗ M / d² ∗ (x − k/n)²
      proof cases
        case lessd
        then have |(f x − f (k/n))| < e/2
          using d x kn by (simp add: abs_minus_commute)
        also have ... ≤ (e/2 + 2 ∗ M / d² ∗ (x − k/n)²)
          using ‹M≥0› d by simp
        finally show ?thesis by simp
      next
        case ged
        then have dle: d² ≤ (x − k/n)²
          by (metis d(1) less_eq_real_def power2_abs power_mono)
        have §: 1 ≤ (x − real k / real n)² / d²
          using dle ‹d>0› by auto
        have |(f x − f (k/n))| ≤ |f x| + |f (k/n)|
          by (rule abs_triangle_ineq4)
        also have ... ≤ M+M
          by (meson M add_mono_thms_linordered_semiring(1) kn x)
        also have ... ≤ 2 ∗ M ∗ ((x − k/n)² / d²)
          using § ‹M≥0› mult_left_mono by fastforce
        also have ... ≤ e/2 + 2 ∗ M / d² ∗ (x − k/n)²
          using e  by simp
        finally show ?thesis .
        qed
    } note ∗ = this
    have |f x − (∑ k≤n. f(k / n) ∗ Bernstein n k x)| ≤ |∑ k≤n. (f x − f(k / n))
∗ Bernstein n k x|
      by (simp add: sum_subtractf sum_distrib_left [symmetric] algebra_simps)
    also have ... ≤ (∑ k≤n. |(f x − f(k / n)) ∗ Bernstein n k x|)
      by (rule sum_abs)
    also have ... ≤ (∑ k≤n. (e/2 + (2 ∗ M / d²) ∗ (x − k / n)²) ∗ Bernstein n
k x)
      using ∗
        by (force simp add: abs_mult Bernstein_nonneg x mult_right_mono intro:
sum_mono)
    also have ... ≤ e/2 + (2 ∗ M) / (d² ∗ n)
     unfolding sum.distrib Rings.semiring_class.distrib_right sum_distrib_left [symmetric]
mult.assoc sum_bern
      using ‹d>0› x by (simp add: divide_simps ‹M≥0› mult_le_one mult_left_le)
    also have ... < e
      using ‹d>0› nbig e ‹n>0›
      apply (simp add: field_split_simps)
      using ed0 by linarith
    finally have |f x − (∑ k≤n. f (real k / real n) ∗ Bernstein n k x)| < e .
  }
  then show ?thesis
    by auto
qed
```

5.6.3 General Stone-Weierstrass theorem

Source: Bruno Brosowski and Frank Deutsch. An Elementary Proof of the Stone-Weierstrass Theorem. Proceedings of the American Mathematical Society Volume 81, Number 1, January 1981. DOI: 10.2307/2043993 https://www.jstor.org/stable/2043993

```
locale function_ring_on =
  fixes R :: ('a::t2_space ⇒ real) set and S :: 'a set
  assumes compact: compact S
  assumes continuous: f ∈ R ⟹ continuous_on S f
  assumes add: f ∈ R ⟹ g ∈ R ⟹ (λx. f x + g x) ∈ R
  assumes mult: f ∈ R ⟹ g ∈ R ⟹ (λx. f x * g x) ∈ R
  assumes const: (λ_. c) ∈ R
  assumes separable: x ∈ S ⟹ y ∈ S ⟹ x ≠ y ⟹ ∃f∈R. f x ≠ f y

begin
  lemma minus: f ∈ R ⟹ (λx. - f x) ∈ R
    by (frule mult [OF const [of -1]]) simp

  lemma diff: f ∈ R ⟹ g ∈ R ⟹ (λx. f x - g x) ∈ R
    unfolding diff_conv_add_uminus by (metis add minus)

  lemma power: f ∈ R ⟹ (λx. f x ^n) ∈ R
    by (induct n) (auto simp: const mult)

  lemma sum: ⟦finite I; ⋀i. i ∈ I ⟹ f i ∈ R⟧ ⟹ (λx. ∑ i ∈ I. f i x) ∈ R
    by (induct I rule: finite_induct; simp add: const add)

  lemma prod: ⟦finite I; ⋀i. i ∈ I ⟹ f i ∈ R⟧ ⟹ (λx. ∏ i ∈ I. f i x) ∈ R
    by (induct I rule: finite_induct; simp add: const mult)

  definition normf :: ('a::t2_space ⇒ real) ⇒ real
    where normf f ≡ SUP x∈S. |f x|

  lemma normf_upper:
    assumes continuous_on S f x ∈ S shows |f x| ≤ normf f
  proof -
    have bdd_above ((λx. |f x|) ` S)
     by (simp add: assms(1) bounded_imp_bdd_above compact compact_continuous_image
compact_imp_bounded continuous_on_rabs)
    then show ?thesis
      using assms cSUP_upper normf_def by fastforce
  qed

  lemma normf_least: S ≠ {} ⟹ (⋀x. x ∈ S ⟹ |f x| ≤ M) ⟹ normf f ≤ M
    by (simp add: normf_def cSUP_least)

end
```

lemma (**in** *function_ring_on*) *one*:
  assumes *U*: *open U* **and** *t0*: *t0* ∈ *S t0* ∈ *U* **and** *t1*: *t1* ∈ *S*−*U*
    shows ∃ *V*. *open V* ∧ *t0* ∈ *V* ∧ *S* ∩ *V* ⊆ *U* ∧
      (∀ *e*>*0*. ∃ *f* ∈ *R*. *f* ‘ *S* ⊆ {*0*..*1*} ∧ (∀ *t* ∈ *S* ∩ *V*. *f t* < *e*) ∧ (∀ *t* ∈ *S* − *U*. *f t* > *1* − *e*))
proof −
  have ∃ *pt* ∈ *R*. *pt t0* = *0* ∧ *pt t* > *0* ∧ *pt* ‘ *S* ⊆ {*0*..*1*} **if** *t*: *t* ∈ *S* − *U* **for** *t*
  proof −
    have *t* ≠ *t0* **using** *t t0* **by** *auto*
    then obtain *g* **where** *g*: *g* ∈ *R g t* ≠ *g t0*
      using *separable t0* **by** (*metis Diff_subset subset_eq t*)
    define *h* **where** [*abs_def*]: *h x* = *g x* − *g t0* **for** *x*
    have *h* ∈ *R*
      unfolding *h_def* **by** (*fast intro*: *g const diff*)
    then have *hsq*: (λ*w*. (*h w*)^2) ∈ *R*
      by (*simp add*: *power2_eq_square mult*)
    have *h t* ≠ *h t0*
      by (*simp add*: *h_def g*)
    then have *h t* ≠ *0*
      by (*simp add*: *h_def*)
    then have *ht2*: *0* < (*h t*) ˆ*2*
      by *simp*
    also have ... ≤ *normf* (λ*w*. (*h w*)^2)
      using *t normf_upper* [**where** *x*=*t*] *continuous* [*OF hsq*] **by** *force*
    finally have *nfp*: *0* < *normf* (λ*w*. (*h w*)^2) **.**
    define *p* **where** [*abs_def*]: *p x* = (*1* / *normf* (λ*w*. (*h w*)^2)) ∗ (*h x*) ˆ*2* **for** *x*
    have *p* ∈ *R*
      unfolding *p_def* **by** (*fast intro*: *hsq const mult*)
    moreover have *p t0* = *0*
      by (*simp add*: *p_def h_def*)
    moreover have *p t* > *0*
      using *nfp ht2* **by** (*simp add*: *p_def*)
    moreover have ⋀*x*. *x* ∈ *S* ⟹ *p x* ∈ {*0*..*1*}
      using *nfp normf_upper* [*OF continuous* [*OF hsq*]] **by** (*auto simp*: *p_def*)
    ultimately show ∃ *pt* ∈ *R*. *pt t0* = *0* ∧ *pt t* > *0* ∧ *pt* ‘ *S* ⊆ {*0*..*1*}
      by *auto*
  qed
  then obtain *pf* **where** *pf*: ⋀*t*. *t* ∈ *S*−*U* ⟹ *pf t* ∈ *R* ∧ *pf t t0* = *0* ∧ *pf t t* > *0*
        and *pf01*: ⋀*t*. *t* ∈ *S*−*U* ⟹ *pf t* ‘ *S* ⊆ {*0*..*1*}
    by *metis*
  have *com_sU*: *compact* (*S*−*U*)
    using *compact closed_Int_compact U* **by** (*simp add*: *Diff_eq compact_Int_closed open_closed*)
  have ⋀*t*. *t* ∈ *S*−*U* ⟹ ∃ *A*. *open A* ∧ *A* ∩ *S* = {*x*∈*S*. *0* < *pf t x*}
    apply (*rule open_Collect_positive*)
    by (*metis pf continuous*)
  then obtain *Uf* **where** *Uf*: ⋀*t*. *t* ∈ *S*−*U* ⟹ *open* (*Uf t*) ∧ (*Uf t*) ∩ *S* = {*x*∈*S*. *0* < *pf t x*}

```
    by metis
  then have open_Uf: ⋀t. t ∈ S−U ⟹ open (Uf t)
    by blast
  have tUft: ⋀t. t ∈ S−U ⟹ t ∈ Uf t
    using pf Uf by blast
  then have *: S−U ⊆ (⋃x ∈ S−U. Uf x)
    by blast
  obtain subU where subU: subU ⊆ S − U finite subU S − U ⊆ (⋃x ∈ subU.
Uf x)
    by (blast intro: that compactE_image [OF com_sU open_Uf *])
  then have [simp]: subU ≠ {}
    using t1 by auto
  then have cardp: card subU > 0 using subU
    by (simp add: card_gt_0_iff)
  define p where [abs_def]: p x = (1 / card subU) * (∑t ∈ subU. pf t x) for x
  have pR: p ∈ R
    unfolding p_def using subU pf by (fast intro: pf const mult sum)
  have pt0 [simp]: p t0 = 0
    using subU pf by (auto simp: p_def intro: sum.neutral)
  have pt_pos: p t > 0 if t: t ∈ S−U for t
  proof −
    obtain i where i: i ∈ subU t ∈ Uf i using subU t by blast
    show ?thesis
      using subU i t
      apply (clarsimp simp: p_def field_split_simps)
      apply (rule sum_pos2 [OF ‹finite subU›])
      using Uf t pf01 apply auto
      apply (force elim!: subsetCE)
      done
  qed
  have p01: p x ∈ {0..1} if t: x ∈ S for x
  proof −
    have 0 ≤ p x
      using subU cardp t pf01
      by (fastforce simp add: p_def field_split_simps intro: sum_nonneg)
    moreover have p x ≤ 1
      using subU cardp t
      apply (simp add: p_def field_split_simps)
      apply (rule sum_bounded_above [where 'a=real and K=1, simplified])
      using pf01 by force
    ultimately show ?thesis
      by auto
  qed
  have compact (p ` (S−U))
    by (meson Diff_subset com_sU compact_continuous_image continuous continu-
ous_on_subset pR)
  then have open (− (p ` (S−U)))
    by (simp add: compact_imp_closed open_Compl)
  moreover have 0 ∈ − (p ` (S−U))
```

by (*metis* (*no_types*) *ComplI image_iff not_less_iff_gr_or_eq pt_pos*)
ultimately obtain *delta0* **where** *delta0*: *delta0* > *0* *ball 0 delta0* ⊆ − (*p* ‘ (*S*−*U*))
by (*auto simp*: *elim*!: *openE*)
then have *pt_delta*: ⋀*x*. *x* ∈ *S*−*U* ⟹ *p x* ≥ *delta0*
by (*force simp*: *ball_def dist_norm dest*: *p01*)
define *δ* **where** *δ* = *delta0*/*2*
have *delta0* ≤ *1* **using** *delta0 p01* [*of t1*] *t1*
by (*force simp*: *ball_def dist_norm dest*: *p01*)
with *delta0* **have** *δ01*: *0* < *δ* *δ* < *1*
by (*auto simp*: *δ_def*)
have *pt_δ*: ⋀*x*. *x* ∈ *S*−*U* ⟹ *p x* ≥ *δ*
using *pt_delta delta0* **by** (*force simp*: *δ_def*)
have ∃ *A*. *open A* ∧ *A* ∩ *S* = {*x*∈*S*. *p x* < *δ*/*2*}
by (*rule open_Collect_less_Int* [*OF continuous* [*OF pR*] *continuous_on_const*])
then obtain *V* **where** *V*: *open V V* ∩ *S* = {*x*∈*S*. *p x* < *δ*/*2*}
by *blast*
define *k* **where** *k* = *nat*⌊*1*/*δ*⌋ + *1*
have *k*>*0* **by** (*simp add*: *k_def*)
have *k*−*1* ≤ *1*/*δ*
using *δ01* **by** (*simp add*: *k_def*)
with *δ01* **have** *k* ≤ (*1*+*δ*)/*δ*
by (*auto simp*: *algebra_simps add_divide_distrib*)
also have ... < *2*/*δ*
using *δ01* **by** (*auto simp*: *field_split_simps*)
finally have *k2δ*: *k* < *2*/*δ* **.**
have *1*/*δ* < *k*
using *δ01* **unfolding** *k_def* **by** *linarith*
with *δ01 k2δ* **have** *kδ*: *1* < *k*∗*δ* *k*∗*δ* < *2*
by (*auto simp*: *field_split_simps*)
define *q* **where** [*abs_def*]: *q n t* = (*1* − *p t*ˆ*n*)ˆ(*k*ˆ*n*) **for** *n t*
have *qR*: *q n* ∈ *R* **for** *n*
by (*simp add*: *q_def const diff power pR*)
have *q01*: ⋀*n t*. *t* ∈ *S* ⟹ *q n t* ∈ {*0*..*1*}
using *p01* **by** (*simp add*: *q_def power_le_one algebra_simps*)
have *qt0* [*simp*]: ⋀*n*. *n*>*0* ⟹ *q n t0* = *1*
using *t0 pf* **by** (*simp add*: *q_def power_0_left*)
{ **fix** *t* **and** *n*::*nat*
assume *t*: *t* ∈ *S* ∩ *V*
with ‹*k*>*0*› *V* **have** *k* ∗ *p t* < *k* ∗ *δ* / *2*
by *force*
then have *1* − (*k* ∗ *δ* / *2*)ˆ*n* ≤ *1* − (*k* ∗ *p t*)ˆ*n*
using ‹*k*>*0*› *p01 t* **by** (*simp add*: *power_mono*)
also have ... ≤ *q n t*
using *Bernoulli_inequality* [*of* − ((*p t*)ˆ*n*) *k*ˆ*n*]
apply (*simp add*: *q_def*)
by (*metis IntE atLeastAtMost_iff p01 power_le_one power_mult_distrib t*)
finally have *1* − (*k* ∗ *δ* / *2*)ˆ*n* ≤ *q n t* **.**
} **note** *limitV* = *this*

```
{ fix t and n::nat
  assume t: t ∈ S − U
  with ‹k>0› U have k ∗ δ ≤ k ∗ p t
    by (simp add: pt_δ)
  with kδ have kpt: 1 < k ∗ p t
    by (blast intro: less_le_trans)
  have ptn_pos: 0 < p tˆn
    using pt_pos [OF t] by simp
  have ptn_le: p tˆn ≤ 1
    by (meson DiffE atLeastAtMost_iff p01 power_le_one t)
  have q n t = (1/(kˆn ∗ (p t)ˆn)) ∗ (1 − p tˆn)ˆ(kˆn) ∗ kˆn ∗ (p t)ˆn
    using pt_pos [OF t] ‹k>0› by (simp add: q_def)
  also have ... ≤ (1/(k ∗ (p t))ˆn) ∗ (1 − p tˆn)ˆ(kˆn) ∗ (1 + kˆn ∗ (p t)ˆn)
    using pt_pos [OF t] ‹k>0›
    by (simp add: divide_simps mult_left_mono ptn_le)
  also have ... ≤ (1/(k ∗ (p t))ˆn) ∗ (1 − p tˆn)ˆ(kˆn) ∗ (1 + (p t)ˆn)ˆ(kˆn)
  proof (rule mult_left_mono [OF Bernoulli_inequality])
    show 0 ≤ 1 / (real k ∗ p t)ˆn ∗ (1 − p tˆn)ˆkˆn
      using ptn_pos ptn_le by (auto simp: power_mult_distrib)
  qed (use ptn_pos in auto)
  also have ... = (1/(k ∗ (p t))ˆn) ∗ (1 − p tˆ(2∗n))ˆ(kˆn)
    using pt_pos [OF t] ‹k>0›
   by (simp add: algebra_simps power_mult power2_eq_square flip: power_mult_distrib)
  also have ... ≤ (1/(k ∗ (p t))ˆn) ∗ 1
    using pt_pos ‹k>0› p01 power_le_one t
    by (intro mult_left_mono [OF power_le_one]) auto
  also have ... ≤ (1 / (k∗δ))ˆn
    using ‹k>0› δ01  power_mono pt_δ t
    by (fastforce simp: field_simps)
  finally have q n t ≤ (1 / (real k ∗ δ))ˆn  .
} note limitNonU = this
define NN
  where NN e = 1 + nat ⌈max (ln e / ln (real k ∗ δ / 2)) (− ln e / ln (real k
∗ δ))⌉ for e
have NN: of_nat (NN e) > ln e / ln (real k ∗ δ / 2)  of_nat (NN e) > − ln e /
ln (real k ∗ δ)
           if 0<e for e
    unfolding NN_def  by linarith+
  have NN1: (k ∗ δ / 2)ˆNN e < e if e>0 for e
  proof −
    have ln ((real k ∗ δ / 2)ˆNN e) = real (NN e) ∗ ln (real k ∗ δ / 2)
      by (simp add: ‹δ>0› ‹0 < k› ln_realpow)
    also have ... < ln e
      using NN kδ that by (force simp add: field_simps)
    finally show ?thesis
      by (simp add: ‹δ>0› ‹0 < k› that)
  qed
have NN0: (1/(k∗δ))ˆ(NN e) < e if e>0 for e
proof −
```

have *0 < ln (real k) + ln δ*
using *δ01(1) ‹0 < k› kδ(1) ln_gt_zero ln_mult* **by** *fastforce*
then have *real (NN e) ∗ ln (1 / (real k ∗ δ)) < ln e*
using *kδ(1) NN(2) [of e] that* **by** *(simp add: ln_div divide_simps)*
then have *exp (real (NN e) ∗ ln (1 / (real k ∗ δ))) < e*
by *(metis exp_less_mono exp_ln that)*
then show *?thesis*
by *(simp add: δ01(1) ‹0 < k› exp_of_nat_mult)*
qed
{ fix *t* **and** *e::real*
assume *e>0*
have *t ∈ S ∩ V ⟹ 1 − q (NN e) t < e t ∈ S − U ⟹ q (NN e) t < e*
proof −
assume *t: t ∈ S ∩ V*
show *1 − q (NN e) t < e*
by *(metis add.commute diff_le_eq not_le limitV [OF t] less_le_trans [OF NN1 [OF ‹e>0›]])*
next
assume *t: t ∈ S − U*
show *q (NN e) t < e*
using *limitNonU [OF t] less_le_trans [OF NN0 [OF ‹e>0›]] not_le* **by** *blast*
qed
} then have *⋀e. e > 0 ⟹ ∃f∈R. f ‘ S ⊆ {0..1} ∧ (∀t ∈ S ∩ V. f t < e) ∧ (∀t ∈ S − U. 1 − e < f t)*
using *q01*
by *(rule_tac x=λx. 1 − q (NN e) x* **in** *bexI) (auto simp: algebra_simps intro: diff const qR)*
moreover have *t0V: t0 ∈ V S ∩ V ⊆ U*
using *pt_δ t0 U V δ01* **by** *fastforce+*
ultimately show *?thesis* **using** *V t0V*
by *blast*
qed

Non-trivial case, with A and B both non-empty

lemma (in *function_ring_on) two_special*:
assumes *A: closed A A ⊆ S a ∈ A*
and *B: closed B B ⊆ S b ∈ B*
and *disj: A ∩ B = {}*
and *e: 0 < e e < 1*
shows *∃f ∈ R. f ‘ S ⊆ {0..1} ∧ (∀x ∈ A. f x < e) ∧ (∀x ∈ B. f x > 1 − e)*
proof −
{ fix *w*
assume *w ∈ A*
then have *open (− B) b ∈ S w ∉ B w ∈ S*
using *assms* **by** *auto*
then have *∃V. open V ∧ w ∈ V ∧ S ∩ V ⊆ −B ∧ (∀e>0. ∃f ∈ R. f ‘ S ⊆ {0..1} ∧ (∀x ∈ S ∩ V. f x < e) ∧ (∀x ∈ S ∩ B. f x > 1 − e))*
using *one [of −B w b] assms ‹w ∈ A›* **by** *simp*

}
then obtain *Vf* **where** *Vf*:
$\bigwedge w.\ w \in A \Longrightarrow open\ (Vf\ w) \wedge w \in Vf\ w \wedge S \cap Vf\ w \subseteq -B \wedge$
$(\forall e>0.\ \exists f \in R.\ f\ `\ S \subseteq \{0..1\} \wedge (\forall x \in S \cap Vf\ w.\ f\ x < e)$
$\wedge (\forall x \in S \cap B.\ f\ x > 1 - e))$
by *metis*
then have *open_Vf*: $\bigwedge w.\ w \in A \Longrightarrow open\ (Vf\ w)$
by *blast*
have *tVft*: $\bigwedge w.\ w \in A \Longrightarrow w \in Vf\ w$
using *Vf* **by** *blast*
then have *sum_max_0*: $A \subseteq (\bigcup x \in A.\ Vf\ x)$
by *blast*
have *com_A*: *compact A* **using** *A*
by (*metis compact compact_Int_closed inf.absorb_iff2*)
obtain *subA* **where** *subA*: $subA \subseteq A$ *finite subA* $A \subseteq (\bigcup x \in subA.\ Vf\ x)$
by (*blast intro*: *that compactE_image* [*OF com_A open_Vf sum_max_0*])
then have [*simp*]: $subA \neq \{\}$
using ‹$a \in A$› **by** *auto*
then have *cardp*: *card subA* > 0 **using** *subA*
by (*simp add*: *card_gt_0_iff*)
have $\bigwedge w.\ w \in A \Longrightarrow \exists f \in R.\ f\ `\ S \subseteq \{0..1\} \wedge (\forall x \in S \cap Vf\ w.\ f\ x < e\ /\ card$
$subA) \wedge (\forall x \in S \cap B.\ f\ x > 1 - e\ /\ card\ subA)$
using *Vf e cardp* **by** *simp*
then obtain *ff* **where** *ff*:
$\bigwedge w.\ w \in A \Longrightarrow ff\ w \in R \wedge ff\ w\ `\ S \subseteq \{0..1\} \wedge$
$(\forall x \in S \cap Vf\ w.\ ff\ w\ x < e\ /\ card\ subA) \wedge (\forall x \in S \cap B.\ ff$
$w\ x > 1 - e\ /\ card\ subA)$
by *metis*
define *pff* **where** [*abs_def*]: $pff\ x = (\prod w \in subA.\ ff\ w\ x)$ **for** *x*
have *pffR*: $pff \in R$
unfolding *pff_def* **using** *subA ff* **by** (*auto simp*: *intro*: *prod*)
moreover
have *pff01*: $pff\ x \in \{0..1\}$ **if** *t*: $x \in S$ **for** *x*
proof −
have $0 \leq pff\ x$
using *subA cardp t ff*
by (*fastforce simp*: *pff_def field_split_simps sum_nonneg intro*: *prod_nonneg*)
moreover have $pff\ x \leq 1$
using *subA cardp t ff*
by (*fastforce simp add*: *pff_def field_split_simps sum_nonneg intro*: *prod_mono*
[**where** $g = \lambda x.\ 1$, *simplified*])
ultimately show *?thesis*
by *auto*
qed
moreover
{ **fix** *v x*
assume *v*: $v \in subA$ **and** *x*: $x \in Vf\ v\ x \in S$
from *subA v* **have** $pff\ x = ff\ v\ x * (\prod w \in subA - \{v\}.\ ff\ w\ x)$
unfolding *pff_def* **by** (*metis prod.remove*)

also have ... ≤ *ff v x* ∗ *1*
proof −
have ⋀*i*. *i* ∈ *subA* − {*v*} ⟹ *0* ≤ *ff i x* ∧ *ff i x* ≤ *1*
by (*metis Diff_subset atLeastAtMost_iff ff image_subset_iff subA*(*1*) *subsetD x*(*2*))
moreover have *0* ≤ *ff v x*
using *ff subA*(*1*) *v x*(*2*) **by** *fastforce*
ultimately show *?thesis*
by (*metis mult_left_mono prod_mono* [**where** *g* = λ*x*. *1*, *simplified*])
qed
also have ... < *e* / *card subA*
using *ff subA*(*1*) *v x* **by** *auto*
also have ... ≤ *e*
using *cardp e* **by** (*simp add*: *field_split_simps*)
finally have *pff x* < *e* **.**
}
then have ⋀*x*. *x* ∈ *A* ⟹ *pff x* < *e*
using *A Vf subA* **by** (*metis UN_E contra_subsetD*)
moreover
{ fix *x*
assume *x*: *x* ∈ *B*
then have *x* ∈ *S*
using *B* **by** *auto*
have *1* − *e* ≤ (*1* − *e* / *card subA*) ^*card subA*
using *Bernoulli_inequality* [*of* −*e* / *card subA card subA*] *e cardp*
by (*auto simp*: *field_simps*)
also have ... = (∏ *w* ∈ *subA*. *1* − *e* / *card subA*)
by (*simp add*: *subA*(*2*))
also have ... < *pff x*
proof −
have ⋀*i*. *i* ∈ *subA* ⟹ *e* / *real* (*card subA*) ≤ *1* ∧ *1* − *e* / *real* (*card subA*) < *ff i x*
using *e* ‹*B* ⊆ *S*› *ff subA*(*1*) *x* **by** (*force simp*: *field_split_simps*)
then show *?thesis*
using *prod_mono_strict* [**where** *f* = λ*x*. *1* − *e* / *card subA*] *subA*(*2*) **by** (*force simp add*: *pff_def*)
qed
finally have *1* − *e* < *pff x* **.**
}
ultimately show *?thesis* **by** *blast*
qed

lemma (**in** *function_ring_on*) *two*:
assumes *A*: *closed A A* ⊆ *S*
and *B*: *closed B B* ⊆ *S*
and *disj*: *A* ∩ *B* = {}
and *e*: *0* < *e e* < *1*
shows ∃*f* ∈ *R*. *f* ‘ *S* ⊆ {*0*..*1*} ∧ (∀ *x* ∈ *A*. *f x* < *e*) ∧ (∀ *x* ∈ *B*. *f x* > *1* − *e*)
proof (*cases A* ≠ {} ∧ *B* ≠ {})

```
  case True then show ?thesis
    using assms
    by (force simp flip: ex_in_conv intro!: two_special)
next
  case False
  then consider A={} | B={} by force
  then show ?thesis
  proof cases
    case 1
    with e show ?thesis
      by (rule_tac x=λx. 1 in bexI) (auto simp: const)
  next
    case 2
    with e show ?thesis
      by (rule_tac x=λx. 0 in bexI) (auto simp: const)
  qed
qed
```

The special case where f is non-negative and $e < (1::'a) / (3::'a)$

```
lemma (in function_ring_on) Stone_Weierstrass_special:
  assumes f: continuous_on S f and fpos: ⋀x. x ∈ S ⟹ f x ≥ 0
    and e: 0 < e e < 1/3
  shows ∃ g ∈ R. ∀ x∈S. |f x − g x| < 2∗e
proof −
  define n where n = 1 + nat ⌈normf f / e⌉
  define A where A j = {x ∈ S. f x ≤ (j − 1/3)∗e} for j :: nat
  define B where B j = {x ∈ S. f x ≥ (j + 1/3)∗e} for j :: nat
  have ngt: (n−1) ∗ e ≥ normf f
    using e pos_divide_le_eq real_nat_ceiling_ge[of normf f / e]
    by (fastforce simp add: divide_simps n_def)
  moreover have n≥1
    by (simp_all add: n_def)
  ultimately have ge_fx: (n−1) ∗ e ≥ f x if x ∈ S for x
    using f normf_upper that by fastforce
  have closed S
    by (simp add: compact compact_imp_closed)
  { fix j
    have closed (A j) A j ⊆ S
      using ‹closed S› continuous_on_closed_Collect_le [OF f continuous_on_const]
      by (simp_all add: A_def Collect_restrict)
    moreover have closed (B j) B j ⊆ S
      using ‹closed S› continuous_on_closed_Collect_le [OF continuous_on_const f]
      by (simp_all add: B_def Collect_restrict)
    moreover have (A j) ∩ (B j) = {}
      using e by (auto simp: A_def B_def field_simps)
    ultimately have ∃f ∈ R. f ‘ S ⊆ {0..1} ∧ (∀x ∈ A j. f x < e/n) ∧ (∀x ∈
B j. f x > 1 − e/n)
      using e ‹1 ≤ n› by (auto intro: two)
  }
```

then obtain *xf* **where** *xfR*: $\bigwedge j$. *xf j* ∈ *R* **and** *xf01*: $\bigwedge j$. *xf j* ‘ *S* ⊆ {*0..1*}
and *xfA*: $\bigwedge x\ j$. *x* ∈ *A j* ⟹ *xf j x* < *e/n*
and *xfB*: $\bigwedge x\ j$. *x* ∈ *B j* ⟹ *xf j x* > *1* − *e/n*
by *metis*
define *g* **where** [*abs_def*]: *g x* = *e* ∗ (∑ *i*≤*n*. *xf i x*) **for** *x*
have *gR*: *g* ∈ *R*
unfolding *g_def* **by** (*fast intro*: *mult const sum xfR*)
have *gge0*: $\bigwedge x$. *x* ∈ *S* ⟹ *g x* ≥ *0*
using *e xf01* **by** (*simp add*: *g_def zero_le_mult_iff image_subset_iff sum_nonneg*)
have *A0*: *A 0* = {}
using *fpos e* **by** (*fastforce simp*: *A_def*)
have *An*: *A n* = *S*
using *e ngt* ‹*n*≥*1*› *f normf_upper* **by** (*fastforce simp*: *A_def field_simps of_nat_diff*)
have *Asub*: *A j* ⊆ *A i* **if** *i*≥*j* **for** *i j*
using *e that* **by** (*force simp*: *A_def intro*: *order_trans*)
{ fix *t*
assume *t*: *t* ∈ *S*
define *j* **where** *j* = (*LEAST j*. *t* ∈ *A j*)
have *jn*: *j* ≤ *n*
using *t An* **by** (*simp add*: *Least_le j_def*)
have *Aj*: *t* ∈ *A j*
using *t An* **by** (*fastforce simp add*: *j_def intro*: *LeastI*)
then have *Ai*: *t* ∈ *A i* **if** *i*≥*j* **for** *i*
using *Asub* [*OF that*] **by** *blast*
then have *fj1*: *f t* ≤ (*j* − *1/3*)∗*e*
by (*simp add*: *A_def*)
then have *Anj*: *t* ∉ *A i* **if** *i*<*j* **for** *i*
using *Aj* ‹*i*<*j*› *not_less_Least* **by** (*fastforce simp add*: *j_def*)
have *j1*: *1* ≤ *j*
using *A0 Aj j_def not_less_eq_eq* **by** (*fastforce simp add*: *j_def*)
then have *Anj*: *t* ∉ *A* (*j*−*1*)
using *Least_le* **by** (*fastforce simp add*: *j_def*)
then have *fj2*: (*j* − *4/3*)∗*e* < *f t*
using *j1 t* **by** (*simp add*: *A_def of_nat_diff*)
have *xf_le1*: $\bigwedge i$. *xf i t* ≤ *1*
using *xf01 t* **by** *force*
have *g t* = *e* ∗ (∑ *i*≤*n*. *xf i t*)
by (*simp add*: *g_def flip*: *distrib_left*)
also have ... = *e* ∗ (∑ *i* ∈ {..<*j*} ∪ {*j..n*}. *xf i t*)
by (*simp add*: *ivl_disj_un_one*(*4*) *jn*)
also have ... = *e* ∗ (∑ *i*<*j*. *xf i t*) + *e* ∗ (∑ *i*=*j..n*. *xf i t*)
by (*simp add*: *distrib_left ivl_disj_int sum.union_disjoint*)
also have ... ≤ *e*∗*j* + *e* ∗ ((*Suc n* − *j*)∗*e/n*)
proof (*intro add_mono mult_left_mono*)
show (∑ *i*<*j*. *xf i t*) ≤ *j*
by (*rule sum_bounded_above* [*OF xf_le1*, **where** *A* = *lessThan j*, *simplified*])
have *xf i t* ≤ *e/n* **if** *i*≥*j* **for** *i*
using *xfA* [*OF Ai*] *that* **by** (*simp add*: *less_eq_real_def*)
then show (∑ *i* = *j..n*. *xf i t*) ≤ *real* (*Suc n* − *j*) ∗ *e* / *real n*

```
        using sum_bounded_above [of {j..n} λi. xf i t]
        by fastforce
    qed (use e in auto)
    also have ... ≤ j*e + e*(n − j + 1)*e/n
      using ‹1 ≤ n› e  by (simp add: field_simps del: of_nat_Suc)
    also have ... ≤ j*e + e*e
      using ‹1 ≤ n› e j1 by (simp add: field_simps del: of_nat_Suc)
    also have ... < (j + 1/3)*e
      using e by (auto simp: field_simps)
    finally have gj1: g t < (j + 1 / 3) * e .
    have gj2: (j − 4/3)*e < g t
    proof (cases 2 ≤ j)
      case False
      then have j=1 using j1 by simp
      with t gge0 e show ?thesis by force
    next
      case True
      then have (j − 4/3)*e < (j−1)*e − e^2
        using e by (auto simp: of_nat_diff algebra_simps power2_eq_square)
      also have ... < (j−1)*e − ((j − 1)/n) * e^2
        using e True jn by (simp add: power2_eq_square field_simps)
      also have ... = e * (j−1) * (1 − e/n)
        by (simp add: power2_eq_square field_simps)
      also have ... ≤ e * (∑ i≤j−2. xf i t)
      proof −
        { fix i
          assume i+2 ≤ j
          then obtain d where i+2+d = j
            using le_Suc_ex that by blast
          then have t ∈ B i
            using Anj e ge_fx [OF t] ‹1 ≤ n› fpos [OF t] t
            unfolding A_def B_def
             by (auto simp add: field_simps of_nat_diff not_le intro: order_trans [of _
e * 2 + e * d * 3 + e * i * 3])
          then have xf i t > 1 − e/n
            by (rule xfB)
        }
         moreover have real (j − Suc 0) * (1 − e / real n) ≤ real (card {..j −
2}) * (1 − e / real n)
          using Suc_diff_le True by fastforce
        ultimately show ?thesis
           using e True by (auto intro: order_trans [OF _ sum_bounded_below [OF
less_imp_le]])
      qed
      also have ... ≤ g t
        using jn e xf01 t
         by (auto intro!: Groups_Big.sum_mono2 simp add: g_def zero_le_mult_iff
image_subset_iff sum_nonneg)
      finally show ?thesis .
```

```
    qed
    have |f t − g t| < 2 ∗ e
      using fj1 fj2 gj1 gj2 by (simp add: abs_less_iff field_simps)
  }
  then show ?thesis
    by (rule_tac x=g in bexI) (auto intro: gR)
qed
```

The "unpretentious" formulation

```
proposition (in function_ring_on) Stone_Weierstrass_basic:
  assumes f: continuous_on S f and e: e > 0
  shows ∃ g ∈ R. ∀ x∈S. |f x − g x| < e
proof −
  have ∃ g ∈ R. ∀ x∈S. |(f x + normf f) − g x| < 2 ∗ min (e/2) (1/4)
  proof (rule Stone_Weierstrass_special)
    show continuous_on S (λx. f x + normf f)
     by (force intro: Limits.continuous_on_add [OF f Topological_Spaces.continuous_on_const])
    show ⋀x. x ∈ S =⇒ 0 ≤ f x + normf f
      using normf_upper [OF f] by force
  qed (use e in auto)
  then obtain g where g ∈ R ∀ x∈S. |g x − (f x + normf f)| < e
    by force
  then show ?thesis
     by (rule_tac x=λx. g x − normf f in bexI) (auto simp: algebra_simps intro:
diff const)
qed
```

```
theorem (in function_ring_on) Stone_Weierstrass:
  assumes f: continuous_on S f
  shows ∃ F∈UNIV → R. LIM n sequentially. F n :> uniformly_on S f
proof −
  define h where h ≡ λn::nat. SOME g. g ∈ R ∧ (∀ x∈S. |f x − g x| < 1 / (1
+ n))
  show ?thesis
  proof
    { fix e::real
      assume e: 0 < e
      then obtain N::nat where N: 0 < N 0 < inverse N inverse N < e
        by (auto simp: real_arch_inverse [of e])
      { fix n :: nat and x :: 'a and g :: 'a ⇒ real
        assume n: N ≤ n  ∀ x∈S. |f x − g x| < 1 / (1 + real n)
        assume x: x ∈ S
        have ¬ real (Suc n) < inverse e
          using ‹N ≤ n› N using less_imp_inverse_less by force
        then have 1 / (1 + real n) ≤ e
          using e by (simp add: field_simps)
        then have |f x − g x| < e
          using n(2) x by auto
```

```
    }
    then have ∀F n in sequentially. ∀x∈S. |f x − h n x| < e
      unfolding h_def
        by (force intro: someI2_bex [OF Stone_Weierstrass_basic [OF f]] eventu-
ally_sequentiallyI [of N])
  }
  then show uniform_limit S h f sequentially
    unfolding uniform_limit_iff by (auto simp: dist_norm abs_minus_commute)
  show h ∈ UNIV → R
      unfolding h_def by (force intro: someI2_bex [OF Stone_Weierstrass_basic
[OF f]])
 qed
qed
```

A HOL Light formulation

```
corollary Stone_Weierstrass_HOL:
 fixes R :: ('a::t2_space ⇒ real) set and S :: 'a set
 assumes compact S ⋀c. P(λx. c::real)
         ⋀f. P f ⟹ continuous_on S f
         ⋀f g. P(f) ∧ P(g) ⟹ P(λx. f x + g x) ⋀f g. P(f) ∧ P(g) ⟹ P(λx. f
x ∗ g x)
         ⋀x y. x ∈ S ∧ y ∈ S ∧ x ≠ y ⟹ ∃f. P(f) ∧ f x ≠ f y
         continuous_on S f
      0 < e
   shows ∃g. P(g) ∧ (∀x ∈ S. |f x − g x| < e)
proof −
 interpret PR: function_ring_on Collect P
   by unfold_locales (use assms in auto)
 show ?thesis
   using PR.Stone_Weierstrass_basic [OF ‹continuous_on S f› ‹0 < e›]
   by blast
qed
```

5.6.4 Polynomial functions

```
inductive real_polynomial_function :: ('a::real_normed_vector ⇒ real) ⇒ bool where
   linear: bounded_linear f ⟹ real_polynomial_function f
 | const: real_polynomial_function (λx. c)
 | add: ⟦real_polynomial_function f; real_polynomial_function g⟧ ⟹ real_polynomial_function
(λx. f x + g x)
 | mult: ⟦real_polynomial_function f; real_polynomial_function g⟧ ⟹ real_polynomial_function
(λx. f x ∗ g x)

declare real_polynomial_function.intros [intro]

definition polynomial_function :: ('a::real_normed_vector ⇒ 'b::real_normed_vector)
⇒ bool
 where
  polynomial_function p ≡ (∀f. bounded_linear f ⟶ real_polynomial_function (f
```

o p))

lemma *real_polynomial_function_eq*: *real_polynomial_function p* = *polynomial_function p*
unfolding *polynomial_function_def*
proof
  assume *real_polynomial_function p*
  then show $\forall f$. *bounded_linear f* $\longrightarrow$ *real_polynomial_function* ($f \circ p$)
  proof (*induction p rule*: *real_polynomial_function.induct*)
    case (*linear h*) **then show** *?case*
      by (*auto simp*: *bounded_linear_compose real_polynomial_function.linear*)
  next
    case (*const h*) **then show** *?case*
      by (*simp add*: *real_polynomial_function.const*)
  next
    case (*add h*) **then show** *?case*
     by (*force simp add*: *bounded_linear_def linear_add real_polynomial_function.add*)
  next
    case (*mult h*) **then show** *?case*
      by (*force simp add*: *real_bounded_linear const real_polynomial_function.mult*)
  qed
next
 assume [*rule_format*, *OF bounded_linear_ident*]: $\forall f$. *bounded_linear f* $\longrightarrow$ *real_polynomial_function* ($f \circ p$)
  then show *real_polynomial_function p*
    by (*simp add*: *o_def*)
qed

lemma *polynomial_function_const* [*iff*]: *polynomial_function* ($\lambda x.\ c$)
  by (*simp add*: *polynomial_function_def o_def const*)

lemma *polynomial_function_bounded_linear*:
  bounded_linear f $\Longrightarrow$ *polynomial_function f*
 by (*simp add*: *polynomial_function_def o_def bounded_linear_compose real_polynomial_function.linear*)

lemma *polynomial_function_id* [*iff*]: *polynomial_function*($\lambda x.\ x$)
  by (*simp add*: *polynomial_function_bounded_linear*)

lemma *polynomial_function_add* [*intro*]:
    $[\![$*polynomial_function f*; *polynomial_function g*$]\!]$ $\Longrightarrow$ *polynomial_function* ($\lambda x.\ f\ x + g\ x$)
 by (*auto simp*: *polynomial_function_def bounded_linear_def linear_add real_polynomial_function.add o_def*)

lemma *polynomial_function_mult* [*intro*]:
  assumes *f*: *polynomial_function f* **and** *g*: *polynomial_function g*
  shows *polynomial_function* ($\lambda x.\ f\ x *_R g\ x$)
proof –
  have *real_polynomial_function* ($\lambda x.\ h\ (g\ x)$) **if** *bounded_linear h* **for** *h*

using *g that* **unfolding** *polynomial_function_def o_def bounded_linear_def*
by (*auto simp*: *real_polynomial_function_eq*)
moreover have *real_polynomial_function f*
by (*simp add*: *f real_polynomial_function_eq*)
ultimately show *?thesis*
unfolding *polynomial_function_def bounded_linear_def o_def*
by (*auto simp*: *linear.scaleR*)
qed

lemma *polynomial_function_cmul* [*intro*]:
assumes *f*: *polynomial_function f*
shows *polynomial_function* ($\lambda x.\ c *_R f\ x$)
by (*rule polynomial_function_mult* [*OF polynomial_function_const f*])

lemma *polynomial_function_minus* [*intro*]:
assumes *f*: *polynomial_function f*
shows *polynomial_function* ($\lambda x.\ - f\ x$)
using *polynomial_function_cmul* [*OF f*, *of* -1] **by** *simp*

lemma *polynomial_function_diff* [*intro*]:
⟦*polynomial_function f*; *polynomial_function g*⟧ ⟹ *polynomial_function* ($\lambda x.\ f$
$x - g\ x$)
unfolding *add_uminus_conv_diff* [*symmetric*]
by (*metis polynomial_function_add polynomial_function_minus*)

lemma *polynomial_function_sum* [*intro*]:
⟦*finite I*; $\bigwedge i.\ i \in I$ ⟹ *polynomial_function* ($\lambda x.\ f\ x\ i$)⟧ ⟹ *polynomial_function*
($\lambda x.$ *sum* ($f\ x$) I)
by (*induct I rule*: *finite_induct*) *auto*

lemma *real_polynomial_function_minus* [*intro*]:
real_polynomial_function f ⟹ *real_polynomial_function* ($\lambda x.\ - f\ x$)
using *polynomial_function_minus* [*of f*]
by (*simp add*: *real_polynomial_function_eq*)

lemma *real_polynomial_function_diff* [*intro*]:
⟦*real_polynomial_function f*; *real_polynomial_function g*⟧ ⟹ *real_polynomial_function*
($\lambda x.\ f\ x - g\ x$)
using *polynomial_function_diff* [*of f*]
by (*simp add*: *real_polynomial_function_eq*)

lemma *real_polynomial_function_sum* [*intro*]:
⟦*finite I*; $\bigwedge i.\ i \in I$ ⟹ *real_polynomial_function* ($\lambda x.\ f\ x\ i$)⟧ ⟹ *real_polynomial_function*
($\lambda x.$ *sum* ($f\ x$) I)
using *polynomial_function_sum* [*of I f*]
by (*simp add*: *real_polynomial_function_eq*)

lemma *real_polynomial_function_power* [*intro*]:
real_polynomial_function f ⟹ *real_polynomial_function* ($\lambda x.\ f\ x\ \hat{}\ n$)

by (*induct n*) (*simp_all add*: *const mult*)

lemma *real_polynomial_function_compose* [*intro*]:
assumes *f*: *polynomial_function f* **and** *g*: *real_polynomial_function g*
shows *real_polynomial_function* (*g o f*)
using *g*
proof (*induction g rule*: *real_polynomial_function.induct*)
case (*linear f*)
then show *?case*
using *f polynomial_function_def* **by** *blast*
next
case (*add f g*)
then show *?case*
using *f add* **by** (*auto simp*: *polynomial_function_def*)
next
case (*mult f g*)
then show *?case*
using *f mult* **by** (*auto simp*: *polynomial_function_def*)
qed *auto*

lemma *polynomial_function_compose* [*intro*]:
assumes *f*: *polynomial_function f* **and** *g*: *polynomial_function g*
shows *polynomial_function* (*g o f*)
using *g real_polynomial_function_compose* [*OF f*]
by (*auto simp*: *polynomial_function_def o_def*)

lemma *sum_max_0*:
fixes *x*::*real*
shows ($\sum$ *i*≤*max m n*. *x*ˆ*i* ∗ (*if i* ≤ *m then a i else 0*)) = ($\sum$ *i*≤*m*. *x*ˆ*i* ∗ *a i*)
proof −
have ($\sum$ *i*≤*max m n*. *x*ˆ*i* ∗ (*if i* ≤ *m then a i else 0*)) = ($\sum$ *i*≤*max m n*. (*if i* ≤ *m then x*ˆ*i* ∗ *a i else 0*))
by (*auto simp*: *algebra_simps intro*: *sum.cong*)
also have ... = ($\sum$ *i*≤*m*. (*if i* ≤ *m then x*ˆ*i* ∗ *a i else 0*))
by (*rule sum.mono_neutral_right*) *auto*
also have ... = ($\sum$ *i*≤*m*. *x*ˆ*i* ∗ *a i*)
by (*auto simp*: *algebra_simps intro*: *sum.cong*)
finally show *?thesis* **.**
qed

lemma *real_polynomial_function_imp_sum*:
assumes *real_polynomial_function f*
shows ∃ *a n*::*nat*. *f* = (λ*x*. $\sum$ *i*≤*n*. *a i* ∗ *x*ˆ*i*)
using *assms*
proof (*induct f*)
case (*linear f*)
then obtain *c* **where** *f*: *f* = (λ*x*. *x* ∗ *c*)
by (*auto simp add*: *real_bounded_linear*)
have *x* ∗ *c* = ($\sum$ *i*≤*1*. (*if i* = *0 then 0 else c*) ∗ *x*ˆ*i*) **for** *x*

```
    by (simp add: mult_ac)
  with f show ?case
    by fastforce
next
  case (const c)
  have c = (∑i≤0. c * x^i) for x
    by auto
  then show ?case
    by fastforce
  case (add f1 f2)
  then obtain a1 n1 a2 n2 where
    f1 = (λx. ∑i≤n1. a1 i * x^i) f2 = (λx. ∑i≤n2. a2 i * x^i)
    by auto
  then have f1 x + f2 x = (∑i≤max n1 n2. ((if i ≤ n1 then a1 i else 0) + (if i
≤ n2 then a2 i else 0)) * x^i)
      for x
    using sum_max_0 [where m=n1 and n=n2] sum_max_0 [where m=n2 and
n=n1]
    by (simp add: sum.distrib algebra_simps max.commute)
  then show ?case
    by force
  case (mult f1 f2)
  then obtain a1 n1 a2 n2 where
    f1 = (λx. ∑i≤n1. a1 i * x^i) f2 = (λx. ∑i≤n2. a2 i * x^i)
    by auto
  then obtain b1 b2 where
    f1 = (λx. ∑i≤n1. b1 i * x^i) f2 = (λx. ∑i≤n2. b2 i * x^i)
    b1 = (λi. if i≤n1 then a1 i else 0) b2 = (λi. if i≤n2 then a2 i else 0)
    by auto
  then have f1 x * f2 x = (∑i≤n1 + n2. (∑k≤i. b1 k * b2 (i − k)) * x ^ i)
for x
   using polynomial_product [of n1 b1 n2 b2] by (simp add: Set_Interval.atLeast0AtMost)
  then show ?case
    by force
qed

lemma real_polynomial_function_iff_sum:
     real_polynomial_function f ⟷ (∃a n. f = (λx. ∑i≤n. a i * x^i)) (is ?lhs
= ?rhs)
proof
  assume ?lhs then show ?rhs
    by (metis real_polynomial_function_imp_sum)
next
  assume ?rhs then show ?lhs
   by (auto simp: linear mult const real_polynomial_function_power real_polynomial_function_sum)
qed

lemma polynomial_function_iff_Basis_inner:
  fixes f :: 'a::real_normed_vector ⇒ 'b::euclidean_space
```

shows *polynomial_function f* ⟷ (∀ *b*∈*Basis*. *real_polynomial_function* (λ*x*. *inner* (*f x*) *b*))
(**is** *?lhs* = *?rhs*)
unfolding *polynomial_function_def*
proof (*intro iffI allI impI*)
assume ∀ *h*. *bounded_linear h* ⟶ *real_polynomial_function* (*h* ∘ *f*)
then show *?rhs*
by (*force simp add*: *bounded_linear_inner_left o_def*)
next
fix *h* :: ′*b* ⇒ *real*
assume *rp*: ∀ *b*∈*Basis*. *real_polynomial_function* (λ*x*. *f x* · *b*) **and** *h*: *bounded_linear h*
have *real_polynomial_function* (*h* ∘ (λ*x*. ∑ *b*∈*Basis*. (*f x* · *b*) $*_R$ *b*))
using *rp*
by (*force simp*: *real_polynomial_function_eq polynomial_function_mult intro*!: *real_polynomial_function_compose* [*OF* _ *linear* [*OF h*]])
then show *real_polynomial_function* (*h* ∘ *f*)
by (*simp add*: *euclidean_representation_sum_fun*)
qed

5.6.5 Stone-Weierstrass theorem for polynomial functions

First, we need to show that they are continuous, differentiable and separable.

lemma *continuous_real_polymonial_function*:
assumes *real_polynomial_function f*
shows *continuous* (*at x*) *f*
using *assms*
by (*induct f*) (*auto simp*: *linear_continuous_at*)

lemma *continuous_polymonial_function*:
fixes *f* :: ′*a*::*real_normed_vector* ⇒ ′*b*::*euclidean_space*
assumes *polynomial_function f*
shows *continuous* (*at x*) *f*
proof (*rule euclidean_isCont*)
show ⋀*b*. *b* ∈ *Basis* ⟹ *isCont* (λ*x*. (*f x* · *b*) $*_R$ *b*) *x*
using *assms continuous_real_polymonial_function*
by (*force simp*: *polynomial_function_iff_Basis_inner intro*: *isCont_scaleR*)
qed

lemma *continuous_on_polymonial_function*:
fixes *f* :: ′*a*::*real_normed_vector* ⇒ ′*b*::*euclidean_space*
assumes *polynomial_function f*
shows *continuous_on S f*
using *continuous_polymonial_function* [*OF assms*] *continuous_at_imp_continuous_on*
by *blast*

lemma *has_real_derivative_polynomial_function*:
assumes *real_polynomial_function p*
shows ∃ *p*′. *real_polynomial_function p*′ ∧

(∀ *x*. (*p has_real_derivative* (*p′ x*)) (*at x*))
using *assms*
proof (*induct p*)
case (*linear p*)
then show *?case*
by (*force simp*: *real_bounded_linear const intro*!: *derivative_eq_intros*)
next
case (*const c*)
show *?case*
by (*rule_tac x*=*λx*. *0* **in** *exI*) *auto*
case (*add f1 f2*)
then obtain *p1 p2* **where**
real_polynomial_function p1 ⋀*x*. (*f1 has_real_derivative p1 x*) (*at x*)
real_polynomial_function p2 ⋀*x*. (*f2 has_real_derivative p2 x*) (*at x*)
by *auto*
then show *?case*
by (*rule_tac x*=*λx*. *p1 x* + *p2 x* **in** *exI*) (*auto intro*!: *derivative_eq_intros*)
case (*mult f1 f2*)
then obtain *p1 p2* **where**
real_polynomial_function p1 ⋀*x*. (*f1 has_real_derivative p1 x*) (*at x*)
real_polynomial_function p2 ⋀*x*. (*f2 has_real_derivative p2 x*) (*at x*)
by *auto*
then show *?case*
using *mult*
by (*rule_tac x*=*λx*. *f1 x* ∗ *p2 x* + *f2 x* ∗ *p1 x* **in** *exI*) (*auto intro*!: *derivative_eq_intros*)
qed

lemma *has_vector_derivative_polynomial_function*:
fixes *p* :: *real* ⇒ ′*a*::*euclidean_space*
assumes *polynomial_function p*
obtains *p′* **where** *polynomial_function p′* ⋀*x*. (*p has_vector_derivative* (*p′ x*)) (*at x*)
proof −
{ **fix** *b* :: ′*a*
assume *b* ∈ *Basis*
then
obtain *p′* **where** *p′*: *real_polynomial_function p′* **and** *pd*: ⋀*x*. ((*λx*. *p x* · *b*) *has_real_derivative p′ x*) (*at x*)
using *assms* [*unfolded polynomial_function_iff_Basis_inner*] *has_real_derivative_polynomial_function*
by *blast*
have *polynomial_function* (*λx*. *p′ x* $*_R$ *b*)
using ‹*b* ∈ *Basis*› *p′ const* [**where** ′*a*=*real* **and** *c*=*0*]
by (*simp add*: *polynomial_function_iff_Basis_inner inner_Basis*)
then have ∃ *q*. *polynomial_function q* ∧ (∀ *x*. ((*λu*. (*p u* · *b*) $*_R$ *b*) *has_vector_derivative q x*) (*at x*))
by (*fastforce intro*: *derivative_eq_intros pd*)
}
then obtain *qf* **where** *qf*:

$\bigwedge b$. $b \in$ *Basis* $\Longrightarrow$ *polynomial_function* (*qf b*)
$\bigwedge b\ x$. $b \in$ *Basis* $\Longrightarrow$ ((λu. (*p u* $\cdot$ *b*) $*_R$ *b*) *has_vector_derivative qf b x*) (*at x*)
by *metis*
show *?thesis*
proof
show $\bigwedge x$. (*p has_vector_derivative* ($\sum$ *b*$\in$*Basis*. *qf b x*)) (*at x*)
apply (*subst euclidean_representation_sum_fun* [*of p*, *symmetric*])
by (*auto intro*: *has_vector_derivative_sum qf*)
qed (*force intro*: *qf*)
qed

lemma *real_polynomial_function_separable*:
fixes *x* :: *'a*::*euclidean_space*
assumes $x \neq y$ **shows** $\exists f$. *real_polynomial_function f* $\wedge$ *f x* $\neq$ *f y*
proof −
have *real_polynomial_function* (λu. $\sum$ *b*$\in$*Basis*. (*inner* (*x*−*u*) *b*) ˆ*2*)
proof (*rule real_polynomial_function_sum*)
show $\bigwedge i$. $i \in$ *Basis* $\Longrightarrow$ *real_polynomial_function* (λu. ((*x* − *u*) $\cdot$ *i*)^2)
by (*auto simp*: *algebra_simps real_polynomial_function_diff const linear bounded_linear_inner_left*)
qed *auto*
moreover have ($\sum$ *b*$\in$*Basis*. ((*x* − *y*) $\cdot$ *b*)^2) $\neq$ *0*
using *assms* **by** (*force simp add*: *euclidean_eq_iff* [*of x y*] *sum_nonneg_eq_0_iff algebra_simps*)
ultimately show *?thesis*
by *auto*
qed

lemma *Stone_Weierstrass_real_polynomial_function*:
fixes *f* :: *'a*::*euclidean_space* $\Rightarrow$ *real*
assumes *compact S continuous_on S f 0* < *e*
obtains *g* **where** *real_polynomial_function g* $\bigwedge x$. $x \in S \Longrightarrow |f\ x - g\ x| < e$
proof −
interpret *PR*: *function_ring_on Collect real_polynomial_function*
proof *unfold_locales*
qed (*use assms continuous_on_polymonial_function real_polynomial_function_eq*
in ‹*auto intro*: *real_polynomial_function_separable*›)
show *?thesis*
using *PR.Stone_Weierstrass_basic* [*OF* ‹*continuous_on S f*› ‹*0* < *e*›] *that* **by** *blast*
qed

theorem *Stone_Weierstrass_polynomial_function*:
fixes *f* :: *'a*::*euclidean_space* $\Rightarrow$ *'b*::*euclidean_space*
assumes *S*: *compact S*
and *f*: *continuous_on S f*
and *e*: *0* < *e*
shows $\exists g$. *polynomial_function g* $\wedge$ ($\forall x \in S$. *norm*(*f x* − *g x*) < *e*)
proof −
{ **fix** *b* :: *'b*

```
    assume b ∈ Basis
    have ∃p. real_polynomial_function p ∧ (∀x ∈ S. |f x · b − p x| < e / DIM('b))
    proof (rule Stone_Weierstrass_real_polynomial_function [OF S _, of λx. f x · b
e / card Basis])
      show continuous_on S (λx. f x · b)
        using f by (auto intro: continuous_intros)
    qed (use e in auto)
  }
  then obtain pf where pf:
      ⋀b. b ∈ Basis ⟹ real_polynomial_function (pf b) ∧ (∀x ∈ S. |f x · b − pf b
x| < e / DIM('b))
    by metis
  let ?g = λx. ∑b∈Basis. pf b x *R b
  { fix x
    assume x ∈ S
    have norm (∑b∈Basis. (f x · b) *R b − pf b x *R b) ≤ (∑b∈Basis. norm
((f x · b) *R b − pf b x *R b))
      by (rule norm_sum)
    also have ... < of_nat DIM('b) * (e / DIM('b))
    proof (rule sum_bounded_above_strict)
      show ⋀i. i ∈ Basis ⟹ norm ((f x · i) *R i − pf i x *R i) < e / real
DIM('b)
        by (simp add: Real_Vector_Spaces.scaleR_diff_left [symmetric] pf ‹x ∈ S›)
    qed (rule DIM_positive)
    also have ... = e
      by (simp add: field_simps)
    finally have norm (∑b∈Basis. (f x · b) *R b − pf b x *R b) < e .
  }
  then have ∀x∈S. norm ((∑b∈Basis. (f x · b) *R b) − ?g x) < e
    by (auto simp flip: sum_subtractf)
  moreover
  have polynomial_function ?g
    using pf by (simp add: polynomial_function_sum polynomial_function_mult
real_polynomial_function_eq)
  ultimately show ?thesis
    using euclidean_representation_sum_fun [of f] by (metis (no_types, lifting))
qed

proposition Stone_Weierstrass_uniform_limit:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes S: compact S
    and f: continuous_on S f
  obtains g where uniform_limit S g f sequentially ⋀n. polynomial_function (g n)
proof −
  have pos: inverse (Suc n) > 0 for n by auto
  obtain g where g: ⋀n. polynomial_function (g n) ⋀x n. x ∈ S ⟹ norm(f x −
g n x) < inverse (Suc n)
    using Stone_Weierstrass_polynomial_function[OF S f pos]
    by metis
```

have *uniform_limit S g f sequentially*
proof (*rule uniform_limitI*)
fix *e*::*real* **assume** *0 < e*
with *LIMSEQ_inverse_real_of_nat* **have** ∀$_F$ *n in sequentially. inverse* (*Suc n*) < *e*
by (*rule order_tendstoD*)
moreover have ∀$_F$ *n in sequentially.* ∀*x*∈*S*. *dist* (*g n x*) (*f x*) < *inverse* (*Suc n*)
using *g* **by** (*simp add*: *dist_norm norm_minus_commute*)
ultimately show ∀$_F$ *n in sequentially.* ∀*x*∈*S*. *dist* (*g n x*) (*f x*) < *e*
by (*eventually_elim*) *auto*
qed
then show *?thesis* **using** *g*(*1*) **..**
qed

5.6.6 Polynomial functions as paths

One application is to pick a smooth approximation to a path, or just pick a smooth path anyway in an open connected set

lemma *path_polynomial_function*:
fixes *g* :: *real* ⇒ *'b*::*euclidean_space*
shows *polynomial_function g* ⟹ *path g*
by (*simp add*: *path_def continuous_on_polymonial_function*)

lemma *path_approx_polynomial_function*:
fixes *g* :: *real* ⇒ *'b*::*euclidean_space*
assumes *path g 0 < e*
obtains *p* **where** *polynomial_function p pathstart p = pathstart g pathfinish p = pathfinish g*
⋀*t*. *t* ∈ {*0*..*1*} ⟹ *norm*(*p t* − *g t*) < *e*
proof −
obtain *q* **where** *poq*: *polynomial_function q* **and** *noq*: ⋀*x*. *x* ∈ {*0*..*1*} ⟹ *norm* (*g x* − *q x*) < *e*/*4*
using *Stone_Weierstrass_polynomial_function* [*of* {*0*..*1*} *g e*/*4*] *assms*
by (*auto simp*: *path_def*)
define *pf* **where** *pf* ≡ λ*t*. *q t* + (*g 0* − *q 0*) + *t* $*_R$ (*g 1* − *q 1* − (*g 0* − *q 0*))
show *thesis*
proof
show *polynomial_function pf*
by (*force simp add*: *poq pf_def*)
show *norm* (*pf t* − *g t*) < *e*
if *t* ∈ {*0*..*1*} **for** *t*
proof −
have ∗: *norm* (((*q t* − *g t*) + (*g 0* − *q 0*)) + (*t* $*_R$ (*g 1* − *q 1*) + *t* $*_R$ (*q 0* − *g 0*))) < (*e*/*4* + *e*/*4*) + (*e*/*4*+*e*/*4*)
proof (*intro Real_Vector_Spaces.norm_add_less*)
show *norm* (*q t* − *g t*) < *e* / *4*
by (*metis noq norm_minus_commute that*)
show *norm* (*t* $*_R$ (*g 1* − *q 1*)) < *e* / *4*

```
          using noq that le_less_trans [OF mult_left_le_one_le noq]
          by auto
        show norm (t *R (q 0 − g 0)) < e / 4
          using noq that le_less_trans [OF mult_left_le_one_le noq]
          by simp (metis norm_minus_commute order_refl zero_le_one)
      qed (use noq norm_minus_commute that in auto)
      then show ?thesis
        by (auto simp add: algebra_simps pf_def)
    qed
  qed (auto simp add: path_defs pf_def)
qed

proposition connected_open_polynomial_connected:
  fixes S :: 'a::euclidean_space set
  assumes S: open S connected S
      and x ∈ S y ∈ S
    shows ∃ g. polynomial_function g ∧ path_image g ⊆ S ∧ pathstart g = x ∧
pathfinish g = y
proof −
  have path_connected S using assms
    by (simp add: connected_open_path_connected)
  with ‹x ∈ S› ‹y ∈ S› obtain p where p: path p path_image p ⊆ S pathstart p
= x pathfinish p = y
    by (force simp: path_connected_def)
  have ∃ e. 0 < e ∧ (∀ x ∈ path_image p. ball x e ⊆ S)
  proof (cases S = UNIV)
    case True then show ?thesis
      by (simp add: gt_ex)
  next
    case False
    show ?thesis
    proof (intro exI conjI ballI)
      show ⋀x. x ∈ path_image p ⟹ ball x (setdist (path_image p) (−S)) ⊆ S
        using setdist_le_dist [of _ path_image p _ −S] by fastforce
      show 0 < setdist (path_image p) (− S)
        using S p False
          by (fastforce simp add: setdist_gt_0_compact_closed compact_path_image
open_closed)
    qed
  qed
  then obtain e where 0 < eand eb: ⋀x. x ∈ path_image p ⟹ ball x e ⊆ S
    by auto
  obtain pf where polynomial_function pf and pf: pathstart pf = pathstart p
pathfinish pf = pathfinish p
                  and pf_e: ⋀t. t ∈ {0..1} ⟹ norm(pf t − p t) < e
    using path_approx_polynomial_function [OF ‹path p› ‹0 < e›] by blast
  show ?thesis
  proof (intro exI conjI)
    show polynomial_function pf
```

by *fact*
show *pathstart pf* = *x pathfinish pf* = *y*
by (*simp_all add*: *p pf*)
show *path_image pf* ⊆ *S*
unfolding *path_image_def*
proof *clarsimp*
fix *x'*::*real*
assume *0* ≤ *x' x'* ≤ *1*
then have *dist* (*p x'*) (*pf x'*) < *e*
by (*metis atLeastAtMost_iff dist_commute dist_norm pf_e*)
then show *pf x'* ∈ *S*
by (*metis* ‹*0* ≤ *x'*› ‹*x'* ≤ *1*› *atLeastAtMost_iff eb imageI mem_ball path_image_def subset_iff*)
qed
qed
qed

lemma *differentiable_componentwise_within*:
f differentiable (*at a within S*) ⟷
(∀ *i* ∈ *Basis*. (λ*x*. *f x* · *i*) *differentiable at a within S*)
proof −
{ **assume** ∀ *i*∈*Basis*. ∃ *D*. ((λ*x*. *f x* · *i*) *has_derivative D*) (*at a within S*)
then obtain *f'* **where** *f'*:
⋀*i*. *i* ∈ *Basis* ⟹ ((λ*x*. *f x* · *i*) *has_derivative f' i*) (*at a within S*)
by *metis*
have *eq*: (λ*x*. (∑*j*∈*Basis*. *f' j x* $*_R$ *j*) · *i*) = *f' i* **if** *i* ∈ *Basis* **for** *i*
using *that* **by** (*simp add*: *inner_add_left inner_add_right*)
have ∃ *D*. ∀ *i*∈*Basis*. ((λ*x*. *f x* · *i*) *has_derivative* (λ*x*. *D x* · *i*)) (*at a within S*)
apply (*rule_tac x*=λ*x*::'*a*. (∑*j*∈*Basis*. *f' j x* $*_R$ *j*) :: '*b* **in** *exI*)
apply (*simp add*: *eq f'*)
done
}
then show *?thesis*
apply (*simp add*: *differentiable_def*)
using *has_derivative_componentwise_within*
by *blast*
qed

lemma *polynomial_function_inner* [*intro*]:
fixes *i* :: '*a*::*euclidean_space*
shows *polynomial_function g* ⟹ *polynomial_function* (λ*x*. *g x* · *i*)
apply (*subst euclidean_representation* [**where** *x*=*i*, *symmetric*])
apply (*force simp*: *inner_sum_right polynomial_function_iff_Basis_inner polynomial_function_sum*)
done

Differentiability of real and vector polynomial functions.

lemma *differentiable_at_real_polynomial_function*:
real_polynomial_function f ⟹ *f differentiable* (*at a within S*)

by (*induction f rule*: *real_polynomial_function.induct*)
(*simp_all add*: *bounded_linear_imp_differentiable*)

lemma *differentiable_on_real_polynomial_function*:
real_polynomial_function p ⟹ *p differentiable_on S*
by (*simp add*: *differentiable_at_imp_differentiable_on differentiable_at_real_polynomial_function*)

lemma *differentiable_at_polynomial_function*:
fixes *f* :: _ ⇒ 'a::*euclidean_space*
shows *polynomial_function f* ⟹ *f differentiable* (*at a within S*)
by (*metis differentiable_at_real_polynomial_function polynomial_function_iff_Basis_inner differentiable_componentwise_within*)

lemma *differentiable_on_polynomial_function*:
fixes *f* :: _ ⇒ 'a::*euclidean_space*
shows *polynomial_function f* ⟹ *f differentiable_on S*
by (*simp add*: *differentiable_at_polynomial_function differentiable_on_def*)

lemma *vector_eq_dot_span*:
assumes *x* ∈ *span B y* ∈ *span B* **and** *i*: ⋀*i*. *i* ∈ *B* ⟹ *i* · *x* = *i* · *y*
shows *x* = *y*
proof −
have ⋀*i*. *i* ∈ *B* ⟹ *orthogonal* (*x* − *y*) *i*
by (*simp add*: *i inner_commute inner_diff_right orthogonal_def*)
moreover have *x* − *y* ∈ *span B*
by (*simp add*: *assms span_diff*)
ultimately have *x* − *y* = *0*
using *orthogonal_to_span orthogonal_self* **by** *blast*
then show *?thesis* **by** *simp*
qed

lemma *orthonormal_basis_expand*:
assumes *B*: *pairwise orthogonal B*
and *1*: ⋀*i*. *i* ∈ *B* ⟹ *norm i* = *1*
and *x* ∈ *span B*
and *finite B*
shows (∑*i*∈*B*. (*x* · *i*) $*_R$ *i*) = *x*
proof (*rule vector_eq_dot_span* [*OF* _ ‹*x* ∈ *span B*›])
show (∑*i*∈*B*. (*x* · *i*) $*_R$ *i*) ∈ *span B*
by (*simp add*: *span_clauses span_sum*)
show *i* · (∑*i*∈*B*. (*x* · *i*) $*_R$ *i*) = *i* · *x* **if** *i* ∈ *B* **for** *i*
proof −
have [*simp*]: *i* · *j* = (*if j* = *i then 1 else 0*) **if** *j* ∈ *B* **for** *j*
using *B 1 that* ‹*i* ∈ *B*›
by (*force simp*: *norm_eq_1 orthogonal_def pairwise_def*)
have *i* · (∑*i*∈*B*. (*x* · *i*) $*_R$ *i*) = (∑*j*∈*B*. *x* · *j* ∗ (*i* · *j*))
by (*simp add*: *inner_sum_right*)
also have ... = (∑*j*∈*B*. *if j* = *i then x* · *i else 0*)
by (*rule sum.cong*; *simp*)

also have ... = $i \cdot x$
  by (*simp add*: ‹*finite B*› *that inner_commute*)
 finally show *?thesis* **.**
 qed
qed

theorem *Stone_Weierstrass_polynomial_function_subspace*:
 fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
 assumes *compact S*
  and *contf*: *continuous_on S f*
  and $0 < e$
  and *subspace T f ‘ S* ⊆ *T*
 obtains *g* **where** *polynomial_function g g ‘ S* ⊆ *T*
  $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x - g\ x) < e$
proof −
 obtain *B* **where** *B* ⊆ *T* **and** *orthB*: *pairwise orthogonal B*
  and *B1*: $\bigwedge x.\ x \in B \Longrightarrow norm\ x = 1$
  and *independent B* **and** *cardB*: *card B* = *dim T*
  and *spanB*: *span B* = *T*
 using *orthonormal_basis_subspace* ‹*subspace T*› **by** *metis*
 then have *finite B*
  by (*simp add*: *independent_imp_finite*)
 then obtain *n::nat* **and** *b* **where** *B* = *b ‘* {*i*. *i* < *n*} *inj_on b* {*i*. *i* < *n*}
  using *finite_imp_nat_seg_image_inj_on* **by** *metis*
 with *cardB* **have** *n* = *card B dim T* = *n*
  by (*auto simp*: *card_image*)
 have *fx*: $(\sum i\in B.\ (f\ x \cdot i) *_R\ i) = f\ x$ **if** $x \in S$ **for** *x*
  by (*metis* (*no_types*, *lifting*) *B1* ‹*finite B*› *assms*(*5*) *image_subset_iff orthB orthonormal_basis_expand spanB sum.cong that*)
 have *cont*: *continuous_on S* $(\lambda x.\ \sum i\in B.\ (f\ x \cdot i) *_R\ i)$
  by (*intro continuous_intros contf*)
 obtain *g* **where** *polynomial_function g*
  and *g*: $\bigwedge x.\ x \in S \Longrightarrow norm\ ((\sum i\in B.\ (f\ x \cdot i) *_R\ i) - g\ x) < e\ /\ (n+2)$
 using *Stone_Weierstrass_polynomial_function* [*OF* ‹*compact S*› *cont*, *of e* / *real* (*n* + *2*)] ‹$0 < e$›
  by *auto*
 with *fx* **have** *g*: $\bigwedge x.\ x \in S \Longrightarrow norm\ (f\ x - g\ x) < e\ /\ (n+2)$
  by *auto*
 show *?thesis*
 proof
  show *polynomial_function* $(\lambda x.\ \sum i\in B.\ (g\ x \cdot i) *_R\ i)$
   using ‹*polynomial_function g*› **by** (*force intro*: ‹*finite B*›)
  show $(\lambda x.\ \sum i\in B.\ (g\ x \cdot i) *_R\ i)$ ‘ *S* ⊆ *T*
   using ‹*B* ⊆ *T*›
   by (*blast intro*: *subspace_sum subspace_mul* ‹*subspace T*›)
  show *norm* $(f\ x - (\sum i\in B.\ (g\ x \cdot i) *_R\ i)) < e$ **if** $x \in S$ **for** *x*
  proof −

have *orth'*: *pairwise* ($\lambda i\ j$. *orthogonal* (($f\ x \cdot i$) $*_R$ i − ($g\ x \cdot i$) $*_R$ i)
(($f\ x \cdot j$) $*_R$ j − ($g\ x \cdot j$) $*_R$ j)) B
by (*auto simp*: *orthogonal_def inner_diff_right inner_diff_left intro*: *pairwise_mono* [*OF orthB*])
then have (*norm* ($\sum i{\in}B$. ($f\ x \cdot i$) $*_R$ i − ($g\ x \cdot i$) $*_R$ i))^2 =
($\sum i{\in}B$. (*norm* (($f\ x \cdot i$) $*_R$ i − ($g\ x \cdot i$) $*_R$ i))^2)
by (*simp add*: *norm_sum_Pythagorean* [*OF* ‹*finite B*› *orth'*])
also have ... = ($\sum i{\in}B$. (*norm* (((f x − g x) · i) $*_R$ i))^2)
by (*simp add*: *algebra_simps*)
also have ... ≤ ($\sum i{\in}B$. (*norm* (f x − g x))^2)
proof −
have $\bigwedge i$. $i \in B \Longrightarrow ((f\ x - g\ x) \cdot i)^2 \le (norm\ (f\ x - g\ x))^2$
by (*metis B1 Cauchy_Schwarz_ineq inner_commute mult.left_neutral norm_eq_1 power2_norm_eq_inner*)
then show *?thesis*
by (*intro sum_mono*) (*simp add*: *sum_mono B1*)
qed
also have ... = *n* ∗ *norm* (f x − g x) ˆ*2*
by (*simp add*: ‹*n* = *card B*›)
also have ... ≤ *n* ∗ (*e* / (*n*+*2*)) ˆ*2*
proof (*rule mult_left_mono*)
show (*norm* (f x − g x))^2 ≤ (*e* / *real* (*n* + *2*))^2
by (*meson dual_order.order_iff_strict g norm_ge_zero power_mono that*)
qed *auto*
also have ... ≤ *e* ˆ*2* / (*n*+*2*)
using ‹*0* < *e*› **by** (*simp add*: *divide_simps power2_eq_square*)
also have ... < *e* ˆ*2*
using ‹*0* < *e*› **by** (*simp add*: *divide_simps*)
finally have (*norm* ($\sum i{\in}B$. ($f\ x \cdot i$) $*_R$ i − ($g\ x \cdot i$) $*_R$ i))^2 < *e* ˆ*2* .
then have (*norm* ($\sum i{\in}B$. ($f\ x \cdot i$) $*_R$ i − ($g\ x \cdot i$) $*_R$ i)) < *e*
by (*simp add*: ‹*0* < *e*› *norm_lt_square power2_norm_eq_inner*)
then show *?thesis*
using *fx that* **by** (*simp add*: *sum_subtractf*)
qed
qed
qed

hide_fact *linear add mult const*

end

Chapter 6

Measure and Integration Theory

theory *Sigma_Algebra*
imports
  Complex_Main
  HOL−Library.Countable_Set
  HOL−Library.FuncSet
  HOL−Library.Indicator_Function
  HOL−Library.Extended_Nonnegative_Real
  HOL−Library.Disjoint_Sets
begin

6.1 Sigma Algebra

Sigma algebras are an elementary concept in measure theory. To measure — that is to integrate — functions, we first have to measure sets. Unfortunately, when dealing with a large universe, it is often not possible to consistently assign a measure to every subset. Therefore it is necessary to define the set of measurable subsets of the universe. A sigma algebra is such a set that has three very natural and desirable properties.

6.1.1 Families of sets

locale *subset_class* =
  fixes Ω :: *'a set* **and** M :: *'a set set*
  assumes *space_closed*: $M \subseteq Pow\ \Omega$

lemma (**in** *subset_class*) *sets_into_space*: $x \in M \Longrightarrow x \subseteq \Omega$
  by (*metis PowD contra_subsetD space_closed*)

Semiring of sets

locale *semiring_of_sets* = *subset_class* +

assumes *empty_sets*[*iff*]: $\{\} \in M$
assumes *Int*[*intro*]: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a \cap b \in M$
assumes *Diff_cover*:
$\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow \exists\, C{\subseteq}M.\ \textit{finite}\ C \wedge \textit{disjoint}\ C \wedge a - b = \bigcup C$

lemma (**in** *semiring_of_sets*) *finite_INT*[*intro*]:
assumes *finite* $I\ I \neq \{\}\ \bigwedge i.\ i \in I \Longrightarrow A\ i \in M$
shows $(\bigcap i{\in}I.\ A\ i) \in M$
using *assms* **by** (*induct rule*: *finite_ne_induct*) *auto*

lemma (**in** *semiring_of_sets*) *Int_space_eq1* [*simp*]: $x \in M \Longrightarrow \Omega \cap x = x$
by (*metis Int_absorb1 sets_into_space*)

lemma (**in** *semiring_of_sets*) *Int_space_eq2* [*simp*]: $x \in M \Longrightarrow x \cap \Omega = x$
by (*metis Int_absorb2 sets_into_space*)

lemma (**in** *semiring_of_sets*) *sets_Collect_conj*:
assumes $\{x{\in}\Omega.\ P\ x\} \in M\ \{x{\in}\Omega.\ Q\ x\} \in M$
shows $\{x{\in}\Omega.\ Q\ x \wedge P\ x\} \in M$
proof −
have $\{x{\in}\Omega.\ Q\ x \wedge P\ x\} = \{x{\in}\Omega.\ Q\ x\} \cap \{x{\in}\Omega.\ P\ x\}$
by *auto*
with *assms* **show** *?thesis* **by** *auto*
qed

lemma (**in** *semiring_of_sets*) *sets_Collect_finite_All'*:
assumes $\bigwedge i.\ i \in S \Longrightarrow \{x{\in}\Omega.\ P\ i\ x\} \in M$ *finite* $S\ S \neq \{\}$
shows $\{x{\in}\Omega.\ \forall\, i{\in}S.\ P\ i\ x\} \in M$
proof −
have $\{x{\in}\Omega.\ \forall\, i{\in}S.\ P\ i\ x\} = (\bigcap i{\in}S.\ \{x{\in}\Omega.\ P\ i\ x\})$
using ‹$S \neq \{\}$› **by** *auto*
with *assms* **show** *?thesis* **by** *auto*
qed

Ring of sets

locale *ring_of_sets* = *semiring_of_sets* +
assumes *Un* [*intro*]: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a \cup b \in M$

lemma (**in** *ring_of_sets*) *finite_Union* [*intro*]:
finite $X \Longrightarrow X \subseteq M \Longrightarrow \bigcup X \in M$
by (*induct set*: *finite*) (*auto simp add*: *Un*)

lemma (**in** *ring_of_sets*) *finite_UN*[*intro*]:
assumes *finite* I **and** $\bigwedge i.\ i \in I \Longrightarrow A\ i \in M$
shows $(\bigcup i{\in}I.\ A\ i) \in M$
using *assms* **by** *induct auto*

lemma (**in** *ring_of_sets*) *Diff* [*intro*]:

assumes $a \in M$ $b \in M$ **shows** $a - b \in M$
using *Diff_cover*[*OF assms*] **by** *auto*

lemma *ring_of_setsI*:
assumes *space_closed*: $M \subseteq Pow\ \Omega$
assumes *empty_sets*[*iff*]: $\{\} \in M$
assumes *Un*[*intro*]: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a \cup b \in M$
assumes *Diff*[*intro*]: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a - b \in M$
shows *ring_of_sets* Ω M
proof
fix a b **assume** *ab*: $a \in M$ $b \in M$
from *ab* **show** $\exists\, C \subseteq M.$ *finite* $C \wedge$ *disjoint* $C \wedge a - b = \bigcup C$
by (*intro exI*[*of* _ $\{a - b\}$]) (*auto simp*: *disjoint_def*)
have $a \cap b = a - (a - b)$ **by** *auto*
also have $\ldots \in M$ **using** *ab* **by** *auto*
finally show $a \cap b \in M$ **.**
qed *fact+*

lemma *ring_of_sets_iff*: *ring_of_sets* Ω $M \longleftrightarrow M \subseteq Pow\ \Omega \wedge \{\} \in M \wedge (\forall\, a \in M.\ \forall\, b \in M.\ a \cup b \in M) \wedge (\forall\, a \in M.\ \forall\, b \in M.\ a - b \in M)$
proof
assume *ring_of_sets* Ω M
then interpret *ring_of_sets* Ω M **.**
show $M \subseteq Pow\ \Omega \wedge \{\} \in M \wedge (\forall\, a \in M.\ \forall\, b \in M.\ a \cup b \in M) \wedge (\forall\, a \in M.\ \forall\, b \in M.\ a - b \in M)$
using *space_closed* **by** *auto*
qed (*auto intro*!: *ring_of_setsI*)

lemma (**in** *ring_of_sets*) *insert_in_sets*:
assumes $\{x\} \in M$ $A \in M$ **shows** *insert* x $A \in M$
proof –
have $\{x\} \cup A \in M$ **using** *assms* **by** (*rule Un*)
thus *?thesis* **by** *auto*
qed

lemma (**in** *ring_of_sets*) *sets_Collect_disj*:
assumes $\{x \in \Omega.\ P\ x\} \in M$ $\{x \in \Omega.\ Q\ x\} \in M$
shows $\{x \in \Omega.\ Q\ x \vee P\ x\} \in M$
proof –
have $\{x \in \Omega.\ Q\ x \vee P\ x\} = \{x \in \Omega.\ Q\ x\} \cup \{x \in \Omega.\ P\ x\}$
by *auto*
with *assms* **show** *?thesis* **by** *auto*
qed

lemma (**in** *ring_of_sets*) *sets_Collect_finite_Ex*:
assumes $\bigwedge i.\ i \in S \Longrightarrow \{x \in \Omega.\ P\ i\ x\} \in M$ *finite* S
shows $\{x \in \Omega.\ \exists\, i \in S.\ P\ i\ x\} \in M$
proof –
have $\{x \in \Omega.\ \exists\, i \in S.\ P\ i\ x\} = (\bigcup\, i \in S.\ \{x \in \Omega.\ P\ i\ x\})$

  by *auto*
 with *assms* **show** *?thesis* **by** *auto*
qed

Algebra of sets

locale *algebra* = *ring_of_sets* +
 assumes *top* [*iff*]: Ω ∈ *M*

lemma (**in** *algebra*) *compl_sets* [*intro*]:
 a ∈ *M* ⟹ Ω − *a* ∈ *M*
 by *auto*

proposition *algebra_iff_Un*:
 algebra Ω *M* ⟷
  M ⊆ *Pow* Ω ∧
  {} ∈ *M* ∧
  (∀ *a* ∈ *M*. Ω − *a* ∈ *M*) ∧
  (∀ *a* ∈ *M*. ∀ *b* ∈ *M*. *a* ∪ *b* ∈ *M*) (**is** _ ⟷ *?Un*)
proof
 assume *algebra* Ω *M*
 then interpret *algebra* Ω *M* **.**
 show *?Un* **using** *sets_into_space* **by** *auto*
next
 assume *?Un*
 then have Ω ∈ *M* **by** *auto*
 interpret *ring_of_sets* Ω *M*
 proof (*rule ring_of_setsI*)
  show Ω: *M* ⊆ *Pow* Ω {} ∈ *M*
   using ‹*?Un*› **by** *auto*
  fix *a b* **assume** *a*: *a* ∈ *M* **and** *b*: *b* ∈ *M*
  then show *a* ∪ *b* ∈ *M* **using** ‹*?Un*› **by** *auto*
  have *a* − *b* = Ω − ((Ω − *a*) ∪ *b*)
   using Ω *a b* **by** *auto*
  then show *a* − *b* ∈ *M*
   using *a b* ‹*?Un*› **by** *auto*
 qed
 show *algebra* Ω *M* **proof qed** *fact*
qed

proposition *algebra_iff_Int*:
  algebra Ω *M* ⟷
   M ⊆ *Pow* Ω & {} ∈ *M* &
   (∀ *a* ∈ *M*. Ω − *a* ∈ *M*) &
   (∀ *a* ∈ *M*. ∀ *b* ∈ *M*. *a* ∩ *b* ∈ *M*) (**is** _ ⟷ *?Int*)
proof
 assume *algebra* Ω *M*
 then interpret *algebra* Ω *M* **.**
 show *?Int* **using** *sets_into_space* **by** *auto*

next
  assume *?Int*
  show *algebra* Ω *M*
  proof (*unfold algebra_iff_Un*, *intro conjI ballI*)
    show Ω: $M \subseteq Pow\ \Omega$ $\{\} \in M$
      using ‹*?Int*› **by** *auto*
    from ‹*?Int*› **show** $\bigwedge a.\ a \in M \Longrightarrow \Omega - a \in M$ **by** *auto*
    fix *a b* **assume** *M*: $a \in M$ $b \in M$
    hence $a \cup b = \Omega - ((\Omega - a) \cap (\Omega - b))$
      using Ω **by** *blast*
    also have ... $\in M$
      using *M* ‹*?Int*› **by** *auto*
    finally show $a \cup b \in M$ **.**
  qed
qed

lemma (**in** *algebra*) *sets_Collect_neg*:
  assumes $\{x{\in}\Omega.\ P\ x\} \in M$
  shows $\{x{\in}\Omega.\ \neg\ P\ x\} \in M$
proof –
  have $\{x{\in}\Omega.\ \neg\ P\ x\} = \Omega - \{x{\in}\Omega.\ P\ x\}$ **by** *auto*
  with *assms* **show** *?thesis* **by** *auto*
qed

lemma (**in** *algebra*) *sets_Collect_imp*:
  $\{x{\in}\Omega.\ P\ x\} \in M \Longrightarrow \{x{\in}\Omega.\ Q\ x\} \in M \Longrightarrow \{x{\in}\Omega.\ Q\ x \longrightarrow P\ x\} \in M$
  unfolding *imp_conv_disj* **by** (*intro sets_Collect_disj sets_Collect_neg*)

lemma (**in** *algebra*) *sets_Collect_const*:
  $\{x{\in}\Omega.\ P\} \in M$
  by (*cases P*) *auto*

lemma *algebra_single_set*:
  $X \subseteq S \Longrightarrow$ *algebra* S $\{ \{\}, X, S - X, S \}$
  by (*auto simp*: *algebra_iff_Int*)

Restricted algebras

abbreviation (**in** *algebra*)
  restricted_space $A \equiv ((\cap)\ A)$ ‘ M

lemma (**in** *algebra*) *restricted_algebra*:
  assumes $A \in M$ **shows** *algebra A* (*restricted_space A*)
  using *assms* **by** (*auto simp*: *algebra_iff_Int*)

Sigma Algebras

locale *sigma_algebra* = *algebra* +
  assumes *countable_nat_UN* [*intro*]: $\bigwedge A.\ range\ A \subseteq M \Longrightarrow (\bigcup i{::}nat.\ A\ i) \in M$

lemma (**in** *algebra*) *is_sigma_algebra*:
  assumes *finite M*
  shows *sigma_algebra* Ω *M*
proof
  fix $A :: nat \Rightarrow 'a\ set$ **assume** $range\ A \subseteq M$
  then have $(\bigcup i.\ A\ i) = (\bigcup s \in M \cap range\ A.\ s)$
    by *auto*
  also have $(\bigcup s \in M \cap range\ A.\ s) \in M$
    using ‹*finite M*› **by** *auto*
  finally show $(\bigcup i.\ A\ i) \in M$ **.**
qed

lemma *countable_UN_eq*:
  fixes $A :: 'i::countable \Rightarrow 'a\ set$
  shows $(range\ A \subseteq M \longrightarrow (\bigcup i.\ A\ i) \in M) \longleftrightarrow$
    $(range\ (A \circ from\_nat) \subseteq M \longrightarrow (\bigcup i.\ (A \circ from\_nat)\ i) \in M)$
proof −
  let $?A' = A \circ from\_nat$
  have $*$: $(\bigcup i.\ ?A'\ i) = (\bigcup i.\ A\ i)$ (**is** $?l = ?r$)
  proof *safe*
    fix *x i* **assume** $x \in A\ i$ **thus** $x \in ?l$
      by (*auto intro*!: *exI*[*of* _ *to_nat i*])
  next
    fix *x i* **assume** $x \in ?A'\ i$ **thus** $x \in ?r$
      by (*auto intro*!: *exI*[*of* _ *from_nat i*])
  qed
  have *A* ‘ *range from_nat* = *range A*
    using *surj_from_nat* **by** *simp*
  then have $**$: $range\ ?A' = range\ A$
    by (*simp only*: *image_comp* [*symmetric*])
  show *?thesis* **unfolding** $*$ $**$ **..**
qed

lemma (**in** *sigma_algebra*) *countable_Union* [*intro*]:
  assumes *countable X* $X \subseteq M$ **shows** $\bigcup X \in M$
proof *cases*
  assume $X \neq \{\}$
  hence $\bigcup X = (\bigcup n.\ from\_nat\_into\ X\ n)$
    using *assms* **by** (*auto cong del*: *SUP_cong*)
  also have $\ldots \in M$ **using** *assms*
    by (*auto intro*!: *countable_nat_UN*) (*metis* ‹$X \neq \{\}$› *from_nat_into subsetD*)
  finally show *?thesis* **.**
qed *simp*

lemma (**in** *sigma_algebra*) *countable_UN*[*intro*]:
  fixes $A :: 'i::countable \Rightarrow 'a\ set$
  assumes $A`X \subseteq M$
  shows $(\bigcup x \in X.\ A\ x) \in M$
proof −

let *?A* = λi. *if* $i \in X$ *then* *A i else* {}
from *assms* **have** *range ?A* $\subseteq M$ **by** *auto*
with *countable_nat_UN*[*of ?A* $\circ$ *from_nat*] *countable_UN_eq*[*of ?A M*]
have $(\bigcup x.\ ?A\ x) \in M$ **by** *auto*
moreover have $(\bigcup x.\ ?A\ x) = (\bigcup x{\in}X.\ A\ x)$ **by** (*auto split*: *if_split_asm*)
ultimately show *?thesis* **by** *simp*
qed

lemma (**in** *sigma_algebra*) *countable_UN'*:
fixes *A* :: *'i* $\Rightarrow$ *'a set*
assumes *X*: *countable X*
assumes *A*: *A'X* $\subseteq M$
shows $(\bigcup x{\in}X.\ A\ x) \in M$
proof –
have $(\bigcup x{\in}X.\ A\ x) = (\bigcup i{\in}to\_nat\_on\ X\ `\ X.\ A\ (from\_nat\_into\ X\ i))$
using *X* **by** *auto*
also have ... $\in M$
using *A X*
by (*intro countable_UN*) *auto*
finally show *?thesis* **.**
qed

lemma (**in** *sigma_algebra*) *countable_UN''*:
$[\![$ *countable X*; $\bigwedge x\ y.\ x \in X \Longrightarrow A\ x \in M$ $]\!] \Longrightarrow (\bigcup x{\in}X.\ A\ x) \in M$
by(*erule countable_UN'*)(*auto*)

lemma (**in** *sigma_algebra*) *countable_INT* [*intro*]:
fixes *A* :: *'i::countable* $\Rightarrow$ *'a set*
assumes *A*: *A'X* $\subseteq M$ $X \neq \{\}$
shows $(\bigcap i{\in}X.\ A\ i) \in M$
proof –
from *A* **have** $\forall i{\in}X.\ A\ i \in M$ **by** *fast*
hence $\Omega - (\bigcup i{\in}X.\ \Omega - A\ i) \in M$ **by** *blast*
moreover
have $(\bigcap i{\in}X.\ A\ i) = \Omega - (\bigcup i{\in}X.\ \Omega - A\ i)$ **using** *space_closed A*
by *blast*
ultimately show *?thesis* **by** *metis*
qed

lemma (**in** *sigma_algebra*) *countable_INT'*:
fixes *A* :: *'i* $\Rightarrow$ *'a set*
assumes *X*: *countable X* $X \neq \{\}$
assumes *A*: *A'X* $\subseteq M$
shows $(\bigcap x{\in}X.\ A\ x) \in M$
proof –
have $(\bigcap x{\in}X.\ A\ x) = (\bigcap i{\in}to\_nat\_on\ X\ `\ X.\ A\ (from\_nat\_into\ X\ i))$
using *X* **by** *auto*
also have ... $\in M$
using *A X*

by (*intro countable_INT*) *auto*
finally show *?thesis* **.**
qed

lemma (**in** *sigma_algebra*) *countable_INT''*:
$UNIV \in M \Longrightarrow countable\ I \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow F\ i \in M) \Longrightarrow (\bigcap i{\in}I.\ F\ i) \in M$
by (*cases* $I = \{\}$) (*auto intro*: *countable_INT'*)

lemma (**in** *sigma_algebra*) *countable*:
assumes $\bigwedge a.\ a \in A \Longrightarrow \{a\} \in M$ *countable A*
shows $A \in M$
proof −
have $(\bigcup a{\in}A.\ \{a\}) \in M$
using *assms* **by** (*intro countable_UN'*) *auto*
also have $(\bigcup a{\in}A.\ \{a\}) = A$ **by** *auto*
finally show *?thesis* **by** *auto*
qed

lemma *ring_of_sets_Pow*: *ring_of_sets sp* (*Pow sp*)
by (*auto simp*: *ring_of_sets_iff*)

lemma *algebra_Pow*: *algebra sp* (*Pow sp*)
by (*auto simp*: *algebra_iff_Un*)

lemma *sigma_algebra_iff*:
sigma_algebra Ω *M* $\longleftrightarrow$
algebra Ω *M* $\wedge$ $(\forall A.\ range\ A \subseteq M \longrightarrow (\bigcup i{::}nat.\ A\ i) \in M)$
by (*simp add*: *sigma_algebra_def sigma_algebra_axioms_def*)

lemma *sigma_algebra_Pow*: *sigma_algebra sp* (*Pow sp*)
by (*auto simp*: *sigma_algebra_iff algebra_iff_Int*)

lemma (**in** *sigma_algebra*) *sets_Collect_countable_All*:
assumes $\bigwedge i.\ \{x{\in}\Omega.\ P\ i\ x\} \in M$
shows $\{x{\in}\Omega.\ \forall i{::}'i{::}countable.\ P\ i\ x\} \in M$
proof −
have $\{x{\in}\Omega.\ \forall i{::}'i{::}countable.\ P\ i\ x\} = (\bigcap i.\ \{x{\in}\Omega.\ P\ i\ x\})$ **by** *auto*
with *assms* **show** *?thesis* **by** *auto*
qed

lemma (**in** *sigma_algebra*) *sets_Collect_countable_Ex*:
assumes $\bigwedge i.\ \{x{\in}\Omega.\ P\ i\ x\} \in M$
shows $\{x{\in}\Omega.\ \exists i{::}'i{::}countable.\ P\ i\ x\} \in M$
proof −
have $\{x{\in}\Omega.\ \exists i{::}'i{::}countable.\ P\ i\ x\} = (\bigcup i.\ \{x{\in}\Omega.\ P\ i\ x\})$ **by** *auto*
with *assms* **show** *?thesis* **by** *auto*
qed

lemma (**in** *sigma_algebra*) *sets_Collect_countable_Ex'*:

assumes $\bigwedge i.\ i \in I \Longrightarrow \{x{\in}\Omega.\ P\ i\ x\} \in M$
assumes *countable I*
shows $\{x{\in}\Omega.\ \exists\, i{\in}I.\ P\ i\ x\} \in M$
proof −
have $\{x{\in}\Omega.\ \exists\, i{\in}I.\ P\ i\ x\} = (\bigcup i{\in}I.\ \{x{\in}\Omega.\ P\ i\ x\})$ **by** *auto*
with *assms* **show** *?thesis*
by (*auto intro*!: *countable_UN'*)
qed

lemma (**in** *sigma_algebra*) *sets_Collect_countable_All'*:
assumes $\bigwedge i.\ i \in I \Longrightarrow \{x{\in}\Omega.\ P\ i\ x\} \in M$
assumes *countable I*
shows $\{x{\in}\Omega.\ \forall\, i{\in}I.\ P\ i\ x\} \in M$
proof −
have $\{x{\in}\Omega.\ \forall\, i{\in}I.\ P\ i\ x\} = (\bigcap i{\in}I.\ \{x{\in}\Omega.\ P\ i\ x\}) \cap \Omega$ **by** *auto*
with *assms* **show** *?thesis*
by (*cases* $I = \{\}$) (*auto intro*!: *countable_INT'*)
qed

lemma (**in** *sigma_algebra*) *sets_Collect_countable_Ex1'*:
assumes $\bigwedge i.\ i \in I \Longrightarrow \{x{\in}\Omega.\ P\ i\ x\} \in M$
assumes *countable I*
shows $\{x{\in}\Omega.\ \exists!i{\in}I.\ P\ i\ x\} \in M$
proof −
have $\{x{\in}\Omega.\ \exists!i{\in}I.\ P\ i\ x\} = \{x{\in}\Omega.\ \exists\, i{\in}I.\ P\ i\ x \wedge (\forall\, j{\in}I.\ P\ j\ x \longrightarrow i = j)\}$
by *auto*
with *assms* **show** *?thesis*
by (*auto intro*!: *sets_Collect_countable_All' sets_Collect_countable_Ex' sets_Collect_conj sets_Collect_imp sets_Collect_const*)
qed

lemmas (**in** *sigma_algebra*) *sets_Collect* =
sets_Collect_imp sets_Collect_disj sets_Collect_conj sets_Collect_neg sets_Collect_const sets_Collect_countable_All sets_Collect_countable_Ex sets_Collect_countable_All

lemma (**in** *sigma_algebra*) *sets_Collect_countable_Ball*:
assumes $\bigwedge i.\ \{x{\in}\Omega.\ P\ i\ x\} \in M$
shows $\{x{\in}\Omega.\ \forall\, i{::}'i{::}countable{\in}X.\ P\ i\ x\} \in M$
unfolding *Ball_def* **by** (*intro sets_Collect assms*)

lemma (**in** *sigma_algebra*) *sets_Collect_countable_Bex*:
assumes $\bigwedge i.\ \{x{\in}\Omega.\ P\ i\ x\} \in M$
shows $\{x{\in}\Omega.\ \exists\, i{::}'i{::}countable{\in}X.\ P\ i\ x\} \in M$
unfolding *Bex_def* **by** (*intro sets_Collect assms*)

lemma *sigma_algebra_single_set*:
assumes $X \subseteq S$
shows *sigma_algebra* S $\{\ \{\},\ X,\ S - X,\ S\ \}$
using *algebra.is_sigma_algebra*[*OF algebra_single_set*[*OF* ‹$X \subseteq S$›]] **by** *simp*

Binary Unions

definition *binary* :: $'a \Rightarrow 'a \Rightarrow nat \Rightarrow 'a$
  where *binary a b* = $(\lambda x.\ b)(0 := a)$

lemma *range_binary_eq*: *range*(*binary a b*) = $\{a,b\}$
  by (*auto simp add*: *binary_def*)

lemma *Un_range_binary*: $a \cup b = (\bigcup i::nat.\ binary\ a\ b\ i)$
  by (*simp add*: *range_binary_eq cong del*: *SUP_cong_simp*)

lemma *Int_range_binary*: $a \cap b = (\bigcap i::nat.\ binary\ a\ b\ i)$
  by (*simp add*: *range_binary_eq cong del*: *INF_cong_simp*)

lemma *sigma_algebra_iff2*:
  sigma_algebra Ω $M \longleftrightarrow$
    $M \subseteq Pow\ \Omega \wedge \{\} \in M \wedge (\forall s \in M.\ \Omega - s \in M)$
    $\wedge\ (\forall A.\ range\ A \subseteq M \longrightarrow (\bigcup\ i::nat.\ A\ i) \in M)$ (**is** $?P \longleftrightarrow\ ?R \wedge\ ?S \wedge\ ?V \wedge$
$?W$)
proof
  assume $?P$
  then interpret *sigma_algebra* Ω M **.**
  from *space_closed* **show** $?R \wedge\ ?S \wedge\ ?V \wedge\ ?W$
    by *auto*
next
  assume $?R \wedge\ ?S \wedge\ ?V \wedge\ ?W$
  then have $?R\ ?S\ ?V\ ?W$
    by *simp_all*
  show $?P$
  proof (*rule sigma_algebra.intro*)
    show *sigma_algebra_axioms M*
      by *standard* (*use* ‹?W› **in** *simp*)
    from ‹?W› **have** $*$: $range\ (binary\ a\ b) \subseteq M \Longrightarrow \bigcup\ (range\ (binary\ a\ b)) \in M$
for *a b*
      by *auto*
    show *algebra* Ω *M*
      unfolding *algebra_iff_Un* **using** ‹?R› ‹?S› ‹?V› $*$
      by (*auto simp add*: *range_binary_eq*)
  qed
qed

Initial Sigma Algebra

Sigma algebras can naturally be created as the closure of any set of M with regard to the properties just postulated.

inductive_set *sigma_sets* :: $'a\ set \Rightarrow 'a\ set\ set \Rightarrow 'a\ set\ set$
  for *sp* :: $'a\ set$ **and** *A* :: $'a\ set\ set$
  where
    Basic[*intro*, *simp*]: $a \in A \Longrightarrow a \in sigma\_sets\ sp\ A$

| *Empty*: {} ∈ *sigma_sets sp A*
| *Compl*: *a* ∈ *sigma_sets sp A* ⟹ *sp* − *a* ∈ *sigma_sets sp A*
| *Union*: (⋀*i*::*nat*. *a i* ∈ *sigma_sets sp A*) ⟹ (⋃*i*. *a i*) ∈ *sigma_sets sp A*

lemma (**in** *sigma_algebra*) *sigma_sets_subset*:
assumes *a*: *a* ⊆ *M*
shows *sigma_sets* Ω *a* ⊆ *M*
proof
fix *x*
assume *x* ∈ *sigma_sets* Ω *a*
from *this* **show** *x* ∈ *M*
by (*induct rule*: *sigma_sets.induct*, *auto*) (*metis a subsetD*)
qed

lemma *sigma_sets_into_sp*: *A* ⊆ *Pow sp* ⟹ *x* ∈ *sigma_sets sp A* ⟹ *x* ⊆ *sp*
by (*erule sigma_sets.induct*, *auto*)

lemma *sigma_algebra_sigma_sets*:
a ⊆ *Pow* Ω ⟹ *sigma_algebra* Ω (*sigma_sets* Ω *a*)
by (*auto simp add*: *sigma_algebra_iff2 dest*: *sigma_sets_into_sp*
intro!: *sigma_sets.Union sigma_sets.Empty sigma_sets.Compl*)

lemma *sigma_sets_least_sigma_algebra*:
assumes *A* ⊆ *Pow S*
shows *sigma_sets S A* = ⋂{*B*. *A* ⊆ *B* ∧ *sigma_algebra S B*}
proof *safe*
fix *B X* **assume** *A* ⊆ *B* **and** *sa*: *sigma_algebra S B*
and *X*: *X* ∈ *sigma_sets S A*
from *sigma_algebra.sigma_sets_subset*[*OF sa*, *simplified*, *OF* ‹*A* ⊆ *B*›] *X*
show *X* ∈ *B* **by** *auto*
next
fix *X* **assume** *X* ∈ ⋂{*B*. *A* ⊆ *B* ∧ *sigma_algebra S B*}
then have [*intro*!]: ⋀*B*. *A* ⊆ *B* ⟹ *sigma_algebra S B* ⟹ *X* ∈ *B*
by *simp*
have *A* ⊆ *sigma_sets S A* **using** *assms* **by** *auto*
moreover have *sigma_algebra S* (*sigma_sets S A*)
using *assms* **by** (*intro sigma_algebra_sigma_sets*[*of A*]) *auto*
ultimately show *X* ∈ *sigma_sets S A* **by** *auto*
qed

lemma *sigma_sets_top*: *sp* ∈ *sigma_sets sp A*
by (*metis Diff_empty sigma_sets.Compl sigma_sets.Empty*)

lemma *binary_in_sigma_sets*:
binary a b i ∈ *sigma_sets sp A* **if** *a* ∈ *sigma_sets sp A* **and** *b* ∈ *sigma_sets sp A*
using *that* **by** (*simp add*: *binary_def*)

lemma *sigma_sets_Un*:
a ∪ *b* ∈ *sigma_sets sp A* **if** *a* ∈ *sigma_sets sp A* **and** *b* ∈ *sigma_sets sp A*

using *that* **by** (*simp add*: *Un_range_binary binary_in_sigma_sets Union*)

lemma *sigma_sets_Inter*:
assumes *Asb*: $A \subseteq Pow\ sp$
shows $(\bigwedge i{::}nat.\ a\ i \in sigma\_sets\ sp\ A) \Longrightarrow (\bigcap i.\ a\ i) \in sigma\_sets\ sp\ A$
proof −
assume *ai*: $\bigwedge i{::}nat.\ a\ i \in sigma\_sets\ sp\ A$
hence $\bigwedge i{::}nat.\ sp-(a\ i) \in sigma\_sets\ sp\ A$
by (*rule sigma_sets.Compl*)
hence $(\bigcup i.\ sp-(a\ i)) \in sigma\_sets\ sp\ A$
by (*rule sigma_sets.Union*)
hence $sp-(\bigcup i.\ sp-(a\ i)) \in sigma\_sets\ sp\ A$
by (*rule sigma_sets.Compl*)
also have $sp-(\bigcup i.\ sp-(a\ i)) = sp\ Int\ (\bigcap i.\ a\ i)$
by *auto*
also have ... = $(\bigcap i.\ a\ i)$ **using** *ai*
by (*blast dest*: *sigma_sets_into_sp* [*OF Asb*])
finally show *?thesis* **.**
qed

lemma *sigma_sets_INTER*:
assumes *Asb*: $A \subseteq Pow\ sp$
and *ai*: $\bigwedge i{::}nat.\ i \in S \Longrightarrow a\ i \in sigma\_sets\ sp\ A$ **and** *non*: $S \neq \{\}$
shows $(\bigcap i{\in}S.\ a\ i) \in sigma\_sets\ sp\ A$
proof −
from *ai* **have** $\bigwedge i.\ (if\ i{\in}S\ then\ a\ i\ else\ sp) \in sigma\_sets\ sp\ A$
by (*simp add*: *sigma_sets.intros*(*2*−) *sigma_sets_top*)
hence $(\bigcap i.\ (if\ i{\in}S\ then\ a\ i\ else\ sp)) \in sigma\_sets\ sp\ A$
by (*rule sigma_sets_Inter* [*OF Asb*])
also have $(\bigcap i.\ (if\ i{\in}S\ then\ a\ i\ else\ sp)) = (\bigcap i{\in}S.\ a\ i)$
by *auto* (*metis ai non sigma_sets_into_sp subset_empty subset_iff Asb*)+
finally show *?thesis* **.**
qed

lemma *sigma_sets_UNION*:
$countable\ B \Longrightarrow (\bigwedge b.\ b \in B \Longrightarrow b \in sigma\_sets\ X\ A) \Longrightarrow \bigcup\ B \in sigma\_sets\ X\ A$
using *from_nat_into* [*of B*] *range_from_nat_into* [*of B*] *sigma_sets.Union* [*of from_nat_into B X A*]
by (*cases* $B = \{\}$) (*simp_all add*: *sigma_sets.Empty cong del*: *SUP_cong*)

lemma (**in** *sigma_algebra*) *sigma_sets_eq*:
$sigma\_sets\ \Omega\ M = M$
proof
show $M \subseteq sigma\_sets\ \Omega\ M$
by (*metis Set.subsetI sigma_sets.Basic*)
next
show $sigma\_sets\ \Omega\ M \subseteq M$
by (*metis sigma_sets_subset subset_refl*)

qed

lemma *sigma_sets_eqI*:
assumes *A*: $\bigwedge a.\ a \in A \Longrightarrow a \in sigma\_sets\ M\ B$
assumes *B*: $\bigwedge b.\ b \in B \Longrightarrow b \in sigma\_sets\ M\ A$
shows $sigma\_sets\ M\ A = sigma\_sets\ M\ B$
proof (*intro set_eqI iffI*)
fix *a* **assume** $a \in sigma\_sets\ M\ A$
from *this A* **show** $a \in sigma\_sets\ M\ B$
by *induct* (*auto intro*!: *sigma_sets.intros*(*2*−) *del*: *sigma_sets.Basic*)
next
fix *b* **assume** $b \in sigma\_sets\ M\ B$
from *this B* **show** $b \in sigma\_sets\ M\ A$
by *induct* (*auto intro*!: *sigma_sets.intros*(*2*−) *del*: *sigma_sets.Basic*)
qed

lemma *sigma_sets_subseteq*: **assumes** $A \subseteq B$ **shows** $sigma\_sets\ X\ A \subseteq sigma\_sets\ X\ B$
proof
fix *x* **assume** $x \in sigma\_sets\ X\ A$ **then show** $x \in sigma\_sets\ X\ B$
by *induct* (*insert* ‹$A \subseteq B$›, *auto intro*: *sigma_sets.intros*(*2*−))
qed

lemma *sigma_sets_mono*: **assumes** $A \subseteq sigma\_sets\ X\ B$ **shows** $sigma\_sets\ X\ A \subseteq sigma\_sets\ X\ B$
proof
fix *x* **assume** $x \in sigma\_sets\ X\ A$ **then show** $x \in sigma\_sets\ X\ B$
by *induct* (*insert* ‹$A \subseteq sigma\_sets\ X\ B$›, *auto intro*: *sigma_sets.intros*(*2*−))
qed

lemma *sigma_sets_mono'*: **assumes** $A \subseteq B$ **shows** $sigma\_sets\ X\ A \subseteq sigma\_sets\ X\ B$
proof
fix *x* **assume** $x \in sigma\_sets\ X\ A$ **then show** $x \in sigma\_sets\ X\ B$
by *induct* (*insert* ‹$A \subseteq B$›, *auto intro*: *sigma_sets.intros*(*2*−))
qed

lemma *sigma_sets_superset_generator*: $A \subseteq sigma\_sets\ X\ A$
by (*auto intro*: *sigma_sets.Basic*)

lemma (**in** *sigma_algebra*) *restriction_in_sets*:
fixes $A :: nat \Rightarrow 'a\ set$
assumes $S \in M$
and ∗: $range\ A \subseteq (\lambda A.\ S \cap A)\ `\ M$ (**is** $\_ \subseteq ?r$)
shows $range\ A \subseteq M\ (\bigcup i.\ A\ i) \in (\lambda A.\ S \cap A)\ `\ M$
proof −
{ **fix** *i* **have** $A\ i \in ?r$ **using** ∗ **by** *auto*
hence $\exists B.\ A\ i = B \cap S \wedge B \in M$ **by** *auto*
hence $A\ i \subseteq S\ A\ i \in M$ **using** ‹$S \in M$› **by** *auto* }

thus *range* $A \subseteq M$ $(\bigcup i.\ A\ i) \in (\lambda A.\ S \cap A)$ ‘ M
  by (*auto intro*!: *image_eqI*[*of* _ _ $(\bigcup i.\ A\ i)$])
qed

lemma (**in** *sigma_algebra*) *restricted_sigma_algebra*:
  assumes $S \in M$
  shows *sigma_algebra S* (*restricted_space S*)
  unfolding *sigma_algebra_def sigma_algebra_axioms_def*
proof *safe*
  show *algebra S* (*restricted_space S*) **using** *restricted_algebra*[*OF assms*] **.**
next
  fix *A* :: *nat* ⇒ *'a set* **assume** *range* $A \subseteq$ *restricted_space S*
  from *restriction_in_sets*[*OF assms this*[*simplified*]]
  show $(\bigcup i.\ A\ i) \in$ *restricted_space S* **by** *simp*
qed

lemma *sigma_sets_Int*:
  assumes $A \in$ *sigma_sets sp st* $A \subseteq sp$
  shows $(\cap)$ *A* ‘ *sigma_sets sp st* = *sigma_sets A* (($\cap$) *A* ‘ *st*)
proof (*intro equalityI subsetI*)
  fix *x* **assume** $x \in (\cap)$ *A* ‘ *sigma_sets sp st*
  then obtain *y* **where** $y \in$ *sigma_sets sp st* $x = y \cap A$ **by** *auto*
  then have $x \in$ *sigma_sets* ($A \cap sp$) (($\cap$) *A* ‘ *st*)
  proof (*induct arbitrary*: *x*)
    case (*Compl a*)
    then show *?case*
      by (*force intro*!: *sigma_sets.Compl simp*: *Diff_Int_distrib ac_simps*)
  next
    case (*Union a*)
    then show *?case*
      by (*auto intro*!: *sigma_sets.Union*
          simp add: *UN_extend_simps simp del*: *UN_simps*)
  qed (*auto intro*!: *sigma_sets.intros*(*2*−))
  then show $x \in$ *sigma_sets A* (($\cap$) *A* ‘ *st*)
    using ‹$A \subseteq sp$› **by** (*simp add*: *Int_absorb2*)
next
  fix *x* **assume** $x \in$ *sigma_sets A* (($\cap$) *A* ‘ *st*)
  then show $x \in (\cap)$ *A* ‘ *sigma_sets sp st*
  proof *induct*
    case (*Compl a*)
    then obtain *x* **where** $a = A \cap x$ $x \in$ *sigma_sets sp st* **by** *auto*
    then show *?case* **using** ‹$A \subseteq sp$›
      by (*force simp add*: *image_iff intro*!: *bexI*[*of* _ *sp* − *x*] *sigma_sets.Compl*)
  next
    case (*Union a*)
    then have $\forall i.\ \exists x.\ x \in$ *sigma_sets sp st* $\wedge$ $a\ i = A \cap x$
      by (*auto simp*: *image_iff Bex_def*)
    from *choice*[*OF this*] **guess** *f* **..**
    then show *?case*

```
    by (auto intro!: bexI[of _ (⋃x. f x)] sigma_sets.Union
          simp add: image_iff)
  qed (auto intro!: sigma_sets.intros(2−))
qed

lemma sigma_sets_empty_eq: sigma_sets A {} = {{}, A}
proof (intro set_eqI iffI)
  fix a assume a ∈ sigma_sets A {} then show a ∈ {{}, A}
    by induct blast+
qed (auto intro: sigma_sets.Empty sigma_sets_top)

lemma sigma_sets_single[simp]: sigma_sets A {A} = {{}, A}
proof (intro set_eqI iffI)
  fix x assume x ∈ sigma_sets A {A}
  then show x ∈ {{}, A}
    by induct blast+
next
  fix x assume x ∈ {{}, A}
  then show x ∈ sigma_sets A {A}
    by (auto intro: sigma_sets.Empty sigma_sets_top)
qed

lemma sigma_sets_sigma_sets_eq:
  M ⊆ Pow S =⇒ sigma_sets S (sigma_sets S M) = sigma_sets S M
  by (rule sigma_algebra.sigma_sets_eq[OF sigma_algebra_sigma_sets, of M S]) auto

lemma sigma_sets_singleton:
  assumes X ⊆ S
  shows sigma_sets S { X } = { {}, X, S − X, S }
proof −
  interpret sigma_algebra S { {}, X, S − X, S }
    by (rule sigma_algebra_single_set) fact
  have sigma_sets S { X } ⊆ sigma_sets S { {}, X, S − X, S }
    by (rule sigma_sets_subseteq) simp
  moreover have . . . = { {}, X, S − X, S }
    using sigma_sets_eq by simp
  moreover
  { fix A assume A ∈ { {}, X, S − X, S }
    then have A ∈ sigma_sets S { X }
      by (auto intro: sigma_sets.intros(2−) sigma_sets_top) }
  ultimately have sigma_sets S { X } = sigma_sets S { {}, X, S − X, S }
    by (intro antisym) auto
  with sigma_sets_eq show ?thesis by simp
qed

lemma restricted_sigma:
  assumes S: S ∈ sigma_sets Ω M and M: M ⊆ Pow Ω
  shows algebra.restricted_space (sigma_sets Ω M) S =
    sigma_sets S (algebra.restricted_space M S)
```

```
proof -
  from S sigma_sets_into_sp[OF M]
  have S ∈ sigma_sets Ω M S ⊆ Ω by auto
  from sigma_sets_Int[OF this]
  show ?thesis by simp
qed

lemma sigma_sets_vimage_commute:
  assumes X: X ∈ Ω → Ω'
  shows {X -` A ∩ Ω |A. A ∈ sigma_sets Ω' M'}
      = sigma_sets Ω {X -` A ∩ Ω |A. A ∈ M'} (is ?L = ?R)
proof
  show ?L ⊆ ?R
  proof clarify
    fix A assume A ∈ sigma_sets Ω' M'
    then show X -` A ∩ Ω ∈ ?R
    proof induct
      case Empty then show ?case
        by (auto intro!: sigma_sets.Empty)
    next
      case (Compl B)
      have [simp]: X -` (Ω' - B) ∩ Ω = Ω - (X -` B ∩ Ω)
        by (auto simp add: funcset_mem [OF X])
      with Compl show ?case
        by (auto intro!: sigma_sets.Compl)
    next
      case (Union F)
      then show ?case
        by (auto simp add: vimage_UN UN_extend_simps(4) simp del: UN_simps
                intro!: sigma_sets.Union)
    qed auto
  qed
  show ?R ⊆ ?L
  proof clarify
    fix A assume A ∈ ?R
    then show ∃B. A = X -` B ∩ Ω ∧ B ∈ sigma_sets Ω' M'
    proof induct
      case (Basic B) then show ?case by auto
    next
      case Empty then show ?case
        by (auto intro!: sigma_sets.Empty exI[of _ {}])
    next
      case (Compl B)
      then obtain A where A: B = X -` A ∩ Ω A ∈ sigma_sets Ω' M' by auto
      then have [simp]: Ω - B = X -` (Ω' - A) ∩ Ω
        by (auto simp add: funcset_mem [OF X])
      with A(2) show ?case
        by (auto intro: sigma_sets.Compl)
    next
```

```
      case (Union F)
      then have ∀ i. ∃ B. F i = X −‘ B ∩ Ω ∧ B ∈ sigma_sets Ω' M' by auto
      from choice[OF this] guess A .. note A = this
      with A show ?case
        by (auto simp: vimage_UN[symmetric] intro: sigma_sets.Union)
    qed
  qed
qed

lemma (in ring_of_sets) UNION_in_sets:
  fixes A:: nat ⇒ 'a set
  assumes A: range A ⊆ M
  shows (⋃ i∈{0..<n}. A i) ∈ M
proof (induct n)
  case 0 show ?case by simp
next
  case (Suc n)
  thus ?case
    by (simp add: atLeastLessThanSuc) (metis A Un UNIV_I image_subset_iff)
qed

lemma (in ring_of_sets) range_disjointed_sets:
  assumes A: range A ⊆ M
  shows range (disjointed A) ⊆ M
proof (auto simp add: disjointed_def)
  fix n
  show A n − (⋃ i∈{0..<n}. A i) ∈ M using UNION_in_sets
    by (metis A Diff UNIV_I image_subset_iff)
qed

lemma (in algebra) range_disjointed_sets':
  range A ⊆ M ⟹ range (disjointed A) ⊆ M
  using range_disjointed_sets .

lemma sigma_algebra_disjoint_iff:
  sigma_algebra Ω M ⟷ algebra Ω M ∧
    (∀ A. range A ⊆ M ⟶ disjoint_family A ⟶ (⋃ i::nat. A i) ∈ M)
proof (auto simp add: sigma_algebra_iff)
  fix A :: nat ⇒ 'a set
  assume M: algebra Ω M
     and A: range A ⊆ M
     and UnA: ∀ A. range A ⊆ M ⟶ disjoint_family A ⟶ (⋃ i::nat. A i) ∈ M
  hence range (disjointed A) ⊆ M ⟶
        disjoint_family (disjointed A) ⟶
        (⋃ i. disjointed A i) ∈ M by blast
  hence (⋃ i. disjointed A i) ∈ M
   by (simp add: algebra.range_disjointed_sets'[of Ω] M A disjoint_family_disjointed)
  thus (⋃ i::nat. A i) ∈ M by (simp add: UN_disjointed_eq)
qed
```

Ring generated by a semiring

definition (**in** *semiring_of_sets*) *generated_ring* :: *'a set set* **where**
generated_ring = { ⋃ *C* | *C*. *C* ⊆ *M* ∧ *finite C* ∧ *disjoint C* }

lemma (**in** *semiring_of_sets*) *generated_ringE*[*elim?*]:
assumes *a* ∈ *generated_ring*
obtains *C* **where** *finite C disjoint C C* ⊆ *M a* = ⋃ *C*
using *assms* **unfolding** *generated_ring_def* **by** *auto*

lemma (**in** *semiring_of_sets*) *generated_ringI*[*intro?*]:
assumes *finite C disjoint C C* ⊆ *M a* = ⋃ *C*
shows *a* ∈ *generated_ring*
using *assms* **unfolding** *generated_ring_def* **by** *auto*

lemma (**in** *semiring_of_sets*) *generated_ringI_Basic*:
A ∈ *M* ⟹ *A* ∈ *generated_ring*
by (*rule generated_ringI*[*of* {*A*}]) (*auto simp*: *disjoint_def*)

lemma (**in** *semiring_of_sets*) *generated_ring_disjoint_Un*[*intro*]:
assumes *a*: *a* ∈ *generated_ring* **and** *b*: *b* ∈ *generated_ring*
and *a* ∩ *b* = {}
shows *a* ∪ *b* ∈ *generated_ring*
proof −
from *a* **guess** *Ca* .. **note** *Ca* = *this*
from *b* **guess** *Cb* .. **note** *Cb* = *this*
show *?thesis*
proof
show *disjoint* (*Ca* ∪ *Cb*)
using ‹*a* ∩ *b* = {}› *Ca Cb* **by** (*auto intro*!: *disjoint_union*)
qed (*insert Ca Cb*, *auto*)
qed

lemma (**in** *semiring_of_sets*) *generated_ring_empty*: {} ∈ *generated_ring*
by (*auto simp*: *generated_ring_def disjoint_def*)

lemma (**in** *semiring_of_sets*) *generated_ring_disjoint_Union*:
assumes *finite A* **shows** *A* ⊆ *generated_ring* ⟹ *disjoint A* ⟹ ⋃ *A* ∈ *generated_ring*
using *assms* **by** (*induct A*) (*auto simp*: *disjoint_def intro*!: *generated_ring_disjoint_Un generated_ring_empty*)

lemma (**in** *semiring_of_sets*) *generated_ring_disjoint_UNION*:
finite I ⟹ *disjoint* (*A* ' *I*) ⟹ (⋀*i*. *i* ∈ *I* ⟹ *A i* ∈ *generated_ring*) ⟹ ⋃(*A* ' *I*) ∈ *generated_ring*
by (*intro generated_ring_disjoint_Union*) *auto*

lemma (**in** *semiring_of_sets*) *generated_ring_Int*:
assumes *a*: *a* ∈ *generated_ring* **and** *b*: *b* ∈ *generated_ring*
shows *a* ∩ *b* ∈ *generated_ring*

```
proof –
  from a guess Ca .. note Ca = this
  from b guess Cb .. note Cb = this
  define C where C = (λ(a,b). a ∩ b)‘ (Ca×Cb)
  show ?thesis
  proof
    show disjoint C
    proof (simp add: disjoint_def C_def, intro ballI impI)
      fix a1 b1 a2 b2 assume sets: a1 ∈ Ca b1 ∈ Cb a2 ∈ Ca b2 ∈ Cb
      assume a1 ∩ b1 ≠ a2 ∩ b2
      then have a1 ≠ a2 ∨ b1 ≠ b2 by auto
      then show (a1 ∩ b1) ∩ (a2 ∩ b2) = {}
      proof
        assume a1 ≠ a2
        with sets Ca have a1 ∩ a2 = {}
          by (auto simp: disjoint_def)
        then show ?thesis by auto
      next
        assume b1 ≠ b2
        with sets Cb have b1 ∩ b2 = {}
          by (auto simp: disjoint_def)
        then show ?thesis by auto
      qed
    qed
  qed (insert Ca Cb, auto simp: C_def)
qed

lemma (in semiring_of_sets) generated_ring_Inter:
  assumes finite A A ≠ {} shows A ⊆ generated_ring ⟹ ⋂A ∈ generated_ring
  using assms by (induct A rule: finite_ne_induct) (auto intro: generated_ring_Int)

lemma (in semiring_of_sets) generated_ring_INTER:
  finite I ⟹ I ≠ {} ⟹ (⋀i. i ∈ I ⟹ A i ∈ generated_ring) ⟹ ⋂(A ‘ I) ∈
generated_ring
  by (intro generated_ring_Inter) auto

lemma (in semiring_of_sets) generating_ring:
  ring_of_sets Ω generated_ring
proof (rule ring_of_setsI)
  let ?R = generated_ring
  show ?R ⊆ Pow Ω
    using sets_into_space by (auto simp: generated_ring_def generated_ring_empty)
  show {} ∈ ?R by (rule generated_ring_empty)

  { fix a assume a: a ∈ ?R then guess Ca .. note Ca = this
    fix b assume b: b ∈ ?R then guess Cb .. note Cb = this

    show a − b ∈ ?R
    proof cases
```

assume *Cb* = {} **with** *Cb* ‹*a* ∈ *?R*› **show** *?thesis*
  by *simp*
next
  assume *Cb* ≠ {}
  with *Ca Cb* **have** $a - b = (\bigcup a' \in Ca. \bigcap b' \in Cb.\ a' - b')$ **by** *auto*
  also have ... ∈ *?R*
  proof (*intro generated_ring_INTER generated_ring_disjoint_UNION*)
    fix *a b* **assume** *a* ∈ *Ca* *b* ∈ *Cb*
    with *Ca Cb Diff_cover*[*of a b*] **show** *a* − *b* ∈ *?R*
      by (*auto simp add*: *generated_ring_def*)
        (*metis DiffI Diff_eq_empty_iff empty_iff*)
  next
    show *disjoint* $((\lambda a'. \bigcap b' \in Cb.\ a' - b')$'*Ca*)
      using *Ca* **by** (*auto simp add*: *disjoint_def* ‹*Cb* ≠ {}›)
  next
    show *finite Ca finite Cb Cb* ≠ {} **by** *fact+*
  qed
  finally show *a* − *b* ∈ *?R* **.**
qed }
note *Diff* = *this*

fix *a b* **assume** *sets*: *a* ∈ *?R* *b* ∈ *?R*
have $a \cup b = (a - b) \cup (a \cap b) \cup (b - a)$ **by** *auto*
also have ... ∈ *?R*
  by (*intro sets generated_ring_disjoint_Un generated_ring_Int Diff*) *auto*
finally show *a* ∪ *b* ∈ *?R* **.**
qed

lemma (**in** *semiring_of_sets*) *sigma_sets_generated_ring_eq*: *sigma_sets* Ω *generated_ring* = *sigma_sets* Ω *M*
proof
  interpret *M*: *sigma_algebra* Ω *sigma_sets* Ω *M*
    using *space_closed* **by** (*rule sigma_algebra_sigma_sets*)
  show *sigma_sets* Ω *generated_ring* ⊆ *sigma_sets* Ω *M*
    by (*blast intro*!: *sigma_sets_mono elim*: *generated_ringE*)
qed (*auto intro*!: *generated_ringI_Basic sigma_sets_mono*)

A Two-Element Series

definition *binaryset* :: *'a set* ⇒ *'a set* ⇒ *nat* ⇒ *'a set*
  where *binaryset A B* = (λ*x*. {})(*0* := *A*, *Suc 0* := *B*)

lemma *range_binaryset_eq*: *range*(*binaryset A B*) = {*A*,*B*,{}}
  apply (*simp add*: *binaryset_def*)
  apply (*rule set_eqI*)
  apply (*auto simp add*: *image_iff*)
  done

lemma *UN_binaryset_eq*: $(\bigcup i.\ binaryset\ A\ B\ i) = A \cup B$

by (*simp add*: *range_binaryset_eq cong del*: *SUP_cong_simp*)

Closed CDI

definition *closed_cdi* :: *'a set* ⇒ *'a set set* ⇒ *bool* **where**
closed_cdi Ω *M* ⟷
M ⊆ *Pow* Ω &
(∀ *s* ∈ *M*. Ω − *s* ∈ *M*) &
(∀ *A*. (*range A* ⊆ *M*) & (*A 0* = {}) & (∀ *n*. *A n* ⊆ *A* (*Suc n*)) ⟶
(⋃ *i*. *A i*) ∈ *M*) &
(∀ *A*. (*range A* ⊆ *M*) & *disjoint_family A* ⟶ (⋃ *i*::*nat*. *A i*) ∈ *M*)

inductive_set
smallest_ccdi_sets :: *'a set* ⇒ *'a set set* ⇒ *'a set set*
for Ω *M*
where
Basic [*intro*]:
a ∈ *M* ⟹ *a* ∈ *smallest_ccdi_sets* Ω *M*
| *Compl* [*intro*]:
a ∈ *smallest_ccdi_sets* Ω *M* ⟹ Ω − *a* ∈ *smallest_ccdi_sets* Ω *M*
| *Inc*:
range A ∈ *Pow*(*smallest_ccdi_sets* Ω *M*) ⟹ *A 0* = {} ⟹ (⋀*n*. *A n* ⊆ *A*
(*Suc n*))
⟹ (⋃ *i*. *A i*) ∈ *smallest_ccdi_sets* Ω *M*
| *Disj*:
range A ∈ *Pow*(*smallest_ccdi_sets* Ω *M*) ⟹ *disjoint_family A*
⟹ (⋃ *i*::*nat*. *A i*) ∈ *smallest_ccdi_sets* Ω *M*

lemma (**in** *subset_class*) *smallest_closed_cdi1*: *M* ⊆ *smallest_ccdi_sets* Ω *M*
by *auto*

lemma (**in** *subset_class*) *smallest_ccdi_sets*: *smallest_ccdi_sets* Ω *M* ⊆ *Pow* Ω
apply (*rule subsetI*)
apply (*erule smallest_ccdi_sets.induct*)
apply (*auto intro*: *range_subsetD dest*: *sets_into_space*)
done

lemma (**in** *subset_class*) *smallest_closed_cdi2*: *closed_cdi* Ω (*smallest_ccdi_sets* Ω
M)
apply (*auto simp add*: *closed_cdi_def smallest_ccdi_sets*)
apply (*blast intro*: *smallest_ccdi_sets.Inc smallest_ccdi_sets.Disj*) +
done

lemma *closed_cdi_subset*: *closed_cdi* Ω *M* ⟹ *M* ⊆ *Pow* Ω
by (*simp add*: *closed_cdi_def*)

lemma *closed_cdi_Compl*: *closed_cdi* Ω *M* ⟹ *s* ∈ *M* ⟹ Ω − *s* ∈ *M*
by (*simp add*: *closed_cdi_def*)

lemma *closed_cdi_Inc*:
closed_cdi Ω *M* $\Longrightarrow$ *range A* $\subseteq$ *M* $\Longrightarrow$ *A 0* = {} $\Longrightarrow$ (!!*n*. *A n* $\subseteq$ *A* (*Suc n*)) $\Longrightarrow$ ($\bigcup$*i*. *A i*) $\in$ *M*
by (*simp add*: *closed_cdi_def*)

lemma *closed_cdi_Disj*:
closed_cdi Ω *M* $\Longrightarrow$ *range A* $\subseteq$ *M* $\Longrightarrow$ *disjoint_family A* $\Longrightarrow$ ($\bigcup$*i*::*nat*. *A i*) $\in$ *M*
by (*simp add*: *closed_cdi_def*)

lemma *closed_cdi_Un*:
assumes *cdi*: *closed_cdi* Ω *M* **and** *empty*: {} $\in$ *M*
and *A*: *A* $\in$ *M* **and** *B*: *B* $\in$ *M*
and *disj*: *A* $\cap$ *B* = {}
shows *A* $\cup$ *B* $\in$ *M*
proof –
have *ra*: *range* (*binaryset A B*) $\subseteq$ *M*
by (*simp add*: *range_binaryset_eq empty A B*)
have *di*: *disjoint_family* (*binaryset A B*) **using** *disj*
by (*simp add*: *disjoint_family_on_def binaryset_def Int_commute*)
from *closed_cdi_Disj* [*OF cdi ra di*]
show *?thesis*
by (*simp add*: *UN_binaryset_eq*)
qed

lemma (**in** *algebra*) *smallest_ccdi_sets_Un*:
assumes *A*: *A* $\in$ *smallest_ccdi_sets* Ω *M* **and** *B*: *B* $\in$ *smallest_ccdi_sets* Ω *M*
and *disj*: *A* $\cap$ *B* = {}
shows *A* $\cup$ *B* $\in$ *smallest_ccdi_sets* Ω *M*
proof –
have *ra*: *range* (*binaryset A B*) $\in$ *Pow* (*smallest_ccdi_sets* Ω *M*)
by (*simp add*: *range_binaryset_eq A B smallest_ccdi_sets.Basic*)
have *di*: *disjoint_family* (*binaryset A B*) **using** *disj*
by (*simp add*: *disjoint_family_on_def binaryset_def Int_commute*)
from *Disj* [*OF ra di*]
show *?thesis*
by (*simp add*: *UN_binaryset_eq*)
qed

lemma (**in** *algebra*) *smallest_ccdi_sets_Int1*:
assumes *a*: *a* $\in$ *M*
shows *b* $\in$ *smallest_ccdi_sets* Ω *M* $\Longrightarrow$ *a* $\cap$ *b* $\in$ *smallest_ccdi_sets* Ω *M*
proof (*induct rule*: *smallest_ccdi_sets.induct*)
case (*Basic x*)
thus *?case*
by (*metis a Int smallest_ccdi_sets.Basic*)
next
case (*Compl x*)
have $a \cap (\Omega - x) = \Omega - ((\Omega - a) \cup (a \cap x))$
by *blast*

also have ... ∈ *smallest_ccdi_sets* Ω *M*
  by (*metis smallest_ccdi_sets.Compl a Compl*(*2*) *Diff_Int2 Diff_Int_distrib2 Diff_disjoint Int_Diff Int_empty_right smallest_ccdi_sets_Un smallest_ccdi_sets.Basic smallest_ccdi_sets.Compl*)
finally show *?case* **.**
next
  case (*Inc A*)
  have *1*: (⋃*i*. (λ*i*. *a* ∩ *A i*) *i*) = *a* ∩ (⋃*i*. *A i*)
    by *blast*
  have *range* (λ*i*. *a* ∩ *A i*) ∈ *Pow*(*smallest_ccdi_sets* Ω *M*) **using** *Inc*
    by *blast*
  moreover have (λ*i*. *a* ∩ *A i*) *0* = {}
    by (*simp add*: *Inc*)
  moreover have !!*n*. (λ*i*. *a* ∩ *A i*) *n* ⊆ (λ*i*. *a* ∩ *A i*) (*Suc n*) **using** *Inc*
    by *blast*
  ultimately have *2*: (⋃*i*. (λ*i*. *a* ∩ *A i*) *i*) ∈ *smallest_ccdi_sets* Ω *M*
    by (*rule smallest_ccdi_sets.Inc*)
  show *?case*
    by (*metis 1 2*)
next
  case (*Disj A*)
  have *1*: (⋃*i*. (λ*i*. *a* ∩ *A i*) *i*) = *a* ∩ (⋃*i*. *A i*)
    by *blast*
  have *range* (λ*i*. *a* ∩ *A i*) ∈ *Pow*(*smallest_ccdi_sets* Ω *M*) **using** *Disj*
    by *blast*
  moreover have *disjoint_family* (λ*i*. *a* ∩ *A i*) **using** *Disj*
    by (*auto simp add*: *disjoint_family_on_def*)
  ultimately have *2*: (⋃*i*. (λ*i*. *a* ∩ *A i*) *i*) ∈ *smallest_ccdi_sets* Ω *M*
    by (*rule smallest_ccdi_sets.Disj*)
  show *?case*
    by (*metis 1 2*)
qed

lemma (**in** *algebra*) *smallest_ccdi_sets_Int*:
  assumes *b*: *b* ∈ *smallest_ccdi_sets* Ω *M*
  shows *a* ∈ *smallest_ccdi_sets* Ω *M* ⟹ *a* ∩ *b* ∈ *smallest_ccdi_sets* Ω *M*
proof (*induct rule*: *smallest_ccdi_sets.induct*)
  case (*Basic x*)
  thus *?case*
    by (*metis b smallest_ccdi_sets_Int1*)
next
  case (*Compl x*)
  have (Ω − *x*) ∩ *b* = Ω − (*x* ∩ *b* ∪ (Ω − *b*))
    by *blast*
  also have ... ∈ *smallest_ccdi_sets* Ω *M*
    by (*metis Compl*(*2*) *Diff_disjoint Int_Diff Int_commute Int_empty_right b smallest_ccdi_sets.Compl smallest_ccdi_sets_Un*)
  finally show *?case* **.**

next
  case (*Inc A*)
  have *1*: $(\bigcup i.\ (\lambda i.\ A\ i \cap b)\ i) = (\bigcup i.\ A\ i) \cap b$
    by *blast*
  have *range* $(\lambda i.\ A\ i \cap b) \in Pow(smallest\_ccdi\_sets\ \Omega\ M)$ **using** *Inc*
    by *blast*
  moreover have $(\lambda i.\ A\ i \cap b)\ 0 = \{\}$
    by (*simp add*: *Inc*)
  moreover have $!!n.\ (\lambda i.\ A\ i \cap b)\ n \subseteq (\lambda i.\ A\ i \cap b)\ (Suc\ n)$ **using** *Inc*
    by *blast*
  ultimately have *2*: $(\bigcup i.\ (\lambda i.\ A\ i \cap b)\ i) \in smallest\_ccdi\_sets\ \Omega\ M$
    by (*rule smallest_ccdi_sets.Inc*)
  show *?case*
    by (*metis 1 2*)
next
  case (*Disj A*)
  have *1*: $(\bigcup i.\ (\lambda i.\ A\ i \cap b)\ i) = (\bigcup i.\ A\ i) \cap b$
    by *blast*
  have *range* $(\lambda i.\ A\ i \cap b) \in Pow(smallest\_ccdi\_sets\ \Omega\ M)$ **using** *Disj*
    by *blast*
  moreover have *disjoint_family* $(\lambda i.\ A\ i \cap b)$ **using** *Disj*
    by (*auto simp add*: *disjoint_family_on_def*)
  ultimately have *2*: $(\bigcup i.\ (\lambda i.\ A\ i \cap b)\ i) \in smallest\_ccdi\_sets\ \Omega\ M$
    by (*rule smallest_ccdi_sets.Disj*)
  show *?case*
    by (*metis 1 2*)
qed

lemma (**in** *algebra*) *sigma_property_disjoint_lemma*:
  assumes *sbC*: $M \subseteq C$
      and *ccdi*: *closed_cdi* Ω *C*
  shows *sigma_sets* $\Omega\ M \subseteq C$
proof −
  have *smallest_ccdi_sets* $\Omega\ M \in \{B\ .\ M \subseteq B \wedge sigma\_algebra\ \Omega\ B\}$
    apply (*auto simp add*: *sigma_algebra_disjoint_iff algebra_iff_Int smallest_ccdi_sets_Int*)
    apply (*metis Union_Pow_eq Union_upper subsetD smallest_ccdi_sets*)
    apply (*blast intro*: *smallest_ccdi_sets.Disj*)
    done
  hence *sigma_sets* $(\Omega)\ (M) \subseteq smallest\_ccdi\_sets\ \Omega\ M$
    by *clarsimp*
      (*drule sigma_algebra.sigma_sets_subset* [**where** *a*=*M*], *auto*)
  also have ... $\subseteq C$
    proof
      fix *x*
      assume *x*: $x \in smallest\_ccdi\_sets\ \Omega\ M$
      thus $x \in C$
        proof (*induct rule*: *smallest_ccdi_sets.induct*)
          case (*Basic x*)

```
        thus ?case
          by (metis Basic subsetD sbC)
      next
        case (Compl x)
        thus ?case
          by (blast intro: closed_cdi_Compl [OF ccdi, simplified])
      next
        case (Inc A)
        thus ?case
             by (auto intro: closed_cdi_Inc [OF ccdi, simplified])
      next
        case (Disj A)
        thus ?case
             by (auto intro: closed_cdi_Disj [OF ccdi, simplified])
      qed
    qed
  finally show ?thesis .
qed

lemma (in algebra) sigma_property_disjoint:
  assumes sbC: M ⊆ C
      and compl: !!s. s ∈ C ∩ sigma_sets (Ω) (M) ⟹ Ω − s ∈ C
      and inc: !!A. range A ⊆ C ∩ sigma_sets (Ω) (M)
                    ⟹ A 0 = {} ⟹ (!!n. A n ⊆ A (Suc n))
                    ⟹ (⋃ i. A i) ∈ C
      and disj: !!A. range A ⊆ C ∩ sigma_sets (Ω) (M)
                     ⟹ disjoint_family A ⟹ (⋃ i::nat. A i) ∈ C
  shows sigma_sets (Ω) (M) ⊆ C
proof −
  have sigma_sets (Ω) (M) ⊆ C ∩ sigma_sets (Ω) (M)
    proof (rule sigma_property_disjoint_lemma)
      show M ⊆ C ∩ sigma_sets (Ω) (M)
        by (metis Int_greatest Set.subsetI sbC sigma_sets.Basic)
    next
      show closed_cdi Ω (C ∩ sigma_sets (Ω) (M))
        by (simp add: closed_cdi_def compl inc disj)
           (metis PowI Set.subsetI le_infI2 sigma_sets_into_sp space_closed
             IntE sigma_sets.Compl range_subsetD sigma_sets.Union)
    qed
  thus ?thesis
    by blast
qed
```

Dynkin systems

```
locale Dynkin_system = subset_class +
  assumes space: Ω ∈ M
    and   compl[intro!]: ⋀A. A ∈ M ⟹ Ω − A ∈ M
    and   UN[intro!]: ⋀A. disjoint_family A ⟹ range A ⊆ M
```

$\Longrightarrow (\bigcup i::nat.\ A\ i) \in M$

lemma (**in** *Dynkin_system*) *empty*[*intro*, *simp*]: $\{\} \in M$
  using *space compl*[*of* Ω] **by** *simp*

lemma (**in** *Dynkin_system*) *diff*:
  assumes *sets*: $D \in M$ $E \in M$ **and** $D \subseteq E$
  shows $E - D \in M$
proof –
  let *?f* = $\lambda x.$ *if* $x = 0$ *then* D *else if* $x = Suc\ 0$ *then* $\Omega - E$ *else* $\{\}$
  have *range ?f* = $\{D, \Omega - E, \{\}\}$
    by (*auto simp*: *image_iff*)
  moreover have $D \cup (\Omega - E) = (\bigcup i.\ ?f\ i)$
    by (*auto simp*: *image_iff split*: *if_split_asm*)
  moreover
  have *disjoint_family ?f* **unfolding** *disjoint_family_on_def*
    using ‹$D \in M$›[*THEN sets_into_space*] ‹$D \subseteq E$› **by** *auto*
  ultimately have $\Omega - (D \cup (\Omega - E)) \in M$
    using *sets UN* **by** *auto fastforce*
  also have $\Omega - (D \cup (\Omega - E)) = E - D$
    using *assms sets_into_space* **by** *auto*
  finally show *?thesis* **.**
qed

lemma *Dynkin_systemI*:
  assumes $\bigwedge A.\ A \in M \Longrightarrow A \subseteq \Omega$ $\Omega \in M$
  assumes $\bigwedge A.\ A \in M \Longrightarrow \Omega - A \in M$
  assumes $\bigwedge A.$ *disjoint_family* $A \Longrightarrow$ *range* $A \subseteq M$
    $\Longrightarrow (\bigcup i::nat.\ A\ i) \in M$
  shows *Dynkin_system* Ω M
  using *assms* **by** (*auto simp*: *Dynkin_system_def Dynkin_system_axioms_def subset_class_def*)

lemma *Dynkin_systemI'*:
  assumes *1*: $\bigwedge A.\ A \in M \Longrightarrow A \subseteq \Omega$
  assumes *empty*: $\{\} \in M$
  assumes *Diff*: $\bigwedge A.\ A \in M \Longrightarrow \Omega - A \in M$
  assumes *2*: $\bigwedge A.$ *disjoint_family* $A \Longrightarrow$ *range* $A \subseteq M$
    $\Longrightarrow (\bigcup i::nat.\ A\ i) \in M$
  shows *Dynkin_system* Ω M
proof –
  from *Diff*[*OF empty*] **have** $\Omega \in M$ **by** *auto*
  from *1 this Diff 2* **show** *?thesis*
    by (*intro Dynkin_systemI*) *auto*
qed

lemma *Dynkin_system_trivial*:
  shows *Dynkin_system* A (*Pow* A)
  by (*rule Dynkin_systemI*) *auto*

lemma *sigma_algebra_imp_Dynkin_system*:
  assumes *sigma_algebra* Ω *M* **shows** *Dynkin_system* Ω *M*
proof −
  interpret *sigma_algebra* Ω *M* **by** *fact*
  show *?thesis* **using** *sets_into_space* **by** (*fastforce intro*!: *Dynkin_systemI*)
qed

Intersection sets systems

definition *Int_stable* :: *'a set set* ⇒ *bool* **where**
Int_stable M ⟷ (∀ *a* ∈ *M*. ∀ *b* ∈ *M*. *a* ∩ *b* ∈ *M*)

lemma (**in** *algebra*) *Int_stable*: *Int_stable M*
  unfolding *Int_stable_def* **by** *auto*

lemma *Int_stableI_image*:
  (⋀*i j*. *i* ∈ *I* ⟹ *j* ∈ *I* ⟹ ∃ *k*∈*I*. *A i* ∩ *A j* = *A k*) ⟹ *Int_stable* (*A* ' *I*)
  by (*auto simp*: *Int_stable_def image_def*)

lemma *Int_stableI*:
  (⋀*a b*. *a* ∈ *A* ⟹ *b* ∈ *A* ⟹ *a* ∩ *b* ∈ *A*) ⟹ *Int_stable A*
  unfolding *Int_stable_def* **by** *auto*

lemma *Int_stableD*:
  Int_stable M ⟹ *a* ∈ *M* ⟹ *b* ∈ *M* ⟹ *a* ∩ *b* ∈ *M*
  unfolding *Int_stable_def* **by** *auto*

lemma (**in** *Dynkin_system*) *sigma_algebra_eq_Int_stable*:
  sigma_algebra Ω *M* ⟷ *Int_stable M*
proof
  assume *sigma_algebra* Ω *M* **then show** *Int_stable M*
    unfolding *sigma_algebra_def* **using** *algebra.Int_stable* **by** *auto*
next
  assume *Int_stable M*
  show *sigma_algebra* Ω *M*
    unfolding *sigma_algebra_disjoint_iff algebra_iff_Un*
  proof (*intro conjI ballI allI impI*)
    show *M* ⊆ *Pow* (Ω) **using** *sets_into_space* **by** *auto*
  next
    fix *A B* **assume** *A* ∈ *M B* ∈ *M*
    then have *A* ∪ *B* = Ω − ((Ω − *A*) ∩ (Ω − *B*))
          Ω − *A* ∈ *M* Ω − *B* ∈ *M*
      using *sets_into_space* **by** *auto*
    then show *A* ∪ *B* ∈ *M*
      using ⟨*Int_stable M*⟩ **unfolding** *Int_stable_def* **by** *auto*
  qed *auto*
qed

Smallest Dynkin systems

definition *Dynkin* :: *'a set* ⇒ *'a set set* ⇒ *'a set set* **where**
  Dynkin Ω *M* = (⋂{*D*. *Dynkin_system* Ω *D* ∧ *M* ⊆ *D*})

lemma *Dynkin_system_Dynkin*:
  assumes *M* ⊆ *Pow* (Ω)
  shows *Dynkin_system* Ω (*Dynkin* Ω *M*)
proof (*rule Dynkin_systemI*)
  fix *A* **assume** *A* ∈ *Dynkin* Ω *M*
  moreover
  { fix *D* **assume** *A* ∈ *D* **and** *d*: *Dynkin_system* Ω *D*
    then have *A* ⊆ Ω **by** (*auto simp*: *Dynkin_system_def subset_class_def*) **}**
  moreover have {*D*. *Dynkin_system* Ω *D* ∧ *M* ⊆ *D*} ≠ {}
    using *assms Dynkin_system_trivial* **by** *fastforce*
  ultimately show *A* ⊆ Ω
    unfolding *Dynkin_def* **using** *assms*
    by *auto*
next
  show Ω ∈ *Dynkin* Ω *M*
    unfolding *Dynkin_def* **using** *Dynkin_system.space* **by** *fastforce*
next
  fix *A* **assume** *A* ∈ *Dynkin* Ω *M*
  then show Ω − *A* ∈ *Dynkin* Ω *M*
    unfolding *Dynkin_def* **using** *Dynkin_system.compl* **by** *force*
next
  fix *A* :: *nat* ⇒ *'a set*
  assume *A*: *disjoint_family A range A* ⊆ *Dynkin* Ω *M*
  show (⋃*i*. *A i*) ∈ *Dynkin* Ω *M* **unfolding** *Dynkin_def*
  proof (*simp*, *safe*)
    fix *D* **assume** *Dynkin_system* Ω *D M* ⊆ *D*
    with *A* **have** (⋃*i*. *A i*) ∈ *D*
      by (*intro Dynkin_system.UN*) (*auto simp*: *Dynkin_def*)
    then show (⋃*i*. *A i*) ∈ *D* **by** *auto*
  qed
qed

lemma *Dynkin_Basic*[*intro*]: *A* ∈ *M* ⟹ *A* ∈ *Dynkin* Ω *M*
  unfolding *Dynkin_def* **by** *auto*

lemma (**in** *Dynkin_system*) *restricted_Dynkin_system*:
  assumes *D* ∈ *M*
  shows *Dynkin_system* Ω {*Q*. *Q* ⊆ Ω ∧ *Q* ∩ *D* ∈ *M*}
proof (*rule Dynkin_systemI*, *simp_all*)
  have Ω ∩ *D* = *D*
    using ‹*D* ∈ *M*› *sets_into_space* **by** *auto*
  then show Ω ∩ *D* ∈ *M*
    using ‹*D* ∈ *M*› **by** *auto*
next
  fix *A* **assume** *A* ⊆ Ω ∧ *A* ∩ *D* ∈ *M*

moreover have $(\Omega - A) \cap D = (\Omega - (A \cap D)) - (\Omega - D)$
  by *auto*
ultimately show $(\Omega - A) \cap D \in M$
  using ‹$D \in M$› **by** (*auto intro*: *diff*)
next
fix A :: *nat* ⇒ *'a set*
assume *disjoint_family A range A* $\subseteq \{Q.\ Q \subseteq \Omega \wedge Q \cap D \in M\}$
then have $\bigwedge i.\ A\ i \subseteq \Omega$ *disjoint_family* $(\lambda i.\ A\ i \cap D)$
  range $(\lambda i.\ A\ i \cap D) \subseteq M$ $(\bigcup x.\ A\ x) \cap D = (\bigcup x.\ A\ x \cap D)$
  by ((*fastforce simp*: *disjoint_family_on_def*)+)
then show $(\bigcup x.\ A\ x) \subseteq \Omega \wedge (\bigcup x.\ A\ x) \cap D \in M$
  by (*auto simp del*: *UN_simps*)
qed

lemma (**in** *Dynkin_system*) *Dynkin_subset*:
assumes $N \subseteq M$
shows *Dynkin* Ω $N \subseteq M$
proof −
have *Dynkin_system* Ω M **..**
then have *Dynkin_system* Ω M
  using *assms* **unfolding** *Dynkin_system_def Dynkin_system_axioms_def subset_class_def* **by** *simp*
with ‹$N \subseteq M$› **show** *?thesis* **by** (*auto simp add*: *Dynkin_def*)
qed

lemma *sigma_eq_Dynkin*:
assumes *sets*: $M \subseteq$ *Pow* Ω
assumes *Int_stable M*
shows *sigma_sets* Ω M = *Dynkin* Ω M
proof −
have *Dynkin* Ω $M \subseteq$ *sigma_sets* (Ω) (M)
  using *sigma_algebra_imp_Dynkin_system*
  unfolding *Dynkin_def sigma_sets_least_sigma_algebra*[*OF sets*] **by** *auto*
moreover
interpret *Dynkin_system* Ω *Dynkin* Ω M
  using *Dynkin_system_Dynkin*[*OF sets*] **.**
have *sigma_algebra* Ω (*Dynkin* Ω M)
  unfolding *sigma_algebra_eq_Int_stable Int_stable_def*
proof (*intro* **ballI**)
  fix A B **assume** $A \in$ *Dynkin* Ω M $B \in$ *Dynkin* Ω M
  let *?D* = $\lambda E.\ \{Q.\ Q \subseteq \Omega \wedge Q \cap E \in$ *Dynkin* Ω $M\}$
  have $M \subseteq$ *?D B*
  proof
    fix E **assume** $E \in M$
    then have $M \subseteq$ *?D E* $E \in$ *Dynkin* Ω M
      using *sets_into_space* ‹*Int_stable M*› **by** (*auto simp*: *Int_stable_def*)
    then have *Dynkin* Ω $M \subseteq$ *?D E*
      using *restricted_Dynkin_system* ‹$E \in$ *Dynkin* Ω M›
      by (*intro Dynkin_system.Dynkin_subset*) *simp_all*

```
      then have B ∈ ?D E
        using ‹B ∈ Dynkin Ω M› by auto
      then have E ∩ B ∈ Dynkin Ω M
        by (subst Int_commute) simp
      then show E ∈ ?D B
        using sets ‹E ∈ M› by auto
    qed
    then have Dynkin Ω M ⊆ ?D B
      using restricted_Dynkin_system ‹B ∈ Dynkin Ω M›
      by (intro Dynkin_system.Dynkin_subset) simp_all
    then show A ∩ B ∈ Dynkin Ω M
      using ‹A ∈ Dynkin Ω M› sets_into_space by auto
  qed
  from sigma_algebra.sigma_sets_subset[OF this, of M]
  have sigma_sets (Ω) (M) ⊆ Dynkin Ω M by auto
  ultimately have sigma_sets (Ω) (M) = Dynkin Ω M by auto
  then show ?thesis
    by (auto simp: Dynkin_def)
qed

lemma (in Dynkin_system) Dynkin_idem:
  Dynkin Ω M = M
proof −
  have Dynkin Ω M = M
  proof
    show M ⊆ Dynkin Ω M
      using Dynkin_Basic by auto
    show Dynkin Ω M ⊆ M
      by (intro Dynkin_subset) auto
  qed
  then show ?thesis
    by (auto simp: Dynkin_def)
qed

lemma (in Dynkin_system) Dynkin_lemma:
  assumes Int_stable E
  and E: E ⊆ M M ⊆ sigma_sets Ω E
  shows sigma_sets Ω E = M
proof −
  have E ⊆ Pow Ω
    using E sets_into_space by force
  then have ∗: sigma_sets Ω E = Dynkin Ω E
    using ‹Int_stable E› by (rule sigma_eq_Dynkin)
  then have Dynkin Ω E = M
    using assms Dynkin_subset[OF E(1)] by simp
  with ∗ show ?thesis
    using assms by (auto simp: Dynkin_def)
qed
```

Induction rule for intersection-stable generators

The reason to introduce Dynkin-systems is the following induction rules for σ-algebras generated by a generator closed under intersection.

proposition *sigma_sets_induct_disjoint*[*consumes 3*, *case_names basic empty compl union*]:
  assumes *Int_stable G*
    and *closed*: $G \subseteq Pow\ \Omega$
    and *A*: $A \in sigma\_sets\ \Omega\ G$
  assumes *basic*: $\bigwedge A.\ A \in G \Longrightarrow P\ A$
    and *empty*: $P\ \{\}$
    and *compl*: $\bigwedge A.\ A \in sigma\_sets\ \Omega\ G \Longrightarrow P\ A \Longrightarrow P\ (\Omega - A)$
    and *union*: $\bigwedge A.\ disjoint\_family\ A \Longrightarrow range\ A \subseteq sigma\_sets\ \Omega\ G \Longrightarrow (\bigwedge i.\ P\ (A\ i)) \Longrightarrow P\ (\bigcup i{::}nat.\ A\ i)$
  shows *P A*
proof –
  let $?D = \{\ A \in sigma\_sets\ \Omega\ G.\ P\ A\ \}$
  interpret *sigma_algebra* Ω *sigma_sets* Ω *G*
    using *closed* **by** (*rule sigma_algebra_sigma_sets*)
  from *compl*[*OF _ empty*] *closed* **have** *space*: *P* Ω **by** *simp*
  interpret *Dynkin_system* Ω *?D*
    by *standard* (*auto dest*: *sets_into_space intro*!: *space compl union*)
  have *sigma_sets* Ω *G* = *?D*
    by (*rule Dynkin_lemma*) (*auto simp*: *basic* ‹*Int_stable G*›)
  with *A* **show** *?thesis* **by** *auto*
qed

6.1.2 Measure type

definition *positive* :: $'a\ set\ set \Rightarrow ('a\ set \Rightarrow ennreal) \Rightarrow bool$ **where**
  $positive\ M\ \mu \longleftrightarrow \mu\ \{\} = 0$

definition *countably_additive* :: $'a\ set\ set \Rightarrow ('a\ set \Rightarrow ennreal) \Rightarrow bool$ **where**
$countably\_additive\ M\ f \longleftrightarrow$
  $(\forall A.\ range\ A \subseteq M \longrightarrow disjoint\_family\ A \longrightarrow (\bigcup i.\ A\ i) \in M \longrightarrow$
    $(\sum i.\ f\ (A\ i)) = f\ (\bigcup i.\ A\ i))$

definition *measure_space* :: $'a\ set \Rightarrow 'a\ set\ set \Rightarrow ('a\ set \Rightarrow ennreal) \Rightarrow bool$ **where**
$measure\_space\ \Omega\ A\ \mu \longleftrightarrow$
  $sigma\_algebra\ \Omega\ A \wedge positive\ A\ \mu \wedge countably\_additive\ A\ \mu$

typedef *'a measure* =
  $\{(\Omega{::}'a\ set,\ A,\ \mu).\ (\forall a{\in}{-}A.\ \mu\ a = 0) \wedge measure\_space\ \Omega\ A\ \mu\ \}$
proof
  have *sigma_algebra UNIV* {{}, *UNIV*}
    by (*auto simp*: *sigma_algebra_iff2*)
  then show $(UNIV,\ \{\{\},\ UNIV\},\ \lambda A.\ 0) \in \{(\Omega,\ A,\ \mu).\ (\forall a{\in}{-}A.\ \mu\ a = 0) \wedge measure\_space\ \Omega\ A\ \mu\}$

by (*auto simp*: *measure_space_def positive_def countably_additive_def*)
qed

definition *space* :: *'a measure* ⇒ *'a set* **where**
space M = *fst* (*Rep_measure M*)

definition *sets* :: *'a measure* ⇒ *'a set set* **where**
sets M = *fst* (*snd* (*Rep_measure M*))

definition *emeasure* :: *'a measure* ⇒ *'a set* ⇒ *ennreal* **where**
emeasure M = *snd* (*snd* (*Rep_measure M*))

definition *measure* :: *'a measure* ⇒ *'a set* ⇒ *real* **where**
measure M A = *enn2real* (*emeasure M A*)

declare [[*coercion sets*]]

declare [[*coercion measure*]]

declare [[*coercion emeasure*]]

lemma *measure_space*: *measure_space* (*space M*) (*sets M*) (*emeasure M*)
by (*cases M*) (*auto simp*: *space_def sets_def emeasure_def Abs_measure_inverse*)

interpretation *sets*: *sigma_algebra space M sets M* **for** *M* :: *'a measure*
using *measure_space*[*of M*] **by** (*auto simp*: *measure_space_def*)

definition *measure_of* :: *'a set* ⇒ *'a set set* ⇒ (*'a set* ⇒ *ennreal*) ⇒ *'a measure*
where
measure_of Ω *A* μ =
Abs_measure (Ω, *if A* ⊆ *Pow* Ω *then sigma_sets* Ω *A else* {{}, Ω},
λ*a*. *if a* ∈ *sigma_sets* Ω *A* ∧ *measure_space* Ω (*sigma_sets* Ω *A*) μ *then* μ *a else*
0)

abbreviation *sigma* Ω *A* ≡ *measure_of* Ω *A* (λ*x*. *0*)

lemma *measure_space_0*: *A* ⊆ *Pow* Ω ⟹ *measure_space* Ω (*sigma_sets* Ω *A*) (λ*x*.
0)
unfolding *measure_space_def*
by (*auto intro*!: *sigma_algebra_sigma_sets simp*: *positive_def countably_additive_def*)

lemma *sigma_algebra_trivial*: *sigma_algebra* Ω {{}, Ω}
by *unfold_locales*(*fastforce intro*: *exI*[**where** *x*={{}}] *exI*[**where** *x*={Ω}])+

lemma *measure_space_0'*: *measure_space* Ω {{}, Ω} (λ*x*. *0*)
by(*simp add*: *measure_space_def positive_def countably_additive_def sigma_algebra_trivial*)

lemma *measure_space_closed*:
assumes *measure_space* Ω *M* μ

shows $M \subseteq$ *Pow* Ω
proof –
interpret *sigma_algebra* Ω M **using** *assms* **by**(*simp add*: *measure_space_def*)
show *?thesis* **by**(*rule space_closed*)
qed

lemma (**in** *ring_of_sets*) *positive_cong_eq*:
$(\bigwedge a.\ a \in M \Longrightarrow \mu'\ a = \mu\ a) \Longrightarrow$ *positive* $M\ \mu'$ = *positive* $M\ \mu$
by (*auto simp add*: *positive_def*)

lemma (**in** *sigma_algebra*) *countably_additive_eq*:
$(\bigwedge a.\ a \in M \Longrightarrow \mu'\ a = \mu\ a) \Longrightarrow$ *countably_additive* $M\ \mu'$ = *countably_additive* $M\ \mu$
unfolding *countably_additive_def*
by (*intro arg_cong*[**where** *f=All*] *ext*) (*auto simp add*: *countably_additive_def subset_eq*)

lemma *measure_space_eq*:
assumes *closed*: $A \subseteq$ *Pow* Ω **and** *eq*: $\bigwedge a.\ a \in$ *sigma_sets* $\Omega\ A \Longrightarrow \mu\ a = \mu'\ a$
shows *measure_space* Ω (*sigma_sets* $\Omega\ A$) μ = *measure_space* Ω (*sigma_sets* Ω A) μ'
proof –
interpret *sigma_algebra* Ω *sigma_sets* $\Omega\ A$ **using** *closed* **by** (*rule sigma_algebra_sigma_sets*)
from *positive_cong_eq*[*OF eq*, *of* $\lambda i.\ i$] *countably_additive_eq*[*OF eq*, *of* $\lambda i.\ i$]
show *?thesis*
by (*auto simp*: *measure_space_def*)
qed

lemma *measure_of_eq*:
assumes *closed*: $A \subseteq$ *Pow* Ω **and** *eq*: $(\bigwedge a.\ a \in$ *sigma_sets* $\Omega\ A \Longrightarrow \mu\ a = \mu'\ a)$
shows *measure_of* $\Omega\ A\ \mu$ = *measure_of* $\Omega\ A\ \mu'$
proof –
have *measure_space* Ω (*sigma_sets* $\Omega\ A$) μ = *measure_space* Ω (*sigma_sets* $\Omega\ A$) μ'
using *assms* **by** (*rule measure_space_eq*)
with *eq* **show** *?thesis*
by (*auto simp add*: *measure_of_def intro*!: *arg_cong*[**where** *f=Abs_measure*])
qed

lemma
shows *space_measure_of_conv*: *space* (*measure_of* $\Omega\ A\ \mu$) = Ω (**is** *?space*)
and *sets_measure_of_conv*:
sets (*measure_of* $\Omega\ A\ \mu$) = (*if* $A \subseteq$ *Pow* Ω *then sigma_sets* $\Omega\ A$ *else* $\{\{\}, \Omega\}$) (**is** *?sets*)
and *emeasure_measure_of_conv*:
emeasure (*measure_of* $\Omega\ A\ \mu$) =
(λB. *if* $B \in$ *sigma_sets* $\Omega\ A \wedge$ *measure_space* Ω (*sigma_sets* $\Omega\ A$) μ *then* $\mu\ B$ *else* *0*) (**is** *?emeasure*)
proof –

have *?space* ∧ *?sets* ∧ *?emeasure*
proof(*cases measure_space* Ω (*sigma_sets* Ω *A*) μ)
case *True*
from *measure_space_closed*[*OF this*] *sigma_sets_superset_generator*[*of A* Ω]
have *A* ⊆ *Pow* Ω **by** *simp*
hence *measure_space* Ω (*sigma_sets* Ω *A*) μ = *measure_space* Ω (*sigma_sets* Ω *A*)
(λ*a*. *if a* ∈ *sigma_sets* Ω *A then* μ *a else 0*)
by(*rule measure_space_eq*) *auto*
with *True* ‹*A* ⊆ *Pow* Ω› **show** *?thesis*
by(*simp add*: *measure_of_def space_def sets_def emeasure_def Abs_measure_inverse*)
next
case *False* **thus** *?thesis*
by(*cases A* ⊆ *Pow* Ω)(*simp_all add*: *Abs_measure_inverse measure_of_def sets_def space_def emeasure_def measure_space_0 measure_space_0′*)
qed
thus *?space ?sets ?emeasure* **by** *simp_all*
qed

lemma [*simp*]:
assumes *A*: *A* ⊆ *Pow* Ω
shows *sets_measure_of*: *sets* (*measure_of* Ω *A* μ) = *sigma_sets* Ω *A*
and *space_measure_of*: *space* (*measure_of* Ω *A* μ) = Ω
using *assms*
by(*simp_all add*: *sets_measure_of_conv space_measure_of_conv*)

lemma *space_in_measure_of*[*simp*]: Ω ∈ *sets* (*measure_of* Ω *M* μ)
by (*subst sets_measure_of_conv*) (*auto simp*: *sigma_sets_top*)

lemma (**in** *sigma_algebra*) *sets_measure_of_eq*[*simp*]: *sets* (*measure_of* Ω *M* μ) = *M*
using *space_closed* **by** (*auto intro*!: *sigma_sets_eq*)

lemma (**in** *sigma_algebra*) *space_measure_of_eq*[*simp*]: *space* (*measure_of* Ω *M* μ) = Ω
by (*rule space_measure_of_conv*)

lemma *measure_of_subset*: *M* ⊆ *Pow* Ω ⟹ *M′* ⊆ *M* ⟹ *sets* (*measure_of* Ω *M′* μ) ⊆ *sets* (*measure_of* Ω *M* μ′)
by (*auto intro*!: *sigma_sets_subseteq*)

lemma *emeasure_sigma*: *emeasure* (*sigma* Ω *A*) = (λ*x*. *0*)
unfolding *measure_of_def emeasure_def*
by (*subst Abs_measure_inverse*)
(*auto simp*: *measure_space_def positive_def countably_additive_def*
intro!: *sigma_algebra_sigma_sets sigma_algebra_trivial*)

lemma *sigma_sets_mono″*:
assumes *A* ∈ *sigma_sets C D*

assumes $B \subseteq D$
assumes $D \subseteq Pow\ C$
shows *sigma_sets* $A\ B \subseteq$ *sigma_sets* $C\ D$
proof
fix x **assume** $x \in$ *sigma_sets* $A\ B$
thus $x \in$ *sigma_sets* $C\ D$
proof *induct*
case (*Basic a*) **with** *assms* **have** $a \in D$ **by** *auto*
thus *?case* **..**
next
case *Empty* **show** *?case* **by** (*rule sigma_sets.Empty*)
next
from *assms* **have** $A \in$ *sets* (*sigma* $C\ D$) **by** (*subst sets_measure_of*[*OF* ‹$D \subseteq$ *Pow* C›])
moreover case (*Compl a*) **hence** $a \in$ *sets* (*sigma* $C\ D$) **by** (*subst sets_measure_of*[*OF* ‹$D \subseteq$ *Pow* C›])
ultimately have $A - a \in$ *sets* (*sigma* $C\ D$) **..**
thus *?case* **by** (*subst* (*asm*) *sets_measure_of*[*OF* ‹$D \subseteq$ *Pow* C›])
next
case (*Union a*)
thus *?case* **by** (*intro sigma_sets.Union*)
qed
qed

lemma *in_measure_of*[*intro*, *simp*]: $M \subseteq Pow\ \Omega \Longrightarrow A \in M \Longrightarrow A \in$ *sets* (*measure_of* $\Omega\ M\ \mu$)
by *auto*

lemma *space_empty_iff*: *space* $N = \{\} \longleftrightarrow$ *sets* $N = \{\{\}\}$
by (*metis Pow_empty Sup_bot_conv*(*1*) *cSup_singleton empty_iff sets.sigma_sets_eq sets.space_closed sigma_sets_top subset_singletonD*)

Constructing simple *'a measure*

proposition *emeasure_measure_of*:
assumes *M*: $M =$ *measure_of* $\Omega\ A\ \mu$
assumes *ms*: $A \subseteq Pow\ \Omega$ *positive* (*sets* M) μ *countably_additive* (*sets* M) μ
assumes *X*: $X \in$ *sets* M
shows *emeasure* $M\ X = \mu\ X$
proof −
interpret *sigma_algebra* Ω *sigma_sets* $\Omega\ A$ **by** (*rule sigma_algebra_sigma_sets*) *fact*
have *measure_space* Ω (*sigma_sets* $\Omega\ A$) μ
using *ms M* **by** (*simp add*: *measure_space_def sigma_algebra_sigma_sets*)
thus *?thesis* **using** *X ms*
by(*simp add*: *M emeasure_measure_of_conv sets_measure_of_conv*)
qed

lemma *emeasure_measure_of_sigma*:

assumes *ms*: *sigma_algebra* Ω *M positive M* μ *countably_additive M* μ
assumes *A*: *A* ∈ *M*
shows *emeasure* (*measure_of* Ω *M* μ) *A* = μ *A*
proof −
interpret *sigma_algebra* Ω *M* **by** *fact*
have *measure_space* Ω (*sigma_sets* Ω *M*) μ
using *ms sigma_sets_eq* **by** (*simp add*: *measure_space_def*)
thus *?thesis* **by**(*simp add*: *emeasure_measure_of_conv A*)
qed

lemma *measure_cases*[*cases type*: *measure*]:
obtains (*measure*) Ω *A* μ **where** *x* = *Abs_measure* (Ω, *A*, μ) ∀ *a*∈−*A*. μ *a* = *0* *measure_space* Ω *A* μ
by *atomize_elim* (*cases x*, *auto*)

lemma *sets_le_imp_space_le*: *sets A* ⊆ *sets B* ⟹ *space A* ⊆ *space B*
by (*auto dest*: *sets.sets_into_space*)

lemma *sets_eq_imp_space_eq*: *sets M* = *sets M'* ⟹ *space M* = *space M'*
by (*auto intro*!: *antisym sets_le_imp_space_le*)

lemma *emeasure_notin_sets*: *A* ∉ *sets M* ⟹ *emeasure M A* = *0*
by (*cases M*) (*auto simp*: *sets_def emeasure_def Abs_measure_inverse measure_space_def*)

lemma *emeasure_neq_0_sets*: *emeasure M A* ≠ *0* ⟹ *A* ∈ *sets M*
using *emeasure_notin_sets*[*of A M*] **by** *blast*

lemma *measure_notin_sets*: *A* ∉ *sets M* ⟹ *measure M A* = *0*
by (*simp add*: *measure_def emeasure_notin_sets zero_ennreal.rep_eq*)

lemma *measure_eqI*:
fixes *M N* :: *'a measure*
assumes *sets M* = *sets N* **and** *eq*: ⋀*A*. *A* ∈ *sets M* ⟹ *emeasure M A* = *emeasure N A*
shows *M* = *N*
proof (*cases M N rule*: *measure_cases*[*case_product measure_cases*])
case (*measure_measure* Ω *A* μ Ω' *A'* μ')
interpret *M*: *sigma_algebra* Ω *A* **using** *measure_measure* **by** (*auto simp*: *measure_space_def*)
interpret *N*: *sigma_algebra* Ω' *A'* **using** *measure_measure* **by** (*auto simp*: *measure_space_def*)
have *A* = *sets M A'* = *sets N*
using *measure_measure* **by** (*simp_all add*: *sets_def Abs_measure_inverse*)
with ‹*sets M* = *sets N*› **have** *AA'*: *A* = *A'* **by** *simp*
moreover from *M.top N.top M.space_closed N.space_closed AA'* **have** Ω = Ω' **by** *auto*
moreover { **fix** *B* **have** μ *B* = μ' *B*
proof *cases*
assume *B* ∈ *A*

with *eq* ‹A = *sets* M› **have** *emeasure* M B = *emeasure* N B **by** *simp*
with *measure_measure* **show** μ B = μ' B
by (*simp add*: *emeasure_def Abs_measure_inverse*)
next
assume $B \notin A$
with ‹A = *sets* M› ‹A' = *sets* N› ‹$A = A'$› **have** $B \notin$ *sets* M $B \notin$ *sets* N
by *auto*
then have *emeasure* M B = *0* *emeasure* N B = *0*
by (*simp_all add*: *emeasure_notin_sets*)
with *measure_measure* **show** μ B = μ' B
by (*simp add*: *emeasure_def Abs_measure_inverse*)
qed **}**
then have $\mu = \mu'$ **by** *auto*
ultimately show $M = N$
by (*simp add*: *measure_measure*)
qed

lemma *sigma_eqI*:
assumes [*simp*]: $M \subseteq$ *Pow* Ω $N \subseteq$ *Pow* Ω *sigma_sets* Ω M = *sigma_sets* Ω N
shows *sigma* Ω M = *sigma* Ω N
by (*rule measure_eqI*) (*simp_all add*: *emeasure_sigma*)

Measurable functions

definition *measurable* :: *'a measure* ⇒ *'b measure* ⇒ (*'a* ⇒ *'b*) *set*
(**infixr** $\rightarrow_M$ *60*) **where**
measurable A B = {$f \in$ *space* $A \rightarrow$ *space* B. $\forall y \in$ *sets* B. f −‘ $y \cap$ *space* $A \in$ *sets* A}

lemma *measurableI*:
($\bigwedge x$. $x \in$ *space* $M \Longrightarrow f\, x \in$ *space* N) $\Longrightarrow$ ($\bigwedge A$. $A \in$ *sets* $N \Longrightarrow f$ −‘ $A \cap$ *space* $M \in$ *sets* M) $\Longrightarrow$
$f \in$ *measurable* M N
by (*auto simp*: *measurable_def*)

lemma *measurable_space*:
$f \in$ *measurable* M $A \Longrightarrow x \in$ *space* $M \Longrightarrow f\, x \in$ *space* A
unfolding *measurable_def* **by** *auto*

lemma *measurable_sets*:
$f \in$ *measurable* M $A \Longrightarrow S \in$ *sets* $A \Longrightarrow f$ −‘ $S \cap$ *space* $M \in$ *sets* M
unfolding *measurable_def* **by** *auto*

lemma *measurable_sets_Collect*:
assumes f: $f \in$ *measurable* M N **and** P: {x∈*space* N. P x} ∈ *sets* N **shows** {x∈*space* M. P (f x)} ∈ *sets* M
proof −
have f −‘ {$x \in$ *space* N. P x} ∩ *space* M = {x∈*space* M. P (f x)}
using *measurable_space*[*OF* f] **by** *auto*

with *measurable_sets*[*OF f P*] **show** *?thesis*
by *simp*
qed

lemma *measurable_sigma_sets*:
assumes *B*: *sets N* = *sigma_sets* Ω *A A* ⊆ *Pow* Ω
and *f*: *f* ∈ *space M* → Ω
and *ba*: ⋀*y*. *y* ∈ *A* ⟹ (*f* −‘ *y*) ∩ *space M* ∈ *sets M*
shows *f* ∈ *measurable M N*
proof −
interpret *A*: *sigma_algebra* Ω *sigma_sets* Ω *A* **using** *B*(*2*) **by** (*rule sigma_algebra_sigma_sets*)
from *B sets.top*[*of N*] *A.top sets.space_closed*[*of N*] *A.space_closed* **have** Ω: Ω = *space N* **by** *force*

{ **fix** *X* **assume** *X* ∈ *sigma_sets* Ω *A*
then have *f* −‘ *X* ∩ *space M* ∈ *sets M* ∧ *X* ⊆ Ω
proof *induct*
case (*Basic a*) **then show** *?case*
by (*auto simp add*: *ba*) (*metis B*(*2*) *subsetD PowD*)
next
case (*Compl a*)
have [*simp*]: *f* −‘ Ω ∩ *space M* = *space M*
by (*auto simp add*: *funcset_mem* [*OF f*])
then show *?case*
by (*auto simp add*: *vimage_Diff Diff_Int_distrib2 sets.compl_sets Compl*)
next
case (*Union a*)
then show *?case*
by (*simp add*: *vimage_UN*, *simp only*: *UN_extend_simps*(*4*)) *blast*
qed *auto* }
with *f* **show** *?thesis*
by (*auto simp add*: *measurable_def B* Ω)
qed

lemma *measurable_measure_of*:
assumes *B*: *N* ⊆ *Pow* Ω
and *f*: *f* ∈ *space M* → Ω
and *ba*: ⋀*y*. *y* ∈ *N* ⟹ (*f* −‘ *y*) ∩ *space M* ∈ *sets M*
shows *f* ∈ *measurable M* (*measure_of* Ω *N* μ)
proof −
have *sets* (*measure_of* Ω *N* μ) = *sigma_sets* Ω *N*
using *B* **by** (*rule sets_measure_of*)
from *this assms* **show** *?thesis* **by** (*rule measurable_sigma_sets*)
qed

lemma *measurable_iff_measure_of*:
assumes *N* ⊆ *Pow* Ω *f* ∈ *space M* → Ω
shows *f* ∈ *measurable M* (*measure_of* Ω *N* μ) ⟷ (∀ *A*∈*N*. *f* −‘ *A* ∩ *space M* ∈ *sets M*)

by (*metis assms in_measure_of measurable_measure_of assms measurable_sets*)

lemma *measurable_cong_sets*:
assumes *sets*: *sets M* = *sets M'* *sets N* = *sets N'*
shows *measurable M N* = *measurable M' N'*
using *sets*[*THEN sets_eq_imp_space_eq*] *sets* **by** (*simp add*: *measurable_def*)

lemma *measurable_cong*:
assumes $\bigwedge w.\ w \in space\ M \Longrightarrow f\ w = g\ w$
shows $f \in measurable\ M\ M' \longleftrightarrow g \in measurable\ M\ M'$
unfolding *measurable_def* **using** *assms*
by (*simp cong*: *vimage_inter_cong Pi_cong*)

lemma *measurable_cong'*:
assumes $\bigwedge w.\ w \in space\ M$ =simp=> $f\ w = g\ w$
shows $f \in measurable\ M\ M' \longleftrightarrow g \in measurable\ M\ M'$
unfolding *measurable_def* **using** *assms*
by (*simp cong*: *vimage_inter_cong Pi_cong add*: *simp_implies_def*)

lemma *measurable_cong_simp*:
$M = N \Longrightarrow M' = N' \Longrightarrow (\bigwedge w.\ w \in space\ M \Longrightarrow f\ w = g\ w) \Longrightarrow$
$f \in measurable\ M\ M' \longleftrightarrow g \in measurable\ N\ N'$
by (*metis measurable_cong*)

lemma *measurable_compose*:
assumes *f*: $f \in measurable\ M\ N$ **and** *g*: $g \in measurable\ N\ L$
shows $(\lambda x.\ g\ (f\ x)) \in measurable\ M\ L$
proof −
have $\bigwedge A.\ (\lambda x.\ g\ (f\ x))$ −‘ $A \cap space\ M = f$ −‘ $(g$ −‘ $A \cap space\ N) \cap space\ M$
using *measurable_space*[*OF f*] **by** *auto*
with *measurable_space*[*OF f*] *measurable_space*[*OF g*] **show** *?thesis*
by (*auto intro*: *measurable_sets*[*OF f*] *measurable_sets*[*OF g*]
simp del: *vimage_Int simp add*: *measurable_def*)
qed

lemma *measurable_comp*:
$f \in measurable\ M\ N \Longrightarrow g \in measurable\ N\ L \Longrightarrow g \circ f \in measurable\ M\ L$
using *measurable_compose*[*of f M N g L*] **by** (*simp add*: *comp_def*)

lemma *measurable_const*:
$c \in space\ M' \Longrightarrow (\lambda x.\ c) \in measurable\ M\ M'$
by (*auto simp add*: *measurable_def*)

lemma *measurable_ident*: $id \in measurable\ M\ M$
by (*auto simp add*: *measurable_def*)

lemma *measurable_id*: $(\lambda x.\ x) \in measurable\ M\ M$
by (*simp add*: *measurable_def*)

lemma *measurable_ident_sets*:
assumes *eq*: *sets M = sets M'* **shows** $(\lambda x.\ x) \in$ *measurable M M'*
using *measurable_ident[of M]*
unfolding *id_def measurable_def eq sets_eq_imp_space_eq[OF eq]* **.**

lemma *sets_Least*:
assumes *meas*: $\bigwedge i{::}nat.\ \{x{\in}space\ M.\ P\ i\ x\} \in M$
shows $(\lambda x.\ LEAST\ j.\ P\ j\ x)\ -`\ A \cap space\ M \in sets\ M$
proof –
{ fix *i* **have** $(\lambda x.\ LEAST\ j.\ P\ j\ x)\ -`\ \{i\} \cap space\ M \in sets\ M$
proof *cases*
assume *i*: $(LEAST\ j.\ False) = i$
have $(\lambda x.\ LEAST\ j.\ P\ j\ x)\ -`\ \{i\} \cap space\ M =$
$\{x{\in}space\ M.\ P\ i\ x\} \cap (space\ M - (\bigcup j{<}i.\ \{x{\in}space\ M.\ P\ j\ x\})) \cup (space\ M - (\bigcup i.\ \{x{\in}space\ M.\ P\ i\ x\}))$
by (*simp add*: *set_eq_iff*, *safe*)
(*insert i*, *auto dest*: *Least_le intro*: *LeastI intro*!: *Least_equality*)
with *meas* **show** *?thesis*
by (*auto intro*!: *sets.Int*)
next
assume *i*: $(LEAST\ j.\ False) \neq i$
then have $(\lambda x.\ LEAST\ j.\ P\ j\ x)\ -`\ \{i\} \cap space\ M =$
$\{x{\in}space\ M.\ P\ i\ x\} \cap (space\ M - (\bigcup j{<}i.\ \{x{\in}space\ M.\ P\ j\ x\}))$
proof (*simp add*: *set_eq_iff*, *safe*)
fix *x* **assume** *neq*: $(LEAST\ j.\ False) \neq (LEAST\ j.\ P\ j\ x)$
have $\exists j.\ P\ j\ x$
by (*rule ccontr*) (*insert neq*, *auto*)
then show $P\ (LEAST\ j.\ P\ j\ x)\ x$ **by** (*rule LeastI_ex*)
qed (*auto dest*: *Least_le intro*!: *Least_equality*)
with *meas* **show** *?thesis*
by *auto*
qed }
then have $(\bigcup i{\in}A.\ (\lambda x.\ LEAST\ j.\ P\ j\ x)\ -`\ \{i\} \cap space\ M) \in sets\ M$
by (*intro sets.countable_UN*) *auto*
moreover have $(\bigcup i{\in}A.\ (\lambda x.\ LEAST\ j.\ P\ j\ x)\ -`\ \{i\} \cap space\ M) =$
$(\lambda x.\ LEAST\ j.\ P\ j\ x)\ -`\ A \cap space\ M$ **by** *auto*
ultimately show *?thesis* **by** *auto*
qed

lemma *measurable_mono1*:
$M' \subseteq Pow\ \Omega \Longrightarrow M \subseteq M' \Longrightarrow$
measurable (*measure_of* Ω *M* μ) *N* $\subseteq$ *measurable* (*measure_of* Ω *M'* μ') *N*
using *measure_of_subset[of M' Ω M]* **by** (*auto simp add*: *measurable_def*)

Counting space

definition *count_space* :: *'a set* ⇒ *'a measure* **where**
count_space Ω = *measure_of* Ω (*Pow* Ω) (λA. *if finite A then of_nat* (*card A*) *else* ∞)

lemma
  shows *space_count_space*[*simp*]: *space* (*count_space* Ω) = Ω
    and *sets_count_space*[*simp*]: *sets* (*count_space* Ω) = *Pow* Ω
  using *sigma_sets_into_sp*[*of Pow* Ω Ω]
  by (*auto simp*: *count_space_def*)

lemma *measurable_count_space_eq1*[*simp*]:
  f ∈ *measurable* (*count_space A*) *M* ⟷ *f* ∈ *A* → *space M*
 unfolding *measurable_def* **by** *simp*

lemma *measurable_compose_countable'*:
  assumes *f*: ⋀*i*. *i* ∈ *I* ⟹ (λ*x*. *f i x*) ∈ *measurable M N*
  and *g*: *g* ∈ *measurable M* (*count_space I*) **and** *I*: *countable I*
  shows (λ*x*. *f* (*g x*) *x*) ∈ *measurable M N*
  unfolding *measurable_def*
proof *safe*
  fix *x* **assume** *x* ∈ *space M* **then show** *f* (*g x*) *x* ∈ *space N*
    using *measurable_space*[*OF f*] *g*[*THEN measurable_space*] **by** *auto*
next
  fix *A* **assume** *A*: *A* ∈ *sets N*
  have (λ*x*. *f* (*g x*) *x*) −' *A* ∩ *space M* = (⋃*i*∈*I*. (*g* −' {*i*} ∩ *space M*) ∩ (*f i* −' *A* ∩ *space M*))
    using *measurable_space*[*OF g*] **by** *auto*
  also have ... ∈ *sets M*
    using *f*[*THEN measurable_sets*, *OF _ A*] *g*[*THEN measurable_sets*]
     by (*auto intro*!: *sets.countable_UN' I intro*: *sets.Int*[*OF measurable_sets measurable_sets*])
  finally show (λ*x*. *f* (*g x*) *x*) −' *A* ∩ *space M* ∈ *sets M* .
qed

lemma *measurable_count_space_eq_countable*:
  assumes *countable A*
  shows *f* ∈ *measurable M* (*count_space A*) ⟷ (*f* ∈ *space M* → *A* ∧ (∀ *a*∈*A*. *f* −' {*a*} ∩ *space M* ∈ *sets M*))
proof −
  { **fix** *X* **assume** *X* ⊆ *A f* ∈ *space M* → *A*
    with ‹*countable A*› **have** *f* −' *X* ∩ *space M* = (⋃*a*∈*X*. *f* −' {*a*} ∩ *space M*) *countable X*
      by (*auto dest*: *countable_subset*)
    moreover assume ∀ *a*∈*A*. *f* −' {*a*} ∩ *space M* ∈ *sets M*
    ultimately have *f* −' *X* ∩ *space M* ∈ *sets M*
      using ‹*X* ⊆ *A*› **by** (*auto intro*!: *sets.countable_UN' simp del*: *UN_simps*) }
  then show *?thesis*
    unfolding *measurable_def* **by** *auto*
qed

lemma *measurable_count_space_eq2*:
  finite A ⟹ *f* ∈ *measurable M* (*count_space A*) ⟷ (*f* ∈ *space M* → *A* ∧ (∀ *a*∈*A*.

f −‘ {a} ∩ *space M* ∈ *sets M*))
by (*intro measurable_count_space_eq_countable countable_finite*)

lemma *measurable_count_space_eq2_countable*:
fixes f :: ′*a* => ′*c*::*countable*
shows f ∈ *measurable M* (*count_space A*) ⟷ (f ∈ *space M* → *A* ∧ (∀ *a*∈*A*. f −‘ {a} ∩ *space M* ∈ *sets M*))
by (*intro measurable_count_space_eq_countable countableI_type*)

lemma *measurable_compose_countable*:
assumes f: ⋀*i*::′*i*::*countable*. (λ*x*. *f i x*) ∈ *measurable M N* **and** g: g ∈ *measurable M* (*count_space UNIV*)
shows (λ*x*. *f* (*g x*) *x*) ∈ *measurable M N*
by (*rule measurable_compose_countable*′[*OF assms*]) *auto*

lemma *measurable_count_space_const*:
(λ*x*. *c*) ∈ *measurable M* (*count_space UNIV*)
by (*simp add*: *measurable_const*)

lemma *measurable_count_space*:
f ∈ *measurable* (*count_space A*) (*count_space UNIV*)
by *simp*

lemma *measurable_compose_rev*:
assumes f: f ∈ *measurable L N* **and** g: g ∈ *measurable M L*
shows (λ*x*. *f* (*g x*)) ∈ *measurable M N*
using *measurable_compose*[*OF g f*] **.**

lemma *measurable_empty_iff*:
space N = {} ⟹ f ∈ *measurable M N* ⟷ *space M* = {}
by (*auto simp add*: *measurable_def Pi_iff*)

Extend measure

definition *extend_measure* :: ′*a set* ⇒ ′*b set* ⇒ (′*b* ⇒ ′*a set*) ⇒ (′*b* ⇒ *ennreal*) ⇒ ′*a measure*
where
extend_measure Ω *I G* μ =
(*if* (∃ μ′. (∀ *i*∈*I*. μ′ (*G i*) = μ *i*) ∧ *measure_space* Ω (*sigma_sets* Ω (*G*‘*I*)) μ′) ∧ ¬ (∀ *i*∈*I*. μ *i* = *0*)
then measure_of Ω (*G*‘*I*) (*SOME* μ′. (∀ *i*∈*I*. μ′ (*G i*) = μ *i*) ∧ *measure_space* Ω (*sigma_sets* Ω (*G*‘*I*)) μ′)
else measure_of Ω (*G*‘*I*) (λ_. *0*))

lemma *space_extend_measure*: *G* ‘ *I* ⊆ *Pow* Ω ⟹ *space* (*extend_measure* Ω *I G* μ) = Ω
unfolding *extend_measure_def* **by** *simp*

lemma *sets_extend_measure*: *G* ‘ *I* ⊆ *Pow* Ω ⟹ *sets* (*extend_measure* Ω *I G* μ)

= *sigma_sets* Ω (*G*‘*I*)
  unfolding *extend_measure_def* **by** *simp*

lemma *emeasure_extend_measure*:
  assumes *M*: *M* = *extend_measure* Ω *I G μ*
    and *eq*: ⋀*i*. *i* ∈ *I* ⟹ *μ'* (*G i*) = *μ i*
    and *ms*: *G* ‘ *I* ⊆ *Pow* Ω *positive* (*sets M*) *μ'* *countably_additive* (*sets M*) *μ'*
    and *i* ∈ *I*
  shows *emeasure M* (*G i*) = *μ i*
proof *cases*
  assume ∗: (∀ *i*∈*I*. *μ i* = *0*)
  with *M* **have** *M_eq*: *M* = *measure_of* Ω (*G*‘*I*) (λ_. *0*)
   by (*simp add*: *extend_measure_def*)
  from *measure_space_0*[*OF ms*(*1*)] *ms* ‹*i*∈*I*›
  have *emeasure M* (*G i*) = *0*
    by (*intro emeasure_measure_of*[*OF M_eq*]) (*auto simp add*: *M measure_space_def sets_extend_measure*)
  with ‹*i*∈*I*› ∗ **show** *?thesis*
    by *simp*
next
  define *P* **where** *P μ'* ⟷ (∀ *i*∈*I*. *μ'* (*G i*) = *μ i*) ∧ *measure_space* Ω (*sigma_sets* Ω (*G*‘*I*)) *μ'* **for** *μ'*
  assume ¬ (∀ *i*∈*I*. *μ i* = *0*)
  moreover
  have *measure_space* (*space M*) (*sets M*) *μ'*
    using *ms* **unfolding** *measure_space_def* **by** *auto standard*
  with *ms eq* **have** ∃ *μ'*. *P μ'*
    unfolding *P_def*
   by (*intro exI*[*of* _ *μ'*]) (*auto simp add*: *M space_extend_measure sets_extend_measure*)
  ultimately have *M_eq*: *M* = *measure_of* Ω (*G*‘*I*) (*Eps P*)
    by (*simp add*: *M extend_measure_def P_def*[*symmetric*])

  from ‹∃ *μ'*. *P μ'*› **have** *P*: *P* (*Eps P*) **by** (*rule someI_ex*)
  show *emeasure M* (*G i*) = *μ i*
  proof (*subst emeasure_measure_of*[*OF M_eq*])
    have *sets_M*: *sets M* = *sigma_sets* Ω (*G*‘*I*)
      using *M_eq ms* **by** (*auto simp*: *sets_extend_measure*)
    then show *G i* ∈ *sets M* **using** ‹*i* ∈ *I*› **by** *auto*
    show *positive* (*sets M*) (*Eps P*) *countably_additive* (*sets M*) (*Eps P*) *Eps P* (*G i*) = *μ i*
      using *P* ‹*i*∈*I*› **by** (*auto simp add*: *sets_M measure_space_def P_def*)
  qed *fact*
qed

lemma *emeasure_extend_measure_Pair*:
  assumes *M*: *M* = *extend_measure* Ω {(*i*, *j*). *I i j*} (λ(*i*, *j*). *G i j*) (λ(*i*, *j*). *μ i j*)
    and *eq*: ⋀*i j*. *I i j* ⟹ *μ'* (*G i j*) = *μ i j*
    and *ms*: ⋀*i j*. *I i j* ⟹ *G i j* ∈ *Pow* Ω *positive* (*sets M*) *μ'* *countably_additive*

(*sets M*) μ'
  and *I i j*
shows *emeasure M* (*G i j*) = μ *i j*
using *emeasure_extend_measure*[*OF M* _ _ *ms*(*2*,*3*), *of* (*i*,*j*)] *eq ms*(*1*) ‹*I i j*›
by (*auto simp*: *subset_eq*)

6.1.3 The smallest σ-algebra regarding a function

definition *vimage_algebra* :: *'a set* ⇒ (*'a* ⇒ *'b*) ⇒ *'b measure* ⇒ *'a measure* **where**
vimage_algebra X f M = *sigma X* {*f* −‘ *A* ∩ *X* | *A*. *A* ∈ *sets M*}

lemma *space_vimage_algebra*[*simp*]: *space* (*vimage_algebra X f M*) = *X*
unfolding *vimage_algebra_def* **by** (*rule space_measure_of*) *auto*

lemma *sets_vimage_algebra*: *sets* (*vimage_algebra X f M*) = *sigma_sets X* {*f* −‘ *A* ∩ *X* | *A*. *A* ∈ *sets M*}
unfolding *vimage_algebra_def* **by** (*rule sets_measure_of*) *auto*

lemma *sets_vimage_algebra2*:
f ∈ *X* → *space M* ⟹ *sets* (*vimage_algebra X f M*) = {*f* −‘ *A* ∩ *X* | *A*. *A* ∈ *sets M*}
using *sigma_sets_vimage_commute*[*of f X space M sets M*]
unfolding *sets_vimage_algebra sets.sigma_sets_eq* **by** *simp*

lemma *sets_vimage_algebra_cong*: *sets M* = *sets N* ⟹ *sets* (*vimage_algebra X f M*) = *sets* (*vimage_algebra X f N*)
by (*simp add*: *sets_vimage_algebra*)

lemma *vimage_algebra_cong*:
assumes *X* = *Y*
assumes ⋀*x*. *x* ∈ *Y* ⟹ *f x* = *g x*
assumes *sets M* = *sets N*
shows *vimage_algebra X f M* = *vimage_algebra Y g N*
by (*auto simp*: *vimage_algebra_def assms intro*!: *arg_cong2*[**where** *f*=*sigma*])

lemma *in_vimage_algebra*: *A* ∈ *sets M* ⟹ *f* −‘ *A* ∩ *X* ∈ *sets* (*vimage_algebra X f M*)
by (*auto simp*: *vimage_algebra_def*)

lemma *sets_image_in_sets*:
assumes *N*: *space N* = *X*
assumes *f*: *f* ∈ *measurable N M*
shows *sets* (*vimage_algebra X f M*) ⊆ *sets N*
unfolding *sets_vimage_algebra N*[*symmetric*]
by (*rule sets.sigma_sets_subset*) (*auto intro*!: *measurable_sets f*)

lemma *measurable_vimage_algebra1*: *f* ∈ *X* → *space M* ⟹ *f* ∈ *measurable* (*vimage_algebra X f M*) *M*
unfolding *measurable_def* **by** (*auto intro*: *in_vimage_algebra*)

lemma *measurable_vimage_algebra2*:
assumes *g*: $g \in space\ N \to X$ **and** *f*: $(\lambda x.\ f\ (g\ x)) \in measurable\ N\ M$
shows $g \in measurable\ N\ (vimage\_algebra\ X\ f\ M)$
unfolding *vimage_algebra_def*
proof (*rule measurable_measure_of*)
fix *A* **assume** $A \in \{f -` A \cap X \mid A.\ A \in sets\ M\}$
then obtain *Y* **where** *Y*: $Y \in sets\ M$ **and** *A*: $A = f -` Y \cap X$
by *auto*
then have $g -` A \cap space\ N = (\lambda x.\ f\ (g\ x)) -` Y \cap space\ N$
using *g* **by** *auto*
also have $\ldots \in sets\ N$
using *f Y* **by** (*rule measurable_sets*)
finally show $g -` A \cap space\ N \in sets\ N$ **.**
qed (*insert g, auto*)

lemma *vimage_algebra_sigma*:
assumes *X*: $X \subseteq Pow\ \Omega'$ **and** *f*: $f \in \Omega \to \Omega'$
shows $vimage\_algebra\ \Omega\ f\ (sigma\ \Omega'\ X) = sigma\ \Omega\ \{f -` A \cap \Omega \mid A.\ A \in X\ \}$
(**is** $?V = ?S$)
proof (*rule measure_eqI*)
have Ω: $\{f -` A \cap \Omega \mid A.\ A \in X\} \subseteq Pow\ \Omega$ **by** *auto*
show $sets\ ?V = sets\ ?S$
using *sigma_sets_vimage_commute*[*OF f*, *of X*]
by (*simp add*: *space_measure_of_conv f sets_vimage_algebra2 Ω X*)
qed (*simp add*: *vimage_algebra_def emeasure_sigma*)

lemma *vimage_algebra_vimage_algebra_eq*:
assumes ∗: $f \in X \to Y\ g \in Y \to space\ M$
shows $vimage\_algebra\ X\ f\ (vimage\_algebra\ Y\ g\ M) = vimage\_algebra\ X\ (\lambda x.\ g\ (f\ x))\ M$
(**is** $?VV = ?V$)
proof (*rule measure_eqI*)
have $(\lambda x.\ g\ (f\ x)) \in X \to space\ M\ \bigwedge A.\ A \cap f -` Y \cap X = A \cap X$
using ∗ **by** *auto*
with ∗ **show** $sets\ ?VV = sets\ ?V$
by (*simp add*: *sets_vimage_algebra2 vimage_comp comp_def flip*: *ex_simps*)
qed (*simp add*: *vimage_algebra_def emeasure_sigma*)

Restricted Space Sigma Algebra

definition *restrict_space* :: *'a measure* ⇒ *'a set* ⇒ *'a measure* **where**
$restrict\_space\ M\ \Omega = measure\_of\ (\Omega \cap space\ M)\ (((\cap)\ \Omega)\ `\ sets\ M)\ (emeasure\ M)$

lemma *space_restrict_space*: $space\ (restrict\_space\ M\ \Omega) = \Omega \cap space\ M$
using *sets.sets_into_space* **unfolding** *restrict_space_def* **by** (*subst space_measure_of*) *auto*

lemma *space_restrict_space2* [*simp*]: $\Omega \in$ *sets M* $\Longrightarrow$ *space* (*restrict_space M* Ω) $= \Omega$
  by (*simp add*: *space_restrict_space sets.sets_into_space*)

lemma *sets_restrict_space*: *sets* (*restrict_space M* Ω) = (($\cap$) Ω) ' *sets M*
  unfolding *restrict_space_def*
proof (*subst sets_measure_of*)
  show ($\cap$) Ω ' *sets M* $\subseteq$ *Pow* ($\Omega \cap$ *space M*)
    by (*auto dest*: *sets.sets_into_space*)
  have *sigma_sets* ($\Omega \cap$ *space M*) {(($\lambda x.\ x$) −' *X*) $\cap$ ($\Omega \cap$ *space M*) | *X*. *X* $\in$ *sets M*} =
    ($\lambda X.\ X \cap$ ($\Omega \cap$ *space M*)) ' *sets M*
    by (*subst sigma_sets_vimage_commute*[*symmetric*, **where** Ω' = *space M*])
      (*auto simp add*: *sets.sigma_sets_eq*)
  moreover have {(($\lambda x.\ x$) −' *X*) $\cap$ ($\Omega \cap$ *space M*) | *X*. *X* $\in$ *sets M*} = ($\lambda X.\ X$ $\cap$ ($\Omega \cap$ *space M*)) ' *sets M*
    by *auto*
  moreover have ($\lambda X.\ X \cap$ ($\Omega \cap$ *space M*)) ' *sets M* = (($\cap$) Ω) ' *sets M*
    by (*intro image_cong*) (*auto dest*: *sets.sets_into_space*)
  ultimately show *sigma_sets* ($\Omega \cap$ *space M*) (($\cap$) Ω ' *sets M*) = ($\cap$) Ω ' *sets M*
    by *simp*
qed

lemma *restrict_space_sets_cong*:
  A = *B* $\Longrightarrow$ *sets M* = *sets N* $\Longrightarrow$ *sets* (*restrict_space M A*) = *sets* (*restrict_space N B*)
  by (*auto simp*: *sets_restrict_space*)

lemma *sets_restrict_space_count_space* :
  sets (*restrict_space* (*count_space A*) *B*) = *sets* (*count_space* (*A* $\cap$ *B*))
by(*auto simp add*: *sets_restrict_space*)

lemma *sets_restrict_UNIV*[*simp*]: *sets* (*restrict_space M UNIV*) = *sets M*
  by (*auto simp add*: *sets_restrict_space*)

lemma *sets_restrict_restrict_space*:
  sets (*restrict_space* (*restrict_space M A*) *B*) = *sets* (*restrict_space M* (*A* $\cap$ *B*))
  unfolding *sets_restrict_space image_comp* **by** (*intro image_cong*) *auto*

lemma *sets_restrict_space_iff*:
  $\Omega \cap$ *space M* $\in$ *sets M* $\Longrightarrow$ *A* $\in$ *sets* (*restrict_space M* Ω) $\longleftrightarrow$ (*A* $\subseteq \Omega \wedge$ *A* $\in$ *sets M*)
proof (*subst sets_restrict_space*, *safe*)
  fix *A* **assume** $\Omega \cap$ *space M* $\in$ *sets M* **and** *A*: *A* $\in$ *sets M*
  then have ($\Omega \cap$ *space M*) $\cap$ *A* $\in$ *sets M*
    by *rule*
  also have ($\Omega \cap$ *space M*) $\cap$ *A* = $\Omega \cap$ *A*
    using *sets.sets_into_space*[*OF A*] **by** *auto*
  finally show $\Omega \cap$ *A* $\in$ *sets M*

by *auto*
qed *auto*

lemma *sets_restrict_space_cong*: *sets M* = *sets N* ⟹ *sets* (*restrict_space M* Ω) = *sets* (*restrict_space N* Ω)
by (*simp add*: *sets_restrict_space*)

lemma *restrict_space_eq_vimage_algebra*:
Ω ⊆ *space M* ⟹ *sets* (*restrict_space M* Ω) = *sets* (*vimage_algebra* Ω (λ*x*. *x*) *M*)
unfolding *restrict_space_def*
apply (*subst sets_measure_of*)
apply (*auto simp add*: *image_subset_iff dest*: *sets.sets_into_space*) []
apply (*auto simp add*: *sets_vimage_algebra intro*!: *arg_cong2*[**where** *f*=*sigma_sets*])
done

lemma *sets_Collect_restrict_space_iff*:
assumes *S* ∈ *sets M*
shows {*x*∈*space* (*restrict_space M S*). *P x*} ∈ *sets* (*restrict_space M S*) ⟷ {*x*∈*space M*. *x* ∈ *S* ∧ *P x*} ∈ *sets M*
proof –
have {*x*∈*S*. *P x*} = {*x*∈*space M*. *x* ∈ *S* ∧ *P x*}
using *sets.sets_into_space*[*OF assms*] **by** *auto*
then show *?thesis*
by (*subst sets_restrict_space_iff*) (*auto simp add*: *space_restrict_space assms*)
qed

lemma *measurable_restrict_space1*:
assumes *f*: *f* ∈ *measurable M N*
shows *f* ∈ *measurable* (*restrict_space M* Ω) *N*
unfolding *measurable_def*
proof (*intro CollectI conjI ballI*)
show *sp*: *f* ∈ *space* (*restrict_space M* Ω) → *space N*
using *measurable_space*[*OF f*] **by** (*auto simp*: *space_restrict_space*)

fix *A* **assume** *A* ∈ *sets N*
have *f* –' *A* ∩ *space* (*restrict_space M* Ω) = (*f* –' *A* ∩ *space M*) ∩ (Ω ∩ *space M*)
by (*auto simp*: *space_restrict_space*)
also have ... ∈ *sets* (*restrict_space M* Ω)
unfolding *sets_restrict_space*
using *measurable_sets*[*OF f* ‹*A* ∈ *sets N*›] **by** *blast*
finally show *f* –' *A* ∩ *space* (*restrict_space M* Ω) ∈ *sets* (*restrict_space M* Ω) **.**
qed

lemma *measurable_restrict_space2_iff*:
f ∈ *measurable M* (*restrict_space N* Ω) ⟷ (*f* ∈ *measurable M N* ∧ *f* ∈ *space M* → Ω)
proof –
have ⋀*A*. *f* ∈ *space M* → Ω ⟹ *f* –' Ω ∩ *f* –' *A* ∩ *space M* = *f* –' *A* ∩ *space*

M
  by *auto*
 then show *?thesis*
  by (*auto simp*: *measurable_def space_restrict_space Pi_Int*[*symmetric*] *sets_restrict_space*)
qed

lemma *measurable_restrict_space2*:
 $f \in space\ M \to \Omega \Longrightarrow f \in measurable\ M\ N \Longrightarrow f \in measurable\ M\ (restrict\_space\ N\ \Omega)$
 by (*simp add*: *measurable_restrict_space2_iff*)

lemma *measurable_piecewise_restrict*:
 assumes *I*: *countable C*
  and *X*: $\bigwedge\Omega.\ \Omega \in C \Longrightarrow \Omega \cap space\ M \in sets\ M$ $space\ M \subseteq \bigcup C$
  and *f*: $\bigwedge\Omega.\ \Omega \in C \Longrightarrow f \in measurable\ (restrict\_space\ M\ \Omega)\ N$
 shows $f \in measurable\ M\ N$
proof (*rule measurableI*)
 fix *x* **assume** $x \in space\ M$
 with *X* **obtain** Ω **where** $\Omega \in C$ $x \in \Omega$ $x \in space\ M$ **by** *auto*
 then show $f\ x \in space\ N$
  by (*auto simp*: *space_restrict_space intro*: *f measurable_space*)
next
 fix *A* **assume** *A*: $A \in sets\ N$
 have $f -` A \cap space\ M = (\bigcup\Omega\in C.\ (f -` A \cap (\Omega \cap space\ M)))$
  using *X* **by** (*auto simp*: *subset_eq*)
 also have $\ldots \in sets\ M$
  using *measurable_sets*[*OF f A*] *X I*
  by (*intro sets.countable_UN'*) (*auto simp*: *sets_restrict_space_iff space_restrict_space*)
 finally show $f -` A \cap space\ M \in sets\ M$ **.**
qed

lemma *measurable_piecewise_restrict_iff*:
 $countable\ C \Longrightarrow (\bigwedge\Omega.\ \Omega \in C \Longrightarrow \Omega \cap space\ M \in sets\ M) \Longrightarrow space\ M \subseteq (\bigcup C) \Longrightarrow$
  $f \in measurable\ M\ N \longleftrightarrow (\forall\Omega\in C.\ f \in measurable\ (restrict\_space\ M\ \Omega)\ N)$
 by (*auto intro*: *measurable_piecewise_restrict measurable_restrict_space1*)

lemma *measurable_If_restrict_space_iff*:
 $\{x\in space\ M.\ P\ x\} \in sets\ M \Longrightarrow$
  $(\lambda x.\ if\ P\ x\ then\ f\ x\ else\ g\ x) \in measurable\ M\ N \longleftrightarrow$
  $(f \in measurable\ (restrict\_space\ M\ \{x.\ P\ x\})\ N \wedge g \in measurable\ (restrict\_space\ M\ \{x.\ \neg\ P\ x\})\ N)$
 by (*subst measurable_piecewise_restrict_iff*[**where** $C=\{\{x.\ P\ x\},\ \{x.\ \neg\ P\ x\}\}$])
  (*auto simp*: *Int_def sets.sets_Collect_neg space_restrict_space conj_commute*[*of* _ $x \in space\ M$ **for** *x*]
   cong: *measurable_cong'*)

lemma *measurable_If*:
 $f \in measurable\ M\ M' \Longrightarrow g \in measurable\ M\ M' \Longrightarrow \{x\in space\ M.\ P\ x\} \in sets$

M ⟹
  (λ*x*. *if P x then f x else g x*) ∈ *measurable M M′*
 unfolding *measurable_If_restrict_space_iff* **by** (*auto intro*: *measurable_restrict_space1*)

lemma *measurable_If_set*:
  assumes *measure*: *f* ∈ *measurable M M′ g* ∈ *measurable M M′*
  assumes *P*: *A* ∩ *space M* ∈ *sets M*
  shows (λ*x*. *if x* ∈ *A then f x else g x*) ∈ *measurable M M′*
proof (*rule measurable_If*[*OF measure*])
  have {*x* ∈ *space M*. *x* ∈ *A*} = *A* ∩ *space M* **by** *auto*
  thus {*x* ∈ *space M*. *x* ∈ *A*} ∈ *sets M* **using** ‹*A* ∩ *space M* ∈ *sets M*› **by** *auto*
qed

lemma *measurable_restrict_space_iff*:
  Ω ∩ *space M* ∈ *sets M* ⟹ *c* ∈ *space N* ⟹
    f ∈ *measurable* (*restrict_space M* Ω) *N* ⟷ (λ*x*. *if x* ∈ Ω *then f x else c*) ∈ *measurable M N*
  by (*subst measurable_If_restrict_space_iff*)
    (*simp_all add*: *Int_def conj_commute measurable_const*)

lemma *restrict_space_singleton*: {*x*} ∈ *sets M* ⟹ *sets* (*restrict_space M* {*x*}) = *sets* (*count_space* {*x*})
  using *sets_restrict_space_iff*[*of* {*x*} *M*]
  by (*auto simp add*: *sets_restrict_space_iff dest*!: *subset_singletonD*)

lemma *measurable_restrict_countable*:
  assumes *X*[*intro*]: *countable X*
  assumes *sets*[*simp*]: ⋀*x*. *x* ∈ *X* ⟹ {*x*} ∈ *sets M*
  assumes *space*[*simp*]: ⋀*x*. *x* ∈ *X* ⟹ *f x* ∈ *space N*
  assumes *f*: *f* ∈ *measurable* (*restrict_space M* (− *X*)) *N*
  shows *f* ∈ *measurable M N*
  using *f sets.countable*[*OF sets X*]
  by (*intro measurable_piecewise_restrict*[**where** *M*=*M* **and** *C*={− *X*} ∪ ((λ*x*. {*x*}) ‘ *X*)])
    (*auto simp*: *Diff_Int_distrib2 Compl_eq_Diff_UNIV Int_insert_left sets.Diff restrict_space_singleton*
      simp del: *sets_count_space cong*: *measurable_cong_sets*)

lemma *measurable_discrete_difference*:
  assumes *f*: *f* ∈ *measurable M N*
  assumes *X*: *countable X* ⋀*x*. *x* ∈ *X* ⟹ {*x*} ∈ *sets M* ⋀*x*. *x* ∈ *X* ⟹ *g x* ∈ *space N*
  assumes *eq*: ⋀*x*. *x* ∈ *space M* ⟹ *x* ∉ *X* ⟹ *f x* = *g x*
  shows *g* ∈ *measurable M N*
  by (*rule measurable_restrict_countable*[*OF X*])
    (*auto simp*: *eq*[*symmetric*] *space_restrict_space cong*: *measurable_cong′ intro*: *f measurable_restrict_space1*)

lemma *measurable_count_space_extend*: *A* ⊆ *B* ⟹ *f* ∈ *space M* → *A* ⟹ *f* ∈ *M*

$\rightarrow_M$ *count_space B* $\Longrightarrow$ *f* $\in$ *M* $\rightarrow_M$ *count_space A*
  by (*auto simp*: *measurable_def*)

end

6.2 Measurability Prover

theory *Measurable*
  imports
    Sigma_Algebra
    HOL−Library.Order_Continuity
begin

lemma (**in** *algebra*) *sets_Collect_finite_All*:
  assumes $\bigwedge i.\ i \in S \Longrightarrow \{x{\in}\Omega.\ P\ i\ x\} \in M$ *finite S*
  shows $\{x{\in}\Omega.\ \forall i{\in}S.\ P\ i\ x\} \in M$
proof −
  have $\{x{\in}\Omega.\ \forall i{\in}S.\ P\ i\ x\} = (\textit{if}\ S = \{\}\ \textit{then}\ \Omega\ \textit{else}\ \bigcap i{\in}S.\ \{x{\in}\Omega.\ P\ i\ x\})$
    by *auto*
  with *assms* **show** *?thesis* **by** (*auto intro*!: *sets_Collect_finite_All′*)
qed

abbreviation *pred M P* $\equiv$ *P* $\in$ *measurable M* (*count_space* (*UNIV*::*bool set*))

lemma *pred_def*: *pred M P* $\longleftrightarrow$ $\{x{\in}\textit{space}\ M.\ P\ x\} \in \textit{sets}\ M$
proof
  assume *pred M P*
  then have $P -` \{\textit{True}\} \cap \textit{space}\ M \in \textit{sets}\ M$
    by (*auto simp*: *measurable_count_space_eq2*)
  also have $P -` \{\textit{True}\} \cap \textit{space}\ M = \{x{\in}\textit{space}\ M.\ P\ x\}$ **by** *auto*
  finally show $\{x{\in}\textit{space}\ M.\ P\ x\} \in \textit{sets}\ M$ **.**
next
  assume *P*: $\{x{\in}\textit{space}\ M.\ P\ x\} \in \textit{sets}\ M$
  moreover
  { fix *X*
    have $X \in \textit{Pow}\ (\textit{UNIV} :: \textit{bool set})$ **by** *simp*
    then have $P -` X \cap \textit{space}\ M = \{x{\in}\textit{space}\ M.\ ((X = \{\textit{True}\} \longrightarrow P\ x) \wedge (X = \{\textit{False}\} \longrightarrow \neg\ P\ x) \wedge X \neq \{\})\}$
      unfolding *UNIV_bool Pow_insert Pow_empty* **by** *auto*
    then have $P -` X \cap \textit{space}\ M \in \textit{sets}\ M$
     by (*auto intro*!: *sets.sets_Collect_neg sets.sets_Collect_imp sets.sets_Collect_conj sets.sets_Collect_const P*) **}**
  then show *pred M P*
    by (*auto simp*: *measurable_def*)
qed

lemma *pred_sets1*: $\{x{\in}\textit{space}\ M.\ P\ x\} \in \textit{sets}\ M \Longrightarrow f \in \textit{measurable}\ N\ M \Longrightarrow$ *pred N* $(\lambda x.\ P\ (f\ x))$

by (*rule measurable_compose*[**where** *f*=*f* **and** *N*=*M*]) (*auto simp*: *pred_def*)

lemma *pred_sets2*: $A \in sets\ N \Longrightarrow f \in measurable\ M\ N \Longrightarrow pred\ M\ (\lambda x.\ f\ x \in A)$
by (*rule measurable_compose*[**where** *f*=*f* **and** *N*=*N*]) (*auto simp*: *pred_def Int_def*[*symmetric*])

ML_file ‹*measurable.ML*›

attribute_setup *measurable* = ‹
Scan.lift (
(*Args.add* >> *K true* || *Args.del* >> *K false* || *Scan.succeed true*) −−
Scan.optional (*Args.parens* (
Scan.optional (*Args.$$$ raw* >> *K true*) *false* −−
Scan.optional (*Args.$$$ generic* >> *K Measurable.Generic*) *Measurable.Concrete*))
(*false*, *Measurable.Concrete*) >>
Measurable.measurable_thm_attr)
› *declaration of measurability theorems*

attribute_setup *measurable_dest* = *Measurable.dest_thm_attr*
add dest rule to measurability prover

attribute_setup *measurable_cong* = *Measurable.cong_thm_attr*
add congurence rules to measurability prover

method_setup *measurable* = ‹ *Scan.lift* (*Scan.succeed* (*METHOD o Measurable.measurable_tac*))
›
measurability prover

simproc_setup *measurable* (*A* ∈ *sets M* | *f* ∈ *measurable M N*) = ‹*K Measurable.simproc*›

setup ‹
Global_Theory.add_thms_dynamic (***binding*** ‹measurable›, *Measurable.get_all*)
›

declare
pred_sets1[*measurable_dest*]
pred_sets2[*measurable_dest*]
sets.sets_into_space[*measurable_dest*]

declare
sets.top[*measurable*]
sets.empty_sets[*measurable* (*raw*)]
sets.Un[*measurable* (*raw*)]
sets.Diff[*measurable* (*raw*)]

declare
measurable_count_space[*measurable* (*raw*)]
measurable_ident[*measurable* (*raw*)]

measurable_id[*measurable* (*raw*)]
measurable_const[*measurable* (*raw*)]
measurable_If [*measurable* (*raw*)]
measurable_comp[*measurable* (*raw*)]
measurable_sets[*measurable* (*raw*)]

declare *measurable_cong_sets*[*measurable_cong*]
declare *sets_restrict_space_cong*[*measurable_cong*]
declare *sets_restrict_UNIV* [*measurable_cong*]

lemma *predE*[*measurable* (*raw*)]:
pred M P $\Longrightarrow$ {*x*∈*space M*. *P x*} ∈ *sets M*
unfolding *pred_def* **.**

lemma *pred_intros_imp'*[*measurable* (*raw*)]:
(*K* $\Longrightarrow$ *pred M* (λ*x*. *P x*)) $\Longrightarrow$ *pred M* (λ*x*. *K* ⟶ *P x*)
by (*cases K*) *auto*

lemma *pred_intros_conj1'*[*measurable* (*raw*)]:
(*K* $\Longrightarrow$ *pred M* (λ*x*. *P x*)) $\Longrightarrow$ *pred M* (λ*x*. *K* ∧ *P x*)
by (*cases K*) *auto*

lemma *pred_intros_conj2'*[*measurable* (*raw*)]:
(*K* $\Longrightarrow$ *pred M* (λ*x*. *P x*)) $\Longrightarrow$ *pred M* (λ*x*. *P x* ∧ *K*)
by (*cases K*) *auto*

lemma *pred_intros_disj1'*[*measurable* (*raw*)]:
(¬ *K* $\Longrightarrow$ *pred M* (λ*x*. *P x*)) $\Longrightarrow$ *pred M* (λ*x*. *K* ∨ *P x*)
by (*cases K*) *auto*

lemma *pred_intros_disj2'*[*measurable* (*raw*)]:
(¬ *K* $\Longrightarrow$ *pred M* (λ*x*. *P x*)) $\Longrightarrow$ *pred M* (λ*x*. *P x* ∨ *K*)
by (*cases K*) *auto*

lemma *pred_intros_logic*[*measurable* (*raw*)]:
pred M (λ*x*. *x* ∈ *space M*)
pred M (λ*x*. *P x*) $\Longrightarrow$ *pred M* (λ*x*. ¬ *P x*)
pred M (λ*x*. *Q x*) $\Longrightarrow$ *pred M* (λ*x*. *P x*) $\Longrightarrow$ *pred M* (λ*x*. *Q x* ∧ *P x*)
pred M (λ*x*. *Q x*) $\Longrightarrow$ *pred M* (λ*x*. *P x*) $\Longrightarrow$ *pred M* (λ*x*. *Q x* ⟶ *P x*)
pred M (λ*x*. *Q x*) $\Longrightarrow$ *pred M* (λ*x*. *P x*) $\Longrightarrow$ *pred M* (λ*x*. *Q x* ∨ *P x*)
pred M (λ*x*. *Q x*) $\Longrightarrow$ *pred M* (λ*x*. *P x*) $\Longrightarrow$ *pred M* (λ*x*. *Q x* = *P x*)
pred M (λ*x*. *f x* ∈ *UNIV*)
pred M (λ*x*. *f x* ∈ {})
pred M (λ*x*. *P'* (*f x*) *x*) $\Longrightarrow$ *pred M* (λ*x*. *f x* ∈ {*y*. *P' y x*})
pred M (λ*x*. *f x* ∈ (*B x*)) $\Longrightarrow$ *pred M* (λ*x*. *f x* ∈ − (*B x*))
pred M (λ*x*. *f x* ∈ (*A x*)) $\Longrightarrow$ *pred M* (λ*x*. *f x* ∈ (*B x*)) $\Longrightarrow$ *pred M* (λ*x*. *f x* ∈ (*A x*) − (*B x*))
pred M (λ*x*. *f x* ∈ (*A x*)) $\Longrightarrow$ *pred M* (λ*x*. *f x* ∈ (*B x*)) $\Longrightarrow$ *pred M* (λ*x*. *f x* ∈ (*A x*) ∩ (*B x*))

pred M $(\lambda x.\ f\ x \in (A\ x)) \Longrightarrow$ *pred M* $(\lambda x.\ f\ x \in (B\ x)) \Longrightarrow$ *pred M* $(\lambda x.\ f\ x \in (A\ x) \cup (B\ x))$
pred M $(\lambda x.\ g\ x\ (f\ x) \in (X\ x)) \Longrightarrow$ *pred M* $(\lambda x.\ f\ x \in (g\ x) -\text{`}\ (X\ x))$
by (*auto simp*: *iff_conv_conj_imp pred_def*)

lemma *pred_intros_countable*[*measurable* (*raw*)]:
fixes $P :: 'a \Rightarrow 'i :: countable \Rightarrow bool$
shows
$(\bigwedge i.\ pred\ M\ (\lambda x.\ P\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ \forall i.\ P\ x\ i)$
$(\bigwedge i.\ pred\ M\ (\lambda x.\ P\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ \exists i.\ P\ x\ i)$
by (*auto intro*!: *sets.sets_Collect_countable_All sets.sets_Collect_countable_Ex simp*: *pred_def*)

lemma *pred_intros_countable_bounded*[*measurable* (*raw*)]:
fixes $X :: 'i :: countable\ set$
shows
$(\bigwedge i.\ i \in X \Longrightarrow pred\ M\ (\lambda x.\ x \in N\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ x \in (\bigcap i{\in}X.\ N\ x\ i))$
$(\bigwedge i.\ i \in X \Longrightarrow pred\ M\ (\lambda x.\ x \in N\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ x \in (\bigcup i{\in}X.\ N\ x\ i))$
$(\bigwedge i.\ i \in X \Longrightarrow pred\ M\ (\lambda x.\ P\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ \forall i{\in}X.\ P\ x\ i)$
$(\bigwedge i.\ i \in X \Longrightarrow pred\ M\ (\lambda x.\ P\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ \exists i{\in}X.\ P\ x\ i)$
by *simp_all* (*auto simp*: *Bex_def Ball_def*)

lemma *pred_intros_finite*[*measurable* (*raw*)]:
$finite\ I \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow pred\ M\ (\lambda x.\ x \in N\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ x \in (\bigcap i{\in}I.\ N\ x\ i))$
$finite\ I \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow pred\ M\ (\lambda x.\ x \in N\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ x \in (\bigcup i{\in}I.\ N\ x\ i))$
$finite\ I \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow pred\ M\ (\lambda x.\ P\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ \forall i{\in}I.\ P\ x\ i)$
$finite\ I \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow pred\ M\ (\lambda x.\ P\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ \exists i{\in}I.\ P\ x\ i)$
by (*auto intro*!: *sets.sets_Collect_finite_Ex sets.sets_Collect_finite_All simp*: *iff_conv_conj_imp pred_def*)

lemma *countable_Un_Int*[*measurable* (*raw*)]:
$(\bigwedge i :: 'i :: countable.\ i \in I \Longrightarrow N\ i \in sets\ M) \Longrightarrow (\bigcup i{\in}I.\ N\ i) \in sets\ M$
$I \neq \{\} \Longrightarrow (\bigwedge i :: 'i :: countable.\ i \in I \Longrightarrow N\ i \in sets\ M) \Longrightarrow (\bigcap i{\in}I.\ N\ i) \in sets\ M$
by *auto*

declare
finite_UN[*measurable* (*raw*)]
finite_INT[*measurable* (*raw*)]

lemma *sets_Int_pred*[*measurable* (*raw*)]:
assumes *space*: $A \cap B \subseteq space\ M$ **and** [*measurable*]: $pred\ M\ (\lambda x.\ x \in A)$ $pred\ M\ (\lambda x.\ x \in B)$
shows $A \cap B \in sets\ M$
proof –
have $\{x{\in}space\ M.\ x \in A \cap B\} \in sets\ M$ **by** *auto*
also have $\{x{\in}space\ M.\ x \in A \cap B\} = A \cap B$

using *space* **by** *auto*
finally show *?thesis* **.**
qed

lemma [*measurable* (*raw generic*)]:
assumes *f*: *f* ∈ *measurable M N* **and** *c*: *c* ∈ *space N* ⟹ {*c*} ∈ *sets N*
shows *pred_eq_const1*: *pred M* (λ*x*. *f x* = *c*)
and *pred_eq_const2*: *pred M* (λ*x*. *c* = *f x*)
proof –
show *pred M* (λ*x*. *f x* = *c*)
proof *cases*
assume *c* ∈ *space N*
with *measurable_sets*[*OF f c*] **show** *?thesis*
by (*auto simp*: *Int_def conj_commute pred_def*)
next
assume *c* ∉ *space N*
with *f*[*THEN measurable_space*] **have** {*x* ∈ *space M*. *f x* = *c*} = {} **by** *auto*
then show *?thesis* **by** (*auto simp*: *pred_def cong*: *conj_cong*)
qed
then show *pred M* (λ*x*. *c* = *f x*)
by (*simp add*: *eq_commute*)
qed

lemma *pred_count_space_const1*[*measurable* (*raw*)]:
f ∈ *measurable M* (*count_space UNIV*) ⟹ *Measurable.pred M* (λ*x*. *f x* = *c*)
by (*intro pred_eq_const1*[**where** *N*=*count_space UNIV*]) (*auto*)

lemma *pred_count_space_const2*[*measurable* (*raw*)]:
f ∈ *measurable M* (*count_space UNIV*) ⟹ *Measurable.pred M* (λ*x*. *c* = *f x*)
by (*intro pred_eq_const2*[**where** *N*=*count_space UNIV*]) (*auto*)

lemma *pred_le_const*[*measurable* (*raw generic*)]:
assumes *f*: *f* ∈ *measurable M N* **and** *c*: {.. *c*} ∈ *sets N* **shows** *pred M* (λ*x*. *f x* ≤ *c*)
using *measurable_sets*[*OF f c*]
by (*auto simp*: *Int_def conj_commute eq_commute pred_def*)

lemma *pred_const_le*[*measurable* (*raw generic*)]:
assumes *f*: *f* ∈ *measurable M N* **and** *c*: {*c* ..} ∈ *sets N* **shows** *pred M* (λ*x*. *c* ≤ *f x*)
using *measurable_sets*[*OF f c*]
by (*auto simp*: *Int_def conj_commute eq_commute pred_def*)

lemma *pred_less_const*[*measurable* (*raw generic*)]:
assumes *f*: *f* ∈ *measurable M N* **and** *c*: {..< *c*} ∈ *sets N* **shows** *pred M* (λ*x*. *f x* < *c*)
using *measurable_sets*[*OF f c*]
by (*auto simp*: *Int_def conj_commute eq_commute pred_def*)

lemma *pred_const_less*[*measurable* (*raw generic*)]:
assumes *f*: $f \in$ *measurable M N* **and** *c*: {*c* <..} $\in$ *sets N* **shows** *pred M* (λx. $c < f\ x$)
using *measurable_sets*[*OF f c*]
by (*auto simp*: *Int_def conj_commute eq_commute pred_def*)

declare
sets.Int[*measurable* (*raw*)]

lemma *pred_in_If*[*measurable* (*raw*)]:
($P \Longrightarrow$ *pred M* (λx. $x \in A\ x$)) $\Longrightarrow$ ($\neg$ $P \Longrightarrow$ *pred M* (λx. $x \in B\ x$)) $\Longrightarrow$
pred M (λx. $x \in$ (*if P then A x else B x*))
by *auto*

lemma *sets_range*[*measurable_dest*]:
A ' $I \subseteq$ *sets M* $\Longrightarrow$ $i \in I \Longrightarrow$ $A\ i \in$ *sets M*
by *auto*

lemma *pred_sets_range*[*measurable_dest*]:
A ' $I \subseteq$ *sets N* $\Longrightarrow$ $i \in I \Longrightarrow$ $f \in$ *measurable M N* $\Longrightarrow$ *pred M* (λx. $f\ x \in A\ i$)
using *pred_sets2*[*OF sets_range*] **by** *auto*

lemma *sets_All*[*measurable_dest*]:
$\forall i$. $A\ i \in$ *sets* ($M\ i$) $\Longrightarrow$ $A\ i \in$ *sets* ($M\ i$)
by *auto*

lemma *pred_sets_All*[*measurable_dest*]:
$\forall i$. $A\ i \in$ *sets* ($N\ i$) $\Longrightarrow$ $f \in$ *measurable M* ($N\ i$) $\Longrightarrow$ *pred M* (λx. $f\ x \in A\ i$)
using *pred_sets2*[*OF sets_All*, *of A N f*] **by** *auto*

lemma *sets_Ball*[*measurable_dest*]:
$\forall i{\in}I$. $A\ i \in$ *sets* ($M\ i$) $\Longrightarrow$ $i{\in}I \Longrightarrow$ $A\ i \in$ *sets* ($M\ i$)
by *auto*

lemma *pred_sets_Ball*[*measurable_dest*]:
$\forall i{\in}I$. $A\ i \in$ *sets* ($N\ i$) $\Longrightarrow$ $i{\in}I \Longrightarrow$ $f \in$ *measurable M* ($N\ i$) $\Longrightarrow$ *pred M* (λx. $f\ x \in A\ i$)
using *pred_sets2*[*OF sets_Ball*, *of _ _ _ f*] **by** *auto*

lemma *measurable_finite*[*measurable* (*raw*)]:
fixes *S* :: *'a* $\Rightarrow$ *nat set*
assumes [*measurable*]: $\bigwedge i$. {$x{\in}$*space M*. $i \in S\ x$} $\in$ *sets M*
shows *pred M* (λx. *finite* ($S\ x$))
unfolding *finite_nat_set_iff_bounded* **by** (*simp add*: *Ball_def*)

lemma *measurable_Least*[*measurable*]:
assumes [*measurable*]: ($\bigwedge i$::*nat*. (λx. *P i x*) $\in$ *measurable M* (*count_space UNIV*))
shows (λx. *LEAST i*. *P i x*) $\in$ *measurable M* (*count_space UNIV*)
unfolding *measurable_def* **by** (*safe intro*!: *sets_Least*) *simp_all*

lemma *measurable_Max_nat*[*measurable* (*raw*)]:
  fixes *P* :: *nat* ⇒ *'a* ⇒ *bool*
  assumes [*measurable*]: ⋀*i*. *Measurable.pred M* (*P i*)
  shows (λ*x*. *Max* {*i*. *P i x*}) ∈ *measurable M* (*count_space UNIV*)
  unfolding *measurable_count_space_eq2_countable*
proof *safe*
  fix *n*

  { fix *x* **assume** ∀ *i*. ∃ *n*≥*i*. *P n x*
    then have *infinite* {*i*. *P i x*}
      unfolding *infinite_nat_iff_unbounded_le* **by** *auto*
    then have *Max* {*i*. *P i x*} = *the None*
      by (*rule Max.infinite*) **}**
  note *1* = *this*

  { fix *x i j* **assume** *P i x* ∀ *n*≥*j*. ¬ *P n x*
    then have *finite* {*i*. *P i x*}
      by (*auto simp*: *subset_eq not_le*[*symmetric*] *finite_nat_iff_bounded*)
    with ‹*P i x*› **have** *P* (*Max* {*i*. *P i x*}) *x i* ≤ *Max* {*i*. *P i x*} *finite* {*i*. *P i x*}
      using *Max_in*[*of* {*i*. *P i x*}] **by** *auto* **}**
  note *2* = *this*

  have (λ*x*. *Max* {*i*. *P i x*}) −‘ {*n*} ∩ *space M* = {*x*∈*space M*. *Max* {*i*. *P i x*} =
n}
    by *auto*
  also have ... =
    {*x*∈*space M*. *if* (∀ *i*. ∃ *n*≥*i*. *P n x*) *then the None* = *n else*
      if (∃ *i*. *P i x*) *then P n x* ∧ (∀ *i*>*n*. ¬ *P i x*)
      else Max {} = *n*}
    by (*intro arg_cong*[**where** *f*=*Collect*] *ext conj_cong*)
      (*auto simp add*: *1 2 not_le*[*symmetric*] *intro*!: *Max_eqI*)
  also have ... ∈ *sets M*
    by *measurable*
  finally show (λ*x*. *Max* {*i*. *P i x*}) −‘ {*n*} ∩ *space M* ∈ *sets M* **.**
qed *simp*

lemma *measurable_Min_nat*[*measurable* (*raw*)]:
  fixes *P* :: *nat* ⇒ *'a* ⇒ *bool*
  assumes [*measurable*]: ⋀*i*. *Measurable.pred M* (*P i*)
  shows (λ*x*. *Min* {*i*. *P i x*}) ∈ *measurable M* (*count_space UNIV*)
  unfolding *measurable_count_space_eq2_countable*
proof *safe*
  fix *n*

  { fix *x* **assume** ∀ *i*. ∃ *n*≥*i*. *P n x*
    then have *infinite* {*i*. *P i x*}
      unfolding *infinite_nat_iff_unbounded_le* **by** *auto*
    then have *Min* {*i*. *P i x*} = *the None*

```
    by (rule Min.infinite) }
  note 1 = this

  { fix x i j assume P i x ∀n≥j. ¬ P n x
    then have finite {i. P i x}
      by (auto simp: subset_eq not_le[symmetric] finite_nat_iff_bounded)
    with ‹P i x› have P (Min {i. P i x}) x Min {i. P i x} ≤ i finite {i. P i x}
      using Min_in[of {i. P i x}] by auto }
  note 2 = this

  have (λx. Min {i. P i x}) −` {n} ∩ space M = {x∈space M. Min {i. P i x} =
n}
    by auto
  also have ... =
    {x∈space M. if (∀i. ∃n≥i. P n x) then the None = n else
      if (∃i. P i x) then P n x ∧ (∀i<n. ¬ P i x)
      else Min {} = n}
    by (intro arg_cong[where f=Collect] ext conj_cong)
       (auto simp add: 1 2 not_le[symmetric] intro!: Min_eqI)
  also have ... ∈ sets M
    by measurable
  finally show (λx. Min {i. P i x}) −` {n} ∩ space M ∈ sets M .
qed simp

lemma measurable_count_space_insert[measurable (raw)]:
  s ∈ S ⟹ A ∈ sets (count_space S) ⟹ insert s A ∈ sets (count_space S)
  by simp

lemma sets_UNIV [measurable (raw)]: A ∈ sets (count_space UNIV)
  by simp

lemma measurable_card[measurable]:
  fixes S :: 'a ⇒ nat set
  assumes [measurable]: ⋀i. {x∈space M. i ∈ S x} ∈ sets M
  shows (λx. card (S x)) ∈ measurable M (count_space UNIV)
  unfolding measurable_count_space_eq2_countable
proof safe
  fix n show (λx. card (S x)) −` {n} ∩ space M ∈ sets M
  proof (cases n)
    case 0
    then have (λx. card (S x)) −` {n} ∩ space M = {x∈space M. infinite (S x)
∨ (∀i. i ∉ S x)}
      by auto
    also have ... ∈ sets M
      by measurable
    finally show ?thesis .
  next
    case (Suc i)
    then have (λx. card (S x)) −` {n} ∩ space M =
```

($\bigcup F{\in}\{A{\in}\{A.$ *finite* $A\}.$ *card* $A = n\}.$ $\{x{\in}$*space* $M.$ $(\forall\, i.\ i \in S\ x \longleftrightarrow i \in F)\})$
unfolding *set_eq_iff*[*symmetric*] *Collect_bex_eq*[*symmetric*] **by** (*auto intro*: *card_ge_0_finite*)
also have ... ∈ *sets M*
by (*intro sets.countable_UN′ countable_Collect countable_Collect_finite*) *auto*
finally show *?thesis* **.**
qed
qed *rule*

lemma *measurable_pred_countable*[*measurable* (*raw*)]:
assumes *countable X*
shows
$(\bigwedge i.\ i \in X \Longrightarrow$ *Measurable.pred M* $(\lambda x.\ P\ x\ i)) \Longrightarrow$ *Measurable.pred M* $(\lambda x.\ \forall\, i{\in}X.\ P\ x\ i)$
$(\bigwedge i.\ i \in X \Longrightarrow$ *Measurable.pred M* $(\lambda x.\ P\ x\ i)) \Longrightarrow$ *Measurable.pred M* $(\lambda x.\ \exists\, i{\in}X.\ P\ x\ i)$
unfolding *pred_def*
by (*auto intro*!: *sets.sets_Collect_countable_All′ sets.sets_Collect_countable_Ex′ assms*)

6.2.1 Measurability for (co)inductive predicates

lemma *measurable_bot*[*measurable*]: *bot* ∈ *measurable M* (*count_space UNIV*)
by (*simp add*: *bot_fun_def*)

lemma *measurable_top*[*measurable*]: *top* ∈ *measurable M* (*count_space UNIV*)
by (*simp add*: *top_fun_def*)

lemma *measurable_SUP*[*measurable*]:
fixes *F* :: ′*i* ⇒ ′*a* ⇒ ′*b*::{*complete_lattice*, *countable*}
assumes [*simp*]: *countable I*
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow F\ i \in$ *measurable M* (*count_space UNIV*)
shows $(\lambda x.$ *SUP* $i{\in}I.\ F\ i\ x) \in$ *measurable M* (*count_space UNIV*)
unfolding *measurable_count_space_eq2_countable*
proof (*safe intro*!: *UNIV_I*)
fix *a*
have $(\lambda x.$ *SUP* $i{\in}I.\ F\ i\ x)$ −‘ $\{a\} \cap$ *space M* =
$\{x{\in}$*space* $M.\ (\forall\, i{\in}I.\ F\ i\ x \le a) \wedge (\forall\, b.\ (\forall\, i{\in}I.\ F\ i\ x \le b) \longrightarrow a \le b)\}$
unfolding *SUP_le_iff*[*symmetric*] **by** *auto*
also have ... ∈ *sets M*
by *measurable*
finally show $(\lambda x.$ *SUP* $i{\in}I.\ F\ i\ x)$ −‘ $\{a\} \cap$ *space M* ∈ *sets M* **.**
qed

lemma *measurable_INF*[*measurable*]:
fixes *F* :: ′*i* ⇒ ′*a* ⇒ ′*b*::{*complete_lattice*, *countable*}
assumes [*simp*]: *countable I*
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow F\ i \in$ *measurable M* (*count_space UNIV*)

shows (λx. *INF* i∈I. F i x) ∈ *measurable* M (*count_space UNIV*)
unfolding *measurable_count_space_eq2_countable*
proof (*safe intro*!: *UNIV_I*)
fix a
have (λx. *INF* i∈I. F i x) −‘ {a} ∩ *space* M =
{x∈*space* M. (∀ i∈I. a ≤ F i x) ∧ (∀ b. (∀ i∈I. b ≤ F i x) ⟶ b ≤ a)}
unfolding *le_INF_iff*[*symmetric*] **by** *auto*
also have ... ∈ *sets* M
by *measurable*
finally show (λx. *INF* i∈I. F i x) −‘ {a} ∩ *space* M ∈ *sets* M **.**
qed

lemma *measurable_lfp_coinduct*[*consumes 1*, *case_names continuity step*]:
fixes F :: ('a ⇒ 'b) ⇒ ('a ⇒ 'b::{*complete_lattice*, *countable*})
assumes P M
assumes F: *sup_continuous* F
assumes ∗: ⋀M A. P M ⟹ (⋀N. P N ⟹ A ∈ *measurable* N (*count_space* *UNIV*)) ⟹ F A ∈ *measurable* M (*count_space UNIV*)
shows *lfp* F ∈ *measurable* M (*count_space UNIV*)
proof −
{ fix i **from** ‹P M› **have** ((F ^^ i) *bot*) ∈ *measurable* M (*count_space UNIV*)
by (*induct i arbitrary*: M) (*auto intro*!: ∗) **}**
then have (λx. *SUP* i. (F ^^ i) *bot* x) ∈ *measurable* M (*count_space UNIV*)
by *measurable*
also have (λx. *SUP* i. (F ^^ i) *bot* x) = *lfp* F
by (*subst sup_continuous_lfp*) (*auto intro*: F *simp*: *image_comp*)
finally show *?thesis* **.**
qed

lemma *measurable_lfp*:
fixes F :: ('a ⇒ 'b) ⇒ ('a ⇒ 'b::{*complete_lattice*, *countable*})
assumes F: *sup_continuous* F
assumes ∗: ⋀A. A ∈ *measurable* M (*count_space UNIV*) ⟹ F A ∈ *measurable* M (*count_space UNIV*)
shows *lfp* F ∈ *measurable* M (*count_space UNIV*)
by (*coinduction rule*: *measurable_lfp_coinduct*[*OF* _ F]) (*blast intro*: ∗)

lemma *measurable_gfp_coinduct*[*consumes 1*, *case_names continuity step*]:
fixes F :: ('a ⇒ 'b) ⇒ ('a ⇒ 'b::{*complete_lattice*, *countable*})
assumes P M
assumes F: *inf_continuous* F
assumes ∗: ⋀M A. P M ⟹ (⋀N. P N ⟹ A ∈ *measurable* N (*count_space* *UNIV*)) ⟹ F A ∈ *measurable* M (*count_space UNIV*)
shows *gfp* F ∈ *measurable* M (*count_space UNIV*)
proof −
{ fix i **from** ‹P M› **have** ((F ^^ i) *top*) ∈ *measurable* M (*count_space UNIV*)
by (*induct i arbitrary*: M) (*auto intro*!: ∗) **}**
then have (λx. *INF* i. (F ^^ i) *top* x) ∈ *measurable* M (*count_space UNIV*)
by *measurable*

also have ($\lambda x.$ *INF i*. (*F* ^^ *i*) *top x*) = *gfp F*
by (*subst inf_continuous_gfp*) (*auto intro*: *F simp*: *image_comp*)
finally show *?thesis* **.**
qed

lemma *measurable_gfp*:
fixes *F* :: ('*a* ⇒ '*b*) ⇒ ('*a* ⇒ '*b*::{*complete_lattice*, *countable*})
assumes *F*: *inf_continuous F*
assumes ∗: ⋀*A*. *A* ∈ *measurable M* (*count_space UNIV*) ⟹ *F A* ∈ *measurable M* (*count_space UNIV*)
shows *gfp F* ∈ *measurable M* (*count_space UNIV*)
by (*coinduction rule*: *measurable_gfp_coinduct*[*OF* _ *F*]) (*blast intro*: ∗)

lemma *measurable_lfp2_coinduct*[*consumes 1*, *case_names continuity step*]:
fixes *F* :: ('*a* ⇒ '*c* ⇒ '*b*) ⇒ ('*a* ⇒ '*c* ⇒ '*b*::{*complete_lattice*, *countable*})
assumes *P M s*
assumes *F*: *sup_continuous F*
assumes ∗: ⋀*M A s*. *P M s* ⟹ (⋀*N t*. *P N t* ⟹ *A t* ∈ *measurable N* (*count_space UNIV*)) ⟹ *F A s* ∈ *measurable M* (*count_space UNIV*)
shows *lfp F s* ∈ *measurable M* (*count_space UNIV*)
proof −
{ **fix** *i* **from** ‹*P M s*› **have** ($\lambda x.$ (*F* ^^ *i*) *bot s x*) ∈ *measurable M* (*count_space UNIV*)
by (*induct i arbitrary*: *M s*) (*auto intro*!: ∗) }
then have ($\lambda x.$ *SUP i*. (*F* ^^ *i*) *bot s x*) ∈ *measurable M* (*count_space UNIV*)
by *measurable*
also have ($\lambda x.$ *SUP i*. (*F* ^^ *i*) *bot s x*) = *lfp F s*
by (*subst sup_continuous_lfp*) (*auto simp*: *F simp*: *image_comp*)
finally show *?thesis* **.**
qed

lemma *measurable_gfp2_coinduct*[*consumes 1*, *case_names continuity step*]:
fixes *F* :: ('*a* ⇒ '*c* ⇒ '*b*) ⇒ ('*a* ⇒ '*c* ⇒ '*b*::{*complete_lattice*, *countable*})
assumes *P M s*
assumes *F*: *inf_continuous F*
assumes ∗: ⋀*M A s*. *P M s* ⟹ (⋀*N t*. *P N t* ⟹ *A t* ∈ *measurable N* (*count_space UNIV*)) ⟹ *F A s* ∈ *measurable M* (*count_space UNIV*)
shows *gfp F s* ∈ *measurable M* (*count_space UNIV*)
proof −
{ **fix** *i* **from** ‹*P M s*› **have** ($\lambda x.$ (*F* ^^ *i*) *top s x*) ∈ *measurable M* (*count_space UNIV*)
by (*induct i arbitrary*: *M s*) (*auto intro*!: ∗) }
then have ($\lambda x.$ *INF i*. (*F* ^^ *i*) *top s x*) ∈ *measurable M* (*count_space UNIV*)
by *measurable*
also have ($\lambda x.$ *INF i*. (*F* ^^ *i*) *top s x*) = *gfp F s*
by (*subst inf_continuous_gfp*) (*auto simp*: *F simp*: *image_comp*)
finally show *?thesis* **.**
qed

lemma *measurable_enat_coinduct*:
fixes f :: $'a \Rightarrow enat$
assumes $R\ f$
assumes $*$: $\bigwedge f.\ R\ f \Longrightarrow \exists g\ h\ i\ P.\ R\ g \wedge f = (\lambda x.\ if\ P\ x\ then\ h\ x\ else\ eSuc\ (g\ (i\ x))) \wedge$
Measurable.pred $M\ P \wedge$
$i \in measurable\ M\ M \wedge$
$h \in measurable\ M\ (count\_space\ UNIV)$
shows $f \in measurable\ M\ (count\_space\ UNIV)$
proof (*simp add*: *measurable_count_space_eq2_countable*, *rule*)
fix a :: *enat*
have $f -` \{a\} \cap space\ M = \{x{\in}space\ M.\ f\ x = a\}$
by *auto*
{ **fix** i :: *nat*
from ‹$R\ f$› **have** *Measurable.pred* $M\ (\lambda x.\ f\ x = enat\ i)$
proof (*induction i arbitrary*: f)
case *0*
from $*$[*OF this*] **obtain** $g\ h\ i\ P$
where f: $f = (\lambda x.\ if\ P\ x\ then\ h\ x\ else\ eSuc\ (g\ (i\ x)))$ **and**
[*measurable*]: *Measurable.pred* $M\ P$ $i \in measurable\ M\ M$ $h \in measurable\ M\ (count\_space\ UNIV)$
by *auto*
have *Measurable.pred* $M\ (\lambda x.\ P\ x \wedge h\ x = 0)$
by *measurable*
also have $(\lambda x.\ P\ x \wedge h\ x = 0) = (\lambda x.\ f\ x = enat\ 0)$
by (*auto simp*: *f zero_enat_def*[*symmetric*])
finally show *?case* .
next
case (*Suc n*)
from $*$[*OF Suc.prems*] **obtain** $g\ h\ i\ P$
where f: $f = (\lambda x.\ if\ P\ x\ then\ h\ x\ else\ eSuc\ (g\ (i\ x)))$ **and** $R\ g$ **and**
M[*measurable*]: *Measurable.pred* $M\ P$ $i \in measurable\ M\ M$ $h \in measurable\ M\ (count\_space\ UNIV)$
by *auto*
have $(\lambda x.\ f\ x = enat\ (Suc\ n)) =$
$(\lambda x.\ (P\ x \longrightarrow h\ x = enat\ (Suc\ n)) \wedge (\neg\ P\ x \longrightarrow g\ (i\ x) = enat\ n))$
by (*auto simp*: *f zero_enat_def*[*symmetric*] *eSuc_enat*[*symmetric*])
also have *Measurable.pred* M ...
by (*intro pred_intros_logic measurable_compose*[*OF M*(*2*)] *Suc* ‹$R\ g$›) *measurable*
finally show *?case* .
qed
then have $f -` \{enat\ i\} \cap space\ M \in sets\ M$
by (*simp add*: *pred_def Int_def conj_commute*) }
note *fin* = *this*
show $f -` \{a\} \cap space\ M \in sets\ M$
proof (*cases a*)
case *infinity*
then have $f -` \{a\} \cap space\ M = space\ M - (\bigcup n.\ f -` \{enat\ n\} \cap space\ M)$

by *auto*
also have ... ∈ *sets M*
by (*intro sets.Diff sets.top sets.Un sets.countable_UN*) (*auto intro*!: *fin*)
finally show *?thesis* **.**
qed (*simp add*: *fin*)
qed

lemma *measurable_THE*:
fixes $P :: 'a \Rightarrow 'b \Rightarrow bool$
assumes [*measurable*]: $\bigwedge i.$ *Measurable.pred M* (*P i*)
assumes *I*[*simp*]: *countable I* $\bigwedge i\ x.\ x \in space\ M \Longrightarrow P\ i\ x \Longrightarrow i \in I$
assumes *unique*: $\bigwedge x\ i\ j.\ x \in space\ M \Longrightarrow P\ i\ x \Longrightarrow P\ j\ x \Longrightarrow i = j$
shows ($\lambda x.$ *THE i. P i x*) ∈ *measurable M* (*count_space UNIV*)
unfolding *measurable_def*
proof *safe*
fix *X*
define *f* **where** *f x* = (*THE i. P i x*) **for** *x*
define *undef* **where** *undef* = (*THE i::'a. False*)
{ fix *i x* **assume** *x* ∈ *space M P i x* **then have** *f x* = *i*
unfolding *f_def* **using** *unique* **by** *auto* **}**
note *f_eq* = *this*
{ fix *x* **assume** *x* ∈ *space M* ∀ *i*∈*I*. ¬ *P i x*
then have $\bigwedge i.\ \neg\ P\ i\ x$
using *I*(*2*)[*of x*] **by** *auto*
then have *f x* = *undef*
by (*auto simp*: *undef_def f_def*) **}**
then have *f* −‘ *X* ∩ *space M* = ($\bigcup$*i*∈*I* ∩ *X*. {*x*∈*space M*. *P i x*}) ∪
(*if undef* ∈ *X then space M* − ($\bigcup$*i*∈*I*. {*x*∈*space M*. *P i x*}) *else* {})
by (*auto dest*: *f_eq*)
also have ... ∈ *sets M*
by (*auto intro*!: *sets.Diff sets.countable_UN'*)
finally show *f* −‘ *X* ∩ *space M* ∈ *sets M* **.**
qed *simp*

lemma *measurable_Ex1*[*measurable* (*raw*)]:
assumes [*simp*]: *countable I* **and** [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow$ *Measurable.pred M* (*P i*)
shows *Measurable.pred M* ($\lambda x.\ \exists!i\in I.\ P\ i\ x$)
unfolding *bex1_def* **by** *measurable*

lemma *measurable_Sup_nat*[*measurable* (*raw*)]:
fixes $F :: 'a \Rightarrow nat\ set$
assumes [*measurable*]: $\bigwedge i.$ *Measurable.pred M* ($\lambda x.\ i \in F\ x$)
shows ($\lambda x.$ *Sup* (*F x*)) ∈ $M \rightarrow_M$ *count_space UNIV*
proof (*clarsimp simp add*: *measurable_count_space_eq2_countable*)
fix *a*
have *F_empty_iff*: *F x* = {} ⟷ (∀ *i*. *i* ∉ *F x*) **for** *x*
by *auto*
have *Measurable.pred M* ($\lambda x.$ *if finite* (*F x*) *then if F x* = {} *then a* = *0*

```
    else a ∈ F x ∧ (∀j. j ∈ F x ⟶ j ≤ a) else a = the None)
    unfolding finite_nat_set_iff_bounded Ball_def F_empty_iff by measurable
  moreover have (λx. Sup (F x)) −‘ {a} ∩ space M =
    {x∈space M. if finite (F x) then if F x = {} then a = 0
      else a ∈ F x ∧ (∀j. j ∈ F x ⟶ j ≤ a) else a = the None}
    by (intro set_eqI)
       (auto simp: Sup_nat_def Max.infinite intro!: Max_in Max_eqI)
  ultimately show (λx. Sup (F x)) −‘ {a} ∩ space M ∈ sets M
    by auto
qed

lemma measurable_if_split[measurable (raw)]:
  (c ⟹ Measurable.pred M f) ⟹ (¬ c ⟹ Measurable.pred M g) ⟹
   Measurable.pred M (if c then f else g)
  by simp

lemma pred_restrict_space:
  assumes S ∈ sets M
  shows Measurable.pred (restrict_space M S) P ⟷ Measurable.pred M (λx. x ∈
S ∧ P x)
  unfolding pred_def sets_Collect_restrict_space_iff[OF assms] ..

lemma measurable_predpow[measurable]:
  assumes Measurable.pred M T
  assumes ⋀Q. Measurable.pred M Q ⟹ Measurable.pred M (R Q)
  shows Measurable.pred M ((R ^^ n) T)
  by (induct n) (auto intro: assms)

lemma measurable_compose_countable_restrict:
  assumes P: countable {i. P i}
    and f: f ∈ M →M count_space UNIV
    and Q: ⋀i. P i ⟹ pred M (Q i)
  shows pred M (λx. P (f x) ∧ Q (f x) x)
proof −
  have P_f: {x ∈ space M. P (f x)} ∈ sets M
    unfolding pred_def[symmetric] by (rule measurable_compose[OF f]) simp
  have pred (restrict_space M {x∈space M. P (f x)}) (λx. Q (f x) x)
  proof (rule measurable_compose_countable'[where g=f, OF _ _ P])
    show f ∈ restrict_space M {x∈space M. P (f x)} →M count_space {i. P i}
      by (rule measurable_count_space_extend[OF subset_UNIV])
         (auto simp: space_restrict_space intro!: measurable_restrict_space1 f)
  qed (auto intro!: measurable_restrict_space1 Q)
  then show ?thesis
    unfolding pred_restrict_space[OF P_f] by (simp cong: measurable_cong)
qed

lemma measurable_limsup [measurable (raw)]:
  assumes [measurable]: ⋀n. A n ∈ sets M
  shows limsup A ∈ sets M
```

by (*subst limsup_INF_SUP*, *auto*)

lemma *measurable_liminf* [*measurable* (*raw*)]:
  assumes [*measurable*]: $\bigwedge n.\ A\ n \in sets\ M$
  shows *liminf* $A \in sets\ M$
by (*subst liminf_SUP_INF*, *auto*)

lemma *measurable_case_enat*[*measurable* (*raw*)]:
  assumes *f*: $f \in M \rightarrow_M$ *count_space UNIV* **and** *g*: $\bigwedge i.\ g\ i \in M \rightarrow_M N$ **and** *h*: $h \in M \rightarrow_M N$
  shows ($\lambda x.$ *case f x of enat i* $\Rightarrow g\ i\ x \mid \infty \Rightarrow h\ x) \in M \rightarrow_M N$
  apply (*rule measurable_compose_countable*[*OF _ f*])
  subgoal for *i*
    by (*cases i*) (*auto intro*: *g h*)
  done

hide_const (**open**) *pred*

end

6.3 Measure Spaces

theory *Measure_Space*
imports
  Measurable HOL−Library.Extended_Nonnegative_Real
begin

6.3.1 Relate extended reals and the indicator function

lemma *suminf_cmult_indicator*:
  fixes f :: *nat* $\Rightarrow$ *ennreal*
  assumes *disjoint_family A* $x \in A\ i$
  shows ($\sum n.\ f\ n * indicator\ (A\ n)\ x) = f\ i$
proof −
  have ∗∗: $\bigwedge n.\ f\ n * indicator\ (A\ n)\ x = ($*if* $n = i$ *then* $f\ n$ *else* 0 :: *ennreal*)
    using ‹$x \in A\ i$› *assms* **unfolding** *disjoint_family_on_def indicator_def* **by** *auto*
  then have $\bigwedge n.\ (\sum j<n.\ f\ j * indicator\ (A\ j)\ x) = ($*if* $i < n$ *then* $f\ i$ *else* 0 :: *ennreal*)
    by (*auto simp*: *sum.If_cases*)
  moreover have (*SUP n. if* $i < n$ *then f i else* 0) = (*f i* :: *ennreal*)
  proof (*rule SUP_eqI*)
    fix *y* :: *ennreal* **assume** $\bigwedge n.\ n \in UNIV \Longrightarrow ($*if* $i < n$ *then f i else* 0) $\le y$
    from *this*[*of Suc i*] **show** $f\ i \le y$ **by** *auto*
  qed (*insert assms*, *simp*)
  ultimately show *?thesis* **using** *assms*
    by (*subst suminf_eq_SUP*) (*auto simp*: *indicator_def*)
qed

lemma *suminf_indicator*:

assumes *disjoint_family A*
shows ($\sum$ *n. indicator (A n) x :: ennreal*) = *indicator* ($\bigcup$ *i. A i*) *x*
proof *cases*
assume ∗: *x* ∈ ($\bigcup$ *i. A i*)
then obtain *i* **where** *x* ∈ *A i* **by** *auto*
from *suminf_cmult_indicator*[*OF assms*(*1*), *OF* ‹*x* ∈ *A i*›, *of* λ*k. 1*]
show *?thesis* **using** ∗ **by** *simp*
qed *simp*

lemma *sum_indicator_disjoint_family*:
fixes *f* :: ′*d* ⇒ ′*e*::*semiring_1*
assumes *d*: *disjoint_family_on A P* **and** *x* ∈ *A j* **and** *finite P* **and** *j* ∈ *P*
shows ($\sum$ *i*∈*P. f i* ∗ *indicator* (*A i*) *x*) = *f j*
proof −
have *P* ∩ {*i. x* ∈ *A i*} = {*j*}
using *d* ‹*x* ∈ *A j*› ‹*j* ∈ *P*› **unfolding** *disjoint_family_on_def*
by *auto*
thus *?thesis*
unfolding *indicator_def*
by (*simp add*: *if_distrib sum.If_cases*[*OF* ‹*finite P*›])
qed

The type for emeasure spaces is already defined in *HOL−Analysis.Sigma_Algebra*, as it is also used to represent sigma algebras (with an arbitrary emeasure).

6.3.2 Extend binary sets

lemma *LIMSEQ_binaryset*:
assumes *f*: *f* {} = *0*
shows (λ*n*. $\sum$ *i*<*n. f* (*binaryset A B i*)) ⟶ *f A* + *f B*
proof −
have (λ*n*. $\sum$ *i* < *Suc* (*Suc n*). *f* (*binaryset A B i*)) = (λ*n. f A* + *f B*)
proof
fix *n*
show ($\sum$ *i* < *Suc* (*Suc n*). *f* (*binaryset A B i*)) = *f A* + *f B*
by (*induct n*) (*auto simp add*: *binaryset_def f*)
qed
moreover
have ... ⟶ *f A* + *f B* **by** (*rule tendsto_const*)
ultimately
have (λ*n*. $\sum$ *i*< *Suc* (*Suc n*). *f* (*binaryset A B i*)) ⟶ *f A* + *f B*
by *metis*
hence (λ*n*. $\sum$ *i*< *n*+*2*. *f* (*binaryset A B i*)) ⟶ *f A* + *f B*
by *simp*
thus *?thesis* **by** (*rule LIMSEQ_offset* [**where** *k*=*2*])
qed

lemma *binaryset_sums*:
assumes *f*: *f* {} = *0*

shows $(\lambda n.\ f\ (binaryset\ A\ B\ n))$ *sums* $(f\ A + f\ B)$
by (*simp add*: *sums_def LIMSEQ_binaryset* [**where** *f*=*f*, *OF f*] *atLeast0LessThan*)

lemma *suminf_binaryset_eq*:
fixes $f :: 'a\ set \Rightarrow 'b::\{comm\_monoid\_add,\ t2\_space\}$
shows $f\ \{\} = 0 \Longrightarrow (\sum n.\ f\ (binaryset\ A\ B\ n)) = f\ A + f\ B$
by (*metis binaryset_sums sums_unique*)

6.3.3 Properties of a premeasure μ

The definitions for *positive* and *countably_additive* should be here, by they are necessary to define *'a measure* in *HOL−Analysis.Sigma_Algebra*.

definition *subadditive* **where**
subadditive M f $\longleftrightarrow (\forall x{\in}M.\ \forall y{\in}M.\ x \cap y = \{\} \longrightarrow f\ (x \cup y) \le f\ x + f\ y)$

lemma *subadditiveD*: *subadditive M f* $\Longrightarrow x \cap y = \{\} \Longrightarrow x \in M \Longrightarrow y \in M \Longrightarrow f\ (x \cup y) \le f\ x + f\ y$
by (*auto simp add*: *subadditive_def*)

definition *countably_subadditive* **where**
countably_subadditive M f $\longleftrightarrow$
$(\forall A.\ range\ A \subseteq M \longrightarrow disjoint\_family\ A \longrightarrow (\bigcup i.\ A\ i) \in M \longrightarrow (f\ (\bigcup i.\ A\ i) \le (\sum i.\ f\ (A\ i))))$

lemma (**in** *ring_of_sets*) *countably_subadditive_subadditive*:
fixes $f :: 'a\ set \Rightarrow ennreal$
assumes *f*: *positive M f* **and** *cs*: *countably_subadditive M f*
shows *subadditive M f*
proof (*auto simp add*: *subadditive_def*)
fix *x y*
assume *x*: $x \in M$ **and** *y*: $y \in M$ **and** $x \cap y = \{\}$
hence *disjoint_family* (*binaryset x y*)
by (*auto simp add*: *disjoint_family_on_def binaryset_def*)
hence $range\ (binaryset\ x\ y) \subseteq M \longrightarrow$
$(\bigcup i.\ binaryset\ x\ y\ i) \in M \longrightarrow$
$f\ (\bigcup i.\ binaryset\ x\ y\ i) \le (\sum n.\ f\ (binaryset\ x\ y\ n))$
using *cs* **by** (*auto simp add*: *countably_subadditive_def*)
hence $\{x,y,\{\}\} \subseteq M \longrightarrow x \cup y \in M \longrightarrow$
$f\ (x \cup y) \le (\sum n.\ f\ (binaryset\ x\ y\ n))$
by (*simp add*: *range_binaryset_eq UN_binaryset_eq*)
thus $f\ (x \cup y) \le f\ x + f\ y$ **using** *f x y*
by (*auto simp add*: *Un o_def suminf_binaryset_eq positive_def*)
qed

definition *additive* **where**
additive M $\mu \longleftrightarrow (\forall x{\in}M.\ \forall y{\in}M.\ x \cap y = \{\} \longrightarrow \mu\ (x \cup y) = \mu\ x + \mu\ y)$

definition *increasing* **where**
increasing M $\mu \longleftrightarrow (\forall x{\in}M.\ \forall y{\in}M.\ x \subseteq y \longrightarrow \mu\ x \le \mu\ y)$

lemma *positiveD1*: *positive M f* ⟹ *f* {} = *0* **by** (*auto simp*: *positive_def*)

lemma *positiveD_empty*:
  positive M f ⟹ *f* {} = *0*
  by (*auto simp add*: *positive_def*)

lemma *additiveD*:
  additive M f ⟹ *x* ∩ *y* = {} ⟹ *x* ∈ *M* ⟹ *y* ∈ *M* ⟹ *f* (*x* ∪ *y*) = *f x* + *f y*
  by (*auto simp add*: *additive_def*)

lemma *increasingD*:
  increasing M f ⟹ *x* ⊆ *y* ⟹ *x*∈*M* ⟹ *y*∈*M* ⟹ *f x* ≤ *f y*
  by (*auto simp add*: *increasing_def*)

lemma *countably_additiveI*[*case_names countably*]:
  (⋀*A*. *range A* ⊆ *M* ⟹ *disjoint_family A* ⟹ (⋃*i*. *A i*) ∈ *M* ⟹ (∑*i*. *f* (*A i*)) = *f* (⋃*i*. *A i*))
  ⟹ *countably_additive M f*
  by (*simp add*: *countably_additive_def*)

lemma (**in** *ring_of_sets*) *disjointed_additive*:
  assumes *f*: *positive M f additive M f* **and** *A*: *range A* ⊆ *M incseq A*
  shows (∑*i*≤*n*. *f* (*disjointed A i*)) = *f* (*A n*)
proof (*induct n*)
  case (*Suc n*)
  then have (∑*i*≤*Suc n*. *f* (*disjointed A i*)) = *f* (*A n*) + *f* (*disjointed A* (*Suc n*))
    by *simp*
  also have ... = *f* (*A n* ∪ *disjointed A* (*Suc n*))
    using *A* **by** (*subst f*(*2*)[*THEN additiveD*]) (*auto simp*: *disjointed_mono*)
  also have *A n* ∪ *disjointed A* (*Suc n*) = *A* (*Suc n*)
    using ‹*incseq A*› **by** (*auto dest*: *incseq_SucD simp*: *disjointed_mono*)
  finally show *?case* **.**
qed *simp*

lemma (**in** *ring_of_sets*) *additive_sum*:
  fixes *A*:: *'i* ⇒ *'a set*
  assumes *f*: *positive M f* **and** *ad*: *additive M f* **and** *finite S*
    and *A*: *A`S* ⊆ *M*
    and *disj*: *disjoint_family_on A S*
  shows (∑*i*∈*S*. *f* (*A i*)) = *f* (⋃*i*∈*S*. *A i*)
  using ‹*finite S*› *disj A*
proof *induct*
  case *empty* **show** *?case* **using** *f* **by** (*simp add*: *positive_def*)
next
  case (*insert s S*)
  then have *A s* ∩ (⋃*i*∈*S*. *A i*) = {}
    by (*auto simp add*: *disjoint_family_on_def neq_iff*)

moreover
have $A\ s \in M$ **using** *insert* **by** *blast*
moreover have $(\bigcup i{\in}S.\ A\ i) \in M$
using *insert* ‹*finite S*› **by** *auto*
ultimately have $f\ (A\ s \cup (\bigcup i{\in}S.\ A\ i)) = f\ (A\ s) + f(\bigcup i{\in}S.\ A\ i)$
using *ad UNION_in_sets A* **by** (*auto simp add*: *additive_def*)
with *insert* **show** *?case* **using** *ad disjoint_family_on_mono*[*of S insert s S A*]
by (*auto simp add*: *additive_def subset_insertI*)
qed

lemma (**in** *ring_of_sets*) *additive_increasing*:
fixes f :: $'a\ set \Rightarrow ennreal$
assumes *posf*: *positive M f* **and** *addf*: *additive M f*
shows *increasing M f*
proof (*auto simp add*: *increasing_def*)
fix *x y*
assume *xy*: $x \in M\ y \in M\ x \subseteq y$
then have $y - x \in M$ **by** *auto*
then have $f\,x + 0 \le f\,x + f\ (y{-}x)$ **by** (*intro add_left_mono zero_le*)
also have ... $= f\ (x \cup (y{-}x))$ **using** *addf*
by (*auto simp add*: *additive_def*) (*metis Diff_disjoint Un_Diff_cancel Diff xy*(*1*,*2*))
also have ... $= f\ y$
by (*metis Un_Diff_cancel Un_absorb1 xy*(*3*))
finally show $f\,x \le f\,y$ **by** *simp*
qed

lemma (**in** *ring_of_sets*) *subadditive*:
fixes f :: $'a\ set \Rightarrow ennreal$
assumes *f*: *positive M f additive M f* **and** *A*: $A`S \subseteq M$ **and** *S*: *finite S*
shows $f\ (\bigcup i{\in}S.\ A\ i) \le (\sum i{\in}S.\ f\ (A\ i))$
using *S A*
proof (*induct S*)
case *empty* **thus** *?case* **using** *f* **by** (*auto simp*: *positive_def*)
next
case (*insert x F*)
hence *in_M*: $A\ x \in M\ (\bigcup i{\in}F.\ A\ i) \in M\ (\bigcup i{\in}F.\ A\ i) - A\ x \in M$ **using** *A* **by** *force+*
have *subs*: $(\bigcup i{\in}F.\ A\ i) - A\ x \subseteq (\bigcup i{\in}F.\ A\ i)$ **by** *auto*
have $(\bigcup i{\in}(insert\ x\ F).\ A\ i) = A\ x \cup ((\bigcup i{\in}F.\ A\ i) - A\ x)$ **by** *auto*
hence $f\ (\bigcup i{\in}(insert\ x\ F).\ A\ i) = f\ (A\ x \cup ((\bigcup i{\in}F.\ A\ i) - A\ x))$
by *simp*
also have $\ldots = f\ (A\ x) + f\ ((\bigcup i{\in}F.\ A\ i) - A\ x)$
using *f*(*2*) **by** (*rule additiveD*) (*insert in_M*, *auto*)
also have $\ldots \le f\ (A\ x) + f\ (\bigcup i{\in}F.\ A\ i)$
using *additive_increasing*[*OF f*] *in_M subs* **by** (*auto simp*: *increasing_def intro*: *add_left_mono*)
also have $\ldots \le f\ (A\ x) + (\sum i{\in}F.\ f\ (A\ i))$ **using** *insert* **by** (*auto intro*: *add_left_mono*)
finally show $f\ (\bigcup i{\in}(insert\ x\ F).\ A\ i) \le (\sum i{\in}(insert\ x\ F).\ f\ (A\ i))$ **using**

insert **by** *simp*
qed

lemma (**in** *ring_of_sets*) *countably_additive_additive*:
fixes *f* :: *'a set* ⇒ *ennreal*
assumes *posf*: *positive M f* **and** *ca*: *countably_additive M f*
shows *additive M f*
proof (*auto simp add*: *additive_def*)
fix *x y*
assume *x*: $x \in M$ **and** *y*: $y \in M$ **and** $x \cap y = \{\}$
hence *disjoint_family* (*binaryset x y*)
by (*auto simp add*: *disjoint_family_on_def binaryset_def*)
hence *range* (*binaryset x y*) ⊆ *M* ⟶
(⋃ *i*. *binaryset x y i*) ∈ *M* ⟶
f (⋃ *i*. *binaryset x y i*) = (∑ *n*. *f* (*binaryset x y n*))
using *ca*
by (*simp add*: *countably_additive_def*)
hence {*x*,*y*,{}} ⊆ *M* ⟶ *x* ∪ *y* ∈ *M* ⟶
f (*x* ∪ *y*) = (∑ *n*. *f* (*binaryset x y n*))
by (*simp add*: *range_binaryset_eq UN_binaryset_eq*)
thus *f* (*x* ∪ *y*) = *f x* + *f y* **using** *posf x y*
by (*auto simp add*: *Un suminf_binaryset_eq positive_def*)
qed

lemma (**in** *algebra*) *increasing_additive_bound*:
fixes *A*:: *nat* ⇒ *'a set* **and** *f* :: *'a set* ⇒ *ennreal*
assumes *f*: *positive M f* **and** *ad*: *additive M f*
and *inc*: *increasing M f*
and *A*: *range A* ⊆ *M*
and *disj*: *disjoint_family A*
shows (∑ *i*. *f* (*A i*)) ≤ *f* Ω
proof (*safe intro*!: *suminf_le_const*)
fix *N*
note *disj_N* = *disjoint_family_on_mono*[*OF* _ *disj*, *of* {..<*N*}]
have (∑ *i*<*N*. *f* (*A i*)) = *f* (⋃ *i*∈{..<*N*}. *A i*)
using *A* **by** (*intro additive_sum* [*OF f ad* _ _]) (*auto simp*: *disj_N*)
also have ... ≤ *f* Ω **using** *space_closed A*
by (*intro increasingD*[*OF inc*] *finite_UN*) *auto*
finally show (∑ *i*<*N*. *f* (*A i*)) ≤ *f* Ω **by** *simp*
qed (*insert f A*, *auto simp*: *positive_def*)

lemma (**in** *ring_of_sets*) *countably_additiveI_finite*:
fixes μ :: *'a set* ⇒ *ennreal*
assumes *finite* Ω *positive M* μ *additive M* μ
shows *countably_additive M* μ
proof (*rule countably_additiveI*)
fix *F* :: *nat* ⇒ *'a set* **assume** *F*: *range F* ⊆ *M* (⋃ *i*. *F i*) ∈ *M* **and** *disj*: *disjoint_family F*

have $\forall i \in \{i.\ F\ i \neq \{\}\}.\ \exists x.\ x \in F\ i$ **by** *auto*
from *bchoice*[*OF this*] **obtain** *f* **where** *f*: $\bigwedge i.\ F\ i \neq \{\} \Longrightarrow f\ i \in F\ i$ **by** *auto*

have *inj_f*: *inj_on f* $\{i.\ F\ i \neq \{\}\}$
proof (*rule inj_onI*, *simp*)
fix *i j a b* **assume** $*$: $f\ i = f\ j\ F\ i \neq \{\}\ F\ j \neq \{\}$
then have $f\ i \in F\ i\ f\ j \in F\ j$ **using** *f* **by** *force+*
with *disj* $*$ **show** $i = j$ **by** (*auto simp*: *disjoint_family_on_def*)
qed
have *finite* $(\bigcup i.\ F\ i)$
by (*metis F(2) assms(1) infinite_super sets_into_space*)

have *F_subset*: $\{i.\ \mu\ (F\ i) \neq 0\} \subseteq \{i.\ F\ i \neq \{\}\}$
by (*auto simp*: *positiveD_empty*[*OF* ‹*positive M* μ›])
moreover have *fin_not_empty*: *finite* $\{i.\ F\ i \neq \{\}\}$
proof (*rule finite_imageD*)
from *f* **have** $f`\{i.\ F\ i \neq \{\}\} \subseteq (\bigcup i.\ F\ i)$ **by** *auto*
then show *finite* $(f`\{i.\ F\ i \neq \{\}\})$
by (*rule finite_subset*) *fact*
qed *fact*
ultimately have *fin_not_0*: *finite* $\{i.\ \mu\ (F\ i) \neq 0\}$
by (*rule finite_subset*)

have *disj_not_empty*: *disjoint_family_on F* $\{i.\ F\ i \neq \{\}\}$
using *disj* **by** (*auto simp*: *disjoint_family_on_def*)

from *fin_not_0* **have** $(\sum i.\ \mu\ (F\ i)) = (\sum i\ |\ \mu\ (F\ i) \neq 0.\ \mu\ (F\ i))$
by (*rule suminf_finite*) *auto*
also have $\ldots = (\sum i\ |\ F\ i \neq \{\}.\ \mu\ (F\ i))$
using *fin_not_empty F_subset* **by** (*rule sum.mono_neutral_left*) *auto*
also have $\ldots = \mu\ (\bigcup i \in \{i.\ F\ i \neq \{\}\}.\ F\ i)$
using ‹*positive M* μ› ‹*additive M* μ› *fin_not_empty disj_not_empty F* **by** (*intro additive_sum*) *auto*
also have $\ldots = \mu\ (\bigcup i.\ F\ i)$
by (*rule arg_cong*[**where** $f=\mu$]) *auto*
finally show $(\sum i.\ \mu\ (F\ i)) = \mu\ (\bigcup i.\ F\ i)$ **.**
qed

lemma (**in** *ring_of_sets*) *countably_additive_iff_continuous_from_below*:
fixes *f* :: *'a set* $\Rightarrow$ *ennreal*
assumes *f*: *positive M f additive M f*
shows *countably_additive M f* $\longleftrightarrow$
$(\forall A.\ range\ A \subseteq M \longrightarrow incseq\ A \longrightarrow (\bigcup i.\ A\ i) \in M \longrightarrow (\lambda i.\ f\ (A\ i)) \longrightarrow f\ (\bigcup i.\ A\ i))$
unfolding *countably_additive_def*
proof *safe*
assume *count_sum*: $\forall A.\ range\ A \subseteq M \longrightarrow disjoint\_family\ A \longrightarrow \bigcup(A\ `\ UNIV) \in M \longrightarrow (\sum i.\ f\ (A\ i)) = f\ (\bigcup(A\ `\ UNIV))$
fix *A* :: *nat* $\Rightarrow$ *'a set* **assume** *A*: $range\ A \subseteq M\ incseq\ A\ (\bigcup i.\ A\ i) \in M$

then have *dA*: *range* (*disjointed A*) ⊆ *M* **by** (*auto simp*: *range_disjointed_sets*)
with *count_sum*[*THEN spec*, *of disjointed A*] *A*(*3*)
have *f_UN*: ($\sum$ *i*. *f* (*disjointed A i*)) = *f* ($\bigcup$ *i*. *A i*)
by (*auto simp*: *UN_disjointed_eq disjoint_family_disjointed*)
moreover have (λ*n*. ($\sum$ *i*<*n*. *f* (*disjointed A i*))) ⟶ ($\sum$ *i*. *f* (*disjointed A i*))
using *f*(*1*)[*unfolded positive_def*] *dA*
by (*auto intro*!: *summable_LIMSEQ*)
from *LIMSEQ_Suc*[*OF this*]
have (λ*n*. ($\sum$ *i*≤*n*. *f* (*disjointed A i*))) ⟶ ($\sum$ *i*. *f* (*disjointed A i*))
unfolding *lessThan_Suc_atMost* **.**
moreover have ⋀*n*. ($\sum$ *i*≤*n*. *f* (*disjointed A i*)) = *f* (*A n*)
using *disjointed_additive*[*OF f A*(*1*,*2*)] **.**
ultimately show (λ*i*. *f* (*A i*)) ⟶ *f* ($\bigcup$ *i*. *A i*) **by** *simp*
next
assume *cont*: ∀ *A*. *range A* ⊆ *M* ⟶ *incseq A* ⟶ ($\bigcup$ *i*. *A i*) ∈ *M* ⟶ (λ*i*. *f* (*A i*)) ⟶ *f* ($\bigcup$ *i*. *A i*)
fix *A* :: *nat* ⇒ *'a set* **assume** *A*: *range A* ⊆ *M disjoint_family A* ($\bigcup$ *i*. *A i*) ∈ *M*
have ∗: ($\bigcup$ *n*. ($\bigcup$ *i*<*n*. *A i*)) = ($\bigcup$ *i*. *A i*) **by** *auto*
have (λ*n*. *f* ($\bigcup$ *i*<*n*. *A i*)) ⟶ *f* ($\bigcup$ *i*. *A i*)
proof (*unfold* ∗[*symmetric*], *intro cont*[*rule_format*])
show *range* (λ*i*. $\bigcup$ *i*<*i*. *A i*) ⊆ *M* ($\bigcup$ *i*. $\bigcup$ *i*<*i*. *A i*) ∈ *M*
using *A* ∗ **by** *auto*
qed (*force intro*!: *incseq_SucI*)
moreover have ⋀*n*. *f* ($\bigcup$ *i*<*n*. *A i*) = ($\sum$ *i*<*n*. *f* (*A i*))
using *A*
by (*intro additive_sum*[*OF f*, *of _ A*, *symmetric*])
(*auto intro*: *disjoint_family_on_mono*[**where** *B*=*UNIV*])
ultimately
have (λ*i*. *f* (*A i*)) *sums f* ($\bigcup$ *i*. *A i*)
unfolding *sums_def* **by** *simp*
from *sums_unique*[*OF this*]
show ($\sum$ *i*. *f* (*A i*)) = *f* ($\bigcup$ *i*. *A i*) **by** *simp*
qed

lemma (**in** *ring_of_sets*) *continuous_from_above_iff_empty_continuous*:
fixes *f* :: *'a set* ⇒ *ennreal*
assumes *f*: *positive M f additive M f*
shows (∀ *A*. *range A* ⊆ *M* ⟶ *decseq A* ⟶ ($\bigcap$ *i*. *A i*) ∈ *M* ⟶ (∀ *i*. *f* (*A i*) ≠ ∞) ⟶ (λ*i*. *f* (*A i*)) ⟶ *f* ($\bigcap$ *i*. *A i*))
⟷ (∀ *A*. *range A* ⊆ *M* ⟶ *decseq A* ⟶ ($\bigcap$ *i*. *A i*) = {} ⟶ (∀ *i*. *f* (*A i*) ≠ ∞) ⟶ (λ*i*. *f* (*A i*)) ⟶ *0*)
proof *safe*
assume *cont*: (∀ *A*. *range A* ⊆ *M* ⟶ *decseq A* ⟶ ($\bigcap$ *i*. *A i*) ∈ *M* ⟶ (∀ *i*. *f* (*A i*) ≠ ∞) ⟶ (λ*i*. *f* (*A i*)) ⟶ *f* ($\bigcap$ *i*. *A i*))
fix *A* :: *nat* ⇒ *'a set* **assume** *A*: *range A* ⊆ *M decseq A* ($\bigcap$ *i*. *A i*) = {} ∀ *i*. *f* (*A i*) ≠ ∞
with *cont*[*THEN spec*, *of A*] **show** (λ*i*. *f* (*A i*)) ⟶ *0*
using ‹*positive M f*›[*unfolded positive_def*] **by** *auto*

next
assume *cont*: $\forall A.\ range\ A \subseteq M \longrightarrow decseq\ A \longrightarrow (\bigcap i.\ A\ i) = \{\} \longrightarrow (\forall i.\ f\ (A\ i) \neq \infty) \longrightarrow (\lambda i.\ f\ (A\ i)) \longrightarrow 0$
fix $A :: nat \Rightarrow 'a\ set$ **assume** A: $range\ A \subseteq M\ decseq\ A\ (\bigcap i.\ A\ i) \in M\ \forall i.\ f\ (A\ i) \neq \infty$

have *f_mono*: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a \subseteq b \Longrightarrow f\ a \leq f\ b$
using *additive_increasing*[*OF f*] **unfolding** *increasing_def* **by** *simp*

have *decseq_fA*: *decseq* $(\lambda i.\ f\ (A\ i))$
using *A* **by** (*auto simp*: *decseq_def intro*!: *f_mono*)
have *decseq*: *decseq* $(\lambda i.\ A\ i - (\bigcap i.\ A\ i))$
using *A* **by** (*auto simp*: *decseq_def*)
then have *decseq_f*: *decseq* $(\lambda i.\ f\ (A\ i - (\bigcap i.\ A\ i)))$
using *A* **unfolding** *decseq_def* **by** (*auto intro*!: *f_mono Diff*)
have $f\ (\bigcap x.\ A\ x) \leq f\ (A\ 0)$
using *A* **by** (*auto intro*!: *f_mono*)
then have *f_Int_fin*: $f\ (\bigcap x.\ A\ x) \neq \infty$
using *A* **by** (*auto simp*: *top_unique*)
{ fix *i*
have $f\ (A\ i - (\bigcap i.\ A\ i)) \leq f\ (A\ i)$ **using** *A* **by** (*auto intro*!: *f_mono*)
then have $f\ (A\ i - (\bigcap i.\ A\ i)) \neq \infty$
using *A* **by** (*auto simp*: *top_unique*) **}**
note *f_fin* = *this*
have $(\lambda i.\ f\ (A\ i - (\bigcap i.\ A\ i))) \longrightarrow 0$
proof (*intro cont*[*rule_format*, *OF _ decseq _ f_fin*])
show $range\ (\lambda i.\ A\ i - (\bigcap i.\ A\ i)) \subseteq M\ (\bigcap i.\ A\ i - (\bigcap i.\ A\ i)) = \{\}$
using *A* **by** *auto*
qed
from *INF_Lim*[*OF decseq_f this*]
have $(INF\ n.\ f\ (A\ n - (\bigcap i.\ A\ i))) = 0$ **.**
moreover have $(INF\ n.\ f\ (\bigcap i.\ A\ i)) = f\ (\bigcap i.\ A\ i)$
by *auto*
ultimately have $(INF\ n.\ f\ (A\ n - (\bigcap i.\ A\ i)) + f\ (\bigcap i.\ A\ i)) = 0 + f\ (\bigcap i.\ A\ i)$
using *A*(*4*) *f_fin f_Int_fin*
by (*subst INF_ennreal_add_const*) (*auto simp*: *decseq_f*)
moreover {
fix *n*
have $f\ (A\ n - (\bigcap i.\ A\ i)) + f\ (\bigcap i.\ A\ i) = f\ ((A\ n - (\bigcap i.\ A\ i)) \cup (\bigcap i.\ A\ i))$
using *A* **by** (*subst f*(*2*)[*THEN additiveD*]) *auto*
also have $(A\ n - (\bigcap i.\ A\ i)) \cup (\bigcap i.\ A\ i) = A\ n$
by *auto*
finally have $f\ (A\ n - (\bigcap i.\ A\ i)) + f\ (\bigcap i.\ A\ i) = f\ (A\ n)$ **. }**
ultimately have $(INF\ n.\ f\ (A\ n)) = f\ (\bigcap i.\ A\ i)$
by *simp*
with *LIMSEQ_INF*[*OF decseq_fA*]
show $(\lambda i.\ f\ (A\ i)) \longrightarrow f\ (\bigcap i.\ A\ i)$ **by** *simp*

qed

lemma (**in** *ring_of_sets*) *empty_continuous_imp_continuous_from_below*:
fixes f :: *'a set* ⇒ *ennreal*
assumes f: *positive M f additive M f* $\forall A \in M.\ f\ A \neq \infty$
assumes *cont*: $\forall A.$ *range A* $\subseteq M \longrightarrow$ *decseq A* $\longrightarrow (\bigcap i.\ A\ i) = \{\} \longrightarrow (\lambda i.\ f\ (A\ i)) \longrightarrow 0$
assumes A: *range A* $\subseteq M$ *incseq A* $(\bigcup i.\ A\ i) \in M$
shows $(\lambda i.\ f\ (A\ i)) \longrightarrow f\ (\bigcup i.\ A\ i)$
proof −
from A **have** $(\lambda i.\ f\ ((\bigcup i.\ A\ i) - A\ i)) \longrightarrow 0$
by (*intro cont*[*rule_format*]) (*auto simp*: *decseq_def incseq_def*)
moreover
{ **fix** i
have $f\ ((\bigcup i.\ A\ i) - A\ i \cup A\ i) = f\ ((\bigcup i.\ A\ i) - A\ i) + f\ (A\ i)$
using A **by** (*intro f*(*2*)[*THEN additiveD*]) *auto*
also have $((\bigcup i.\ A\ i) - A\ i) \cup A\ i = (\bigcup i.\ A\ i)$
by *auto*
finally have $f\ ((\bigcup i.\ A\ i) - A\ i) = f\ (\bigcup i.\ A\ i) - f\ (A\ i)$
using *f*(*3*)[*rule_format*, *of A i*] *A* **by** (*auto simp*: *ennreal_add_diff_cancel subset_eq*) **}**
moreover have $\forall_F$ *i in sequentially*. $f\ (A\ i) \le f\ (\bigcup i.\ A\ i)$
using *increasingD*[*OF additive_increasing*[*OF f*(*1*, *2*)], *of A* _ $\bigcup i.\ A\ i$] *A*
by (*auto intro*!: *always_eventually simp*: *subset_eq*)
ultimately show $(\lambda i.\ f\ (A\ i)) \longrightarrow f\ (\bigcup i.\ A\ i)$
by (*auto intro*: *ennreal_tendsto_const_minus*)
qed

lemma (**in** *ring_of_sets*) *empty_continuous_imp_countably_additive*:
fixes f :: *'a set* ⇒ *ennreal*
assumes f: *positive M f additive M f* **and** *fin*: $\forall A \in M.\ f\ A \neq \infty$
assumes *cont*: $\bigwedge A.$ *range A* $\subseteq M \Longrightarrow$ *decseq A* $\Longrightarrow (\bigcap i.\ A\ i) = \{\} \Longrightarrow (\lambda i.\ f\ (A\ i)) \longrightarrow 0$
shows *countably_additive M f*
using *countably_additive_iff_continuous_from_below*[*OF f*]
using *empty_continuous_imp_continuous_from_below*[*OF f fin*] *cont*
by *blast*

6.3.4 Properties of *emeasure*

lemma *emeasure_positive*: *positive* (*sets M*) (*emeasure M*)
by (*cases M*) (*auto simp*: *sets_def emeasure_def Abs_measure_inverse measure_space_def*)

lemma *emeasure_empty*[*simp*, *intro*]: *emeasure M* {} = *0*
using *emeasure_positive*[*of M*] **by** (*simp add*: *positive_def*)

lemma *emeasure_single_in_space*: *emeasure M* $\{x\} \neq 0 \Longrightarrow x \in$ *space M*
using *emeasure_notin_sets*[*of* $\{x\}$ *M*] **by** (*auto dest*: *sets.sets_into_space zero_less_iff_neq_zero*[*THEN iffD2*])

lemma *emeasure_countably_additive*: *countably_additive* (*sets M*) (*emeasure M*)
by (*cases M*) (*auto simp*: *sets_def emeasure_def Abs_measure_inverse measure_space_def*)

lemma *suminf_emeasure*:
range A ⊆ *sets M* ⟹ *disjoint_family A* ⟹ ($\sum$ *i*. *emeasure M* (*A i*)) = *emeasure M* ($\bigcup$ *i*. *A i*)
using *sets.countable_UN*[*of A UNIV M*] *emeasure_countably_additive*[*of M*]
by (*simp add*: *countably_additive_def*)

lemma *sums_emeasure*:
disjoint_family F ⟹ ($\bigwedge$*i*. *F i* ∈ *sets M*) ⟹ (λ*i*. *emeasure M* (*F i*)) *sums emeasure M* ($\bigcup$ *i*. *F i*)
unfolding *sums_iff* **by** (*intro conjI suminf_emeasure*) *auto*

lemma *emeasure_additive*: *additive* (*sets M*) (*emeasure M*)
by (*metis sets.countably_additive_additive emeasure_positive emeasure_countably_additive*)

lemma *plus_emeasure*:
a ∈ *sets M* ⟹ *b* ∈ *sets M* ⟹ *a* ∩ *b* = {} ⟹ *emeasure M a* + *emeasure M b* = *emeasure M* (*a* ∪ *b*)
using *additiveD*[*OF emeasure_additive*] **..**

lemma *emeasure_Un*:
A ∈ *sets M* ⟹ *B* ∈ *sets M* ⟹ *emeasure M* (*A* ∪ *B*) = *emeasure M A* + *emeasure M* (*B* − *A*)
using *plus_emeasure*[*of A M B* − *A*] **by** *auto*

lemma *emeasure_Un_Int*:
assumes *A* ∈ *sets M B* ∈ *sets M*
shows *emeasure M A* + *emeasure M B* = *emeasure M* (*A* ∪ *B*) + *emeasure M* (*A* ∩ *B*)
proof −
have *A* = (*A*−*B*) ∪ (*A* ∩ *B*) **by** *auto*
then have *emeasure M A* = *emeasure M* (*A*−*B*) + *emeasure M* (*A* ∩ *B*)
by (*metis Diff_Diff_Int Diff_disjoint assms plus_emeasure sets.Diff*)
moreover have *A* ∪ *B* = (*A*−*B*) ∪ *B* **by** *auto*
then have *emeasure M* (*A* ∪ *B*) = *emeasure M* (*A*−*B*) + *emeasure M B*
by (*metis Diff_disjoint Int_commute assms plus_emeasure sets.Diff*)
ultimately show *?thesis* **by** (*metis add.assoc add.commute*)
qed

lemma *sum_emeasure*:
F‘I ⊆ *sets M* ⟹ *disjoint_family_on F I* ⟹ *finite I* ⟹
($\sum$*i*∈*I*. *emeasure M* (*F i*)) = *emeasure M* ($\bigcup$*i*∈*I*. *F i*)
by (*metis sets.additive_sum emeasure_positive emeasure_additive*)

lemma *emeasure_mono*:
a ⊆ *b* ⟹ *b* ∈ *sets M* ⟹ *emeasure M a* ≤ *emeasure M b*

by (*metis zero_le sets.additive_increasing emeasure_additive emeasure_notin_sets emeasure_positive increasingD*)

lemma *emeasure_space*:
emeasure M A $\leq$ *emeasure M* (*space M*)
by (*metis emeasure_mono emeasure_notin_sets sets.sets_into_space sets.top zero_le*)

lemma *emeasure_Diff*:
assumes *finite*: *emeasure M B* $\neq \infty$
and [*measurable*]: *A* $\in$ *sets M B* $\in$ *sets M* **and** *B* $\subseteq$ *A*
shows *emeasure M* (*A* $-$ *B*) = *emeasure M A* $-$ *emeasure M B*
proof $-$
have (*A* $-$ *B*) $\cup$ *B* = *A* **using** ‹*B* $\subseteq$ *A*› **by** *auto*
then have *emeasure M A* = *emeasure M* ((*A* $-$ *B*) $\cup$ *B*) **by** *simp*
also have ... = *emeasure M* (*A* $-$ *B*) + *emeasure M B*
by (*subst plus_emeasure*[*symmetric*]) *auto*
finally show *emeasure M* (*A* $-$ *B*) = *emeasure M A* $-$ *emeasure M B*
using *finite* **by** *simp*
qed

lemma *emeasure_compl*:
s $\in$ *sets M* $\Longrightarrow$ *emeasure M s* $\neq \infty \Longrightarrow$ *emeasure M* (*space M* $-$ *s*) = *emeasure M* (*space M*) $-$ *emeasure M s*
by (*rule emeasure_Diff*) (*auto dest*: *sets.sets_into_space*)

lemma *Lim_emeasure_incseq*:
range A $\subseteq$ *sets M* $\Longrightarrow$ *incseq A* $\Longrightarrow$ (λ*i*. (*emeasure M* (*A i*))) $\longlongrightarrow$ *emeasure M* ($\bigcup$*i*. *A i*)
using *emeasure_countably_additive*
by (*auto simp add*: *sets.countably_additive_iff_continuous_from_below emeasure_positive emeasure_additive*)

lemma *incseq_emeasure*:
assumes *range B* $\subseteq$ *sets M incseq B*
shows *incseq* (λ*i*. *emeasure M* (*B i*))
using *assms* **by** (*auto simp*: *incseq_def intro*!: *emeasure_mono*)

lemma *SUP_emeasure_incseq*:
assumes *A*: *range A* $\subseteq$ *sets M incseq A*
shows (*SUP n*. *emeasure M* (*A n*)) = *emeasure M* ($\bigcup$*i*. *A i*)
using *LIMSEQ_SUP*[*OF incseq_emeasure*, *OF A*] *Lim_emeasure_incseq*[*OF A*]
by (*simp add*: *LIMSEQ_unique*)

lemma *decseq_emeasure*:
assumes *range B* $\subseteq$ *sets M decseq B*
shows *decseq* (λ*i*. *emeasure M* (*B i*))
using *assms* **by** (*auto simp*: *decseq_def intro*!: *emeasure_mono*)

lemma *INF_emeasure_decseq*:

assumes *A*: *range A* ⊆ *sets M* **and** *decseq A*
and *finite*: $\bigwedge i$. *emeasure M* (*A i*) ≠ ∞
shows (*INF n*. *emeasure M* (*A n*)) = *emeasure M* ($\bigcap i$. *A i*)
proof −
have *le_MI*: *emeasure M* ($\bigcap i$. *A i*) ≤ *emeasure M* (*A 0*)
using *A* **by** (*auto intro*!: *emeasure_mono*)
hence ∗: *emeasure M* ($\bigcap i$. *A i*) ≠ ∞ **using** *finite*[*of 0*] **by** (*auto simp*: *top_unique*)

have *emeasure M* (*A 0*) − (*INF n*. *emeasure M* (*A n*)) = (*SUP n*. *emeasure M* (*A 0*) − *emeasure M* (*A n*))
by (*simp add*: *ennreal_INF_const_minus*)
also have ... = (*SUP n*. *emeasure M* (*A 0* − *A n*))
using *A finite* ‹*decseq A*›[*unfolded decseq_def*] **by** (*subst emeasure_Diff*) *auto*
also have ... = *emeasure M* ($\bigcup i$. *A 0* − *A i*)
proof (*rule SUP_emeasure_incseq*)
show *range* (λ*n*. *A 0* − *A n*) ⊆ *sets M*
using *A* **by** *auto*
show *incseq* (λ*n*. *A 0* − *A n*)
using ‹*decseq A*› **by** (*auto simp add*: *incseq_def decseq_def*)
qed
also have ... = *emeasure M* (*A 0*) − *emeasure M* ($\bigcap i$. *A i*)
using *A finite* ∗ **by** (*simp*, *subst emeasure_Diff*) *auto*
finally show *?thesis*
by (*rule ennreal_minus_cancel*[*rotated 3*])
(*insert finite A*, *auto intro*: *INF_lower emeasure_mono*)
qed

lemma *INF_emeasure_decseq′*:
assumes *A*: $\bigwedge i$. *A i* ∈ *sets M* **and** *decseq A*
and *finite*: ∃ *i*. *emeasure M* (*A i*) ≠ ∞
shows (*INF n*. *emeasure M* (*A n*)) = *emeasure M* ($\bigcap i$. *A i*)
proof −
from *finite* **obtain** *i* **where** *i*: *emeasure M* (*A i*) < ∞
by (*auto simp*: *less_top*)
have *fin*: *i* ≤ *j* ⟹ *emeasure M* (*A j*) < ∞ **for** *j*
by (*rule le_less_trans*[*OF emeasure_mono i*]) (*auto intro*!: *decseqD*[*OF* ‹*decseq A*›] *A*)

have (*INF n*. *emeasure M* (*A n*)) = (*INF n*. *emeasure M* (*A* (*n* + *i*)))
proof (*rule INF_eq*)
show ∃ *j*∈*UNIV*. *emeasure M* (*A* (*j* + *i*)) ≤ *emeasure M* (*A i′*) **for** *i′*
by (*intro bexI*[*of _ i′*] *emeasure_mono decseqD*[*OF* ‹*decseq A*›] *A*) *auto*
qed *auto*
also have ... = *emeasure M* (*INF n*. (*A* (*n* + *i*)))
using *A* ‹*decseq A*› *fin* **by** (*intro INF_emeasure_decseq*) (*auto simp*: *decseq_def less_top*)
also have (*INF n*. (*A* (*n* + *i*))) = (*INF n*. *A n*)
by (*meson INF_eq UNIV_I assms*(*2*) *decseqD le_add1*)
finally show *?thesis* **.**

qed

lemma *emeasure_INT_decseq_subset*:
  fixes F :: *nat* ⇒ *'a set*
  assumes I: $I \neq \{\}$ **and** F: $\bigwedge i\ j.\ i \in I \Longrightarrow j \in I \Longrightarrow i \leq j \Longrightarrow F\ j \subseteq F\ i$
  assumes *F_sets*[*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow F\ i \in$ *sets M*
    and *fin*: $\bigwedge i.\ i \in I \Longrightarrow$ *emeasure M* $(F\ i) \neq \infty$
  shows *emeasure M* $(\bigcap i{\in}I.\ F\ i) = (INF\ i{\in}I.\ emeasure\ M\ (F\ i))$
proof *cases*
  assume *finite I*
  have $(\bigcap i{\in}I.\ F\ i) = F\ (Max\ I)$
    using *I* ‹*finite I*› **by** (*intro antisym INF_lower INF_greatest F*) *auto*
  moreover have $(INF\ i{\in}I.\ emeasure\ M\ (F\ i)) = emeasure\ M\ (F\ (Max\ I))$
    using *I* ‹*finite I*› **by** (*intro antisym INF_lower INF_greatest F emeasure_mono*)
auto
  ultimately show *?thesis*
    by *simp*
next
  assume *infinite I*
  define *L* **where** $L\ n = (LEAST\ i.\ i \in I \wedge i \geq n)$ **for** *n*
  have *L*: $L\ n \in I \wedge n \leq L\ n$ **for** *n*
    unfolding *L_def*
  proof (*rule LeastI_ex*)
    show $\exists x.\ x \in I \wedge n \leq x$
      using ‹*infinite I*› *finite_subset*[*of I* {..< *n*}]
      by (*rule_tac ccontr*) (*auto simp*: *not_le*)
  qed
  have *L_eq*[*simp*]: $i \in I \Longrightarrow L\ i = i$ **for** *i*
    unfolding *L_def* **by** (*intro Least_equality*) *auto*
  have *L_mono*: $i \leq j \Longrightarrow L\ i \leq L\ j$ **for** *i j*
    using *L*[*of j*] **unfolding** *L_def* **by** (*intro Least_le*) (*auto simp*: *L_def*)

  have *emeasure M* $(\bigcap i.\ F\ (L\ i)) = (INF\ i.\ emeasure\ M\ (F\ (L\ i)))$
  proof (*intro INF_emeasure_decseq*[*symmetric*])
    show *decseq* $(\lambda i.\ F\ (L\ i))$
      using *L* **by** (*intro antimonoI F L_mono*) *auto*
  qed (*insert L fin*, *auto*)
  also have ... $= (INF\ i{\in}I.\ emeasure\ M\ (F\ i))$
  proof (*intro antisym INF_greatest*)
    show $i \in I \Longrightarrow (INF\ i.\ emeasure\ M\ (F\ (L\ i))) \leq emeasure\ M\ (F\ i)$ **for** *i*
      by (*intro INF_lower2*[*of i*]) *auto*
  qed (*insert L*, *auto intro*: *INF_lower*)
  also have $(\bigcap i.\ F\ (L\ i)) = (\bigcap i{\in}I.\ F\ i)$
  proof (*intro antisym INF_greatest*)
    show $i \in I \Longrightarrow (\bigcap i.\ F\ (L\ i)) \subseteq F\ i$ **for** *i*
      by (*intro INF_lower2*[*of i*]) *auto*
  qed (*insert L*, *auto*)
  finally show *?thesis* **.**
qed

lemma *Lim_emeasure_decseq*:
assumes *A*: *range A* ⊆ *sets M decseq A* **and** *fin*: $\bigwedge i$. *emeasure M* (*A i*) ≠ ∞
shows (λ*i*. *emeasure M* (*A i*)) ⟶ *emeasure M* ($\bigcap$*i*. *A i*)
using *LIMSEQ_INF*[*OF decseq_emeasure*, *OF A*]
using *INF_emeasure_decseq*[*OF A fin*] **by** *simp*

lemma *emeasure_lfp'*[*consumes 1*, *case_names cont measurable*]:
assumes *P M*
assumes *cont*: *sup_continuous F*
assumes ∗: $\bigwedge$*M A*. *P M* ⟹ ($\bigwedge$*N*. *P N* ⟹ *Measurable.pred N A*) ⟹ *Measurable.pred M* (*F A*)
shows *emeasure M* {*x*∈*space M*. *lfp F x*} = (*SUP i*. *emeasure M* {*x*∈*space M*. (*F* ^^ *i*) (λ*x*. *False*) *x*})
proof −
have *emeasure M* {*x*∈*space M*. *lfp F x*} = *emeasure M* ($\bigcup$*i*. {*x*∈*space M*. (*F* ^^ *i*) (λ*x*. *False*) *x*})
using *sup_continuous_lfp*[*OF cont*] **by** (*auto simp add*: *bot_fun_def intro*!: *arg_cong2*[**where** *f*=*emeasure*])
moreover { **fix** *i* **from** ‹*P M*› **have** {*x*∈*space M*. (*F* ^^ *i*) (λ*x*. *False*) *x*} ∈ *sets M*
by (*induct i arbitrary*: *M*) (*auto simp add*: *pred_def*[*symmetric*] *intro*: ∗) }
moreover have *incseq* (λ*i*. {*x*∈*space M*. (*F* ^^ *i*) (λ*x*. *False*) *x*})
proof (*rule incseq_SucI*)
fix *i*
have (*F* ^^ *i*) (λ*x*. *False*) ≤ (*F* ^^ (*Suc i*)) (λ*x*. *False*)
proof (*induct i*)
case *0* **show** *?case* **by** (*simp add*: *le_fun_def*)
next
case *Suc* **thus** *?case* **using** *monoD*[*OF sup_continuous_mono*[*OF cont*] *Suc*] **by** *auto*
qed
then show {*x* ∈ *space M*. (*F* ^^ *i*) (λ*x*. *False*) *x*} ⊆ {*x* ∈ *space M*. (*F* ^^ *Suc i*) (λ*x*. *False*) *x*}
by *auto*
qed
ultimately show *?thesis*
by (*subst SUP_emeasure_incseq*) *auto*
qed

lemma *emeasure_lfp*:
assumes [*simp*]: $\bigwedge$*s*. *sets* (*M s*) = *sets N*
assumes *cont*: *sup_continuous F sup_continuous f*
assumes *meas*: $\bigwedge$*P*. *Measurable.pred N P* ⟹ *Measurable.pred N* (*F P*)
assumes *iter*: $\bigwedge$*P s*. *Measurable.pred N P* ⟹ *P* ≤ *lfp F* ⟹ *emeasure* (*M s*) {*x*∈*space N*. *F P x*} = *f* (λ*s*. *emeasure* (*M s*) {*x*∈*space N*. *P x*}) *s*
shows *emeasure* (*M s*) {*x*∈*space N*. *lfp F x*} = *lfp f s*
proof (*subst lfp_transfer_bounded*[**where** α=λ*F s*. *emeasure* (*M s*) {*x*∈*space N*. *F x*} **and** *g*=*f* **and** *f*=*F* **and** *P*=*Measurable.pred N*, *symmetric*])

fix C **assume** *incseq* C $\bigwedge i$. *Measurable.pred* N $(C\ i)$
then show $(\lambda s.$ *emeasure* $(M\ s)$ $\{x \in$ *space* $N.$ $(SUP\ i.\ C\ i)\ x\}) = (SUP\ i.\ (\lambda s.$ *emeasure* $(M\ s)$ $\{x \in$ *space* $N.$ $C\ i\ x\}))$
unfolding *SUP_apply*[*abs_def*]
by (*subst SUP_emeasure_incseq*) (*auto simp*: *mono_def fun_eq_iff intro*!: *arg_cong2*[**where** *f*=*emeasure*])
qed (*auto simp add*: *iter le_fun_def SUP_apply*[*abs_def*] *intro*!: *meas cont*)

lemma *emeasure_subadditive_finite*:
finite $I \Longrightarrow A$ ' $I \subseteq$ *sets* $M \Longrightarrow$ *emeasure* M $(\bigcup i{\in}I.\ A\ i) \le (\sum i{\in}I.$ *emeasure* $M\ (A\ i))$
by (*rule sets.subadditive*[*OF emeasure_positive emeasure_additive*]) *auto*

lemma *emeasure_subadditive*:
$A \in$ *sets* $M \Longrightarrow B \in$ *sets* $M \Longrightarrow$ *emeasure* M $(A \cup B) \le$ *emeasure* M A + *emeasure* M B
using *emeasure_subadditive_finite*[*of* {*True*, *False*} λ*True* $\Rightarrow A$ | *False* $\Rightarrow B$ M]
by *simp*

lemma *emeasure_subadditive_countably*:
assumes *range* $f \subseteq$ *sets* M
shows *emeasure* M $(\bigcup i.\ f\ i) \le (\sum i.$ *emeasure* $M\ (f\ i))$
proof –
have *emeasure* M $(\bigcup i.\ f\ i) =$ *emeasure* M $(\bigcup i.$ *disjointed* $f\ i)$
unfolding *UN_disjointed_eq* ..
also have ... $= (\sum i.$ *emeasure* M (*disjointed* $f\ i$))
using *sets.range_disjointed_sets*[*OF assms*] *suminf_emeasure*[*of disjointed f*]
by (*simp add*: *disjoint_family_disjointed comp_def*)
also have ... $\le (\sum i.$ *emeasure* $M\ (f\ i))$
using *sets.range_disjointed_sets*[*OF assms*] *assms*
by (*auto intro*!: *suminf_le emeasure_mono disjointed_subset*)
finally show *?thesis* .
qed

lemma *emeasure_insert*:
assumes *sets*: $\{x\} \in$ *sets* M $A \in$ *sets* M **and** $x \notin A$
shows *emeasure* M (*insert* x A) = *emeasure* M $\{x\}$ + *emeasure* M A
proof –
have $\{x\} \cap A = \{\}$ **using** ‹$x \notin A$› **by** *auto*
from *plus_emeasure*[*OF sets this*] **show** *?thesis* **by** *simp*
qed

lemma *emeasure_insert_ne*:
$A \neq \{\} \Longrightarrow \{x\} \in$ *sets* $M \Longrightarrow A \in$ *sets* $M \Longrightarrow x \notin A \Longrightarrow$ *emeasure* M (*insert* x A) = *emeasure* M $\{x\}$ + *emeasure* M A
by (*rule emeasure_insert*)

lemma *emeasure_eq_sum_singleton*:
assumes *finite* S $\bigwedge x.\ x \in S \Longrightarrow \{x\} \in$ *sets* M

```
  shows emeasure M S = (∑ x∈S. emeasure M {x})
  using sum_emeasure[of λx. {x} S M] assms
  by (auto simp: disjoint_family_on_def subset_eq)

lemma sum_emeasure_cover:
  assumes finite S and A ∈ sets M and br_in_M: B ` S ⊆ sets M
  assumes A: A ⊆ (⋃ i∈S. B i)
  assumes disj: disjoint_family_on B S
  shows emeasure M A = (∑ i∈S. emeasure M (A ∩ (B i)))
proof −
  have (∑ i∈S. emeasure M (A ∩ (B i))) = emeasure M (⋃ i∈S. A ∩ (B i))
  proof (rule sum_emeasure)
    show disjoint_family_on (λi. A ∩ B i) S
      using ‹disjoint_family_on B S›
      unfolding disjoint_family_on_def by auto
  qed (insert assms, auto)
  also have (⋃ i∈S. A ∩ (B i)) = A
    using A by auto
  finally show ?thesis by simp
qed

lemma emeasure_eq_0:
  N ∈ sets M ⟹ emeasure M N = 0 ⟹ K ⊆ N ⟹ emeasure M K = 0
  by (metis emeasure_mono order_eq_iff zero_le)

lemma emeasure_UN_eq_0:
  assumes ⋀i::nat. emeasure M (N i) = 0 and range N ⊆ sets M
  shows emeasure M (⋃ i. N i) = 0
proof −
  have emeasure M (⋃ i. N i) ≤ 0
    using emeasure_subadditive_countably[OF assms(2)] assms(1) by simp
  then show ?thesis
    by (auto intro: antisym zero_le)
qed

lemma measure_eqI_finite:
  assumes [simp]: sets M = Pow A sets N = Pow A and finite A
  assumes eq: ⋀a. a ∈ A ⟹ emeasure M {a} = emeasure N {a}
  shows M = N
proof (rule measure_eqI)
  fix X assume X ∈ sets M
  then have X: X ⊆ A by auto
  then have emeasure M X = (∑ a∈X. emeasure M {a})
    using ‹finite A› by (subst emeasure_eq_sum_singleton) (auto dest: finite_subset)
  also have ... = (∑ a∈X. emeasure N {a})
    using X eq by (auto intro!: sum.cong)
  also have ... = emeasure N X
    using X ‹finite A› by (subst emeasure_eq_sum_singleton) (auto dest: finite_subset)
  finally show emeasure M X = emeasure N X .
```

qed *simp*

lemma *measure_eqI_generator_eq*:
  fixes *M N* :: *'a measure* **and** *E* :: *'a set set* **and** *A* :: *nat* ⇒ *'a set*
  assumes *Int_stable E E* ⊆ *Pow* Ω
  and *eq*: ⋀*X*. *X* ∈ *E* ⟹ *emeasure M X* = *emeasure N X*
  and *M*: *sets M* = *sigma_sets* Ω *E*
  and *N*: *sets N* = *sigma_sets* Ω *E*
  and *A*: *range A* ⊆ *E* (⋃*i*. *A i*) = Ω ⋀*i*. *emeasure M* (*A i*) ≠ ∞
  shows *M* = *N*
proof −
  let *?μ* = *emeasure M* **and** *?ν* = *emeasure N*
  interpret *S*: *sigma_algebra* Ω *sigma_sets* Ω *E* **by** (*rule sigma_algebra_sigma_sets*)
fact
  have *space M* = Ω
    using *sets.top*[*of M*] *sets.space_closed*[*of M*] *S.top S.space_closed* ‹*sets M* = *sigma_sets* Ω *E*›
    by *blast*

  { **fix** *F D* **assume** *F* ∈ *E* **and** *?μ F* ≠ ∞
    then have [*intro*]: *F* ∈ *sigma_sets* Ω *E* **by** *auto*
    have *?ν F* ≠ ∞ **using** ‹*?μ F* ≠ ∞› ‹*F* ∈ *E*› *eq* **by** *simp*
    assume *D* ∈ *sets M*
    with ‹*Int_stable E*› ‹*E* ⊆ *Pow* Ω› **have** *emeasure M* (*F* ∩ *D*) = *emeasure N* (*F* ∩ *D*)
      unfolding *M*
    proof (*induct rule*: *sigma_sets_induct_disjoint*)
      case (*basic A*)
        then have *F* ∩ *A* ∈ *E* **using** ‹*Int_stable E*› ‹*F* ∈ *E*› **by** (*auto simp*: *Int_stable_def*)
      then show *?case* **using** *eq* **by** *auto*
    next
      case *empty* **then show** *?case* **by** *simp*
    next
      case (*compl A*)
      then have ∗∗: *F* ∩ (Ω − *A*) = *F* − (*F* ∩ *A*)
        and [*intro*]: *F* ∩ *A* ∈ *sigma_sets* Ω *E*
        using ‹*F* ∈ *E*› *S.sets_into_space* **by** (*auto simp*: *M*)
      have *?ν* (*F* ∩ *A*) ≤ *?ν F* **by** (*auto intro*!: *emeasure_mono simp*: *M N*)
      then have *?ν* (*F* ∩ *A*) ≠ ∞ **using** ‹*?ν F* ≠ ∞› **by** (*auto simp*: *top_unique*)
      have *?μ* (*F* ∩ *A*) ≤ *?μ F* **by** (*auto intro*!: *emeasure_mono simp*: *M N*)
      then have *?μ* (*F* ∩ *A*) ≠ ∞ **using** ‹*?μ F* ≠ ∞› **by** (*auto simp*: *top_unique*)
      then have *?μ* (*F* ∩ (Ω − *A*)) = *?μ F* − *?μ* (*F* ∩ *A*) **unfolding** ∗∗
        using ‹*F* ∩ *A* ∈ *sigma_sets* Ω *E*› **by** (*auto intro*!: *emeasure_Diff simp*: *M N*)
      also have ... = *?ν F* − *?ν* (*F* ∩ *A*) **using** *eq* ‹*F* ∈ *E*› *compl* **by** *simp*
      also have ... = *?ν* (*F* ∩ (Ω − *A*)) **unfolding** ∗∗
        using ‹*F* ∩ *A* ∈ *sigma_sets* Ω *E*› ‹*?ν* (*F* ∩ *A*) ≠ ∞›
        by (*auto intro*!: *emeasure_Diff*[*symmetric*] *simp*: *M N*)

finally show *?case*
using ‹*space M* = Ω› **by** *auto*
next
case (*union A*)
then have *?μ* (⋃*x*. *F* ∩ *A x*) = *?ν* (⋃*x*. *F* ∩ *A x*)
by (*subst* (*1 2*) *suminf_emeasure*[*symmetric*]) (*auto simp*: *disjoint_family_on_def subset_eq M N*)
with *A* **show** *?case*
by *auto*
qed }
note ∗ = *this*
show *M* = *N*
proof (*rule measure_eqI*)
show *sets M* = *sets N*
using *M N* **by** *simp*
have [*simp*, *intro*]: ⋀*i*. *A i* ∈ *sets M*
using *A*(*1*) **by** (*auto simp*: *subset_eq M*)
fix *F* **assume** *F* ∈ *sets M*
let *?D* = *disjointed* (λ*i*. *F* ∩ *A i*)
from ‹*space M* = Ω› **have** *F_eq*: *F* = (⋃*i*. *?D i*)
using ‹*F* ∈ *sets M*›[*THEN sets.sets_into_space*] *A*(*2*)[*symmetric*] **by** (*auto simp*: *UN_disjointed_eq*)
have [*simp*, *intro*]: ⋀*i*. *?D i* ∈ *sets M*
using *sets.range_disjointed_sets*[*of* λ*i*. *F* ∩ *A i M*] ‹*F* ∈ *sets M*›
by (*auto simp*: *subset_eq*)
have *disjoint_family ?D*
by (*auto simp*: *disjoint_family_disjointed*)
moreover
have (∑*i*. *emeasure M* (*?D i*)) = (∑*i*. *emeasure N* (*?D i*))
proof (*intro arg_cong*[**where** *f*=*suminf*] *ext*)
fix *i*
have *A i* ∩ *?D i* = *?D i*
by (*auto simp*: *disjointed_def*)
then show *emeasure M* (*?D i*) = *emeasure N* (*?D i*)
using ∗[*of A i ?D i*, *OF* _ *A*(*3*)] *A*(*1*) **by** *auto*
qed
ultimately show *emeasure M F* = *emeasure N F*
by (*simp add*: *image_subset_iff* ‹*sets M* = *sets N*›[*symmetric*] *F_eq*[*symmetric*] *suminf_emeasure*)
qed
qed

lemma *space_empty*: *space M* = {} ⟹ *M* = *count_space* {}
by (*rule measure_eqI*) (*simp_all add*: *space_empty_iff*)

lemma *measure_eqI_generator_eq_countable*:
fixes *M N* :: *'a measure* **and** *E* :: *'a set set* **and** *A* :: *'a set set*
assumes *E*: *Int_stable E E* ⊆ *Pow* Ω ⋀*X*. *X* ∈ *E* ⟹ *emeasure M X* = *emeasure N X*

and *sets*: *sets M* = *sigma_sets* Ω *E sets N* = *sigma_sets* Ω *E*
and *A*: *A* ⊆ *E* (⋃*A*) = Ω *countable A* ⋀*a*. *a* ∈ *A* ⟹ *emeasure M a* ≠ ∞
shows *M* = *N*
proof *cases*
assume Ω = {}
have ∗: *sigma_sets* Ω *E* = *sets* (*sigma* Ω *E*)
using *E*(*2*) **by** *simp*
have *space M* = Ω *space N* = Ω
using *sets E*(*2*) **unfolding** ∗ **by** (*auto dest*: *sets_eq_imp_space_eq simp del*: *sets_measure_of*)
then show *M* = *N*
unfolding ‹Ω = {}› **by** (*auto dest*: *space_empty*)
next
assume Ω ≠ {} **with** ‹⋃*A* = Ω› **have** *A* ≠ {} **by** *auto*
from *this* ‹*countable A*› **have** *rng*: *range* (*from_nat_into A*) = *A*
by (*rule range_from_nat_into*)
show *M* = *N*
proof (*rule measure_eqI_generator_eq*[*OF E sets*])
show *range* (*from_nat_into A*) ⊆ *E*
unfolding *rng* **using** ‹*A* ⊆ *E*› .
show (⋃*i*. *from_nat_into A i*) = Ω
unfolding *rng* **using** ‹⋃*A* = Ω› .
show *emeasure M* (*from_nat_into A i*) ≠ ∞ **for** *i*
using *rng* **by** (*intro A*) *auto*
qed
qed

lemma *measure_of_of_measure*: *measure_of* (*space M*) (*sets M*) (*emeasure M*) = *M*
proof (*intro measure_eqI emeasure_measure_of_sigma*)
show *sigma_algebra* (*space M*) (*sets M*) ..
show *positive* (*sets M*) (*emeasure M*)
by (*simp add*: *positive_def*)
show *countably_additive* (*sets M*) (*emeasure M*)
by (*simp add*: *emeasure_countably_additive*)
qed *simp_all*

6.3.5 μ-null sets

definition *null_sets* :: *'a measure* ⇒ *'a set set* **where**
null_sets M = {*N*∈*sets M*. *emeasure M N* = *0*}

lemma *null_setsD1*[*dest*]: *A* ∈ *null_sets M* ⟹ *emeasure M A* = *0*
by (*simp add*: *null_sets_def*)

lemma *null_setsD2*[*dest*]: *A* ∈ *null_sets M* ⟹ *A* ∈ *sets M*
unfolding *null_sets_def* **by** *simp*

lemma *null_setsI*[*intro*]: *emeasure M A* = *0* ⟹ *A* ∈ *sets M* ⟹ *A* ∈ *null_sets M*

unfolding *null_sets_def* **by** *simp*

interpretation *null_sets*: *ring_of_sets space M null_sets M* **for** *M*
proof (*rule ring_of_setsI*)
  show *null_sets M* ⊆ *Pow* (*space M*)
    using *sets.sets_into_space* **by** *auto*
  show {} ∈ *null_sets M*
    by *auto*
  fix *A B* **assume** *null_sets*: *A* ∈ *null_sets M B* ∈ *null_sets M*
  then have *sets*: *A* ∈ *sets M B* ∈ *sets M*
    by *auto*
  then have ∗: *emeasure M* (*A* ∪ *B*) ≤ *emeasure M A* + *emeasure M B*
    emeasure M (*A* − *B*) ≤ *emeasure M A*
    by (*auto intro*!: *emeasure_subadditive emeasure_mono*)
  then have *emeasure M B* = *0 emeasure M A* = *0*
    using *null_sets* **by** *auto*
  with *sets* ∗ **show** *A* − *B* ∈ *null_sets M A* ∪ *B* ∈ *null_sets M*
    by (*auto intro*!: *antisym zero_le*)
qed

lemma *UN_from_nat_into*:
  assumes *I*: *countable I I* ≠ {}
  shows (⋃*i*∈*I*. *N i*) = (⋃*i*. *N* (*from_nat_into I i*))
proof −
  have (⋃*i*∈*I*. *N i*) = ⋃(*N* ‘ *range* (*from_nat_into I*))
    using *I* **by** *simp*
  also have . . . = (⋃*i*. (*N* ∘ *from_nat_into I*) *i*)
    by *simp*
  finally show *?thesis* **by** *simp*
qed

lemma *null_sets_UN'*:
  assumes *countable I*
  assumes ⋀*i*. *i* ∈ *I* ⟹ *N i* ∈ *null_sets M*
  shows (⋃*i*∈*I*. *N i*) ∈ *null_sets M*
proof *cases*
  assume *I* = {} **then show** *?thesis* **by** *simp*
next
  assume *I* ≠ {}
  show *?thesis*
  proof (*intro conjI CollectI null_setsI*)
    show (⋃*i*∈*I*. *N i*) ∈ *sets M*
      using *assms* **by** (*intro sets.countable_UN'*) *auto*
    have *emeasure M* (⋃*i*∈*I*. *N i*) ≤ (∑*n*. *emeasure M* (*N* (*from_nat_into I n*)))
      unfolding *UN_from_nat_into*[*OF* ‹*countable I*› ‹*I* ≠ {}›]
      using *assms* ‹*I* ≠ {}› **by** (*intro emeasure_subadditive_countably*) (*auto intro*:
from_nat_into)
    also have (λ*n*. *emeasure M* (*N* (*from_nat_into I n*))) = (λ_. *0*)
      using *assms* ‹*I* ≠ {}› **by** (*auto intro*: *from_nat_into*)

finally show *emeasure M* ($\bigcup$*i*∈*I*. *N i*) = *0*
by (*intro antisym zero_le*) *simp*
qed
qed

lemma *null_sets_UN*[*intro*]:
($\bigwedge$*i*::'*i*::*countable*. *N i* ∈ *null_sets M*) ⟹ ($\bigcup$*i*. *N i*) ∈ *null_sets M*
by (*rule null_sets_UN'*) *auto*

lemma *null_set_Int1*:
assumes *B* ∈ *null_sets M A* ∈ *sets M* **shows** *A* ∩ *B* ∈ *null_sets M*
proof (*intro CollectI conjI null_setsI*)
show *emeasure M* (*A* ∩ *B*) = *0* **using** *assms*
by (*intro emeasure_eq_0*[*of B _ A* ∩ *B*]) *auto*
qed (*insert assms*, *auto*)

lemma *null_set_Int2*:
assumes *B* ∈ *null_sets M A* ∈ *sets M* **shows** *B* ∩ *A* ∈ *null_sets M*
using *assms* **by** (*subst Int_commute*) (*rule null_set_Int1*)

lemma *emeasure_Diff_null_set*:
assumes *B* ∈ *null_sets M A* ∈ *sets M*
shows *emeasure M* (*A* − *B*) = *emeasure M A*
proof −
have ∗: *A* − *B* = (*A* − (*A* ∩ *B*)) **by** *auto*
have *A* ∩ *B* ∈ *null_sets M* **using** *assms* **by** (*rule null_set_Int1*)
then show *?thesis*
unfolding ∗ **using** *assms*
by (*subst emeasure_Diff*) *auto*
qed

lemma *null_set_Diff*:
assumes *B* ∈ *null_sets M A* ∈ *sets M* **shows** *B* − *A* ∈ *null_sets M*
proof (*intro CollectI conjI null_setsI*)
show *emeasure M* (*B* − *A*) = *0* **using** *assms* **by** (*intro emeasure_eq_0*[*of B _ B* − *A*]) *auto*
qed (*insert assms*, *auto*)

lemma *emeasure_Un_null_set*:
assumes *A* ∈ *sets M B* ∈ *null_sets M*
shows *emeasure M* (*A* ∪ *B*) = *emeasure M A*
proof −
have ∗: *A* ∪ *B* = *A* ∪ (*B* − *A*) **by** *auto*
have *B* − *A* ∈ *null_sets M* **using** *assms*(*2*,*1*) **by** (*rule null_set_Diff*)
then show *?thesis*
unfolding ∗ **using** *assms*
by (*subst plus_emeasure*[*symmetric*]) *auto*
qed

lemma *emeasure_Un'*:
assumes $A \in$ *sets* M $B \in$ *sets* M $A \cap B \in$ *null_sets* M
shows *emeasure* M $(A \cup B)$ = *emeasure* M A + *emeasure* M B
proof −
have $A \cup B = A \cup (B - A \cap B)$ **by** *blast*
also have *emeasure* M ... = *emeasure* M A + *emeasure* M $(B - A \cap B)$
using *assms* **by** (*subst plus_emeasure*) *auto*
also have *emeasure* M $(B - A \cap B)$ = *emeasure* M B
using *assms* **by** (*intro emeasure_Diff_null_set*) *auto*
finally show *?thesis* **.**
qed

6.3.6 The almost everywhere filter (i.e. quantifier)

definition *ae_filter* :: *'a measure* ⇒ *'a filter* **where**
ae_filter M = (*INF* N∈*null_sets* M. *principal* (*space* M − N))

abbreviation *almost_everywhere* :: *'a measure* ⇒ (*'a* ⇒ *bool*) ⇒ *bool* **where**
almost_everywhere M P ≡ *eventually* P (*ae_filter* M)

syntax
_almost_everywhere :: *pttrn* ⇒ *'a* ⇒ *bool* ⇒ *bool* (*AE* _ *in* _. _ [*0*,*0*,*10*] *10*)

translations
AE x in M. P ⇌ *CONST almost_everywhere M* (λ*x*. *P*)

abbreviation
set_almost_everywhere A M P ≡ *AE x in M. x* ∈ *A* ⟶ *P x*

syntax
_set_almost_everywhere :: *pttrn* ⇒ *'a set* ⇒ *'a* ⇒ *bool* ⇒ *bool*
(*AE* _∈_ *in* _./ _ [*0*,*0*,*0*,*10*] *10*)

translations
AE x∈*A in M. P* ⇌ *CONST set_almost_everywhere A M* (λ*x*. *P*)

lemma *eventually_ae_filter*: *eventually P* (*ae_filter M*) ⟷ (∃ *N*∈*null_sets M*. {*x* ∈ *space M*. ¬ *P x*} ⊆ *N*)
unfolding *ae_filter_def* **by** (*subst eventually_INF_base*) (*auto simp*: *eventually_principal subset_eq*)

lemma *AE_I'*:
N ∈ *null_sets M* ⟹ {*x*∈*space M*. ¬ *P x*} ⊆ *N* ⟹ (*AE x in M. P x*)
unfolding *eventually_ae_filter* **by** *auto*

lemma *AE_iff_null*:
assumes {*x*∈*space M*. ¬ *P x*} ∈ *sets M* (**is** *?P* ∈ *sets M*)
shows (*AE x in M. P x*) ⟷ {*x*∈*space M*. ¬ *P x*} ∈ *null_sets M*
proof

assume *AE x in M. P x* **then obtain** *N* **where** *N*: *N* ∈ *sets M ?P* ⊆ *N emeasure M N* = *0*
unfolding *eventually_ae_filter* **by** *auto*
have *emeasure M ?P* ≤ *emeasure M N*
using *assms N(1,2)* **by** (*auto intro*: *emeasure_mono*)
then have *emeasure M ?P* = *0*
unfolding ‹*emeasure M N* = *0*› **by** *auto*
then show *?P* ∈ *null_sets M* **using** *assms* **by** *auto*
next
assume *?P* ∈ *null_sets M* **with** *assms* **show** *AE x in M. P x* **by** (*auto intro*: *AE_I'*)
qed

lemma *AE_iff_null_sets*:
N ∈ *sets M* ⟹ *N* ∈ *null_sets M* ⟷ (*AE x in M. x* ∉ *N*)
using *Int_absorb1*[*OF sets.sets_into_space*, *of N M*]
by (*subst AE_iff_null*) (*auto simp*: *Int_def*[*symmetric*])

lemma *AE_not_in*:
N ∈ *null_sets M* ⟹ *AE x in M. x* ∉ *N*
by (*metis AE_iff_null_sets null_setsD2*)

lemma *AE_iff_measurable*:
N ∈ *sets M* ⟹ {*x*∈*space M*. ¬ *P x*} = *N* ⟹ (*AE x in M. P x*) ⟷ *emeasure M N* = *0*
using *AE_iff_null*[*of _ P*] **by** *auto*

lemma *AE_E*[*consumes 1*]:
assumes *AE x in M. P x*
obtains *N* **where** {*x* ∈ *space M*. ¬ *P x*} ⊆ *N emeasure M N* = *0 N* ∈ *sets M*
using *assms* **unfolding** *eventually_ae_filter* **by** *auto*

lemma *AE_E2*:
assumes *AE x in M. P x* {*x*∈*space M*. *P x*} ∈ *sets M*
shows *emeasure M* {*x*∈*space M*. ¬ *P x*} = *0* (**is** *emeasure M ?P* = *0*)
proof −
have {*x*∈*space M*. ¬ *P x*} = *space M* − {*x*∈*space M*. *P x*} **by** *auto*
with *AE_iff_null*[*of M P*] *assms* **show** *?thesis* **by** *auto*
qed

lemma *AE_E3*:
assumes *AE x in M. P x*
obtains *N* **where** ⋀*x*. *x* ∈ *space M* − *N* ⟹ *P x N* ∈ *null_sets M*
using *assms* **unfolding** *eventually_ae_filter* **by** *auto*

lemma *AE_I*:
assumes {*x* ∈ *space M*. ¬ *P x*} ⊆ *N emeasure M N* = *0 N* ∈ *sets M*
shows *AE x in M. P x*
using *assms* **unfolding** *eventually_ae_filter* **by** *auto*

lemma *AE_mp*[*elim*!]:
assumes *AE_P*: *AE x in M. P x* **and** *AE_imp*: *AE x in M. P x* ⟶ *Q x*
shows *AE x in M. Q x*
proof –
from *AE_P* **obtain** *A* **where** *P*: {*x*∈*space M*. ¬ *P x*} ⊆ *A*
and *A*: *A* ∈ *sets M emeasure M A* = *0*
by (*auto elim*!: *AE_E*)

from *AE_imp* **obtain** *B* **where** *imp*: {*x*∈*space M*. *P x* ∧ ¬ *Q x*} ⊆ *B*
and *B*: *B* ∈ *sets M emeasure M B* = *0*
by (*auto elim*!: *AE_E*)

show *?thesis*
proof (*intro AE_I*)
have *emeasure M* (*A* ∪ *B*) ≤ *0*
using *emeasure_subadditive*[*of A M B*] *A B* **by** *auto*
then show *A* ∪ *B* ∈ *sets M emeasure M* (*A* ∪ *B*) = *0*
using *A B* **by** *auto*
show {*x*∈*space M*. ¬ *Q x*} ⊆ *A* ∪ *B*
using *P imp* **by** *auto*
qed
qed

The next lemma is convenient to combine with a lemma whose conclusion is of the form *AE x in M. P x* = *Q x*: for such a lemma, there is no [*symmetric*] variant, but using *AE_symmetric*[*OF*...] will replace it.

lemma
shows *AE_iffI*: *AE x in M. P x* ⟹ *AE x in M. P x* ⟷ *Q x* ⟹ *AE x in M. Q x*
and *AE_disjI1*: *AE x in M. P x* ⟹ *AE x in M. P x* ∨ *Q x*
and *AE_disjI2*: *AE x in M. Q x* ⟹ *AE x in M. P x* ∨ *Q x*
and *AE_conjI*: *AE x in M. P x* ⟹ *AE x in M. Q x* ⟹ *AE x in M. P x* ∧ *Q x*
and *AE_conj_iff*[*simp*]: (*AE x in M. P x* ∧ *Q x*) ⟷ (*AE x in M. P x*) ∧ (*AE x in M. Q x*)
by *auto*

lemma *AE_symmetric*:
assumes *AE x in M. P x* = *Q x*
shows *AE x in M. Q x* = *P x*
using *assms* **by** *auto*

lemma *AE_impI*:
(*P* ⟹ *AE x in M. Q x*) ⟹ *AE x in M. P* ⟶ *Q x*
by *fastforce*

lemma *AE_measure*:
assumes *AE*: *AE x in M. P x* **and** *sets*: {*x*∈*space M*. *P x*} ∈ *sets M* (**is** *?P* ∈

sets M)
  shows *emeasure M* {*x*∈*space M*. *P x*} = *emeasure M* (*space M*)
proof −
  from *AE_E*[*OF AE*] **guess** *N* **.** **note** *N* = *this*
  with *sets* **have** *emeasure M* (*space M*) ≤ *emeasure M* (*?P* ∪ *N*)
    by (*intro emeasure_mono*) *auto*
  also have ... ≤ *emeasure M ?P* + *emeasure M N*
    using *sets N* **by** (*intro emeasure_subadditive*) *auto*
  also have ... = *emeasure M ?P* **using** *N* **by** *simp*
  finally show *emeasure M ?P* = *emeasure M* (*space M*)
    using *emeasure_space*[*of M ?P*] **by** *auto*
qed

lemma *AE_space*: *AE x in M*. *x* ∈ *space M*
  by (*rule AE_I*[**where** *N*={}]) *auto*

lemma *AE_I2*[*simp*, *intro*]:
  (⋀*x*. *x* ∈ *space M* ⟹ *P x*) ⟹ *AE x in M*. *P x*
  using *AE_space* **by** *force*

lemma *AE_Ball_mp*:
  ∀ *x*∈*space M*. *P x* ⟹ *AE x in M*. *P x* ⟶ *Q x* ⟹ *AE x in M*. *Q x*
  by *auto*

lemma *AE_cong*[*cong*]:
  (⋀*x*. *x* ∈ *space M* ⟹ *P x* ⟷ *Q x*) ⟹ (*AE x in M*. *P x*) ⟷ (*AE x in M*. *Q x*)
  by *auto*

lemma *AE_cong_simp*: *M* = *N* ⟹ (⋀*x*. *x* ∈ *space N* =*simp*=> *P x* = *Q x*) ⟹ (*AE x in M*. *P x*) ⟷ (*AE x in N*. *Q x*)
  by (*auto simp*: *simp_implies_def*)

lemma *AE_all_countable*:
  (*AE x in M*. ∀ *i*. *P i x*) ⟷ (∀ *i*::*'i*::*countable*. *AE x in M*. *P i x*)
proof
  assume ∀ *i*. *AE x in M*. *P i x*
  from *this*[*unfolded eventually_ae_filter Bex_def*, *THEN choice*]
  obtain *N* **where** *N*: ⋀*i*. *N i* ∈ *null_sets M* ⋀*i*. {*x*∈*space M*. ¬ *P i x*} ⊆ *N i* **by** *auto*
  have {*x*∈*space M*. ¬ (∀ *i*. *P i x*)} ⊆ (⋃ *i*. {*x*∈*space M*. ¬ *P i x*}) **by** *auto*
  also have ... ⊆ (⋃ *i*. *N i*) **using** *N* **by** *auto*
  finally have {*x*∈*space M*. ¬ (∀ *i*. *P i x*)} ⊆ (⋃ *i*. *N i*) **.**
  moreover from *N* **have** (⋃ *i*. *N i*) ∈ *null_sets M*
    by (*intro null_sets_UN*) *auto*
  ultimately show *AE x in M*. ∀ *i*. *P i x*
    unfolding *eventually_ae_filter* **by** *auto*
qed *auto*

lemma *AE_ball_countable*:
assumes [*intro*]: *countable X*
shows (*AE x in M*. ∀ *y*∈*X*. *P x y*) ⟷ (∀ *y*∈*X*. *AE x in M*. *P x y*)
proof
assume ∀ *y*∈*X*. *AE x in M*. *P x y*
from *this*[*unfolded eventually_ae_filter Bex_def*, *THEN bchoice*]
obtain *N* **where** *N*: ⋀*y*. *y* ∈ *X* ⟹ *N y* ∈ *null_sets M* ⋀*y*. *y* ∈ *X* ⟹ {*x*∈*space M*. ¬ *P x y*} ⊆ *N y*
by *auto*
have {*x*∈*space M*. ¬ (∀ *y*∈*X*. *P x y*)} ⊆ (⋃ *y*∈*X*. {*x*∈*space M*. ¬ *P x y*})
by *auto*
also have ... ⊆ (⋃ *y*∈*X*. *N y*)
using *N* **by** *auto*
finally have {*x*∈*space M*. ¬ (∀ *y*∈*X*. *P x y*)} ⊆ (⋃ *y*∈*X*. *N y*) .
moreover from *N* **have** (⋃ *y*∈*X*. *N y*) ∈ *null_sets M*
by (*intro null_sets_UN'*) *auto*
ultimately show *AE x in M*. ∀ *y*∈*X*. *P x y*
unfolding *eventually_ae_filter* **by** *auto*
qed *auto*

lemma *AE_ball_countable'*:
(⋀*N*. *N* ∈ *I* ⟹ *AE x in M*. *P N x*) ⟹ *countable I* ⟹ *AE x in M*. ∀ *N* ∈ *I*. *P N x*
unfolding *AE_ball_countable* **by** *simp*

lemma *AE_pairwise*: *countable F* ⟹ *pairwise* (λ*A B*. *AE x in M*. *R x A B*) *F* ⟷ (*AE x in M*. *pairwise* (*R x*) *F*)
unfolding *pairwise_alt* **by** (*simp add*: *AE_ball_countable*)

lemma *AE_discrete_difference*:
assumes *X*: *countable X*
assumes *null*: ⋀*x*. *x* ∈ *X* ⟹ *emeasure M* {*x*} = *0*
assumes *sets*: ⋀*x*. *x* ∈ *X* ⟹ {*x*} ∈ *sets M*
shows *AE x in M*. *x* ∉ *X*
proof −
have (⋃*x*∈*X*. {*x*}) ∈ *null_sets M*
using *assms* **by** (*intro null_sets_UN'*) *auto*
from *AE_not_in*[*OF this*] **show** *AE x in M*. *x* ∉ *X*
by *auto*
qed

lemma *AE_finite_all*:
assumes *f*: *finite S* **shows** (*AE x in M*. ∀ *i*∈*S*. *P i x*) ⟷ (∀ *i*∈*S*. *AE x in M*. *P i x*)
using *f* **by** *induct auto*

lemma *AE_finite_allI*:
assumes *finite S*
shows (⋀*s*. *s* ∈ *S* ⟹ *AE x in M*. *Q s x*) ⟹ *AE x in M*. ∀ *s*∈*S*. *Q s x*

using *AE_finite_all*[*OF* ‹*finite S*›] **by** *auto*

lemma *emeasure_mono_AE*:
assumes *imp*: *AE x in M*. $x \in A \longrightarrow x \in B$
and *B*: $B \in$ *sets M*
shows *emeasure M A* $\leq$ *emeasure M B*
proof *cases*
assume *A*: $A \in$ *sets M*
from *imp* **obtain** *N* **where** *N*: $\{x \in$*space M*. $\neg (x \in A \longrightarrow x \in B)\} \subseteq N$ $N \in$ *null_sets M*
by (*auto simp*: *eventually_ae_filter*)
have *emeasure M A* = *emeasure M* $(A - N)$
using *N A* **by** (*subst emeasure_Diff_null_set*) *auto*
also have *emeasure M* $(A - N) \leq$ *emeasure M* $(B - N)$
using *N A B sets.sets_into_space* **by** (*auto intro*!: *emeasure_mono*)
also have *emeasure M* $(B - N)$ = *emeasure M B*
using *N B* **by** (*subst emeasure_Diff_null_set*) *auto*
finally show *?thesis* **.**
qed (*simp add*: *emeasure_notin_sets*)

lemma *emeasure_eq_AE*:
assumes *iff*: *AE x in M*. $x \in A \longleftrightarrow x \in B$
assumes *A*: $A \in$ *sets M* **and** *B*: $B \in$ *sets M*
shows *emeasure M A* = *emeasure M B*
using *assms* **by** (*safe intro*!: *antisym emeasure_mono_AE*) *auto*

lemma *emeasure_Collect_eq_AE*:
AE x in M. *P x* $\longleftrightarrow$ *Q x* $\Longrightarrow$ *Measurable.pred M Q* $\Longrightarrow$ *Measurable.pred M P* $\Longrightarrow$
emeasure M $\{x \in$*space M*. *P x*$\}$ = *emeasure M* $\{x \in$*space M*. *Q x*$\}$
by (*intro emeasure_eq_AE*) *auto*

lemma *emeasure_eq_0_AE*: *AE x in M*. $\neg$ *P x* $\Longrightarrow$ *emeasure M* $\{x \in$*space M*. *P x*$\}$ = *0*
using *AE_iff_measurable*[*OF _ refl*, *of M* λx. $\neg$ *P x*]
by (*cases* $\{x \in$*space M*. *P x*$\} \in$ *sets M*) (*simp_all add*: *emeasure_notin_sets*)

lemma *emeasure_0_AE*:
assumes *emeasure M* (*space M*) = *0*
shows *AE x in M*. *P x*
using *eventually_ae_filter assms* **by** *blast*

lemma *emeasure_add_AE*:
assumes [*measurable*]: $A \in$ *sets M* $B \in$ *sets M* $C \in$ *sets M*
assumes *1*: *AE x in M*. $x \in C \longleftrightarrow x \in A \lor x \in B$
assumes *2*: *AE x in M*. $\neg (x \in A \land x \in B)$
shows *emeasure M C* = *emeasure M A* + *emeasure M B*
proof −
have *emeasure M C* = *emeasure M* $(A \cup B)$

by (*rule emeasure_eq_AE*) (*insert 1*, *auto*)
also have ... = *emeasure M A* + *emeasure M* (*B* − *A*)
by (*subst plus_emeasure*) *auto*
also have *emeasure M* (*B* − *A*) = *emeasure M B*
by (*rule emeasure_eq_AE*) (*insert 2*, *auto*)
finally show *?thesis* **.**
qed

6.3.7 σ-finite Measures

locale *sigma_finite_measure* =
fixes *M* :: *'a measure*
assumes *sigma_finite_countable*:
∃ *A*::*'a set set*. *countable A* ∧ *A* ⊆ *sets M* ∧ (⋃*A*) = *space M* ∧ (∀ *a*∈*A*. *emeasure M a* ≠ ∞)

lemma (**in** *sigma_finite_measure*) *sigma_finite*:
obtains *A* :: *nat* ⇒ *'a set*
where *range A* ⊆ *sets M* (⋃*i*. *A i*) = *space M* ⋀*i*. *emeasure M* (*A i*) ≠ ∞
proof −
obtain *A* :: *'a set set* **where**
[*simp*]: *countable A* **and**
A: *A* ⊆ *sets M* (⋃*A*) = *space M* ⋀*a*. *a* ∈ *A* ⟹ *emeasure M a* ≠ ∞
using *sigma_finite_countable* **by** *metis*
show *thesis*
proof *cases*
assume *A* = {} **with** ‹(⋃*A*) = *space M*› **show** *thesis*
by (*intro that*[*of* λ_. {}]) *auto*
next
assume *A* ≠ {}
show *thesis*
proof
show *range* (*from_nat_into A*) ⊆ *sets M*
using ‹*A* ≠ {}› *A* **by** *auto*
have (⋃*i*. *from_nat_into A i*) = ⋃*A*
using *range_from_nat_into*[*OF* ‹*A* ≠ {}› ‹*countable A*›] **by** *auto*
with *A* **show** (⋃*i*. *from_nat_into A i*) = *space M*
by *auto*
qed (*intro A from_nat_into* ‹*A* ≠ {}›)
qed
qed

lemma (**in** *sigma_finite_measure*) *sigma_finite_disjoint*:
obtains *A* :: *nat* ⇒ *'a set*
where *range A* ⊆ *sets M* (⋃*i*. *A i*) = *space M* ⋀*i*. *emeasure M* (*A i*) ≠ ∞ *disjoint_family A*
proof −
obtain *A* :: *nat* ⇒ *'a set* **where**
range: *range A* ⊆ *sets M* **and**

space: $(\bigcup i.\ A\ i) = space\ M$ **and**
measure: $\bigwedge i.\ emeasure\ M\ (A\ i) \neq \infty$
using *sigma_finite* **by** *blast*
show *thesis*
proof (*rule that*[*of disjointed A*])
show *range* (*disjointed A*) $\subseteq$ *sets M*
by (*rule sets.range_disjointed_sets*[*OF range*])
show $(\bigcup i.\ disjointed\ A\ i) = space\ M$
and *disjoint_family* (*disjointed A*)
using *disjoint_family_disjointed UN_disjointed_eq*[*of A*] *space range*
by *auto*
show $emeasure\ M\ (disjointed\ A\ i) \neq \infty$ **for** *i*
proof −
have $emeasure\ M\ (disjointed\ A\ i) \leq emeasure\ M\ (A\ i)$
using *range disjointed_subset*[*of A i*] **by** (*auto intro*!: *emeasure_mono*)
then show *?thesis* **using** *measure*[*of i*] **by** (*auto simp*: *top_unique*)
qed
qed
qed

lemma (**in** *sigma_finite_measure*) *sigma_finite_incseq*:
obtains $A :: nat \Rightarrow 'a\ set$
where $range\ A \subseteq sets\ M\ (\bigcup i.\ A\ i) = space\ M\ \bigwedge i.\ emeasure\ M\ (A\ i) \neq \infty$
incseq A
proof −
obtain $F :: nat \Rightarrow 'a\ set$ **where**
F: $range\ F \subseteq sets\ M\ (\bigcup i.\ F\ i) = space\ M\ \bigwedge i.\ emeasure\ M\ (F\ i) \neq \infty$
using *sigma_finite* **by** *blast*
show *thesis*
proof (*rule that*[*of* $\lambda n.\ \bigcup i\leq n.\ F\ i$])
show $range\ (\lambda n.\ \bigcup i\leq n.\ F\ i) \subseteq sets\ M$
using *F* **by** (*force simp*: *incseq_def*)
show $(\bigcup n.\ \bigcup i\leq n.\ F\ i) = space\ M$
proof −
from *F* **have** $\bigwedge x.\ x \in space\ M \Longrightarrow \exists i.\ x \in F\ i$ **by** *auto*
with *F* **show** *?thesis* **by** *fastforce*
qed
show $emeasure\ M\ (\bigcup i\leq n.\ F\ i) \neq \infty$ **for** *n*
proof −
have $emeasure\ M\ (\bigcup i\leq n.\ F\ i) \leq (\sum i\leq n.\ emeasure\ M\ (F\ i))$
using *F* **by** (*auto intro*!: *emeasure_subadditive_finite*)
also have $\ldots < \infty$
using *F* **by** (*auto simp*: *sum_Pinfty less_top*)
finally show *?thesis* **by** *simp*
qed
show $incseq\ (\lambda n.\ \bigcup i\leq n.\ F\ i)$
by (*force simp*: *incseq_def*)
qed
qed

lemma (**in** *sigma_finite_measure*) *approx_PInf_emeasure_with_finite*:
  fixes *C*::*real*
  assumes *W_meas*: *W* ∈ *sets M*
    and *W_inf*: *emeasure M W* = ∞
  obtains *Z* **where** *Z* ∈ *sets M Z* ⊆ *W emeasure M Z* < ∞ *emeasure M Z* > *C*
proof −
  obtain *A* :: *nat* ⇒ *'a set*
    where *A*: *range A* ⊆ *sets M* (⋃*i*. *A i*) = *space M* ⋀*i*. *emeasure M* (*A i*) ≠ ∞ *incseq A*
    using *sigma_finite_incseq* **by** *blast*
  define *B* **where** *B* = (λ*i*. *W* ∩ *A i*)
  have *B_meas*: ⋀*i*. *B i* ∈ *sets M* **using** *W_meas* ‹*range A* ⊆ *sets M*› *B_def* **by** *blast*
  have *b*: ⋀*i*. *B i* ⊆ *W* **using** *B_def* **by** *blast*

  { **fix** *i*
    have *emeasure M* (*B i*) ≤ *emeasure M* (*A i*)
      using *A* **by** (*intro emeasure_mono*) (*auto simp*: *B_def*)
    also have *emeasure M* (*A i*) < ∞
      using ‹⋀*i*. *emeasure M* (*A i*) ≠ ∞› **by** (*simp add*: *less_top*)
    finally have *emeasure M* (*B i*) < ∞ **.** }
  note *c* = *this*

  have *W* = (⋃*i*. *B i*) **using** *B_def* ‹(⋃*i*. *A i*) = *space M*› *W_meas* **by** *auto*
  moreover have *incseq B* **using** *B_def* ‹*incseq A*› **by** (*simp add*: *incseq_def subset_eq*)
  ultimately have (λ*i*. *emeasure M* (*B i*)) ⟶ *emeasure M W* **using** *W_meas B_meas*
    by (*simp add*: *B_meas Lim_emeasure_incseq image_subset_iff*)
  then have (λ*i*. *emeasure M* (*B i*)) ⟶ ∞ **using** *W_inf* **by** *simp*
  from *order_tendstoD*(*1*)[*OF this*, *of C*]
  obtain *i* **where** *d*: *emeasure M* (*B i*) > *C*
    by (*auto simp*: *eventually_sequentially*)
  have *B i* ∈ *sets M B i* ⊆ *W emeasure M* (*B i*) < ∞ *emeasure M* (*B i*) > *C*
    using *B_meas b c d* **by** *auto*
  then show *?thesis* **using** *that* **by** *blast*
qed

6.3.8 Measure space induced by distribution of $(\rightarrow_M)$-functions

definition *distr* :: *'a measure* ⇒ *'b measure* ⇒ (*'a* ⇒ *'b*) ⇒ *'b measure* **where**
distr M N f =
  measure_of (*space N*) (*sets N*) (λ*A*. *emeasure M* (*f* −‘ *A* ∩ *space M*))

lemma
  shows *sets_distr*[*simp*, *measurable_cong*]: *sets* (*distr M N f*) = *sets N*
    and *space_distr*[*simp*]: *space* (*distr M N f*) = *space N*
  by (*auto simp*: *distr_def*)

lemma
shows *measurable_distr_eq1*[*simp*]: *measurable* (*distr Mf Nf f*) *Mf'* = *measurable Nf Mf'*
and *measurable_distr_eq2*[*simp*]: *measurable Mg'* (*distr Mg Ng g*) = *measurable Mg' Ng*
by (*auto simp*: *measurable_def*)

lemma *distr_cong*:
$M = K \Longrightarrow$ *sets N* = *sets L* $\Longrightarrow$ ($\bigwedge x$. $x \in$ *space M* $\Longrightarrow f\,x = g\,x$) $\Longrightarrow$ *distr M N f* = *distr K L g*
using *sets_eq_imp_space_eq*[*of N L*] **by** (*simp add*: *distr_def Int_def cong*: *rev_conj_cong*)

lemma *emeasure_distr*:
fixes *f* :: *'a* ⇒ *'b*
assumes *f*: *f* ∈ *measurable M N* **and** *A*: *A* ∈ *sets N*
shows *emeasure* (*distr M N f*) *A* = *emeasure M* (*f* −' *A* ∩ *space M*) (**is** _ = *?μ A*)
unfolding *distr_def*
proof (*rule emeasure_measure_of_sigma*)
show *positive* (*sets N*) *?μ*
by (*auto simp*: *positive_def*)

show *countably_additive* (*sets N*) *?μ*
proof (*intro countably_additiveI*)
fix *A* :: *nat* ⇒ *'b set* **assume** *range A* ⊆ *sets N disjoint_family A*
then have *A*: $\bigwedge i$. *A i* ∈ *sets N* ($\bigcup i$. *A i*) ∈ *sets N* **by** *auto*
then have ∗: *range* (λ*i*. *f* −' (*A i*) ∩ *space M*) ⊆ *sets M*
using *f* **by** (*auto simp*: *measurable_def*)
moreover have ($\bigcup i$. *f* −' *A i* ∩ *space M*) ∈ *sets M*
using ∗ **by** *blast*
moreover have ∗∗: *disjoint_family* (λ*i*. *f* −' *A i* ∩ *space M*)
using ‹*disjoint_family A*› **by** (*auto simp*: *disjoint_family_on_def*)
ultimately show ($\sum i$. *?μ* (*A i*)) = *?μ* ($\bigcup i$. *A i*)
using *suminf_emeasure*[*OF* _ ∗∗] *A f*
by (*auto simp*: *comp_def vimage_UN*)
qed
show *sigma_algebra* (*space N*) (*sets N*) ..
qed *fact*

lemma *emeasure_Collect_distr*:
assumes *X*[*measurable*]: *X* ∈ *measurable M N Measurable.pred N P*
shows *emeasure* (*distr M N X*) {*x*∈*space N*. *P x*} = *emeasure M* {*x*∈*space M*. *P* (*X x*)}
by (*subst emeasure_distr*)
(*auto intro*!: *arg_cong2*[**where** *f*=*emeasure*] *X*(*1*)[*THEN measurable_space*])

lemma *emeasure_lfp2*[*consumes 1*, *case_names cont f measurable*]:
assumes *P M*

assumes *cont*: *sup_continuous F*
assumes *f*: $\bigwedge M.\ P\ M \Longrightarrow f \in measurable\ M'\ M$
assumes $*$: $\bigwedge M\ A.\ P\ M \Longrightarrow (\bigwedge N.\ P\ N \Longrightarrow Measurable.pred\ N\ A) \Longrightarrow Measurable.pred\ M\ (F\ A)$
shows $emeasure\ M'\ \{x{\in}space\ M'.\ lfp\ F\ (f\ x)\} = (SUP\ i.\ emeasure\ M'\ \{x{\in}space\ M'.\ (F\ \hat{}\hat{}\ i)\ (\lambda x.\ False)\ (f\ x)\})$
proof (*subst* (*1 2*) *emeasure_Collect_distr*[*symmetric*, **where** *X*=*f*])
show $f \in measurable\ M'\ M\ \ f \in measurable\ M'\ M$
using *f*[*OF* ‹*P M*›] **by** *auto*
{ fix *i* **show** $Measurable.pred\ M\ ((F\ \hat{}\hat{}\ i)\ (\lambda x.\ False))$
using ‹*P M*› **by** (*induction i arbitrary*: *M*) (*auto intro*!: $*$) **}**
show *Measurable.pred M* (*lfp F*)
using ‹*P M*› *cont* $*$ **by** (*rule measurable_lfp_coinduct*[*of P*])

have $emeasure\ (distr\ M'\ M\ f)\ \{x \in space\ (distr\ M'\ M\ f).\ lfp\ F\ x\} = (SUP\ i.\ emeasure\ (distr\ M'\ M\ f)\ \{x \in space\ (distr\ M'\ M\ f).\ (F\ \hat{}\hat{}\ i)\ (\lambda x.\ False)\ x\})$
using ‹*P M*›
proof (*coinduction arbitrary*: *M rule*: *emeasure_lfp'*)
case (*measurable A N*) **then have** $\bigwedge N.\ P\ N \Longrightarrow Measurable.pred\ (distr\ M'\ N\ f)\ A$
by *metis*
then have $\bigwedge N.\ P\ N \Longrightarrow Measurable.pred\ N\ A$
by *simp*
with ‹*P N*›[*THEN* $*$] **show** *?case*
by *auto*
qed *fact*
then show $emeasure\ (distr\ M'\ M\ f)\ \{x \in space\ M.\ lfp\ F\ x\} = (SUP\ i.\ emeasure\ (distr\ M'\ M\ f)\ \{x \in space\ M.\ (F\ \hat{}\hat{}\ i)\ (\lambda x.\ False)\ x\})$
by *simp*
qed

lemma *distr_id*[*simp*]: $distr\ N\ N\ (\lambda x.\ x) = N$
by (*rule measure_eqI*) (*auto simp*: *emeasure_distr*)

lemma *distr_id2*: $sets\ M = sets\ N \Longrightarrow distr\ N\ M\ (\lambda x.\ x) = N$
by (*rule measure_eqI*) (*auto simp*: *emeasure_distr*)

lemma *measure_distr*:
$f \in measurable\ M\ N \Longrightarrow S \in sets\ N \Longrightarrow measure\ (distr\ M\ N\ f)\ S = measure\ M\ (f\ -`\ S \cap space\ M)$
by (*simp add*: *emeasure_distr measure_def*)

lemma *distr_cong_AE*:
assumes *1*: $M = K$ $sets\ N = sets\ L$ **and**
2: $(AE\ x\ in\ M.\ f\ x = g\ x)$ **and** $f \in measurable\ M\ N$ **and** $g \in measurable\ K\ L$
shows $distr\ M\ N\ f = distr\ K\ L\ g$
proof (*rule measure_eqI*)
fix *A* **assume** $A \in sets\ (distr\ M\ N\ f)$

with *assms* **show** *emeasure* (*distr M N f*) *A* = *emeasure* (*distr K L g*) *A*
by (*auto simp add*: *emeasure_distr intro*!: *emeasure_eq_AE measurable_sets*)
qed (*insert 1*, *simp*)

lemma *AE_distrD*:
assumes *f*: *f* ∈ *measurable M M'*
and *AE*: *AE x in distr M M' f*. *P x*
shows *AE x in M*. *P* (*f x*)
proof −
from *AE*[*THEN AE_E*] **guess** *N* **.**
with *f* **show** *?thesis*
unfolding *eventually_ae_filter*
by (*intro bexI*[*of _ f* −‘ *N* ∩ *space M*])
(*auto simp*: *emeasure_distr measurable_def*)
qed

lemma *AE_distr_iff*:
assumes *f*[*measurable*]: *f* ∈ *measurable M N* **and** *P*[*measurable*]: {*x* ∈ *space N*. *P x*} ∈ *sets N*
shows (*AE x in distr M N f*. *P x*) ⟷ (*AE x in M*. *P* (*f x*))
proof (*subst* (*1 2*) *AE_iff_measurable*[*OF _ refl*])
have *f* −‘ {*x*∈*space N*. ¬ *P x*} ∩ *space M* = {*x* ∈ *space M*. ¬ *P* (*f x*)}
using *f*[*THEN measurable_space*] **by** *auto*
then show (*emeasure* (*distr M N f*) {*x* ∈ *space* (*distr M N f*). ¬ *P x*} = *0*) =
(*emeasure M* {*x* ∈ *space M*. ¬ *P* (*f x*)} = *0*)
by (*simp add*: *emeasure_distr*)
qed *auto*

lemma *null_sets_distr_iff*:
f ∈ *measurable M N* ⟹ *A* ∈ *null_sets* (*distr M N f*) ⟷ *f* −‘ *A* ∩ *space M* ∈ *null_sets M* ∧ *A* ∈ *sets N*
by (*auto simp add*: *null_sets_def emeasure_distr*)

proposition *distr_distr*:
g ∈ *measurable N L* ⟹ *f* ∈ *measurable M N* ⟹ *distr* (*distr M N f*) *L g* = *distr M L* (*g* ∘ *f*)
by (*auto simp add*: *emeasure_distr measurable_space*
intro!: *arg_cong*[**where** *f*=*emeasure M*] *measure_eqI*)

6.3.9 Real measure values

lemma *ring_of_finite_sets*: *ring_of_sets* (*space M*) {*A*∈*sets M*. *emeasure M A* ≠ *top*}
proof (*rule ring_of_setsI*)
show *a* ∈ {*A* ∈ *sets M*. *emeasure M A* ≠ *top*} ⟹ *b* ∈ {*A* ∈ *sets M*. *emeasure M A* ≠ *top*} ⟹
a ∪ *b* ∈ {*A* ∈ *sets M*. *emeasure M A* ≠ *top*} **for** *a b*
using *emeasure_subadditive*[*of a M b*] **by** (*auto simp*: *top_unique*)

show $a \in \{A \in sets\ M.\ emeasure\ M\ A \neq top\} \Longrightarrow b \in \{A \in sets\ M.\ emeasure\ M\ A \neq top\} \Longrightarrow$
$a - b \in \{A \in sets\ M.\ emeasure\ M\ A \neq top\}$ **for** *a b*
using *emeasure_mono*[*of a* − *b a M*] **by** (*auto simp*: *top_unique*)
qed (*auto dest*: *sets.sets_into_space*)

lemma *measure_nonneg*[*simp*]: $0 \leq measure\ M\ A$
unfolding *measure_def* **by** *auto*

lemma *measure_nonneg′* [*simp*]: $\neg\ measure\ M\ A < 0$
using *measure_nonneg not_le* **by** *blast*

lemma *zero_less_measure_iff*: $0 < measure\ M\ A \longleftrightarrow measure\ M\ A \neq 0$
using *measure_nonneg*[*of M A*] **by** (*auto simp add*: *le_less*)

lemma *measure_le_0_iff*: $measure\ M\ X \leq 0 \longleftrightarrow measure\ M\ X = 0$
using *measure_nonneg*[*of M X*] **by** *linarith*

lemma *measure_empty*[*simp*]: $measure\ M\ \{\} = 0$
unfolding *measure_def* **by** (*simp add*: *zero_ennreal.rep_eq*)

lemma *emeasure_eq_ennreal_measure*:
$emeasure\ M\ A \neq top \Longrightarrow emeasure\ M\ A = ennreal\ (measure\ M\ A)$
by (*cases emeasure M A rule*: *ennreal_cases*) (*auto simp*: *measure_def*)

lemma *measure_zero_top*: $emeasure\ M\ A = top \Longrightarrow measure\ M\ A = 0$
by (*simp add*: *measure_def*)

lemma *measure_eq_emeasure_eq_ennreal*: $0 \leq x \Longrightarrow emeasure\ M\ A = ennreal\ x \Longrightarrow measure\ M\ A = x$
using *emeasure_eq_ennreal_measure*[*of M A*]
by (*cases* $A \in M$) (*auto simp*: *measure_notin_sets emeasure_notin_sets*)

lemma *enn2real_plus*:$a < top \Longrightarrow b < top \Longrightarrow enn2real\ (a + b) = enn2real\ a + enn2real\ b$
by (*simp add*: *enn2real_def plus_ennreal.rep_eq real_of_ereal_add less_top*
del: *real_of_ereal_enn2ereal*)

lemma *enn2real_sum*:$(\bigwedge i.\ i \in I \Longrightarrow f\ i < top) \Longrightarrow enn2real\ (sum\ f\ I) = sum\ (enn2real \circ f)\ I$
by (*induction I rule*: *infinite_finite_induct*) (*auto simp*: *enn2real_plus*)

lemma *measure_eq_AE*:
assumes *iff*: $AE\ x\ in\ M.\ x \in A \longleftrightarrow x \in B$
assumes *A*: $A \in sets\ M$ **and** *B*: $B \in sets\ M$
shows $measure\ M\ A = measure\ M\ B$
using *assms emeasure_eq_AE*[*OF assms*] **by** (*simp add*: *measure_def*)

lemma *measure_Union*:

emeasure M A $\neq \infty \Longrightarrow$ *emeasure M B* $\neq \infty \Longrightarrow A \in$ *sets M* $\Longrightarrow B \in$ *sets M* $\Longrightarrow A \cap B = \{\} \Longrightarrow$
measure M $(A \cup B) =$ *measure M A* + *measure M B*
by (*simp add*: *measure_def plus_emeasure*[*symmetric*] *enn2real_plus less_top*)

lemma *disjoint_family_on_insert*:
$i \notin I \Longrightarrow$ *disjoint_family_on A* (*insert i I*) $\longleftrightarrow A\ i \cap (\bigcup i{\in}I.\ A\ i) = \{\} \wedge$ *disjoint_family_on A I*
by (*fastforce simp*: *disjoint_family_on_def*)

lemma *measure_finite_Union*:
finite S $\Longrightarrow$ *A'S* $\subseteq$ *sets M* $\Longrightarrow$ *disjoint_family_on A S* $\Longrightarrow (\bigwedge i.\ i \in S \Longrightarrow$ *emeasure M* $(A\ i) \neq \infty) \Longrightarrow$
measure M $(\bigcup i{\in}S.\ A\ i) = (\sum i{\in}S.$ *measure M* $(A\ i))$
by (*induction S rule*: *finite_induct*)
(*auto simp*: *disjoint_family_on_insert measure_Union sum_emeasure*[*symmetric*] *sets.countable_UN'*[*OF countable_finite*])

lemma *measure_Diff*:
assumes *finite*: *emeasure M A* $\neq \infty$
and *measurable*: $A \in$ *sets M* $B \in$ *sets M* $B \subseteq A$
shows *measure M* $(A - B) =$ *measure M A* $-$ *measure M B*
proof –
have *emeasure M* $(A - B) \leq$ *emeasure M A* *emeasure M B* $\leq$ *emeasure M A*
using *measurable* **by** (*auto intro*!: *emeasure_mono*)
hence *measure M* $((A - B) \cup B) =$ *measure M* $(A - B)$ + *measure M B*
using *measurable finite* **by** (*rule_tac measure_Union*) (*auto simp*: *top_unique*)
thus *?thesis* **using** ‹$B \subseteq A$› **by** (*auto simp*: *Un_absorb2*)
qed

lemma *measure_UNION*:
assumes *measurable*: *range A* $\subseteq$ *sets M disjoint_family A*
assumes *finite*: *emeasure M* $(\bigcup i.\ A\ i) \neq \infty$
shows $(\lambda i.$ *measure M* $(A\ i))$ *sums* (*measure M* $(\bigcup i.\ A\ i)$)
proof –
have $(\lambda i.$ *emeasure M* $(A\ i))$ *sums* (*emeasure M* $(\bigcup i.\ A\ i)$)
unfolding *suminf_emeasure*[*OF measurable*, *symmetric*] **by** (*simp add*: *summable_sums*)
moreover
{ **fix** *i*
have *emeasure M* $(A\ i) \leq$ *emeasure M* $(\bigcup i.\ A\ i)$
using *measurable* **by** (*auto intro*!: *emeasure_mono*)
then have *emeasure M* $(A\ i) =$ *ennreal* ((*measure M* $(A\ i)$))
using *finite* **by** (*intro emeasure_eq_ennreal_measure*) (*auto simp*: *top_unique*)
}
ultimately show *?thesis* **using** *finite*
by (*subst* (*asm*) (*2*) *emeasure_eq_ennreal_measure*) *simp_all*
qed

lemma *measure_subadditive*:

assumes *measurable*: $A \in$ *sets* M $B \in$ *sets* M
and *fin*: *emeasure* M $A \neq \infty$ *emeasure* M $B \neq \infty$
shows *measure* M $(A \cup B) \leq$ *measure* M A + *measure* M B
proof −
have *emeasure* M $(A \cup B) \neq \infty$
using *emeasure_subadditive*[*OF measurable*] *fin* **by** (*auto simp*: *top_unique*)
then show (*measure* M $(A \cup B)$) $\leq$ (*measure* M A) + (*measure* M B)
using *emeasure_subadditive*[*OF measurable*] *fin*
apply *simp*
apply (*subst* (*asm*) (*2 3 4*) *emeasure_eq_ennreal_measure*)
apply (*auto simp flip*: *ennreal_plus*)
done
qed

lemma *measure_subadditive_finite*:
assumes *A*: *finite I A'I* $\subseteq$ *sets M* **and** *fin*: $\bigwedge i.\ i \in I \Longrightarrow$ *emeasure* M $(A\ i) \neq$
∞
shows *measure* M $(\bigcup i{\in}I.\ A\ i) \leq (\sum i{\in}I.$ *measure* M $(A\ i))$
proof −
{ **have** *emeasure* M $(\bigcup i{\in}I.\ A\ i) \leq (\sum i{\in}I.$ *emeasure* M $(A\ i))$
using *emeasure_subadditive_finite*[*OF A*] **.**
also have ... $< \infty$
using *fin* **by** (*simp add*: *less_top A*)
finally have *emeasure* M $(\bigcup i{\in}I.\ A\ i) \neq$ *top* **by** *simp* }
note $*$ = *this*
show *?thesis*
using *emeasure_subadditive_finite*[*OF A*] *fin*
unfolding *emeasure_eq_ennreal_measure*[*OF* $*$]
by (*simp_all add*: *sum_nonneg emeasure_eq_ennreal_measure*)
qed

lemma *measure_subadditive_countably*:
assumes *A*: *range A* $\subseteq$ *sets M* **and** *fin*: $(\sum i.$ *emeasure* M $(A\ i)) \neq \infty$
shows *measure* M $(\bigcup i.\ A\ i) \leq (\sum i.$ *measure* M $(A\ i))$
proof −
from *fin* **have** $**$: $\bigwedge i.$ *emeasure* M $(A\ i) \neq$ *top*
using *ennreal_suminf_lessD*[*of* $\lambda i.$ *emeasure* M $(A\ i)$] **by** (*simp add*: *less_top*)
{ **have** *emeasure* M $(\bigcup i.\ A\ i) \leq (\sum i.$ *emeasure* M $(A\ i))$
using *emeasure_subadditive_countably*[*OF A*] **.**
also have ... $< \infty$
using *fin* **by** (*simp add*: *less_top*)
finally have *emeasure* M $(\bigcup i.\ A\ i) \neq$ *top* **by** *simp* }
then have *ennreal* (*measure* M $(\bigcup i.\ A\ i)$) = *emeasure* M $(\bigcup i.\ A\ i)$
by (*rule emeasure_eq_ennreal_measure*[*symmetric*])
also have ... $\leq (\sum i.$ *emeasure* M $(A\ i))$
using *emeasure_subadditive_countably*[*OF A*] **.**
also have ... = *ennreal* $(\sum i.$ *measure* M $(A\ i))$
using *fin* **unfolding** *emeasure_eq_ennreal_measure*[*OF* $**$]
by (*subst suminf_ennreal*) (*auto simp*: $**$)

finally show *?thesis*
  apply (*rule ennreal_le_iff*[*THEN iffD1*, *rotated*])
  apply (*intro suminf_nonneg allI measure_nonneg summable_suminf_not_top*)
  using *fin*
  apply (*simp add*: *emeasure_eq_ennreal_measure*[*OF* ∗∗])
  done
qed

lemma *measure_Un_null_set*: $A \in sets\ M \Longrightarrow B \in null\_sets\ M \Longrightarrow measure\ M\ (A \cup B) = measure\ M\ A$
  by (*simp add*: *measure_def emeasure_Un_null_set*)

lemma *measure_Diff_null_set*: $A \in sets\ M \Longrightarrow B \in null\_sets\ M \Longrightarrow measure\ M\ (A - B) = measure\ M\ A$
  by (*simp add*: *measure_def emeasure_Diff_null_set*)

lemma *measure_eq_sum_singleton*:
  $finite\ S \Longrightarrow (\bigwedge x.\ x \in S \Longrightarrow \{x\} \in sets\ M) \Longrightarrow (\bigwedge x.\ x \in S \Longrightarrow emeasure\ M\ \{x\} \neq \infty) \Longrightarrow$
    $measure\ M\ S = (\sum x{\in}S.\ measure\ M\ \{x\})$
  using *emeasure_eq_sum_singleton*[*of S M*]
  by (*intro measure_eq_emeasure_eq_ennreal*) (*auto simp*: *sum_nonneg emeasure_eq_ennreal_measure*)

lemma *Lim_measure_incseq*:
  assumes *A*: $range\ A \subseteq sets\ M$ *incseq A* **and** *fin*: $emeasure\ M\ (\bigcup i.\ A\ i) \neq \infty$
  shows $(\lambda i.\ measure\ M\ (A\ i)) \longlonglongrightarrow measure\ M\ (\bigcup i.\ A\ i)$
proof (*rule tendsto_ennrealD*)
  have $ennreal\ (measure\ M\ (\bigcup i.\ A\ i)) = emeasure\ M\ (\bigcup i.\ A\ i)$
    using *fin* **by** (*auto simp*: *emeasure_eq_ennreal_measure*)
  moreover have $ennreal\ (measure\ M\ (A\ i)) = emeasure\ M\ (A\ i)$ **for** *i*
    using *assms emeasure_mono*[*of A _* $\bigcup i.\ A\ i\ M$]
   by (*intro emeasure_eq_ennreal_measure*[*symmetric*]) (*auto simp*: *less_top UN_upper intro*: *le_less_trans*)
  ultimately show $(\lambda x.\ ennreal\ (measure\ M\ (A\ x))) \longlonglongrightarrow ennreal\ (measure\ M\ (\bigcup i.\ A\ i))$
    using *A* **by** (*auto intro*!: *Lim_emeasure_incseq*)
qed *auto*

lemma *Lim_measure_decseq*:
  assumes *A*: $range\ A \subseteq sets\ M$ *decseq A* **and** *fin*: $\bigwedge i.\ emeasure\ M\ (A\ i) \neq \infty$
  shows $(\lambda n.\ measure\ M\ (A\ n)) \longlonglongrightarrow measure\ M\ (\bigcap i.\ A\ i)$
proof (*rule tendsto_ennrealD*)
  have $ennreal\ (measure\ M\ (\bigcap i.\ A\ i)) = emeasure\ M\ (\bigcap i.\ A\ i)$
    using *fin*[*of 0*] *A emeasure_mono*[*of* $\bigcap i.\ A\ i\ A\ 0\ M$]
   by (*auto intro*!: *emeasure_eq_ennreal_measure*[*symmetric*] *simp*: *INT_lower less_top intro*: *le_less_trans*)
  moreover have $ennreal\ (measure\ M\ (A\ i)) = emeasure\ M\ (A\ i)$ **for** *i*
    using *A fin*[*of i*] **by** (*intro emeasure_eq_ennreal_measure*[*symmetric*]) *auto*
  ultimately show $(\lambda x.\ ennreal\ (measure\ M\ (A\ x))) \longlonglongrightarrow ennreal\ (measure\ M$

$(\bigcap i.\ A\ i))$
  using *fin A* **by** (*auto intro*!: *Lim_emeasure_decseq*)
qed *auto*

6.3.10 Set of measurable sets with finite measure

definition *fmeasurable* :: *'a measure* ⇒ *'a set set* **where**
fmeasurable M = {*A*∈*sets M*. *emeasure M A* < ∞}

lemma *fmeasurableD*[*dest*, *measurable_dest*]: *A* ∈ *fmeasurable M* ⟹ *A* ∈ *sets M*
 by (*auto simp*: *fmeasurable_def*)

lemma *fmeasurableD2*: *A* ∈ *fmeasurable M* ⟹ *emeasure M A* ≠ *top*
 by (*auto simp*: *fmeasurable_def*)

lemma *fmeasurableI*: *A* ∈ *sets M* ⟹ *emeasure M A* < ∞ ⟹ *A* ∈ *fmeasurable M*
 by (*auto simp*: *fmeasurable_def*)

lemma *fmeasurableI_null_sets*: *A* ∈ *null_sets M* ⟹ *A* ∈ *fmeasurable M*
 by (*auto simp*: *fmeasurable_def*)

lemma *fmeasurableI2*: *A* ∈ *fmeasurable M* ⟹ *B* ⊆ *A* ⟹ *B* ∈ *sets M* ⟹ *B* ∈ *fmeasurable M*
 using *emeasure_mono*[*of B A M*] **by** (*auto simp*: *fmeasurable_def*)

lemma *measure_mono_fmeasurable*:
 A ⊆ *B* ⟹ *A* ∈ *sets M* ⟹ *B* ∈ *fmeasurable M* ⟹ *measure M A* ≤ *measure M B*
 by (*auto simp*: *measure_def fmeasurable_def intro*!: *emeasure_mono enn2real_mono*)

lemma *emeasure_eq_measure2*: *A* ∈ *fmeasurable M* ⟹ *emeasure M A* = *measure M A*
 by (*simp add*: *emeasure_eq_ennreal_measure fmeasurable_def less_top*)

interpretation *fmeasurable*: *ring_of_sets space M fmeasurable M*
proof (*rule ring_of_setsI*)
 show *fmeasurable M* ⊆ *Pow* (*space M*) {} ∈ *fmeasurable M*
  by (*auto simp*: *fmeasurable_def dest*: *sets.sets_into_space*)
 fix *a b* **assume** ∗: *a* ∈ *fmeasurable M b* ∈ *fmeasurable M*
 then have *emeasure M* (*a* ∪ *b*) ≤ *emeasure M a* + *emeasure M b*
  by (*intro emeasure_subadditive*) *auto*
 also have ... < *top*
  using ∗ **by** (*auto simp*: *fmeasurable_def*)
 finally show *a* ∪ *b* ∈ *fmeasurable M*
  using ∗ **by** (*auto intro*: *fmeasurableI*)
 show *a* − *b* ∈ *fmeasurable M*
  using *emeasure_mono*[*of a* − *b a M*] ∗ **by** (*auto simp*: *fmeasurable_def*)
qed

6.3.11 Measurable sets formed by unions and intersections

lemma *fmeasurable_Diff*: $A \in fmeasurable\ M \Longrightarrow B \in sets\ M \Longrightarrow A - B \in fmeasurable\ M$
using *fmeasurableI2*[*of* $A\ M\ A - B$] **by** *auto*

lemma *fmeasurable_Int_fmeasurable*:
$[\![S \in fmeasurable\ M;\ T \in sets\ M]\!] \Longrightarrow (S \cap T) \in fmeasurable\ M$
by (*meson fmeasurableD fmeasurableI2 inf_le1 sets.Int*)

lemma *fmeasurable_UN*:
assumes *countable* $I \bigwedge i.\ i \in I \Longrightarrow F\ i \subseteq A \bigwedge i.\ i \in I \Longrightarrow F\ i \in sets\ M\ A \in fmeasurable\ M$
shows $(\bigcup i{\in}I.\ F\ i) \in fmeasurable\ M$
proof (*rule fmeasurableI2*)
show $A \in fmeasurable\ M\ (\bigcup i{\in}I.\ F\ i) \subseteq A$ **using** *assms* **by** *auto*
show $(\bigcup i{\in}I.\ F\ i) \in sets\ M$
using *assms* **by** (*intro sets.countable_UN'*) *auto*
qed

lemma *fmeasurable_INT*:
assumes *countable* $I\ i \in I \bigwedge i.\ i \in I \Longrightarrow F\ i \in sets\ M\ F\ i \in fmeasurable\ M$
shows $(\bigcap i{\in}I.\ F\ i) \in fmeasurable\ M$
proof (*rule fmeasurableI2*)
show $F\ i \in fmeasurable\ M\ (\bigcap i{\in}I.\ F\ i) \subseteq F\ i$
using *assms* **by** *auto*
show $(\bigcap i{\in}I.\ F\ i) \in sets\ M$
using *assms* **by** (*intro sets.countable_INT'*) *auto*
qed

lemma *measurable_measure_Diff*:
assumes $A \in fmeasurable\ M\ B \in sets\ M\ B \subseteq A$
shows $measure\ M\ (A - B) = measure\ M\ A - measure\ M\ B$
by (*simp add*: *assms fmeasurableD fmeasurableD2 measure_Diff*)

lemma *measurable_Un_null_set*:
assumes $B \in null\_sets\ M$
shows $(A \cup B \in fmeasurable\ M \wedge A \in sets\ M) \longleftrightarrow A \in fmeasurable\ M$
using *assms* **by** (*fastforce simp add*: *fmeasurable.Un fmeasurableI_null_sets intro*: *fmeasurableI2*)

lemma *measurable_Diff_null_set*:
assumes $B \in null\_sets\ M$
shows $(A - B) \in fmeasurable\ M \wedge A \in sets\ M \longleftrightarrow A \in fmeasurable\ M$
using *assms*
by (*metis Un_Diff_cancel2 fmeasurable.Diff fmeasurableD fmeasurableI_null_sets measurable_Un_null_set*)

lemma *fmeasurable_Diff_D*:
assumes *m*: $T - S \in fmeasurable\ M\ S \in fmeasurable\ M$ **and** *sub*: $S \subseteq T$

shows $T \in$ *fmeasurable* M
proof −
have $T = S \cup (T - S)$
using *assms* **by** *blast*
then show *?thesis*
by (*metis m fmeasurable.Un*)
qed

lemma *measure_Un2*:
$A \in$ *fmeasurable* $M \Longrightarrow B \in$ *fmeasurable* $M \Longrightarrow$ *measure* M $(A \cup B) =$ *measure* M A + *measure* M $(B - A)$
using *measure_Union*[*of* M A $B - A$] **by** (*auto simp*: *fmeasurableD2 fmeasurable.Diff*)

lemma *measure_Un3*:
assumes $A \in$ *fmeasurable* M $B \in$ *fmeasurable* M
shows *measure* M $(A \cup B) =$ *measure* M A + *measure* M B − *measure* M $(A \cap B)$
proof −
have *measure* M $(A \cup B) =$ *measure* M A + *measure* M $(B - A)$
using *assms* **by** (*rule measure_Un2*)
also have $B - A = B - (A \cap B)$
by *auto*
also have *measure* M $(B - (A \cap B)) =$ *measure* M B − *measure* M $(A \cap B)$
using *assms* **by** (*intro measure_Diff*) (*auto simp*: *fmeasurable_def*)
finally show *?thesis*
by *simp*
qed

lemma *measure_Un_AE*:
AE x *in* M. $x \notin A \vee x \notin B \Longrightarrow A \in$ *fmeasurable* $M \Longrightarrow B \in$ *fmeasurable* $M \Longrightarrow$
measure M $(A \cup B) =$ *measure* M A + *measure* M B
by (*subst measure_Un2*) (*auto intro*!: *measure_eq_AE*)

lemma *measure_UNION_AE*:
assumes I: *finite* I
shows $(\bigwedge i.\ i \in I \Longrightarrow F\ i \in$ *fmeasurable* $M) \Longrightarrow$ *pairwise* $(\lambda i\ j.$ *AE* x *in* $M.\ x \notin F\ i \vee x \notin F\ j)\ I \Longrightarrow$
measure M $(\bigcup i{\in}I.\ F\ i) = (\sum i{\in}I.$ *measure* M $(F\ i))$
unfolding *AE_pairwise*[*OF countable_finite*, *OF* I]
using I
proof (*induction* I *rule*: *finite_induct*)
case (*insert* x I)
have *measure* M $(F\ x \cup \bigcup(F\ `\ I)) =$ *measure* $M$ $(F\ x)$ + *measure* $M$ $(\bigcup(F\ `\ I))$
by (*rule measure_Un_AE*) (*use insert* **in** ‹*auto simp*: *pairwise_insert*›)
with *insert* **show** *?case*
by (*simp add*: *pairwise_insert*)
qed *simp*

lemma *measure_UNION'*:
finite I $\Longrightarrow$ ($\bigwedge$*i*. *i* $\in$ *I* $\Longrightarrow$ *F i* $\in$ *fmeasurable M*) $\Longrightarrow$ *pairwise* (λ*i j*. *disjnt* (*F i*) (*F j*)) *I* $\Longrightarrow$
measure M ($\bigcup$*i*$\in$*I*. *F i*) = ($\sum$*i*$\in$*I*. *measure M* (*F i*))
by (*intro measure_UNION_AE*) (*auto simp*: *disjnt_def elim*!: *pairwise_mono intro*!: *always_eventually*)

lemma *measure_Union_AE*:
finite F $\Longrightarrow$ ($\bigwedge$*S*. *S* $\in$ *F* $\Longrightarrow$ *S* $\in$ *fmeasurable M*) $\Longrightarrow$ *pairwise* (λ*S T*. *AE x in M*. *x* $\notin$ *S* $\vee$ *x* $\notin$ *T*) *F* $\Longrightarrow$
measure M ($\bigcup$*F*) = ($\sum$*S*$\in$*F*. *measure M S*)
using *measure_UNION_AE*[*of F* λ*x*. *x M*] **by** *simp*

lemma *measure_Union'*:
finite F $\Longrightarrow$ ($\bigwedge$*S*. *S* $\in$ *F* $\Longrightarrow$ *S* $\in$ *fmeasurable M*) $\Longrightarrow$ *pairwise disjnt F* $\Longrightarrow$ *measure M* ($\bigcup$*F*) = ($\sum$*S*$\in$*F*. *measure M S*)
using *measure_UNION'*[*of F* λ*x*. *x M*] **by** *simp*

lemma *measure_Un_le*:
assumes *A* $\in$ *sets M B* $\in$ *sets M* **shows** *measure M* (*A* $\cup$ *B*) $\leq$ *measure M A* + *measure M B*
proof *cases*
assume *A* $\in$ *fmeasurable M* $\wedge$ *B* $\in$ *fmeasurable M*
with *measure_subadditive*[*of A M B*] *assms* **show** *?thesis*
by (*auto simp*: *fmeasurableD2*)
next
assume $\neg$ (*A* $\in$ *fmeasurable M* $\wedge$ *B* $\in$ *fmeasurable M*)
then have *A* $\cup$ *B* $\notin$ *fmeasurable M*
using *fmeasurableI2*[*of A* $\cup$ *B M A*] *fmeasurableI2*[*of A* $\cup$ *B M B*] *assms* **by** *auto*
with *assms* **show** *?thesis*
by (*auto simp*: *fmeasurable_def measure_def less_top*[*symmetric*])
qed

lemma *measure_UNION_le*:
finite I $\Longrightarrow$ ($\bigwedge$*i*. *i* $\in$ *I* $\Longrightarrow$ *F i* $\in$ *sets M*) $\Longrightarrow$ *measure M* ($\bigcup$*i*$\in$*I*. *F i*) $\leq$ ($\sum$*i*$\in$*I*. *measure M* (*F i*))
proof (*induction I rule*: *finite_induct*)
case (*insert i I*)
then have *measure M* ($\bigcup$*i*$\in$*insert i I*. *F i*) = *measure M* (*F i* $\cup$ $\bigcup$ (*F* ‘ *I*))
by *simp*
also from *insert* **have** *measure M* (*F i* $\cup$ $\bigcup$ (*F* ‘ *I*)) $\leq$ *measure M* (*F i*) + *measure M* ($\bigcup$ (*F* ‘ *I*))
by (*intro measure_Un_le sets.finite_Union*) *auto*
also have *measure M* ($\bigcup$*i*$\in$*I*. *F i*) $\leq$ ($\sum$*i*$\in$*I*. *measure M* (*F i*))
using *insert* **by** *auto*
finally show *?case*
using *insert* **by** *simp*

qed *simp*

lemma *measure_Union_le*:
finite F $\Longrightarrow$ ($\bigwedge S.\ S \in F \Longrightarrow S \in$ *sets M*) $\Longrightarrow$ *measure M* ($\bigcup F$) $\leq$ ($\sum S{\in}F.$ *measure M S*)
using *measure_UNION_le*[*of F* $\lambda x.\ x$ *M*] **by** *simp*

Version for indexed union over a countable set

lemma
assumes *countable I* **and** *I*: $\bigwedge i.\ i \in I \Longrightarrow A\ i \in$ *fmeasurable M*
and *bound*: $\bigwedge I'.\ I' \subseteq I \Longrightarrow$ *finite I'* $\Longrightarrow$ *measure M* ($\bigcup i{\in}I'.\ A\ i$) $\leq B$
shows *fmeasurable_UN_bound*: ($\bigcup i{\in}I.\ A\ i$) $\in$ *fmeasurable M* (**is** *?fm*)
and *measure_UN_bound*: *measure M* ($\bigcup i{\in}I.\ A\ i$) $\leq B$ (**is** *?m*)
proof −
have $B \geq 0$
using *bound* **by** *force*
have *?fm* $\wedge$ *?m*
proof *cases*
assume $I = \{\}$
with ‹$B \geq 0$› **show** *?thesis*
by *simp*
next
assume $I \neq \{\}$
have ($\bigcup i{\in}I.\ A\ i$) = ($\bigcup i.$ ($\bigcup n{\leq}i.\ A$ (*from_nat_into I n*)))
by (*subst range_from_nat_into*[*symmetric*, *OF* ‹$I \neq \{\}$› ‹*countable I*›]) *auto*
then have *emeasure M* ($\bigcup i{\in}I.\ A\ i$) = *emeasure M* ($\bigcup i.$ ($\bigcup n{\leq}i.\ A$ (*from_nat_into I n*))) **by** *simp*
also have ... = (*SUP i. emeasure M* ($\bigcup n{\leq}i.\ A$ (*from_nat_into I n*)))
using *I* ‹$I \neq \{\}$›[*THEN from_nat_into*] **by** (*intro SUP_emeasure_incseq*[*symmetric*]) (*fastforce simp*: *incseq_Suc_iff*)+
also have ... $\leq B$
proof (*intro SUP_least*)
fix *i* :: *nat*
have *emeasure M* ($\bigcup n{\leq}i.\ A$ (*from_nat_into I n*)) = *measure M* ($\bigcup n{\leq}i.\ A$ (*from_nat_into I n*))
using *I* ‹$I \neq \{\}$›[*THEN from_nat_into*] **by** (*intro emeasure_eq_measure2 fmeasurable.finite_UN*) *auto*
also have ... = *measure M* ($\bigcup n{\in}$*from_nat_into I* ‘ {..*i*}. *A n*)
by *simp*
also have ... $\leq B$
by (*intro ennreal_leI bound*) (*auto intro*: *from_nat_into*[*OF* ‹$I \neq \{\}$›])
finally show *emeasure M* ($\bigcup n{\leq}i.\ A$ (*from_nat_into I n*)) $\leq$ *ennreal B* **.**
qed
finally have ∗: *emeasure M* ($\bigcup i{\in}I.\ A\ i$) $\leq B$ **.**
then have *?fm*
using *I* ‹*countable I*› **by** (*intro fmeasurableI conjI*) (*auto simp*: *less_top*[*symmetric*] *top_unique*)
with ∗ ‹$0{\leq}B$› **show** *?thesis*
by (*simp add*: *emeasure_eq_measure2*)

```
  qed
  then show ?fm ?m by auto
qed
```

Version for big union of a countable set

```
lemma
  assumes countable 𝒟
    and meas: ⋀D. D ∈ 𝒟 ⟹ D ∈ fmeasurable M
    and bound:  ⋀ℰ. ⟦ℰ ⊆ 𝒟; finite ℰ⟧ ⟹ measure M (⋃ℰ) ≤ B
 shows fmeasurable_Union_bound: ⋃𝒟 ∈ fmeasurable M  (is ?fm)
    and measure_Union_bound: measure M (⋃𝒟) ≤ B     (is ?m)
proof -
  have B ≥ 0
    using bound by force
  have ?fm ∧ ?m
  proof (cases 𝒟 = {})
    case True
    with ‹B ≥ 0› show ?thesis
      by auto
  next
    case False
    then obtain D :: nat ⇒ 'a set where D: 𝒟 = range D
      using ‹countable 𝒟› uncountable_def by force
      have 1: ⋀i. D i ∈ fmeasurable M
        by (simp add: D meas)
      have 2: ⋀I'. finite I' ⟹ measure M (⋃x∈I'. D x) ≤ B
        by (simp add: D bound image_subset_iff)
      show ?thesis
        unfolding D
        by (intro conjI fmeasurable_UN_bound [OF _ 1 2] measure_UN_bound [OF _
1 2]) auto
    qed
    then show ?fm ?m by auto
qed
```

Version for indexed union over the type of naturals

```
lemma
  fixes S :: nat ⇒ 'a set
  assumes S: ⋀i. S i ∈ fmeasurable M and B: ⋀n. measure M (⋃i≤n. S i) ≤ B
  shows fmeasurable_countable_Union: (⋃i. S i) ∈ fmeasurable M
    and measure_countable_Union_le: measure M (⋃i. S i) ≤ B
proof -
  have mB: measure M (⋃i∈I. S i) ≤ B if finite I for I
  proof -
    have (⋃i∈I. S i) ⊆ (⋃i≤Max I. S i)
      using Max_ge that by force
    then have measure M (⋃i∈I. S i) ≤ measure M (⋃i ≤ Max I. S i)
      by (rule measure_mono_fmeasurable) (use S in ‹blast+›)
    then show ?thesis
```

using *B order_trans* **by** *blast*
qed
show $(\bigcup i.\ S\ i) \in$ *fmeasurable M*
by (*auto intro*: *fmeasurable_UN_bound* [*OF* _ *S mB*])
show *measure M* $(\bigcup n.\ S\ n) \leq B$
by (*auto intro*: *measure_UN_bound* [*OF* _ *S mB*])
qed

lemma *measure_diff_le_measure_setdiff*:
assumes $S \in$ *fmeasurable M* $T \in$ *fmeasurable M*
shows *measure M S* − *measure M T* ≤ *measure M* (*S* − *T*)
proof −
have *measure M S* ≤ *measure M* ((*S* − *T*) ∪ *T*)
by (*simp add*: *assms fmeasurable.Un fmeasurableD measure_mono_fmeasurable*)
also have ... ≤ *measure M* (*S* − *T*) + *measure M T*
using *assms* **by** (*blast intro*: *measure_Un_le*)
finally show *?thesis*
by (*simp add*: *algebra_simps*)
qed

lemma *suminf_exist_split2*:
fixes *f* :: *nat* ⇒ ′*a*::*real_normed_vector*
assumes *summable f*
shows $(\lambda n.\ (\sum k.\ f(k+n))) \longrightarrow 0$
by (*subst lim_sequentially*, *auto simp add*: *dist_norm suminf_exist_split*[*OF* _ *assms*])

lemma *emeasure_union_summable*:
assumes [*measurable*]: $\bigwedge n.\ A\ n \in$ *sets M*
and $\bigwedge n.$ *emeasure M* (*A n*) < ∞ *summable* (λ*n*. *measure M* (*A n*))
shows *emeasure M* $(\bigcup n.\ A\ n) < \infty$ *emeasure M* $(\bigcup n.\ A\ n) \leq (\sum n.$ *measure M* (*A n*))
proof −
define *B* **where** $B = (\lambda N.\ (\bigcup n \in \{..<N\}.\ A\ n))$
have [*measurable*]: *B N* ∈ *sets M* **for** *N* **unfolding** *B_def* **by** *auto*
have (λ*N*. *emeasure M* (*B N*)) ⟶ *emeasure M* $(\bigcup N.\ B\ N)$
apply (*rule Lim_emeasure_incseq*) **unfolding** *B_def* **by** (*auto simp add*: *SUP_subset_mono incseq_def*)
moreover have *emeasure M* (*B N*) ≤ *ennreal* $(\sum n.$ *measure M* (*A n*)) **for** *N*
proof −
have ∗: $(\sum n \in \{..<N\}.$ *measure M* (*A n*)) ≤ $(\sum n.$ *measure M* (*A n*))
using *assms*(*3*) *measure_nonneg sum_le_suminf* **by** *blast*

have *emeasure M* (*B N*) ≤ $(\sum n \in \{..<N\}.$ *emeasure M* (*A n*))
unfolding *B_def* **by** (*rule emeasure_subadditive_finite*, *auto*)
also have ... = $(\sum n \in \{..<N\}.$ *ennreal*(*measure M* (*A n*)))
using *assms*(*2*) **by** (*simp add*: *emeasure_eq_ennreal_measure less_top*)
also have ... = *ennreal* $(\sum n \in \{..<N\}.$ *measure M* (*A n*))
by *auto*
also have ... ≤ *ennreal* $(\sum n.$ *measure M* (*A n*))

```
      using * by (auto simp: ennreal_leI)
    finally show ?thesis by simp
  qed
  ultimately have emeasure M (⋃N. B N) ≤ ennreal (∑n. measure M (A n))
    by (simp add: Lim_bounded)
  then show emeasure M (⋃n. A n) ≤ (∑n. measure M (A n))
    unfolding B_def by (metis UN_UN_flatten UN_lessThan_UNIV)
  then show emeasure M (⋃n. A n) < ∞
    by (auto simp: less_top[symmetric] top_unique)
qed

lemma borel_cantelli_limsup1:
  assumes [measurable]: ⋀n. A n ∈ sets M
    and ⋀n. emeasure M (A n) < ∞ summable (λn. measure M (A n))
  shows limsup A ∈ null_sets M
proof -
  have emeasure M (limsup A) ≤ 0
  proof (rule LIMSEQ_le_const)
   have (λn. (∑k. measure M (A (k+n)))) ⟶ 0 by (rule suminf_exist_split2[OF
assms(3)])
    then show (λn. ennreal (∑k. measure M (A (k+n)))) ⟶ 0
      unfolding ennreal_0[symmetric] by (intro tendsto_ennrealI)
    have emeasure M (limsup A) ≤ (∑k. measure M (A (k+n))) for n
    proof -
     have I: (⋃k∈{n..}. A k) = (⋃k. A (k+n)) by (auto, metis le_add_diff_inverse2,
fastforce)
      have emeasure M (limsup A) ≤ emeasure M (⋃k∈{n..}. A k)
        by (rule emeasure_mono, auto simp add: limsup_INF_SUP)
      also have ... = emeasure M (⋃k. A (k+n))
        using I by auto
      also have ... ≤ (∑k. measure M (A (k+n)))
        apply (rule emeasure_union_summable)
        using assms summable_ignore_initial_segment[OF assms(3), of n] by auto
      finally show ?thesis by simp
    qed
     then show ∃N. ∀n≥N. emeasure M (limsup A) ≤ (∑k. measure M (A
(k+n)))
      by auto
  qed
  then show ?thesis using assms(1) measurable_limsup by auto
qed

lemma borel_cantelli_AE1:
  assumes [measurable]: ⋀n. A n ∈ sets M
    and ⋀n. emeasure M (A n) < ∞ summable (λn. measure M (A n))
  shows AE x in M. eventually (λn. x ∈ space M - A n) sequentially
proof -
  have AE x in M. x ∉ limsup A
    using borel_cantelli_limsup1[OF assms] unfolding eventually_ae_filter by auto
```

moreover
{
fix *x* **assume** $x \notin$ *limsup A*
then obtain *N* **where** $x \notin (\bigcup n\in\{N..\}.\ A\ n)$ **unfolding** *limsup_INF_SUP* **by** *blast*
then have *eventually* $(\lambda n.\ x \notin A\ n)$ *sequentially* **using** *eventually_sequentially* **by** *auto*
}
ultimately show *?thesis* **by** *auto*
qed

6.3.12 Measure spaces with *emeasure M* (*space M*) $< \infty$

locale *finite_measure* = *sigma_finite_measure M* **for** *M* +
assumes *finite_emeasure_space*: *emeasure M* (*space M*) $\neq$ *top*

lemma *finite_measureI*[*Pure.intro*!]:
emeasure M (*space M*) $\neq \infty \Longrightarrow$ *finite_measure M*
proof qed (*auto intro*!: *exI*[*of* _ {*space M*}])

lemma (**in** *finite_measure*) *emeasure_finite*[*simp*, *intro*]: *emeasure M A* $\neq$ *top*
using *finite_emeasure_space emeasure_space*[*of M A*] **by** (*auto simp*: *top_unique*)

lemma (**in** *finite_measure*) *fmeasurable_eq_sets*: *fmeasurable M* = *sets M*
by (*auto simp*: *fmeasurable_def less_top*[*symmetric*])

lemma (**in** *finite_measure*) *emeasure_eq_measure*: *emeasure M A* = *ennreal* (*measure M A*)
by (*intro emeasure_eq_ennreal_measure*) *simp*

lemma (**in** *finite_measure*) *emeasure_real*: $\exists\, r.\ 0 \leq r \wedge$ *emeasure M A* = *ennreal r*
using *emeasure_finite*[*of A*] **by** (*cases emeasure M A rule*: *ennreal_cases*) *auto*

lemma (**in** *finite_measure*) *bounded_measure*: *measure M A* $\leq$ *measure M* (*space M*)
using *emeasure_space*[*of M A*] *emeasure_real*[*of A*] *emeasure_real*[*of space M*] **by** (*auto simp*: *measure_def*)

lemma (**in** *finite_measure*) *finite_measure_Diff*:
assumes *sets*: $A \in$ *sets M* $B \in$ *sets M* **and** $B \subseteq A$
shows *measure M* $(A - B)$ = *measure M A* $-$ *measure M B*
using *measure_Diff*[*OF* _ *assms*] **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_Union*:
assumes *sets*: $A \in$ *sets M* $B \in$ *sets M* **and** $A \cap B = \{\}$
shows *measure M* $(A \cup B)$ = *measure M A* + *measure M B*
using *measure_Union*[*OF* _ _ *assms*] **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_finite_Union*:
assumes *measurable*: *finite S A‘S* ⊆ *sets M disjoint_family_on A S*
shows *measure M* ($\bigcup$*i*∈*S*. *A i*) = ($\sum$*i*∈*S*. *measure M* (*A i*))
using *measure_finite_Union*[*OF assms*] **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_UNION*:
assumes *A*: *range A* ⊆ *sets M disjoint_family A*
shows (λ*i*. *measure M* (*A i*)) *sums* (*measure M* ($\bigcup$*i*. *A i*))
using *measure_UNION*[*OF A*] **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_mono*:
assumes *A* ⊆ *B B* ∈ *sets M* **shows** *measure M A* ≤ *measure M B*
using *emeasure_mono*[*OF assms*] *emeasure_real*[*of A*] *emeasure_real*[*of B*] **by** (*auto simp*: *measure_def*)

lemma (**in** *finite_measure*) *finite_measure_subadditive*:
assumes *m*: *A* ∈ *sets M B* ∈ *sets M*
shows *measure M* (*A* ∪ *B*) ≤ *measure M A* + *measure M B*
using *measure_subadditive*[*OF m*] **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_subadditive_finite*:
assumes *finite I A‘I* ⊆ *sets M* **shows** *measure M* ($\bigcup$*i*∈*I*. *A i*) ≤ ($\sum$*i*∈*I*. *measure M* (*A i*))
using *measure_subadditive_finite*[*OF assms*] **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_subadditive_countably*:
range A ⊆ *sets M* ⟹ *summable* (λ*i*. *measure M* (*A i*)) ⟹ *measure M* ($\bigcup$*i*. *A i*) ≤ ($\sum$*i*. *measure M* (*A i*))
by (*rule measure_subadditive_countably*)
(*simp_all add*: *ennreal_suminf_neq_top emeasure_eq_measure*)

lemma (**in** *finite_measure*) *finite_measure_eq_sum_singleton*:
assumes *finite S* **and** ∗: $\bigwedge$*x*. *x* ∈ *S* ⟹ {*x*} ∈ *sets M*
shows *measure M S* = ($\sum$*x*∈*S*. *measure M* {*x*})
using *measure_eq_sum_singleton*[*OF assms*] **by** *simp*

lemma (**in** *finite_measure*) *finite_Lim_measure_incseq*:
assumes *A*: *range A* ⊆ *sets M incseq A*
shows (λ*i*. *measure M* (*A i*)) ⟶ *measure M* ($\bigcup$*i*. *A i*)
using *Lim_measure_incseq*[*OF A*] **by** *simp*

lemma (**in** *finite_measure*) *finite_Lim_measure_decseq*:
assumes *A*: *range A* ⊆ *sets M decseq A*
shows (λ*n*. *measure M* (*A n*)) ⟶ *measure M* ($\bigcap$*i*. *A i*)
using *Lim_measure_decseq*[*OF A*] **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_compl*:
assumes *S*: *S* ∈ *sets M*
shows *measure M* (*space M* − *S*) = *measure M* (*space M*) − *measure M S*

using *measure_Diff*[*OF* _ *sets.top S sets.sets_into_space*] *S* **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_mono_AE*:
assumes *imp*: *AE x in M*. $x \in A \longrightarrow x \in B$ **and** *B*: $B \in$ *sets M*
shows *measure M A* $\leq$ *measure M B*
using *assms emeasure_mono_AE*[*OF imp B*]
by (*simp add*: *emeasure_eq_measure*)

lemma (**in** *finite_measure*) *finite_measure_eq_AE*:
assumes *iff*: *AE x in M*. $x \in A \longleftrightarrow x \in B$
assumes *A*: $A \in$ *sets M* **and** *B*: $B \in$ *sets M*
shows *measure M A* = *measure M B*
using *assms emeasure_eq_AE*[*OF assms*] **by** (*simp add*: *emeasure_eq_measure*)

lemma (**in** *finite_measure*) *measure_increasing*: *increasing M* (*measure M*)
by (*auto intro*!: *finite_measure_mono simp*: *increasing_def*)

lemma (**in** *finite_measure*) *measure_zero_union*:
assumes $s \in$ *sets M* $t \in$ *sets M measure M t* = *0*
shows *measure M* ($s \cup t$) = *measure M s*
using *assms*
proof −
have *measure M* ($s \cup t$) $\leq$ *measure M s*
using *finite_measure_subadditive*[*of s t*] *assms* **by** *auto*
moreover have *measure M* ($s \cup t$) $\geq$ *measure M s*
using *assms* **by** (*blast intro*: *finite_measure_mono*)
ultimately show *?thesis* **by** *simp*
qed

lemma (**in** *finite_measure*) *measure_eq_compl*:
assumes $s \in$ *sets M* $t \in$ *sets M*
assumes *measure M* (*space M* − *s*) = *measure M* (*space M* − *t*)
shows *measure M s* = *measure M t*
using *assms finite_measure_compl* **by** *auto*

lemma (**in** *finite_measure*) *measure_eq_bigunion_image*:
assumes *range f* $\subseteq$ *sets M range g* $\subseteq$ *sets M*
assumes *disjoint_family f disjoint_family g*
assumes $\bigwedge$ *n* :: *nat*. *measure M* (*f n*) = *measure M* (*g n*)
shows *measure M* ($\bigcup$ *i*. *f i*) = *measure M* ($\bigcup$ *i*. *g i*)
using *assms*
proof −
have *a*: (λ *i*. *measure M* (*f i*)) *sums* (*measure M* ($\bigcup$ *i*. *f i*))
by (*rule finite_measure_UNION*[*OF assms*(*1*,*3*)])
have *b*: (λ *i*. *measure M* (*g i*)) *sums* (*measure M* ($\bigcup$ *i*. *g i*))
by (*rule finite_measure_UNION*[*OF assms*(*2*,*4*)])
show *?thesis* **using** *sums_unique*[*OF b*] *sums_unique*[*OF a*] *assms* **by** *simp*
qed

lemma (**in** *finite_measure*) *measure_countably_zero*:
  assumes *range c* ⊆ *sets M*
  assumes ⋀ *i*. *measure M* (*c i*) = *0*
  shows *measure M* (⋃ *i* :: *nat*. *c i*) = *0*
proof (*rule antisym*)
  show *measure M* (⋃ *i* :: *nat*. *c i*) ≤ *0*
   using *finite_measure_subadditive_countably*[*OF assms*(*1*)] **by** (*simp add*: *assms*(*2*))
qed *simp*

lemma (**in** *finite_measure*) *measure_space_inter*:
  assumes *events*:*s* ∈ *sets M t* ∈ *sets M*
  assumes *measure M t* = *measure M* (*space M*)
  shows *measure M* (*s* ∩ *t*) = *measure M s*
proof −
  have *measure M* ((*space M* − *s*) ∪ (*space M* − *t*)) = *measure M* (*space M* − *s*)
   using *events assms finite_measure_compl*[*of t*] **by** (*auto intro*!: *measure_zero_union*)
  also have (*space M* − *s*) ∪ (*space M* − *t*) = *space M* − (*s* ∩ *t*)
   by *blast*
  finally show *measure M* (*s* ∩ *t*) = *measure M s*
   using *events* **by** (*auto intro*!: *measure_eq_compl*[*of s* ∩ *t s*])
qed

lemma (**in** *finite_measure*) *measure_equiprobable_finite_unions*:
  assumes *s*: *finite s* ⋀*x*. *x* ∈ *s* ⟹ {*x*} ∈ *sets M*
  assumes ⋀ *x y*. ⟦*x* ∈ *s*; *y* ∈ *s*⟧ ⟹ *measure M* {*x*} = *measure M* {*y*}
  shows *measure M s* = *real* (*card s*) ∗ *measure M* {*SOME x*. *x* ∈ *s*}
proof *cases*
  assume *s* ≠ {}
  then have ∃ *x*. *x* ∈ *s* **by** *blast*
  from *someI_ex*[*OF this*] *assms*
  have *prob_some*: ⋀ *x*. *x* ∈ *s* ⟹ *measure M* {*x*} = *measure M* {*SOME y*. *y* ∈ *s*} **by** *blast*
  have *measure M s* = (∑ *x* ∈ *s*. *measure M* {*x*})
   using *finite_measure_eq_sum_singleton*[*OF s*] **by** *simp*
  also have ... = (∑ *x* ∈ *s*. *measure M* {*SOME y*. *y* ∈ *s*}) **using** *prob_some* **by** *auto*
  also have ... = *real* (*card s*) ∗ *measure M* {(*SOME x*. *x* ∈ *s*)}
   using *sum_constant assms* **by** *simp*
  finally show *?thesis* **by** *simp*
qed *simp*

lemma (**in** *finite_measure*) *measure_real_sum_image_fn*:
  assumes *e* ∈ *sets M*
  assumes ⋀ *x*. *x* ∈ *s* ⟹ *e* ∩ *f x* ∈ *sets M*
  assumes *finite s*
  assumes *disjoint*: ⋀ *x y*. ⟦*x* ∈ *s* ; *y* ∈ *s* ; *x* ≠ *y*⟧ ⟹ *f x* ∩ *f y* = {}
  assumes *upper*: *space M* ⊆ (⋃ *i* ∈ *s*. *f i*)
  shows *measure M e* = (∑ *x* ∈ *s*. *measure M* (*e* ∩ *f x*))

proof –
  have $e \subseteq (\bigcup i{\in}s.\ f\ i)$
    using ‹$e \in sets\ M$› *sets.sets_into_space upper* **by** *blast*
  then have e: $e = (\bigcup i \in s.\ e \cap f\ i)$
    by *auto*
  hence $measure\ M\ e = measure\ M\ (\bigcup i \in s.\ e \cap f\ i)$ **by** *simp*
  also have $\ldots = (\sum x \in s.\ measure\ M\ (e \cap f\ x))$
  proof (*rule finite_measure_finite_Union*)
    show *finite s* **by** *fact*
    show $(\lambda i.\ e \cap f\ i)`s \subseteq sets\ M$ **using** *assms*(*2*) **by** *auto*
    show *disjoint_family_on* $(\lambda i.\ e \cap f\ i)\ s$
      using *disjoint* **by** (*auto simp*: *disjoint_family_on_def*)
  qed
  finally show *?thesis* **.**
qed

lemma (**in** *finite_measure*) *measure_exclude*:
  assumes $A \in sets\ M\ B \in sets\ M$
  assumes $measure\ M\ A = measure\ M\ (space\ M)\ A \cap B = \{\}$
  shows $measure\ M\ B = 0$
  using *measure_space_inter*[*of B A*] *assms* **by** (*auto simp*: *ac_simps*)
lemma (**in** *finite_measure*) *finite_measure_distr*:
  assumes f: $f \in measurable\ M\ M'$
  shows *finite_measure* $(distr\ M\ M'\ f)$
proof (*rule finite_measureI*)
  have $f\ -`\ space\ M' \cap space\ M = space\ M$ **using** f **by** (*auto dest*: *measurable_space*)
  with f **show** $emeasure\ (distr\ M\ M'\ f)\ (space\ (distr\ M\ M'\ f)) \neq \infty$ **by** (*auto simp*: *emeasure_distr*)
qed

lemma *emeasure_gfp*[*consumes 1*, *case_names cont measurable*]:
  assumes *sets*[*simp*]: $\bigwedge s.\ sets\ (M\ s) = sets\ N$
  assumes $\bigwedge s.$ *finite_measure* $(M\ s)$
  assumes *cont*: *inf_continuous F inf_continuous f*
  assumes *meas*: $\bigwedge P.\ Measurable.pred\ N\ P \Longrightarrow Measurable.pred\ N\ (F\ P)$
  assumes *iter*: $\bigwedge P\ s.\ Measurable.pred\ N\ P \Longrightarrow emeasure\ (M\ s)\ \{x{\in}space\ N.\ F\ P\ x\} = f\ (\lambda s.\ emeasure\ (M\ s)\ \{x{\in}space\ N.\ P\ x\})\ s$
  assumes *bound*: $\bigwedge P.\ f\ P \leq f\ (\lambda s.\ emeasure\ (M\ s)\ (space\ (M\ s)))$
  shows $emeasure\ (M\ s)\ \{x{\in}space\ N.\ gfp\ F\ x\} = gfp\ f\ s$
proof (*subst gfp_transfer_bounded*[**where** $\alpha{=}\lambda F\ s.\ emeasure\ (M\ s)\ \{x{\in}space\ N.\ F\ x\}$ **and** $g{=}f$ **and** $f{=}F$ **and**
    $P{=}Measurable.pred\ N$, *symmetric*])
  interpret *finite_measure M s* **for** *s* **by** *fact*
  fix C **assume** *decseq* $C\ \bigwedge i.\ Measurable.pred\ N\ (C\ i)$
  then show $(\lambda s.\ emeasure\ (M\ s)\ \{x \in space\ N.\ (INF\ i.\ C\ i)\ x\}) = (INF\ i.\ (\lambda s.\ emeasure\ (M\ s)\ \{x \in space\ N.\ C\ i\ x\}))$
    unfolding *INF_apply*[*abs_def*]
      by (*subst INF_emeasure_decseq*) (*auto simp*: *antimono_def fun_eq_iff intro*!:

arg_cong2[**where** *f*=*emeasure*])
next
  show *f x* ≤ (λ*s*. *emeasure* (*M s*) {*x* ∈ *space N*. *F top x*}) **for** *x*
    using *bound*[*of x*] *sets_eq_imp_space_eq*[*OF sets*] **by** (*simp add*: *iter*)
qed (*auto simp add*: *iter le_fun_def INF_apply*[*abs_def*] *intro*!: *meas cont*)

6.3.13 Counting space

lemma *strict_monoI_Suc*:
  assumes *ord* [*simp*]: (⋀*n*. *f n* < *f* (*Suc n*)) **shows** *strict_mono f*
  unfolding *strict_mono_def*
proof *safe*
  fix *n m* :: *nat* **assume** *n* < *m* **then show** *f n* < *f m*
    by (*induct m*) (*auto simp*: *less_Suc_eq intro*: *less_trans ord*)
qed

lemma *emeasure_count_space*:
  assumes *X* ⊆ *A* **shows** *emeasure* (*count_space A*) *X* = (*if finite X then of_nat* (*card X*) *else* ∞)
    (**is** _ = *?M X*)
  unfolding *count_space_def*
proof (*rule emeasure_measure_of_sigma*)
  show *X* ∈ *Pow A* **using** ‹*X* ⊆ *A*› **by** *auto*
  show *sigma_algebra A* (*Pow A*) **by** (*rule sigma_algebra_Pow*)
  show *positive*: *positive* (*Pow A*) *?M*
    by (*auto simp*: *positive_def*)
  have *additive*: *additive* (*Pow A*) *?M*
    by (*auto simp*: *card_Un_disjoint additive_def*)

  interpret *ring_of_sets A Pow A*
    by (*rule ring_of_setsI*) *auto*
  show *countably_additive* (*Pow A*) *?M*
    unfolding *countably_additive_iff_continuous_from_below*[*OF positive additive*]
  proof *safe*
    fix *F* :: *nat* ⇒ *'a set* **assume** *incseq F*
    show (λ*i*. *?M* (*F i*)) ⟶ *?M* (⋃*i*. *F i*)
    proof *cases*
      assume ∃*i*. ∀*j*≥*i*. *F i* = *F j*
      then guess *i* **..** **note** *i* = *this*
      { fix *j* **from** *i* ‹*incseq F*› **have** *F j* ⊆ *F i*
          by (*cases i* ≤ *j*) (*auto simp*: *incseq_def*) **}**
      then have *eq*: (⋃*i*. *F i*) = *F i*
        by *auto*
      with *i* **show** *?thesis*
      by (*auto intro*!: *Lim_transform_eventually*[*OF tendsto_const*] *eventually_sequentiallyI*[**where** *c*=*i*])
    next
      assume ¬ (∃*i*. ∀*j*≥*i*. *F i* = *F j*)
      then obtain *f* **where** *f*: ⋀*i*. *i* ≤ *f i* ⋀*i*. *F i* ≠ *F* (*f i*) **by** *metis*

then have $\bigwedge i.\ F\ i \subseteq F\ (f\ i)$ **using** ‹*incseq F*› **by** (*auto simp*: *incseq_def*)
with *f* **have** ∗: $\bigwedge i.\ F\ i \subset F\ (f\ i)$ **by** *auto*

have *incseq* (λ*i*. *?M* (*F i*))
using ‹*incseq F*› **unfolding** *incseq_def* **by** (*auto simp*: *card_mono dest*: *finite_subset*)
then have (λ*i*. *?M* (*F i*)) ⟶ (*SUP n*. *?M* (*F n*))
by (*rule LIMSEQ_SUP*)

moreover have (*SUP n*. *?M* (*F n*)) = *top*
proof (*rule ennreal_SUP_eq_top*)
fix *n* :: *nat* **show** ∃ *k*::*nat*∈*UNIV*. *of_nat n* ≤ *?M* (*F k*)
proof (*induct n*)
case (*Suc n*)
then guess *k* .. **note** *k* = *this*
moreover have *finite* (*F k*) ⟹ *finite* (*F* (*f k*)) ⟹ *card* (*F k*) < *card* (*F* (*f k*))
using ‹*F k* ⊂ *F* (*f k*)› **by** (*simp add*: *psubset_card_mono*)
moreover have *finite* (*F* (*f k*)) ⟹ *finite* (*F k*)
using ‹*k* ≤ *f k*› ‹*incseq F*› **by** (*auto simp*: *incseq_def dest*: *finite_subset*)
ultimately show *?case*
by (*auto intro*!: *exI*[*of* _ *f k*] *simp del*: *of_nat_Suc*)
qed *auto*
qed

moreover
have *inj* (λ*n*. *F* ((*f* ^^ *n*) *0*))
by (*intro strict_mono_imp_inj_on strict_monoI_Suc*) (*simp add*: ∗)
then have *1*: *infinite* (*range* (λ*i*. *F* ((*f* ^^ *i*) *0*)))
by (*rule range_inj_infinite*)
have *infinite* (*Pow* (⋃*i*. *F i*))
by (*rule infinite_super*[*OF* _ *1*]) *auto*
then have *infinite* (⋃*i*. *F i*)
by *auto*
ultimately show *?thesis* **by** (*simp only*:) *simp*

qed
qed
qed

lemma *distr_bij_count_space*:
assumes *f*: *bij_betw f A B*
shows *distr* (*count_space A*) (*count_space B*) *f* = *count_space B*
proof (*rule measure_eqI*)
have *f'*: *f* ∈ *measurable* (*count_space A*) (*count_space B*)
using *f* **unfolding** *Pi_def bij_betw_def* **by** *auto*
fix *X* **assume** *X* ∈ *sets* (*distr* (*count_space A*) (*count_space B*) *f*)
then have *X*: *X* ∈ *sets* (*count_space B*) **by** *auto*
moreover from *X* **have** *f* −‘ *X* ∩ *A* = *the_inv_into A f* ‘ *X*

using *f* **by** (*auto simp*: *bij_betw_def subset_image_iff image_iff the_inv_into_f_f intro*: *the_inv_into_f_f*[*symmetric*])
moreover have *inj_on* (*the_inv_into A f*) *B*
using *X f* **by** (*auto simp*: *bij_betw_def inj_on_the_inv_into*)
with *X* **have** *inj_on* (*the_inv_into A f*) *X*
by (*auto intro*: *subset_inj_on*)
ultimately show *emeasure* (*distr* (*count_space A*) (*count_space B*) *f*) *X* = *emeasure* (*count_space B*) *X*
using *f* **unfolding** *emeasure_distr*[*OF f' X*]
by (*subst* (*1 2*) *emeasure_count_space*) (*auto simp*: *card_image dest*: *finite_imageD*)
qed *simp*

lemma *emeasure_count_space_finite*[*simp*]:
$X \subseteq A \Longrightarrow$ *finite X* $\Longrightarrow$ *emeasure* (*count_space A*) *X* = *of_nat* (*card X*)
using *emeasure_count_space*[*of X A*] **by** *simp*

lemma *emeasure_count_space_infinite*[*simp*]:
$X \subseteq A \Longrightarrow$ *infinite X* $\Longrightarrow$ *emeasure* (*count_space A*) *X* = ∞
using *emeasure_count_space*[*of X A*] **by** *simp*

lemma *measure_count_space*: *measure* (*count_space A*) *X* = (*if* $X \subseteq A$ *then of_nat* (*card X*) *else 0*)
by (*cases finite X*) (*auto simp*: *measure_notin_sets ennreal_of_nat_eq_real_of_nat measure_zero_top measure_eq_emeasure_eq_ennreal*)

lemma *emeasure_count_space_eq_0*:
emeasure (*count_space A*) *X* = *0* $\longleftrightarrow$ ($X \subseteq A \longrightarrow X = \{\}$)
proof *cases*
assume *X*: $X \subseteq A$
then show *?thesis*
proof (*intro iffI impI*)
assume *emeasure* (*count_space A*) *X* = *0*
with *X* **show** $X = \{\}$
by (*subst* (*asm*) *emeasure_count_space*) (*auto split*: *if_split_asm*)
qed *simp*
qed (*simp add*: *emeasure_notin_sets*)

lemma *null_sets_count_space*: *null_sets* (*count_space A*) = $\{ \{\} \}$
unfolding *null_sets_def* **by** (*auto simp add*: *emeasure_count_space_eq_0*)

lemma *AE_count_space*: (*AE x in count_space A*. *P x*) $\longleftrightarrow$ ($\forall x \in A$. *P x*)
unfolding *eventually_ae_filter* **by** (*auto simp add*: *null_sets_count_space*)

lemma *sigma_finite_measure_count_space_countable*:
assumes *A*: *countable A*
shows *sigma_finite_measure* (*count_space A*)
proof qed (*insert A*, *auto intro*!: *exI*[*of _* (λa. $\{a\}$) ‘ *A*])

lemma *sigma_finite_measure_count_space*:

fixes *A* :: *'a::countable set* **shows** *sigma_finite_measure* (*count_space A*)
by (*rule sigma_finite_measure_count_space_countable*) *auto*

lemma *finite_measure_count_space*:
assumes [*simp*]: *finite A*
shows *finite_measure* (*count_space A*)
by *rule simp*

lemma *sigma_finite_measure_count_space_finite*:
assumes *A*: *finite A* **shows** *sigma_finite_measure* (*count_space A*)
proof −
interpret *finite_measure count_space A* **using** *A* **by** (*rule finite_measure_count_space*)
show *sigma_finite_measure* (*count_space A*) **..**
qed

6.3.14 Measure restricted to space

lemma *emeasure_restrict_space*:
assumes Ω ∩ *space M* ∈ *sets M A* ⊆ Ω
shows *emeasure* (*restrict_space M* Ω) *A* = *emeasure M A*
proof (*cases A* ∈ *sets M*)
case *True*
show *?thesis*
proof (*rule emeasure_measure_of*[*OF restrict_space_def*])
show (∩) Ω ' *sets M* ⊆ *Pow* (Ω ∩ *space M*) *A* ∈ *sets* (*restrict_space M* Ω)
using ‹*A* ⊆ Ω› ‹*A* ∈ *sets M*› *sets.space_closed* **by** (*auto simp*: *sets_restrict_space*)
show *positive* (*sets* (*restrict_space M* Ω)) (*emeasure M*)
by (*auto simp*: *positive_def*)
show *countably_additive* (*sets* (*restrict_space M* Ω)) (*emeasure M*)
proof (*rule countably_additiveI*)
fix *A* :: *nat* ⇒ _ **assume** *range A* ⊆ *sets* (*restrict_space M* Ω) *disjoint_family*
A
with *assms* **have** ⋀*i*. *A i* ∈ *sets M* ⋀*i*. *A i* ⊆ *space M disjoint_family A*
by (*fastforce simp*: *sets_restrict_space_iff*[*OF assms*(*1*)] *image_subset_iff*
dest: *sets.sets_into_space*)+
then show (∑ *i*. *emeasure M* (*A i*)) = *emeasure M* (⋃ *i*. *A i*)
by (*subst suminf_emeasure*) (*auto simp*: *disjoint_family_subset*)
qed
qed
next
case *False*
with *assms* **have** *A* ∉ *sets* (*restrict_space M* Ω)
by (*simp add*: *sets_restrict_space_iff*)
with *False* **show** *?thesis*
by (*simp add*: *emeasure_notin_sets*)
qed

lemma *measure_restrict_space*:
assumes Ω ∩ *space M* ∈ *sets M A* ⊆ Ω

shows *measure* (*restrict_space* *M* Ω) *A* = *measure* *M* *A*
using *emeasure_restrict_space*[*OF assms*] **by** (*simp add*: *measure_def*)

lemma *AE_restrict_space_iff*:
assumes Ω ∩ *space* *M* ∈ *sets* *M*
shows (*AE* *x* *in* *restrict_space* *M* Ω. *P* *x*) ⟷ (*AE* *x* *in* *M*. *x* ∈ Ω ⟶ *P* *x*)
proof −
have *ex_cong*: ⋀*P* *Q* *f*. (⋀*x*. *P* *x* ⟹ *Q* *x*) ⟹ (⋀*x*. *Q* *x* ⟹ *P* (*f* *x*)) ⟹ (∃*x*. *P* *x*) ⟷ (∃*x*. *Q* *x*)
by *auto*
{ **fix** *X* **assume** *X*: *X* ∈ *sets* *M* *emeasure* *M* *X* = *0*
then have *emeasure* *M* (Ω ∩ *space* *M* ∩ *X*) ≤ *emeasure* *M* *X*
by (*intro emeasure_mono*) *auto*
then have *emeasure* *M* (Ω ∩ *space* *M* ∩ *X*) = *0*
using *X* **by** (*auto intro*!: *antisym*) }
with *assms* **show** *?thesis*
unfolding *eventually_ae_filter*
by (*auto simp add*: *space_restrict_space null_sets_def sets_restrict_space_iff*
emeasure_restrict_space cong: *conj_cong*
intro!: *ex_cong*[**where** *f*=λ*X*. (Ω ∩ *space* *M*) ∩ *X*])
qed

lemma *restrict_restrict_space*:
assumes *A* ∩ *space* *M* ∈ *sets* *M* *B* ∩ *space* *M* ∈ *sets* *M*
shows *restrict_space* (*restrict_space* *M* *A*) *B* = *restrict_space* *M* (*A* ∩ *B*) (**is** *?l* = *?r*)
proof (*rule measure_eqI*[*symmetric*])
show *sets* *?r* = *sets* *?l*
unfolding *sets_restrict_space image_comp* **by** (*intro image_cong*) *auto*
next
fix *X* **assume** *X* ∈ *sets* (*restrict_space* *M* (*A* ∩ *B*))
then obtain *Y* **where** *Y* ∈ *sets* *M* *X* = *Y* ∩ *A* ∩ *B*
by (*auto simp*: *sets_restrict_space*)
with *assms sets.Int*[*OF assms*] **show** *emeasure* *?r* *X* = *emeasure* *?l* *X*
by (*subst* (*1 2*) *emeasure_restrict_space*)
(*auto simp*: *space_restrict_space sets_restrict_space_iff emeasure_restrict_space ac_simps*)
qed

lemma *restrict_count_space*: *restrict_space* (*count_space* *B*) *A* = *count_space* (*A* ∩ *B*)
proof (*rule measure_eqI*)
show *sets* (*restrict_space* (*count_space* *B*) *A*) = *sets* (*count_space* (*A* ∩ *B*))
by (*subst sets_restrict_space*) *auto*
moreover fix *X* **assume** *X* ∈ *sets* (*restrict_space* (*count_space* *B*) *A*)
ultimately have *X* ⊆ *A* ∩ *B* **by** *auto*
then show *emeasure* (*restrict_space* (*count_space* *B*) *A*) *X* = *emeasure* (*count_space* (*A* ∩ *B*)) *X*
by (*cases finite X*) (*auto simp add*: *emeasure_restrict_space*)

qed

lemma *sigma_finite_measure_restrict_space*:
assumes *sigma_finite_measure M*
and *A*: $A \in$ *sets M*
shows *sigma_finite_measure* (*restrict_space M A*)
proof −
interpret *sigma_finite_measure M* **by** *fact*
from *sigma_finite_countable* **obtain** *C*
where *C*: *countable C* $C \subseteq$ *sets M* ($\bigcup C$) = *space M* $\forall a \in C$. *emeasure M a* $\neq \infty$
by *blast*
let *?C* = ($\cap$) *A* ‘ *C*
from *C* **have** *countable ?C ?C* $\subseteq$ *sets* (*restrict_space M A*) ($\bigcup$ *?C*) = *space* (*restrict_space M A*)
by(*auto simp add*: *sets_restrict_space space_restrict_space*)
moreover {
fix *a*
assume $a \in$ *?C*
then obtain a' **where** $a = A \cap a'$ $a' \in C$ **..**
then have *emeasure* (*restrict_space M A*) $a \leq$ *emeasure M* a'
using *A C* **by**(*auto simp add*: *emeasure_restrict_space intro*: *emeasure_mono*)
also have $\ldots < \infty$ **using** *C*(*4*)[*rule_format*, *of a'*] ‹$a' \in C$› **by** (*simp add*: *less_top*)
finally have *emeasure* (*restrict_space M A*) $a \neq \infty$ **by** *simp* }
ultimately show *?thesis*
by *unfold_locales* (*rule exI conjI*|*assumption*|*blast*)+
qed

lemma *finite_measure_restrict_space*:
assumes *finite_measure M*
and *A*: $A \in$ *sets M*
shows *finite_measure* (*restrict_space M A*)
proof −
interpret *finite_measure M* **by** *fact*
show *?thesis*
by(*rule finite_measureI*)(*simp add*: *emeasure_restrict_space A space_restrict_space*)
qed

lemma *restrict_distr*:
assumes [*measurable*]: $f \in$ *measurable M N*
assumes [*simp*]: $\Omega \cap$ *space N* $\in$ *sets N* **and** *restrict*: $f \in$ *space M* $\rightarrow \Omega$
shows *restrict_space* (*distr M N f*) Ω = *distr M* (*restrict_space N* Ω) *f*
(**is** *?l* = *?r*)
proof (*rule measure_eqI*)
fix *A* **assume** $A \in$ *sets* (*restrict_space* (*distr M N f*) Ω)
with *restrict* **show** *emeasure ?l A* = *emeasure ?r A*
by (*subst emeasure_distr*)
(*auto simp*: *sets_restrict_space_iff emeasure_restrict_space emeasure_distr*

intro!: *measurable_restrict_space2*)
qed (*simp add*: *sets_restrict_space*)

lemma *measure_eqI_restrict_generator*:
assumes *E*: *Int_stable E E* ⊆ *Pow* Ω ⋀*X*. *X* ∈ *E* ⟹ *emeasure M X* = *emeasure N X*
assumes *sets_eq*: *sets M* = *sets N* **and** Ω: Ω ∈ *sets M*
assumes *sets* (*restrict_space M* Ω) = *sigma_sets* Ω *E*
assumes *sets* (*restrict_space N* Ω) = *sigma_sets* Ω *E*
assumes *ae*: *AE x in M*. *x* ∈ Ω *AE x in N*. *x* ∈ Ω
assumes *A*: *countable A A* ≠ {} *A* ⊆ *E* ⋃*A* = Ω ⋀*a*. *a* ∈ *A* ⟹ *emeasure M a* ≠ ∞
shows *M* = *N*
proof (*rule measure_eqI*)
fix *X* **assume** *X*: *X* ∈ *sets M*
then have *emeasure M X* = *emeasure* (*restrict_space M* Ω) (*X* ∩ Ω)
using *ae* Ω **by** (*auto simp add*: *emeasure_restrict_space intro*!: *emeasure_eq_AE*)
also have *restrict_space M* Ω = *restrict_space N* Ω
proof (*rule measure_eqI_generator_eq*)
fix *X* **assume** *X* ∈ *E*
then show *emeasure* (*restrict_space M* Ω) *X* = *emeasure* (*restrict_space N* Ω) *X*
using *E* Ω **by** (*subst* (*1 2*) *emeasure_restrict_space*) (*auto simp*: *sets_eq sets_eq*[*THEN sets_eq_imp_space_eq*])
next
show *range* (*from_nat_into A*) ⊆ *E* (⋃*i*. *from_nat_into A i*) = Ω
using *A* **by** (*auto cong del*: *SUP_cong_simp*)
next
fix *i*
have *emeasure* (*restrict_space M* Ω) (*from_nat_into A i*) = *emeasure M* (*from_nat_into A i*)
using *A* Ω **by** (*subst emeasure_restrict_space*)
(*auto simp*: *sets_eq sets_eq*[*THEN sets_eq_imp_space_eq*] *intro*: *from_nat_into*)
with *A* **show** *emeasure* (*restrict_space M* Ω) (*from_nat_into A i*) ≠ ∞
by (*auto intro*: *from_nat_into*)
qed *fact*+
also have *emeasure* (*restrict_space N* Ω) (*X* ∩ Ω) = *emeasure N X*
using *X ae* Ω **by** (*auto simp add*: *emeasure_restrict_space sets_eq intro*!: *emeasure_eq_AE*)
finally show *emeasure M X* = *emeasure N X* **.**
qed *fact*

6.3.15 Null measure

definition *null_measure* :: *'a measure* ⇒ *'a measure* **where**
null_measure M = *sigma* (*space M*) (*sets M*)

lemma *space_null_measure*[*simp*]: *space* (*null_measure M*) = *space M*

by (*simp add*: *null_measure_def*)

lemma *sets_null_measure*[*simp*, *measurable_cong*]: *sets* (*null_measure M*) = *sets M*
by (*simp add*: *null_measure_def*)

lemma *emeasure_null_measure*[*simp*]: *emeasure* (*null_measure M*) *X* = *0*
by (*cases X* ∈ *sets M*, *rule emeasure_measure_of*)
(*auto simp*: *positive_def countably_additive_def emeasure_notin_sets null_measure_def dest*: *sets.sets_into_space*)

lemma *measure_null_measure*[*simp*]: *measure* (*null_measure M*) *X* = *0*
by (*intro measure_eq_emeasure_eq_ennreal*) *auto*

lemma *null_measure_idem* [*simp*]: *null_measure* (*null_measure M*) = *null_measure M*
by(*rule measure_eqI*) *simp_all*

6.3.16 Scaling a measure

definition *scale_measure* :: *ennreal* ⇒ *'a measure* ⇒ *'a measure* **where**
scale_measure r M = *measure_of* (*space M*) (*sets M*) (λ*A*. *r* ∗ *emeasure M A*)

lemma *space_scale_measure*: *space* (*scale_measure r M*) = *space M*
by (*simp add*: *scale_measure_def*)

lemma *sets_scale_measure* [*simp*, *measurable_cong*]: *sets* (*scale_measure r M*) = *sets M*
by (*simp add*: *scale_measure_def*)

lemma *emeasure_scale_measure* [*simp*]:
emeasure (*scale_measure r M*) *A* = *r* ∗ *emeasure M A*
(**is** _ = *?μ A*)
proof(*cases A* ∈ *sets M*)
case *True*
show *?thesis* **unfolding** *scale_measure_def*
proof(*rule emeasure_measure_of_sigma*)
show *sigma_algebra* (*space M*) (*sets M*) **..**
show *positive* (*sets M*) *?μ* **by** (*simp add*: *positive_def*)
show *countably_additive* (*sets M*) *?μ*
proof (*rule countably_additiveI*)
fix *A* :: *nat* ⇒ _ **assume** ∗: *range A* ⊆ *sets M disjoint_family A*
have (∑ *i*. *?μ* (*A i*)) = *r* ∗ (∑ *i*. *emeasure M* (*A i*))
by *simp*
also have ... = *?μ* (⋃ *i*. *A i*) **using** ∗ **by**(*simp add*: *suminf_emeasure*)
finally show (∑ *i*. *?μ* (*A i*)) = *?μ* (⋃ *i*. *A i*) **.**
qed
qed(*fact True*)
qed(*simp add*: *emeasure_notin_sets*)

lemma *scale_measure_1* [*simp*]: *scale_measure 1 M = M*
  by(*rule measure_eqI*) *simp_all*

lemma *scale_measure_0*[*simp*]: *scale_measure 0 M = null_measure M*
  by(*rule measure_eqI*) *simp_all*

lemma *measure_scale_measure* [*simp*]: *0 ≤ r ⟹ measure* (*scale_measure r M*) *A = r ∗ measure M A*
  using *emeasure_scale_measure*[*of r M A*]
    emeasure_eq_ennreal_measure[*of M A*]
    measure_eq_emeasure_eq_ennreal[*of _ scale_measure r M A*]
  by (*cases emeasure* (*scale_measure r M*) *A = top*)
    (*auto simp del*: *emeasure_scale_measure*
      simp: *ennreal_top_eq_mult_iff ennreal_mult_eq_top_iff measure_zero_top ennreal_mult*[*symmetric*])

lemma *scale_scale_measure* [*simp*]:
  scale_measure r (*scale_measure r′ M*) = *scale_measure* (*r ∗ r′*) *M*
  by (*rule measure_eqI*) (*simp_all add*: *max_def mult.assoc*)

lemma *scale_null_measure* [*simp*]: *scale_measure r* (*null_measure M*) = *null_measure M*
  by (*rule measure_eqI*) *simp_all*

6.3.17 Complete lattice structure on measures

lemma (**in** *finite_measure*) *finite_measure_Diff′*:
  A ∈ sets M ⟹ B ∈ sets M ⟹ measure M (*A − B*) = *measure M A − measure M* (*A ∩ B*)
  using *finite_measure_Diff*[*of A A ∩ B*] **by** (*auto simp*: *Diff_Int*)

lemma (**in** *finite_measure*) *finite_measure_Union′*:
  A ∈ sets M ⟹ B ∈ sets M ⟹ measure M (*A ∪ B*) = *measure M A + measure M* (*B − A*)
  using *finite_measure_Union*[*of A B − A*] **by** *auto*

lemma *finite_unsigned_Hahn_decomposition*:
  assumes *finite_measure M finite_measure N* **and** [*simp*]: *sets N = sets M*
  shows *∃ Y∈sets M.* (*∀ X∈sets M. X ⊆ Y ⟶ N X ≤ M X*) *∧* (*∀ X∈sets M. X ∩ Y =* {} *⟶ M X ≤ N X*)
proof −
  interpret *M*: *finite_measure M* **by** *fact*
  interpret *N*: *finite_measure N* **by** *fact*

  define *d* **where** *d X = measure M X − measure N X* **for** *X*

  have [*intro*]: *bdd_above* (*d‘sets M*)
    using *sets.sets_into_space*[*of _ M*]
    by (*intro bdd_aboveI*[**where** *M=measure M* (*space M*)])

(*auto simp*: *d_def field_simps subset_eq intro*!: *add_increasing M.finite_measure_mono*)

define γ **where** $\gamma = (SUP\ X{\in}sets\ M.\ d\ X)$
have *le_*γ[*intro*]: $X \in sets\ M \Longrightarrow d\ X \leq \gamma$ **for** X
by (*auto simp*: γ*_def intro*!: *cSUP_upper*)

have $\exists f.\ \forall n.\ f\ n \in sets\ M \wedge d\ (f\ n) > \gamma - 1\ /\ 2\hat{}n$
proof (*intro choice_iff*[*THEN iffD1*] *allI*)
fix n
have $\exists X{\in}sets\ M.\ \gamma - 1\ /\ 2\hat{}n < d\ X$
unfolding γ*_def* **by** (*intro less_cSUP_iff*[*THEN iffD1*]) *auto*
then show $\exists y.\ y \in sets\ M \wedge \gamma - 1\ /\ 2\ \hat{}\ n < d\ y$
by *auto*
qed
then obtain E **where** [*measurable*]: $E\ n \in sets\ M$ **and** E: $d\ (E\ n) > \gamma - 1\ /\ 2\hat{}n$ **for** n
by *auto*

define F **where** $F\ m\ n = (if\ m \leq n\ then\ \bigcap i{\in}\{m..n\}.\ E\ i\ else\ space\ M)$ **for** $m\ n$

have [*measurable*]: $m \leq n \Longrightarrow F\ m\ n \in sets\ M$ **for** $m\ n$
by (*auto simp*: *F_def*)

have *1*: $\gamma - 2\ /\ 2\ \hat{}\ m + 1\ /\ 2\ \hat{}\ n \leq d\ (F\ m\ n)$ **if** $m \leq n$ **for** $m\ n$
using *that*
proof (*induct rule*: *dec_induct*)
case *base* **with** E[*of m*] **show** *?case*
by (*simp add*: *F_def field_simps*)
next
case (*step i*)
have *F_Suc*: $F\ m\ (Suc\ i) = F\ m\ i \cap E\ (Suc\ i)$
using ‹$m \leq i$› **by** (*auto simp*: *F_def le_Suc_eq*)

have $\gamma + (\gamma - 2\ /\ 2\hat{}m + 1\ /\ 2\ \hat{}\ Suc\ i) \leq (\gamma - 1\ /\ 2\hat{}Suc\ i) + (\gamma - 2\ /\ 2\hat{}m + 1\ /\ 2\hat{}i)$
by (*simp add*: *field_simps*)
also have $\ldots \leq d\ (E\ (Suc\ i)) + d\ (F\ m\ i)$
using E[*of Suc i*] **by** (*intro add_mono step*) *auto*
also have $\ldots = d\ (E\ (Suc\ i)) + d\ (F\ m\ i - E\ (Suc\ i)) + d\ (F\ m\ (Suc\ i))$
using ‹$m \leq i$› **by** (*simp add*: *d_def field_simps F_Suc M.finite_measure_Diff' N.finite_measure_Diff'*)
also have $\ldots = d\ (E\ (Suc\ i) \cup F\ m\ i) + d\ (F\ m\ (Suc\ i))$
using ‹$m \leq i$› **by** (*simp add*: *d_def field_simps M.finite_measure_Union' N.finite_measure_Union'*)
also have $\ldots \leq \gamma + d\ (F\ m\ (Suc\ i))$
using ‹$m \leq i$› **by** *auto*
finally show *?case*
by *auto*

qed

define F' **where** $F'\ m = (\bigcap i{\in}\{m..\}.\ E\ i)$ **for** m
have F'_eq: $F'\ m = (\bigcap i.\ F\ m\ (i + m))$ **for** m
  by (*fastforce simp*: *le_iff_add*[*of m*] *F'_def F_def*)

have [*measurable*]: $F'\ m \in sets\ M$ **for** m
  by (*auto simp*: *F'_def*)

have γ_le: $\gamma - 0 \leq d\ (\bigcup m.\ F'\ m)$
proof (*rule LIMSEQ_le*)
  show $(\lambda n.\ \gamma - 2\ /\ 2\ \hat{}\ n) \longrightarrow \gamma - 0$
    by (*intro tendsto_intros LIMSEQ_divide_realpow_zero*) *auto*
  have *incseq* F'
    by (*auto simp*: *incseq_def F'_def*)
  then show $(\lambda m.\ d\ (F'\ m)) \longrightarrow d\ (\bigcup m.\ F'\ m)$
    unfolding *d_def*
   by (*intro tendsto_diff M.finite_Lim_measure_incseq N.finite_Lim_measure_incseq*)
auto

  have $\gamma - 2\ /\ 2\ \hat{}\ m + 0 \leq d\ (F'\ m)$ **for** m
  proof (*rule LIMSEQ_le*)
    have $*$: *decseq* $(\lambda n.\ F\ m\ (n + m))$
      by (*auto simp*: *decseq_def F_def*)
    show $(\lambda n.\ d\ (F\ m\ n)) \longrightarrow d\ (F'\ m)$
      unfolding *d_def F'_eq*
      by (*rule LIMSEQ_offset*[**where** *k=m*])
      (*auto intro*!: *tendsto_diff M.finite_Lim_measure_decseq N.finite_Lim_measure_decseq*
$*$)
    show $(\lambda n.\ \gamma - 2\ /\ 2\ \hat{}\ m + 1\ /\ 2\ \hat{}\ n) \longrightarrow \gamma - 2\ /\ 2\ \hat{}\ m + 0$
      by (*intro tendsto_add LIMSEQ_divide_realpow_zero tendsto_const*) *auto*
    show $\exists N.\ \forall n{\geq}N.\ \gamma - 2\ /\ 2\ \hat{}\ m + 1\ /\ 2\ \hat{}\ n \leq d\ (F\ m\ n)$
      using *1*[*of m*] **by** (*intro exI*[*of _ m*]) *auto*
  qed
  then show $\exists N.\ \forall n{\geq}N.\ \gamma - 2\ /\ 2\ \hat{}\ n \leq d\ (F'\ n)$
    by *auto*
qed

show *?thesis*
proof (*safe intro*!: *bexI*[*of _* $\bigcup m.\ F'\ m$])
  fix X **assume** [*measurable*]: $X \in sets\ M$ **and** X: $X \subseteq (\bigcup m.\ F'\ m)$
  have $d\ (\bigcup m.\ F'\ m) - d\ X = d\ ((\bigcup m.\ F'\ m) - X)$
    using X **by** (*auto simp*: *d_def M.finite_measure_Diff N.finite_measure_Diff*)
  also have $\ldots \leq \gamma$
    by *auto*
  finally have $0 \leq d\ X$
    using γ_le **by** *auto*
  then show *emeasure N X* $\leq$ *emeasure M X*
    by (*auto simp*: *d_def M.emeasure_eq_measure N.emeasure_eq_measure*)

next
fix *X* **assume** [*measurable*]: $X \in$ *sets M* **and** *X*: $X \cap (\bigcup m.\ F'\ m) = \{\}$
then have $d\ (\bigcup m.\ F'\ m) + d\ X = d\ (X \cup (\bigcup m.\ F'\ m))$
by (*auto simp*: *d_def M.finite_measure_Union N.finite_measure_Union*)
also have $\ldots \leq \gamma$
by *auto*
finally have $d\ X \leq 0$
using *γ_le* **by** *auto*
then show *emeasure M X* ≤ *emeasure N X*
by (*auto simp*: *d_def M.emeasure_eq_measure N.emeasure_eq_measure*)
qed *auto*
qed

proposition *unsigned_Hahn_decomposition*:
assumes [*simp*]: *sets N* = *sets M* **and** [*measurable*]: *A* ∈ *sets M*
and [*simp*]: *emeasure M A* ≠ *top emeasure N A* ≠ *top*
shows ∃ *Y*∈*sets M*. *Y* ⊆ *A* ∧ (∀ *X*∈*sets M*. *X* ⊆ *Y* ⟶ *N X* ≤ *M X*) ∧ (∀ *X*∈*sets M*. *X* ⊆ *A* ⟶ *X* ∩ *Y* = {} ⟶ *M X* ≤ *N X*)
proof −
have ∃ *Y*∈*sets* (*restrict_space M A*).
(∀ *X*∈*sets* (*restrict_space M A*). *X* ⊆ *Y* ⟶ (*restrict_space N A*) *X* ≤ (*restrict_space M A*) *X*) ∧
(∀ *X*∈*sets* (*restrict_space M A*). *X* ∩ *Y* = {} ⟶ (*restrict_space M A*) *X* ≤ (*restrict_space N A*) *X*)
proof (*rule finite_unsigned_Hahn_decomposition*)
show *finite_measure* (*restrict_space M A*) *finite_measure* (*restrict_space N A*)
by (*auto simp*: *space_restrict_space emeasure_restrict_space less_top intro*!: *finite_measureI*)
qed (*simp add*: *sets_restrict_space*)
then guess *Y* **..**
then show *?thesis*
apply (*intro bexI*[*of _ Y*] *conjI ballI conjI*)
apply (*simp_all add*: *sets_restrict_space emeasure_restrict_space*)
apply *safe*
subgoal for *X Z*
by (*erule ballE*[*of _ _ X*]) (*auto simp add*: *Int_absorb1*)
subgoal for *X Z*
by (*erule ballE*[*of _ _ X*]) (*auto simp add*: *Int_absorb1 ac_simps*)
apply *auto*
done
qed

Define a lexicographical order on *measure*, in the order space, sets and measure. The parts of the lexicographical order are point-wise ordered.

instantiation *measure* :: (*type*) *order_bot*
begin

inductive *less_eq_measure* :: *'a measure* ⇒ *'a measure* ⇒ *bool* **where**
space M ⊂ *space N* ⟹ *less_eq_measure M N*

| *space M* = *space N* ⟹ *sets M* ⊂ *sets N* ⟹ *less_eq_measure M N*
| *space M* = *space N* ⟹ *sets M* = *sets N* ⟹ *emeasure M* ≤ *emeasure N* ⟹ *less_eq_measure M N*

lemma *le_measure_iff*:
  M ≤ *N* ⟷ (*if space M* = *space N then*
    if sets M = *sets N then emeasure M* ≤ *emeasure N else sets M* ⊆ *sets N else space M* ⊆ *space N*)
  by (*auto elim*: *less_eq_measure.cases intro*: *less_eq_measure.intros*)

definition *less_measure* :: *'a measure* ⇒ *'a measure* ⇒ *bool* **where**
  less_measure M N ⟷ (*M* ≤ *N* ∧ ¬ *N* ≤ *M*)

definition *bot_measure* :: *'a measure* **where**
  bot_measure = *sigma* {} {}

lemma
  shows *space_bot*[*simp*]: *space bot* = {}
    and *sets_bot*[*simp*]: *sets bot* = {{}}
    and *emeasure_bot*[*simp*]: *emeasure bot X* = *0*
  by (*auto simp*: *bot_measure_def sigma_sets_empty_eq emeasure_sigma*)

instance
proof *standard*
  show *bot* ≤ *a* **for** *a* :: *'a measure*
   by (*simp add*: *le_measure_iff bot_measure_def sigma_sets_empty_eq emeasure_sigma le_fun_def*)
qed (*auto simp*: *le_measure_iff less_measure_def split*: *if_split_asm intro*: *measure_eqI*)

end

proposition *le_measure*: *sets M* = *sets N* ⟹ *M* ≤ *N* ⟷ (∀ *A*∈*sets M*. *emeasure M A* ≤ *emeasure N A*)
  apply −
  apply (*auto simp*: *le_measure_iff le_fun_def dest*: *sets_eq_imp_space_eq*)
  subgoal for *X*
    by (*cases X* ∈ *sets M*) (*auto simp*: *emeasure_notin_sets*)
  done

definition *sup_measure'* :: *'a measure* ⇒ *'a measure* ⇒ *'a measure* **where**
sup_measure' A B =
  measure_of (*space A*) (*sets A*)
    (λ*X*. *SUP Y*∈*sets A*. *emeasure A* (*X* ∩ *Y*) + *emeasure B* (*X* ∩ − *Y*))

lemma assumes [*simp*]: *sets B* = *sets A*
  shows *space_sup_measure'*[*simp*]: *space* (*sup_measure' A B*) = *space A*
    and *sets_sup_measure'*[*simp*]: *sets* (*sup_measure' A B*) = *sets A*
  using *sets_eq_imp_space_eq*[*OF assms*] **by** (*simp_all add*: *sup_measure'_def*)

lemma *emeasure_sup_measure'*:
assumes *sets_eq*[*simp*]: *sets B* = *sets A* **and** [*simp*, *intro*]: *X* ∈ *sets A*
shows *emeasure* (*sup_measure' A B*) *X* = (*SUP Y*∈*sets A*. *emeasure A* (*X* ∩ *Y*) + *emeasure B* (*X* ∩ − *Y*))
(**is** _ = *?S X*)
proof −
note *sets_eq_imp_space_eq*[*OF sets_eq*, *simp*]
show *?thesis*
using *sup_measure'_def*
proof (*rule emeasure_measure_of*)
let *?d* = λ*X Y*. *emeasure A* (*X* ∩ *Y*) + *emeasure B* (*X* ∩ − *Y*)
show *countably_additive* (*sets* (*sup_measure' A B*)) (λ*X*. *SUP Y* ∈ *sets A*. *emeasure A* (*X* ∩ *Y*) + *emeasure B* (*X* ∩ − *Y*))
proof (*rule countably_additiveI*, *goal_cases*)
case (*1 X*)
then have [*measurable*]: ⋀*i*. *X i* ∈ *sets A* **and** *disjoint_family X*
by *auto*
have *disjoint*: *disjoint_family* (λ*i*. *X i* ∩ *Y*) *disjoint_family* (λ*i*. *X i* − *Y*) **for** *Y*
by (*auto intro*: *disjoint_family_on_bisimulation* [*OF* ‹*disjoint_family X*›, *simplified*])
have (∑ *i*. *?S* (*X i*)) = (*SUP Y*∈*sets A*. ∑ *i*. *?d* (*X i*) *Y*)
proof (*rule ennreal_suminf_SUP_eq_directed*)
fix *J* :: *nat set* **and** *a b* **assume** *finite J* **and** [*measurable*]: *a* ∈ *sets A b* ∈ *sets A*
have ∃ *c*∈*sets A*. *c* ⊆ *X i* ∧ (∀ *a*∈*sets A*. *?d* (*X i*) *a* ≤ *?d* (*X i*) *c*) **for** *i*
proof *cases*
assume *emeasure A* (*X i*) = *top* ∨ *emeasure B* (*X i*) = *top*
then show *?thesis*
proof
assume *emeasure A* (*X i*) = *top* **then show** *?thesis*
by (*intro bexI*[*of* _ *X i*]) *auto*
next
assume *emeasure B* (*X i*) = *top* **then show** *?thesis*
by (*intro bexI*[*of* _ {}]) *auto*
qed
next
assume *finite*: ¬ (*emeasure A* (*X i*) = *top* ∨ *emeasure B* (*X i*) = *top*)
then have ∃ *Y*∈*sets A*. *Y* ⊆ *X i* ∧ (∀ *C*∈*sets A*. *C* ⊆ *Y* ⟶ *B C* ≤ *A C*) ∧ (∀ *C*∈*sets A*. *C* ⊆ *X i* ⟶ *C* ∩ *Y* = {} ⟶ *A C* ≤ *B C*)
using *unsigned_Hahn_decomposition*[*of B A X i*] **by** *simp*
then obtain *Y* **where** [*measurable*]: *Y* ∈ *sets A* **and** [*simp*]: *Y* ⊆ *X i*
and *B_le_A*: ⋀*C*. *C* ∈ *sets A* ⟹ *C* ⊆ *Y* ⟹ *B C* ≤ *A C*
and *A_le_B*: ⋀*C*. *C* ∈ *sets A* ⟹ *C* ⊆ *X i* ⟹ *C* ∩ *Y* = {} ⟹ *A C* ≤ *B C*
by *auto*

show *?thesis*
proof (*intro bexI*[*of* _ *Y*] *ballI conjI*)

```
          fix a assume [measurable]: a ∈ sets A
          have *: (X i ∩ a ∩ Y ∪ (X i ∩ a − Y)) = X i ∩ a (X i − a) ∩ Y ∪
(X i − a − Y) = X i ∩ − a
            for a Y by auto
          then have ?d (X i) a =
            (A (X i ∩ a ∩ Y) + A (X i ∩ a ∩ − Y)) + (B (X i ∩ − a ∩ Y) +
B (X i ∩ − a ∩ − Y))
            by (subst (1 2) plus_emeasure) (auto simp: Diff_eq[symmetric])
          also have ... ≤ (A (X i ∩ a ∩ Y) + B (X i ∩ a ∩ − Y)) + (A (X i
∩ − a ∩ Y) + B (X i ∩ − a ∩ − Y))
        by (intro add_mono order_refl B_le_A A_le_B) (auto simp: Diff_eq[symmetric])
          also have ... ≤ (A (X i ∩ Y ∩ a) + A (X i ∩ Y ∩ − a)) + (B (X i
∩ − Y ∩ a) + B (X i ∩ − Y ∩ − a))
            by (simp add: ac_simps)
          also have ... ≤ A (X i ∩ Y) + B (X i ∩ − Y)
            by (subst (1 2) plus_emeasure) (auto simp: Diff_eq[symmetric] *)
          finally show ?d (X i) a ≤ ?d (X i) Y .
        qed auto
      qed
      then obtain C where [measurable]: C i ∈ sets A and C i ⊆ X i
        and C: ⋀a. a ∈ sets A ⟹ ?d (X i) a ≤ ?d (X i) (C i) for i
        by metis
      have *: X i ∩ (⋃i. C i) = X i ∩ C i for i
      proof safe
        fix x j assume x ∈ X i x ∈ C j
        moreover have i = j ∨ X i ∩ X j = {}
          using ‹disjoint_family X› by (auto simp: disjoint_family_on_def)
        ultimately show x ∈ C i
          using ‹C i ⊆ X i› ‹C j ⊆ X j› by auto
      qed auto
      have **: X i ∩ − (⋃i. C i) = X i ∩ − C i for i
      proof safe
        fix x j assume x ∈ X i x ∉ C i x ∈ C j
        moreover have i = j ∨ X i ∩ X j = {}
          using ‹disjoint_family X› by (auto simp: disjoint_family_on_def)
        ultimately show False
          using ‹C i ⊆ X i› ‹C j ⊆ X j› by auto
      qed auto
     show ∃c∈sets A. ∀i∈J. ?d (X i) a ≤ ?d (X i) c ∧ ?d (X i) b ≤ ?d (X i) c
        apply (intro bexI[of _ ⋃i. C i])
        unfolding * **
        apply (intro C ballI conjI)
        apply auto
        done
    qed
    also have ... = ?S (⋃i. X i)
      apply (simp only: UN_extend_simps(4))
      apply (rule arg_cong [of _ _ Sup])
      apply (rule image_cong)
```

apply (*fact refl*)
using *disjoint*
apply (*auto simp add*: *suminf_add* [*symmetric*] *Diff_eq* [*symmetric*] *image_subset_iff suminf_emeasure simp del*: *UN_simps*)
done
finally show ($\sum i.$ *?S* (*X i*)) = *?S* ($\bigcup i.$ *X i*) **.**
qed
qed (*auto dest*: *sets.sets_into_space simp*: *positive_def intro*!: *SUP_const*)
qed

lemma *le_emeasure_sup_measure'1*:
assumes *sets B* = *sets A X* ∈ *sets A* **shows** *emeasure A X* ≤ *emeasure* (*sup_measure' A B*) *X*
by (*subst emeasure_sup_measure'*[*OF assms*]) (*auto intro*!: *SUP_upper2*[*of X*] *assms*)

lemma *le_emeasure_sup_measure'2*:
assumes *sets B* = *sets A X* ∈ *sets A* **shows** *emeasure B X* ≤ *emeasure* (*sup_measure' A B*) *X*
by (*subst emeasure_sup_measure'*[*OF assms*]) (*auto intro*!: *SUP_upper2*[*of* {}] *assms*)

lemma *emeasure_sup_measure'_le2*:
assumes [*simp*]: *sets B* = *sets C sets A* = *sets C* **and** [*measurable*]: *X* ∈ *sets C*
assumes *A*: $\bigwedge Y.$ *Y* ⊆ *X* ⟹ *Y* ∈ *sets A* ⟹ *emeasure A Y* ≤ *emeasure C Y*
assumes *B*: $\bigwedge Y.$ *Y* ⊆ *X* ⟹ *Y* ∈ *sets A* ⟹ *emeasure B Y* ≤ *emeasure C Y*
shows *emeasure* (*sup_measure' A B*) *X* ≤ *emeasure C X*
proof (*subst emeasure_sup_measure'*)
show (*SUP Y*∈*sets A*. *emeasure A* (*X* ∩ *Y*) + *emeasure B* (*X* ∩ − *Y*)) ≤ *emeasure C X*
unfolding ⟨*sets A* = *sets C*⟩
proof (*intro SUP_least*)
fix *Y* **assume** [*measurable*]: *Y* ∈ *sets C*
have [*simp*]: *X* ∩ *Y* ∪ (*X* − *Y*) = *X*
by *auto*
have *emeasure A* (*X* ∩ *Y*) + *emeasure B* (*X* ∩ − *Y*) ≤ *emeasure C* (*X* ∩ *Y*) + *emeasure C* (*X* ∩ − *Y*)
by (*intro add_mono A B*) (*auto simp*: *Diff_eq*[*symmetric*])
also have ... = *emeasure C X*
by (*subst plus_emeasure*) (*auto simp*: *Diff_eq*[*symmetric*])
finally show *emeasure A* (*X* ∩ *Y*) + *emeasure B* (*X* ∩ − *Y*) ≤ *emeasure C X* **.**
qed
qed *simp_all*

definition *sup_lexord* :: *'a* ⇒ *'a* ⇒ (*'a* ⇒ *'b*::*order*) ⇒ *'a* ⇒ *'a* ⇒ *'a* **where**
sup_lexord A B k s c =
(*if k A* = *k B then c else*
if ¬ *k A* ≤ *k B* ∧ ¬ *k B* ≤ *k A then s else*

```
  if k B ≤ k A then A else B)

lemma sup_lexord:
  (k A < k B ⟹ P B) ⟹ (k B < k A ⟹ P A) ⟹ (k A = k B ⟹ P c) ⟹
    (¬ k B ≤ k A ⟹ ¬ k A ≤ k B ⟹ P s) ⟹ P (sup_lexord A B k s c)
  by (auto simp: sup_lexord_def)

lemmas le_sup_lexord = sup_lexord[where P=λa. c ≤ a for c]

lemma sup_lexord1: k A = k B ⟹ sup_lexord A B k s c = c
  by (simp add: sup_lexord_def)

lemma sup_lexord_commute: sup_lexord A B k s c = sup_lexord B A k s c
  by (auto simp: sup_lexord_def)

lemma sigma_sets_le_sets_iff: (sigma_sets (space x) 𝒜 ⊆ sets x) = (𝒜 ⊆ sets x)
  using sets.sigma_sets_subset[of 𝒜 x] by auto

lemma sigma_le_iff: 𝒜 ⊆ Pow Ω ⟹ sigma Ω 𝒜 ≤ x ⟷ (Ω ⊆ space x ∧ (space
x = Ω ⟶ 𝒜 ⊆ sets x))
  by (cases Ω = space x)
    (simp_all add: eq_commute[of _ sets x] le_measure_iff emeasure_sigma le_fun_def
                  sigma_sets_superset_generator sigma_sets_le_sets_iff)

instantiation measure :: (type) semilattice_sup
begin

definition sup_measure :: 'a measure ⇒ 'a measure ⇒ 'a measure where
  sup_measure A B =
    sup_lexord A B space (sigma (space A ∪ space B) {})
     (sup_lexord A B sets (sigma (space A) (sets A ∪ sets B)) (sup_measure' A B))

instance
proof
  fix x y z :: 'a measure
  show x ≤ sup x y
    unfolding sup_measure_def
  proof (intro le_sup_lexord)
    assume space x = space y
    then have *: sets x ∪ sets y ⊆ Pow (space x)
      using sets.space_closed by auto
    assume ¬ sets y ⊆ sets x ¬ sets x ⊆ sets y
    then have sets x ⊂ sets x ∪ sets y
      by auto
    also have ... ≤ sigma (space x) (sets x ∪ sets y)
      by (subst sets_measure_of[OF *]) (rule sigma_sets_superset_generator)
    finally show x ≤ sigma (space x) (sets x ∪ sets y)
      by (simp add: space_measure_of[OF *] less_eq_measure.intros(2))
  next
```

assume ¬ *space y* ⊆ *space x* ¬ *space x* ⊆ *space y*
then show *x* ≤ *sigma* (*space x* ∪ *space y*) {}
by (*intro less_eq_measure.intros*) *auto*
next
assume *sets x* = *sets y* **then show** *x* ≤ *sup_measure′ x y*
by (*simp add*: *le_measure le_emeasure_sup_measure′1*)
qed (*auto intro*: *less_eq_measure.intros*)
show *y* ≤ *sup x y*
unfolding *sup_measure_def*
proof (*intro le_sup_lexord*)
assume ∗∗: *space x* = *space y*
then have ∗: *sets x* ∪ *sets y* ⊆ *Pow* (*space y*)
using *sets.space_closed* **by** *auto*
assume ¬ *sets y* ⊆ *sets x* ¬ *sets x* ⊆ *sets y*
then have *sets y* ⊂ *sets x* ∪ *sets y*
by *auto*
also have ... ≤ *sigma* (*space y*) (*sets x* ∪ *sets y*)
by (*subst sets_measure_of*[*OF* ∗]) (*rule sigma_sets_superset_generator*)
finally show *y* ≤ *sigma* (*space x*) (*sets x* ∪ *sets y*)
by (*simp add*: ∗∗ *space_measure_of*[*OF* ∗] *less_eq_measure.intros*(*2*))
next
assume ¬ *space y* ⊆ *space x* ¬ *space x* ⊆ *space y*
then show *y* ≤ *sigma* (*space x* ∪ *space y*) {}
by (*intro less_eq_measure.intros*) *auto*
next
assume *sets x* = *sets y* **then show** *y* ≤ *sup_measure′ x y*
by (*simp add*: *le_measure le_emeasure_sup_measure′2*)
qed (*auto intro*: *less_eq_measure.intros*)
show *x* ≤ *y* ⟹ *z* ≤ *y* ⟹ *sup x z* ≤ *y*
unfolding *sup_measure_def*
proof (*intro sup_lexord*[**where** *P*=λ*x*. *x* ≤ *y*])
assume *x* ≤ *y z* ≤ *y* **and** [*simp*]: *space x* = *space z sets x* = *sets z*
from ‹*x* ≤ *y*› **show** *sup_measure′ x z* ≤ *y*
proof *cases*
case *1* **then show** *?thesis*
by (*intro less_eq_measure.intros*(*1*)) *simp*
next
case *2* **then show** *?thesis*
by (*intro less_eq_measure.intros*(*2*)) *simp_all*
next
case *3* **with** ‹*z* ≤ *y*› ‹*x* ≤ *y*› **show** *?thesis*
by (*auto simp add*: *le_measure intro*!: *emeasure_sup_measure′_le2*)
qed
next
assume ∗∗: *x* ≤ *y z* ≤ *y space x* = *space z* ¬ *sets z* ⊆ *sets x* ¬ *sets x* ⊆ *sets z*
then have ∗: *sets x* ∪ *sets z* ⊆ *Pow* (*space x*)
using *sets.space_closed* **by** *auto*
show *sigma* (*space x*) (*sets x* ∪ *sets z*) ≤ *y*
unfolding *sigma_le_iff*[*OF* ∗] **using** ∗∗ **by** (*auto simp*: *le_measure_iff split*:

if_split_asm)
  next
    assume $x \le y$ $z \le y$ $\neg$ *space* $z \subseteq$ *space* x $\neg$ *space* $x \subseteq$ *space* z
    then have *space* $x \subseteq$ *space* y *space* $z \subseteq$ *space* y
      by (*auto simp*: *le_measure_iff split*: *if_split_asm*)
    then show *sigma* (*space* $x \cup$ *space* z) {} $\le y$
      by (*simp add*: *sigma_le_iff*)
  qed
qed

end

lemma *space_empty_eq_bot*: *space* a = {} $\longleftrightarrow$ a = *bot*
  using *space_empty*[*of a*] **by** (*auto intro*!: *measure_eqI*)

lemma *sets_eq_iff_bounded*: $A \le B \Longrightarrow B \le C \Longrightarrow$ *sets* A = *sets* $C \Longrightarrow$ *sets* B = *sets* A
  by (*auto dest*: *sets_eq_imp_space_eq simp add*: *le_measure_iff split*: *if_split_asm*)

lemma *sets_sup*: *sets* A = *sets* $M \Longrightarrow$ *sets* B = *sets* $M \Longrightarrow$ *sets* (*sup* A B) = *sets* M
  by (*auto simp add*: *sup_measure_def sup_lexord_def dest*: *sets_eq_imp_space_eq*)

lemma *le_measureD1*: $A \le B \Longrightarrow$ *space* $A \le$ *space* B
  by (*auto simp*: *le_measure_iff split*: *if_split_asm*)

lemma *le_measureD2*: $A \le B \Longrightarrow$ *space* A = *space* $B \Longrightarrow$ *sets* $A \le$ *sets* B
  by (*auto simp*: *le_measure_iff split*: *if_split_asm*)

lemma *le_measureD3*: $A \le B \Longrightarrow$ *sets* A = *sets* $B \Longrightarrow$ *emeasure* A $X \le$ *emeasure* B X
  by (*auto simp*: *le_measure_iff le_fun_def dest*: *sets_eq_imp_space_eq split*: *if_split_asm*)

lemma *UN_space_closed*: $\bigcup$(*sets* ‘ S) $\subseteq$ *Pow* ($\bigcup$(*space* ‘ S))
  using *sets.space_closed* **by** *auto*

definition
  Sup_lexord :: ('*a* $\Rightarrow$ '*b*::*complete_lattice*) $\Rightarrow$ ('*a set* $\Rightarrow$ '*a*) $\Rightarrow$ ('*a set* $\Rightarrow$ '*a*) $\Rightarrow$ '*a set* $\Rightarrow$ '*a*
where
  Sup_lexord k c s A =
  (*let U* = (*SUP a*$\in$*A*. *k a*)
   in if $\exists$ *a*$\in$*A*. *k a* = *U then c* {*a*$\in$*A*. *k a* = *U*} *else s A*)

lemma *Sup_lexord*:
  ($\bigwedge a$ S. $a \in A \Longrightarrow k$ a = (*SUP* $a\in A$. k a) $\Longrightarrow S$ = {$a'\in A$. k a' = k a} $\Longrightarrow P$ (c S)) $\Longrightarrow$ (($\bigwedge a$. $a \in A \Longrightarrow k$ $a \ne$ (*SUP* $a\in A$. k a)) $\Longrightarrow P$ (s A)) $\Longrightarrow$
    P (*Sup_lexord* k c s A)
  by (*auto simp*: *Sup_lexord_def Let_def*)

lemma *Sup_lexord1*:
  assumes *A*: $A \neq \{\}$ $(\bigwedge a.\ a \in A \Longrightarrow k\ a = (\bigcup a{\in}A.\ k\ a))$ $P\ (c\ A)$
  shows P (*Sup_lexord* $k\ c\ s\ A$)
  unfolding *Sup_lexord_def Let_def*
proof (*clarsimp*, *safe*)
  show $\forall a{\in}A.\ k\ a \neq (\bigcup x{\in}A.\ k\ x) \Longrightarrow P\ (s\ A)$
    by (*metis assms*(*1*,*2*) *ex_in_conv*)
next
  fix *a* **assume** $a \in A$ $k\ a = (\bigcup x{\in}A.\ k\ x)$
  then have $\{a \in A.\ k\ a = (\bigcup x{\in}A.\ k\ x)\} = \{a \in A.\ k\ a = k\ a\}$
    by (*metis A*(*2*)[*symmetric*])
  then show $P\ (c\ \{a \in A.\ k\ a = (\bigcup x{\in}A.\ k\ x)\})$
    by (*simp add*: *A*(*3*))
qed

instantiation *measure* :: (*type*) *complete_lattice*
begin

interpretation *sup_measure*: *comm_monoid_set sup bot* :: *'a measure*
  by *standard* (*auto intro*!: *antisym*)

lemma *sup_measure_F_mono'*:
  finite J ⟹ *finite I* ⟹ *sup_measure.F id I* ≤ *sup_measure.F id* (*I* ∪ *J*)
proof (*induction J rule*: *finite_induct*)
  case *empty* **then show** *?case*
    by *simp*
next
  case (*insert i J*)
  show *?case*
  proof *cases*
    assume $i \in I$ **with** *insert* **show** *?thesis*
      by (*auto simp*: *insert_absorb*)
  next
    assume $i \notin I$
    have *sup_measure.F id I* ≤ *sup_measure.F id* (*I* ∪ *J*)
      by (*intro insert*)
    also have ... ≤ *sup_measure.F id* (*insert i* (*I* ∪ *J*))
      using *insert* ‹$i \notin I$› **by** (*subst sup_measure.insert*) *auto*
    finally show *?thesis*
      by *auto*
  qed
qed

lemma *sup_measure_F_mono*: *finite I* ⟹ *J* ⊆ *I* ⟹ *sup_measure.F id J* ≤ *sup_measure.F id I*
  using *sup_measure_F_mono'*[*of I J*] **by** (*auto simp*: *finite_subset Un_absorb1*)

lemma *sets_sup_measure_F*:

finite I ⟹ *I* ≠ {} ⟹ (⋀*i*. *i* ∈ *I* ⟹ *sets i* = *sets M*) ⟹ *sets* (*sup_measure.F id I*) = *sets M*
by (*induction I rule*: *finite_ne_induct*) (*simp_all add*: *sets_sup*)

definition *Sup_measure'* :: *'a measure set* ⇒ *'a measure* **where**
Sup_measure' M =
measure_of (⋃*a*∈*M*. *space a*) (⋃*a*∈*M*. *sets a*)
(λ*X*. (*SUP P*∈{*P*. *finite P* ∧ *P* ⊆ *M* }. *sup_measure.F id P X*))

lemma *space_Sup_measure'2*: *space* (*Sup_measure' M*) = (⋃*m*∈*M*. *space m*)
unfolding *Sup_measure'_def* **by** (*intro space_measure_of*[*OF UN_space_closed*])

lemma *sets_Sup_measure'2*: *sets* (*Sup_measure' M*) = *sigma_sets* (⋃*m*∈*M*. *space m*) (⋃*m*∈*M*. *sets m*)
unfolding *Sup_measure'_def* **by** (*intro sets_measure_of*[*OF UN_space_closed*])

lemma *sets_Sup_measure'*:
assumes *sets_eq*[*simp*]: ⋀*m*. *m* ∈ *M* ⟹ *sets m* = *sets A* **and** *M* ≠ {}
shows *sets* (*Sup_measure' M*) = *sets A*
using *sets_eq*[*THEN sets_eq_imp_space_eq*, *simp*] ‹*M* ≠ {}› **by** (*simp add*: *Sup_measure'_def*)

lemma *space_Sup_measure'*:
assumes *sets_eq*[*simp*]: ⋀*m*. *m* ∈ *M* ⟹ *sets m* = *sets A* **and** *M* ≠ {}
shows *space* (*Sup_measure' M*) = *space A*
using *sets_eq*[*THEN sets_eq_imp_space_eq*, *simp*] ‹*M* ≠ {}›
by (*simp add*: *Sup_measure'_def*)

lemma *emeasure_Sup_measure'*:
assumes *sets_eq*[*simp*]: ⋀*m*. *m* ∈ *M* ⟹ *sets m* = *sets A* **and** *X* ∈ *sets A M* ≠ {}
shows *emeasure* (*Sup_measure' M*) *X* = (*SUP P*∈{*P*. *finite P* ∧ *P* ⊆ *M*}. *sup_measure.F id P X*)
(**is** _ = *?S X*)
using *Sup_measure'_def*
proof (*rule emeasure_measure_of*)
note *sets_eq*[*THEN sets_eq_imp_space_eq*, *simp*]
have ∗: *sets* (*Sup_measure' M*) = *sets A space* (*Sup_measure' M*) = *space A*
using ‹*M* ≠ {}› **by** (*simp_all add*: *Sup_measure'_def*)
let *?μ* = *sup_measure.F id*
show *countably_additive* (*sets* (*Sup_measure' M*)) *?S*
proof (*rule countably_additiveI*, *goal_cases*)
case (*1 F*)
then have ∗∗: *range F* ⊆ *sets A*
by (*auto simp*: ∗)
show (∑*i*. *?S* (*F i*)) = *?S* (⋃*i*. *F i*)
proof (*subst ennreal_suminf_SUP_eq_directed*)
fix *i j* **and** *N* :: *nat set* **assume** *ij*: *i* ∈ {*P*. *finite P* ∧ *P* ⊆ *M*} *j* ∈ {*P*. *finite P* ∧ *P* ⊆ *M*}
have (*i* ≠ {} ⟶ *sets* (*?μ i*) = *sets A*) ∧ (*j* ≠ {} ⟶ *sets* (*?μ j*) = *sets A*)

∧
    (*i* ≠ {} ∨ *j* ≠ {} ⟶ *sets* (*?μ* (*i* ∪ *j*)) = *sets A*)
    using *ij* **by** (*intro impI sets_sup_measure_F conjI*) *auto*
   then have *?μ j* (*F n*) ≤ *?μ* (*i* ∪ *j*) (*F n*) ∧ *?μ i* (*F n*) ≤ *?μ* (*i* ∪ *j*) (*F n*)
for *n*
    using *ij*
    by (*cases i* = {}; *cases j* = {})
     (*auto intro*!: *le_measureD3 sup_measure_F_mono simp*: *sets_sup_measure_F*
       simp del: *id_apply*)
   with *ij* **show** ∃ *k*∈{*P*. *finite P* ∧ *P* ⊆ *M*}. ∀ *n*∈*N*. *?μ i* (*F n*) ≤ *?μ k* (*F n*)
∧ *?μ j* (*F n*) ≤ *?μ k* (*F n*)
    by (*safe intro*!: *bexI*[*of* _ *i* ∪ *j*]) *auto*
  next
   show (*SUP P* ∈ {*P*. *finite P* ∧ *P* ⊆ *M*}. ∑ *n*. *?μ P* (*F n*)) = (*SUP P* ∈
{*P*. *finite P* ∧ *P* ⊆ *M*}. *?μ P* (⋃(*F* ‘ *UNIV*)))
   proof (*intro arg_cong* [*of* _ _ *Sup*] *image_cong refl*)
    fix *i* **assume** *i*: *i* ∈ {*P*. *finite P* ∧ *P* ⊆ *M*}
    show (∑ *n*. *?μ i* (*F n*)) = *?μ i* (⋃(*F* ‘ *UNIV*))
    proof *cases*
     assume *i* ≠ {} **with** *i* ∗∗ **show** *?thesis*
      apply (*intro suminf_emeasure* ‹*disjoint_family F*›)
      apply (*subst sets_sup_measure_F*[*OF* _ _ *sets_eq*])
      apply *auto*
      done
    qed *simp*
   qed
  qed
 qed
 show *positive* (*sets* (*Sup_measure′ M*)) *?S*
  by (*auto simp*: *positive_def bot_ennreal*[*symmetric*])
 show *X* ∈ *sets* (*Sup_measure′ M*)
  using *assms* ∗ **by** *auto*
qed (*rule UN_space_closed*)

definition *Sup_measure* :: *′a measure set* ⇒ *′a measure* **where**
Sup_measure =
 Sup_lexord space
  (*Sup_lexord sets Sup_measure′*
   (λ*U*. *sigma* (⋃ *u*∈*U*. *space u*) (⋃ *u*∈*U*. *sets u*)))
  (λ*U*. *sigma* (⋃ *u*∈*U*. *space u*) {})

definition *Inf_measure* :: *′a measure set* ⇒ *′a measure* **where**
 Inf_measure A = *Sup* {*x*. ∀ *a*∈*A*. *x* ≤ *a*}

definition *inf_measure* :: *′a measure* ⇒ *′a measure* ⇒ *′a measure* **where**
 inf_measure a b = *Inf* {*a*, *b*}

definition *top_measure* :: *′a measure* **where**
 top_measure = *Inf* {}

instance
proof
  note *UN_space_closed* [*simp*]
  show *upper*: $x \leq Sup\ A$ **if** x: $x \in A$ **for** x :: *'a measure* **and** A
    unfolding *Sup_measure_def*
  proof (*intro Sup_lexord*[**where** $P=\lambda y.\ x \leq y$])
    assume $\bigwedge a.\ a \in A \Longrightarrow space\ a \neq (\bigcup a\in A.\ space\ a)$
    from *this*[*OF* ‹$x \in A$›] ‹$x \in A$› **show** $x \leq sigma\ (\bigcup a\in A.\ space\ a)\ \{\}$
      by (*intro less_eq_measure.intros*) *auto*
  next
    fix a S **assume** $a \in A$ **and** a: $space\ a = (\bigcup a\in A.\ space\ a)$ **and** S: $S = \{a' \in A.\ space\ a' = space\ a\}$
      and *neq*: $\bigwedge aa.\ aa \in S \Longrightarrow sets\ aa \neq (\bigcup a\in S.\ sets\ a)$
    have *sp_a*: $space\ a = (\bigcup(space\ `\ S))$
      using ‹$a\in A$› **by** (*auto simp*: S)
    show $x \leq sigma\ (\bigcup(space\ `\ S))\ (\bigcup(sets\ `\ S))$
    proof *cases*
      assume [*simp*]: $space\ x = space\ a$
      have $sets\ x \subset (\bigcup a\in S.\ sets\ a)$
        using ‹$x\in A$› *neq*[*of* x] **by** (*auto simp*: S)
      also have ... $\subseteq sigma\_sets\ (\bigcup x\in S.\ space\ x)\ (\bigcup x\in S.\ sets\ x)$
        by (*rule sigma_sets_superset_generator*)
      finally show *?thesis*
        by (*intro less_eq_measure.intros*(*2*)) (*simp_all add*: *sp_a*)
    next
      assume $space\ x \neq space\ a$
      moreover have $space\ x \leq space\ a$
        unfolding a **using** ‹$x\in A$› **by** *auto*
      ultimately show *?thesis*
        by (*intro less_eq_measure.intros*) (*simp add*: *less_le sp_a*)
    qed
  next
    fix a b S S' **assume** $a \in A$ **and** a: $space\ a = (\bigcup a\in A.\ space\ a)$ **and** S: $S = \{a' \in A.\ space\ a' = space\ a\}$
      and $b \in S$ **and** b: $sets\ b = (\bigcup a\in S.\ sets\ a)$ **and** S': $S' = \{a' \in S.\ sets\ a' = sets\ b\}$
    then have $S' \neq \{\}$ $space\ b = space\ a$
      by *auto*
    have *sets_eq*: $\bigwedge x.\ x \in S' \Longrightarrow sets\ x = sets\ b$
      by (*auto simp*: S')
    note *sets_eq*[*THEN sets_eq_imp_space_eq*, *simp*]
    have $*$: $sets\ (Sup\_measure'\ S') = sets\ b$ $space\ (Sup\_measure'\ S') = space\ b$
      using ‹$S' \neq \{\}$› **by** (*simp_all add*: *Sup_measure'_def sets_eq*)
    show $x \leq Sup\_measure'\ S'$
    proof *cases*
      assume $x \in S$
      with ‹$b \in S$› **have** $space\ x = space\ b$
        by (*simp add*: S)

show *?thesis*
proof *cases*
assume $x \in S'$
show $x \leq$ *Sup_measure'* S'
proof (*intro le_measure*[*THEN iffD2*] *ballI*)
show *sets* $x =$ *sets* (*Sup_measure'* S')
using ‹$x \in S'$› ∗ **by** (*simp add*: S')
fix X **assume** $X \in$ *sets* x
show *emeasure* x $X \leq$ *emeasure* (*Sup_measure'* S') X
proof (*subst emeasure_Sup_measure'*[*OF* _ ‹$X \in$ *sets* x›])
show *emeasure* x $X \leq$ (*SUP* $P \in \{P.$ *finite* $P \wedge P \subseteq S'\}$. *emeasure* (*sup_measure.F id* P) X)
using ‹$x \in S'$› **by** (*intro SUP_upper2*[**where** $i=\{x\}$]) *auto*
qed (*insert* ‹$x \in S'$› S', *auto*)
qed
next
assume $x \notin S'$
then have *sets* $x \neq$ *sets* b
using ‹$x \in S$› **by** (*auto simp*: S')
moreover have *sets* $x \leq$ *sets* b
using ‹$x \in S$› **unfolding** b **by** *auto*
ultimately show *?thesis*
using ∗ ‹$x \in S$›
by (*intro less_eq_measure.intros*(*2*))
(*simp_all add*: ∗ ‹*space* $x =$ *space* b› *less_le*)
qed
next
assume $x \notin S$
with ‹$x \in A$› ‹$x \notin S$› ‹*space* $b =$ *space* a› **show** *?thesis*
by (*intro less_eq_measure.intros*)
(*simp_all add*: ∗ *less_le* a *SUP_upper* S)
qed
qed
show *least*: *Sup* $A \leq x$ **if** x: $\bigwedge z.\ z \in A \Longrightarrow z \leq x$ **for** x :: *'a measure* **and** A
unfolding *Sup_measure_def*
proof (*intro Sup_lexord*[**where** $P=\lambda y.\ y \leq x$])
assume $\bigwedge a.\ a \in A \Longrightarrow$ *space* $a \neq (\bigcup a \in A.$ *space* $a)$
show *sigma* ($\bigcup$(*space* ‘ A)) {} $\leq x$
using x[*THEN le_measureD1*] **by** (*subst sigma_le_iff*) *auto*
next
fix a S **assume** $a \in A$ *space* $a = (\bigcup a \in A.$ *space* $a)$ **and** S: $S = \{a' \in A.$ *space* $a' =$ *space* $a\}$
$\bigwedge a.\ a \in S \Longrightarrow$ *sets* $a \neq (\bigcup a \in S.$ *sets* $a)$
have $\bigcup$(*space* ‘ S) $\subseteq$ *space* x
using S *le_measureD1*[*OF* x] **by** *auto*
moreover
have $\bigcup$(*space* ‘ S) $=$ *space* a
using ‹$a \in A$› S **by** *auto*
then have *space* $x = \bigcup$(*space* ‘ S) $\Longrightarrow \bigcup$(*sets* ‘ S) $\subseteq$ *sets* x

```
      using ‹a ∈ A› le_measureD2[OF x] by (auto simp: S)
    ultimately show sigma (⋃(space ‘ S)) (⋃(sets ‘ S)) ≤ x
      by (subst sigma_le_iff) simp_all
  next
    fix a b S S' assume a ∈ A and a: space a = (⋃a∈A. space a) and S: S =
{a' ∈ A. space a' = space a}
      and b ∈ S and b: sets b = (⋃a∈S. sets a) and S': S' = {a' ∈ S. sets a' =
sets b}
    then have S' ≠ {} space b = space a
      by auto
    have sets_eq: ⋀x. x ∈ S' ⟹ sets x = sets b
      by (auto simp: S')
    note sets_eq[THEN sets_eq_imp_space_eq, simp]
    have *: sets (Sup_measure' S') = sets b space (Sup_measure' S') = space b
      using ‹S' ≠ {}› by (simp_all add: Sup_measure'_def sets_eq)
    show Sup_measure' S' ≤ x
    proof cases
      assume space x = space a
      show ?thesis
      proof cases
        assume **: sets x = sets b
        show ?thesis
        proof (intro le_measure[THEN iffD2] ballI)
          show ***: sets (Sup_measure' S') = sets x
            by (simp add: * **)
          fix X assume X ∈ sets (Sup_measure' S')
          show emeasure (Sup_measure' S') X ≤ emeasure x X
            unfolding ***
          proof (subst emeasure_Sup_measure'[OF _ ‹X ∈ sets (Sup_measure' S')›])
             show (SUP P ∈ {P. finite P ∧ P ⊆ S'}. emeasure (sup_measure.F id
P) X) ≤ emeasure x X
            proof (safe intro!: SUP_least)
              fix P assume P: finite P P ⊆ S'
              show emeasure (sup_measure.F id P) X ≤ emeasure x X
              proof cases
                assume P = {} then show ?thesis
                  by auto
              next
                assume P ≠ {}
                from P have finite P P ⊆ A
                  unfolding S' S by (simp_all add: subset_eq)
                then have sup_measure.F id P ≤ x
                  by (induction P) (auto simp: x)
                moreover have sets (sup_measure.F id P) = sets x
                  using ‹finite P› ‹P ≠ {}› ‹P ⊆ S'› ‹sets x = sets b›
                  by (intro sets_sup_measure_F) (auto simp: S')
               ultimately show emeasure (sup_measure.F id P) X ≤ emeasure x X
                  by (rule le_measureD3)
              qed
```

```
          qed
          show m ∈ S' ⟹ sets m = sets (Sup_measure' S') for m
            unfolding * by (simp add: S')
        qed fact
      qed
    next
      assume sets x ≠ sets b
      moreover have sets b ≤ sets x
        unfolding b S using x[THEN le_measureD2] ‹space x = space a› by auto
      ultimately show Sup_measure' S' ≤ x
        using ‹space x = space a› ‹b ∈ S›
        by (intro less_eq_measure.intros(2)) (simp_all add: * S)
    qed
  next
    assume space x ≠ space a
    then have space a < space x
      using le_measureD1[OF x[OF ‹a∈A›]] by auto
    then show Sup_measure' S' ≤ x
      by (intro less_eq_measure.intros) (simp add: * ‹space b = space a›)
  qed
qed
show Sup {} = (bot::'a measure) Inf {} = (top::'a measure)
  by (auto intro!: antisym least simp: top_measure_def)
show lower: x ∈ A ⟹ Inf A ≤ x for x :: 'a measure and A
  unfolding Inf_measure_def by (intro least) auto
show greatest: (⋀z. z ∈ A ⟹ x ≤ z) ⟹ x ≤ Inf A for x :: 'a measure and A
  unfolding Inf_measure_def by (intro upper) auto
show inf x y ≤ x inf x y ≤ y x ≤ y ⟹ x ≤ z ⟹ x ≤ inf y z for x y z :: 'a
measure
  by (auto simp: inf_measure_def intro!: lower greatest)
qed

end

lemma sets_SUP:
  assumes ⋀x. x ∈ I ⟹ sets (M x) = sets N
  shows I ≠ {} ⟹ sets (SUP i∈I. M i) = sets N
  unfolding Sup_measure_def
  using assms assms[THEN sets_eq_imp_space_eq]
    sets_Sup_measure'[where A=N and M=M`I]
  by (intro Sup_lexord1[where P=λx. sets x = sets N]) auto

lemma emeasure_SUP:
  assumes sets: ⋀i. i ∈ I ⟹ sets (M i) = sets N X ∈ sets N I ≠ {}
  shows emeasure (SUP i∈I. M i) X = (SUP J∈{J. J ≠ {} ∧ finite J ∧ J ⊆ I}.
emeasure (SUP i∈J. M i) X)
proof -
  interpret sup_measure: comm_monoid_set sup bot :: 'b measure
    by standard (auto intro!: antisym)
```

have *eq*: *finite J* ⟹ *sup_measure.F id J* = (*SUP i*∈*J*. *i*) **for** *J* :: *'b measure set*
  by (*induction J rule*: *finite_induct*) *auto*
have *1*: *J* ≠ {} ⟹ *J* ⊆ *I* ⟹ *sets* (*SUP x*∈*J*. *M x*) = *sets N* **for** *J*
  by (*intro sets_SUP sets*) (*auto*)
from ‹*I* ≠ {}› **obtain** *i* **where** *i*∈*I* **by** *auto*
have *Sup_measure'* (*M'I*) *X* = (*SUP P*∈{*P*. *finite P* ∧ *P* ⊆ *M'I*}. *sup_measure.F id P X*)
  using *sets* **by** (*intro emeasure_Sup_measure'*) *auto*
also have *Sup_measure'* (*M'I*) = (*SUP i*∈*I*. *M i*)
  unfolding *Sup_measure_def* **using** ‹*I* ≠ {}› *sets sets*(*1*)[*THEN sets_eq_imp_space_eq*]
  by (*intro Sup_lexord1*[**where** *P*=λ*x*. _ = *x*]) *auto*
also have (*SUP P*∈{*P*. *finite P* ∧ *P* ⊆ *M'I*}. *sup_measure.F id P X*) =
  (*SUP J*∈{*J*. *J* ≠ {} ∧ *finite J* ∧ *J* ⊆ *I*}. (*SUP i*∈*J*. *M i*) *X*)
proof (*intro SUP_eq*)
  fix *J* **assume** *J* ∈ {*P*. *finite P* ∧ *P* ⊆ *M'I*}
  then obtain *J'* **where** *J'*: *J'* ⊆ *I finite J'* **and** *J*: *J* = *M'J'* **and** *finite J*
    using *finite_subset_image*[*of J M I*] **by** *auto*
  show ∃*j*∈{*J*. *J* ≠ {} ∧ *finite J* ∧ *J* ⊆ *I*}. *sup_measure.F id J X* ≤ (*SUP i*∈*j*. *M i*) *X*
  proof *cases*
    assume *J'* = {} **with** ‹*i* ∈ *I*› **show** *?thesis*
      by (*auto simp add*: *J*)
  next
    assume *J'* ≠ {} **with** *J J'* **show** *?thesis*
      by (*intro bexI*[*of* _ *J'*]) (*auto simp add*: *eq simp del*: *id_apply*)
  qed
next
  fix *J* **assume** *J*: *J* ∈ {*P*. *P* ≠ {} ∧ *finite P* ∧ *P* ⊆ *I*}
  show ∃*J'*∈{*J*. *finite J* ∧ *J* ⊆ *M'I*}. (*SUP i*∈*J*. *M i*) *X* ≤ *sup_measure.F id J' X*
    using *J* **by** (*intro bexI*[*of* _ *M'J*]) (*auto simp add*: *eq simp del*: *id_apply*)
qed
finally show *?thesis* .
qed

lemma *emeasure_SUP_chain*:
  assumes *sets*: ⋀*i*. *i* ∈ *A* ⟹ *sets* (*M i*) = *sets N X* ∈ *sets N*
  assumes *ch*: *Complete_Partial_Order.chain* (≤) (*M* ' *A*) **and** *A* ≠ {}
  shows *emeasure* (*SUP i*∈*A*. *M i*) *X* = (*SUP i*∈*A*. *emeasure* (*M i*) *X*)
proof (*subst emeasure_SUP*[*OF sets* ‹*A* ≠ {}›])
  show (*SUP J*∈{*J*. *J* ≠ {} ∧ *finite J* ∧ *J* ⊆ *A*}. *emeasure* (*Sup* (*M* ' *J*)) *X*) = (*SUP i*∈*A*. *emeasure* (*M i*) *X*)
  proof (*rule SUP_eq*)
    fix *J* **assume** *J* ∈ {*J*. *J* ≠ {} ∧ *finite J* ∧ *J* ⊆ *A*}
    then have *J*: *Complete_Partial_Order.chain* (≤) (*M* ' *J*) *finite J J* ≠ {} **and** *J* ⊆ *A*
      using *ch*[*THEN chain_subset*, *of M'J*] **by** *auto*
    with *in_chain_finite*[*OF J*(*1*)] **obtain** *j* **where** *j* ∈ *J* (*SUP j*∈*J*. *M j*) = *M j*
      by *auto*

with ‹$J \subseteq A$› **show** $\exists j{\in}A.$ *emeasure* (*Sup* (*M* ‘ *J*)) *X* $\leq$ *emeasure* (*M j*) *X*
by *auto*
next
fix *j* **assume** $j{\in}A$ **then show** $\exists i{\in}\{J.\ J \neq \{\} \wedge$ *finite* $J \wedge J \subseteq A\}$. *emeasure* (*M j*) *X* $\leq$ *emeasure* (*Sup* (*M* ‘ *i*)) *X*
by (*intro bexI*[*of* _ {*j*}]) *auto*
qed
qed

Supremum of a set of σ-algebras

lemma *space_Sup_eq_UN*: *space* (*Sup M*) = ($\bigcup x{\in}M.$ *space x*)
unfolding *Sup_measure_def*
apply (*intro Sup_lexord*[**where** *P*=λ*x*. *space x* = _])
apply (*subst space_Sup_measure'2*)
apply *auto* []
apply (*subst space_measure_of*[*OF UN_space_closed*])
apply *auto*
done

lemma *sets_Sup_eq*:
assumes ∗: $\bigwedge m.\ m \in M \Longrightarrow$ *space m* = *X* **and** $M \neq \{\}$
shows *sets* (*Sup M*) = *sigma_sets X* ($\bigcup x{\in}M.$ *sets x*)
unfolding *Sup_measure_def*
apply (*rule Sup_lexord1*)
apply *fact*
apply (*simp add*: *assms*)
apply (*rule Sup_lexord*)
subgoal premises *that* **for** *a S*
unfolding *that*(*3*) *that*(*2*)[*symmetric*]
using *that*(*1*)
apply (*subst sets_Sup_measure'2*)
apply (*intro arg_cong2*[**where** *f*=*sigma_sets*])
apply (*auto simp*: ∗)
done
apply (*subst sets_measure_of*[*OF UN_space_closed*])
apply (*simp add*: *assms*)
done

lemma *in_sets_Sup*: ($\bigwedge m.\ m \in M \Longrightarrow$ *space m* = *X*) $\Longrightarrow m \in M \Longrightarrow A \in$ *sets m* $\Longrightarrow A \in$ *sets* (*Sup M*)
by (*subst sets_Sup_eq*[**where** *X*=*X*]) *auto*

lemma *Sup_lexord_rel*:
assumes $\bigwedge i.\ i \in I \Longrightarrow k\ (A\ i) = k\ (B\ i)$
R (*c* (*A* ‘ {*a* ∈ *I*. *k* (*B a*) = (*SUP x*∈*I*. *k* (*B x*))})) (*c* (*B* ‘ {*a* ∈ *I*. *k* (*B a*) = (*SUP x*∈*I*. *k* (*B x*))}))
R (*s* (*A*‘*I*)) (*s* (*B*‘*I*))
shows *R* (*Sup_lexord k c s* (*A*‘*I*)) (*Sup_lexord k c s* (*B*‘*I*))

proof –
have *A* ‘ {*a* ∈ *I*. *k* (*B a*) = (*SUP x*∈*I*. *k* (*B x*))} = {*a* ∈ *A* ‘ *I*. *k a* = (*SUP x*∈*I*. *k* (*B x*))}
using *assms*(*1*) **by** *auto*
moreover have *B* ‘ {*a* ∈ *I*. *k* (*B a*) = (*SUP x*∈*I*. *k* (*B x*))} = {*a* ∈ *B* ‘ *I*. *k a* = (*SUP x*∈*I*. *k* (*B x*))}
by *auto*
ultimately show *?thesis*
using *assms* **by** (*auto simp*: *Sup_lexord_def Let_def image_comp*)
qed

lemma *sets_SUP_cong*:
assumes *eq*: ⋀*i*. *i* ∈ *I* ⟹ *sets* (*M i*) = *sets* (*N i*) **shows** *sets* (*SUP i*∈*I*. *M i*) = *sets* (*SUP i*∈*I*. *N i*)
unfolding *Sup_measure_def*
using *eq eq*[*THEN sets_eq_imp_space_eq*]
apply (*intro Sup_lexord_rel*[**where** *R*=λ*x y*. *sets x* = *sets y*])
apply *simp*
apply *simp*
apply (*simp add*: *sets_Sup_measure'2*)
apply (*intro arg_cong2*[**where** *f*=λ*x y*. *sets* (*sigma x y*)])
apply *auto*
done

lemma *sets_Sup_in_sets*:
assumes *M* ≠ {}
assumes ⋀*m*. *m* ∈ *M* ⟹ *space m* = *space N*
assumes ⋀*m*. *m* ∈ *M* ⟹ *sets m* ⊆ *sets N*
shows *sets* (*Sup M*) ⊆ *sets N*
proof –
have ∗: ⋃(*space* ‘ *M*) = *space N*
using *assms* **by** *auto*
show *?thesis*
unfolding ∗ **using** *assms* **by** (*subst sets_Sup_eq*[*of M space N*]) (*auto intro*!: *sets.sigma_sets_subset*)
qed

lemma *measurable_Sup1*:
assumes *m*: *m* ∈ *M* **and** *f*: *f* ∈ *measurable m N*
and *const_space*: ⋀*m n*. *m* ∈ *M* ⟹ *n* ∈ *M* ⟹ *space m* = *space n*
shows *f* ∈ *measurable* (*Sup M*) *N*
proof –
have *space* (*Sup M*) = *space m*
using *m* **by** (*auto simp add*: *space_Sup_eq_UN dest*: *const_space*)
then show *?thesis*
using *m f* **unfolding** *measurable_def* **by** (*auto intro*: *in_sets_Sup*[*OF const_space*])
qed

lemma *measurable_Sup2*:

```
  assumes M: M ≠ {}
  assumes f: ⋀m. m ∈ M ⟹ f ∈ measurable N m
    and const_space: ⋀m n. m ∈ M ⟹ n ∈ M ⟹ space m = space n
  shows f ∈ measurable N (Sup M)
proof –
  from M obtain m where m ∈ M by auto
  have space_eq: ⋀n. n ∈ M ⟹ space n = space m
    by (intro const_space ‹m ∈ M›)
  have f ∈ measurable N (sigma (⋃m∈M. space m) (⋃m∈M. sets m))
  proof (rule measurable_measure_of)
    show f ∈ space N → ⋃(space ‘ M)
      using measurable_space[OF f] M by auto
  qed (auto intro: measurable_sets f dest: sets.sets_into_space)
  also have measurable N (sigma (⋃m∈M. space m) (⋃m∈M. sets m)) = measurable N (Sup M)
    apply (intro measurable_cong_sets refl)
    apply (subst sets_Sup_eq[OF space_eq M])
    apply simp
    apply (subst sets_measure_of[OF UN_space_closed])
    apply (simp add: space_eq M)
    done
  finally show ?thesis .
qed

lemma measurable_SUP2:
  I ≠ {} ⟹ (⋀i. i ∈ I ⟹ f ∈ measurable N (M i)) ⟹
    (⋀i j. i ∈ I ⟹ j ∈ I ⟹ space (M i) = space (M j)) ⟹ f ∈ measurable N (SUP i∈I. M i)
  by (auto intro!: measurable_Sup2)

lemma sets_Sup_sigma:
  assumes [simp]: M ≠ {} and M: ⋀m. m ∈ M ⟹ m ⊆ Pow Ω
  shows sets (SUP m∈M. sigma Ω m) = sets (sigma Ω (⋃M))
proof –
  { fix a m assume a ∈ sigma_sets Ω m m ∈ M
    then have a ∈ sigma_sets Ω (⋃M)
     by induction (auto intro: sigma_sets.intros(2–)) }
  then show sets (SUP m∈M. sigma Ω m) = sets (sigma Ω (⋃M))
    apply (subst sets_Sup_eq[where X=Ω])
    apply (auto simp add: M) []
    apply auto []
    apply (simp add: space_measure_of_conv M Union_least)
    apply (rule sigma_sets_eqI)
    apply auto
    done
qed

lemma Sup_sigma:
  assumes [simp]: M ≠ {} and M: ⋀m. m ∈ M ⟹ m ⊆ Pow Ω
```

```
  shows (SUP m∈M. sigma Ω m) = (sigma Ω (⋃M))
proof (intro antisym SUP_least)
  have *: ⋃M ⊆ Pow Ω
    using M by auto
  show sigma Ω (⋃M) ≤ (SUP m∈M. sigma Ω m)
  proof (intro less_eq_measure.intros(3))
    show space (sigma Ω (⋃M)) = space (SUP m∈M. sigma Ω m)
      sets (sigma Ω (⋃M)) = sets (SUP m∈M. sigma Ω m)
     using sets_Sup_sigma[OF assms] sets_Sup_sigma[OF assms, THEN sets_eq_imp_space_eq]
      by auto
  qed (simp add: emeasure_sigma le_fun_def)
  fix m assume m ∈ M then show sigma Ω m ≤ sigma Ω (⋃M)
    by (subst sigma_le_iff) (auto simp add: M *)
qed

lemma SUP_sigma_sigma:
  M ≠ {} ⟹ (⋀m. m ∈ M ⟹ f m ⊆ Pow Ω) ⟹ (SUP m∈M. sigma Ω (f m))
= sigma Ω (⋃m∈M. f m)
  using Sup_sigma[of f‘M Ω] by (auto simp: image_comp)

lemma sets_vimage_Sup_eq:
  assumes *: M ≠ {} f ∈ X → Y ⋀m. m ∈ M ⟹ space m = Y
  shows sets (vimage_algebra X f (Sup M)) = sets (SUP m ∈ M. vimage_algebra
X f m)
  (is ?IS = ?SI)
proof
  show ?IS ⊆ ?SI
    apply (intro sets_image_in_sets measurable_Sup2)
    apply (simp add: space_Sup_eq_UN *)
    apply (simp add: *)
    apply (intro measurable_Sup1)
    apply (rule imageI)
    apply assumption
    apply (rule measurable_vimage_algebra1)
    apply (auto simp: *)
    done
  show ?SI ⊆ ?IS
    apply (intro sets_Sup_in_sets)
    apply (auto simp: *) []
    apply (auto simp: *) []
    apply (elim imageE)
    apply simp
    apply (rule sets_image_in_sets)
    apply simp
    apply (simp add: measurable_def)
    apply (simp add: * space_Sup_eq_UN sets_vimage_algebra2)
    apply (auto intro: in_sets_Sup[OF *(3)])
    done
qed
```

lemma *restrict_space_eq_vimage_algebra'*:
sets (*restrict_space M* Ω) = *sets* (*vimage_algebra* (Ω ∩ *space M*) (λ*x*. *x*) *M*)
proof −
have ∗: {*A* ∩ (Ω ∩ *space M*) |*A*. *A* ∈ *sets M*} = {*A* ∩ Ω |*A*. *A* ∈ *sets M*}
using *sets.sets_into_space*[*of* _ *M*] **by** *blast*

show *?thesis*
unfolding *restrict_space_def*
by (*subst sets_measure_of*)
(*auto simp add*: *image_subset_iff sets_vimage_algebra* ∗ *dest*: *sets.sets_into_space intro*!: *arg_cong2*[**where** *f*=*sigma_sets*])
qed

lemma *sigma_le_sets*:
assumes [*simp*]: *A* ⊆ *Pow X* **shows** *sets* (*sigma X A*) ⊆ *sets N* ⟷ *X* ∈ *sets N* ∧ *A* ⊆ *sets N*
proof
have *X* ∈ *sigma_sets X A A* ⊆ *sigma_sets X A*
by (*auto intro*: *sigma_sets_top*)
moreover assume *sets* (*sigma X A*) ⊆ *sets N*
ultimately show *X* ∈ *sets N* ∧ *A* ⊆ *sets N*
by *auto*
next
assume ∗: *X* ∈ *sets N* ∧ *A* ⊆ *sets N*
{ fix *Y* **assume** *Y* ∈ *sigma_sets X A* **from** *this* ∗ **have** *Y* ∈ *sets N*
by *induction auto* **}**
then show *sets* (*sigma X A*) ⊆ *sets N*
by *auto*
qed

lemma *measurable_iff_sets*:
f ∈ *measurable M N* ⟷ (*f* ∈ *space M* → *space N* ∧ *sets* (*vimage_algebra* (*space M*) *f N*) ⊆ *sets M*)
proof −
have ∗: {*f* −‘ *A* ∩ *space M* |*A*. *A* ∈ *sets N*} ⊆ *Pow* (*space M*)
by *auto*
show *?thesis*
unfolding *measurable_def*
by (*auto simp add*: *vimage_algebra_def sigma_le_sets*[*OF* ∗])
qed

lemma *sets_vimage_algebra_space*: *X* ∈ *sets* (*vimage_algebra X f M*)
using *sets.top*[*of vimage_algebra X f M*] **by** *simp*

lemma *measurable_mono*:
assumes *N*: *sets N'* ≤ *sets N space N* = *space N'*
assumes *M*: *sets M* ≤ *sets M' space M* = *space M'*
shows *measurable M N* ⊆ *measurable M' N'*

unfolding *measurable_def*
proof *safe*
fix *f A* **assume** *f* ∈ *space M* → *space N A* ∈ *sets N′*
moreover assume ∀ *y*∈*sets N*. *f* −‘ *y* ∩ *space M* ∈ *sets M* **note** *this*[*THEN bspec*, *of A*]
ultimately show *f* −‘ *A* ∩ *space M′* ∈ *sets M′*
using *assms* **by** *auto*
qed (*insert N M*, *auto*)

lemma *measurable_Sup_measurable*:
assumes *f*: *f* ∈ *space N* → *A*
shows *f* ∈ *measurable N* (*Sup* {*M*. *space M* = *A* ∧ *f* ∈ *measurable N M*})
proof (*rule measurable_Sup2*)
show {*M*. *space M* = *A* ∧ *f* ∈ *measurable N M*} ≠ {}
using *f* **unfolding** *ex_in_conv*[*symmetric*]
by (*intro exI*[*of* _ *sigma A* {}]) (*auto intro*!: *measurable_measure_of*)
qed *auto*

lemma (**in** *sigma_algebra*) *sigma_sets_subset′*:
assumes *a*: *a* ⊆ *M* Ω′ ∈ *M*
shows *sigma_sets* Ω′ *a* ⊆ *M*
proof
show *x* ∈ *M* **if** *x*: *x* ∈ *sigma_sets* Ω′ *a* **for** *x*
using *x* **by** (*induct rule*: *sigma_sets.induct*) (*insert a*, *auto*)
qed

lemma *in_sets_SUP*: *i* ∈ *I* ⟹ (⋀*i*. *i* ∈ *I* ⟹ *space* (*M i*) = *Y*) ⟹ *X* ∈ *sets* (*M i*) ⟹ *X* ∈ *sets* (*SUP i*∈*I*. *M i*)
by (*intro in_sets_Sup*[**where** *X*=*Y*]) *auto*

lemma *measurable_SUP1*:
i ∈ *I* ⟹ *f* ∈ *measurable* (*M i*) *N* ⟹ (⋀*m n*. *m* ∈ *I* ⟹ *n* ∈ *I* ⟹ *space* (*M m*) = *space* (*M n*)) ⟹
f ∈ *measurable* (*SUP i*∈*I*. *M i*) *N*
by (*auto intro*: *measurable_Sup1*)

lemma *sets_image_in_sets′*:
assumes *X*: *X* ∈ *sets N*
assumes *f*: ⋀*A*. *A* ∈ *sets M* ⟹ *f* −‘ *A* ∩ *X* ∈ *sets N*
shows *sets* (*vimage_algebra X f M*) ⊆ *sets N*
unfolding *sets_vimage_algebra*
by (*rule sets.sigma_sets_subset′*) (*auto intro*!: *measurable_sets X f*)

lemma *mono_vimage_algebra*:
sets M ≤ *sets N* ⟹ *sets* (*vimage_algebra X f M*) ⊆ *sets* (*vimage_algebra X f N*)
using *sets.top*[*of sigma X* {*f* −‘ *A* ∩ *X* |*A*. *A* ∈ *sets N*}]
unfolding *vimage_algebra_def*
apply (*subst* (*asm*) *space_measure_of*)
apply *auto* []

apply (*subst sigma_le_sets*)
apply *auto*
done

lemma *mono_restrict_space*: *sets M* ≤ *sets N* ⟹ *sets* (*restrict_space M X*) ⊆ *sets* (*restrict_space N X*)
unfolding *sets_restrict_space* **by** (*rule image_mono*)

lemma *sets_eq_bot*: *sets M* = {{}} ⟷ *M* = *bot*
apply *safe*
apply (*intro measure_eqI*)
apply *auto*
done

lemma *sets_eq_bot2*: {{}} = *sets M* ⟷ *M* = *bot*
using *sets_eq_bot*[*of M*] **by** *blast*

lemma (**in** *finite_measure*) *countable_support*:
countable {*x*. *measure M* {*x*} ≠ *0*}
proof *cases*
assume *measure M* (*space M*) = *0*
with *bounded_measure measure_le_0_iff* **have** {*x*. *measure M* {*x*} ≠ *0*} = {}
by *auto*
then show *?thesis*
by *simp*
next
let *?M* = *measure M* (*space M*) **and** *?m* = λ*x*. *measure M* {*x*}
assume *?M* ≠ *0*
then have ∗: {*x*. *?m x* ≠ *0*} = (⋃*n*. {*x*. *?M* / *Suc n* < *?m x*})
using *reals_Archimedean*[*of ?m x* / *?M* **for** *x*]
by (*auto simp*: *field_simps not_le*[*symmetric*] *divide_le_0_iff measure_le_0_iff*)
have ∗∗: ⋀*n*. *finite* {*x*. *?M* / *Suc n* < *?m x*}
proof (*rule ccontr*)
fix *n* **assume** *infinite* {*x*. *?M* / *Suc n* < *?m x*} (**is** *infinite ?X*)
then obtain *X* **where** *finite X card X* = *Suc* (*Suc n*) *X* ⊆ *?X*
by (*metis infinite_arbitrarily_large*)
from *this*(*3*) **have** ∗: ⋀*x*. *x* ∈ *X* ⟹ *?M* / *Suc n* ≤ *?m x*
by *auto*
{ fix *x* **assume** *x* ∈ *X*
from ‹*?M* ≠ *0*› ∗[*OF this*] **have** *?m x* ≠ *0* **by** (*auto simp*: *field_simps measure_le_0_iff*)
then have {*x*} ∈ *sets M* **by** (*auto dest*: *measure_notin_sets*) **}**
note *singleton_sets* = *this*
have *?M* < (∑*x*∈*X*. *?M* / *Suc n*)
using ‹*?M* ≠ *0*›
by (*simp add*: ‹*card X* = *Suc* (*Suc n*)› *field_simps less_le*)
also have ... ≤ (∑*x*∈*X*. *?m x*)
by (*rule sum_mono*) *fact*

also have ... = *measure* M ($\bigcup x{\in}X.\ \{x\}$)
  using *singleton_sets* ‹*finite* X›
 by (*intro finite_measure_finite_Union*[*symmetric*]) (*auto simp*: *disjoint_family_on_def*)
 finally have *?M* < *measure* M ($\bigcup x{\in}X.\ \{x\}$) **.**
 moreover have *measure* M ($\bigcup x{\in}X.\ \{x\}$) $\leq$ *?M*
 using *singleton_sets*[*THEN sets.sets_into_space*] **by** (*intro finite_measure_mono*) *auto*
 ultimately show *False* **by** *simp*
 qed
 show *?thesis*
 unfolding * **by** (*intro countable_UN countableI_type countable_finite*[*OF* **])
qed

end

6.4 Ordered Euclidean Space

theory *Ordered_Euclidean_Space*
imports
 Convex_Euclidean_Space
 HOL−Library.Product_Order
begin

An ordering on euclidean spaces that will allow us to talk about intervals

class *ordered_euclidean_space* = *ord* + *inf* + *sup* + *abs* + *Inf* + *Sup* + *euclidean_space* +
 assumes *eucl_le*: $x \leq y \longleftrightarrow (\forall i{\in}Basis.\ x \cdot i \leq y \cdot i)$
 assumes *eucl_less_le_not_le*: $x < y \longleftrightarrow x \leq y \wedge \neg\, y \leq x$
 assumes *eucl_inf*: $inf\ x\ y = (\sum i{\in}Basis.\ inf\ (x \cdot i)\ (y \cdot i) *_R i)$
 assumes *eucl_sup*: $sup\ x\ y = (\sum i{\in}Basis.\ sup\ (x \cdot i)\ (y \cdot i) *_R i)$
 assumes *eucl_Inf*: $Inf\ X = (\sum i{\in}Basis.\ (INF\ x{\in}X.\ x \cdot i) *_R i)$
 assumes *eucl_Sup*: $Sup\ X = (\sum i{\in}Basis.\ (SUP\ x{\in}X.\ x \cdot i) *_R i)$
 assumes *eucl_abs*: $|x| = (\sum i{\in}Basis.\ |x \cdot i| *_R i)$
begin

subclass *order*
 by *standard*
  (*auto simp*: *eucl_le eucl_less_le_not_le intro*!: *euclidean_eqI antisym intro*: *order.trans*)

subclass *ordered_ab_group_add_abs*
 by *standard* (*auto simp*: *eucl_le inner_add_left eucl_abs abs_leI*)

subclass *ordered_real_vector*
 by *standard* (*auto simp*: *eucl_le intro*!: *mult_left_mono mult_right_mono*)

subclass *lattice*
 by *standard* (*auto simp*: *eucl_inf eucl_sup eucl_le*)

subclass *distrib_lattice*
 by *standard* (*auto simp*: *eucl_inf eucl_sup sup_inf_distrib1 intro*!: *euclidean_eqI*)

subclass *conditionally_complete_lattice*
proof
 fix $z::'a$ **and** $X::'a$ *set*
 assume $X \neq \{\}$
 hence $\bigwedge i.\ (\lambda x.\ x \cdot i)$ ‘ $X \neq \{\}$ **by** *simp*
 thus $(\bigwedge x.\ x \in X \Longrightarrow z \leq x) \Longrightarrow z \leq Inf\ X$ $(\bigwedge x.\ x \in X \Longrightarrow x \leq z) \Longrightarrow Sup\ X \leq z$
  by (*auto simp*: *eucl_Inf eucl_Sup eucl_le*
   intro!: *cInf_greatest cSup_least*)
qed (*force intro*!: *cInf_lower cSup_upper*
  simp: *bdd_below_def bdd_above_def preorder_class.bdd_below_def preorder_class.bdd_above_def*
   eucl_Inf eucl_Sup eucl_le)+

lemma *inner_Basis_inf_left*: $i \in Basis \Longrightarrow inf\ x\ y \cdot i = inf\ (x \cdot i)\ (y \cdot i)$
 and *inner_Basis_sup_left*: $i \in Basis \Longrightarrow sup\ x\ y \cdot i = sup\ (x \cdot i)\ (y \cdot i)$
 by (*simp_all add*: *eucl_inf eucl_sup inner_sum_left inner_Basis if_distrib*
  cong: *if_cong*)

lemma *inner_Basis_INF_left*: $i \in Basis \Longrightarrow (INF\ x{\in}X.\ f\ x) \cdot i = (INF\ x{\in}X.\ f\ x \cdot i)$
 and *inner_Basis_SUP_left*: $i \in Basis \Longrightarrow (SUP\ x{\in}X.\ f\ x) \cdot i = (SUP\ x{\in}X.\ f\ x \cdot i)$
 using *eucl_Sup* [*of f* ‘ *X*] *eucl_Inf* [*of f* ‘ *X*] **by** (*simp_all add*: *image_comp*)

lemma *abs_inner*: $i \in Basis \Longrightarrow |x| \cdot i = |x \cdot i|$
 by (*auto simp*: *eucl_abs*)

lemma
 abs_scaleR: $|a *_R b| = |a| *_R |b|$
 by (*auto simp*: *eucl_abs abs_mult intro*!: *euclidean_eqI*)

lemma *interval_inner_leI*:
 assumes $x \in \{a\ ..\ b\}$ $0 \leq i$
 shows $a{\cdot}i \leq x{\cdot}i$ $x{\cdot}i \leq b{\cdot}i$
 using *assms*
 unfolding *euclidean_inner*[*of a i*] *euclidean_inner*[*of x i*] *euclidean_inner*[*of b i*]
 by (*auto intro*!: *ordered_comm_monoid_add_class.sum_mono mult_right_mono simp*: *eucl_le*)

lemma *inner_nonneg_nonneg*:
 shows $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow 0 \leq a \cdot b$
 using *interval_inner_leI*[*of a 0 a b*]
 by *auto*

lemma *inner_Basis_mono*:
 shows $a \leq b \Longrightarrow c \in Basis \Longrightarrow a \cdot c \leq b \cdot c$

by (*simp add*: *eucl_le*)

lemma *Basis_nonneg*[*intro*, *simp*]: $i \in Basis \Longrightarrow 0 \leq i$
by (*auto simp*: *eucl_le inner_Basis*)

lemma *Sup_eq_maximum_componentwise*:
fixes s::'*a set*
assumes *i*: $\bigwedge b.\ b \in Basis \Longrightarrow X \cdot b = i\ b \cdot b$
assumes *sup*: $\bigwedge b\ x.\ b \in Basis \Longrightarrow x \in s \Longrightarrow x \cdot b \leq X \cdot b$
assumes *i_s*: $\bigwedge b.\ b \in Basis \Longrightarrow (i\ b \cdot b) \in (\lambda x.\ x \cdot b)$ ‘ s
shows *Sup* $s = X$
using *assms*
unfolding *eucl_Sup euclidean_representation_sum*
by (*auto intro*!: *conditionally_complete_lattice_class.cSup_eq_maximum*)

lemma *Inf_eq_minimum_componentwise*:
assumes *i*: $\bigwedge b.\ b \in Basis \Longrightarrow X \cdot b = i\ b \cdot b$
assumes *sup*: $\bigwedge b\ x.\ b \in Basis \Longrightarrow x \in s \Longrightarrow X \cdot b \leq x \cdot b$
assumes *i_s*: $\bigwedge b.\ b \in Basis \Longrightarrow (i\ b \cdot b) \in (\lambda x.\ x \cdot b)$ ‘ s
shows *Inf* $s = X$
using *assms*
unfolding *eucl_Inf euclidean_representation_sum*
by (*auto intro*!: *conditionally_complete_lattice_class.cInf_eq_minimum*)

end

proposition *compact_attains_Inf_componentwise*:
fixes b::'*a*::*ordered_euclidean_space*
assumes $b \in Basis$ **assumes** $X \neq \{\}$ *compact X*
obtains x **where** $x \in X\ x \cdot b = Inf\ X \cdot b\ \bigwedge y.\ y \in X \Longrightarrow x \cdot b \leq y \cdot b$
proof *atomize_elim*
let *?proj* = $(\lambda x.\ x \cdot b)$ ‘ X
from *assms* **have** *compact ?proj ?proj* $\neq \{\}$
by (*auto intro*!: *compact_continuous_image continuous_intros*)
from *compact_attains_inf*[*OF this*]
obtain $s\ x$
where *s*: $s \in (\lambda x.\ x \cdot b)$ ‘ $X\ \bigwedge t.\ t \in (\lambda x.\ x \cdot b)$ ‘ $X \Longrightarrow s \leq t$
and *x*: $x \in X\ s = x \cdot b\ \bigwedge y.\ y \in X \Longrightarrow x \cdot b \leq y \cdot b$
by *auto*
hence *Inf ?proj* $= x \cdot b$
by (*auto intro*!: *conditionally_complete_lattice_class.cInf_eq_minimum*)
hence $x \cdot b = Inf\ X \cdot b$
by (*auto simp*: *eucl_Inf inner_sum_left inner_Basis if_distrib* ‹$b \in Basis$›
cong: *if_cong*)
with x **show** $\exists x.\ x \in X \wedge x \cdot b = Inf\ X \cdot b \wedge (\forall y.\ y \in X \longrightarrow x \cdot b \leq y \cdot b)$
by *blast*
qed

proposition

compact_attains_Sup_componentwise:
fixes b::'*a*::*ordered_euclidean_space*
assumes $b \in Basis$ **assumes** $X \neq \{\}$ *compact X*
obtains x **where** $x \in X$ $x \cdot b = Sup\ X \cdot b$ $\bigwedge y.\ y \in X \Longrightarrow y \cdot b \leq x \cdot b$
proof *atomize_elim*
let *?proj* = $(\lambda x.\ x \cdot b)$ ' X
from *assms* **have** *compact ?proj ?proj* $\neq \{\}$
by (*auto intro*!: *compact_continuous_image continuous_intros*)
from *compact_attains_sup*[*OF this*]
obtain *s x*
where *s*: $s \in (\lambda x.\ x \cdot b)$ ' X $\bigwedge t.\ t \in (\lambda x.\ x \cdot b)$ ' $X \Longrightarrow t \leq s$
and *x*: $x \in X$ $s = x \cdot b$ $\bigwedge y.\ y \in X \Longrightarrow y \cdot b \leq x \cdot b$
by *auto*
hence *Sup ?proj* = $x \cdot b$
by (*auto intro*!: *cSup_eq_maximum*)
hence $x \cdot b = Sup\ X \cdot b$
by (*auto simp*: *eucl_Sup*[**where** '*a*='*a*] *inner_sum_left inner_Basis if_distrib* ‹b $\in$ *Basis*›
cong: *if_cong*)
with *x* **show** $\exists x.\ x \in X \wedge x \cdot b = Sup\ X \cdot b \wedge (\forall y.\ y \in X \longrightarrow y \cdot b \leq x \cdot b)$
by *blast*
qed

lemma *tendsto_sup*[*tendsto_intros*]:
fixes X :: '*a* $\Rightarrow$ '*b*::*ordered_euclidean_space*
assumes $(X \longrightarrow x)$ *net* $(Y \longrightarrow y)$ *net*
shows $((\lambda i.\ sup\ (X\ i)\ (Y\ i)) \longrightarrow sup\ x\ y)$ *net*
unfolding *sup_max eucl_sup* **by** (*intro assms tendsto_intros*)

lemma *tendsto_inf*[*tendsto_intros*]:
fixes X :: '*a* $\Rightarrow$ '*b*::*ordered_euclidean_space*
assumes $(X \longrightarrow x)$ *net* $(Y \longrightarrow y)$ *net*
shows $((\lambda i.\ inf\ (X\ i)\ (Y\ i)) \longrightarrow inf\ x\ y)$ *net*
unfolding *inf_min eucl_inf* **by** (*intro assms tendsto_intros*)

lemma *tendsto_componentwise_max*:
assumes *f*: $(f \longrightarrow l)$ *F* **and** *g*: $(g \longrightarrow m)$ *F*
shows $((\lambda x.\ (\sum i \in Basis.\ max\ (f\ x \cdot i)\ (g\ x \cdot i) *_R i)) \longrightarrow (\sum i \in Basis.\ max\ (l \cdot i)\ (m \cdot i) *_R i))$ *F*
by (*intro tendsto_intros assms*)

lemma *tendsto_componentwise_min*:
assumes *f*: $(f \longrightarrow l)$ *F* **and** *g*: $(g \longrightarrow m)$ *F*
shows $((\lambda x.\ (\sum i \in Basis.\ min\ (f\ x \cdot i)\ (g\ x \cdot i) *_R i)) \longrightarrow (\sum i \in Basis.\ min\ (l \cdot i)\ (m \cdot i) *_R i))$ *F*
by (*intro tendsto_intros assms*)

lemma (**in** *order*) *atLeastatMost_empty'*[*simp*]:
$(\neg\ a \leq b) \Longrightarrow \{a..b\} = \{\}$

by (*auto*)

instance *real* :: *ordered_euclidean_space*
by *standard auto*

lemma *in_Basis_prod_iff*:
fixes *i*::'*a*::*euclidean_space*∗'*b*::*euclidean_space*
shows $i \in Basis \longleftrightarrow fst\ i = 0 \wedge snd\ i \in Basis \vee snd\ i = 0 \wedge fst\ i \in Basis$
by (*cases i*) (*auto simp*: *Basis_prod_def*)

instantiation *prod* :: (*abs*, *abs*) *abs*
begin

definition |*x*| = (|*fst x*|, |*snd x*|)

instance ..

end

instance *prod* :: (*ordered_euclidean_space*, *ordered_euclidean_space*) *ordered_euclidean_space*
by *standard*
(*auto intro*!: *add_mono simp add*: *euclidean_representation_sum*' *Ball_def inner_prod_def*
in_Basis_prod_iff inner_Basis_inf_left inner_Basis_sup_left inner_Basis_INF_left Inf_prod_def
inner_Basis_SUP_left Sup_prod_def less_prod_def less_eq_prod_def eucl_le[**where** '*a*='*a*]
eucl_le[**where** '*a*='*b*] *abs_prod_def abs_inner*)

Instantiation for intervals on *ordered_euclidean_space*

proposition
fixes *a* :: '*a*::*ordered_euclidean_space*
shows *cbox_interval*: *cbox a b* = {*a*..*b*}
and *interval_cbox*: {*a*..*b*} = *cbox a b*
and *eucl_le_atMost*: {*x*. ∀*i*∈*Basis*. *x* · *i* <= *a* · *i*} = {..*a*}
and *eucl_le_atLeast*: {*x*. ∀*i*∈*Basis*. *a* · *i* <= *x* · *i*} = {*a*..}
by (*auto simp*: *eucl_le*[**where** '*a*='*a*] *eucl_less_def box_def cbox_def*)

lemma *sums_vec_nth* :
assumes *f sums a*
shows (λ*x*. *f x* $ *i*) *sums a* $ *i*
using *assms* **unfolding** *sums_def*
by (*auto dest*: *tendsto_vec_nth* [**where** *i*=*i*])

lemma *summable_vec_nth* :
assumes *summable f*
shows *summable* (λ*x*. *f x* $ *i*)
using *assms* **unfolding** *summable_def*
by (*blast intro*: *sums_vec_nth*)

lemma *closed_eucl_atLeastAtMost*[*simp*, *intro*]:
fixes *a* :: *'a*::*ordered_euclidean_space*
shows *closed* {*a*..*b*}
by (*simp add*: *cbox_interval*[*symmetric*] *closed_cbox*)

lemma *closed_eucl_atMost*[*simp*, *intro*]:
fixes *a* :: *'a*::*ordered_euclidean_space*
shows *closed* {..*a*}
by (*simp add*: *closed_interval_left eucl_le_atMost*[*symmetric*])

lemma *closed_eucl_atLeast*[*simp*, *intro*]:
fixes *a* :: *'a*::*ordered_euclidean_space*
shows *closed* {*a*..}
by (*simp add*: *closed_interval_right eucl_le_atLeast*[*symmetric*])

lemma *bounded_closed_interval* [*simp*]:
fixes *a* :: *'a*::*ordered_euclidean_space*
shows *bounded* {*a* .. *b*}
using *bounded_cbox*[*of a b*]
by (*metis interval_cbox*)

lemma *convex_closed_interval* [*simp*]:
fixes *a* :: *'a*::*ordered_euclidean_space*
shows *convex* {*a* .. *b*}
using *convex_box*[*of a b*]
by (*metis interval_cbox*)

lemma *image_smult_interval*:($\lambda x.\ m *_R (x::\_::ordered\_euclidean\_space)$) ' {*a* .. *b*} =
(*if* {*a* .. *b*} = {} *then* {} *else if 0* ≤ *m then* {$m *_R a$ .. $m *_R b$} *else* {$m *_R b$ .. $m *_R a$})
using *image_smult_cbox*[*of m a b*]
by (*simp add*: *cbox_interval*)

lemma [*simp*]:
fixes *a b*::*'a*::*ordered_euclidean_space*
shows *is_interval_ic*: *is_interval* {..*a*}
and *is_interval_ci*: *is_interval* {*a*..}
and *is_interval_cc*: *is_interval* {*b*..*a*}
by (*force simp*: *is_interval_def eucl_le*[**where** *'a*=*'a*])+

lemma *connected_interval* [*simp*]:
fixes *a b*::*'a*::*ordered_euclidean_space*
shows *connected* {*a*..*b*}
using *is_interval_cc is_interval_connected* **by** *blast*

lemma *compact_interval* [*simp*]:
fixes *a b*::*'a*::*ordered_euclidean_space*

shows *compact* {*a* .. *b*}
by (*metis compact_cbox interval_cbox*)

no_notation
eucl_less (**infix** <*e* *50*)

lemma *One_nonneg*: *0* ≤ (∑ *Basis*::'*a*::*ordered_euclidean_space*)
by (*auto intro*: *sum_nonneg*)

lemma
fixes *a b*::'*a*::*ordered_euclidean_space*
shows *bdd_above_cbox*[*intro*, *simp*]: *bdd_above* (*cbox a b*)
and *bdd_below_cbox*[*intro*, *simp*]: *bdd_below* (*cbox a b*)
and *bdd_above_box*[*intro*, *simp*]: *bdd_above* (*box a b*)
and *bdd_below_box*[*intro*, *simp*]: *bdd_below* (*box a b*)
unfolding *atomize_conj*
by (*metis bdd_above_Icc bdd_above_mono bdd_below_Icc bdd_below_mono bounded_box bounded_subset_cbox_symmetric interval_cbox*)

instantiation *vec* :: (*ordered_euclidean_space*, *finite*) *ordered_euclidean_space*
begin

definition *inf x y* = (χ *i*. *inf* (*x* $ *i*) (*y* $ *i*))
definition *sup x y* = (χ *i*. *sup* (*x* $ *i*) (*y* $ *i*))
definition *Inf X* = (χ *i*. (*INF x*∈*X*. *x* $ *i*))
definition *Sup X* = (χ *i*. (*SUP x*∈*X*. *x* $ *i*))
definition |*x*| = (χ *i*. |*x* $ *i*|)

instance
apply *standard*
unfolding *euclidean_representation_sum'*
apply (*auto simp*: *less_eq_vec_def inf_vec_def sup_vec_def Inf_vec_def Sup_vec_def inner_axis*
Basis_vec_def inner_Basis_inf_left inner_Basis_sup_left inner_Basis_INF_left
inner_Basis_SUP_left eucl_le[**where** '*a*='*a*] *less_le_not_le abs_vec_def abs_inner*)
done

end

end

6.5 Borel Space

theory *Borel_Space*
imports
Measurable Derivative Ordered_Euclidean_Space Extended_Real_Limits
begin

lemma *is_interval_real_ereal_oo*: *is_interval* (*real_of_ereal* ` {*N*<..<*M*::*ereal*})

by (*auto simp*: *real_atLeastGreaterThan_eq*)

lemma *sets_Collect_eventually_sequentially*[*measurable*]:
($\bigwedge$*i*. {*x*∈*space M*. *P x i*} ∈ *sets M*) ⟹ {*x*∈*space M*. *eventually* (*P x*) *sequentially*} ∈ *sets M*
unfolding *eventually_sequentially* **by** *simp*

lemma *topological_basis_trivial*: *topological_basis* {*A*. *open A*}
by (*auto simp*: *topological_basis_def*)

proposition *open_prod_generated*: *open* = *generate_topology* {*A* × *B* | *A B*. *open A* ∧ *open B*}
proof −
have {*A* × *B* :: (′*a* × ′*b*) *set* | *A B*. *open A* ∧ *open B*} = ((λ(*a*, *b*). *a* × *b*) ‘ ({*A*. *open A*} × {*A*. *open A*}))
by *auto*
then show *?thesis*
by (*auto intro*: *topological_basis_prod topological_basis_trivial topological_basis_imp_subbasis*)
qed

proposition *mono_on_imp_deriv_nonneg*:
assumes *mono*: *mono_on f A* **and** *deriv*: (*f has_real_derivative D*) (*at x*)
assumes *x* ∈ *interior A*
shows *D* ≥ *0*
proof (*rule tendsto_lowerbound*)
let *?A′* = (λ*y*. *y* − *x*) ‘ *interior A*
from *deriv* **show** ((λ*h*. (*f* (*x* + *h*) − *f x*) / *h*) ⟶ *D*) (*at 0*)
by (*simp add*: *field_has_derivative_at has_field_derivative_def*)
from *mono* **have** *mono′*: *mono_on f* (*interior A*) **by** (*rule mono_on_subset*) (*rule interior_subset*)

show *eventually* (λ*h*. (*f* (*x* + *h*) − *f x*) / *h* ≥ *0*) (*at 0*)
proof (*subst eventually_at_topological*, *intro exI conjI ballI impI*)
have *open* (*interior A*) **by** *simp*
hence *open* ((+) (−*x*) ‘ *interior A*) **by** (*rule open_translation*)
also have ((+) (−*x*) ‘ *interior A*) = *?A′* **by** *auto*
finally show *open ?A′* **.**
next
from ‹*x* ∈ *interior A*› **show** *0* ∈ *?A′* **by** *auto*
next
fix *h* **assume** *h* ∈ *?A′*
hence *x* + *h* ∈ *interior A* **by** *auto*
with *mono′* **and** ‹*x* ∈ *interior A*› **show** (*f* (*x* + *h*) − *f x*) / *h* ≥ *0*
by (*cases h rule*: *linorder_cases*[*of _ 0*])
(*simp_all add*: *divide_nonpos_neg divide_nonneg_pos mono_onD field_simps*)
qed
qed *simp*

proposition *mono_on_ctble_discont*:

```
  fixes f :: real ⇒ real
  fixes A :: real set
  assumes mono_on f A
  shows countable {a∈A. ¬ continuous (at a within A) f}
proof –
  have mono: ⋀x y. x ∈ A ⟹ y ∈ A ⟹ x ≤ y ⟹ f x ≤ f y
    using ‹mono_on f A› by (simp add: mono_on_def)
  have ∀ a ∈ {a∈A. ¬ continuous (at a within A) f}. ∃ q :: nat × rat.
      (fst q = 0 ∧ of_rat (snd q) < f a ∧ (∀ x ∈ A. x < a ⟶ f x < of_rat (snd
q))) ∨
      (fst q = 1 ∧ of_rat (snd q) > f a ∧ (∀ x ∈ A. x > a ⟶ f x > of_rat (snd
q)))
  proof (clarsimp simp del: One_nat_def)
    fix a assume a ∈ A assume ¬ continuous (at a within A) f
    thus ∃ q1 q2.
          q1 = 0 ∧ real_of_rat q2 < f a ∧ (∀ x∈A. x < a ⟶ f x < real_of_rat q2)
∨
          q1 = 1 ∧ f a < real_of_rat q2 ∧ (∀ x∈A. a < x ⟶ real_of_rat q2 < f x)
    proof (auto simp add: continuous_within order_tendsto_iff eventually_at)
      fix l assume l < f a
      then obtain q2 where q2: l < of_rat q2 of_rat q2 < f a
        using of_rat_dense by blast
      assume * [rule_format]: ∀ d>0. ∃ x∈A. x ≠ a ∧ dist x a < d ∧ ¬ l < f x
      from q2 have real_of_rat q2 < f a ∧ (∀ x∈A. x < a ⟶ f x < real_of_rat q2)
      proof auto
        fix x assume x ∈ A x < a
        with q2 *[of a − x] show f x < real_of_rat q2
          apply (auto simp add: dist_real_def not_less)
          apply (subgoal_tac f x ≤ f xa)
          by (auto intro: mono)
      qed
      thus ?thesis by auto
    next
      fix u assume u > f a
      then obtain q2 where q2: f a < of_rat q2 of_rat q2 < u
        using of_rat_dense by blast
      assume *[rule_format]: ∀ d>0. ∃ x∈A. x ≠ a ∧ dist x a < d ∧ ¬ u > f x
      from q2 have real_of_rat q2 > f a ∧ (∀ x∈A. x > a ⟶ f x > real_of_rat q2)
      proof auto
        fix x assume x ∈ A x > a
        with q2 *[of x − a] show f x > real_of_rat q2
          apply (auto simp add: dist_real_def)
          apply (subgoal_tac f x ≥ f xa)
          by (auto intro: mono)
      qed
      thus ?thesis by auto
    qed
  qed
  hence ∃ g :: real ⇒ nat × rat . ∀ a ∈ {a∈A. ¬ continuous (at a within A) f}.
```

(*fst* (*g a*) = *0* ∧ *of_rat* (*snd* (*g a*)) < *f a* ∧ (∀ *x* ∈ *A*. *x* < *a* ⟶ *f x* < *of_rat* (*snd* (*g a*)))) |
(*fst* (*g a*) = *1* ∧ *of_rat* (*snd* (*g a*)) > *f a* ∧ (∀ *x* ∈ *A*. *x* > *a* ⟶ *f x* > *of_rat* (*snd* (*g a*))))
by (*rule bchoice*)
then guess *g* **..**
hence *g*: ⋀*a x*. *a* ∈ *A* ⟹ ¬ *continuous* (*at a within A*) *f* ⟹ *x* ∈ *A* ⟹
(*fst* (*g a*) = *0* ∧ *of_rat* (*snd* (*g a*)) < *f a* ∧ (*x* < *a* ⟶ *f x* < *of_rat* (*snd* (*g a*)))) |
(*fst* (*g a*) = *1* ∧ *of_rat* (*snd* (*g a*)) > *f a* ∧ (*x* > *a* ⟶ *f x* > *of_rat* (*snd* (*g a*))))
by *auto*
have *inj_on g* {*a*∈*A*. ¬ *continuous* (*at a within A*) *f*}
proof (*auto simp add*: *inj_on_def*)
fix *w z*
assume *1*: *w* ∈ *A* **and** *2*: ¬ *continuous* (*at w within A*) *f* **and**
3: *z* ∈ *A* **and** *4*: ¬ *continuous* (*at z within A*) *f* **and**
5: *g w* = *g z*
from *g* [*OF 1 2 3*] *g* [*OF 3 4 1*] *5*
show *w* = *z* **by** *auto*
qed
thus *?thesis*
by (*rule countableI'*)
qed

lemma *mono_on_ctble_discont_open*:
fixes *f* :: *real* ⇒ *real*
fixes *A* :: *real set*
assumes *open A mono_on f A*
shows *countable* {*a*∈*A*. ¬*isCont f a*}
proof −
have {*a*∈*A*. ¬*isCont f a*} = {*a*∈*A*. ¬(*continuous* (*at a within A*) *f*)}
by (*auto simp add*: *continuous_within_open* [*OF _* ‹*open A*›])
thus *?thesis*
apply (*elim ssubst*)
by (*rule mono_on_ctble_discont*, *rule assms*)
qed

lemma *mono_ctble_discont*:
fixes *f* :: *real* ⇒ *real*
assumes *mono f*
shows *countable* {*a*. ¬ *isCont f a*}
using *assms mono_on_ctble_discont* [*of f UNIV*] **unfolding** *mono_on_def mono_def*
by *auto*

lemma *has_real_derivative_imp_continuous_on*:
assumes ⋀*x*. *x* ∈ *A* ⟹ (*f has_real_derivative f' x*) (*at x*)
shows *continuous_on A f*
apply (*intro differentiable_imp_continuous_on*, *unfold differentiable_on_def*)

using *assms differentiable_at_withinI real_differentiable_def* **by** *blast*

lemma *continuous_interval_vimage_Int*:
assumes *continuous_on* $\{a::real..b\}$ g **and** *mono*: $\bigwedge x\ y.\ a \leq x \Longrightarrow x \leq y \Longrightarrow y \leq b \Longrightarrow g\ x \leq g\ y$
assumes $a \leq b$ $(c::real) \leq d$ $\{c..d\} \subseteq \{g\ a..g\ b\}$
obtains c' d' **where** $\{a..b\} \cap g -` \{c..d\} = \{c'..d'\}$ $c' \leq d'$ $g\ c' = c$ $g\ d' = d$
proof−
let $?A = \{a..b\} \cap g -` \{c..d\}$
from *IVT'*[*of g a c b*, *OF* _ _ ‹$a \leq b$› *assms*(*1*)] *assms*(*4*,*5*)
obtain c'' **where** c'': $c'' \in ?A$ $g\ c'' = c$ **by** *auto*
from *IVT'*[*of g a d b*, *OF* _ _ ‹$a \leq b$› *assms*(*1*)] *assms*(*4*,*5*)
obtain d'' **where** d'': $d'' \in ?A$ $g\ d'' = d$ **by** *auto*
hence [*simp*]: $?A \neq \{\}$ **by** *blast*

define c' **where** $c' = Inf\ ?A$
define d' **where** $d' = Sup\ ?A$
have $?A \subseteq \{c'..d'\}$ **unfolding** *c'_def d'_def*
by (*intro subsetI*) (*auto intro*: *cInf_lower cSup_upper*)
moreover from *assms* **have** *closed ?A*
using *continuous_on_closed_vimage*[*of* $\{a..b\}$ *g*] **by** (*subst Int_commute*) *simp*
hence *c'd'_in_set*: $c' \in ?A$ $d' \in ?A$ **unfolding** *c'_def d'_def*
by ((*intro closed_contains_Inf closed_contains_Sup*, *simp_all*)[])+
hence $\{c'..d'\} \subseteq ?A$ **using** *assms*
by (*intro subsetI*)
(*auto intro*!: *order_trans*[*of c g c' g x* **for** *x*] *order_trans*[*of g x g d' d* **for** *x*]
intro!: *mono*)
moreover have $c' \leq d'$ **using** *c'd'_in_set*(*2*) **unfolding** *c'_def* **by** (*intro cInf_lower*) *auto*
moreover have $g\ c' \leq c$ $g\ d' \geq d$
apply (*insert c'' d'' c'd'_in_set*)
apply (*subst c''*(*2*)[*symmetric*])
apply (*auto simp*: *c'_def intro*!: *mono cInf_lower c''*) []
apply (*subst d''*(*2*)[*symmetric*])
apply (*auto simp*: *d'_def intro*!: *mono cSup_upper d''*) []
done
with *c'd'_in_set* **have** $g\ c' = c$ $g\ d' = d$ **by** *auto*
ultimately show *?thesis* **using** *that* **by** *blast*
qed

6.5.1 Generic Borel spaces

definition (**in** *topological_space*) *borel* :: *'a measure* **where**
borel = *sigma UNIV* $\{S.\ open\ S\}$

abbreviation *borel_measurable M* $\equiv$ *measurable M borel*

lemma *in_borel_measurable*:
$f \in$ *borel_measurable M* $\longleftrightarrow$

(∀ *S* ∈ *sigma_sets UNIV* {*S*. *open S*}. *f* −‘ *S* ∩ *space M* ∈ *sets M*)
by (*auto simp add*: *measurable_def borel_def*)

lemma *in_borel_measurable_borel*:
f ∈ *borel_measurable M* ⟷
(∀ *S* ∈ *sets borel*.
f −‘ *S* ∩ *space M* ∈ *sets M*)
by (*auto simp add*: *measurable_def borel_def*)

lemma *space_borel*[*simp*]: *space borel* = *UNIV*
unfolding *borel_def* **by** *auto*

lemma *space_in_borel*[*measurable*]: *UNIV* ∈ *sets borel*
unfolding *borel_def* **by** *auto*

lemma *sets_borel*: *sets borel* = *sigma_sets UNIV* {*S*. *open S*}
unfolding *borel_def* **by** (*rule sets_measure_of*) *simp*

lemma *measurable_sets_borel*:
⟦*f* ∈ *measurable borel M*; *A* ∈ *sets M*⟧ ⟹ *f* −‘ *A* ∈ *sets borel*
by (*drule* (*1*) *measurable_sets*) *simp*

lemma *pred_Collect_borel*[*measurable* (*raw*)]: *Measurable.pred borel P* ⟹ {*x*. *P x*} ∈ *sets borel*
unfolding *borel_def pred_def* **by** *auto*

lemma *borel_open*[*measurable* (*raw generic*)]:
assumes *open A* **shows** *A* ∈ *sets borel*
proof −
have *A* ∈ {*S*. *open S*} **unfolding** *mem_Collect_eq* **using** *assms* **.**
thus *?thesis* **unfolding** *borel_def* **by** *auto*
qed

lemma *borel_closed*[*measurable* (*raw generic*)]:
assumes *closed A* **shows** *A* ∈ *sets borel*
proof −
have *space borel* − (− *A*) ∈ *sets borel*
using *assms* **unfolding** *closed_def* **by** (*blast intro*: *borel_open*)
thus *?thesis* **by** *simp*
qed

lemma *borel_singleton*[*measurable*]:
A ∈ *sets borel* ⟹ *insert x A* ∈ *sets* (*borel* :: ′*a*::*t1_space measure*)
unfolding *insert_def* **by** (*rule sets.Un*) *auto*

lemma *sets_borel_eq_count_space*: *sets* (*borel* :: ′*a*::{*countable*, *t2_space*} *measure*) = *count_space UNIV*
proof −
have (⋃ *a*∈*A*. {*a*}) ∈ *sets borel* **for** *A* :: ′*a set*

```
    by (intro sets.countable_UN') auto
  then show ?thesis
    by auto
qed

lemma borel_comp[measurable]: A ∈ sets borel ⟹ − A ∈ sets borel
  unfolding Compl_eq_Diff_UNIV by simp

lemma borel_measurable_vimage:
  fixes f :: 'a ⇒ 'x::t2_space
  assumes borel[measurable]: f ∈ borel_measurable M
  shows f −` {x} ∩ space M ∈ sets M
  by simp

lemma borel_measurableI:
  fixes f :: 'a ⇒ 'x::topological_space
  assumes ⋀S. open S ⟹ f −` S ∩ space M ∈ sets M
  shows f ∈ borel_measurable M
  unfolding borel_def
proof (rule measurable_measure_of, simp_all)
  fix S :: 'x set assume open S thus f −` S ∩ space M ∈ sets M
    using assms[of S] by simp
qed

lemma borel_measurable_const:
  (λx. c) ∈ borel_measurable M
  by auto

lemma borel_measurable_indicator:
  assumes A: A ∈ sets M
  shows indicator A ∈ borel_measurable M
  unfolding indicator_def [abs_def] using A
  by (auto intro!: measurable_If_set)

lemma borel_measurable_count_space[measurable (raw)]:
  f ∈ borel_measurable (count_space S)
  unfolding measurable_def by auto

lemma borel_measurable_indicator'[measurable (raw)]:
  assumes [measurable]: {x∈space M. f x ∈ A x} ∈ sets M
  shows (λx. indicator (A x) (f x)) ∈ borel_measurable M
  unfolding indicator_def[abs_def]
  by (auto intro!: measurable_If)

lemma borel_measurable_indicator_iff:
  (indicator A :: 'a ⇒ 'x::{t1_space, zero_neq_one}) ∈ borel_measurable M ⟷ A
∩ space M ∈ sets M
    (is ?I ∈ borel_measurable M ⟷ _)
proof
```

assume *?I* ∈ *borel_measurable M*
then have *?I* −‘ {*1*} ∩ *space M* ∈ *sets M*
unfolding *measurable_def* **by** *auto*
also have *?I* −‘ {*1*} ∩ *space M* = *A* ∩ *space M*
unfolding *indicator_def* [*abs_def*] **by** *auto*
finally show *A* ∩ *space M* ∈ *sets M* **.**
next
assume *A* ∩ *space M* ∈ *sets M*
moreover have *?I* ∈ *borel_measurable M* ⟷
(*indicator* (*A* ∩ *space M*) :: ′*a* ⇒ ′*x*) ∈ *borel_measurable M*
by (*intro measurable_cong*) (*auto simp*: *indicator_def*)
ultimately show *?I* ∈ *borel_measurable M* **by** *auto*
qed

lemma *borel_measurable_subalgebra*:
assumes *sets N* ⊆ *sets M space N* = *space M f* ∈ *borel_measurable N*
shows *f* ∈ *borel_measurable M*
using *assms* **unfolding** *measurable_def* **by** *auto*

lemma *borel_measurable_restrict_space_iff_ereal*:
fixes *f* :: ′*a* ⇒ *ereal*
assumes Ω[*measurable*, *simp*]: Ω ∩ *space M* ∈ *sets M*
shows *f* ∈ *borel_measurable* (*restrict_space M* Ω) ⟷
(λ*x*. *f x* ∗ *indicator* Ω *x*) ∈ *borel_measurable M*
by (*subst measurable_restrict_space_iff*)
(*auto simp*: *indicator_def if_distrib*[**where** *f*=λ*x*. *a* ∗ *x* **for** *a*] *cong del*: *if_weak_cong*)

lemma *borel_measurable_restrict_space_iff_ennreal*:
fixes *f* :: ′*a* ⇒ *ennreal*
assumes Ω[*measurable*, *simp*]: Ω ∩ *space M* ∈ *sets M*
shows *f* ∈ *borel_measurable* (*restrict_space M* Ω) ⟷
(λ*x*. *f x* ∗ *indicator* Ω *x*) ∈ *borel_measurable M*
by (*subst measurable_restrict_space_iff*)
(*auto simp*: *indicator_def if_distrib*[**where** *f*=λ*x*. *a* ∗ *x* **for** *a*] *cong del*: *if_weak_cong*)

lemma *borel_measurable_restrict_space_iff*:
fixes *f* :: ′*a* ⇒ ′*b*::*real_normed_vector*
assumes Ω[*measurable*, *simp*]: Ω ∩ *space M* ∈ *sets M*
shows *f* ∈ *borel_measurable* (*restrict_space M* Ω) ⟷
(λ*x*. *indicator* Ω *x* $*_R$ *f x*) ∈ *borel_measurable M*
by (*subst measurable_restrict_space_iff*)
(*auto simp*: *indicator_def if_distrib*[**where** *f*=λ*x*. *x* $*_R$ *a* **for** *a*] *ac_simps*
cong del: *if_weak_cong*)

lemma *cbox_borel*[*measurable*]: *cbox a b* ∈ *sets borel*
by (*auto intro*: *borel_closed*)

lemma *box_borel*[*measurable*]: *box a b* ∈ *sets borel*
  by (*auto intro*: *borel_open*)

lemma *borel_compact*: *compact* (*A*::′*a*::*t2_space set*) ⟹ *A* ∈ *sets borel*
  by (*auto intro*: *borel_closed dest*!: *compact_imp_closed*)

lemma *borel_sigma_sets_subset*:
  A ⊆ *sets borel* ⟹ *sigma_sets UNIV A* ⊆ *sets borel*
  using *sets.sigma_sets_subset*[*of A borel*] **by** *simp*

lemma *borel_eq_sigmaI1*:
  fixes *F* :: ′*i* ⇒ ′*a*::*topological_space set* **and** *X* :: ′*a*::*topological_space set set*
  assumes *borel_eq*: *borel* = *sigma UNIV X*
  assumes *X*: ⋀*x*. *x* ∈ *X* ⟹ *x* ∈ *sets* (*sigma UNIV* (*F* ‘ *A*))
  assumes *F*: ⋀*i*. *i* ∈ *A* ⟹ *F i* ∈ *sets borel*
  shows *borel* = *sigma UNIV* (*F* ‘ *A*)
  unfolding *borel_def*
proof (*intro sigma_eqI antisym*)
  have *borel_rev_eq*: *sigma_sets UNIV* {*S*::′*a set*. *open S*} = *sets borel*
    unfolding *borel_def* **by** *simp*
  also have ... = *sigma_sets UNIV X*
    unfolding *borel_eq* **by** *simp*
  also have ... ⊆ *sigma_sets UNIV* (*F*‘*A*)
   using *X* **by** (*intro sigma_algebra.sigma_sets_subset*[*OF sigma_algebra_sigma_sets*])
auto
  finally show *sigma_sets UNIV* {*S*. *open S*} ⊆ *sigma_sets UNIV* (*F*‘*A*) **.**
  show *sigma_sets UNIV* (*F*‘*A*) ⊆ *sigma_sets UNIV* {*S*. *open S*}
    unfolding *borel_rev_eq* **using** *F* **by** (*intro borel_sigma_sets_subset*) *auto*
qed *auto*

lemma *borel_eq_sigmaI2*:
  fixes *F* :: ′*i* ⇒ ′*j* ⇒ ′*a*::*topological_space set*
    and *G* :: ′*l* ⇒ ′*k* ⇒ ′*a*::*topological_space set*
  assumes *borel_eq*: *borel* = *sigma UNIV* ((λ(*i*, *j*). *G i j*)‘*B*)
  assumes *X*: ⋀*i j*. (*i*, *j*) ∈ *B* ⟹ *G i j* ∈ *sets* (*sigma UNIV* ((λ(*i*, *j*). *F i j*) ‘
A))
  assumes *F*: ⋀*i j*. (*i*, *j*) ∈ *A* ⟹ *F i j* ∈ *sets borel*
  shows *borel* = *sigma UNIV* ((λ(*i*, *j*). *F i j*) ‘ *A*)
  using *assms*
   by (*intro borel_eq_sigmaI1*[**where** *X*=(λ(*i*, *j*). *G i j*) ‘ *B* **and** *F*=(λ(*i*, *j*). *F i*
j)]) *auto*

lemma *borel_eq_sigmaI3*:
  fixes *F* :: ′*i* ⇒ ′*j* ⇒ ′*a*::*topological_space set* **and** *X* :: ′*a*::*topological_space set*
set
  assumes *borel_eq*: *borel* = *sigma UNIV X*
  assumes *X*: ⋀*x*. *x* ∈ *X* ⟹ *x* ∈ *sets* (*sigma UNIV* ((λ(*i*, *j*). *F i j*) ‘ *A*))
  assumes *F*: ⋀*i j*. (*i*, *j*) ∈ *A* ⟹ *F i j* ∈ *sets borel*
  shows *borel* = *sigma UNIV* ((λ(*i*, *j*). *F i j*) ‘ *A*)

using *assms* **by** (*intro borel_eq_sigmaI1*[**where** *X*=*X* **and** *F*=(λ(*i*, *j*). *F i j*)]) *auto*

lemma *borel_eq_sigmaI4*:
fixes *F* :: ′*i* ⇒ ′*a*::*topological_space set*
and *G* :: ′*l* ⇒ ′*k* ⇒ ′*a*::*topological_space set*
assumes *borel_eq*: *borel* = *sigma UNIV* ((λ(*i*, *j*). *G i j*)‘*A*)
assumes *X*: ⋀*i j*. (*i*, *j*) ∈ *A* ⟹ *G i j* ∈ *sets* (*sigma UNIV* (*range F*))
assumes *F*: ⋀*i*. *F i* ∈ *sets borel*
shows *borel* = *sigma UNIV* (*range F*)
using *assms* **by** (*intro borel_eq_sigmaI1*[**where** *X*=(λ(*i*, *j*). *G i j*) ‘ *A* **and** *F*=*F*]) *auto*

lemma *borel_eq_sigmaI5*:
fixes *F* :: ′*i* ⇒ ′*j* ⇒ ′*a*::*topological_space set* **and** *G* :: ′*l* ⇒ ′*a*::*topological_space set*
assumes *borel_eq*: *borel* = *sigma UNIV* (*range G*)
assumes *X*: ⋀*i*. *G i* ∈ *sets* (*sigma UNIV* (*range* (λ(*i*, *j*). *F i j*)))
assumes *F*: ⋀*i j*. *F i j* ∈ *sets borel*
shows *borel* = *sigma UNIV* (*range* (λ(*i*, *j*). *F i j*))
using *assms* **by** (*intro borel_eq_sigmaI1*[**where** *X*=*range G* **and** *F*=(λ(*i*, *j*). *F i j*)]) *auto*

theorem *second_countable_borel_measurable*:
fixes *X* :: ′*a*::*second_countable_topology set set*
assumes *eq*: *open* = *generate_topology X*
shows *borel* = *sigma UNIV X*
unfolding *borel_def*
proof (*intro sigma_eqI sigma_sets_eqI*)
interpret *X*: *sigma_algebra UNIV sigma_sets UNIV X*
by (*rule sigma_algebra_sigma_sets*) *simp*

fix *S* :: ′*a set* **assume** *S* ∈ *Collect open*
then have *generate_topology X S*
by (*auto simp*: *eq*)
then show *S* ∈ *sigma_sets UNIV X*
proof *induction*
case (*UN K*)
then have *K*: ⋀*k*. *k* ∈ *K* ⟹ *open k*
unfolding *eq* **by** *auto*
from *ex_countable_basis* **obtain** *B* :: ′*a set set* **where**
B: ⋀*b*. *b* ∈ *B* ⟹ *open b* ⋀*X*. *open X* ⟹ ∃ *b*⊆*B*. (⋃ *b*) = *X* **and** *countable B*
by (*auto simp*: *topological_basis_def*)
from *B*(*2*)[*OF K*] **obtain** *m* **where** *m*: ⋀*k*. *k* ∈ *K* ⟹ *m k* ⊆ *B* ⋀*k*. *k* ∈ *K* ⟹ ⋃(*m k*) = *k*
by *metis*
define *U* **where** *U* = (⋃*k*∈*K*. *m k*)
with *m* **have** *countable U*

```
      by (intro countable_subset[OF _ ‹countable B›]) auto
    have ⋃ U = (⋃ A∈U. A) by simp
    also have ... = ⋃ K
      unfolding U_def UN_simps by (simp add: m)
    finally have ⋃ U = ⋃ K .

    have ∀ b∈U. ∃ k∈K. b ⊆ k
      using m by (auto simp: U_def)
    then obtain u where u: ⋀b. b ∈ U ⟹ u b ∈ K and ⋀b. b ∈ U ⟹ b ⊆ u
b
      by metis
    then have (⋃ b∈U. u b) ⊆ ⋃ K ⋃ U ⊆ (⋃ b∈U. u b)
      by auto
    then have ⋃ K = (⋃ b∈U. u b)
      unfolding ‹⋃ U = ⋃ K› by auto
    also have ... ∈ sigma_sets UNIV X
      using u UN by (intro X.countable_UN' ‹countable U›) auto
    finally show ⋃ K ∈ sigma_sets UNIV X .
  qed auto
qed (auto simp: eq intro: generate_topology.Basis)

lemma borel_eq_closed: borel = sigma UNIV (Collect closed)
  unfolding borel_def
proof (intro sigma_eqI sigma_sets_eqI, safe)
  fix x :: 'a set assume open x
  hence x = UNIV − (UNIV − x) by auto
  also have ... ∈ sigma_sets UNIV (Collect closed)
    by (force intro: sigma_sets.Compl simp: ‹open x›)
  finally show x ∈ sigma_sets UNIV (Collect closed) by simp
next
  fix x :: 'a set assume closed x
  hence x = UNIV − (UNIV − x) by auto
  also have ... ∈ sigma_sets UNIV (Collect open)
    by (force intro: sigma_sets.Compl simp: ‹closed x›)
  finally show x ∈ sigma_sets UNIV (Collect open) by simp
qed simp_all

proposition borel_eq_countable_basis:
  fixes B::'a::topological_space set set
  assumes countable B
  assumes topological_basis B
  shows borel = sigma UNIV B
  unfolding borel_def
proof (intro sigma_eqI sigma_sets_eqI, safe)
  interpret countable_basis open B using assms by (rule countable_basis_openI)
  fix X::'a set assume open X
  from open_countable_basisE[OF this] obtain B' where B': B' ⊆ B X = ⋃ B'
.
  then show X ∈ sigma_sets UNIV B
```

by (*blast intro*: *sigma_sets_UNION* ‹*countable B*› *countable_subset*)
next
fix *b* **assume** *b* ∈ *B*
hence *open b* **by** (*rule topological_basis_open*[*OF assms*(*2*)])
thus *b* ∈ *sigma_sets UNIV* (*Collect open*) **by** *auto*
qed *simp_all*

lemma *borel_measurable_continuous_on_restrict*:
fixes *f* :: *'a*::*topological_space* ⇒ *'b*::*topological_space*
assumes *f*: *continuous_on A f*
shows *f* ∈ *borel_measurable* (*restrict_space borel A*)
proof (*rule borel_measurableI*)
fix *S* :: *'b set* **assume** *open S*
with *f* **obtain** *T* **where** *f* −' *S* ∩ *A* = *T* ∩ *A open T*
by (*metis continuous_on_open_invariant*)
then show *f* −' *S* ∩ *space* (*restrict_space borel A*) ∈ *sets* (*restrict_space borel A*)
by (*force simp add*: *sets_restrict_space space_restrict_space*)
qed

lemma *borel_measurable_continuous_onI*: *continuous_on UNIV f* ⟹ *f* ∈ *borel_measurable borel*
by (*drule borel_measurable_continuous_on_restrict*) *simp*

lemma *borel_measurable_continuous_on_if*:
A ∈ *sets borel* ⟹ *continuous_on A f* ⟹ *continuous_on* (− *A*) *g* ⟹
(λ*x*. *if x* ∈ *A then f x else g x*) ∈ *borel_measurable borel*
by (*auto simp add*: *measurable_If_restrict_space_iff Collect_neg_eq*
intro!: *borel_measurable_continuous_on_restrict*)

lemma *borel_measurable_continuous_countable_exceptions*:
fixes *f* :: *'a*::*t1_space* ⇒ *'b*::*topological_space*
assumes *X*: *countable X*
assumes *continuous_on* (− *X*) *f*
shows *f* ∈ *borel_measurable borel*
proof (*rule measurable_discrete_difference*[*OF _ X*])
have *X* ∈ *sets borel*
by (*rule sets.countable*[*OF _ X*]) *auto*
then show (λ*x*. *if x* ∈ *X then undefined else f x*) ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_on_if assms continuous_intros*)
qed *auto*

lemma *borel_measurable_continuous_on*:
assumes *f*: *continuous_on UNIV f* **and** *g*: *g* ∈ *borel_measurable M*
shows (λ*x*. *f* (*g x*)) ∈ *borel_measurable M*
using *measurable_comp*[*OF g borel_measurable_continuous_onI*[*OF f*]] **by** (*simp add*: *comp_def*)

lemma *borel_measurable_continuous_on_indicator*:
fixes *f g* :: *'a*::*topological_space* ⇒ *'b*::*real_normed_vector*

shows *A* ∈ *sets borel* ⟹ *continuous_on A f* ⟹ (λ*x*. *indicator A x* $*_R$ *f x*) ∈ *borel_measurable borel*
by (*subst borel_measurable_restrict_space_iff*[*symmetric*])
(*auto intro*: *borel_measurable_continuous_on_restrict*)

lemma *borel_measurable_Pair*[*measurable* (*raw*)]:
fixes *f* :: *'a* ⇒ *'b*::*second_countable_topology* **and** *g* :: *'a* ⇒ *'c*::*second_countable_topology*
assumes *f*[*measurable*]: *f* ∈ *borel_measurable M*
assumes *g*[*measurable*]: *g* ∈ *borel_measurable M*
shows (λ*x*. (*f x*, *g x*)) ∈ *borel_measurable M*
proof (*subst borel_eq_countable_basis*)
let *?B* = *SOME B*::*'b set set*. *countable B* ∧ *topological_basis B*
let *?C* = *SOME B*::*'c set set*. *countable B* ∧ *topological_basis B*
let *?P* = (λ(*b*, *c*). *b* × *c*) ‘ (*?B* × *?C*)
show *countable ?P topological_basis ?P*
by (*auto intro*!: *countable_basis topological_basis_prod is_basis*)

show (λ*x*. (*f x*, *g x*)) ∈ *measurable M* (*sigma UNIV ?P*)
proof (*rule measurable_measure_of*)
fix *S* **assume** *S* ∈ *?P*
then obtain *b c* **where** *b* ∈ *?B c* ∈ *?C* **and** *S*: *S* = *b* × *c* **by** *auto*
then have *borel*: *open b open c*
by (*auto intro*: *is_basis topological_basis_open*)
have (λ*x*. (*f x*, *g x*)) −‘ *S* ∩ *space M* = (*f* −‘ *b* ∩ *space M*) ∩ (*g* −‘ *c* ∩ *space M*)
unfolding *S* **by** *auto*
also have ... ∈ *sets M*
using *borel* **by** *simp*
finally show (λ*x*. (*f x*, *g x*)) −‘ *S* ∩ *space M* ∈ *sets M* **.**
qed *auto*
qed

lemma *borel_measurable_continuous_Pair*:
fixes *f* :: *'a* ⇒ *'b*::*second_countable_topology* **and** *g* :: *'a* ⇒ *'c*::*second_countable_topology*
assumes [*measurable*]: *f* ∈ *borel_measurable M*
assumes [*measurable*]: *g* ∈ *borel_measurable M*
assumes *H*: *continuous_on UNIV* (λ*x*. *H* (*fst x*) (*snd x*))
shows (λ*x*. *H* (*f x*) (*g x*)) ∈ *borel_measurable M*
proof −
have *eq*: (λ*x*. *H* (*f x*) (*g x*)) = (λ*x*. (λ*x*. *H* (*fst x*) (*snd x*)) (*f x*, *g x*)) **by** *auto*
show *?thesis*
unfolding *eq* **by** (*rule borel_measurable_continuous_on*[*OF H*]) *auto*
qed

6.5.2 Borel spaces on order topologies

lemma [*measurable*]:
fixes *a b* :: *'a*::*linorder_topology*
shows *lessThan_borel*: {..< *a*} ∈ *sets borel*

and *greaterThan_borel*: $\{a <..\} \in$ *sets borel*
and *greaterThanLessThan_borel*: $\{a<..<b\} \in$ *sets borel*
and *atMost_borel*: $\{..a\} \in$ *sets borel*
and *atLeast_borel*: $\{a..\} \in$ *sets borel*
and *atLeastAtMost_borel*: $\{a..b\} \in$ *sets borel*
and *greaterThanAtMost_borel*: $\{a<..b\} \in$ *sets borel*
and *atLeastLessThan_borel*: $\{a..<b\} \in$ *sets borel*
unfolding *greaterThanAtMost_def atLeastLessThan_def*
by (*blast intro*: *borel_open borel_closed open_lessThan open_greaterThan open_greaterThanLessThan closed_atMost closed_atLeast closed_atLeastAtMost*)+

lemma *borel_Iio*:
borel = *sigma UNIV* (*range lessThan* :: *'a*::{*linorder_topology*, *second_countable_topology*} *set set*)
unfolding *second_countable_borel_measurable*[*OF open_generated_order*]
proof (*intro sigma_eqI sigma_sets_eqI*)
from *countable_dense_setE* **guess** *D* :: *'a set* **. note** *D* = *this*

interpret *L*: *sigma_algebra UNIV sigma_sets UNIV* (*range lessThan*)
by (*rule sigma_algebra_sigma_sets*) *simp*

fix *A* :: *'a set* **assume** $A \in$ *range lessThan* $\cup$ *range greaterThan*
then obtain *y* **where** $A = \{y <..\} \vee A = \{..< y\}$
by *blast*
then show $A \in$ *sigma_sets UNIV* (*range lessThan*)
proof
assume *A*: $A = \{y <..\}$
show *?thesis*
proof *cases*
assume $\forall x>y.\ \exists d.\ y < d \wedge d < x$
with *D*(*2*)[*of* $\{y <..< x\}$ **for** *x*] **have** $\forall x>y.\ \exists d\in D.\ y < d \wedge d < x$
by (*auto simp*: *set_eq_iff*)
then have $A = UNIV - (\bigcap d\in\{d\in D.\ y < d\}.\ \{..< d\})$
by (*auto simp*: *A*) (*metis less_asym*)
also have $\ldots \in$ *sigma_sets UNIV* (*range lessThan*)
using *D*(*1*) **by** (*intro L.Diff L.top L.countable_INT''*) *auto*
finally show *?thesis* **.**
next
assume $\neg (\forall x>y.\ \exists d.\ y < d \wedge d < x)$
then obtain *x* **where** $y < x\ \ \bigwedge d.\ y < d \Longrightarrow \neg\ d < x$
by *auto*
then have $A = UNIV - \{..< x\}$
unfolding *A* **by** (*auto simp*: *not_less*[*symmetric*])
also have $\ldots \in$ *sigma_sets UNIV* (*range lessThan*)
by *auto*
finally show *?thesis* **.**
qed
qed *auto*
qed *auto*

```
lemma borel_Ioi:
  borel = sigma UNIV (range greaterThan :: 'a::{linorder_topology, second_countable_topology}
set set)
  unfolding second_countable_borel_measurable[OF open_generated_order]
proof (intro sigma_eqI sigma_sets_eqI)
  from countable_dense_setE guess D :: 'a set . note D = this

  interpret L: sigma_algebra UNIV sigma_sets UNIV (range greaterThan)
    by (rule sigma_algebra_sigma_sets) simp

  fix A :: 'a set assume A ∈ range lessThan ∪ range greaterThan
  then obtain y where A = {y <..} ∨ A = {..< y}
    by blast
  then show A ∈ sigma_sets UNIV (range greaterThan)
  proof
    assume A: A = {..< y}
    show ?thesis
    proof cases
      assume ∀x<y. ∃d. x < d ∧ d < y
      with D(2)[of {x <..< y} for x] have ∀x<y. ∃d∈D. x < d ∧ d < y
        by (auto simp: set_eq_iff)
      then have A = UNIV − (⋂d∈{d∈D. d < y}. {d <..})
        by (auto simp: A) (metis less_asym)
      also have ... ∈ sigma_sets UNIV (range greaterThan)
        using D(1) by (intro L.Diff L.top L.countable_INT'') auto
      finally show ?thesis .
    next
      assume ¬ (∀x<y. ∃d. x < d ∧ d < y)
      then obtain x where x < y ⋀d. y > d ⟹ x ≥ d
        by (auto simp: not_less[symmetric])
      then have A = UNIV − {x <..}
        unfolding A Compl_eq_Diff_UNIV[symmetric] by auto
      also have ... ∈ sigma_sets UNIV (range greaterThan)
        by auto
      finally show ?thesis .
    qed
  qed auto
qed auto

lemma borel_measurableI_less:
  fixes f :: 'a ⇒ 'b::{linorder_topology, second_countable_topology}
  shows (⋀y. {x∈space M. f x < y} ∈ sets M) ⟹ f ∈ borel_measurable M
  unfolding borel_Iio
  by (rule measurable_measure_of) (auto simp: Int_def conj_commute)

lemma borel_measurableI_greater:
  fixes f :: 'a ⇒ 'b::{linorder_topology, second_countable_topology}
  shows (⋀y. {x∈space M. y < f x} ∈ sets M) ⟹ f ∈ borel_measurable M
```

unfolding *borel_Ioi*
by (*rule measurable_measure_of*) (*auto simp*: *Int_def conj_commute*)

lemma *borel_measurableI_le*:
fixes f :: $'a \Rightarrow 'b$::{*linorder_topology*, *second_countable_topology*}
shows $(\bigwedge y.\ \{x{\in}space\ M.\ f\ x \le y\} \in sets\ M) \Longrightarrow f \in borel\_measurable\ M$
by (*rule borel_measurableI_greater*) (*auto simp*: *not_le*[*symmetric*])

lemma *borel_measurableI_ge*:
fixes f :: $'a \Rightarrow 'b$::{*linorder_topology*, *second_countable_topology*}
shows $(\bigwedge y.\ \{x{\in}space\ M.\ y \le f\ x\} \in sets\ M) \Longrightarrow f \in borel\_measurable\ M$
by (*rule borel_measurableI_less*) (*auto simp*: *not_le*[*symmetric*])

lemma *borel_measurable_less*[*measurable*]:
fixes f :: $'a \Rightarrow 'b$::{*second_countable_topology*, *linorder_topology*}
assumes $f \in borel\_measurable\ M$
assumes $g \in borel\_measurable\ M$
shows $\{w \in space\ M.\ f\ w < g\ w\} \in sets\ M$
proof –
have $\{w \in space\ M.\ f\ w < g\ w\} = (\lambda x.\ (f\ x,\ g\ x))\ -`\ \{x.\ fst\ x < snd\ x\} \cap space$
M
by *auto*
also have $\ldots \in sets\ M$
by (*intro measurable_sets*[*OF borel_measurable_Pair borel_open*, *OF assms open_Collect_less*]
continuous_intros)
finally show *?thesis* **.**
qed

lemma
fixes f :: $'a \Rightarrow 'b$::{*second_countable_topology*, *linorder_topology*}
assumes f[*measurable*]: $f \in borel\_measurable\ M$
assumes g[*measurable*]: $g \in borel\_measurable\ M$
shows *borel_measurable_le*[*measurable*]: $\{w \in space\ M.\ f\ w \le g\ w\} \in sets\ M$
and *borel_measurable_eq*[*measurable*]: $\{w \in space\ M.\ f\ w = g\ w\} \in sets\ M$
and *borel_measurable_neq*: $\{w \in space\ M.\ f\ w \ne g\ w\} \in sets\ M$
unfolding *eq_iff not_less*[*symmetric*]
by *measurable*

lemma *borel_measurable_SUP*[*measurable* (*raw*)]:
fixes F :: _ ⇒ _ ⇒ _::{*complete_linorder*, *linorder_topology*, *second_countable_topology*}
assumes [*simp*]: *countable I*
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow F\ i \in borel\_measurable\ M$
shows $(\lambda x.\ SUP\ i{\in}I.\ F\ i\ x) \in borel\_measurable\ M$
by (*rule borel_measurableI_greater*) (*simp add*: *less_SUP_iff*)

lemma *borel_measurable_INF*[*measurable* (*raw*)]:
fixes F :: _ ⇒ _ ⇒ _::{*complete_linorder*, *linorder_topology*, *second_countable_topology*}
assumes [*simp*]: *countable I*
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow F\ i \in borel\_measurable\ M$

shows (λx. *INF i∈I. F i x*) ∈ *borel_measurable M*
by (*rule borel_measurableI_less*) (*simp add*: *INF_less_iff*)

lemma *borel_measurable_cSUP*[*measurable* (*raw*)]:
fixes *F* :: _ ⇒ _ ⇒ ′*a*::{*conditionally_complete_linorder*, *linorder_topology*, *second_countable_topology*}
assumes [*simp*]: *countable I*
assumes [*measurable*]: ⋀*i*. *i* ∈ *I* ⟹ *F i* ∈ *borel_measurable M*
assumes *bdd*: ⋀*x*. *x* ∈ *space M* ⟹ *bdd_above* ((λ*i*. *F i x*) ‘ *I*)
shows (λ*x*. *SUP i*∈*I*. *F i x*) ∈ *borel_measurable M*
proof *cases*
assume *I* = {} **then show** *?thesis*
unfolding ‹*I* = {}› *image_empty* **by** *simp*
next
assume *I* ≠ {}
show *?thesis*
proof (*rule borel_measurableI_le*)
fix *y*
have {*x* ∈ *space M*. ∀ *i*∈*I*. *F i x* ≤ *y*} ∈ *sets M*
by *measurable*
also have {*x* ∈ *space M*. ∀ *i*∈*I*. *F i x* ≤ *y*} = {*x* ∈ *space M*. (*SUP i*∈*I*. *F i x*) ≤ *y*}
by (*simp add*: *cSUP_le_iff* ‹*I* ≠ {}› *bdd cong*: *conj_cong*)
finally show {*x* ∈ *space M*. (*SUP i*∈*I*. *F i x*) ≤ *y*} ∈ *sets M* **.**
qed
qed

lemma *borel_measurable_cINF*[*measurable* (*raw*)]:
fixes *F* :: _ ⇒ _ ⇒ ′*a*::{*conditionally_complete_linorder*, *linorder_topology*, *second_countable_topology*}
assumes [*simp*]: *countable I*
assumes [*measurable*]: ⋀*i*. *i* ∈ *I* ⟹ *F i* ∈ *borel_measurable M*
assumes *bdd*: ⋀*x*. *x* ∈ *space M* ⟹ *bdd_below* ((λ*i*. *F i x*) ‘ *I*)
shows (λ*x*. *INF i*∈*I*. *F i x*) ∈ *borel_measurable M*
proof *cases*
assume *I* = {} **then show** *?thesis*
unfolding ‹*I* = {}› *image_empty* **by** *simp*
next
assume *I* ≠ {}
show *?thesis*
proof (*rule borel_measurableI_ge*)
fix *y*
have {*x* ∈ *space M*. ∀ *i*∈*I*. *y* ≤ *F i x*} ∈ *sets M*
by *measurable*
also have {*x* ∈ *space M*. ∀ *i*∈*I*. *y* ≤ *F i x*} = {*x* ∈ *space M*. *y* ≤ (*INF i*∈*I*. *F i x*)}
by (*simp add*: *le_cINF_iff* ‹*I* ≠ {}› *bdd cong*: *conj_cong*)
finally show {*x* ∈ *space M*. *y* ≤ (*INF i*∈*I*. *F i x*)} ∈ *sets M* **.**
qed

qed

lemma *borel_measurable_lfp*[*consumes 1*, *case_names continuity step*]:
fixes F :: ($'a \Rightarrow 'b$) $\Rightarrow$ ($'a \Rightarrow 'b$::{*complete_linorder*, *linorder_topology*, *second_countable_topology*})
assumes *sup_continuous F*
assumes $*$: $\bigwedge f$. $f \in$ *borel_measurable M* $\Longrightarrow$ $F\ f \in$ *borel_measurable M*
shows *lfp F* $\in$ *borel_measurable M*
proof –
{ fix *i* **have** ((*F* ^^ *i*) *bot*) $\in$ *borel_measurable M*
by (*induct i*) (*auto intro*!: $*$) **}**
then have (λx. *SUP i*. (*F* ^^ *i*) *bot x*) $\in$ *borel_measurable M*
by *measurable*
also have (λx. *SUP i*. (*F* ^^ *i*) *bot x*) = (*SUP i*. (*F* ^^ *i*) *bot*)
by (*auto simp add*: *image_comp*)
also have (*SUP i*. (*F* ^^ *i*) *bot*) = *lfp F*
by (*rule sup_continuous_lfp*[*symmetric*]) *fact*
finally show *?thesis* **.**
qed

lemma *borel_measurable_gfp*[*consumes 1*, *case_names continuity step*]:
fixes F :: ($'a \Rightarrow 'b$) $\Rightarrow$ ($'a \Rightarrow 'b$::{*complete_linorder*, *linorder_topology*, *second_countable_topology*})
assumes *inf_continuous F*
assumes $*$: $\bigwedge f$. $f \in$ *borel_measurable M* $\Longrightarrow$ $F\ f \in$ *borel_measurable M*
shows *gfp F* $\in$ *borel_measurable M*
proof –
{ fix *i* **have** ((*F* ^^ *i*) *top*) $\in$ *borel_measurable M*
by (*induct i*) (*auto intro*!: $*$ *simp*: *bot_fun_def*) **}**
then have (λx. *INF i*. (*F* ^^ *i*) *top x*) $\in$ *borel_measurable M*
by *measurable*
also have (λx. *INF i*. (*F* ^^ *i*) *top x*) = (*INF i*. (*F* ^^ *i*) *top*)
by (*auto simp add*: *image_comp*)
also have ... = *gfp F*
by (*rule inf_continuous_gfp*[*symmetric*]) *fact*
finally show *?thesis* **.**
qed

lemma *borel_measurable_max*[*measurable* (*raw*)]:
$f \in$ *borel_measurable M* $\Longrightarrow$ $g \in$ *borel_measurable M* $\Longrightarrow$ (λx. *max* (*g x*) (*f x*) :: $'b$::{*second_countable_topology*, *linorder_topology*}) $\in$ *borel_measurable M*
by (*rule borel_measurableI_less*) *simp*

lemma *borel_measurable_min*[*measurable* (*raw*)]:
$f \in$ *borel_measurable M* $\Longrightarrow$ $g \in$ *borel_measurable M* $\Longrightarrow$ (λx. *min* (*g x*) (*f x*) :: $'b$::{*second_countable_topology*, *linorder_topology*}) $\in$ *borel_measurable M*
by (*rule borel_measurableI_greater*) *simp*

lemma *borel_measurable_Min*[*measurable* (*raw*)]:

finite I ⟹ (⋀*i*. *i* ∈ *I* ⟹ *f i* ∈ *borel_measurable M*) ⟹ (λ*x*. *Min* ((λ*i*. *f i* *x*)'*I*) :: '*b*::{*second_countable_topology*, *linorder_topology*}) ∈ *borel_measurable M*
proof (*induct I rule*: *finite_induct*)
case (*insert i I*) **then show** *?case*
by (*cases I* = {}) *auto*
qed *auto*

lemma *borel_measurable_Max*[*measurable* (*raw*)]:
finite I ⟹ (⋀*i*. *i* ∈ *I* ⟹ *f i* ∈ *borel_measurable M*) ⟹ (λ*x*. *Max* ((λ*i*. *f i* *x*)'*I*) :: '*b*::{*second_countable_topology*, *linorder_topology*}) ∈ *borel_measurable M*
proof (*induct I rule*: *finite_induct*)
case (*insert i I*) **then show** *?case*
by (*cases I* = {}) *auto*
qed *auto*

lemma *borel_measurable_sup*[*measurable* (*raw*)]:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ (λ*x*. *sup* (*g x*) (*f x*) :: '*b*::{*lattice*, *second_countable_topology*, *linorder_topology*}) ∈ *borel_measurable M*
unfolding *sup_max* **by** *measurable*

lemma *borel_measurable_inf*[*measurable* (*raw*)]:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ (λ*x*. *inf* (*g x*) (*f x*) :: '*b*::{*lattice*, *second_countable_topology*, *linorder_topology*}) ∈ *borel_measurable M*
unfolding *inf_min* **by** *measurable*

lemma [*measurable* (*raw*)]:
fixes *f* :: *nat* ⇒ '*a* ⇒ '*b*::{*complete_linorder*, *second_countable_topology*, *linorder_topology*}
assumes ⋀*i*. *f i* ∈ *borel_measurable M*
shows *borel_measurable_liminf*: (λ*x*. *liminf* (λ*i*. *f i x*)) ∈ *borel_measurable M*
and *borel_measurable_limsup*: (λ*x*. *limsup* (λ*i*. *f i x*)) ∈ *borel_measurable M*
unfolding *liminf_SUP_INF limsup_INF_SUP* **using** *assms* **by** *auto*

lemma *measurable_convergent*[*measurable* (*raw*)]:
fixes *f* :: *nat* ⇒ '*a* ⇒ '*b*::{*complete_linorder*, *second_countable_topology*, *linorder_topology*}
assumes [*measurable*]: ⋀*i*. *f i* ∈ *borel_measurable M*
shows *Measurable.pred M* (λ*x*. *convergent* (λ*i*. *f i x*))
unfolding *convergent_ereal* **by** *measurable*

lemma *sets_Collect_convergent*[*measurable*]:
fixes *f* :: *nat* ⇒ '*a* ⇒ '*b*::{*complete_linorder*, *second_countable_topology*, *linorder_topology*}
assumes *f*[*measurable*]: ⋀*i*. *f i* ∈ *borel_measurable M*
shows {*x*∈*space M*. *convergent* (λ*i*. *f i x*)} ∈ *sets M*
by *measurable*

lemma *borel_measurable_lim*[*measurable* (*raw*)]:
fixes *f* :: *nat* ⇒ '*a* ⇒ '*b*::{*complete_linorder*, *second_countable_topology*, *linorder_topology*}
assumes [*measurable*]: ⋀*i*. *f i* ∈ *borel_measurable M*
shows (λ*x*. *lim* (λ*i*. *f i x*)) ∈ *borel_measurable M*
proof –

have $\bigwedge x.$ *lim* $(\lambda i.\ f\ i\ x) =$ (*if convergent* $(\lambda i.\ f\ i\ x)$ *then limsup* $(\lambda i.\ f\ i\ x)$ *else* (*THE i. False*))
  by (*simp add*: *lim_def convergent_def convergent_limsup_cl*)
 then show *?thesis*
  by *simp*
qed

lemma *borel_measurable_LIMSEQ_order*:
 fixes $u :: nat \Rightarrow 'a \Rightarrow 'b$::{*complete_linorder*, *second_countable_topology*, *linorder_topology*}
 assumes u': $\bigwedge x.\ x \in space\ M \Longrightarrow (\lambda i.\ u\ i\ x) \longrightarrow u'\ x$
 and u: $\bigwedge i.\ u\ i \in$ *borel_measurable M*
 shows $u' \in$ *borel_measurable M*
proof −
 have $\bigwedge x.\ x \in space\ M \Longrightarrow u'\ x =$ *liminf* $(\lambda n.\ u\ n\ x)$
  using u' **by** (*simp add*: *lim_imp_Liminf*[*symmetric*])
 with u **show** *?thesis* **by** (*simp cong*: *measurable_cong*)
qed

6.5.3 Borel spaces on topological monoids

lemma *borel_measurable_add*[*measurable* (*raw*)]:
 fixes $f\ g :: 'a \Rightarrow 'b$::{*second_countable_topology*, *topological_monoid_add*}
 assumes f: $f \in$ *borel_measurable M*
 assumes g: $g \in$ *borel_measurable M*
 shows $(\lambda x.\ f\ x + g\ x) \in$ *borel_measurable M*
 using *f g* **by** (*rule borel_measurable_continuous_Pair*) (*intro continuous_intros*)

lemma *borel_measurable_sum*[*measurable* (*raw*)]:
 fixes $f :: 'c \Rightarrow 'a \Rightarrow 'b$::{*second_countable_topology*, *topological_comm_monoid_add*}
 assumes $\bigwedge i.\ i \in S \Longrightarrow f\ i \in$ *borel_measurable M*
 shows $(\lambda x.\ \sum i{\in}S.\ f\ i\ x) \in$ *borel_measurable M*
proof *cases*
 assume *finite S*
 thus *?thesis* **using** *assms* **by** *induct auto*
qed *simp*

lemma *borel_measurable_suminf_order*[*measurable* (*raw*)]:
 fixes $f :: nat \Rightarrow 'a \Rightarrow 'b$::{*complete_linorder*, *second_countable_topology*, *linorder_topology*, *topological_comm_monoid_add*}
 assumes f[*measurable*]: $\bigwedge i.\ f\ i \in$ *borel_measurable M*
 shows $(\lambda x.$ *suminf* $(\lambda i.\ f\ i\ x)) \in$ *borel_measurable M*
 unfolding *suminf_def sums_def*[*abs_def*] *lim_def*[*symmetric*] **by** *simp*

6.5.4 Borel spaces on Euclidean spaces

lemma *borel_measurable_inner*[*measurable* (*raw*)]:
 fixes $f\ g :: 'a \Rightarrow 'b$::{*second_countable_topology*, *real_inner*}
 assumes $f \in$ *borel_measurable M*
 assumes $g \in$ *borel_measurable M*
 shows $(\lambda x.\ f\ x \cdot g\ x) \in$ *borel_measurable M*

```
  using assms
  by (rule borel_measurable_continuous_Pair) (intro continuous_intros)

notation
  eucl_less (infix <e 50)

lemma box_oc: {x. a <e x ∧ x ≤ b} = {x. a <e x} ∩ {..b}
  and box_co: {x. a ≤ x ∧ x <e b} = {a..} ∩ {x. x <e b}
  by auto

lemma eucl_ivals[measurable]:
  fixes a b :: 'a::ordered_euclidean_space
  shows {x. x <e a} ∈ sets borel
    and {x. a <e x} ∈ sets borel
    and {..a} ∈ sets borel
    and {a..} ∈ sets borel
    and {a..b} ∈ sets borel
    and  {x. a <e x ∧ x ≤ b} ∈ sets borel
    and {x. a ≤ x ∧  x <e b} ∈ sets borel
  unfolding box_oc box_co
  by (auto intro: borel_open borel_closed)

lemma
  fixes i :: 'a::{second_countable_topology, real_inner}
  shows hafspace_less_borel: {x. a < x • i} ∈ sets borel
    and hafspace_greater_borel: {x. x • i < a} ∈ sets borel
    and hafspace_less_eq_borel: {x. a ≤ x • i} ∈ sets borel
    and hafspace_greater_eq_borel: {x. x • i ≤ a} ∈ sets borel
  by simp_all

lemma borel_eq_box:
  borel = sigma UNIV (range (λ (a, b). box a b :: 'a :: euclidean_space set))
    (is _ = ?SIGMA)
proof (rule borel_eq_sigmaI1[OF borel_def])
  fix M :: 'a set assume M ∈ {S. open S}
  then have open M by simp
  show M ∈ ?SIGMA
    apply (subst open_UNION_box[OF ‹open M›])
    apply (safe intro!: sets.countable_UN' countable_PiE countable_Collect)
    apply (auto intro: countable_rat)
    done
qed (auto simp: box_def)

lemma halfspace_gt_in_halfspace:
  assumes i: i ∈ A
  shows {x::'a. a < x • i} ∈
   sigma_sets UNIV ((λ (a, i). {x::'a::euclidean_space. x • i < a}) ` (UNIV × A))
  (is ?set ∈ ?SIGMA)
proof -
```

interpret *sigma_algebra UNIV ?SIGMA*
by (*intro sigma_algebra_sigma_sets*) *simp_all*
have ∗: *?set* = (⋃ *n*. *UNIV* − {*x*::′*a*. *x* · *i* < *a* + *1* / *real* (*Suc n*)})
proof (*safe*, *simp_all add*: *not_less del*: *of_nat_Suc*)
fix *x* :: ′*a* **assume** *a* < *x* · *i*
with *reals_Archimedean*[*of x* · *i* − *a*]
obtain *n* **where** *a* + *1* / *real* (*Suc n*) < *x* · *i*
by (*auto simp*: *field_simps*)
then show ∃ *n*. *a* + *1* / *real* (*Suc n*) ≤ *x* · *i*
by (*blast intro*: *less_imp_le*)
next
fix *x n*
have *a* < *a* + *1* / *real* (*Suc n*) **by** *auto*
also assume . . . ≤ *x*
finally show *a* < *x* **.**
qed
show *?set* ∈ *?SIGMA* **unfolding** ∗
by (*auto intro*!: *Diff sigma_sets_Inter i*)
qed

lemma *borel_eq_halfspace_less*:
borel = *sigma UNIV* ((λ(*a*, *i*). {*x*::′*a*::*euclidean_space*. *x* · *i* < *a*}) ‘ (*UNIV* × *Basis*))
(**is** _ = *?SIGMA*)
proof (*rule borel_eq_sigmaI2*[*OF borel_eq_box*])
fix *a b* :: ′*a*
have *box a b* = {*x*∈*space ?SIGMA*. ∀ *i*∈*Basis*. *a* · *i* < *x* · *i* ∧ *x* · *i* < *b* · *i*}
by (*auto simp*: *box_def*)
also have . . . ∈ *sets ?SIGMA*
by (*intro sets.sets_Collect_conj sets.sets_Collect_finite_All sets.sets_Collect_const*)
(*auto intro*!: *halfspace_gt_in_halfspace countable_PiE countable_rat*)
finally show *box a b* ∈ *sets ?SIGMA* **.**
qed *auto*

lemma *borel_eq_halfspace_le*:
borel = *sigma UNIV* ((λ (*a*, *i*). {*x*::′*a*::*euclidean_space*. *x* · *i* ≤ *a*}) ‘ (*UNIV* × *Basis*))
(**is** _ = *?SIGMA*)
proof (*rule borel_eq_sigmaI2*[*OF borel_eq_halfspace_less*])
fix *a* :: *real* **and** *i* :: ′*a* **assume** (*a*, *i*) ∈ *UNIV* × *Basis*
then have *i*: *i* ∈ *Basis* **by** *auto*
have ∗: {*x*::′*a*. *x*·*i* < *a*} = (⋃ *n*. {*x*. *x*·*i* ≤ *a* − *1*/*real* (*Suc n*)})
proof (*safe*, *simp_all del*: *of_nat_Suc*)
fix *x*::′*a* **assume** ∗: *x*·*i* < *a*
with *reals_Archimedean*[*of a* − *x*·*i*]
obtain *n* **where** *x* · *i* < *a* − *1* / (*real* (*Suc n*))
by (*auto simp*: *field_simps*)
then show ∃ *n*. *x* · *i* ≤ *a* − *1* / (*real* (*Suc n*))
by (*blast intro*: *less_imp_le*)

```
  next
    fix x::'a and n
    assume x·i ≤ a − 1 / real (Suc n)
    also have ... < a by auto
    finally show x·i < a .
  qed
  show {x. x·i < a} ∈ ?SIGMA unfolding *
    by (intro sets.countable_UN) (auto intro: i)
qed auto

lemma borel_eq_halfspace_ge:
  borel = sigma UNIV ((λ (a, i). {x::'a::euclidean_space. a ≤ x · i}) ‘ (UNIV ×
Basis))
  (is _ = ?SIGMA)
proof (rule borel_eq_sigmaI2[OF borel_eq_halfspace_less])
  fix a :: real and i :: 'a assume i: (a, i) ∈ UNIV × Basis
  have *: {x::'a. x·i < a} = space ?SIGMA − {x::'a. a ≤ x·i} by auto
  show {x. x·i < a} ∈ ?SIGMA unfolding *
    using i by (intro sets.compl_sets) auto
qed auto

lemma borel_eq_halfspace_greater:
  borel = sigma UNIV ((λ (a, i). {x::'a::euclidean_space. a < x · i}) ‘ (UNIV ×
Basis))
  (is _ = ?SIGMA)
proof (rule borel_eq_sigmaI2[OF borel_eq_halfspace_le])
  fix a :: real and i :: 'a assume (a, i) ∈ (UNIV × Basis)
  then have i: i ∈ Basis by auto
  have *: {x::'a. x·i ≤ a} = space ?SIGMA − {x::'a. a < x·i} by auto
  show {x. x·i ≤ a} ∈ ?SIGMA unfolding *
    by (intro sets.compl_sets) (auto intro: i)
qed auto

lemma borel_eq_atMost:
  borel = sigma UNIV (range (λa. {..a::'a::ordered_euclidean_space}))
  (is _ = ?SIGMA)
proof (rule borel_eq_sigmaI4[OF borel_eq_halfspace_le])
  fix a :: real and i :: 'a assume (a, i) ∈ UNIV × Basis
  then have i ∈ Basis by auto
  then have *: {x::'a. x·i ≤ a} = (⋃k::nat. {.. (∑n∈Basis. (if n = i then a else
real k)*R n)})
  proof (safe, simp_all add: eucl_le[where 'a='a] split: if_split_asm)
    fix x :: 'a
    from real_arch_simple[of Max ((λi. x·i)‘Basis)] guess k::nat ..
    then have ⋀i. i ∈ Basis ⟹ x·i ≤ real k
      by (subst (asm) Max_le_iff) auto
    then show ∃k::nat. ∀ia∈Basis. ia ≠ i ⟶ x · ia ≤ real k
      by (auto intro!: exI[of _ k])
  qed
```

show {*x*. *x*·*i* ≤ *a*} ∈ *?SIGMA* **unfolding** ∗
  by (*intro sets.countable_UN*) *auto*
qed *auto*

lemma *borel_eq_greaterThan*:
  borel = *sigma UNIV* (*range* (λ*a*::′*a*::*ordered_euclidean_space*. {*x*. *a* <*e* *x*}))
  (**is** _ = *?SIGMA*)
proof (*rule borel_eq_sigmaI4*[*OF borel_eq_halfspace_le*])
  fix *a* :: *real* **and** *i* :: ′*a* **assume** (*a*, *i*) ∈ *UNIV* × *Basis*
  then have *i*: *i* ∈ *Basis* **by** *auto*
  have {*x*::′*a*. *x*·*i* ≤ *a*} = *UNIV* − {*x*::′*a*. *a* < *x*·*i*} **by** *auto*
  also have ∗: {*x*::′*a*. *a* < *x*·*i*} =
    (⋃*k*::*nat*. {*x*. (∑*n*∈*Basis*. (*if n* = *i then a else* −*real k*) ∗*R* *n*) <*e* *x*}) **using**
i
  proof (*safe*, *simp_all add*: *eucl_less_def split*: *if_split_asm*)
    fix *x* :: ′*a*
    from *reals_Archimedean2*[*of Max* ((λ*i*. −*x*·*i*)‘*Basis*)]
    guess *k*::*nat* .. **note** *k* = *this*
    { **fix** *i* :: ′*a* **assume** *i* ∈ *Basis*
      then have −*x*·*i* < *real k*
        using *k* **by** (*subst* (*asm*) *Max_less_iff*) *auto*
      then have − *real k* < *x*·*i* **by** *simp* }
    then show ∃*k*::*nat*. ∀*ia*∈*Basis*. *ia* ≠ *i* ⟶ −*real k* < *x* · *ia*
      by (*auto intro*!: *exI*[*of* _ *k*])
  qed
  finally show {*x*. *x*·*i* ≤ *a*} ∈ *?SIGMA*
    apply (*simp only*:)
    apply (*intro sets.countable_UN sets.Diff*)
    apply (*auto intro*: *sigma_sets_top*)
    done
qed *auto*

lemma *borel_eq_lessThan*:
  borel = *sigma UNIV* (*range* (λ*a*::′*a*::*ordered_euclidean_space*. {*x*. *x* <*e* *a*}))
  (**is** _ = *?SIGMA*)
proof (*rule borel_eq_sigmaI4*[*OF borel_eq_halfspace_ge*])
  fix *a* :: *real* **and** *i* :: ′*a* **assume** (*a*, *i*) ∈ *UNIV* × *Basis*
  then have *i*: *i* ∈ *Basis* **by** *auto*
  have {*x*::′*a*. *a* ≤ *x*·*i*} = *UNIV* − {*x*::′*a*. *x*·*i* < *a*} **by** *auto*
  also have ∗: {*x*::′*a*. *x*·*i* < *a*} = (⋃*k*::*nat*. {*x*. *x* <*e* (∑*n*∈*Basis*. (*if n* = *i then*
a else real k) ∗*R* *n*)}) **using** ‹*i*∈ *Basis*›
  proof (*safe*, *simp_all add*: *eucl_less_def split*: *if_split_asm*)
    fix *x* :: ′*a*
    from *reals_Archimedean2*[*of Max* ((λ*i*. *x*·*i*)‘*Basis*)]
    guess *k*::*nat* .. **note** *k* = *this*
    { **fix** *i* :: ′*a* **assume** *i* ∈ *Basis*
      then have *x*·*i* < *real k*
        using *k* **by** (*subst* (*asm*) *Max_less_iff*) *auto*
      then have *x*·*i* < *real k* **by** *simp* }

```
    then show ∃k::nat. ∀ia∈Basis. ia ≠ i ⟶ x · ia < real k
      by (auto intro!: exI[of _ k])
  qed
  finally show {x. a ≤ x•i} ∈ ?SIGMA
    apply (simp only:)
    apply (intro sets.countable_UN sets.Diff)
    apply (auto intro: sigma_sets_top )
    done
qed auto

lemma borel_eq_atLeastAtMost:
  borel = sigma UNIV (range (λ(a,b). {a..b} ::'a::ordered_euclidean_space set))
  (is _ = ?SIGMA)
proof (rule borel_eq_sigmaI5[OF borel_eq_atMost])
  fix a::'a
  have *: {..a} = (⋃n::nat. {− real n *R One .. a})
  proof (safe, simp_all add: eucl_le[where 'a='a])
    fix x :: 'a
    from real_arch_simple[of Max ((λi. − x•i)`Basis)]
    guess k::nat .. note k = this
    { fix i :: 'a assume i ∈ Basis
      with k have − x•i ≤ real k
        by (subst (asm) Max_le_iff) (auto simp: field_simps)
      then have − real k ≤ x•i by simp }
    then show ∃n::nat. ∀i∈Basis. − real n ≤ x · i
      by (auto intro!: exI[of _ k])
  qed
  show {..a} ∈ ?SIGMA unfolding *
    by (intro sets.countable_UN)
       (auto intro!: sigma_sets_top)
qed auto

lemma borel_set_induct[consumes 1, case_names empty interval compl union]:
  assumes A ∈ sets borel
  assumes empty: P {} and int: ⋀a b. a ≤ b ⟹ P {a..b} and compl: ⋀A. A ∈
sets borel ⟹ P A ⟹ P (−A) and
          un: ⋀f. disjoint_family f ⟹ (⋀i. f i ∈ sets borel) ⟹ (⋀i. P (f i)) ⟹
P (⋃i::nat. f i)
  shows P (A::real set)
proof −
  let ?G = range (λ(a,b). {a..b::real})
  have Int_stable ?G ?G ⊆ Pow UNIV A ∈ sigma_sets UNIV ?G
      using assms(1) by (auto simp add: borel_eq_atLeastAtMost Int_stable_def)
  thus ?thesis
  proof (induction rule: sigma_sets_induct_disjoint)
    case (union f)
    from union.hyps(2) have ⋀i. f i ∈ sets borel by (auto simp: borel_eq_atLeastAtMost)
      with union show ?case by (auto intro: un)
  next
```

case (*basic A*)
then obtain *a b* **where** $A = \{a .. b\}$ **by** *auto*
then show *?case*
by (*cases* $a \le b$) (*auto intro*: *int empty*)
qed (*auto intro*: *empty compl simp*: *Compl_eq_Diff_UNIV*[*symmetric*] *borel_eq_atLeastAtMost*)
qed

lemma *borel_sigma_sets_Ioc*: *borel* = *sigma UNIV* (*range* ($\lambda(a, b)$. {*a* <.. *b*::*real*}))
proof (*rule borel_eq_sigmaI5*[*OF borel_eq_atMost*])
fix *i* :: *real*
have {..*i*} = ($\bigcup$*j*::*nat*. {−*j* <.. *i*})
by (*auto simp*: *minus_less_iff reals_Archimedean2*)
also have ... ∈ *sets* (*sigma UNIV* (*range* ($\lambda(i, j)$. {*i*<..*j*})))
by (*intro sets.countable_nat_UN*) *auto*
finally show {..*i*} ∈ *sets* (*sigma UNIV* (*range* ($\lambda(i, j)$. {*i*<..*j*}))) **.**
qed *simp*

lemma *eucl_lessThan*: {*x*::*real*. *x* <*e a*} = *lessThan a*
by (*simp add*: *eucl_less_def lessThan_def*)

lemma *borel_eq_atLeastLessThan*:
borel = *sigma UNIV* (*range* ($\lambda(a, b)$. {*a* ..< *b* :: *real*})) (**is** _ = *?SIGMA*)
proof (*rule borel_eq_sigmaI5*[*OF borel_eq_lessThan*])
have *move_uminus*: $\bigwedge$*x y*::*real*. $-x \le y \longleftrightarrow -y \le x$ **by** *auto*
fix *x* :: *real*
have {..<*x*} = ($\bigcup$*i*::*nat*. {−*real i* ..< *x*})
by (*auto simp*: *move_uminus real_arch_simple*)
then show {*y*. *y* <*e x*} ∈ *?SIGMA*
by (*auto intro*: *sigma_sets.intros*(*2*−) *simp*: *eucl_lessThan*)
qed *auto*

lemma *borel_measurable_halfspacesI*:
fixes *f* :: *'a* ⇒ *'c*::*euclidean_space*
assumes *F*: *borel* = *sigma UNIV* (*F* ' (*UNIV* × *Basis*))
and *S_eq*: $\bigwedge$*a i*. *S a i* = *f* −' *F* (*a*,*i*) ∩ *space M*
shows *f* ∈ *borel_measurable M* = (∀ *i*∈*Basis*. ∀ *a*::*real*. *S a i* ∈ *sets M*)
proof *safe*
fix *a* :: *real* **and** *i* :: *'b* **assume** *i*: *i* ∈ *Basis* **and** *f*: *f* ∈ *borel_measurable M*
then show *S a i* ∈ *sets M* **unfolding** *assms*
by (*auto intro*!: *measurable_sets simp*: *assms*(*1*))
next
assume *a*: ∀ *i*∈*Basis*. ∀ *a*. *S a i* ∈ *sets M*
then show *f* ∈ *borel_measurable M*
by (*auto intro*!: *measurable_measure_of simp*: *S_eq F*)
qed

lemma *borel_measurable_iff_halfspace_le*:
fixes *f* :: *'a* ⇒ *'c*::*euclidean_space*
shows *f* ∈ *borel_measurable M* = (∀ *i*∈*Basis*. ∀ *a*. {*w* ∈ *space M*. *f w* • *i* ≤ *a*}

$\in$ *sets M*)
by (*rule borel_measurable_halfspacesI*[*OF borel_eq_halfspace_le*]) *auto*

lemma *borel_measurable_iff_halfspace_less*:
fixes f :: $'a \Rightarrow 'c$::*euclidean_space*
shows $f \in$ *borel_measurable M* $\longleftrightarrow$ ($\forall i \in Basis. \forall a. \{w \in space\ M. f\ w \cdot i < a\}$ $\in$ *sets M*)
by (*rule borel_measurable_halfspacesI*[*OF borel_eq_halfspace_less*]) *auto*

lemma *borel_measurable_iff_halfspace_ge*:
fixes f :: $'a \Rightarrow 'c$::*euclidean_space*
shows $f \in$ *borel_measurable M* = ($\forall i \in Basis. \forall a. \{w \in space\ M. a \leq f\ w \cdot i\}$ $\in$ *sets M*)
by (*rule borel_measurable_halfspacesI*[*OF borel_eq_halfspace_ge*]) *auto*

lemma *borel_measurable_iff_halfspace_greater*:
fixes f :: $'a \Rightarrow 'c$::*euclidean_space*
shows $f \in$ *borel_measurable M* $\longleftrightarrow$ ($\forall i \in Basis. \forall a. \{w \in space\ M. a < f\ w \cdot i\} \in$ *sets M*)
by (*rule borel_measurable_halfspacesI*[*OF borel_eq_halfspace_greater*]) *auto*

lemma *borel_measurable_iff_le*:
(f::$'a \Rightarrow real$) $\in$ *borel_measurable M* = ($\forall a. \{w \in space\ M. f\ w \leq a\} \in$ *sets M*)
using *borel_measurable_iff_halfspace_le*[**where** $'c$=*real*] **by** *simp*

lemma *borel_measurable_iff_less*:
(f::$'a \Rightarrow real$) $\in$ *borel_measurable M* = ($\forall a. \{w \in space\ M. f\ w < a\} \in$ *sets M*)
using *borel_measurable_iff_halfspace_less*[**where** $'c$=*real*] **by** *simp*

lemma *borel_measurable_iff_ge*:
(f::$'a \Rightarrow real$) $\in$ *borel_measurable M* = ($\forall a. \{w \in space\ M. a \leq f\ w\} \in$ *sets M*)
using *borel_measurable_iff_halfspace_ge*[**where** $'c$=*real*]
by *simp*

lemma *borel_measurable_iff_greater*:
(f::$'a \Rightarrow real$) $\in$ *borel_measurable M* = ($\forall a. \{w \in space\ M. a < f\ w\} \in$ *sets M*)
using *borel_measurable_iff_halfspace_greater*[**where** $'c$=*real*] **by** *simp*

lemma *borel_measurable_euclidean_space*:
fixes f :: $'a \Rightarrow 'c$::*euclidean_space*
shows $f \in$ *borel_measurable M* $\longleftrightarrow$ ($\forall i \in Basis.$ ($\lambda x. f\ x \cdot i$) $\in$ *borel_measurable M*)
proof *safe*
assume f: $\forall i \in Basis.$ ($\lambda x. f\ x \cdot i$) $\in$ *borel_measurable M*
then show $f \in$ *borel_measurable M*
by (*subst borel_measurable_iff_halfspace_le*) *auto*
qed *auto*

6.5.5 Borel measurable operators

lemma *borel_measurable_norm*[*measurable*]: *norm* ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_sgn* [*measurable*]: (*sgn*::′*a*::*real_normed_vector* ⇒ ′*a*) ∈ *borel_measurable borel*
by (*rule borel_measurable_continuous_countable_exceptions*[**where** *X*={*0*}])
(*auto intro*!: *continuous_on_sgn continuous_on_id*)

lemma *borel_measurable_uminus*[*measurable* (*raw*)]:
fixes *g* :: ′*a* ⇒ ′*b*::{*second_countable_topology*, *real_normed_vector*}
assumes *g*: *g* ∈ *borel_measurable M*
shows (λ*x*. − *g x*) ∈ *borel_measurable M*
by (*rule borel_measurable_continuous_on*[*OF* _ *g*]) (*intro continuous_intros*)

lemma *borel_measurable_diff*[*measurable* (*raw*)]:
fixes *f* :: ′*a* ⇒ ′*b*::{*second_countable_topology*, *real_normed_vector*}
assumes *f*: *f* ∈ *borel_measurable M*
assumes *g*: *g* ∈ *borel_measurable M*
shows (λ*x*. *f x* − *g x*) ∈ *borel_measurable M*
using *borel_measurable_add* [*of f M* − *g*] *assms* **by** (*simp add*: *fun_Compl_def*)

lemma *borel_measurable_times*[*measurable* (*raw*)]:
fixes *f* :: ′*a* ⇒ ′*b*::{*second_countable_topology*, *real_normed_algebra*}
assumes *f*: *f* ∈ *borel_measurable M*
assumes *g*: *g* ∈ *borel_measurable M*
shows (λ*x*. *f x* ∗ *g x*) ∈ *borel_measurable M*
using *f g* **by** (*rule borel_measurable_continuous_Pair*) (*intro continuous_intros*)

lemma *borel_measurable_prod*[*measurable* (*raw*)]:
fixes *f* :: ′*c* ⇒ ′*a* ⇒ ′*b*::{*second_countable_topology*, *real_normed_field*}
assumes ⋀*i*. *i* ∈ *S* ⟹ *f i* ∈ *borel_measurable M*
shows (λ*x*. ∏ *i*∈*S*. *f i x*) ∈ *borel_measurable M*
proof *cases*
assume *finite S*
thus *?thesis* **using** *assms* **by** *induct auto*
qed *simp*

lemma *borel_measurable_dist*[*measurable* (*raw*)]:
fixes *g f* :: ′*a* ⇒ ′*b*::{*second_countable_topology*, *metric_space*}
assumes *f*: *f* ∈ *borel_measurable M*
assumes *g*: *g* ∈ *borel_measurable M*
shows (λ*x*. *dist* (*f x*) (*g x*)) ∈ *borel_measurable M*
using *f g* **by** (*rule borel_measurable_continuous_Pair*) (*intro continuous_intros*)

lemma *borel_measurable_scaleR*[*measurable* (*raw*)]:
fixes *g* :: ′*a* ⇒ ′*b*::{*second_countable_topology*, *real_normed_vector*}
assumes *f*: *f* ∈ *borel_measurable M*
assumes *g*: *g* ∈ *borel_measurable M*

shows (λx. f x $*_R$ g x) ∈ borel_measurable M
using f g **by** (rule borel_measurable_continuous_Pair) (intro continuous_intros)

lemma borel_measurable_uminus_eq [simp]:
fixes f :: 'a ⇒ 'b::{second_countable_topology, real_normed_vector}
shows (λx. − f x) ∈ borel_measurable M ⟷ f ∈ borel_measurable M (**is** ?l = ?r)
proof
assume ?l **from** borel_measurable_uminus[OF this] **show** ?r **by** simp
qed auto

lemma affine_borel_measurable_vector:
fixes f :: 'a ⇒ 'x::real_normed_vector
assumes f ∈ borel_measurable M
shows (λx. a + b $*_R$ f x) ∈ borel_measurable M
proof (rule borel_measurableI)
fix S :: 'x set **assume** open S
show (λx. a + b $*_R$ f x) −' S ∩ space M ∈ sets M
proof cases
assume b ≠ 0
with ‹open S› **have** open ((λx. (− a + x) $/_R$ b) ' S) (**is** open ?S)
using open_affinity [of S inverse b − a $/_R$ b]
by (auto simp: algebra_simps)
hence ?S ∈ sets borel **by** auto
moreover
from ‹b ≠ 0› **have** (λx. a + b $*_R$ f x) −' S = f −' ?S
apply auto **by** (rule_tac x=a + b $*_R$ f x **in** image_eqI, simp_all)
ultimately show ?thesis **using** assms **unfolding** in_borel_measurable_borel
by auto
qed simp
qed

lemma borel_measurable_const_scaleR[measurable (raw)]:
f ∈ borel_measurable M ⟹ (λx. b $*_R$ f x ::'a::real_normed_vector) ∈ borel_measurable M
using affine_borel_measurable_vector[of f M 0 b] **by** simp

lemma borel_measurable_const_add[measurable (raw)]:
f ∈ borel_measurable M ⟹ (λx. a + f x ::'a::real_normed_vector) ∈ borel_measurable M
using affine_borel_measurable_vector[of f M a 1] **by** simp

lemma borel_measurable_inverse[measurable (raw)]:
fixes f :: 'a ⇒ 'b::real_normed_div_algebra
assumes f: f ∈ borel_measurable M
shows (λx. inverse (f x)) ∈ borel_measurable M
apply (rule measurable_compose[OF f])
apply (rule borel_measurable_continuous_countable_exceptions[of {0}])
apply (auto intro!: continuous_on_inverse continuous_on_id)

done

lemma *borel_measurable_divide*[*measurable* (*raw*)]:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹
(λ*x*. *f x* / *g x*::′*b*::{*second_countable_topology*, *real_normed_div_algebra*}) ∈ *borel_measurable M*
by (*simp add*: *divide_inverse*)

lemma *borel_measurable_abs*[*measurable* (*raw*)]:
f ∈ *borel_measurable M* ⟹ (λ*x*. |*f x* :: *real*|) ∈ *borel_measurable M*
unfolding *abs_real_def* **by** *simp*

lemma *borel_measurable_nth*[*measurable* (*raw*)]:
(λ*x*::*real*^′*n*. *x* $ *i*) ∈ *borel_measurable borel*
by (*simp add*: *cart_eq_inner_axis*)

lemma *convex_measurable*:
fixes *A* :: ′*a* :: *euclidean_space set*
shows *X* ∈ *borel_measurable M* ⟹ *X* ‘ *space M* ⊆ *A* ⟹ *open A* ⟹ *convex_on A q* ⟹
(λ*x*. *q* (*X x*)) ∈ *borel_measurable M*
by (*rule measurable_compose*[**where** *f*=*X* **and** *N*=*restrict_space borel A*])
(*auto intro*!: *borel_measurable_continuous_on_restrict convex_on_continuous measurable_restrict_space2*)

lemma *borel_measurable_ln*[*measurable* (*raw*)]:
assumes *f*: *f* ∈ *borel_measurable M*
shows (λ*x*. *ln* (*f x* :: *real*)) ∈ *borel_measurable M*
apply (*rule measurable_compose*[*OF f*])
apply (*rule borel_measurable_continuous_countable_exceptions*[*of* {*0*}])
apply (*auto intro*!: *continuous_on_ln continuous_on_id*)
done

lemma *borel_measurable_log*[*measurable* (*raw*)]:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ (λ*x*. *log* (*g x*) (*f x*)) ∈ *borel_measurable M*
unfolding *log_def* **by** *auto*

lemma *borel_measurable_exp*[*measurable*]:
(*exp*::′*a*::{*real_normed_field*,*banach*}⇒′*a*) ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_at_imp_continuous_on ballI isCont_exp*)

lemma *measurable_real_floor*[*measurable*]:
(*floor* :: *real* ⇒ *int*) ∈ *measurable borel* (*count_space UNIV*)
proof −
have ⋀*a x*. ⌊*x*⌋ = *a* ⟷ (*real_of_int a* ≤ *x* ∧ *x* < *real_of_int* (*a* + *1*))
by (*auto intro*: *floor_eq2*)
then show *?thesis*

by (*auto simp*: *vimage_def measurable_count_space_eq2_countable*)
qed

lemma *measurable_real_ceiling*[*measurable*]:
(*ceiling* :: *real* ⇒ *int*) ∈ *measurable borel* (*count_space UNIV*)
unfolding *ceiling_def*[*abs_def*] **by** *simp*

lemma *borel_measurable_real_floor*: (λ*x*::*real*. *real_of_int* ⌊*x*⌋) ∈ *borel_measurable borel*
by *simp*

lemma *borel_measurable_root* [*measurable*]: *root n* ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_sqrt* [*measurable*]: *sqrt* ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_power* [*measurable* (*raw*)]:
fixes *f* :: _ ⇒ '*b*::{*power*,*real_normed_algebra*}
assumes *f*: *f* ∈ *borel_measurable M*
shows (λ*x*. (*f x*) ^ *n*) ∈ *borel_measurable M*
by (*intro borel_measurable_continuous_on* [*OF* _ *f*] *continuous_intros*)

lemma *borel_measurable_Re* [*measurable*]: *Re* ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_Im* [*measurable*]: *Im* ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_of_real* [*measurable*]: (*of_real* :: _ ⇒ (_::*real_normed_algebra*)) ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_sin* [*measurable*]: (*sin* :: _ ⇒ (_::{*real_normed_field*,*banach*})) ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_cos* [*measurable*]: (*cos* :: _ ⇒ (_::{*real_normed_field*,*banach*})) ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_arctan* [*measurable*]: *arctan* ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_complex_iff*:
f ∈ *borel_measurable M* ⟷
(λ*x*. *Re* (*f x*)) ∈ *borel_measurable M* ∧ (λ*x*. *Im* (*f x*)) ∈ *borel_measurable M*
apply *auto*
apply (*subst fun_complex_eq*)

apply (*intro borel_measurable_add*)
apply *auto*
done

lemma *powr_real_measurable* [*measurable*]:
assumes $f \in$ *measurable M borel* $g \in$ *measurable M borel*
shows (λx. *f x powr g x* :: *real*) $\in$ *measurable M borel*
using *assms* **by** (*simp_all add*: *powr_def*)

lemma *measurable_of_bool*[*measurable*]: *of_bool* $\in$ *count_space UNIV* $\rightarrow_M$ *borel*
by *simp*

6.5.6 Borel space on the extended reals

lemma *borel_measurable_ereal*[*measurable* (*raw*)]:
assumes *f*: $f \in$ *borel_measurable M* **shows** (λx. *ereal* (*f x*)) $\in$ *borel_measurable M*
using *continuous_on_ereal f* **by** (*rule borel_measurable_continuous_on*) (*rule continuous_on_id*)

lemma *borel_measurable_real_of_ereal*[*measurable* (*raw*)]:
fixes *f* :: *'a* $\Rightarrow$ *ereal*
assumes *f*: $f \in$ *borel_measurable M*
shows (λx. *real_of_ereal* (*f x*)) $\in$ *borel_measurable M*
apply (*rule measurable_compose*[*OF f*])
apply (*rule borel_measurable_continuous_countable_exceptions*[*of* $\{\infty, -\infty\}$])
apply (*auto intro*: *continuous_on_real simp*: *Compl_eq_Diff_UNIV*)
done

lemma *borel_measurable_ereal_cases*:
fixes *f* :: *'a* $\Rightarrow$ *ereal*
assumes *f*: $f \in$ *borel_measurable M*
assumes *H*: (λx. *H* (*ereal* (*real_of_ereal* (*f x*)))) $\in$ *borel_measurable M*
shows (λx. *H* (*f x*)) $\in$ *borel_measurable M*
proof −
let *?F* = λx. *if f x* = ∞ *then H* ∞ *else if f x* = − ∞ *then H* ($-\infty$) *else H* (*ereal* (*real_of_ereal* (*f x*)))
{ fix *x* **have** *H* (*f x*) = *?F x* **by** (*cases f x*) *auto* **}**
with *f H* **show** *?thesis* **by** *simp*
qed

lemma
fixes *f* :: *'a* $\Rightarrow$ *ereal* **assumes** *f*[*measurable*]: $f \in$ *borel_measurable M*
shows *borel_measurable_ereal_abs*[*measurable*(*raw*)]: (λx. $|f x|$) $\in$ *borel_measurable M*
and *borel_measurable_ereal_inverse*[*measurable*(*raw*)]: (λx. *inverse* (*f x*) :: *ereal*) $\in$ *borel_measurable M*
and *borel_measurable_uminus_ereal*[*measurable*(*raw*)]: (λx. − *f x* :: *ereal*) $\in$ *borel_measurable M*

by (*auto simp del*: *abs_real_of_ereal simp*: *borel_measurable_ereal_cases*[*OF f*] *measurable_If*)

lemma *borel_measurable_uminus_eq_ereal*[*simp*]:
$(\lambda x. - f\ x :: ereal) \in borel\_measurable\ M \longleftrightarrow f \in borel\_measurable\ M$ (**is** *?l* = *?r*)
proof
assume *?l* **from** *borel_measurable_uminus_ereal*[*OF this*] **show** *?r* **by** *simp*
qed *auto*

lemma *set_Collect_ereal2*:
fixes $f\ g :: 'a \Rightarrow ereal$
assumes f: $f \in borel\_measurable\ M$
assumes g: $g \in borel\_measurable\ M$
assumes H: $\{x \in space\ M.\ H\ (ereal\ (real\_of\_ereal\ (f\ x)))\ (ereal\ (real\_of\_ereal\ (g\ x)))\} \in sets\ M$
$\{x \in space\ borel.\ H\ (-\infty)\ (ereal\ x)\} \in sets\ borel$
$\{x \in space\ borel.\ H\ (\infty)\ (ereal\ x)\} \in sets\ borel$
$\{x \in space\ borel.\ H\ (ereal\ x)\ (-\infty)\} \in sets\ borel$
$\{x \in space\ borel.\ H\ (ereal\ x)\ (\infty)\} \in sets\ borel$
shows $\{x \in space\ M.\ H\ (f\ x)\ (g\ x)\} \in sets\ M$
proof −
let $?G = \lambda y\ x.$ *if* $g\ x = \infty$ *then* $H\ y\ \infty$ *else if* $g\ x = -\infty$ *then* $H\ y\ (-\infty)$ *else* $H\ y\ (ereal\ (real\_of\_ereal\ (g\ x)))$
let $?F = \lambda x.$ *if* $f\ x = \infty$ *then* $?G\ \infty\ x$ *else if* $f\ x = -\infty$ *then* $?G\ (-\infty)\ x$ *else* $?G\ (ereal\ (real\_of\_ereal\ (f\ x)))\ x$
{ fix *x* **have** $H\ (f\ x)\ (g\ x) = ?F\ x$ **by** (*cases f x g x rule*: *ereal2_cases*) *auto* **}**
note ∗ = *this*
from *assms* **show** *?thesis*
by (*subst* ∗) (*simp del*: *space_borel split del*: *if_split*)
qed

lemma *borel_measurable_ereal_iff*:
shows $(\lambda x.\ ereal\ (f\ x)) \in borel\_measurable\ M \longleftrightarrow f \in borel\_measurable\ M$
proof
assume $(\lambda x.\ ereal\ (f\ x)) \in borel\_measurable\ M$
from *borel_measurable_real_of_ereal*[*OF this*]
show $f \in borel\_measurable\ M$ **by** *auto*
qed *auto*

lemma *borel_measurable_erealD*[*measurable_dest*]:
$(\lambda x.\ ereal\ (f\ x)) \in borel\_measurable\ M \Longrightarrow g \in measurable\ N\ M \Longrightarrow (\lambda x.\ f\ (g\ x)) \in borel\_measurable\ N$
unfolding *borel_measurable_ereal_iff* **by** *simp*

theorem *borel_measurable_ereal_iff_real*:
fixes $f :: 'a \Rightarrow ereal$
shows $f \in borel\_measurable\ M \longleftrightarrow$
$((\lambda x.\ real\_of\_ereal\ (f\ x)) \in borel\_measurable\ M \wedge f -\text{`}\ \{\infty\} \cap space\ M \in sets$

$M \wedge f$ −‘ {−∞} ∩ *space M* ∈ *sets M*)
proof *safe*
assume ∗: (λ*x*. *real_of_ereal* (*f x*)) ∈ *borel_measurable M f* −‘ {∞} ∩ *space M* ∈ *sets M f* −‘ {−∞} ∩ *space M* ∈ *sets M*
have *f* −‘ {∞} ∩ *space M* = {*x*∈*space M*. *f x* = ∞} *f* −‘ {−∞} ∩ *space M* = {*x*∈*space M*. *f x* = −∞} **by** *auto*
with ∗ **have** ∗∗: {*x*∈*space M*. *f x* = ∞} ∈ *sets M* {*x*∈*space M*. *f x* = −∞} ∈ *sets M* **by** *simp_all*
let *?f* = λ*x*. *if f x* = ∞ *then* ∞ *else if f x* = −∞ *then* −∞ *else ereal* (*real_of_ereal* (*f x*))
have *?f* ∈ *borel_measurable M* **using** ∗ ∗∗ **by** (*intro measurable_If*) *auto*
also have *?f* = *f* **by** (*auto simp*: *fun_eq_iff ereal_real*)
finally show *f* ∈ *borel_measurable M* **.**
qed *simp_all*

lemma *borel_measurable_ereal_iff_Iio*:
(*f*::′*a* ⇒ *ereal*) ∈ *borel_measurable M* ⟷ (∀ *a*. *f* −‘ {..< *a*} ∩ *space M* ∈ *sets M*)
by (*auto simp*: *borel_Iio measurable_iff_measure_of*)

lemma *borel_measurable_ereal_iff_Ioi*:
(*f*::′*a* ⇒ *ereal*) ∈ *borel_measurable M* ⟷ (∀ *a*. *f* −‘ {*a* <..} ∩ *space M* ∈ *sets M*)
by (*auto simp*: *borel_Ioi measurable_iff_measure_of*)

lemma *vimage_sets_compl_iff*:
f −‘ *A* ∩ *space M* ∈ *sets M* ⟷ *f* −‘ (− *A*) ∩ *space M* ∈ *sets M*
proof −
{ fix *A* **assume** *f* −‘ *A* ∩ *space M* ∈ *sets M*
moreover have *f* −‘ (− *A*) ∩ *space M* = *space M* − *f* −‘ *A* ∩ *space M* **by** *auto*
ultimately have *f* −‘ (− *A*) ∩ *space M* ∈ *sets M* **by** *auto* **}**
from *this*[*of A*] *this*[*of* −*A*] **show** *?thesis*
by (*metis double_complement*)
qed

lemma *borel_measurable_iff_Iic_ereal*:
(*f*::′*a*⇒*ereal*) ∈ *borel_measurable M* ⟷ (∀ *a*. *f* −‘ {..*a*} ∩ *space M* ∈ *sets M*)
unfolding *borel_measurable_ereal_iff_Ioi vimage_sets_compl_iff*[**where** *A*={*a* <..} **for** *a*] **by** *simp*

lemma *borel_measurable_iff_Ici_ereal*:
(*f*::′*a* ⇒ *ereal*) ∈ *borel_measurable M* ⟷ (∀ *a*. *f* −‘ {*a*..} ∩ *space M* ∈ *sets M*)
unfolding *borel_measurable_ereal_iff_Iio vimage_sets_compl_iff*[**where** *A*={..< *a*} **for** *a*] **by** *simp*

lemma *borel_measurable_ereal2*:
fixes *f g* :: ′*a* ⇒ *ereal*
assumes *f*: *f* ∈ *borel_measurable M*

assumes *g*: *g* ∈ *borel_measurable M*
assumes *H*: (λ*x*. *H* (*ereal* (*real_of_ereal* (*f x*))) (*ereal* (*real_of_ereal* (*g x*)))) ∈ *borel_measurable M*
(λ*x*. *H* (−∞) (*ereal* (*real_of_ereal* (*g x*)))) ∈ *borel_measurable M*
(λ*x*. *H* (∞) (*ereal* (*real_of_ereal* (*g x*)))) ∈ *borel_measurable M*
(λ*x*. *H* (*ereal* (*real_of_ereal* (*f x*))) (−∞)) ∈ *borel_measurable M*
(λ*x*. *H* (*ereal* (*real_of_ereal* (*f x*))) (∞)) ∈ *borel_measurable M*
shows (λ*x*. *H* (*f x*) (*g x*)) ∈ *borel_measurable M*
proof −
let *?G* = λ*y x*. *if g x* = ∞ *then H y* ∞ *else if g x* = − ∞ *then H y* (−∞) *else H y* (*ereal* (*real_of_ereal* (*g x*)))
let *?F* = λ*x*. *if f x* = ∞ *then ?G* ∞ *x else if f x* = − ∞ *then ?G* (−∞) *x else ?G* (*ereal* (*real_of_ereal* (*f x*))) *x*
{ fix *x* **have** *H* (*f x*) (*g x*) = *?F x* **by** (*cases f x g x rule*: *ereal2_cases*) *auto* **}**
note ∗ = *this*
from *assms* **show** *?thesis* **unfolding** ∗ **by** *simp*
qed

lemma [*measurable*(*raw*)]:
fixes *f* :: ′*a* ⇒ *ereal*
assumes [*measurable*]: *f* ∈ *borel_measurable M g* ∈ *borel_measurable M*
shows *borel_measurable_ereal_add*: (λ*x*. *f x* + *g x*) ∈ *borel_measurable M*
and *borel_measurable_ereal_times*: (λ*x*. *f x* ∗ *g x*) ∈ *borel_measurable M*
by (*simp_all add*: *borel_measurable_ereal2*)

lemma [*measurable*(*raw*)]:
fixes *f g* :: ′*a* ⇒ *ereal*
assumes *f* ∈ *borel_measurable M*
assumes *g* ∈ *borel_measurable M*
shows *borel_measurable_ereal_diff*: (λ*x*. *f x* − *g x*) ∈ *borel_measurable M*
and *borel_measurable_ereal_divide*: (λ*x*. *f x* / *g x*) ∈ *borel_measurable M*
using *assms* **by** (*simp_all add*: *minus_ereal_def divide_ereal_def*)

lemma *borel_measurable_ereal_sum*[*measurable* (*raw*)]:
fixes *f* :: ′*c* ⇒ ′*a* ⇒ *ereal*
assumes ⋀*i*. *i* ∈ *S* ⟹ *f i* ∈ *borel_measurable M*
shows (λ*x*. ∑ *i*∈*S*. *f i x*) ∈ *borel_measurable M*
using *assms* **by** (*induction S rule*: *infinite_finite_induct*) *auto*

lemma *borel_measurable_ereal_prod*[*measurable* (*raw*)]:
fixes *f* :: ′*c* ⇒ ′*a* ⇒ *ereal*
assumes ⋀*i*. *i* ∈ *S* ⟹ *f i* ∈ *borel_measurable M*
shows (λ*x*. ∏ *i*∈*S*. *f i x*) ∈ *borel_measurable M*
using *assms* **by** (*induction S rule*: *infinite_finite_induct*) *auto*

lemma *borel_measurable_extreal_suminf*[*measurable* (*raw*)]:
fixes *f* :: *nat* ⇒ ′*a* ⇒ *ereal*
assumes [*measurable*]: ⋀*i*. *f i* ∈ *borel_measurable M*
shows (λ*x*. (∑ *i*. *f i x*)) ∈ *borel_measurable M*

unfolding *suminf_def sums_def*[*abs_def*] *lim_def*[*symmetric*] **by** *simp*

6.5.7 Borel space on the extended non-negative reals

ennreal is a topological monoid, so no rules for plus are required, also all order statements are usually done on type classes.

lemma *measurable_enn2ereal*[*measurable*]: *enn2ereal* ∈ *borel* $\rightarrow_M$ *borel*
by (*intro borel_measurable_continuous_onI continuous_on_enn2ereal*)

lemma *measurable_e2ennreal*[*measurable*]: *e2ennreal* ∈ *borel* $\rightarrow_M$ *borel*
by (*intro borel_measurable_continuous_onI continuous_on_e2ennreal*)

lemma *borel_measurable_enn2real*[*measurable* (*raw*)]:
f ∈ *M* $\rightarrow_M$ *borel* ⟹ (λ*x*. *enn2real* (*f x*)) ∈ *M* $\rightarrow_M$ *borel*
unfolding *enn2real_def*[*abs_def*] **by** *measurable*

definition [*simp*]: *is_borel f M* ⟷ *f* ∈ *borel_measurable M*

lemma *is_borel_transfer*[*transfer_rule*]: *rel_fun* (*rel_fun* (=) *pcr_ennreal*) (=) *is_borel is_borel*
unfolding *is_borel_def*[*abs_def*]
proof (*safe intro*!: *rel_funI ext dest*!: *rel_fun_eq_pcr_ennreal*[*THEN iffD1*])
fix *f* **and** *M* :: ′*a measure*
show *f* ∈ *borel_measurable M* **if** *f*: *enn2ereal* ∘ *f* ∈ *borel_measurable M*
using *measurable_compose*[*OF f measurable_e2ennreal*] **by** *simp*
qed *simp*

context
includes *ennreal.lifting*
begin

lemma *measurable_ennreal*[*measurable*]: *ennreal* ∈ *borel* $\rightarrow_M$ *borel*
unfolding *is_borel_def*[*symmetric*]
by *transfer simp*

lemma *borel_measurable_ennreal_iff* [*simp*]:
assumes [*simp*]: ⋀*x*. *x* ∈ *space M* ⟹ *0* ≤ *f x*
shows (λ*x*. *ennreal* (*f x*)) ∈ *M* $\rightarrow_M$ *borel* ⟷ *f* ∈ *M* $\rightarrow_M$ *borel*
proof *safe*
assume (λ*x*. *ennreal* (*f x*)) ∈ *M* $\rightarrow_M$ *borel*
then have (λ*x*. *enn2real* (*ennreal* (*f x*))) ∈ *M* $\rightarrow_M$ *borel*
by *measurable*
then show *f* ∈ *M* $\rightarrow_M$ *borel*
by (*rule measurable_cong*[*THEN iffD1*, *rotated*]) *auto*
qed *measurable*

lemma *borel_measurable_times_ennreal*[*measurable* (*raw*)]:
fixes *f g* :: ′*a* ⇒ *ennreal*
shows *f* ∈ *M* $\rightarrow_M$ *borel* ⟹ *g* ∈ *M* $\rightarrow_M$ *borel* ⟹ (λ*x*. *f x* ∗ *g x*) ∈ *M* $\rightarrow_M$

borel
  unfolding *is_borel_def*[*symmetric*] **by** *transfer simp*

lemma *borel_measurable_inverse_ennreal*[*measurable* (*raw*)]:
  fixes f :: $'a \Rightarrow$ *ennreal*
  shows $f \in M \to_M$ *borel* $\Longrightarrow$ ($\lambda x.$ *inverse* ($f\ x$)) $\in M \to_M$ *borel*
  unfolding *is_borel_def*[*symmetric*] **by** *transfer simp*

lemma *borel_measurable_divide_ennreal*[*measurable* (*raw*)]:
  fixes f :: $'a \Rightarrow$ *ennreal*
  shows $f \in M \to_M$ *borel* $\Longrightarrow g \in M \to_M$ *borel* $\Longrightarrow$ ($\lambda x.\ f\ x\ /\ g\ x$) $\in M \to_M$
borel
  unfolding *divide_ennreal_def* **by** *simp*

lemma *borel_measurable_minus_ennreal*[*measurable* (*raw*)]:
  fixes f :: $'a \Rightarrow$ *ennreal*
  shows $f \in M \to_M$ *borel* $\Longrightarrow g \in M \to_M$ *borel* $\Longrightarrow$ ($\lambda x.\ f\ x - g\ x$) $\in M \to_M$
borel
  unfolding *is_borel_def*[*symmetric*] **by** *transfer simp*

lemma *borel_measurable_prod_ennreal*[*measurable* (*raw*)]:
  fixes f :: $'c \Rightarrow 'a \Rightarrow$ *ennreal*
  assumes $\bigwedge i.\ i \in S \Longrightarrow f\ i \in$ *borel_measurable M*
  shows ($\lambda x.\ \prod i{\in}S.\ f\ i\ x$) $\in$ *borel_measurable M*
  using *assms* **by** (*induction S rule*: *infinite_finite_induct*) *auto*

end

hide_const (**open**) *is_borel*

6.5.8 LIMSEQ is borel measurable

lemma *borel_measurable_LIMSEQ_real*:
  fixes u :: *nat* $\Rightarrow 'a \Rightarrow$ *real*
  assumes u': $\bigwedge x.\ x \in$ *space M* $\Longrightarrow$ ($\lambda i.\ u\ i\ x$) $\longrightarrow u'\ x$
  and u: $\bigwedge i.\ u\ i \in$ *borel_measurable M*
  shows $u' \in$ *borel_measurable M*
proof –
  have $\bigwedge x.\ x \in$ *space M* $\Longrightarrow$ *liminf* ($\lambda n.$ *ereal* ($u\ n\ x$)) = *ereal* ($u'\ x$)
    using u' **by** (*simp add*: *lim_imp_Liminf*)
  moreover from u **have** ($\lambda x.$ *liminf* ($\lambda n.$ *ereal* ($u\ n\ x$))) $\in$ *borel_measurable M*
    by *auto*
  ultimately show *?thesis* **by** (*simp cong*: *measurable_cong add*: *borel_measurable_ereal_iff*)
qed

lemma *borel_measurable_LIMSEQ_metric*:
  fixes f :: *nat* $\Rightarrow 'a \Rightarrow 'b$ :: *metric_space*
  assumes [*measurable*]: $\bigwedge i.\ f\ i \in$ *borel_measurable M*
  assumes *lim*: $\bigwedge x.\ x \in$ *space M* $\Longrightarrow$ ($\lambda i.\ f\ i\ x$) $\longrightarrow g\ x$

shows *g* ∈ *borel_measurable M*
unfolding *borel_eq_closed*
proof (*safe intro*!: *measurable_measure_of*)
fix *A* :: *'b set* **assume** *closed A*

have [*measurable*]: (λ*x*. *infdist* (*g x*) *A*) ∈ *borel_measurable M*
proof (*rule borel_measurable_LIMSEQ_real*)
show ⋀*x*. *x* ∈ *space M* ⟹ (λ*i*. *infdist* (*f i x*) *A*) ⟶ *infdist* (*g x*) *A*
by (*intro tendsto_infdist lim*)
show ⋀*i*. (λ*x*. *infdist* (*f i x*) *A*) ∈ *borel_measurable M*
by (*intro borel_measurable_continuous_on*[**where** *f*=λ*x*. *infdist x A*]
continuous_at_imp_continuous_on ballI continuous_infdist continuous_ident)
auto
qed

show *g* −‘ *A* ∩ *space M* ∈ *sets M*
proof *cases*
assume *A* ≠ {}
then have ⋀*x*. *infdist x A* = *0* ⟷ *x* ∈ *A*
using ‹*closed A*› **by** (*simp add*: *in_closed_iff_infdist_zero*)
then have *g* −‘ *A* ∩ *space M* = {*x*∈*space M*. *infdist* (*g x*) *A* = *0*}
by *auto*
also have ... ∈ *sets M*
by *measurable*
finally show *?thesis* **.**
qed *simp*
qed *auto*

lemma *sets_Collect_Cauchy*[*measurable*]:
fixes *f* :: *nat* ⇒ *'a* => *'b*::{*metric_space*, *second_countable_topology*}
assumes *f*[*measurable*]: ⋀*i*. *f i* ∈ *borel_measurable M*
shows {*x*∈*space M*. *Cauchy* (λ*i*. *f i x*)} ∈ *sets M*
unfolding *metric_Cauchy_iff2* **using** *f* **by** *auto*

lemma *borel_measurable_lim_metric*[*measurable* (*raw*)]:
fixes *f* :: *nat* ⇒ *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes *f*[*measurable*]: ⋀*i*. *f i* ∈ *borel_measurable M*
shows (λ*x*. *lim* (λ*i*. *f i x*)) ∈ *borel_measurable M*
proof −
define *u'* **where** *u' x* = *lim* (λ*i*. *if Cauchy* (λ*i*. *f i x*) *then f i x else 0*) **for** *x*
then have ∗: ⋀*x*. *lim* (λ*i*. *f i x*) = (*if Cauchy* (λ*i*. *f i x*) *then u' x else* (*THE x*. *False*))
by (*auto simp*: *lim_def convergent_eq_Cauchy*[*symmetric*])
have *u'* ∈ *borel_measurable M*
proof (*rule borel_measurable_LIMSEQ_metric*)
fix *x*
have *convergent* (λ*i*. *if Cauchy* (λ*i*. *f i x*) *then f i x else 0*)
by (*cases Cauchy* (λ*i*. *f i x*))
(*auto simp add*: *convergent_eq_Cauchy*[*symmetric*] *convergent_def*)

```
    then show (λi. if Cauchy (λi. f i x) then f i x else 0) ⟶ u' x
      unfolding u'_def
      by (rule convergent_LIMSEQ_iff[THEN iffD1])
  qed measurable
  then show ?thesis
    unfolding * by measurable
qed

lemma borel_measurable_suminf[measurable (raw)]:
  fixes f :: nat ⇒ 'a ⇒ 'b::{banach, second_countable_topology}
  assumes f[measurable]: ⋀i. f i ∈ borel_measurable M
  shows (λx. suminf (λi. f i x)) ∈ borel_measurable M
  unfolding suminf_def sums_def[abs_def] lim_def[symmetric] by simp

lemma Collect_closed_imp_pred_borel: closed {x. P x} ⟹ Measurable.pred borel P
  by (simp add: pred_def)


lemma isCont_borel_pred[measurable]:
  fixes f :: 'b::metric_space ⇒ 'a::metric_space
  shows Measurable.pred borel (isCont f)
proof (subst measurable_cong)
  let ?I = λj. inverse(real (Suc j))
  show isCont f x = (∀i. ∃j. ∀y z. dist x y < ?I j ∧ dist x z < ?I j ⟶ dist (f
y) (f z) ≤ ?I i) for x
    unfolding continuous_at_eps_delta
  proof safe
    fix i assume ∀e>0. ∃d>0. ∀y. dist y x < d ⟶ dist (f y) (f x) < e
    moreover have 0 < ?I i / 2
      by simp
    ultimately obtain d where d: 0 < d ⋀y. dist x y < d ⟹ dist (f y) (f x) <
?I i / 2
      by (metis dist_commute)
    then obtain j where j: ?I j < d
      by (metis reals_Archimedean)

    show ∃j. ∀y z. dist x y < ?I j ∧ dist x z < ?I j ⟶ dist (f y) (f z) ≤ ?I i
    proof (safe intro!: exI[where x=j])
      fix y z assume *: dist x y < ?I j dist x z < ?I j
      have dist (f y) (f z) ≤ dist (f y) (f x) + dist (f z) (f x)
        by (rule dist_triangle2)
      also have ... < ?I i / 2 + ?I i / 2
        by (intro add_strict_mono d less_trans[OF _ j] *)
      also have ... ≤ ?I i
        by (simp add: field_simps)
      finally show dist (f y) (f z) ≤ ?I i
        by simp
    qed
  next
```

fix *e*::*real* **assume** *0* < *e*
then obtain *n* **where** *n*: *?I n* < *e*
by (*metis reals_Archimedean*)
assume ∀ *i*. ∃ *j*. ∀ *y z*. *dist x y* < *?I j* ∧ *dist x z* < *?I j* ⟶ *dist* (*f y*) (*f z*) ≤ *?I i*
from *this*[*THEN spec*, *of Suc n*]
obtain *j* **where** *j*: ⋀*y z*. *dist x y* < *?I j* ⟹ *dist x z* < *?I j* ⟹ *dist* (*f y*) (*f z*) ≤ *?I* (*Suc n*)
by *auto*

show ∃ *d*>*0*. ∀ *y*. *dist y x* < *d* ⟶ *dist* (*f y*) (*f x*) < *e*
proof (*safe intro*!: *exI*[*of* _ *?I j*])
fix *y* **assume** *dist y x* < *?I j*
then have *dist* (*f y*) (*f x*) ≤ *?I* (*Suc n*)
by (*intro j*) (*auto simp*: *dist_commute*)
also have *?I* (*Suc n*) < *?I n*
by *simp*
also note *n*
finally show *dist* (*f y*) (*f x*) < *e* **.**
qed *simp*
qed
qed (*intro pred_intros_countable closed_Collect_all closed_Collect_le open_Collect_less Collect_closed_imp_pred_borel closed_Collect_imp open_Collect_conj continuous_intros*)

lemma *isCont_borel*:
fixes *f* :: ′*b*::*metric_space* ⇒ ′*a*::*metric_space*
shows {*x*. *isCont f x*} ∈ *sets borel*
by *simp*

lemma *is_real_interval*:
assumes *S*: *is_interval S*
shows ∃ *a b*::*real*. *S* = {} ∨ *S* = *UNIV* ∨ *S* = {..<*b*} ∨ *S* = {..*b*} ∨ *S* = {*a*<..} ∨ *S* = {*a*..} ∨
S = {*a*<..<*b*} ∨ *S* = {*a*<..*b*} ∨ *S* = {*a*..<*b*} ∨ *S* = {*a*..*b*}
using *S* **unfolding** *is_interval_1* **by** (*blast intro*: *interval_cases*)

lemma *real_interval_borel_measurable*:
assumes *is_interval* (*S*::*real set*)
shows *S* ∈ *sets borel*
proof −
from *assms is_real_interval* **have** ∃ *a b*::*real*. *S* = {} ∨ *S* = *UNIV* ∨ *S* = {..<*b*} ∨ *S* = {..*b*} ∨
S = {*a*<..} ∨ *S* = {*a*..} ∨ *S* = {*a*<..<*b*} ∨ *S* = {*a*<..*b*} ∨ *S* = {*a*..<*b*} ∨ *S* = {*a*..*b*} **by** *auto*
then guess *a* **..**
then guess *b* **..**
thus *?thesis*
by *auto*

qed

The next lemmas hold in any second countable linorder (including ennreal or ereal for instance), but in the current state they are restricted to reals.

lemma *borel_measurable_mono_on_fnc*:
fixes f :: *real* ⇒ *real* **and** A :: *real set*
assumes *mono_on f A*
shows $f \in$ *borel_measurable* (*restrict_space borel A*)
apply (*rule measurable_restrict_countable*[*OF mono_on_ctble_discont*[*OF assms*]])
apply (*auto intro*!: *image_eqI*[**where** x={x} **for** x] *simp*: *sets_restrict_space*)
apply (*auto simp add*: *sets_restrict_restrict_space continuous_on_eq_continuous_within cong*: *measurable_cong_sets intro*!: *borel_measurable_continuous_on_restrict intro*: *continuous_within_subset*)
done

lemma *borel_measurable_piecewise_mono*:
fixes f::*real* ⇒ *real* **and** C::*real set set*
assumes *countable* C $\bigwedge c.\ c \in C \Longrightarrow c \in$ *sets borel* $\bigwedge c.\ c \in C \Longrightarrow$ *mono_on f c* ($\bigcup C$) = *UNIV*
shows $f \in$ *borel_measurable borel*
by (*rule measurable_piecewise_restrict*[*of C*], *auto intro*: *borel_measurable_mono_on_fnc simp*: *assms*)

lemma *borel_measurable_mono*:
fixes f :: *real* ⇒ *real*
shows *mono* $f \Longrightarrow f \in$ *borel_measurable borel*
using *borel_measurable_mono_on_fnc*[*of f UNIV*] **by** (*simp add*: *mono_def mono_on_def*)

lemma *measurable_bdd_below_real*[*measurable* (*raw*)]:
fixes F :: ′*a* ⇒ ′*i* ⇒ *real*
assumes [*simp*]: *countable I* **and** [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow F\ i \in M \rightarrow_M$ *borel*
shows *Measurable.pred M* (λx. *bdd_below* ((λi. $F\ i\ x$)‘I))
proof (*subst measurable_cong*)
show *bdd_below* ((λi. $F\ i\ x$)‘I) ⟷ ($\exists q \in \mathbb{Z}.\ \forall i \in I.\ q \le F\ i\ x$) **for** x
by (*auto simp*: *bdd_below_def intro*!: *bexI*[*of _ of_int* (*floor _*)] *intro*: *order_trans of_int_floor_le*)
show *Measurable.pred M* ($\lambda w.\ \exists q \in \mathbb{Z}.\ \forall i \in I.\ q \le F\ i\ w$)
using *countable_int* **by** *measurable*
qed

lemma *borel_measurable_cINF_real*[*measurable* (*raw*)]:
fixes F :: _ ⇒ _ ⇒ *real*
assumes [*simp*]: *countable I*
assumes *F*[*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow F\ i \in$ *borel_measurable M*
shows (λx. *INF* $i \in I$. $F\ i\ x$) $\in$ *borel_measurable M*
proof (*rule measurable_piecewise_restrict*)
let ?Ω = {$x \in$*space M*. *bdd_below* ((λi. $F\ i\ x$)‘I)}
show *countable* {?Ω, − ?Ω} *space M* ⊆ ⋃{?Ω, − ?Ω} $\bigwedge X.\ X \in$ {?Ω, − ?Ω} ⟹ $X \cap$ *space M* $\in$ *sets M*

by *auto*
fix *X* **assume** *X* ∈ {*?Ω*, − *?Ω*} **then show** (λ*x*. *INF i*∈*I*. *F i x*) ∈ *borel_measurable* (*restrict_space M X*)
proof *safe*
show (λ*x*. *INF i*∈*I*. *F i x*) ∈ *borel_measurable* (*restrict_space M ?Ω*)
by (*intro borel_measurable_cINF measurable_restrict_space1 F*)
(*auto simp*: *space_restrict_space*)
show (λ*x*. *INF i*∈*I*. *F i x*) ∈ *borel_measurable* (*restrict_space M* (−*?Ω*))
proof (*subst measurable_cong*)
fix *x* **assume** *x* ∈ *space* (*restrict_space M* (−*?Ω*))
then have ¬ (∀ *i*∈*I*. − *F i x* ≤ *y*) **for** *y*
by (*auto simp*: *space_restrict_space bdd_above_def bdd_above_uminus*[*symmetric*])
then show (*INF i*∈*I*. *F i x*) = − (*THE x*. *False*)
by (*auto simp*: *space_restrict_space Inf_real_def Sup_real_def Least_def simp del*: *Set.ball_simps*(*10*))
qed *simp*
qed
qed

lemma *borel_Ici*: *borel* = *sigma UNIV* (*range* (λ*x*::*real*. {*x* ..}))
proof (*safe intro*!: *borel_eq_sigmaI1*[*OF borel_Iio*])
fix *x* :: *real*
have *eq*: {..<*x*} = *space* (*sigma UNIV* (*range atLeast*)) − {*x* ..}
by *auto*
show {..<*x*} ∈ *sets* (*sigma UNIV* (*range atLeast*))
unfolding *eq* **by** (*intro sets.compl_sets*) *auto*
qed *auto*

lemma *borel_measurable_pred_less*[*measurable* (*raw*)]:
fixes *f* :: *'a* ⇒ *'b*::{*second_countable_topology*, *linorder_topology*}
shows *f* ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ *Measurable.pred M* (λ*w*. *f w* < *g w*)
unfolding *Measurable.pred_def* **by** (*rule borel_measurable_less*)

no_notation
eucl_less (**infix** <*e 50*)

lemma *borel_measurable_Max2*[*measurable* (*raw*)]:
fixes *f*::_ ⇒ _ ⇒ *'a*::{*second_countable_topology*, *dense_linorder*, *linorder_topology*}
assumes *finite I*
and [*measurable*]: ⋀*i*. *f i* ∈ *borel_measurable M*
shows (λ*x*. *Max*{*f i x* |*i*. *i* ∈ *I*}) ∈ *borel_measurable M*
by (*simp add*: *borel_measurable_Max*[*OF assms*(*1*), **where** *?f*=*f* **and** *?M*=*M*] *Setcompr_eq_image*)

lemma *measurable_compose_n* [*measurable* (*raw*)]:
assumes *T* ∈ *measurable M M*
shows (*T*^^*n*) ∈ *measurable M M*
by (*induction n*, *auto simp add*: *measurable_compose*[*OF _ assms*])

lemma *measurable_real_imp_nat*:
fixes f::'$a \Rightarrow nat$
assumes [*measurable*]: $(\lambda x.\ real(f\ x)) \in borel\_measurable\ M$
shows $f \in measurable\ M\ (count\_space\ UNIV)$
proof −
let $?g = (\lambda x.\ real(f\ x))$
have $\bigwedge(n::nat).\ ?g-`(\{real\ n\}) \cap space\ M = f-`\{n\} \cap space\ M$ **by** *auto*
moreover have $\bigwedge(n::nat).\ ?g-`(\{real\ n\}) \cap space\ M \in sets\ M$ **using** *assms* **by** *measurable*
ultimately have $\bigwedge(n::nat).\ f-`\{n\} \cap space\ M \in sets\ M$ **by** *simp*
then show *?thesis* **using** *measurable_count_space_eq2_countable* **by** *blast*
qed

lemma *measurable_equality_set* [*measurable*]:
fixes $f\ g$::_$\Rightarrow$ 'a::{*second_countable_topology*, *t2_space*}
assumes [*measurable*]: $f \in borel\_measurable\ M\ g \in borel\_measurable\ M$
shows $\{x \in space\ M.\ f\ x = g\ x\} \in sets\ M$

proof −
define A **where** $A = \{x \in space\ M.\ f\ x = g\ x\}$
define B **where** $B = \{y.\ \exists x::'a.\ y = (x,x)\}$
have $A = (\lambda x.\ (f\ x,\ g\ x))-`B \cap space\ M$ **unfolding** *A_def B_def* **by** *auto*
moreover have $(\lambda x.\ (f\ x,\ g\ x)) \in borel\_measurable\ M$ **by** *simp*
moreover have $B \in sets\ borel$ **unfolding** *B_def* **by** (*simp add*: *closed_diagonal*)
ultimately have $A \in sets\ M$ **by** *simp*
then show *?thesis* **unfolding** *A_def* **by** *simp*
qed

lemma *measurable_inequality_set* [*measurable*]:
fixes $f\ g$::_ $\Rightarrow$ 'a::{*second_countable_topology*, *linorder_topology*}
assumes [*measurable*]: $f \in borel\_measurable\ M\ g \in borel\_measurable\ M$
shows $\{x \in space\ M.\ f\ x \le g\ x\} \in sets\ M$
$\{x \in space\ M.\ f\ x < g\ x\} \in sets\ M$
$\{x \in space\ M.\ f\ x \ge g\ x\} \in sets\ M$
$\{x \in space\ M.\ f\ x > g\ x\} \in sets\ M$
proof −
define F **where** $F = (\lambda x.\ (f\ x,\ g\ x))$
have ∗ [*measurable*]: $F \in borel\_measurable\ M$ **unfolding** *F_def* **by** *simp*

have $\{x \in space\ M.\ f\ x \le g\ x\} = F-`\{(x,\ y)\ |\ x\ y.\ x \le y\} \cap space\ M$ **unfolding** *F_def* **by** *auto*
moreover have $\{(x,\ y)\ |\ x\ y.\ x \le (y::'a)\} \in sets\ borel$ **using** *closed_subdiagonal borel_closed* **by** *blast*
ultimately show $\{x \in space\ M.\ f\ x \le g\ x\} \in sets\ M$ **using** ∗ **by** (*metis* (*mono_tags*, *lifting*) *measurable_sets*)

have $\{x \in space\ M.\ f\ x < g\ x\} = F-`\{(x,\ y)\ |\ x\ y.\ x < y\} \cap space\ M$ **unfolding** *F_def* **by** *auto*

moreover have $\{(x, y) \mid x\ y.\ x < (y{::}'a)\} \in$ *sets borel* **using** *open_subdiagonal borel_open* **by** *blast*
ultimately show $\{x \in$ *space* $M.\ f\ x < g\ x\} \in$ *sets* M **using** $*$ **by** (*metis* (*mono_tags*, *lifting*) *measurable_sets*)

have $\{x \in$ *space* $M.\ f\ x \geq g\ x\} = F-`\{(x, y) \mid x\ y.\ x \geq y\} \cap$ *space* M **unfolding** *F_def* **by** *auto*
moreover have $\{(x, y) \mid x\ y.\ x \geq (y{::}'a)\} \in$ *sets borel* **using** *closed_superdiagonal borel_closed* **by** *blast*
ultimately show $\{x \in$ *space* $M.\ f\ x \geq g\ x\} \in$ *sets* M **using** $*$ **by** (*metis* (*mono_tags*, *lifting*) *measurable_sets*)

have $\{x \in$ *space* $M.\ f\ x > g\ x\} = F-`\{(x, y) \mid x\ y.\ x > y\} \cap$ *space* M **unfolding** *F_def* **by** *auto*
moreover have $\{(x, y) \mid x\ y.\ x > (y{::}'a)\} \in$ *sets borel* **using** *open_superdiagonal borel_open* **by** *blast*
ultimately show $\{x \in$ *space* $M.\ f\ x > g\ x\} \in$ *sets* M **using** $*$ **by** (*metis* (*mono_tags*, *lifting*) *measurable_sets*)
qed

proposition *measurable_limit* [*measurable*]:
fixes $f{::}nat \Rightarrow 'a \Rightarrow 'b{::}$*first_countable_topology*
assumes [*measurable*]: $\bigwedge n{::}nat.\ f\ n \in$ *borel_measurable* M
shows *Measurable.pred* M $(\lambda x.\ (\lambda n.\ f\ n\ x) \longrightarrow c)$
proof −
obtain $A :: nat \Rightarrow 'b$ *set* **where** A:
$\bigwedge i.$ *open* $(A\ i)$
$\bigwedge i.\ c \in A\ i$
$\bigwedge S.$ *open* $S \Longrightarrow c \in S \Longrightarrow$ *eventually* $(\lambda i.\ A\ i \subseteq S)$ *sequentially*
by (*rule countable_basis_at_decseq*) *blast*

have [*measurable*]: $\bigwedge N\ i.\ (f\ N)-`(A\ i) \cap$ *space* $M \in$ *sets* M **using** $A(1)$ **by** *auto*
then have *mes*: $(\bigcap i.\ \bigcup n.\ \bigcap N\in\{n..\}.\ (f\ N)-`(A\ i) \cap$ *space* $M) \in$ *sets* M **by** *blast*

have $(u \longrightarrow c) \longleftrightarrow (\forall i.$ *eventually* $(\lambda n.\ u\ n \in A\ i)$ *sequentially*$)$ **for** $u{::}nat \Rightarrow 'b$
proof
assume $u \longrightarrow c$
then have *eventually* $(\lambda n.\ u\ n \in A\ i)$ *sequentially* **for** i **using** $A(1)$[*of i*] $A(2)$[*of i*]
by (*simp add*: *topological_tendstoD*)
then show $(\forall i.$ *eventually* $(\lambda n.\ u\ n \in A\ i)$ *sequentially*$)$ **by** *auto*
next
assume H: $(\forall i.$ *eventually* $(\lambda n.\ u\ n \in A\ i)$ *sequentially*$)$
show $(u \longrightarrow c)$
proof (*rule topological_tendstoI*)
fix S **assume** *open* S $c \in S$
with $A(3)$[*OF this*] **obtain** i **where** $A\ i \subseteq S$

using *eventually_False_sequentially eventually_mono* **by** *blast*
moreover have *eventually* ($\lambda n.\ u\ n \in A\ i$) *sequentially* **using** *H* **by** *simp*
ultimately show $\forall_F\ n$ *in sequentially*. $u\ n \in S$
by (*simp add*: *eventually_mono subset_eq*)
qed
qed
then have $\{x.\ (\lambda n.\ f\ n\ x) \longrightarrow c\} = (\bigcap i.\ \bigcup n.\ \bigcap N\in\{n..\}.\ (f\ N)-`(A\ i))$
by (*auto simp add*: *atLeast_def eventually_at_top_linorder*)
then have $\{x \in space\ M.\ (\lambda n.\ f\ n\ x) \longrightarrow c\} = (\bigcap i.\ \bigcup n.\ \bigcap N\in\{n..\}.\ (f\ N)-`(A\ i) \cap space\ M)$
by *auto*
then have $\{x \in space\ M.\ (\lambda n.\ f\ n\ x) \longrightarrow c\} \in sets\ M$ **using** *mes* **by** *simp*
then show *?thesis* **by** *auto*
qed

lemma *measurable_limit2* [*measurable*]:
fixes *u*::*nat* ⇒ *'a* ⇒ *real*
assumes [*measurable*]: $\bigwedge n.\ u\ n \in borel\_measurable\ M\ v \in borel\_measurable\ M$
shows *Measurable.pred M* ($\lambda x.\ (\lambda n.\ u\ n\ x) \longrightarrow v\ x$)
proof −
define *w* **where** $w = (\lambda n\ x.\ u\ n\ x - v\ x)$
have [*measurable*]: $w\ n \in borel\_measurable\ M$ **for** *n* **unfolding** *w_def* **by** *auto*
have $((\lambda n.\ u\ n\ x) \longrightarrow v\ x) \longleftrightarrow ((\lambda n.\ w\ n\ x) \longrightarrow 0)$ **for** *x*
unfolding *w_def* **using** *Lim_null* **by** *auto*
then show *?thesis* **using** *measurable_limit* **by** *auto*
qed

lemma *measurable_P_restriction* [*measurable* (*raw*)]:
assumes [*measurable*]: *Measurable.pred M P A* ∈ *sets M*
shows $\{x \in A.\ P\ x\} \in sets\ M$
proof −
have $A \subseteq space\ M$ **using** *sets.sets_into_space*[*OF assms*(*2*)].
then have $\{x \in A.\ P\ x\} = A \cap \{x \in space\ M.\ P\ x\}$ **by** *blast*
then show *?thesis* **by** *auto*
qed

lemma *measurable_sum_nat* [*measurable* (*raw*)]:
fixes *f* :: *'c* ⇒ *'a* ⇒ *nat*
assumes $\bigwedge i.\ i \in S \Longrightarrow f\ i \in measurable\ M\ (count\_space\ UNIV)$
shows $(\lambda x.\ \sum i\in S.\ f\ i\ x) \in measurable\ M\ (count\_space\ UNIV)$
proof *cases*
assume *finite S*
then show *?thesis* **using** *assms* **by** *induct auto*
qed *simp*

lemma *measurable_abs_powr* [*measurable*]:
fixes *p*::*real*
assumes [*measurable*]: $f \in borel\_measurable\ M$

shows ($\lambda x.$ $|f\ x|$ *powr* p) $\in$ *borel_measurable M*
by *simp*

The next one is a variation around *measurable_restrict_space*.

lemma *measurable_restrict_space3*:
assumes $f \in$ *measurable M N* **and**
$f \in A \to B$
shows $f \in$ *measurable* (*restrict_space M A*) (*restrict_space N B*)
proof –
have $f \in$ *measurable* (*restrict_space M A*) *N* **using** *assms*(*1*) *measurable_restrict_space1*
by *auto*
then show *?thesis* **by** (*metis Int_iff funcsetI funcset_mem*
measurable_restrict_space2[*of f*, *of restrict_space M A*, *of B*, *of N*] *assms*(*2*)
space_restrict_space)
qed

lemma *measurable_restrict_mono*:
assumes *f*: $f \in$ *restrict_space M A* $\to_M$ *N* **and** $B \subseteq A$
shows $f \in$ *restrict_space M B* $\to_M$ *N*
by (*rule measurable_compose*[*OF measurable_restrict_space3 f*])
(*insert* ‹$B \subseteq A$›, *auto*)

The next one is a variation around *measurable_piecewise_restrict*.

lemma *measurable_piecewise_restrict2*:
assumes [*measurable*]: $\bigwedge n.$ *A n* $\in$ *sets M*
and *space M* = ($\bigcup$(*n*::*nat*). *A n*)
$\bigwedge n.$ $\exists h \in$ *measurable M N*. ($\forall x \in A\ n.$ $f\ x = h\ x$)
shows $f \in$ *measurable M N*
proof (*rule measurableI*)
fix *B* **assume** [*measurable*]: $B \in$ *sets N*
{
fix *n*::*nat*
obtain *h* **where** [*measurable*]: $h \in$ *measurable M N* **and** $\forall x \in A\ n.$ $f\ x = h\ x$
using *assms*(*3*) **by** *blast*
then have $*$: $f-$‘$B \cap A\ n = h-$‘$B \cap A\ n$ **by** *auto*
have $h-$‘$B \cap A\ n = h-$‘$B \cap$ *space M* $\cap A\ n$ **using** *assms*(*2*) *sets.sets_into_space*
by *auto*
then have $h-$‘$B \cap A\ n \in$ *sets M* **by** *simp*
then have $f-$‘$B \cap A\ n \in$ *sets M* **using** $*$ **by** *simp*
}
then have ($\bigcup n.$ $f-$‘$B \cap A\ n$) $\in$ *sets M* **by** *measurable*
moreover have $f-$‘$B \cap$ *space M* = ($\bigcup n.$ $f-$‘$B \cap A\ n$) **using** *assms*(*2*) **by**
blast
ultimately show $f-$‘$B \cap$ *space M* $\in$ *sets M* **by** *simp*
next
fix *x* **assume** $x \in$ *space M*
then obtain *n* **where** $x \in A\ n$ **using** *assms*(*2*) **by** *blast*
obtain *h* **where** [*measurable*]: $h \in$ *measurable M N* **and** $\forall x \in A\ n.$ $f\ x = h\ x$
using *assms*(*3*) **by** *blast*

then have *f x* = *h x* **using** ‹*x* ∈ *A n*› **by** *blast*
moreover have *h x* ∈ *space N* **by** (*metis measurable_space* ‹*x* ∈ *space M*› ‹*h* ∈ *measurable M N*›)
ultimately show *f x* ∈ *space N* **by** *simp*
qed

end

6.6 Lebesgue Integration for Nonnegative Functions

theory *Nonnegative_Lebesgue_Integration*
imports *Measure_Space Borel_Space*
begin

6.6.1 Approximating functions

```
lemma AE_upper_bound_inf_ennreal:
  fixes F G::'a ⇒ ennreal
  assumes ⋀e. (e::real) > 0 ⟹ AE x in M. F x ≤ G x + e
  shows AE x in M. F x ≤ G x
proof -
  have AE x in M. ∀n::nat. F x ≤ G x + ennreal (1 / Suc n)
    using assms by (auto simp: AE_all_countable)
  then show ?thesis
  proof (eventually_elim)
    fix x assume x: ∀n::nat. F x ≤ G x + ennreal (1 / Suc n)
    show F x ≤ G x
    proof (rule ennreal_le_epsilon)
      fix e :: real assume 0 < e
      then obtain n where n: 1 / Suc n < e
        by (blast elim: nat_approx_posE)
      have F x ≤ G x + 1 / Suc n
        using x by simp
      also have ... ≤ G x + e
        using n by (intro add_mono ennreal_leI) auto
      finally show F x ≤ G x + ennreal e .
    qed
  qed
qed

lemma AE_upper_bound_inf:
  fixes F G::'a ⇒ real
  assumes ⋀e. e > 0 ⟹ AE x in M. F x ≤ G x + e
  shows AE x in M. F x ≤ G x
proof -
  have AE x in M. F x ≤ G x + 1/real (n+1) for n::nat
    by (rule assms, auto)
```

then have *AE x in M.* ∀ *n::nat* ∈ *UNIV. F x* ≤ *G x* + *1/real* (*n+1*)
by (*rule AE_ball_countable′*, *auto*)
moreover
{
fix *x* **assume** *i*: ∀ *n::nat* ∈ *UNIV. F x* ≤ *G x* + *1/real* (*n+1*)
have (λ*n. G x* + *1/real* (*n+1*)) ⟶ *G x* + *0*
by (*rule tendsto_add*, *simp*, *rule LIMSEQ_ignore_initial_segment*[*OF lim_1_over_n*, *of 1*])
then have *F x* ≤ *G x* **using** *i LIMSEQ_le_const* **by** *fastforce*
}
ultimately show *?thesis* **by** *auto*
qed

lemma *not_AE_zero_ennreal_E*:
fixes *f::′a* ⇒ *ennreal*
assumes ¬ (*AE x in M. f x* = *0*) **and** [*measurable*]: *f* ∈ *borel_measurable M*
shows ∃ *A*∈*sets M.* ∃ *e::real*>*0. emeasure M A* > *0* ∧ (∀ *x* ∈ *A. f x* ≥ *e*)
proof −
{ assume ¬ (∃ *e::real*>*0.* {*x* ∈ *space M. f x* ≥ *e*} ∉ *null_sets M*)
then have *0* < *e* ⟹ *AE x in M. f x* ≤ *e* **for** *e* :: *real*
by (*auto simp*: *not_le less_imp_le dest*!: *AE_not_in*)
then have *AE x in M. f x* ≤ *0*
by (*intro AE_upper_bound_inf_ennreal*[**where** *G*=λ_. *0*]) *simp*
then have *False*
using *assms* **by** *auto* **}**
then obtain *e::real* **where** *e*: *e* > *0* {*x* ∈ *space M. f x* ≥ *e*} ∉ *null_sets M* **by** *auto*
define *A* **where** *A* = {*x* ∈ *space M. f x* ≥ *e*}
have *1* [*measurable*]: *A* ∈ *sets M* **unfolding** *A_def* **by** *auto*
have *2*: *emeasure M A* > *0*
using *e*(*2*) *A_def* ‹*A* ∈ *sets M*› **by** *auto*
have *3*: ⋀*x. x* ∈ *A* ⟹ *f x* ≥ *e* **unfolding** *A_def* **by** *auto*
show *?thesis* **using** *e*(*1*) *1 2 3* **by** *blast*
qed

lemma *not_AE_zero_E*:
fixes *f::′a* ⇒ *real*
assumes *AE x in M. f x* ≥ *0*
¬(*AE x in M. f x* = *0*)
and [*measurable*]: *f* ∈ *borel_measurable M*
shows ∃ *A e. A* ∈ *sets M* ∧ *e*>*0* ∧ *emeasure M A* > *0* ∧ (∀ *x* ∈ *A. f x* ≥ *e*)
proof −
have ∃ *e. e* > *0* ∧ {*x* ∈ *space M. f x* ≥ *e*} ∉ *null_sets M*
proof (*rule ccontr*)
assume ∗: ¬(∃ *e. e* > *0* ∧ {*x* ∈ *space M. f x* ≥ *e*} ∉ *null_sets M*)
{
fix *e::real* **assume** *e* > *0*
then have {*x* ∈ *space M. f x* ≥ *e*} ∈ *null_sets M* **using** ∗ **by** *blast*
then have *AE x in M. x* ∉ {*x* ∈ *space M. f x* ≥ *e*} **using** *AE_not_in* **by**

```
blast
      then have AE x in M. f x ≤ e by auto
    }
    then have AE x in M. f x ≤ 0 by (rule AE_upper_bound_inf, auto)
    then have AE x in M. f x = 0 using assms(1) by auto
    then show False using assms(2) by auto
  qed
  then obtain e where e: e>0 {x ∈ space M. f x ≥ e} ∉ null_sets M by auto
  define A where A = {x ∈ space M. f x ≥ e}
  have 1 [measurable]: A ∈ sets M unfolding A_def by auto
  have 2: emeasure M A > 0
    using e(2) A_def ‹A ∈ sets M› by auto
  have 3: ⋀x. x ∈ A ⟹ f x ≥ e unfolding A_def by auto
  show ?thesis
    using e(1) 1 2 3 by blast
qed
```

6.6.2 Simple function

Our simple functions are not restricted to nonnegative real numbers. Instead they are just functions with a finite range and are measurable when singleton sets are measurable.

```
definition simple_function M g ⟷
    finite (g ` space M) ∧
    (∀ x ∈ g ` space M. g −` {x} ∩ space M ∈ sets M)

lemma simple_functionD:
  assumes simple_function M g
  shows finite (g ` space M) and g −` X ∩ space M ∈ sets M
proof −
  show finite (g ` space M)
    using assms unfolding simple_function_def by auto
  have g −` X ∩ space M = g −` (X ∩ g`space M) ∩ space M by auto
  also have ... = (⋃x∈X ∩ g`space M. g−`{x} ∩ space M) by auto
  finally show g −` X ∩ space M ∈ sets M using assms
    by (auto simp del: UN_simps simp: simple_function_def)
qed

lemma measurable_simple_function[measurable_dest]:
  simple_function M f ⟹ f ∈ measurable M (count_space UNIV)
  unfolding simple_function_def measurable_def
proof safe
  fix A assume finite (f ` space M) ∀ x∈f ` space M. f −` {x} ∩ space M ∈ sets
M
  then have (⋃x∈f ` space M. if x ∈ A then f −` {x} ∩ space M else {}) ∈ sets
M
    by (intro sets.finite_UN) auto
  also have (⋃x∈f ` space M. if x ∈ A then f −` {x} ∩ space M else {}) = f −`
A ∩ space M
```

by (*auto split*: *if_split_asm*)
finally show $f -` A \cap space\ M \in sets\ M$ **.**
qed *simp*

lemma *borel_measurable_simple_function*:
simple_function M f $\Longrightarrow$ $f \in borel\_measurable\ M$
by (*auto dest*!: *measurable_simple_function simp*: *measurable_def*)

lemma *simple_function_measurable2*[*intro*]:
assumes *simple_function M f simple_function M g*
shows $f -` A \cap g -` B \cap space\ M \in sets\ M$
proof −
have $f -` A \cap g -` B \cap space\ M = (f -` A \cap space\ M) \cap (g -` B \cap space\ M)$
by *auto*
then show *?thesis* **using** *assms*[*THEN simple_functionD*(*2*)] **by** *auto*
qed

lemma *simple_function_indicator_representation*:
fixes $f :: 'a \Rightarrow ennreal$
assumes *f*: *simple_function M f* **and** *x*: $x \in space\ M$
shows $f\ x = (\sum y \in f ` space\ M.\ y * indicator\ (f -` \{y\} \cap space\ M)\ x)$
(**is** *?l* = *?r*)
proof −
have $?r = (\sum y \in f ` space\ M.$
$(if\ y = f\ x\ then\ y * indicator\ (f -` \{y\} \cap space\ M)\ x\ else\ 0))$
by (*auto intro*!: *sum.cong*)
also have ... = $f\ x * indicator\ (f -` \{f\ x\} \cap space\ M)\ x$
using *assms* **by** (*auto dest*: *simple_functionD*)
also have ... = *f x* **using** *x* **by** (*auto simp*: *indicator_def*)
finally show *?thesis* **by** *auto*
qed

lemma *simple_function_notspace*:
simple_function M ($\lambda x.\ h\ x * indicator\ (- space\ M)\ x::ennreal$) (**is** *simple_function M ?h*)
proof −
have $?h ` space\ M \subseteq \{0\}$ **unfolding** *indicator_def* **by** *auto*
hence [*simp*, *intro*]: *finite* (*?h* ` *space M*) **by** (*auto intro*: *finite_subset*)
have $?h -` \{0\} \cap space\ M = space\ M$ **by** *auto*
thus *?thesis* **unfolding** *simple_function_def* **by** (*auto simp add*: *image_constant_conv*)
qed

lemma *simple_function_cong*:
assumes $\bigwedge t.\ t \in space\ M \Longrightarrow f\ t = g\ t$
shows *simple_function M f* $\longleftrightarrow$ *simple_function M g*
proof −
have $\bigwedge x.\ f -` \{x\} \cap space\ M = g -` \{x\} \cap space\ M$
using *assms* **by** *auto*
with *assms* **show** *?thesis*

```
  by (simp add: simple_function_def cong: image_cong)
qed

lemma simple_function_cong_algebra:
 assumes sets N = sets M space N = space M
 shows simple_function M f ⟷ simple_function N f
 unfolding simple_function_def assms ..

lemma simple_function_borel_measurable:
 fixes f :: 'a ⇒ 'x::{t2_space}
 assumes f ∈ borel_measurable M and finite (f ` space M)
 shows simple_function M f
 using assms unfolding simple_function_def
 by (auto intro: borel_measurable_vimage)

lemma simple_function_iff_borel_measurable:
 fixes f :: 'a ⇒ 'x::{t2_space}
 shows simple_function M f ⟷ finite (f ` space M) ∧ f ∈ borel_measurable M
 by (metis borel_measurable_simple_function simple_functionD(1) simple_function_borel_measurable)

lemma simple_function_eq_measurable:
  simple_function M f ⟷ finite (f`space M) ∧ f ∈ measurable M (count_space
UNIV)
 using measurable_simple_function[of M f] by (fastforce simp: simple_function_def)

lemma simple_function_const[intro, simp]:
 simple_function M (λx. c)
 by (auto intro: finite_subset simp: simple_function_def)
lemma simple_function_compose[intro, simp]:
 assumes simple_function M f
 shows simple_function M (g ∘ f)
 unfolding simple_function_def
proof safe
 show finite ((g ∘ f) ` space M)
  using assms unfolding simple_function_def image_comp [symmetric] by auto
next
 fix x assume x ∈ space M
 let ?G = g -` {g (f x)} ∩ (f`space M)
 have *: (g ∘ f) -` {(g ∘ f) x} ∩ space M =
  (⋃x∈?G. f -` {x} ∩ space M) by auto
 show (g ∘ f) -` {(g ∘ f) x} ∩ space M ∈ sets M
  using assms unfolding simple_function_def *
  by (rule_tac sets.finite_UN) auto
qed

lemma simple_function_indicator[intro, simp]:
 assumes A ∈ sets M
 shows simple_function M (indicator A)
proof -
```

have *indicator A ' space M* ⊆ {*0*, *1*} (**is** *?S* ⊆ _)
by (*auto simp*: *indicator_def*)
hence *finite ?S* **by** (*rule finite_subset*) *simp*
moreover have − *A* ∩ *space M* = *space M* − *A* **by** *auto*
ultimately show *?thesis* **unfolding** *simple_function_def*
using *assms* **by** (*auto simp*: *indicator_def* [*abs_def*])
qed

lemma *simple_function_Pair*[*intro*, *simp*]:
assumes *simple_function M f*
assumes *simple_function M g*
shows *simple_function M* (λ*x*. (*f x*, *g x*)) (**is** *simple_function M ?p*)
unfolding *simple_function_def*
proof *safe*
show *finite* (*?p ' space M*)
using *assms* **unfolding** *simple_function_def*
by (*rule_tac finite_subset*[*of _ f'space M* × *g'space M*]) *auto*
next
fix *x* **assume** *x* ∈ *space M*
have (λ*x*. (*f x*, *g x*)) −' {(*f x*, *g x*)} ∩ *space M* =
(*f* −' {*f x*} ∩ *space M*) ∩ (*g* −' {*g x*} ∩ *space M*)
by *auto*
with ‹*x* ∈ *space M*› **show** (λ*x*. (*f x*, *g x*)) −' {(*f x*, *g x*)} ∩ *space M* ∈ *sets M*
using *assms* **unfolding** *simple_function_def* **by** *auto*
qed

lemma *simple_function_compose1*:
assumes *simple_function M f*
shows *simple_function M* (λ*x*. *g* (*f x*))
using *simple_function_compose*[*OF assms*, *of g*]
by (*simp add*: *comp_def*)

lemma *simple_function_compose2*:
assumes *simple_function M f* **and** *simple_function M g*
shows *simple_function M* (λ*x*. *h* (*f x*) (*g x*))
proof −
have *simple_function M* ((λ(*x*, *y*). *h x y*) ∘ (λ*x*. (*f x*, *g x*)))
using *assms* **by** *auto*
thus *?thesis* **by** (*simp_all add*: *comp_def*)
qed

lemmas *simple_function_add*[*intro*, *simp*] = *simple_function_compose2*[**where** *h*=(+)]
and *simple_function_diff*[*intro*, *simp*] = *simple_function_compose2*[**where** *h*=(−)]
and *simple_function_uminus*[*intro*, *simp*] = *simple_function_compose*[**where** *g*=*uminus*]
and *simple_function_mult*[*intro*, *simp*] = *simple_function_compose2*[**where** *h*=(∗)]
and *simple_function_div*[*intro*, *simp*] = *simple_function_compose2*[**where** *h*=(/)]
and *simple_function_inverse*[*intro*, *simp*] = *simple_function_compose*[**where** *g*=*inverse*]
and *simple_function_max*[*intro*, *simp*] = *simple_function_compose2*[**where** *h*=*max*]

lemma *simple_function_sum*[*intro*, *simp*]:
  assumes $\bigwedge i.\ i \in P \Longrightarrow$ *simple_function M* (*f i*)
  shows *simple_function M* ($\lambda x.\ \sum i{\in}P.\ f\ i\ x$)
proof *cases*
  assume *finite P* **from** *this assms* **show** *?thesis* **by** *induct auto*
qed *auto*

lemma *simple_function_ennreal*[*intro*, *simp*]:
  fixes *f g* :: *'a* ⇒ *real* **assumes** *sf*: *simple_function M f*
  shows *simple_function M* (λ*x*. *ennreal* (*f x*))
  by (*rule simple_function_compose1*[*OF sf*])

lemma *simple_function_real_of_nat*[*intro*, *simp*]:
  fixes *f g* :: *'a* ⇒ *nat* **assumes** *sf*: *simple_function M f*
  shows *simple_function M* (λ*x*. *real* (*f x*))
  by (*rule simple_function_compose1*[*OF sf*])

lemma *borel_measurable_implies_simple_function_sequence*:
  fixes *u* :: *'a* ⇒ *ennreal*
  assumes *u*[*measurable*]: *u* ∈ *borel_measurable M*
  shows ∃*f*. *incseq f* ∧ (∀*i*. (∀*x*. *f i x* < *top*) ∧ *simple_function M* (*f i*)) ∧ *u* =
(*SUP i*. *f i*)
proof −
  define *f* **where** [*abs_def*]:
    f i x = *real_of_int* (*floor* (*enn2real* (*min i* (*u x*)) ∗ *2^i*)) / *2^i* **for** *i x*

  have [*simp*]: *0* ≤ *f i x* **for** *i x*
   by (*auto simp*: *f_def intro*!: *divide_nonneg_nonneg mult_nonneg_nonneg enn2real_nonneg*)

  have ∗: *2^n* ∗ *real_of_int x* = *real_of_int* (*2^n* ∗ *x*) **for** *n x*
    by *simp*

  have *real_of_int* ⌊*real i* ∗ *2* ^ *i*⌋ = *real_of_int* ⌊*i* ∗ *2* ^ *i*⌋ **for** *i*
    by (*intro arg_cong*[**where** *f*=*real_of_int*]) *simp*
  then have [*simp*]: *real_of_int* ⌊*real i* ∗ *2* ^ *i*⌋ = *i* ∗ *2* ^ *i* **for** *i*
    unfolding *floor_of_nat* **by** *simp*

  have *incseq f*
  proof (*intro monoI le_funI*)
    fix *m n* :: *nat* **and** *x* **assume** *m* ≤ *n*
    moreover
    { **fix** *d* :: *nat*
      have ⌊*2^d::real*⌋ ∗ ⌊*2^m* ∗ *enn2real* (*min* (*of_nat m*) (*u x*))⌋ ≤
        ⌊*2^d* ∗ (*2^m* ∗ *enn2real* (*min* (*of_nat m*) (*u x*)))⌋
        by (*rule le_mult_floor*) (*auto*)
     also have . . . ≤ ⌊*2^d* ∗ (*2^m* ∗ *enn2real* (*min* (*of_nat d* + *of_nat m*) (*u x*)))⌋
        by (*intro floor_mono mult_mono enn2real_mono min.mono*)
          (*auto simp*: *min_less_iff_disj of_nat_less_top*)
      finally have *f m x* ≤ *f* (*m* + *d*) *x*

unfolding *f_def*
by (*auto simp*: *field_simps power_add* ∗ *simp del*: *of_int_mult*) **}**
ultimately show $f\ m\ x \leq f\ n\ x$
by (*auto simp add*: *le_iff_add*)
qed
then have *inc_f*: *incseq* (λ*i*. *ennreal* (*f i x*)) **for** *x*
by (*auto simp*: *incseq_def le_fun_def*)
then have *incseq* (λ*i x*. *ennreal* (*f i x*))
by (*auto simp*: *incseq_def le_fun_def*)
moreover
have *simple_function M* (*f i*) **for** *i*
proof (*rule simple_function_borel_measurable*)
have ⌊*enn2real* (*min* (*of_nat i*) (*u x*)) ∗ *2* ˆ *i*⌋ ≤ ⌊*int i* ∗ *2* ˆ *i*⌋ **for** *x*
by (*cases u x rule*: *ennreal_cases*)
(*auto split*: *split_min intro*!: *floor_mono*)
then have *f i* ‘ *space M* ⊆ (λ*n*. *real_of_int n* / *2*ˆ*i*) ‘ {*0* .. *of_nat i* ∗ *2*ˆ*i*}
unfolding *floor_of_int* **by** (*auto simp*: *f_def intro*!: *imageI*)
then show *finite* (*f i* ‘ *space M*)
by (*rule finite_subset*) *auto*
show *f i* ∈ *borel_measurable M*
unfolding *f_def enn2real_def* **by** *measurable*
qed
moreover
{ fix *x*
have (*SUP i*. *ennreal* (*f i x*)) = *u x*
proof (*cases u x rule*: *ennreal_cases*)
case *top* **then show** *?thesis*
by (*simp add*: *f_def inf_min*[*symmetric*] *ennreal_of_nat_eq_real_of_nat*[*symmetric*]
ennreal_SUP_of_nat_eq_top)
next
case (*real r*)
obtain *n* **where** *r* ≤ *of_nat n* **using** *real_arch_simple* **by** *auto*
then have *min_eq_r*: ∀$_F$ *x in sequentially*. *min* (*real x*) *r* = *r*
by (*auto simp*: *eventually_sequentially intro*!: *exI*[*of* _ *n*] *split*: *split_min*)

have (λ*i*. *real_of_int* ⌊*min* (*real i*) *r* ∗ *2*ˆ*i*⌋ / *2*ˆ*i*) ⟶ *r*
proof (*rule tendsto_sandwich*)
show (λ*n*. *r* − (*1*/*2*)ˆ*n*) ⟶ *r*
by (*auto intro*!: *tendsto_eq_intros LIMSEQ_power_zero*)
show ∀$_F$ *n in sequentially*. *real_of_int* ⌊*min* (*real n*) *r* ∗ *2* ˆ *n*⌋ / *2* ˆ *n* ≤ *r*
using *min_eq_r* **by** *eventually_elim* (*auto simp*: *field_simps*)
have ∗: *r* ∗ (*2* ˆ *n* ∗ *2* ˆ *n*) ≤ *2*ˆ*n* + *2*ˆ*n* ∗ *real_of_int* ⌊*r* ∗ *2* ˆ *n*⌋ **for** *n*
using *real_of_int_floor_ge_diff_one*[*of r* ∗ *2*ˆ*n*, *THEN mult_left_mono*, *of*
*2*ˆ*n*]
by (*auto simp*: *field_simps*)
show ∀$_F$ *n in sequentially*. *r* − (*1*/*2*)ˆ*n* ≤ *real_of_int* ⌊*min* (*real n*) *r* ∗ *2*
ˆ *n*⌋ / *2* ˆ *n*
using *min_eq_r* **by** *eventually_elim* (*insert* ∗, *auto simp*: *field_simps*)
qed *auto*

```
      then have (λi. ennreal (f i x)) ⟶ ennreal r
        by (simp add: real f_def ennreal_of_nat_eq_real_of_nat min_ennreal)
      from LIMSEQ_unique[OF LIMSEQ_SUP[OF inc_f] this]
      show ?thesis
        by (simp add: real)
    qed }
  ultimately show ?thesis
    by (intro exI [of _ λi x. ennreal (f i x)]) (auto simp add: image_comp)
qed

lemma borel_measurable_implies_simple_function_sequence':
  fixes u :: 'a ⇒ ennreal
  assumes u: u ∈ borel_measurable M
  obtains f where
    ⋀i. simple_function M (f i) incseq f ⋀i x. f i x < top ⋀x. (SUP i. f i x) = u x
  using borel_measurable_implies_simple_function_sequence [OF u]
  by (metis SUP_apply)

lemma simple_function_induct
    [consumes 1, case_names cong set mult add, induct set: simple_function]:
  fixes u :: 'a ⇒ ennreal
  assumes u: simple_function M u
  assumes cong: ⋀f g. simple_function M f ⟹ simple_function M g ⟹ (AE x
in M. f x = g x) ⟹ P f ⟹ P g
  assumes set: ⋀A. A ∈ sets M ⟹ P (indicator A)
  assumes mult: ⋀u c. P u ⟹ P (λx. c * u x)
  assumes add: ⋀u v. P u ⟹ P v ⟹ P (λx. v x + u x)
  shows P u
proof (rule cong)
  from AE_space show AE x in M. (∑y∈u ` space M. y * indicator (u -` {y}
∩ space M) x) = u x
  proof eventually_elim
    fix x assume x: x ∈ space M
    from simple_function_indicator_representation[OF u x]
    show (∑y∈u ` space M. y * indicator (u -` {y} ∩ space M) x) = u x ..
  qed
next
  from u have finite (u ` space M)
    unfolding simple_function_def by auto
  then show P (λx. ∑y∈u ` space M. y * indicator (u -` {y} ∩ space M) x)
  proof induct
    case empty show ?case
      using set[of {}] by (simp add: indicator_def[abs_def])
  qed (auto intro!: add mult set simple_functionD u)
next
  show simple_function M (λx. (∑y∈u ` space M. y * indicator (u -` {y} ∩ space
M) x))
    apply (subst simple_function_cong)
    apply (rule simple_function_indicator_representation[symmetric])
```

apply (*auto intro*: *u*)
done
qed *fact*

lemma *simple_function_induct_nn*[*consumes 1*, *case_names cong set mult add*]:
fixes *u* :: *'a* ⇒ *ennreal*
assumes *u*: *simple_function M u*
assumes *cong*: ⋀*f g*. *simple_function M f* ⟹ *simple_function M g* ⟹ (⋀*x*. *x* ∈ *space M* ⟹ *f x* = *g x*) ⟹ *P f* ⟹ *P g*
assumes *set*: ⋀*A*. *A* ∈ *sets M* ⟹ *P* (*indicator A*)
assumes *mult*: ⋀*u c*. *simple_function M u* ⟹ *P u* ⟹ *P* (λ*x*. *c* ∗ *u x*)
assumes *add*: ⋀*u v*. *simple_function M u* ⟹ *P u* ⟹ *simple_function M v* ⟹ (⋀*x*. *x* ∈ *space M* ⟹ *u x* = *0* ∨ *v x* = *0*) ⟹ *P v* ⟹ *P* (λ*x*. *v x* + *u x*)
shows *P u*
proof −
show *?thesis*
proof (*rule cong*)
fix *x* **assume** *x*: *x* ∈ *space M*
from *simple_function_indicator_representation*[*OF u x*]
show (∑*y*∈*u* ‘ *space M*. *y* ∗ *indicator* (*u* −‘ {*y*} ∩ *space M*) *x*) = *u x* **..**
next
show *simple_function M* (λ*x*. (∑*y*∈*u* ‘ *space M*. *y* ∗ *indicator* (*u* −‘ {*y*} ∩ *space M*) *x*))
apply (*subst simple_function_cong*)
apply (*rule simple_function_indicator_representation*[*symmetric*])
apply (*auto intro*: *u*)
done
next
from *u* **have** *finite* (*u* ‘ *space M*)
unfolding *simple_function_def* **by** *auto*
then show *P* (λ*x*. ∑*y*∈*u* ‘ *space M*. *y* ∗ *indicator* (*u* −‘ {*y*} ∩ *space M*) *x*)
proof *induct*
case *empty* **show** *?case*
using *set*[*of* {}] **by** (*simp add*: *indicator_def*[*abs_def*])
next
case (*insert x S*)
{ **fix** *z* **have** (∑*y*∈*S*. *y* ∗ *indicator* (*u* −‘ {*y*} ∩ *space M*) *z*) = *0* ∨
x ∗ *indicator* (*u* −‘ {*x*} ∩ *space M*) *z* = *0*
using *insert* **by** (*subst sum_eq_0_iff*) (*auto simp*: *indicator_def*) }
note *disj* = *this*
from *insert* **show** *?case*
by (*auto intro*!: *add mult set simple_functionD u simple_function_sum disj*)
qed
qed *fact*
qed

lemma *borel_measurable_induct*
[*consumes 1*, *case_names cong set mult add seq*, *induct set*: *borel_measurable*]:
fixes *u* :: *'a* ⇒ *ennreal*

```
  assumes u: u ∈ borel_measurable M
  assumes cong: ⋀f g. f ∈ borel_measurable M ⟹ g ∈ borel_measurable M ⟹
(⋀x. x ∈ space M ⟹ f x = g x) ⟹ P g ⟹ P f
  assumes set: ⋀A. A ∈ sets M ⟹ P (indicator A)
  assumes mult': ⋀u c. c < top ⟹ u ∈ borel_measurable M ⟹ (⋀x. x ∈ space
M ⟹ u x < top) ⟹ P u ⟹ P (λx. c * u x)
  assumes add: ⋀u v. u ∈ borel_measurable M⟹ (⋀x. x ∈ space M ⟹ u x <
top) ⟹ P u ⟹ v ∈ borel_measurable M ⟹ (⋀x. x ∈ space M ⟹ v x < top)
⟹ (⋀x. x ∈ space M ⟹ u x = 0 ∨ v x = 0) ⟹ P v ⟹ P (λx. v x + u x)
  assumes seq: ⋀U. (⋀i. U i ∈ borel_measurable M) ⟹ (⋀i x. x ∈ space M ⟹
U i x < top) ⟹ (⋀i. P (U i)) ⟹ incseq U ⟹ u = (SUP i. U i) ⟹ P (SUP
i. U i)
  shows P u
  using u
proof (induct rule: borel_measurable_implies_simple_function_sequence')
  fix U assume U: ⋀i. simple_function M (U i) incseq U ⋀i x. U i x < top and
sup: ⋀x. (SUP i. U i x) = u x
  have u_eq: u = (SUP i. U i)
    using u by (auto simp add: image_comp sup)

  have not_inf: ⋀x i. x ∈ space M ⟹ U i x < top
    using U by (auto simp: image_iff eq_commute)

  from U have ⋀i. U i ∈ borel_measurable M
    by (simp add: borel_measurable_simple_function)

  show P u
    unfolding u_eq
  proof (rule seq)
    fix i show P (U i)
      using ‹simple_function M (U i)› not_inf[of _ i]
    proof (induct rule: simple_function_induct_nn)
      case (mult u c)
      show ?case
      proof cases
        assume c = 0 ∨ space M = {} ∨ (∀x∈space M. u x = 0)
        with mult(1) show ?thesis
          by (intro cong[of λx. c * u x indicator {}] set)
            (auto dest!: borel_measurable_simple_function)
      next
        assume ¬ (c = 0 ∨ space M = {} ∨ (∀x∈space M. u x = 0))
        then obtain x where space M ≠ {} and x: x ∈ space M u x ≠ 0 c ≠ 0
          by auto
        with mult(3)[of x] have c < top
          by (auto simp: ennreal_mult_less_top)
        then have u_fin: x' ∈ space M ⟹ u x' < top for x'
          using mult(3)[of x'] ‹c ≠ 0› by (auto simp: ennreal_mult_less_top)
        then have P u
          by (rule mult)
```

with *u_fin* ‹*c* < *top*› *mult*(*1*) **show** *?thesis*
by (*intro mult'*) (*auto dest*!: *borel_measurable_simple_function*)
qed
qed (*auto intro*: *cong intro*!: *set add dest*!: *borel_measurable_simple_function*)
qed *fact*+
qed

lemma *simple_function_If_set*:
assumes *sf*: *simple_function M f simple_function M g* **and** *A*: *A* ∩ *space M* ∈ *sets M*
shows *simple_function M* (λ*x*. *if x* ∈ *A then f x else g x*) (**is** *simple_function M ?IF*)
proof −
define *F* **where** *F x* = *f* −' {*x*} ∩ *space M* **for** *x*
define *G* **where** *G x* = *g* −' {*x*} ∩ *space M* **for** *x*
show *?thesis* **unfolding** *simple_function_def*
proof *safe*
have *?IF* ' *space M* ⊆ *f* ' *space M* ∪ *g* ' *space M* **by** *auto*
from *finite_subset*[*OF this*] *assms*
show *finite* (*?IF* ' *space M*) **unfolding** *simple_function_def* **by** *auto*
next
fix *x* **assume** *x* ∈ *space M*
then have ∗: *?IF* −' {*?IF x*} ∩ *space M* = (*if x* ∈ *A*
then ((*F* (*f x*) ∩ (*A* ∩ *space M*)) ∪ (*G* (*f x*) − (*G* (*f x*) ∩ (*A* ∩ *space M*))))
else ((*F* (*g x*) ∩ (*A* ∩ *space M*)) ∪ (*G* (*g x*) − (*G* (*g x*) ∩ (*A* ∩ *space M*)))))
using *sets.sets_into_space*[*OF A*] **by** (*auto split*: *if_split_asm simp*: *G_def F_def*)
have [*intro*]: ⋀*x*. *F x* ∈ *sets M* ⋀*x*. *G x* ∈ *sets M*
unfolding *F_def G_def* **using** *sf*[*THEN simple_functionD*(*2*)] **by** *auto*
show *?IF* −' {*?IF x*} ∩ *space M* ∈ *sets M* **unfolding** ∗ **using** *A* **by** *auto*
qed
qed

lemma *simple_function_If*:
assumes *sf*: *simple_function M f simple_function M g* **and** *P*: {*x*∈*space M*. *P x*} ∈ *sets M*
shows *simple_function M* (λ*x*. *if P x then f x else g x*)
proof −
have {*x*∈*space M*. *P x*} = {*x*. *P x*} ∩ *space M* **by** *auto*
with *simple_function_If_set*[*OF sf*, *of* {*x*. *P x*}] *P* **show** *?thesis* **by** *simp*
qed

lemma *simple_function_subalgebra*:
assumes *simple_function N f*
and *N_subalgebra*: *sets N* ⊆ *sets M space N* = *space M*
shows *simple_function M f*
using *assms* **unfolding** *simple_function_def* **by** *auto*

lemma *simple_function_comp*:

assumes *T*: $T \in$ *measurable M M′*
and *f*: *simple_function M′ f*
shows *simple_function M* ($\lambda x.\ f\ (T\ x)$)
proof (*intro simple_function_def*[*THEN iffD2*] *conjI ballI*)
have ($\lambda x.\ f\ (T\ x)$) ‘ *space M* $\subseteq$ *f* ‘ *space M′*
using *T* **unfolding** *measurable_def* **by** *auto*
then show *finite* (($\lambda x.\ f\ (T\ x)$) ‘ *space M*)
using *f* **unfolding** *simple_function_def* **by** (*auto intro*: *finite_subset*)
fix *i* **assume** *i*: $i \in$ ($\lambda x.\ f\ (T\ x)$) ‘ *space M*
then have $i \in f$ ‘ *space M′*
using *T* **unfolding** *measurable_def* **by** *auto*
then have *f* −‘ {*i*} ∩ *space M′* ∈ *sets M′*
using *f* **unfolding** *simple_function_def* **by** *auto*
then have *T* −‘ (*f* −‘ {*i*} ∩ *space M′*) ∩ *space M* ∈ *sets M*
using *T* **unfolding** *measurable_def* **by** *auto*
also have *T* −‘ (*f* −‘ {*i*} ∩ *space M′*) ∩ *space M* = ($\lambda x.\ f\ (T\ x)$) −‘ {*i*} ∩ *space M*
using *T* **unfolding** *measurable_def* **by** *auto*
finally show ($\lambda x.\ f\ (T\ x)$) −‘ {*i*} ∩ *space M* ∈ *sets M* .
qed

6.6.3 Simple integral

definition *simple_integral* :: *′a measure* ⇒ (*′a* ⇒ *ennreal*) ⇒ *ennreal* (*integral*^{S})
where
integral^S *M f* = ($\sum x \in f$ ‘ *space M*. $x *$ *emeasure M* (*f* −‘ {*x*} ∩ *space M*))

syntax
_simple_integral :: *pttrn* ⇒ *ennreal* ⇒ *′a measure* ⇒ *ennreal* ($\int^S$ _. _ ∂_ [*60*,*61*] *110*)

translations
$\int^S$ *x*. *f* ∂*M* == *CONST simple_integral M* (%*x*. *f*)

lemma *simple_integral_cong*:
assumes $\bigwedge t.\ t \in$ *space M* ⟹ *f t* = *g t*
shows *integral*^{S} *M f* = *integral*^{S} *M g*
proof −
have *f* ‘ *space M* = *g* ‘ *space M*
$\bigwedge x$. *f* −‘ {*x*} ∩ *space M* = *g* −‘ {*x*} ∩ *space M*
using *assms* **by** (*auto intro*!: *image_eqI*)
thus *?thesis* **unfolding** *simple_integral_def* **by** *simp*
qed

lemma *simple_integral_const*[*simp*]:
($\int^S$*x*. *c* ∂*M*) = *c* ∗ (*emeasure M*) (*space M*)
proof (*cases space M* = {})
case *True* **thus** *?thesis* **unfolding** *simple_integral_def* **by** *simp*
next

case *False* **hence** $(\lambda x.\ c)$ ‘ *space M* = {*c*} **by** *auto*
thus *?thesis* **unfolding** *simple_integral_def* **by** *simp*
qed

lemma *simple_function_partition*:
assumes *f*: *simple_function M f* **and** *g*: *simple_function M g*
assumes *sub*: $\bigwedge x\ y.\ x \in space\ M \Longrightarrow y \in space\ M \Longrightarrow g\ x = g\ y \Longrightarrow f\ x = f\ y$
assumes *v*: $\bigwedge x.\ x \in space\ M \Longrightarrow f\ x = v\ (g\ x)$
shows $integral^S\ M\ f = (\sum y \in g$ ‘ *space M*. *v y* ∗ *emeasure M* {*x*∈*space M*. *g x* = *y*})
(**is** _ = *?r*)
proof −
from *f g* **have** [*simp*]: *finite* (*f*‘*space M*) *finite* (*g*‘*space M*)
by (*auto simp*: *simple_function_def*)
from *f g* **have** [*measurable*]: *f* ∈ *measurable M* (*count_space UNIV*) *g* ∈ *measurable M* (*count_space UNIV*)
by (*auto intro*: *measurable_simple_function*)

{ fix *y* **assume** *y* ∈ *space M*
then have *f* ‘ *space M* ∩ {*i*. ∃ *x*∈*space M*. *i* = *f x* ∧ *g y* = *g x*} = {*v* (*g y*)}
by (*auto cong*: *sub simp*: *v*[*symmetric*]) **}**
note *eq* = *this*

have $integral^S\ M\ f$ =
($\sum$ *y*∈*f*‘*space M*. *y* ∗ ($\sum$ *z*∈*g*‘*space M*.
if ∃ *x*∈*space M*. *y* = *f x* ∧ *z* = *g x then emeasure M* {*x*∈*space M*. *g x* = *z*} *else 0*))
unfolding *simple_integral_def*
proof (*safe intro*!: *sum.cong ennreal_mult_left_cong*)
fix *y* **assume** *y*: *y* ∈ *space M f y* ≠ *0*
have [*simp*]: *g* ‘ *space M* ∩ {*z*. ∃ *x*∈*space M*. *f y* = *f x* ∧ *z* = *g x*} =
{*z*. ∃ *x*∈*space M*. *f y* = *f x* ∧ *z* = *g x*}
by *auto*
have *eq*:($\bigcup$ *i*∈{*z*. ∃ *x*∈*space M*. *f y* = *f x* ∧ *z* = *g x*}. {*x* ∈ *space M*. *g x* = *i*}) =
f −‘ {*f y*} ∩ *space M*
by (*auto simp*: *eq_commute cong*: *sub rev_conj_cong*)
have *finite* (*g*‘*space M*) **by** *simp*
then have *finite* {*z*. ∃ *x*∈*space M*. *f y* = *f x* ∧ *z* = *g x*}
by (*rule rev_finite_subset*) *auto*
then show *emeasure M* (*f* −‘ {*f y*} ∩ *space M*) =
($\sum$ *z*∈*g* ‘ *space M*. *if* ∃ *x*∈*space M*. *f y* = *f x* ∧ *z* = *g x then emeasure M* {*x* ∈ *space M*. *g x* = *z*} *else 0*)
apply (*simp add*: *sum.If_cases*)
apply (*subst sum_emeasure*)
apply (*auto simp*: *disjoint_family_on_def eq*)
done
qed
also have . . . = ($\sum$ *y*∈*f*‘*space M*. ($\sum$ *z*∈*g*‘*space M*.

if $\exists x\in$*space* M. $y = f\ x \wedge z = g\ x$ *then* $y *$ *emeasure* M $\{x\in$*space* M. $g\ x = z\}$ *else* 0))
by (*auto intro*!: *sum.cong simp*: *sum_distrib_left*)
also have ... = *?r*
by (*subst sum.swap*)
(*auto intro*!: *sum.cong simp*: *sum.If_cases scaleR_sum_right*[*symmetric*] *eq*)
finally show *integral*^{S} $M\ f$ = *?r* .
qed

lemma *simple_integral_add*[*simp*]:
assumes f: *simple_function* $M\ f$ **and** $\bigwedge x.\ 0 \leq f\ x$ **and** g: *simple_function* $M\ g$ **and** $\bigwedge x.\ 0 \leq g\ x$
shows ($\int^S x.\ f\ x + g\ x\ \partial M$) = *integral*^{S} $M\ f$ + *integral*^{S} $M\ g$
proof −
have ($\int^S x.\ f\ x + g\ x\ \partial M$) =
($\sum y\in(\lambda x.\ (f\ x,\ g\ x))$‘*space* M. (*fst* y + *snd* y) $*$ *emeasure* M $\{x\in$*space* M. ($f\ x,\ g\ x) = y\}$)
by (*intro simple_function_partition*) (*auto intro*: *f g*)
also have ... = ($\sum y\in(\lambda x.\ (f\ x,\ g\ x))$‘*space* M. *fst* y $*$ *emeasure* M $\{x\in$*space* M. ($f\ x,\ g\ x) = y\}$) +
($\sum y\in(\lambda x.\ (f\ x,\ g\ x))$‘*space* M. *snd* y $*$ *emeasure* M $\{x\in$*space* M. ($f\ x,\ g\ x) = y\}$)
using *assms*(*2*,*4*) **by** (*auto intro*!: *sum.cong distrib_right simp*: *sum.distrib*[*symmetric*])
also have ($\sum y\in(\lambda x.\ (f\ x,\ g\ x))$‘*space* M. *fst* y $*$ *emeasure* M $\{x\in$*space* M. ($f\ x,\ g\ x) = y\}$) = ($\int^S x.\ f\ x\ \partial M$)
by (*intro simple_function_partition*[*symmetric*]) (*auto intro*: *f g*)
also have ($\sum y\in(\lambda x.\ (f\ x,\ g\ x))$‘*space* M. *snd* y $*$ *emeasure* M $\{x\in$*space* M. ($f\ x,\ g\ x) = y\}$) = ($\int^S x.\ g\ x\ \partial M$)
by (*intro simple_function_partition*[*symmetric*]) (*auto intro*: *f g*)
finally show *?thesis* .
qed

lemma *simple_integral_sum*[*simp*]:
assumes $\bigwedge i\ x.\ i \in P \Longrightarrow 0 \leq f\ i\ x$
assumes $\bigwedge i.\ i \in P \Longrightarrow$ *simple_function* M ($f\ i$)
shows ($\int^S x.\ (\sum i\in P.\ f\ i\ x)\ \partial M$) = ($\sum i\in P.$ *integral*^{S} M ($f\ i$))
proof *cases*
assume *finite P*
from *this assms* **show** *?thesis*
by *induct* (*auto*)
qed *auto*

lemma *simple_integral_mult*[*simp*]:
assumes f: *simple_function* $M\ f$
shows ($\int^S x.\ c * f\ x\ \partial M$) = $c *$ *integral*^{S} $M\ f$
proof −
have ($\int^S x.\ c * f\ x\ \partial M$) = ($\sum y\in f$ ‘ *space* M. ($c * y$) $*$ *emeasure* M $\{x\in$*space* M. $f\ x = y\}$)
using f **by** (*intro simple_function_partition*) *auto*

also have . . . = $c * integral^S\ M\ f$
  using *f* **unfolding** *simple_integral_def*
  by (*subst sum_distrib_left*) (*auto simp*: *mult.assoc Int_def conj_commute*)
finally show *?thesis* **.**
qed

lemma *simple_integral_mono_AE*:
assumes *f*[*measurable*]: *simple_function M f* **and** *g*[*measurable*]: *simple_function M g*
and *mono*: *AE x in M. f x* ≤ *g x*
shows $integral^S\ M\ f \leq integral^S\ M\ g$
proof −
let *?μ* = λ*P. emeasure M* {*x*∈*space M. P x*}
have $integral^S\ M\ f$ = (∑ *y*∈(λ*x.* (*f x*, *g x*))'*space M. fst y* ∗ *?μ* (λ*x.* (*f x*, *g x*) = *y*))
  using *f g* **by** (*intro simple_function_partition*) *auto*
also have . . . ≤ (∑ *y*∈(λ*x.* (*f x*, *g x*))'*space M. snd y* ∗ *?μ* (λ*x.* (*f x*, *g x*) = *y*))
proof (*clarsimp intro*!: *sum_mono*)
  fix *x* **assume** *x* ∈ *space M*
  let *?M* = *?μ* (λ*y. f y* = *f x* ∧ *g y* = *g x*)
  show *f x* ∗ *?M* ≤ *g x* ∗ *?M*
  proof *cases*
    assume *?M* ≠ *0*
    then have *0* < *?M*
      by (*simp add*: *less_le*)
    also have . . . ≤ *?μ* (λ*y. f x* ≤ *g x*)
      using *mono* **by** (*intro emeasure_mono_AE*) *auto*
    finally have ¬ ¬ *f x* ≤ *g x*
      by (*intro notI*) *auto*
    then show *?thesis*
      by (*intro mult_right_mono*) *auto*
  qed *simp*
qed
also have . . . = $integral^S\ M\ g$
  using *f g* **by** (*intro simple_function_partition*[*symmetric*]) *auto*
finally show *?thesis* **.**
qed

lemma *simple_integral_mono*:
assumes *simple_function M f* **and** *simple_function M g*
and *mono*: ⋀ *x. x* ∈ *space M* ⟹ *f x* ≤ *g x*
shows $integral^S\ M\ f \leq integral^S\ M\ g$
using *assms* **by** (*intro simple_integral_mono_AE*) *auto*

lemma *simple_integral_cong_AE*:
assumes *simple_function M f* **and** *simple_function M g*
and *AE x in M. f x* = *g x*
shows $integral^S\ M\ f = integral^S\ M\ g$
using *assms* **by** (*auto simp*: *eq_iff intro*!: *simple_integral_mono_AE*)

lemma *simple_integral_cong'*:
  assumes *sf*: *simple_function M f simple_function M g*
  and *mea*: (*emeasure M*) {*x*∈*space M*. *f x* ≠ *g x*} = *0*
  shows *integral*^{S} *M f* = *integral*^{S} *M g*
proof (*intro simple_integral_cong_AE sf AE_I*)
  show (*emeasure M*) {*x*∈*space M*. *f x* ≠ *g x*} = *0* **by** *fact*
  show {*x* ∈ *space M*. *f x* ≠ *g x*} ∈ *sets M*
    using *sf*[*THEN borel_measurable_simple_function*] **by** *auto*
qed *simp*

lemma *simple_integral_indicator*:
  assumes *A*: *A* ∈ *sets M*
  assumes *f*: *simple_function M f*
  shows (∫^{S}*x*. *f x* ∗ *indicator A x* ∂*M*) =
    (∑ *x* ∈ *f* ‘ *space M*. *x* ∗ *emeasure M* (*f* −‘ {*x*} ∩ *space M* ∩ *A*))
proof −
  have *eq*: (λ*x*. (*f x*, *indicator A x*)) ‘ *space M* ∩ {*x*. *snd x* = *1*} = (λ*x*. (*f x*, *1*::*ennreal*))‘*A*
    using *A*[*THEN sets.sets_into_space*] **by** (*auto simp*: *indicator_def image_iff split*: *if_split_asm*)
  have *eq2*: ⋀*x*. *f x* ∉ *f* ‘ *A* ⟹ *f* −‘ {*f x*} ∩ *space M* ∩ *A* = {}
    by (*auto simp*: *image_iff*)

  have (∫^{S}*x*. *f x* ∗ *indicator A x* ∂*M*) =
    (∑ *y*∈(λ*x*. (*f x*, *indicator A x*))‘*space M*. (*fst y* ∗ *snd y*) ∗ *emeasure M* {*x*∈*space M*. (*f x*, *indicator A x*) = *y*})
    using *assms* **by** (*intro simple_function_partition*) *auto*
  also have ... = (∑ *y*∈(λ*x*. (*f x*, *indicator A x*::*ennreal*))‘*space M*.
    if snd y = *1 then fst y* ∗ *emeasure M* (*f* −‘ {*fst y*} ∩ *space M* ∩ *A*) *else 0*)
    by (*auto simp*: *indicator_def split*: *if_split_asm intro*!: *arg_cong2*[**where** *f*=(∗)] *arg_cong2*[**where** *f*=*emeasure*] *sum.cong*)
  also have ... = (∑ *y*∈(λ*x*. (*f x*, *1*::*ennreal*))‘*A*. *fst y* ∗ *emeasure M* (*f* −‘ {*fst y*} ∩ *space M* ∩ *A*))
    using *assms* **by** (*subst sum.If_cases*) (*auto intro*!: *simple_functionD*(*1*) *simp*: *eq*)
  also have ... = (∑ *y*∈*fst*‘(λ*x*. (*f x*, *1*::*ennreal*))‘*A*. *y* ∗ *emeasure M* (*f* −‘ {*y*} ∩ *space M* ∩ *A*))
    by (*subst sum.reindex* [*of fst*]) (*auto simp*: *inj_on_def*)
  also have ... = (∑ *x* ∈ *f* ‘ *space M*. *x* ∗ *emeasure M* (*f* −‘ {*x*} ∩ *space M* ∩ *A*))
    using *A*[*THEN sets.sets_into_space*]
      by (*intro sum.mono_neutral_cong_left simple_functionD f*) (*auto simp*: *image_comp comp_def eq2*)
  finally show *?thesis* .
qed

lemma *simple_integral_indicator_only*[*simp*]:
  assumes *A* ∈ *sets M*

shows *integral*^{S} *M* (*indicator A*) = *emeasure M A*
using *simple_integral_indicator*[*OF assms*, *of* λ*x*. *1*] *sets.sets_into_space*[*OF assms*]
by (*simp_all add*: *image_constant_conv Int_absorb1 split*: *if_split_asm*)

lemma *simple_integral_null_set*:
assumes *simple_function M u* ⋀*x*. *0* ≤ *u x* **and** *N* ∈ *null_sets M*
shows (∫ ^{S}*x*. *u x* ∗ *indicator N x* ∂*M*) = *0*
proof −
have *AE x in M*. *indicator N x* = (*0* :: *ennreal*)
using ‹*N* ∈ *null_sets M*› **by** (*auto simp*: *indicator_def intro*!: *AE_I*[*of* _ _ *N*])
then have (∫ ^{S}*x*. *u x* ∗ *indicator N x* ∂*M*) = (∫ ^{S}*x*. *0* ∂*M*)
using *assms* **apply** (*intro simple_integral_cong_AE*) **by** *auto*
then show *?thesis* **by** *simp*
qed

lemma *simple_integral_cong_AE_mult_indicator*:
assumes *sf*: *simple_function M f* **and** *eq*: *AE x in M*. *x* ∈ *S* **and** *S* ∈ *sets M*
shows *integral*^{S} *M f* = (∫ ^{S}*x*. *f x* ∗ *indicator S x* ∂*M*)
using *assms* **by** (*intro simple_integral_cong_AE*) *auto*

lemma *simple_integral_cmult_indicator*:
assumes *A*: *A* ∈ *sets M*
shows (∫ ^{S}*x*. *c* ∗ *indicator A x* ∂*M*) = *c* ∗ *emeasure M A*
using *simple_integral_mult*[*OF simple_function_indicator*[*OF A*]]
unfolding *simple_integral_indicator_only*[*OF A*] **by** *simp*

lemma *simple_integral_nonneg*:
assumes *f*: *simple_function M f* **and** *ae*: *AE x in M*. *0* ≤ *f x*
shows *0* ≤ *integral*^{S} *M f*
proof −
have *integral*^{S} *M* (λ*x*. *0*) ≤ *integral*^{S} *M f*
using *simple_integral_mono_AE*[*OF* _ *f ae*] **by** *auto*
then show *?thesis* **by** *simp*
qed

6.6.4 Integral on nonnegative functions

definition *nn_integral* :: ′*a measure* ⇒ (′*a* ⇒ *ennreal*) ⇒ *ennreal* (*integral*^{N})
where
integral^N *M f* = (*SUP g* ∈ {*g*. *simple_function M g* ∧ *g* ≤ *f*}. *integral*^{S} *M g*)

syntax
_nn_integral :: *pttrn* ⇒ *ennreal* ⇒ ′*a measure* ⇒ *ennreal* (∫ $^+$((*2* _./ _)/ ∂_)
[*60*,*61*] *110*)

translations
∫ $^+$*x*. *f* ∂*M* == *CONST nn_integral M* (λ*x*. *f*)

lemma *nn_integral_def_finite*:

```
  integral^N M f = (SUP g ∈ {g. simple_function M g ∧ g ≤ f ∧ (∀x. g x < top)}.
integral^S M g)
    (is _ = Sup (?A ‘ ?f))
  unfolding nn_integral_def
proof (safe intro!: antisym SUP_least)
  fix g assume g[measurable]: simple_function M g g ≤ f

  show integral^S M g ≤ Sup (?A ‘ ?f)
  proof cases
    assume ae: AE x in M. g x ≠ top
    let ?G = {x ∈ space M. g x ≠ top}
    have integral^S M g = integral^S M (λx. g x * indicator ?G x)
    proof (rule simple_integral_cong_AE)
      show AE x in M. g x = g x * indicator ?G x
        using ae AE_space by eventually_elim auto
    qed (insert g, auto)
    also have ... ≤ Sup (?A ‘ ?f)
     using g by (intro SUP_upper) (auto simp: le_fun_def less_top split: split_indicator)
    finally show ?thesis .
  next
    assume nAE: ¬ (AE x in M. g x ≠ top)
     then have emeasure M {x∈space M. g x = top} ≠ 0 (is emeasure M ?G ≠
0)
      by (subst (asm) AE_iff_measurable[OF _ refl]) auto
    then have top = (SUP n. (∫^S x. of_nat n * indicator ?G x ∂M))
     by (simp add: ennreal_SUP_of_nat_eq_top ennreal_top_eq_mult_iff SUP_mult_right_ennreal[symmetric])
    also have ... ≤ Sup (?A ‘ ?f)
      using g
      by (safe intro!: SUP_least SUP_upper)
       (auto simp: le_fun_def of_nat_less_top top_unique[symmetric] split: split_indicator
              intro: order_trans[of _ g x f x for x, OF order_trans[of _ top]])
    finally show ?thesis
      by (simp add: top_unique del: SUP_eq_top_iff Sup_eq_top_iff)
  qed
qed (auto intro: SUP_upper)

lemma nn_integral_mono_AE:
  assumes ae: AE x in M. u x ≤ v x shows integral^N M u ≤ integral^N M v
  unfolding nn_integral_def
proof (safe intro!: SUP_mono)
  fix n assume n: simple_function M n n ≤ u
  from ae[THEN AE_E] guess N . note N = this
  then have ae_N: AE x in M. x ∉ N by (auto intro: AE_not_in)
  let ?n = λx. n x * indicator (space M − N) x
  have AE x in M. n x ≤ ?n x simple_function M ?n
    using n N ae_N by auto
  moreover
  { fix x have ?n x ≤ v x
    proof cases
```

assume *x*: $x \in$ *space M* $-$ *N*
with *N* **have** *u x* $\leq$ *v x* **by** *auto*
with *n*(*2*)[*THEN le_funD*, *of x*] *x* **show** *?thesis*
by (*auto simp*: *max_def split*: *if_split_asm*)
qed *simp* **}**
then have *?n* $\leq$ *v* **by** (*auto simp*: *le_funI*)
moreover have *integral*^{S} *M n* $\leq$ *integral*^{S} *M ?n*
using *ae_N N n* **by** (*auto intro*!: *simple_integral_mono_AE*)
ultimately show $\exists$ *m*$\in${*g*. *simple_function M g* $\wedge$ *g* $\leq$ *v*}. *integral*^{S} *M n* $\leq$ *integral*^{S} *M m*
by *force*
qed

lemma *nn_integral_mono*:
($\bigwedge$*x*. $x \in$ *space M* $\Longrightarrow$ *u x* $\leq$ *v x*) $\Longrightarrow$ *integral*^{N} *M u* $\leq$ *integral*^{N} *M v*
by (*auto intro*: *nn_integral_mono_AE*)

lemma *mono_nn_integral*: *mono F* $\Longrightarrow$ *mono* (λ*x*. *integral*^{N} *M* (*F x*))
by (*auto simp add*: *mono_def le_fun_def intro*!: *nn_integral_mono*)

lemma *nn_integral_cong_AE*:
AE x in M. *u x* = *v x* $\Longrightarrow$ *integral*^{N} *M u* = *integral*^{N} *M v*
by (*auto simp*: *eq_iff intro*!: *nn_integral_mono_AE*)

lemma *nn_integral_cong*:
($\bigwedge$*x*. $x \in$ *space M* $\Longrightarrow$ *u x* = *v x*) $\Longrightarrow$ *integral*^{N} *M u* = *integral*^{N} *M v*
by (*auto intro*: *nn_integral_cong_AE*)

lemma *nn_integral_cong_simp*:
($\bigwedge$*x*. $x \in$ *space M* =*simp*=> *u x* = *v x*) $\Longrightarrow$ *integral*^{N} *M u* = *integral*^{N} *M v*
by (*auto intro*: *nn_integral_cong simp*: *simp_implies_def*)

lemma *incseq_nn_integral*:
assumes *incseq f* **shows** *incseq* (λ*i*. *integral*^{N} *M* (*f i*))
proof −
have $\bigwedge$*i x*. *f i x* $\leq$ *f* (*Suc i*) *x*
using *assms* **by** (*auto dest*!: *incseq_SucD simp*: *le_fun_def*)
then show *?thesis*
by (*auto intro*!: *incseq_SucI nn_integral_mono*)
qed

lemma *nn_integral_eq_simple_integral*:
assumes *f*: *simple_function M f* **shows** *integral*^{N} *M f* = *integral*^{S} *M f*
proof −
let *?f* = λ*x*. *f x* $*$ *indicator* (*space M*) *x*
have *f'*: *simple_function M ?f* **using** *f* **by** *auto*
have *integral*^{N} *M ?f* $\leq$ *integral*^{S} *M ?f* **using** *f'*
by (*force intro*!: *SUP_least simple_integral_mono simp*: *le_fun_def nn_integral_def*)
moreover have *integral*^{S} *M ?f* $\leq$ *integral*^{N} *M ?f*

```
    unfolding nn_integral_def
    using f' by (auto intro!: SUP_upper)
  ultimately show ?thesis
    by (simp cong: nn_integral_cong simple_integral_cong)
qed
```

Beppo-Levi monotone convergence theorem

```
lemma nn_integral_monotone_convergence_SUP:
  assumes f: incseq f and [measurable]: ⋀i. f i ∈ borel_measurable M
  shows (∫+ x. (SUP i. f i x) ∂M) = (SUP i. integral^N M (f i))
proof (rule antisym)
  show (∫+ x. (SUP i. f i x) ∂M) ≤ (SUP i. (∫+ x. f i x ∂M))
    unfolding nn_integral_def_finite[of _ λx. SUP i. f i x]
  proof (safe intro!: SUP_least)
    fix u assume sf_u[simp]: simple_function M u and
      u: u ≤ (λx. SUP i. f i x) and u_range: ∀ x. u x < top
    note sf_u[THEN borel_measurable_simple_function, measurable]
    show integral^S M u ≤ (SUP j. ∫+x. f j x ∂M)
    proof (rule ennreal_approx_unit)
      fix a :: ennreal assume a < 1
      let ?au = λx. a * u x

      let ?B = λc i. {x∈space M. ?au x = c ∧ c ≤ f i x}
      have integral^S M ?au = (∑ c∈?au‘space M. c * (SUP i. emeasure M (?B c
i)))
        unfolding simple_integral_def
      proof (intro sum.cong ennreal_mult_left_cong refl)
        fix c assume c ∈ ?au ‘ space M c ≠ 0
        { fix x' assume x': x' ∈ space M ?au x' = c
          with ‹c ≠ 0› u_range have ?au x' < 1 * u x'
            by (intro ennreal_mult_strict_right_mono ‹a < 1›) (auto simp: less_le)
          also have . . . ≤ (SUP i. f i x')
            using u by (auto simp: le_fun_def)
          finally have ∃ i. ?au x' ≤ f i x'
            by (auto simp: less_SUP_iff intro: less_imp_le) }
        then have *: ?au −‘ {c} ∩ space M = (⋃ i. ?B c i)
          by auto
        show emeasure M (?au −‘ {c} ∩ space M) = (SUP i. emeasure M (?B c
i))
          unfolding * using f
          by (intro SUP_emeasure_incseq[symmetric])
            (auto simp: incseq_def le_fun_def intro: order_trans)
      qed
      also have . . . = (SUP i. ∑ c∈?au‘space M. c * emeasure M (?B c i))
        unfolding SUP_mult_left_ennreal using f
        by (intro ennreal_SUP_sum[symmetric])
          (auto intro!: mult_mono emeasure_mono simp: incseq_def le_fun_def intro:
order_trans)
      also have . . . ≤ (SUP i. integral^N M (f i))
```

```
    proof (intro SUP_subset_mono order_refl)
      fix i
      have (∑ c∈?au‘space M. c * emeasure M (?B c i)) =
        (∫ Sx. (a * u x) * indicator {x∈space M. a * u x ≤ f i x} x ∂M)
        by (subst simple_integral_indicator)
        (auto intro!: sum.cong ennreal_mult_left_cong arg_cong2[where f=emeasure])
      also have ... = (∫ +x. (a * u x) * indicator {x∈space M. a * u x ≤ f i x}
x ∂M)
        by (rule nn_integral_eq_simple_integral[symmetric]) simp
      also have ... ≤ (∫ +x. f i x ∂M)
        by (intro nn_integral_mono) (auto split: split_indicator)
      finally show (∑ c∈?au‘space M. c * emeasure M (?B c i)) ≤ (∫ +x. f i x
∂M) .
    qed
    finally show a * integral^S M u ≤ (SUP i. integral^N M (f i))
      by simp
  qed
 qed
qed (auto intro!: SUP_least SUP_upper nn_integral_mono)

lemma sup_continuous_nn_integral[order_continuous_intros]:
 assumes f: ⋀y. sup_continuous (f y)
 assumes [measurable]: ⋀x. (λy. f y x) ∈ borel_measurable M
 shows sup_continuous (λx. (∫ +y. f y x ∂M))
 unfolding sup_continuous_def
proof safe
 fix C :: nat ⇒ 'b assume C: incseq C
 with sup_continuous_mono[OF f] show (∫ + y. f y (Sup (C ‘ UNIV)) ∂M) =
(SUP i. ∫ + y. f y (C i) ∂M)
   unfolding sup_continuousD[OF f C]
  by (subst nn_integral_monotone_convergence_SUP) (auto simp: mono_def le_fun_def)
qed

theorem nn_integral_monotone_convergence_SUP_AE:
 assumes f: ⋀i. AE x in M. f i x ≤ f (Suc i) x ⋀i. f i ∈ borel_measurable M
 shows (∫ + x. (SUP i. f i x) ∂M) = (SUP i. integral^N M (f i))
proof −
 from f have AE x in M. ∀ i. f i x ≤ f (Suc i) x
   by (simp add: AE_all_countable)
 from this[THEN AE_E] guess N . note N = this
 let ?f = λi x. if x ∈ space M − N then f i x else 0
 have f_eq: AE x in M. ∀ i. ?f i x = f i x using N by (auto intro!: AE_I[of _ _
N])
 then have (∫ + x. (SUP i. f i x) ∂M) = (∫ + x. (SUP i. ?f i x) ∂M)
   by (auto intro!: nn_integral_cong_AE)
 also have ... = (SUP i. (∫ + x. ?f i x ∂M))
 proof (rule nn_integral_monotone_convergence_SUP)
   show incseq ?f using N(1) by (force intro!: incseq_SucI le_funI)
   { fix i show (λx. if x ∈ space M − N then f i x else 0) ∈ borel_measurable M
```

```
      using f N(3) by (intro measurable_If_set) auto }
  qed
  also have ... = (SUP i. (∫+ x. f i x ∂M))
   using f_eq by (force intro!: arg_cong[where f = λf. Sup (range f)] nn_integral_cong_AE
ext)
  finally show ?thesis .
qed

lemma nn_integral_monotone_convergence_simple:
  incseq f ⟹ (⋀i. simple_function M (f i)) ⟹ (SUP i. ∫S x. f i x ∂M) = (∫+x.
(SUP i. f i x) ∂M)
  using nn_integral_monotone_convergence_SUP[of f M]
 by (simp add: nn_integral_eq_simple_integral[symmetric] borel_measurable_simple_function)

lemma SUP_simple_integral_sequences:
  assumes f: incseq f ⋀i. simple_function M (f i)
  and g: incseq g ⋀i. simple_function M (g i)
  and eq: AE x in M. (SUP i. f i x) = (SUP i. g i x)
  shows (SUP i. integralS M (f i)) = (SUP i. integralS M (g i))
    (is Sup (?F ` _) = Sup (?G ` _))
proof –
  have (SUP i. integralS M (f i)) = (∫+x. (SUP i. f i x) ∂M)
    using f by (rule nn_integral_monotone_convergence_simple)
  also have ... = (∫+x. (SUP i. g i x) ∂M)
    unfolding eq[THEN nn_integral_cong_AE] ..
  also have ... = (SUP i. ?G i)
    using g by (rule nn_integral_monotone_convergence_simple[symmetric])
  finally show ?thesis by simp
qed

lemma nn_integral_const[simp]: (∫+ x. c ∂M) = c * emeasure M (space M)
  by (subst nn_integral_eq_simple_integral) auto

lemma nn_integral_linear:
  assumes f: f ∈ borel_measurable M and g: g ∈ borel_measurable M
  shows (∫+ x. a * f x + g x ∂M) = a * integralN M f + integralN M g
    (is integralN M ?L = _)
proof –
  from borel_measurable_implies_simple_function_sequence'[OF f(1)] guess u .
  note u = nn_integral_monotone_convergence_simple[OF this(2,1)] this
  from borel_measurable_implies_simple_function_sequence'[OF g(1)] guess v .
  note v = nn_integral_monotone_convergence_simple[OF this(2,1)] this
  let ?L' = λi x. a * u i x + v i x

  have ?L ∈ borel_measurable M using assms by auto
  from borel_measurable_implies_simple_function_sequence'[OF this] guess l .
  note l = nn_integral_monotone_convergence_simple[OF this(2,1)] this

  have inc: incseq (λi. a * integralS M (u i)) incseq (λi. integralS M (v i))
```

using *u v* **by** (*auto simp*: *incseq_Suc_iff le_fun_def intro*!: *add_mono mult_left_mono simple_integral_mono*)

have *l'*: (*SUP i. integral^S M* (*l i*)) = (*SUP i. integral^S M* (*?L' i*))
proof (*rule SUP_simple_integral_sequences*[*OF l*(*3*,*2*)])
show *incseq ?L'* $\bigwedge$*i. simple_function M* (*?L' i*)
using *u v* **unfolding** *incseq_Suc_iff le_fun_def*
by (*auto intro*!: *add_mono mult_left_mono*)
{ fix *x*
have (*SUP i. a* ∗ *u i x* + *v i x*) = *a* ∗ (*SUP i. u i x*) + (*SUP i. v i x*)
using *u*(*3*) *v*(*3*) *u*(*4*)[*of _ x*] *v*(*4*)[*of _ x*] **unfolding** *SUP_mult_left_ennreal*
by (*auto intro*!: *ennreal_SUP_add simp*: *incseq_Suc_iff le_fun_def add_mono mult_left_mono*) **}**
then show *AE x in M*. (*SUP i. l i x*) = (*SUP i. ?L' i x*)
unfolding *l*(*5*) **using** *u*(*5*) *v*(*5*) **by** (*intro AE_I2*) *auto*
qed
also have ... = (*SUP i. a* ∗ *integral^S M* (*u i*) + *integral^S M* (*v i*))
using *u*(*2*) *v*(*2*) **by** *auto*
finally show *?thesis*
unfolding *l*(*5*)[*symmetric*] *l*(*1*)[*symmetric*]
by (*simp add*: *ennreal_SUP_add*[*OF inc*] *v u SUP_mult_left_ennreal*[*symmetric*])
qed

lemma *nn_integral_cmult*: *f* ∈ *borel_measurable M* ⟹ ($\int^+$ *x. c* ∗ *f x* ∂*M*) = *c* ∗ *integral^N M f*
using *nn_integral_linear*[*of f M* λ*x. 0 c*] **by** *simp*

lemma *nn_integral_multc*: *f* ∈ *borel_measurable M* ⟹ ($\int^+$ *x. f x* ∗ *c* ∂*M*) = *integral^N M f* ∗ *c*
unfolding *mult.commute*[*of _ c*] *nn_integral_cmult* **by** *simp*

lemma *nn_integral_divide*: *f* ∈ *borel_measurable M* ⟹ ($\int^+$ *x. f x* / *c* ∂*M*) = ($\int^+$ *x. f x* ∂*M*) / *c*
unfolding *divide_ennreal_def* **by** (*rule nn_integral_multc*)

lemma *nn_integral_indicator*[*simp*]: *A* ∈ *sets M* ⟹ ($\int^+$ *x. indicator A x*∂*M*) = (*emeasure M*) *A*
by (*subst nn_integral_eq_simple_integral*) (*auto simp*: *simple_integral_indicator*)

lemma *nn_integral_cmult_indicator*: *A* ∈ *sets M* ⟹ ($\int^+$ *x. c* ∗ *indicator A x* ∂*M*) = *c* ∗ *emeasure M A*
by (*subst nn_integral_eq_simple_integral*) (*auto*)

lemma *nn_integral_indicator'*:
assumes [*measurable*]: *A* ∩ *space M* ∈ *sets M*
shows ($\int^+$ *x. indicator A x* ∂*M*) = *emeasure M* (*A* ∩ *space M*)
proof −
have ($\int^+$ *x. indicator A x* ∂*M*) = ($\int^+$ *x. indicator* (*A* ∩ *space M*) *x* ∂*M*)
by (*intro nn_integral_cong*) (*simp split*: *split_indicator*)

also have ... = *emeasure M* (*A* ∩ *space M*)
  by *simp*
finally show *?thesis* **.**
qed

lemma *nn_integral_indicator_singleton*[*simp*]:
  assumes [*measurable*]: {*y*} ∈ *sets M* **shows** ($\int^+ x.\ f\ x * indicator\ \{y\}\ x\ \partial M$) = *f y* ∗ *emeasure M* {*y*}
proof −
  have ($\int^+ x.\ f\ x * indicator\ \{y\}\ x\ \partial M$) = ($\int^+ x.\ f\ y * indicator\ \{y\}\ x\ \partial M$)
    by (*auto intro*!: *nn_integral_cong split*: *split_indicator*)
  then show *?thesis*
    by (*simp add*: *nn_integral_cmult*)
qed

lemma *nn_integral_set_ennreal*:
  ($\int^+ x.\ ennreal\ (f\ x) * indicator\ A\ x\ \partial M$) = ($\int^+ x.\ ennreal\ (f\ x * indicator\ A\ x)\ \partial M$)
  by (*rule nn_integral_cong*) (*simp split*: *split_indicator*)

lemma *nn_integral_indicator_singleton'*[*simp*]:
  assumes [*measurable*]: {*y*} ∈ *sets M*
  shows ($\int^+ x.\ ennreal\ (f\ x * indicator\ \{y\}\ x)\ \partial M$) = *f y* ∗ *emeasure M* {*y*}
  by (*subst nn_integral_set_ennreal*[*symmetric*]) (*simp*)

lemma *nn_integral_add*:
  f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ ($\int^+ x.\ f\ x + g\ x\ \partial M$) = $integral^N$ *M f* + $integral^N$ *M g*
  using *nn_integral_linear*[*of f M g 1*] **by** *simp*

lemma *nn_integral_sum*:
  (⋀*i*. *i* ∈ *P* ⟹ *f i* ∈ *borel_measurable M*) ⟹ ($\int^+ x.\ (\sum i{\in}P.\ f\ i\ x)\ \partial M$) = ($\sum i{\in}P.\ integral^N\ M\ (f\ i)$)
  by (*induction P rule*: *infinite_finite_induct*) (*auto simp*: *nn_integral_add*)

theorem *nn_integral_suminf*:
  assumes *f*: ⋀*i*. *f i* ∈ *borel_measurable M*
  shows ($\int^+ x.\ (\sum i.\ f\ i\ x)\ \partial M$) = ($\sum i.\ integral^N\ M\ (f\ i)$)
proof −
  have *all_pos*: *AE x in M*. ∀ *i*. *0* ≤ *f i x*
    using *assms* **by** (*auto simp*: *AE_all_countable*)
  have ($\sum i.\ integral^N\ M\ (f\ i)$) = (*SUP n*. $\sum i{<}n.\ integral^N\ M\ (f\ i)$)
    by (*rule suminf_eq_SUP*)
  also have ... = (*SUP n*. $\int^+ x.\ (\sum i{<}n.\ f\ i\ x)\ \partial M$)
    unfolding *nn_integral_sum*[*OF f*] **..**
  also have ... = $\int^+ x.\ (SUP\ n.\ \sum i{<}n.\ f\ i\ x)\ \partial M$ **using** *f all_pos*
    by (*intro nn_integral_monotone_convergence_SUP_AE*[*symmetric*])
      (*elim AE_mp*, *auto simp*: *sum_nonneg simp del*: *sum.lessThan_Suc intro*!: *AE_I2 sum_mono2*)

also have ... $= \int^+ x.\ (\sum i.\ f\ i\ x)\ \partial M$ **using** *all_pos*
by (*intro nn_integral_cong_AE*) (*auto simp*: *suminf_eq_SUP*)
finally show *?thesis* **by** *simp*
qed

lemma *nn_integral_bound_simple_function*:
assumes *bnd*: $\bigwedge x.\ x \in$ *space M* $\Longrightarrow f\ x < \infty$
assumes *f*[*measurable*]: *simple_function M f*
assumes *supp*: *emeasure M* $\{x \in$ *space M*. $f\ x \neq 0\} < \infty$
shows *nn_integral M f* $< \infty$
proof *cases*
assume *space M* $= \{\}$
then have *nn_integral M f* $= (\int^+ x.\ 0\ \partial M)$
by (*intro nn_integral_cong*) *auto*
then show *?thesis* **by** *simp*
next
assume *space M* $\neq \{\}$
with *simple_functionD*(*1*)[*OF f*] *bnd* **have** *bnd*: $0 \leq$ *Max* (*f'space M*) $\wedge$ *Max* (*f'space M*) $< \infty$
by (*subst Max_less_iff*) (*auto simp*: *Max_ge_iff*)

have *nn_integral M f* $\leq (\int^+ x.$ *Max* (*f'space M*) $*$ *indicator* $\{x \in$ *space M*. $f\ x \neq 0\}\ x\ \partial M)$
proof (*rule nn_integral_mono*)
fix *x* **assume** $x \in$ *space M*
with *f* **show** $f\ x \leq$ *Max* (*f ' space M*) $*$ *indicator* $\{x \in$ *space M*. $f\ x \neq 0\}\ x$
by (*auto split*: *split_indicator intro*!: *Max_ge simple_functionD*)
qed
also have ... $< \infty$
using *bnd supp* **by** (*subst nn_integral_cmult*) (*auto simp*: *ennreal_mult_less_top*)
finally show *?thesis* **.**
qed

theorem *nn_integral_Markov_inequality*:
assumes *u*: $u \in$ *borel_measurable M* **and** $A \in$ *sets M*
shows (*emeasure M*) $(\{x \in$ *space M*. $1 \leq c * u\ x\} \cap A) \leq c * (\int^+ x.\ u\ x *$ *indicator A x* $\partial M)$
(**is** (*emeasure M*) *?A* $\leq$ _ $*$ *?PI*)
proof –
have *?A* $\in$ *sets M*
using ‹$A \in$ *sets M*› *u* **by** *auto*
hence (*emeasure M*) *?A* $= (\int^+ x.$ *indicator ?A x* $\partial M)$
using *nn_integral_indicator* **by** *simp*
also have ... $\leq (\int^+ x.\ c * (u\ x *$ *indicator A x*$)\ \partial M)$
using *u* **by** (*auto intro*!: *nn_integral_mono_AE simp*: *indicator_def*)
also have ... $= c * (\int^+ x.\ u\ x *$ *indicator A x* $\partial M)$
using *assms* **by** (*auto intro*!: *nn_integral_cmult*)
finally show *?thesis* **.**
qed

lemma *nn_integral_noteq_infinite*:
assumes *g*: $g \in$ *borel_measurable M* **and** *integral*^{N} *M g* $\neq \infty$
shows *AE x in M. g x* $\neq \infty$
proof (*rule ccontr*)
assume *c*: $\neg$ (*AE x in M. g x* $\neq \infty$)
have (*emeasure M*) {*x*∈*space M. g x* = ∞} ≠ *0*
using *c g* **by** (*auto simp add*: *AE_iff_null*)
then have *0* < (*emeasure M*) {*x*∈*space M. g x* = ∞}
by (*auto simp*: *zero_less_iff_neq_zero*)
then have ∞ = ∞ ∗ (*emeasure M*) {*x*∈*space M. g x* = ∞}
by (*auto simp*: *ennreal_top_eq_mult_iff*)
also have . . . ≤ ($\int^+$*x*. ∞ ∗ *indicator* {*x*∈*space M. g x* = ∞} *x* ∂*M*)
using *g* **by** (*subst nn_integral_cmult_indicator*) *auto*
also have . . . ≤ *integral*^{N} *M g*
using *assms* **by** (*auto intro*!: *nn_integral_mono_AE simp*: *indicator_def*)
finally show *False*
using ‹*integral*^{N} *M g* ≠ ∞› **by** (*auto simp*: *top_unique*)
qed

lemma *nn_integral_PInf*:
assumes *f*: $f \in$ *borel_measurable M* **and** *not_Inf*: *integral*^{N} *M f* ≠ ∞
shows *emeasure M* (*f* −‘ {∞} ∩ *space M*) = *0*
proof −
have ∞ ∗ *emeasure M* (*f* −‘ {∞} ∩ *space M*) = ($\int^+$ *x*. ∞ ∗ *indicator* (*f* −‘ {∞} ∩ *space M*) *x* ∂*M*)
using *f* **by** (*subst nn_integral_cmult_indicator*) (*auto simp*: *measurable_sets*)
also have . . . ≤ *integral*^{N} *M f*
by (*auto intro*!: *nn_integral_mono simp*: *indicator_def*)
finally have ∞ ∗ (*emeasure M*) (*f* −‘ {∞} ∩ *space M*) ≤ *integral*^{N} *M f*
by *simp*
then show *?thesis*
using *assms* **by** (*auto simp*: *ennreal_top_mult top_unique split*: *if_split_asm*)
qed

lemma *simple_integral_PInf*:
simple_function M f ⟹ *integral*^{S} *M f* ≠ ∞ ⟹ *emeasure M* (*f* −‘ {∞} ∩ *space M*) = *0*
by (*rule nn_integral_PInf*) (*auto simp*: *nn_integral_eq_simple_integral borel_measurable_simple_function*)

lemma *nn_integral_PInf_AE*:
assumes $f \in$ *borel_measurable M integral*^{N} *M f* ≠ ∞ **shows** *AE x in M. f x* ≠ ∞
proof (*rule AE_I*)
show (*emeasure M*) (*f* −‘ {∞} ∩ *space M*) = *0*
by (*rule nn_integral_PInf*[*OF assms*])
show *f* −‘ {∞} ∩ *space M* ∈ *sets M*
using *assms* **by** (*auto intro*: *borel_measurable_vimage*)
qed *auto*

lemma *nn_integral_diff*:
assumes *f*: $f \in$ *borel_measurable M*
and *g*: $g \in$ *borel_measurable M*
and *fin*: $integral^N\ M\ g \neq \infty$
and *mono*: *AE x in M*. $g\ x \leq f\ x$
shows $(\int^+ x.\ f\ x - g\ x\ \partial M) = integral^N\ M\ f - integral^N\ M\ g$
proof −
have *diff*: $(\lambda x.\ f\ x - g\ x) \in$ *borel_measurable M*
using *assms* **by** *auto*
have *AE x in M*. $f\ x = f\ x - g\ x + g\ x$
using *diff_add_cancel_ennreal mono nn_integral_noteq_infinite*[*OF g fin*] *assms*
by *auto*
then have ∗∗: $integral^N\ M\ f = (\int^+ x.\ f\ x - g\ x\ \partial M) + integral^N\ M\ g$
unfolding *nn_integral_add*[*OF diff g*, *symmetric*]
by (*rule nn_integral_cong_AE*)
show *?thesis* **unfolding** ∗∗
using *fin*
by (*cases rule*: *ennreal2_cases*[*of* $\int^+ x.\ f\ x - g\ x\ \partial M\ integral^N\ M\ g$]) *auto*
qed

lemma *nn_integral_mult_bounded_inf*:
assumes *f*: $f \in$ *borel_measurable M* $(\int^+ x.\ f\ x\ \partial M) < \infty$ **and** *c*: $c \neq \infty$ **and** *ae*: *AE x in M*. $g\ x \leq c * f\ x$
shows $(\int^+ x.\ g\ x\ \partial M) < \infty$
proof −
have $(\int^+ x.\ g\ x\ \partial M) \leq (\int^+ x.\ c * f\ x\ \partial M)$
by (*intro nn_integral_mono_AE ae*)
also have $(\int^+ x.\ c * f\ x\ \partial M) < \infty$
using *c f* **by** (*subst nn_integral_cmult*) (*auto simp*: *ennreal_mult_less_top top_unique not_less*)
finally show *?thesis* **.**
qed

Fatou's lemma: convergence theorem on limes inferior

lemma *nn_integral_monotone_convergence_INF_AE'*:
assumes *f*: $\bigwedge i.$ *AE x in M*. $f\ (Suc\ i)\ x \leq f\ i\ x$ **and** [*measurable*]: $\bigwedge i.\ f\ i \in$ *borel_measurable M*
and ∗: $(\int^+ x.\ f\ 0\ x\ \partial M) < \infty$
shows $(\int^+ x.\ (INF\ i.\ f\ i\ x)\ \partial M) = (INF\ i.\ integral^N\ M\ (f\ i))$
proof (*rule ennreal_minus_cancel*)
have $integral^N\ M\ (f\ 0) - (\int^+ x.\ (INF\ i.\ f\ i\ x)\ \partial M) = (\int^+ x.\ f\ 0\ x - (INF\ i.\ f\ i\ x)\ \partial M)$
proof (*rule nn_integral_diff*[*symmetric*])
have $(\int^+ x.\ (INF\ i.\ f\ i\ x)\ \partial M) \leq (\int^+ x.\ f\ 0\ x\ \partial M)$
by (*intro nn_integral_mono INF_lower*) *simp*
with ∗ **show** $(\int^+ x.\ (INF\ i.\ f\ i\ x)\ \partial M) \neq \infty$
by *simp*
qed (*auto intro*: *INF_lower*)

also have ... = ($\int^+ x.$ (*SUP i. f 0 x − f i x*) ∂M)
  by (*simp add: ennreal_INF_const_minus*)
also have ... = (*SUP i.* ($\int^+ x.$ *f 0 x − f i x* ∂M))
proof (*intro nn_integral_monotone_convergence_SUP_AE*)
  show *AE x in M. f 0 x − f i x ≤ f 0 x − f (Suc i) x* **for** *i*
    using *f[of i]* **by** *eventually_elim (auto simp: ennreal_mono_minus)*
qed *simp*
also have ... = (*SUP i. nn_integral M (f 0)* − ($\int^+ x.$ *f i x* ∂M))
proof (*subst nn_integral_diff[symmetric]*)
  fix *i*
  have *dec: AE x in M. ∀ i. f (Suc i) x ≤ f i x*
    unfolding *AE_all_countable* **using** *f* **by** *auto*
  then show *AE x in M. f i x ≤ f 0 x*
    using *dec* **by** *eventually_elim (auto intro: lift_Suc_antimono_le[of λi. f i x 0 i*
for *x*])
  then have ($\int^+ x.$ *f i x* ∂M) ≤ ($\int^+ x.$ *f 0 x* ∂M)
    by (*rule nn_integral_mono_AE*)
  with ∗ **show** ($\int^+ x.$ *f i x* ∂M) ≠ ∞
    by *simp*
qed (*insert f, auto simp: decseq_def le_fun_def*)
finally show $integral^N$ *M (f 0)* − ($\int^+ x.$ (*INF i. f i x*) ∂M) =
  $integral^N$ *M (f 0)* − (*INF i.* $\int^+ x.$ *f i x* ∂M)
  by (*simp add: ennreal_INF_const_minus*)
qed (*insert* ∗, *auto intro!: nn_integral_mono intro: INF_lower*)

theorem *nn_integral_monotone_convergence_INF_AE*:
  fixes *f* :: *nat ⇒ 'a ⇒ ennreal*
  assumes *f*: ⋀*i. AE x in M. f (Suc i) x ≤ f i x*
    and [*measurable*]: ⋀*i. f i ∈ borel_measurable M*
    and *fin*: ($\int^+ x.$ *f i x* ∂M) < ∞
  shows ($\int^+ x.$ (*INF i. f i x*) ∂M) = (*INF i.* $integral^N$ *M (f i)*)
proof −
  { **fix** *f* :: *nat ⇒ ennreal* **and** *j* **assume** *decseq f*
    then have (*INF i. f i*) = (*INF i. f (i + j)*)
      apply (*intro INF_eq*)
      apply (*rule_tac x=i* **in** *bexI*)
      apply (*auto simp: decseq_def le_fun_def*)
      done }
  note *INF_shift = this*
  have *mono: AE x in M. ∀ i. f (Suc i) x ≤ f i x*
    using *f* **by** (*auto simp: AE_all_countable*)
  then have *AE x in M.* (*INF i. f i x*) = (*INF n. f (n + i) x*)
    by *eventually_elim (auto intro!: decseq_SucI INF_shift)*
  then have ($\int^+ x.$ (*INF i. f i x*) ∂M) = ($\int^+ x.$ (*INF n. f (n + i) x*) ∂M)
    by (*rule nn_integral_cong_AE*)
  also have ... = (*INF n.* ($\int^+ x.$ *f (n + i) x* ∂M))
    by (*rule nn_integral_monotone_convergence_INF_AE'*) (*insert assms, auto*)
  also have ... = (*INF n.* ($\int^+ x.$ *f n x* ∂M))
    by (*intro INF_shift[symmetric] decseq_SucI nn_integral_mono_AE f*)

finally show *?thesis* **.**
qed

lemma *nn_integral_monotone_convergence_INF_decseq*:
assumes *f*: *decseq f* **and** $*$: $\bigwedge i.\ f\ i \in borel\_measurable\ M\ (\int^+ x.\ f\ i\ x\ \partial M) < \infty$
shows $(\int^+ x.\ (INF\ i.\ f\ i\ x)\ \partial M) = (INF\ i.\ integral^N\ M\ (f\ i))$
using *nn_integral_monotone_convergence_INF_AE*[*of f M i*, *OF _ $*$*] *f* **by** (*auto simp*: *decseq_Suc_iff le_fun_def*)

theorem *nn_integral_liminf*:
fixes $u :: nat \Rightarrow 'a \Rightarrow ennreal$
assumes *u*: $\bigwedge i.\ u\ i \in borel\_measurable\ M$
shows $(\int^+ x.\ liminf\ (\lambda n.\ u\ n\ x)\ \partial M) \leq liminf\ (\lambda n.\ integral^N\ M\ (u\ n))$
proof −
have $(\int^+ x.\ liminf\ (\lambda n.\ u\ n\ x)\ \partial M) = (SUP\ n.\ \int^+ x.\ (INF\ i\in\{n..\}.\ u\ i\ x)\ \partial M)$
unfolding *liminf_SUP_INF* **using** *u*
by (*intro nn_integral_monotone_convergence_SUP_AE*)
(*auto intro*!: *AE_I2 intro*: *INF_greatest INF_superset_mono*)
also have $\ldots \leq liminf\ (\lambda n.\ integral^N\ M\ (u\ n))$
by (*auto simp*: *liminf_SUP_INF intro*!: *SUP_mono INF_greatest nn_integral_mono INF_lower*)
finally show *?thesis* **.**
qed

theorem *nn_integral_limsup*:
fixes $u :: nat \Rightarrow 'a \Rightarrow ennreal$
assumes [*measurable*]: $\bigwedge i.\ u\ i \in borel\_measurable\ M\ w \in borel\_measurable\ M$
assumes *bounds*: $\bigwedge i.\ AE\ x\ in\ M.\ u\ i\ x \leq w\ x$ **and** *w*: $(\int^+ x.\ w\ x\ \partial M) < \infty$
shows $limsup\ (\lambda n.\ integral^N\ M\ (u\ n)) \leq (\int^+ x.\ limsup\ (\lambda n.\ u\ n\ x)\ \partial M)$
proof −
have *bnd*: $AE\ x\ in\ M.\ \forall i.\ u\ i\ x \leq w\ x$
using *bounds* **by** (*auto simp*: *AE_all_countable*)
then have $(\int^+ x.\ (SUP\ n.\ u\ n\ x)\ \partial M) \leq (\int^+ x.\ w\ x\ \partial M)$
by (*auto intro*!: *nn_integral_mono_AE elim*: *eventually_mono intro*: *SUP_least*)
then have $(\int^+ x.\ limsup\ (\lambda n.\ u\ n\ x)\ \partial M) = (INF\ n.\ \int^+ x.\ (SUP\ i\in\{n..\}.\ u\ i\ x)\ \partial M)$
unfolding *limsup_INF_SUP* **using** *bnd w*
by (*intro nn_integral_monotone_convergence_INF_AE'*)
(*auto intro*!: *AE_I2 intro*: *SUP_least SUP_subset_mono*)
also have $\ldots \geq limsup\ (\lambda n.\ integral^N\ M\ (u\ n))$
by (*auto simp*: *limsup_INF_SUP intro*!: *INF_mono SUP_least exI nn_integral_mono SUP_upper*)
finally (*xtrans*) **show** *?thesis* **.**
qed

lemma *nn_integral_LIMSEQ*:
assumes *f*: *incseq f* $\bigwedge i.\ f\ i \in borel\_measurable\ M$

and u: $\bigwedge x.\ (\lambda i.\ f\ i\ x) \longrightarrow u\ x$
shows $(\lambda n.\ integral^N\ M\ (f\ n)) \longrightarrow integral^N\ M\ u$
proof –
have $(\lambda n.\ integral^N\ M\ (f\ n)) \longrightarrow (SUP\ n.\ integral^N\ M\ (f\ n))$
using f **by** (*intro LIMSEQ_SUP*[*of* $\lambda n.\ integral^N\ M\ (f\ n)$] *incseq_nn_integral*)
also have $(SUP\ n.\ integral^N\ M\ (f\ n)) = integral^N\ M\ (\lambda x.\ SUP\ n.\ f\ n\ x)$
using f **by** (*intro nn_integral_monotone_convergence_SUP*[*symmetric*])
also have $integral^N\ M\ (\lambda x.\ SUP\ n.\ f\ n\ x) = integral^N\ M\ (\lambda x.\ u\ x)$
using f **by** (*subst LIMSEQ_SUP*[*THEN LIMSEQ_unique*, *OF* _ *u*]) (*auto simp*: *incseq_def le_fun_def*)
finally show *?thesis* **.**
qed

theorem *nn_integral_dominated_convergence*:
assumes [*measurable*]:
$\bigwedge i.\ u\ i \in borel\_measurable\ M\ u' \in borel\_measurable\ M\ w \in borel\_measurable\ M$
and *bound*: $\bigwedge j.\ AE\ x\ in\ M.\ u\ j\ x \le w\ x$
and w: $(\int^+ x.\ w\ x\ \partial M) < \infty$
and u': $AE\ x\ in\ M.\ (\lambda i.\ u\ i\ x) \longrightarrow u'\ x$
shows $(\lambda i.\ (\int^+ x.\ u\ i\ x\ \partial M)) \longrightarrow (\int^+ x.\ u'\ x\ \partial M)$
proof –
have $limsup\ (\lambda n.\ integral^N\ M\ (u\ n)) \le (\int^+ x.\ limsup\ (\lambda n.\ u\ n\ x)\ \partial M)$
by (*intro nn_integral_limsup*[*OF* _ _ *bound w*]) *auto*
moreover have $(\int^+ x.\ limsup\ (\lambda n.\ u\ n\ x)\ \partial M) = (\int^+ x.\ u'\ x\ \partial M)$
using u' **by** (*intro nn_integral_cong_AE*, *eventually_elim*) (*metis tendsto_iff_Liminf_eq_Limsup sequentially_bot*)
moreover have $(\int^+ x.\ liminf\ (\lambda n.\ u\ n\ x)\ \partial M) = (\int^+ x.\ u'\ x\ \partial M)$
using u' **by** (*intro nn_integral_cong_AE*, *eventually_elim*) (*metis tendsto_iff_Liminf_eq_Limsup sequentially_bot*)
moreover have $(\int^+ x.\ liminf\ (\lambda n.\ u\ n\ x)\ \partial M) \le liminf\ (\lambda n.\ integral^N\ M\ (u\ n))$
by (*intro nn_integral_liminf*) *auto*
moreover have $liminf\ (\lambda n.\ integral^N\ M\ (u\ n)) \le limsup\ (\lambda n.\ integral^N\ M\ (u\ n))$
by (*intro Liminf_le_Limsup sequentially_bot*)
ultimately show *?thesis*
by (*intro Liminf_eq_Limsup*) *auto*
qed

lemma *inf_continuous_nn_integral*[*order_continuous_intros*]:
assumes f: $\bigwedge y.\ inf\_continuous\ (f\ y)$
assumes [*measurable*]: $\bigwedge x.\ (\lambda y.\ f\ y\ x) \in borel\_measurable\ M$
assumes *bnd*: $\bigwedge x.\ (\int^+ y.\ f\ y\ x\ \partial M) \ne \infty$
shows $inf\_continuous\ (\lambda x.\ (\int^+ y.\ f\ y\ x\ \partial M))$
unfolding *inf_continuous_def*
proof *safe*
fix C :: $nat \Rightarrow 'b$ **assume** C: *decseq* C
then show $(\int^+ y.\ f\ y\ (Inf\ (C\ `\ UNIV))\ \partial M) = (INF\ i.\ \int^+ y.\ f\ y\ (C\ i)\ \partial M)$

```
  using inf_continuous_mono[OF f] bnd
  by (auto simp add: inf_continuousD[OF f C] fun_eq_iff antimono_def mono_def
le_fun_def less_top
        intro!: nn_integral_monotone_convergence_INF_decseq)
qed

lemma nn_integral_null_set:
 assumes N ∈ null_sets M shows (∫+ x. u x * indicator N x ∂M) = 0
proof -
 have (∫+ x. u x * indicator N x ∂M) = (∫+ x. 0 ∂M)
 proof (intro nn_integral_cong_AE AE_I)
  show {x ∈ space M. u x * indicator N x ≠ 0} ⊆ N
   by (auto simp: indicator_def)
  show (emeasure M) N = 0 N ∈ sets M
   using assms by auto
 qed
 then show ?thesis by simp
qed

lemma nn_integral_0_iff:
 assumes u: u ∈ borel_measurable M
 shows integral^N M u = 0 ⟷ emeasure M {x∈space M. u x ≠ 0} = 0
  (is _ ⟷ (emeasure M) ?A = 0)
proof -
 have u_eq: (∫+ x. u x * indicator ?A x ∂M) = integral^N M u
  by (auto intro!: nn_integral_cong simp: indicator_def)
 show ?thesis
 proof
  assume (emeasure M) ?A = 0
  with nn_integral_null_set[of ?A M u] u
  show integral^N M u = 0 by (simp add: u_eq null_sets_def)
 next
  assume *: integral^N M u = 0
  let ?M = λn. {x ∈ space M. 1 ≤ real (n::nat) * u x}
  have 0 = (SUP n. (emeasure M) (?M n ∩ ?A))
  proof -
   { fix n :: nat
    from nn_integral_Markov_inequality[OF u, of ?A of_nat n] u
    have (emeasure M) (?M n ∩ ?A) ≤ 0
     by (simp add: ennreal_of_nat_eq_real_of_nat u_eq *)
    moreover have 0 ≤ (emeasure M) (?M n ∩ ?A) using u by auto
    ultimately have (emeasure M) (?M n ∩ ?A) = 0 by auto }
   thus ?thesis by simp
  qed
  also have ... = (emeasure M) (⋃n. ?M n ∩ ?A)
  proof (safe intro!: SUP_emeasure_incseq)
   fix n show ?M n ∩ ?A ∈ sets M
    using u by (auto intro!: sets.Int)
  next
```

```
    show incseq (λn. {x ∈ space M. 1 ≤ real n ∗ u x} ∩ {x ∈ space M. u x ≠
0})
      proof (safe intro!: incseq_SucI)
        fix n :: nat and x
        assume ∗: 1 ≤ real n ∗ u x
        also have real n ∗ u x ≤ real (Suc n) ∗ u x
          by (auto intro!: mult_right_mono)
        finally show 1 ≤ real (Suc n) ∗ u x by auto
      qed
    qed
    also have . . . = (emeasure M) {x∈space M. 0 < u x}
    proof (safe intro!: arg_cong[where f=(emeasure M)])
      fix x assume 0 < u x and [simp, intro]: x ∈ space M
      show x ∈ (⋃n. ?M n ∩ ?A)
      proof (cases u x rule: ennreal_cases)
        case (real r) with ‹0 < u x› have 0 < r by auto
        obtain j :: nat where 1 / r ≤ real j using real_arch_simple ..
        hence 1 / r ∗ r ≤ real j ∗ r unfolding mult_le_cancel_right using ‹0 < r›
by auto
        hence 1 ≤ real j ∗ r using real ‹0 < r› by auto
        thus ?thesis using ‹0 < r› real
            by (auto simp: ennreal_of_nat_eq_real_of_nat ennreal_1[symmetric] en-
nreal_mult[symmetric]
                  simp del: ennreal_1)
      qed (insert ‹0 < u x›, auto simp: ennreal_mult_top)
    qed (auto simp: zero_less_iff_neq_zero)
    finally show emeasure M ?A = 0
      by (simp add: zero_less_iff_neq_zero)
  qed
qed

lemma nn_integral_0_iff_AE:
  assumes u: u ∈ borel_measurable M
  shows integral^N M u = 0 ⟷ (AE x in M. u x = 0)
proof −
  have sets: {x∈space M. u x ≠ 0} ∈ sets M
    using u by auto
  show integral^N M u = 0 ⟷ (AE x in M. u x = 0)
    using nn_integral_0_iff[of u] AE_iff_null[OF sets] u by auto
qed

lemma AE_iff_nn_integral:
  {x∈space M. P x} ∈ sets M ⟹ (AE x in M. P x) ⟷ integral^N M (indicator
{x. ¬ P x}) = 0
  by (subst nn_integral_0_iff_AE) (auto simp: indicator_def[abs_def])

lemma nn_integral_less:
  assumes [measurable]: f ∈ borel_measurable M g ∈ borel_measurable M
  assumes f: (∫⁺x. f x ∂M) ≠ ∞
```

assumes *ord*: $AE\ x\ in\ M.\ f\ x \leq g\ x$ $\neg$ $(AE\ x\ in\ M.\ g\ x \leq f\ x)$
shows $(\int^+ x.\ f\ x\ \partial M) < (\int^+ x.\ g\ x\ \partial M)$
proof –
have $0 < (\int^+ x.\ g\ x - f\ x\ \partial M)$
proof (*intro order_le_neq_trans notI*)
assume $0 = (\int^+ x.\ g\ x - f\ x\ \partial M)$
then have $AE\ x\ in\ M.\ g\ x - f\ x = 0$
using *nn_integral_0_iff_AE*[*of* $\lambda x.\ g\ x - f\ x\ M$] **by** *simp*
with *ord*(*1*) **have** $AE\ x\ in\ M.\ g\ x \leq f\ x$
by *eventually_elim* (*auto simp*: *ennreal_minus_eq_0*)
with *ord* **show** *False*
by *simp*
qed *simp*
also have $\ldots = (\int^+ x.\ g\ x\ \partial M) - (\int^+ x.\ f\ x\ \partial M)$
using *f* **by** (*subst nn_integral_diff*) (*auto simp*: *ord*)
finally show *?thesis*
using *f* **by** (*auto dest*!: *ennreal_minus_pos_iff*[*rotated*] *simp*: *less_top*)
qed

lemma *nn_integral_subalgebra*:
assumes *f*: $f \in$ *borel_measurable* N
and *N*: *sets* $N \subseteq$ *sets* M *space* N = *space* M $\bigwedge A.\ A \in$ *sets* $N \Longrightarrow$ *emeasure* N A = *emeasure* M A
shows $integral^N\ N\ f = integral^N\ M\ f$
proof –
have [*simp*]: $\bigwedge f :: 'a \Rightarrow ennreal.\ f \in$ *borel_measurable* $N \Longrightarrow f \in$ *borel_measurable* M
using *N* **by** (*auto simp*: *measurable_def*)
have [*simp*]: $\bigwedge P.\ (AE\ x\ in\ N.\ P\ x) \Longrightarrow (AE\ x\ in\ M.\ P\ x)$
using *N* **by** (*auto simp add*: *eventually_ae_filter null_sets_def subset_eq*)
have [*simp*]: $\bigwedge A.\ A \in$ *sets* $N \Longrightarrow A \in$ *sets* M
using *N* **by** *auto*
from *f* **show** *?thesis*
apply *induct*
apply (*simp_all add*: *nn_integral_add nn_integral_cmult nn_integral_monotone_convergence_SUP N image_comp*)
apply (*auto intro*!: *nn_integral_cong cong*: *nn_integral_cong simp*: *N*(*2*)[*symmetric*])
done
qed

lemma *nn_integral_nat_function*:
fixes $f :: 'a \Rightarrow nat$
assumes $f \in$ *measurable* M (*count_space UNIV*)
shows $(\int^+ x.\ of\_nat\ (f\ x)\ \partial M) = (\sum t.\ emeasure\ M\ \{x \in space\ M.\ t < f\ x\})$
proof –
define *F* **where** $F\ i = \{x \in space\ M.\ i < f\ x\}$ **for** *i*
with *assms* **have** [*measurable*]: $\bigwedge i.\ F\ i \in$ *sets* M
by *auto*

{ fix x **assume** $x \in$ *space M*
have (λi. *if* $i < f\ x$ *then 1 else 0*) *sums* (*of_nat* ($f\ x$)::*real*)
using *sums_If_finite*[*of* λi. $i < f\ x$ λ_. *1*::*real*] **by** *simp*
then have (λi. *ennreal* (*if* $i < f\ x$ *then 1 else 0*)) *sums of_nat*($f\ x$)
unfolding *ennreal_of_nat_eq_real_of_nat*
by (*subst sums_ennreal*) *auto*
moreover have $\bigwedge i$. *ennreal* (*if* $i < f\ x$ *then 1 else 0*) = *indicator* ($F\ i$) x
using ‹$x \in$ *space M*› **by** (*simp add*: *one_ennreal_def F_def*)
ultimately have *of_nat* ($f\ x$) = ($\sum i$. *indicator* ($F\ i$) x :: *ennreal*)
by (*simp add*: *sums_iff*) **}**
then have ($\int^+ x$. *of_nat* ($f\ x$) ∂M) = ($\int^+ x$. ($\sum i$. *indicator* ($F\ i$) x) ∂M)
by (*simp cong*: *nn_integral_cong*)
also have ... = ($\sum i$. *emeasure M* ($F\ i$))
by (*simp add*: *nn_integral_suminf*)
finally show *?thesis*
by (*simp add*: *F_def*)
qed

theorem *nn_integral_lfp*:
assumes *sets*[*simp*]: $\bigwedge s$. *sets* ($M\ s$) = *sets N*
assumes f: *sup_continuous f*
assumes g: *sup_continuous g*
assumes *meas*: $\bigwedge F$. $F \in$ *borel_measurable N* $\Longrightarrow$ $f\ F \in$ *borel_measurable N*
assumes *step*: $\bigwedge F\ s$. $F \in$ *borel_measurable N* $\Longrightarrow$ *integral*^{N} ($M\ s$) ($f\ F$) = g (λs. *integral*^{N} ($M\ s$) F) s
shows ($\int^+ \omega$. *lfp f* ω $\partial M\ s$) = *lfp g s*
proof (*subst lfp_transfer_bounded*[**where** α=$\lambda F\ s$. $\int^+ x$. $F\ x$ $\partial M\ s$ **and** g=g **and** f=f **and** P=λf. $f \in$ *borel_measurable N*, *symmetric*])
fix C :: *nat* $\Rightarrow$ *'b* $\Rightarrow$ *ennreal* **assume** *incseq C* $\bigwedge i$. $C\ i \in$ *borel_measurable N*
then show (λs. $\int^+ x$. (*SUP i*. $C\ i$) x $\partial M\ s$) = (*SUP i*. (λs. $\int^+ x$. $C\ i\ x$ $\partial M\ s$))
unfolding *SUP_apply*[*abs_def*]
by (*subst nn_integral_monotone_convergence_SUP*)
(*auto simp*: *mono_def fun_eq_iff intro*!: *arg_cong2*[**where** f=*emeasure*] *cong*: *measurable_cong_sets*)
qed (*auto simp add*: *step le_fun_def SUP_apply*[*abs_def*] *bot_fun_def bot_ennreal intro*!: *meas f g*)

theorem *nn_integral_gfp*:
assumes *sets*[*simp*]: $\bigwedge s$. *sets* ($M\ s$) = *sets N*
assumes f: *inf_continuous f* **and** g: *inf_continuous g*
assumes *meas*: $\bigwedge F$. $F \in$ *borel_measurable N* $\Longrightarrow$ $f\ F \in$ *borel_measurable N*
assumes *bound*: $\bigwedge F\ s$. $F \in$ *borel_measurable N* $\Longrightarrow$ ($\int^+ x$. $f\ F\ x$ $\partial M\ s$) < ∞
assumes *non_zero*: $\bigwedge s$. *emeasure* ($M\ s$) (*space* ($M\ s$)) $\neq$ *0*
assumes *step*: $\bigwedge F\ s$. $F \in$ *borel_measurable N* $\Longrightarrow$ *integral*^{N} ($M\ s$) ($f\ F$) = g (λs. *integral*^{N} ($M\ s$) F) s
shows ($\int^+ \omega$. *gfp f* ω $\partial M\ s$) = *gfp g s*
proof (*subst gfp_transfer_bounded*[**where** α=$\lambda F\ s$. $\int^+ x$. $F\ x$ $\partial M\ s$ **and** g=g **and** f=f
and P=λF. $F \in$ *borel_measurable N* $\wedge$ ($\forall s$. ($\int^+ x$. $F\ x$ $\partial M\ s$) < ∞), *symmetric*])

fix C :: *nat* ⇒ *'b* ⇒ *ennreal* **assume** *decseq C* $\bigwedge i$. *C i* ∈ *borel_measurable N* ∧ (∀ *s*. *integral*^{N} (*M s*) (*C i*) < ∞)
then show (*λs*. ∫$^+$*x*. (*INF i*. *C i*) *x* ∂*M s*) = (*INF i*. (*λs*. ∫$^+$*x*. *C i x* ∂*M s*))
unfolding *INF_apply*[*abs_def*]
by (*subst nn_integral_monotone_convergence_INF_decseq*)
(*auto simp*: *mono_def fun_eq_iff intro*!: *arg_cong2*[**where** *f*=*emeasure*] *cong*: *measurable_cong_sets*)
next
show $\bigwedge x$. *g x* ≤ (*λs*. *integral*^{N} (*M s*) (*f top*))
by (*subst step*)
(*auto simp add*: *top_fun_def less_le non_zero le_fun_def ennreal_top_mult*
cong del: *if_weak_cong intro*!: *monoD*[*OF inf_continuous_mono*[*OF g*], *THEN le_funD*])
next
fix *C* **assume** $\bigwedge i$::*nat*. *C i* ∈ *borel_measurable N* ∧ (∀ *s*. *integral*^{N} (*M s*) (*C i*) < ∞) *decseq C*
with *bound* **show** *Inf* (*C* ‘ *UNIV*) ∈ *borel_measurable N* ∧ (∀ *s*. *integral*^{N} (*M s*) (*Inf* (*C* ‘ *UNIV*)) < ∞)
unfolding *INF_apply*[*abs_def*]
by (*subst nn_integral_monotone_convergence_INF_decseq*)
(*auto simp*: *INF_less_iff cong*: *measurable_cong_sets intro*!: *borel_measurable_INF*)
next
show $\bigwedge x$. *x* ∈ *borel_measurable N* ∧ (∀ *s*. *integral*^{N} (*M s*) *x* < ∞) ⟹
(*λs*. *integral*^{N} (*M s*) (*f x*)) = *g* (*λs*. *integral*^{N} (*M s*) *x*)
by (*subst step*) *auto*
qed (*insert bound*, *auto simp add*: *le_fun_def INF_apply*[*abs_def*] *top_fun_def intro*!: *meas f g*)

6.6.5 Integral under concrete measures

lemma *nn_integral_mono_measure*:
assumes *sets M* = *sets N M* ≤ *N* **shows** *nn_integral M f* ≤ *nn_integral N f*
unfolding *nn_integral_def*
proof (*intro SUP_subset_mono*)
note ‹*sets M* = *sets N*›[*simp*] ‹*sets M* = *sets N*›[*THEN sets_eq_imp_space_eq*, *simp*]
show {*g*. *simple_function M g* ∧ *g* ≤ *f*} ⊆ {*g*. *simple_function N g* ∧ *g* ≤ *f*}
by (*simp add*: *simple_function_def*)
show *integral*^{S} *M x* ≤ *integral*^{S} *N x* **for** *x*
using *le_measureD3*[*OF* ‹*M* ≤ *N*›]
by (*auto simp add*: *simple_integral_def intro*!: *sum_mono mult_mono*)
qed

lemma *nn_integral_empty*:
assumes *space M* = {}
shows *nn_integral M f* = *0*
proof −
have (∫$^+$ *x*. *f x* ∂*M*) = (∫$^+$ *x*. *0* ∂*M*)
by(*rule nn_integral_cong*)(*simp add*: *assms*)

thus *?thesis* **by** *simp*
qed

lemma *nn_integral_bot*[*simp*]: *nn_integral bot f* = *0*
by (*simp add*: *nn_integral_empty*)

Distributions

lemma *nn_integral_distr*:
assumes *T*: *T* ∈ *measurable M M′* **and** *f*: *f* ∈ *borel_measurable* (*distr M M′ T*)
shows *integral*^{N} (*distr M M′ T*) *f* = (∫$^+$ *x*. *f* (*T x*) ∂*M*)
using *f*
proof *induct*
case (*cong f g*)
with *T* **show** *?case*
apply (*subst nn_integral_cong*[*of* _ *f g*])
apply *simp*
apply (*subst nn_integral_cong*[*of* _ λ*x*. *f* (*T x*) λ*x*. *g* (*T x*)])
apply (*simp add*: *measurable_def Pi_iff*)
apply *simp*
done
next
case (*set A*)
then have *eq*: ⋀*x*. *x* ∈ *space M* ⟹ *indicator A* (*T x*) = *indicator* (*T* −‘ *A* ∩ *space M*) *x*
by (*auto simp*: *indicator_def*)
from *set T* **show** *?case*
by (*subst nn_integral_cong*[*OF eq*])
(*auto simp add*: *emeasure_distr intro*!: *nn_integral_indicator*[*symmetric*] *measurable_sets*)
qed (*simp_all add*: *measurable_compose*[*OF T*] *T nn_integral_cmult nn_integral_add nn_integral_monotone_convergence_SUP le_fun_def incseq_def image_comp*)

Counting space

lemma *simple_function_count_space*[*simp*]:
simple_function (*count_space A*) *f* ⟷ *finite* (*f* ‘ *A*)
unfolding *simple_function_def* **by** *simp*

lemma *nn_integral_count_space*:
assumes *A*: *finite* {*a*∈*A*. *0* < *f a*}
shows *integral*^{N} (*count_space A*) *f* = (∑ *a*|*a*∈*A* ∧ *0* < *f a*. *f a*)
proof −
have ∗: (∫$^+$*x*. *max 0* (*f x*) ∂*count_space A*) =
(∫$^+$ *x*. (∑ *a*|*a*∈*A* ∧ *0* < *f a*. *f a* ∗ *indicator* {*a*} *x*) ∂*count_space A*)
by (*auto intro*!: *nn_integral_cong*
simp add: *indicator_def if_distrib sum.If_cases*[*OF A*] *max_def le_less*)
also have ... = (∑ *a*|*a*∈*A* ∧ *0* < *f a*. ∫$^+$ *x*. *f a* ∗ *indicator* {*a*} *x* ∂*count_space A*)

by (*subst nn_integral_sum*) (*simp_all add*: *AE_count_space less_imp_le*)
also have ... = ($\sum$ *a*|*a*∈*A* ∧ *0* < *f a*. *f a*)
by (*auto intro*!: *sum.cong simp*: *one_ennreal_def*[*symmetric*] *max_def*)
finally show *?thesis* **by** (*simp add*: *max.absorb2*)
qed

lemma *nn_integral_count_space_finite*:
finite A ⟹ ($\int^+$*x*. *f x* ∂*count_space A*) = ($\sum$ *a*∈*A*. *f a*)
by (*auto intro*!: *sum.mono_neutral_left simp*: *nn_integral_count_space less_le*)

lemma *nn_integral_count_space'*:
assumes *finite A* ⋀*x*. *x* ∈ *B* ⟹ *x* ∉ *A* ⟹ *f x* = *0 A* ⊆ *B*
shows ($\int^+$*x*. *f x* ∂*count_space B*) = ($\sum$ *x*∈*A*. *f x*)
proof −
have ($\int^+$*x*. *f x* ∂*count_space B*) = ($\sum$ *a* | *a* ∈ *B* ∧ *0* < *f a*. *f a*)
using *assms*(*2*,*3*)
by (*intro nn_integral_count_space finite_subset*[*OF* _ ‹*finite A*›]) (*auto simp*: *less_le*)
also have ... = ($\sum$ *a*∈*A*. *f a*)
using *assms* **by** (*intro sum.mono_neutral_cong_left*) (*auto simp*: *less_le*)
finally show *?thesis* **.**
qed

lemma *nn_integral_bij_count_space*:
assumes *g*: *bij_betw g A B*
shows ($\int^+$*x*. *f* (*g x*) ∂*count_space A*) = ($\int^+$*x*. *f x* ∂*count_space B*)
using *g*[*THEN bij_betw_imp_funcset*]
by (*subst distr_bij_count_space*[*OF g*, *symmetric*])
(*auto intro*!: *nn_integral_distr*[*symmetric*])

lemma *nn_integral_indicator_finite*:
fixes *f* :: *'a* ⇒ *ennreal*
assumes *f*: *finite A* **and** [*measurable*]: ⋀*a*. *a* ∈ *A* ⟹ {*a*} ∈ *sets M*
shows ($\int^+$*x*. *f x* ∗ *indicator A x* ∂*M*) = ($\sum$ *x*∈*A*. *f x* ∗ *emeasure M* {*x*})
proof −
from *f* **have** ($\int^+$*x*. *f x* ∗ *indicator A x* ∂*M*) = ($\int^+$*x*. ($\sum$ *a*∈*A*. *f a* ∗ *indicator* {*a*} *x*) ∂*M*)
by (*intro nn_integral_cong*) (*auto simp*: *indicator_def if_distrib*[**where** *f*=λ*a*. *x* ∗ *a* **for** *x*] *sum.If_cases*)
also have ... = ($\sum$ *a*∈*A*. *f a* ∗ *emeasure M* {*a*})
by (*subst nn_integral_sum*) *auto*
finally show *?thesis* **.**
qed

lemma *nn_integral_count_space_nat*:
fixes *f* :: *nat* ⇒ *ennreal*
shows ($\int^+$*i*. *f i* ∂*count_space UNIV*) = ($\sum$ *i*. *f i*)
proof −
have ($\int^+$*i*. *f i* ∂*count_space UNIV*) =

(∫⁺i. (∑j. f j ∗ indicator {j} i) ∂count_space UNIV)
proof (intro nn_integral_cong)
fix i
have f i = (∑j∈{i}. f j ∗ indicator {j} i)
by simp
also have ... = (∑j. f j ∗ indicator {j} i)
by (rule suminf_finite[symmetric]) auto
finally show f i = (∑j. f j ∗ indicator {j} i) **.**
qed
also have ... = (∑j. (∫⁺i. f j ∗ indicator {j} i ∂count_space UNIV))
by (rule nn_integral_suminf) auto
finally show ?thesis
by simp
qed

lemma nn_integral_enat_function:
assumes f: f ∈ measurable M (count_space UNIV)
shows (∫⁺ x. ennreal_of_enat (f x) ∂M) = (∑t. emeasure M {x ∈ space M. t < f x})
proof −
define F **where** F i = {x∈space M. i < f x} **for** i :: nat
with assms **have** [measurable]: ⋀i. F i ∈ sets M
by auto

{ **fix** x **assume** x ∈ space M
have (λi::nat. if i < f x then 1 else 0) sums ennreal_of_enat (f x)
using sums_If_finite[of λr. r < f x λ_. 1 :: ennreal]
by (cases f x) (simp_all add: sums_def of_nat_tendsto_top_ennreal)
also have (λi. (if i < f x then 1 else 0)) = (λi. indicator (F i) x)
using ‹x ∈ space M› **by** (simp add: one_ennreal_def F_def fun_eq_iff)
finally have ennreal_of_enat (f x) = (∑i. indicator (F i) x)
by (simp add: sums_iff) }
then have (∫⁺x. ennreal_of_enat (f x) ∂M) = (∫⁺x. (∑i. indicator (F i) x) ∂M)
by (simp cong: nn_integral_cong)
also have ... = (∑i. emeasure M (F i))
by (simp add: nn_integral_suminf)
finally show ?thesis
by (simp add: F_def)
qed

lemma nn_integral_count_space_nn_integral:
fixes f :: 'i ⇒ 'a ⇒ ennreal
assumes countable I **and** [measurable]: ⋀i. i ∈ I ⟹ f i ∈ borel_measurable M
shows (∫⁺x. ∫⁺i. f i x ∂count_space I ∂M) = (∫⁺i. ∫⁺x. f i x ∂M ∂count_space I)
proof cases
assume finite I **then show** ?thesis
by (simp add: nn_integral_count_space_finite nn_integral_sum)

next
  assume *infinite I*
  then have [*simp*]: $I \neq \{\}$
    by *auto*
  note $* =$ *bij_betw_from_nat_into*[*OF* ‹*countable I*› ‹*infinite I*›]
  have $**$: $\bigwedge f. (\bigwedge i.\ 0 \leq f\ i) \Longrightarrow (\int^+ i.\ f\ i\ \partial count\_space\ I) = (\sum n.\ f\ (from\_nat\_into\ I\ n))$
   by (*simp add*: *nn_integral_bij_count_space*[*symmetric*, *OF* $*$] *nn_integral_count_space_nat*)
  show *?thesis*
    by (*simp add*: $**$ *nn_integral_suminf from_nat_into*)
qed

lemma *of_bool_Bex_eq_nn_integral*:
  assumes *unique*: $\bigwedge x\ y.\ x \in X \Longrightarrow y \in X \Longrightarrow P\ x \Longrightarrow P\ y \Longrightarrow x = y$
  shows *of_bool* $(\exists y{\in}X.\ P\ y) = (\int^+ y.\ of\_bool\ (P\ y)\ \partial count\_space\ X)$
proof *cases*
  assume $\exists y{\in}X.\ P\ y$
  then obtain *y* **where** $P\ y\ y \in X$ **by** *auto*
  then show *?thesis*
    by (*subst nn_integral_count_space'*[**where** $A=\{y\}$]) (*auto dest*: *unique*)
qed (*auto cong*: *nn_integral_cong_simp*)

lemma *emeasure_UN_countable*:
  assumes *sets*[*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow X\ i \in sets\ M$ **and** *I*[*simp*]: *countable I*
  assumes *disj*: *disjoint_family_on X I*
  shows *emeasure M* $(\bigcup (X\ `\ I)) = (\int^+ i.\ emeasure\ M\ (X\ i)\ \partial count\_space\ I)$
proof −
  have *eq*: $\bigwedge x.\ indicator\ (\bigcup (X\ `\ I))\ x = \int^+ i.\ indicator\ (X\ i)\ x\ \partial count\_space\ I$
  proof *cases*
    fix *x* **assume** *x*: $x \in \bigcup (X\ `\ I)$
    then obtain *j* **where** *j*: $x \in X\ j\ j \in I$
      by *auto*
    with *disj* **have** $\bigwedge i.\ i \in I \Longrightarrow indicator\ (X\ i)\ x = (indicator\ \{j\}\ i::ennreal)$
      by (*auto simp*: *disjoint_family_on_def split*: *split_indicator*)
    with *x j* **show** *?thesis x*
      by (*simp cong*: *nn_integral_cong_simp*)
  qed (*auto simp*: *nn_integral_0_iff_AE*)

  note *sets.countable_UN'*[*unfolded subset_eq*, *measurable*]
  have *emeasure M* $(\bigcup (X\ `\ I)) = (\int^+ x.\ indicator\ (\bigcup (X\ `\ I))\ x\ \partial M)$
    by *simp*
  also have $\ldots = (\int^+ i.\ \int^+ x.\ indicator\ (X\ i)\ x\ \partial M\ \partial count\_space\ I)$
    by (*simp add*: *eq nn_integral_count_space_nn_integral*)
  finally show *?thesis*
    by (*simp cong*: *nn_integral_cong_simp*)
qed

lemma *emeasure_countable_singleton*:
  assumes *sets*: $\bigwedge x.\ x \in X \Longrightarrow \{x\} \in sets\ M$ **and** *X*: *countable X*

shows *emeasure M X* = ($\int^+ x$. *emeasure M* {*x*} ∂*count_space X*)
proof −
have *emeasure M* (⋃ *i*∈*X*. {*i*}) = ($\int^+ x$. *emeasure M* {*x*} ∂*count_space X*)
using *assms* **by** (*intro emeasure_UN_countable*) (*auto simp*: *disjoint_family_on_def*)
also have (⋃ *i*∈*X*. {*i*}) = *X* **by** *auto*
finally show *?thesis* **.**
qed

lemma *measure_eqI_countable*:
assumes [*simp*]: *sets M* = *Pow A sets N* = *Pow A* **and** *A*: *countable A*
assumes *eq*: ⋀*a*. *a* ∈ *A* ⟹ *emeasure M* {*a*} = *emeasure N* {*a*}
shows *M* = *N*
proof (*rule measure_eqI*)
fix *X* **assume** *X* ∈ *sets M*
then have *X*: *X* ⊆ *A* **by** *auto*
moreover from *A X* **have** *countable X* **by** (*auto dest*: *countable_subset*)
ultimately have
emeasure M X = ($\int^+ a$. *emeasure M* {*a*} ∂*count_space X*)
emeasure N X = ($\int^+ a$. *emeasure N* {*a*} ∂*count_space X*)
by (*auto intro*!: *emeasure_countable_singleton*)
moreover have ($\int^+ a$. *emeasure M* {*a*} ∂*count_space X*) = ($\int^+ a$. *emeasure N* {*a*} ∂*count_space X*)
using *X* **by** (*intro nn_integral_cong eq*) *auto*
ultimately show *emeasure M X* = *emeasure N X*
by *simp*
qed *simp*

lemma *measure_eqI_countable_AE*:
assumes [*simp*]: *sets M* = *UNIV sets N* = *UNIV*
assumes *ae*: *AE x in M*. *x* ∈ Ω *AE x in N*. *x* ∈ Ω **and** [*simp*]: *countable* Ω
assumes *eq*: ⋀*x*. *x* ∈ Ω ⟹ *emeasure M* {*x*} = *emeasure N* {*x*}
shows *M* = *N*
proof (*rule measure_eqI*)
fix *A*
have *emeasure N A* = *emeasure N* {*x*∈Ω. *x* ∈ *A*}
using *ae* **by** (*intro emeasure_eq_AE*) *auto*
also have ... = ($\int^+ x$. *emeasure N* {*x*} ∂*count_space* {*x*∈Ω. *x* ∈ *A*})
by (*intro emeasure_countable_singleton*) *auto*
also have ... = ($\int^+ x$. *emeasure M* {*x*} ∂*count_space* {*x*∈Ω. *x* ∈ *A*})
by (*intro nn_integral_cong eq*[*symmetric*]) *auto*
also have ... = *emeasure M* {*x*∈Ω. *x* ∈ *A*}
by (*intro emeasure_countable_singleton*[*symmetric*]) *auto*
also have ... = *emeasure M A*
using *ae* **by** (*intro emeasure_eq_AE*) *auto*
finally show *emeasure M A* = *emeasure N A* **..**
qed *simp*

lemma *nn_integral_monotone_convergence_SUP_nat*:
fixes *f* :: ′*a* ⇒ *nat* ⇒ *ennreal*

assumes *chain*: $Complete\_Partial\_Order.chain$ ($\leq$) (f ‘ Y)
and *nonempty*: $Y \neq \{\}$
shows ($\int^+ x.$ ($SUP\ i{\in}Y.\ f\ i\ x$) $\partial count\_space\ UNIV$) = ($SUP\ i{\in}Y.$ ($\int^+ x.\ f\ i$ $x\ \partial count\_space\ UNIV$))
(**is** *?lhs* = *?rhs* **is** $integral^N$ *?M* _ = _)
proof (*rule order_class.order.antisym*)
show *?rhs* $\leq$ *?lhs*
by (*auto intro*!: *SUP_least SUP_upper nn_integral_mono*)
next
have $\exists g.$ *incseq g* $\wedge$ *range g* $\subseteq$ ($\lambda i.\ f\ i\ x$) ‘ Y $\wedge$ ($SUP\ i{\in}Y.\ f\ i\ x$) = ($SUP\ i.\ g$ i) **for** x
by (*rule ennreal_Sup_countable_SUP*) (*simp add*: *nonempty*)
then obtain g **where** *incseq*: $\bigwedge x.$ *incseq* ($g\ x$)
and *range*: $\bigwedge x.$ *range* ($g\ x$) $\subseteq$ ($\lambda i.\ f\ i\ x$) ‘ Y
and *sup*: $\bigwedge x.$ ($SUP\ i{\in}Y.\ f\ i\ x$) = ($SUP\ i.\ g\ x\ i$) **by** *moura*
from *incseq* **have** *incseq'*: *incseq* ($\lambda i\ x.\ g\ x\ i$)
by(*blast intro*: *incseq_SucI le_funI dest*: *incseq_SucD*)

have *?lhs* = $\int^+ x.$ ($SUP\ i.\ g\ x\ i$) $\partial ?M$ **by**(*simp add*: *sup*)
also have ... = ($SUP\ i.\ \int^+ x.\ g\ x\ i\ \partial ?M$) **using** *incseq'*
by(*rule nn_integral_monotone_convergence_SUP*) *simp*
also have ... $\leq$ ($SUP\ i{\in}Y.\ \int^+ x.\ f\ i\ x\ \partial ?M$)
proof(*rule SUP_least*)
fix n
have $\bigwedge x.\ \exists i.\ g\ x\ n = f\ i\ x \wedge i \in Y$ **using** *range* **by** *blast*
then obtain I **where** I: $\bigwedge x.\ g\ x\ n = f\ (I\ x)\ x$ $\bigwedge x.\ I\ x \in Y$ **by** *moura*

have ($\int^+ x.\ g\ x\ n\ \partial count\_space\ UNIV$) = ($\sum x.\ g\ x\ n$)
by(*rule nn_integral_count_space_nat*)
also have ... = ($SUP\ m.\ \sum x{<}m.\ g\ x\ n$)
by(*rule suminf_eq_SUP*)
also have ... $\leq$ ($SUP\ i{\in}Y.\ \int^+ x.\ f\ i\ x\ \partial ?M$)
proof(*rule SUP_mono*)
fix m
show $\exists m'{\in}Y.$ ($\sum x{<}m.\ g\ x\ n$) $\leq$ ($\int^+ x.\ f\ m'\ x\ \partial ?M$)
proof(*cases* $m > 0$)
case *False*
thus *?thesis* **using** *nonempty* **by** *auto*
next
case *True*
let *?Y* = I ‘ $\{..{<}m\}$
have f ‘ *?Y* $\subseteq$ f ‘ Y **using** I **by** *auto*
with *chain* **have** *chain'*: $Complete\_Partial\_Order.chain$ ($\leq$) (f ‘ *?Y*) **by**(*rule chain_subset*)
hence *Sup* (f ‘ *?Y*) $\in$ f ‘ *?Y*
by(*rule ccpo_class.in_chain_finite*)(*auto simp add*: *True lessThan_empty_iff*)
then obtain m' **where** $m' < m$ **and** m': ($SUP\ i{\in}?Y.\ f\ i$) = f ($I\ m'$) **by** *auto*
have $I\ m' \in Y$ **using** I **by** *blast*

```
        have (∑x<m. g x n) ≤ (∑x<m. f (I m′) x)
        proof(rule sum_mono)
          fix x
          assume x ∈ {..<m}
          hence x < m by simp
          have g x n = f (I x) x by(simp add: I)
          also have ... ≤ (SUP i∈?Y. f i) x unfolding Sup_fun_def image_image
            using ‹x ∈ {..<m}› by (rule Sup_upper [OF imageI])
          also have ... = f (I m′) x unfolding m′ by simp
          finally show g x n ≤ f (I m′) x .
        qed
        also have ... ≤ (SUP m. (∑x<m. f (I m′) x))
          by(rule SUP_upper) simp
        also have ... = (∑x. f (I m′) x)
          by(rule suminf_eq_SUP[symmetric])
        also have ... = (∫+ x. f (I m′) x ∂?M)
          by(rule nn_integral_count_space_nat[symmetric])
        finally show ?thesis using ‹I m′ ∈ Y› by blast
      qed
    qed
    finally show (∫+ x. g x n ∂count_space UNIV) ≤ ... .
  qed
  finally show ?lhs ≤ ?rhs .
qed

lemma power_series_tendsto_at_left:
  assumes nonneg: ⋀i. 0 ≤ f i and summable: ⋀z. 0 ≤ z ⟹ z < 1 ⟹ summable
(λn. f n * z^n)
   shows ((λz. ennreal (∑n. f n * z^n)) ⟶ (∑n. ennreal (f n))) (at_left
(1::real))
proof (intro tendsto_at_left_sequentially)
  show 0 < (1::real) by simp
  fix S :: nat ⇒ real assume S: ⋀n. S n < 1 ⋀n. 0 < S n S ⟶ 1 incseq S
  then have S_nonneg: ⋀i. 0 ≤ S i by (auto intro: less_imp_le)

  have (λi. (∫+n. f n * S i^n ∂count_space UNIV)) ⟶ (∫+n. ennreal (f n)
∂count_space UNIV)
  proof (rule nn_integral_LIMSEQ)
    show incseq (λi n. ennreal (f n * S i^n))
      using S by (auto intro!: mult_mono power_mono nonneg ennreal_leI
                        simp: incseq_def le_fun_def less_imp_le)
    fix n have (λi. ennreal (f n * S i^n)) ⟶ ennreal (f n * 1^n)
      by (intro tendsto_intros tendsto_ennrealI S)
    then show (λi. ennreal (f n * S i^n)) ⟶ ennreal (f n)
      by simp
  qed (auto simp: S_nonneg intro!: mult_nonneg_nonneg nonneg)
  also have (λi. (∫+n. f n * S i^n ∂count_space UNIV)) = (λi. ∑n. f n * S i^n)
    by (subst nn_integral_count_space_nat)
      (intro ext suminf_ennreal2 mult_nonneg_nonneg nonneg S_nonneg
```

zero_le_power summable S)+
also have $(\int^+ n.\ ennreal\ (f\ n)\ \partial count\_space\ UNIV) = (\sum n.\ ennreal\ (f\ n))$
by (*simp add*: *nn_integral_count_space_nat nonneg*)
finally show $(\lambda n.\ ennreal\ (\sum na.\ f\ na * S\ n\ \hat{}\ na)) \longrightarrow (\sum n.\ ennreal\ (f\ n))$
.
qed

Measures with Restricted Space

lemma *simple_function_restrict_space_ennreal*:
fixes *f* :: *'a* ⇒ *ennreal*
assumes Ω ∩ *space M* ∈ *sets M*
shows *simple_function* (*restrict_space M* Ω) *f* ⟷ *simple_function M* (λ*x*. *f x* ∗ *indicator* Ω *x*)
proof −
{ **assume** *finite* (*f* ' *space* (*restrict_space M* Ω))
then have *finite* (*f* ' *space* (*restrict_space M* Ω) ∪ {*0*}) **by** *simp*
then have *finite* ((λ*x*. *f x* ∗ *indicator* Ω *x*) ' *space M*)
by (*rule rev_finite_subset*) (*auto split*: *split_indicator simp*: *space_restrict_space*)
}
moreover
{ **assume** *finite* ((λ*x*. *f x* ∗ *indicator* Ω *x*) ' *space M*)
then have *finite* (*f* ' *space* (*restrict_space M* Ω))
by (*rule rev_finite_subset*) (*auto split*: *split_indicator simp*: *space_restrict_space*)
}
ultimately show *?thesis*
unfolding
simple_function_iff_borel_measurable borel_measurable_restrict_space_iff_ennreal[*OF assms*]
by *auto*
qed

lemma *simple_function_restrict_space*:
fixes *f* :: *'a* ⇒ *'b*::*real_normed_vector*
assumes Ω ∩ *space M* ∈ *sets M*
shows *simple_function* (*restrict_space M* Ω) *f* ⟷ *simple_function M* (λ*x*. *indicator* Ω $x *_R f\ x$)
proof −
{ **assume** *finite* (*f* ' *space* (*restrict_space M* Ω))
then have *finite* (*f* ' *space* (*restrict_space M* Ω) ∪ {*0*}) **by** *simp*
then have *finite* ((λ*x*. *indicator* Ω $x *_R f\ x$) ' *space M*)
by (*rule rev_finite_subset*) (*auto split*: *split_indicator simp*: *space_restrict_space*)
}
moreover
{ **assume** *finite* ((λ*x*. *indicator* Ω $x *_R f\ x$) ' *space M*)
then have *finite* (*f* ' *space* (*restrict_space M* Ω))
by (*rule rev_finite_subset*) (*auto split*: *split_indicator simp*: *space_restrict_space*)
}
ultimately show *?thesis*

unfolding *simple_function_iff_borel_measurable*
borel_measurable_restrict_space_iff [*OF assms*]
by *auto*
qed

lemma *simple_integral_restrict_space*:
assumes Ω: Ω ∩ *space M* ∈ *sets M simple_function* (*restrict_space M* Ω) *f*
shows *simple_integral* (*restrict_space M* Ω) *f* = *simple_integral M* (λ*x*. *f x* ∗ *indicator* Ω *x*)
using *simple_function_restrict_space_ennreal*[*THEN iffD1*, *OF* Ω, *THEN simple_functionD*(*1*)]
by (*auto simp add*: *space_restrict_space emeasure_restrict_space*[*OF* Ω(*1*)] *le_infI2 simple_integral_def*
split: *split_indicator split_indicator_asm*
intro!: *sum.mono_neutral_cong_left ennreal_mult_left_cong arg_cong2*[**where** *f*=*emeasure*])

lemma *nn_integral_restrict_space*:
assumes Ω[*simp*]: Ω ∩ *space M* ∈ *sets M*
shows *nn_integral* (*restrict_space M* Ω) *f* = *nn_integral M* (λ*x*. *f x* ∗ *indicator* Ω *x*)
proof −
let *?R* = *restrict_space M* Ω **and** *?X* = λ*M f*. {*s*. *simple_function M s* ∧ *s* ≤ *f* ∧ (∀ *x*. *s x* < *top*)}
have *integral*^{S} *?R* ‘ *?X ?R f* = *integral*^{S} *M* ‘ *?X M* (λ*x*. *f x* ∗ *indicator* Ω *x*)
proof (*safe intro*!: *image_eqI*)
fix *s* **assume** *s*: *simple_function ?R s s* ≤ *f* ∀ *x*. *s x* < *top*
from *s* **show** *integral*^{S} (*restrict_space M* Ω) *s* = *integral*^{S} *M* (λ*x*. *s x* ∗ *indicator* Ω *x*)
by (*intro simple_integral_restrict_space*) *auto*
from *s* **show** *simple_function M* (λ*x*. *s x* ∗ *indicator* Ω *x*)
by (*simp add*: *simple_function_restrict_space_ennreal*)
from *s* **show** (λ*x*. *s x* ∗ *indicator* Ω *x*) ≤ (λ*x*. *f x* ∗ *indicator* Ω *x*)
⋀*x*. *s x* ∗ *indicator* Ω *x* < *top*
by (*auto split*: *split_indicator simp*: *le_fun_def image_subset_iff*)
next
fix *s* **assume** *s*: *simple_function M s s* ≤ (λ*x*. *f x* ∗ *indicator* Ω *x*) ∀ *x*. *s x* < *top*
then have *simple_function M* (λ*x*. *s x* ∗ *indicator* (Ω ∩ *space M*) *x*) (**is** *?s'*)
by (*intro simple_function_mult simple_function_indicator*) *auto*
also have *?s'* ⟷ *simple_function M* (λ*x*. *s x* ∗ *indicator* Ω *x*)
by (*rule simple_function_cong*) (*auto split*: *split_indicator*)
finally show *sf*: *simple_function* (*restrict_space M* Ω) *s*
by (*simp add*: *simple_function_restrict_space_ennreal*)

from *s* **have** *s_eq*: *s* = (λ*x*. *s x* ∗ *indicator* Ω *x*)
by (*auto simp add*: *fun_eq_iff le_fun_def image_subset_iff*
split: *split_indicator split_indicator_asm*
intro: *antisym*)

show *integral*^{S} *M s* = *integral*^{S} (*restrict_space M* Ω) *s*
by (*subst s_eq*) (*rule simple_integral_restrict_space*[*symmetric*, *OF* Ω *sf*])
show ⋀*x*. *s x* < *top*
using *s* **by** (*auto simp*: *image_subset_iff*)
from *s* **show** *s* ≤ *f*
by (*subst s_eq*) (*auto simp*: *image_subset_iff le_fun_def split*: *split_indicator split_indicator_asm*)
qed
then show *?thesis*
unfolding *nn_integral_def_finite* **by** (*simp cong del*: *SUP_cong_simp*)
qed

lemma *nn_integral_count_space_indicator*:
assumes *NO_MATCH* (*UNIV*::′*a set*) (*X*::′*a set*)
shows (∫⁺*x*. *f x* ∂*count_space X*) = (∫⁺*x*. *f x* ∗ *indicator X x* ∂*count_space UNIV*)
by (*simp add*: *nn_integral_restrict_space*[*symmetric*] *restrict_count_space*)

lemma *nn_integral_count_space_eq*:
(⋀*x*. *x* ∈ *A* − *B* ⟹ *f x* = *0*) ⟹ (⋀*x*. *x* ∈ *B* − *A* ⟹ *f x* = *0*) ⟹
(∫⁺*x*. *f x* ∂*count_space A*) = (∫⁺*x*. *f x* ∂*count_space B*)
by (*auto simp*: *nn_integral_count_space_indicator intro*!: *nn_integral_cong split*: *split_indicator*)

lemma *nn_integral_ge_point*:
assumes *x* ∈ *A*
shows *p x* ≤ ∫⁺ *x*. *p x* ∂*count_space A*
proof −
from *assms* **have** *p x* ≤ ∫⁺ *x*. *p x* ∂*count_space* {*x*}
by(*auto simp add*: *nn_integral_count_space_finite max_def*)
also have … = ∫⁺ *x*′. *p x*′ ∗ *indicator* {*x*} *x*′ ∂*count_space A*
using *assms* **by**(*auto simp add*: *nn_integral_count_space_indicator indicator_def intro*!: *nn_integral_cong*)
also have … ≤ ∫⁺ *x*. *p x* ∂*count_space A*
by(*rule nn_integral_mono*)(*simp add*: *indicator_def*)
finally show *?thesis* **.**
qed

Measure spaces with an associated density

definition *density* :: ′*a measure* ⇒ (′*a* ⇒ *ennreal*) ⇒ ′*a measure* **where**
density M f = *measure_of* (*space M*) (*sets M*) (λ*A*. ∫⁺ *x*. *f x* ∗ *indicator A x* ∂*M*)

lemma
shows *sets_density*[*simp*, *measurable_cong*]: *sets* (*density M f*) = *sets M*
and *space_density*[*simp*]: *space* (*density M f*) = *space M*
by (*auto simp*: *density_def*)

lemma *space_density_imp*[*measurable_dest*]:
  $\bigwedge x\ M\ f.\ x \in space\ (density\ M\ f) \Longrightarrow x \in space\ M$ **by** *auto*

lemma
  shows *measurable_density_eq1*[*simp*]: $g \in measurable\ (density\ Mg\ f)\ Mg' \longleftrightarrow g \in measurable\ Mg\ Mg'$
    and *measurable_density_eq2*[*simp*]: $h \in measurable\ Mh\ (density\ Mh'\ f) \longleftrightarrow h \in measurable\ Mh\ Mh'$
    and *simple_function_density_eq*[*simp*]: $simple\_function\ (density\ Mu\ f)\ u \longleftrightarrow simple\_function\ Mu\ u$
  unfolding *measurable_def simple_function_def* **by** *simp_all*

lemma *density_cong*: $f \in borel\_measurable\ M \Longrightarrow f' \in borel\_measurable\ M \Longrightarrow$
  $(AE\ x\ in\ M.\ f\ x = f'\ x) \Longrightarrow density\ M\ f = density\ M\ f'$
  unfolding *density_def* **by** (*auto intro*!: *measure_of_eq nn_integral_cong_AE sets.space_closed*)

lemma *emeasure_density*:
  assumes *f*[*measurable*]: $f \in borel\_measurable\ M$ **and** *A*[*measurable*]: $A \in sets\ M$
  shows $emeasure\ (density\ M\ f)\ A = (\int^+ x.\ f\ x * indicator\ A\ x\ \partial M)$
    (**is** _ = *?μ A*)
  unfolding *density_def*
proof (*rule emeasure_measure_of_sigma*)
  show *sigma_algebra* (*space M*) (*sets M*) **..**
  show *positive* (*sets M*) *?μ*
    using *f* **by** (*auto simp*: *positive_def*)
  show *countably_additive* (*sets M*) *?μ*
  proof (*intro countably_additiveI*)
    fix $A :: nat \Rightarrow 'a\ set$ **assume** $range\ A \subseteq sets\ M$
    then have $\bigwedge i.\ A\ i \in sets\ M$ **by** *auto*
    then have $*$: $\bigwedge i.\ (\lambda x.\ f\ x * indicator\ (A\ i)\ x) \in borel\_measurable\ M$
      by *auto*
    assume *disj*: *disjoint_family A*
    then have $(\sum n.\ ?\mu\ (A\ n)) = (\int^+ x.\ (\sum n.\ f\ x * indicator\ (A\ n)\ x)\ \partial M)$
      using *f* $*$ **by** (*subst nn_integral_suminf*) *auto*
    also have $(\int^+ x.\ (\sum n.\ f\ x * indicator\ (A\ n)\ x)\ \partial M) = (\int^+ x.\ f\ x * (\sum n.\ indicator\ (A\ n)\ x)\ \partial M)$
      using *f* **by** (*auto intro*!: *ennreal_suminf_cmult nn_integral_cong_AE*)
    also have $\ldots = (\int^+ x.\ f\ x * indicator\ (\bigcup n.\ A\ n)\ x\ \partial M)$
      unfolding *suminf_indicator*[*OF disj*] **..**
    finally show $(\sum i.\ \int^+ x.\ f\ x * indicator\ (A\ i)\ x\ \partial M) = \int^+ x.\ f\ x * indicator\ (\bigcup i.\ A\ i)\ x\ \partial M$ **.**
  qed
qed *fact*

lemma *null_sets_density_iff*:
  assumes *f*: $f \in borel\_measurable\ M$
  shows $A \in null\_sets\ (density\ M\ f) \longleftrightarrow A \in sets\ M \wedge (AE\ x\ in\ M.\ x \in A \longrightarrow$

f x = 0)
proof –
{ assume *A* ∈ *sets M*
have (∫ +*x*. *f x* ∗ *indicator A x* ∂*M*) = *0* ⟷ *emeasure M* {*x* ∈ *space M*. *f x* ∗ *indicator A x* ≠ *0*} = *0*
using *f* ‹*A* ∈ *sets M*› **by** (*intro nn_integral_0_iff*) *auto*
also have ... ⟷ (*AE x in M*. *f x* ∗ *indicator A x* = *0*)
using *f* ‹*A* ∈ *sets M*› **by** (*intro AE_iff_measurable*[*OF* _ *refl*, *symmetric*]) *auto*
also have (*AE x in M*. *f x* ∗ *indicator A x* = *0*) ⟷ (*AE x in M*. *x* ∈ *A* ⟶ *f x* ≤ *0*)
by (*auto simp add*: *indicator_def max_def split*: *if_split_asm*)
finally have (∫ +*x*. *f x* ∗ *indicator A x* ∂*M*) = *0* ⟷ (*AE x in M*. *x* ∈ *A* ⟶ *f x* ≤ *0*) **. }**
with *f* **show** *?thesis*
by (*simp add*: *null_sets_def emeasure_density cong*: *conj_cong*)
qed

lemma *AE_density*:
assumes *f*: *f* ∈ *borel_measurable M*
shows (*AE x in density M f*. *P x*) ⟷ (*AE x in M*. *0* < *f x* ⟶ *P x*)
proof
assume *AE x in density M f*. *P x*
with *f* **obtain** *N* **where** {*x* ∈ *space M*. ¬ *P x*} ⊆ *N N* ∈ *sets M* **and** *ae*: *AE x in M*. *x* ∈ *N* ⟶ *f x* = *0*
by (*auto simp*: *eventually_ae_filter null_sets_density_iff*)
then have *AE x in M*. *x* ∉ *N* ⟶ *P x* **by** *auto*
with *ae* **show** *AE x in M*. *0* < *f x* ⟶ *P x*
by (*rule eventually_elim2*) *auto*
next
fix *N* **assume** *ae*: *AE x in M*. *0* < *f x* ⟶ *P x*
then obtain *N* **where** {*x* ∈ *space M*. ¬ (*0* < *f x* ⟶ *P x*)} ⊆ *N N* ∈ *null_sets M*
by (*auto simp*: *eventually_ae_filter*)
then have ∗: {*x* ∈ *space* (*density M f*). ¬ *P x*} ⊆ *N* ∪ {*x*∈*space M*. *f x* = *0*}
N ∪ {*x*∈*space M*. *f x* = *0*} ∈ *sets M* **and** *ae2*: *AE x in M*. *x* ∉ *N*
using *f* **by** (*auto simp*: *subset_eq zero_less_iff_neq_zero intro*!: *AE_not_in*)
show *AE x in density M f*. *P x*
using *ae2*
unfolding *eventually_ae_filter*[*of* _ *density M f*] *Bex_def null_sets_density_iff* [*OF f*]
by (*intro exI*[*of* _ *N* ∪ {*x*∈*space M*. *f x* = *0*}] *conjI* ∗) (*auto elim*: *eventually_elim2*)
qed

lemma *nn_integral_density*:
assumes *f*: *f* ∈ *borel_measurable M*
assumes *g*: *g* ∈ *borel_measurable M*
shows *integral*^{N} (*density M f*) *g* = (∫ + *x*. *f x* ∗ *g x* ∂*M*)
using *g* **proof** *induct*

```
  case (cong u v)
  then show ?case
    apply (subst nn_integral_cong[OF cong(3)])
    apply (simp_all cong: nn_integral_cong)
    done
next
  case (set A) then show ?case
    by (simp add: emeasure_density f)
next
  case (mult u c)
  moreover have ⋀x. f x * (c * u x) = c * (f x * u x) by (simp add: field_simps)
  ultimately show ?case
    using f by (simp add: nn_integral_cmult)
next
  case (add u v)
  then have ⋀x. f x * (v x + u x) = f x * v x + f x * u x
    by (simp add: distrib_left)
  with add f show ?case
    by (auto simp add: nn_integral_add intro!: nn_integral_add[symmetric])
next
  case (seq U)
  have eq: AE x in M. f x * (SUP i. U i x) = (SUP i. f x * U i x)
    by eventually_elim (simp add: SUP_mult_left_ennreal seq)
  from seq f show ?case
    apply (simp add: nn_integral_monotone_convergence_SUP image_comp)
    apply (subst nn_integral_cong_AE[OF eq])
    apply (subst nn_integral_monotone_convergence_SUP_AE)
    apply (auto simp: incseq_def le_fun_def intro!: mult_left_mono)
    done
qed

lemma density_distr:
  assumes [measurable]: f ∈ borel_measurable N X ∈ measurable M N
  shows density (distr M N X) f = distr (density M (λx. f (X x))) N X
  by (intro measure_eqI)
     (auto simp add: emeasure_density nn_integral_distr emeasure_distr
           split: split_indicator intro!: nn_integral_cong)

lemma emeasure_restricted:
  assumes S: S ∈ sets M and X: X ∈ sets M
  shows emeasure (density M (indicator S)) X = emeasure M (S ∩ X)
proof -
  have emeasure (density M (indicator S)) X = (∫+x. indicator S x * indicator
X x ∂M)
    using S X by (simp add: emeasure_density)
  also have ... = (∫+x. indicator (S ∩ X) x ∂M)
    by (auto intro!: nn_integral_cong simp: indicator_def)
  also have ... = emeasure M (S ∩ X)
    using S X by (simp add: sets.Int)
```

finally show *?thesis* **.**
qed

lemma *measure_restricted*:
S ∈ *sets M* ⟹ *X* ∈ *sets M* ⟹ *measure* (*density M* (*indicator S*)) *X* = *measure M* (*S* ∩ *X*)
by (*simp add*: *emeasure_restricted measure_def*)

lemma (**in** *finite_measure*) *finite_measure_restricted*:
S ∈ *sets M* ⟹ *finite_measure* (*density M* (*indicator S*))
by *standard* (*simp add*: *emeasure_restricted*)

lemma *emeasure_density_const*:
A ∈ *sets M* ⟹ *emeasure* (*density M* (λ_. *c*)) *A* = *c* ∗ *emeasure M A*
by (*auto simp*: *nn_integral_cmult_indicator emeasure_density*)

lemma *measure_density_const*:
A ∈ *sets M* ⟹ *c* ≠ ∞ ⟹ *measure* (*density M* (λ_. *c*)) *A* = *enn2real c* ∗ *measure M A*
by (*auto simp*: *emeasure_density_const measure_def enn2real_mult*)

lemma *density_density_eq*:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹
density (*density M f*) *g* = *density M* (λ*x*. *f x* ∗ *g x*)
by (*auto intro*!: *measure_eqI simp*: *emeasure_density nn_integral_density ac_simps*)

lemma *distr_density_distr*:
assumes *T*: *T* ∈ *measurable M M′* **and** *T′*: *T′* ∈ *measurable M′ M*
and *inv*: ∀ *x*∈*space M*. *T′* (*T x*) = *x*
assumes *f*: *f* ∈ *borel_measurable M′*
shows *distr* (*density* (*distr M M′ T*) *f*) *M T′* = *density M* (*f* ∘ *T*) (**is** *?R* = *?L*)
proof (*rule measure_eqI*)
fix *A* **assume** *A*: *A* ∈ *sets ?R*
{ fix *x* **assume** *x* ∈ *space M*
with *sets.sets_into_space*[*OF A*]
have *indicator* (*T′* −‘ *A* ∩ *space M′*) (*T x*) = (*indicator A x* :: *ennreal*)
using *T inv* **by** (*auto simp*: *indicator_def measurable_space*) **}**
with *A T T′ f* **show** *emeasure ?R A* = *emeasure ?L A*
by (*simp add*: *measurable_comp emeasure_density emeasure_distr*
nn_integral_distr measurable_sets cong: *nn_integral_cong*)
qed *simp*

lemma *density_density_divide*:
fixes *f g* :: ′*a* ⇒ *real*
assumes *f*: *f* ∈ *borel_measurable M AE x in M*. *0* ≤ *f x*
assumes *g*: *g* ∈ *borel_measurable M AE x in M*. *0* ≤ *g x*
assumes *ac*: *AE x in M*. *f x* = *0* ⟶ *g x* = *0*
shows *density* (*density M f*) (λ*x*. *g x* / *f x*) = *density M g*

proof −
  have *density M g = density M (λx. ennreal (f x) ∗ ennreal (g x / f x))*
   using *f g ac* **by** (*auto intro*!: *density_cong measurable_If simp*: *ennreal_mult*[*symmetric*])
  then show *?thesis*
    using *f g* **by** (*subst density_density_eq*) *auto*
qed

lemma *density_1*: *density M* (*λ_. 1*) = *M*
  by (*intro measure_eqI*) (*auto simp*: *emeasure_density*)

lemma *emeasure_density_add*:
  assumes *X*: *X* ∈ *sets M*
  assumes *Mf*[*measurable*]: *f* ∈ *borel_measurable M*
  assumes *Mg*[*measurable*]: *g* ∈ *borel_measurable M*
  shows *emeasure* (*density M f*) *X* + *emeasure* (*density M g*) *X* =
        emeasure (*density M* (*λx. f x + g x*)) *X*
  using *assms*
  apply (*subst* (*1 2 3*) *emeasure_density*, *simp_all*) []
  apply (*subst nn_integral_add*[*symmetric*], *simp_all*) []
  apply (*intro nn_integral_cong*, *simp split*: *split_indicator*)
  done

Point measure

definition *point_measure* :: *′a set* ⇒ (*′a* ⇒ *ennreal*) ⇒ *′a measure* **where**
  point_measure A f = *density* (*count_space A*) *f*

lemma
  shows *space_point_measure*: *space* (*point_measure A f*) = *A*
    and *sets_point_measure*: *sets* (*point_measure A f*) = *Pow A*
  by (*auto simp*: *point_measure_def*)

lemma *sets_point_measure_count_space*[*measurable_cong*]: *sets* (*point_measure A f*)
= *sets* (*count_space A*)
  by (*simp add*: *sets_point_measure*)

lemma *measurable_point_measure_eq1*[*simp*]:
  g ∈ *measurable* (*point_measure A f*) *M* ⟷ *g* ∈ *A* → *space M*
  unfolding *point_measure_def* **by** *simp*

lemma *measurable_point_measure_eq2_finite*[*simp*]:
  finite A ⟹
   g ∈ *measurable M* (*point_measure A f*) ⟷
   (*g* ∈ *space M* → *A* ∧ (∀ *a*∈*A*. *g* −‘ {*a*} ∩ *space M* ∈ *sets M*))
  unfolding *point_measure_def* **by** (*simp add*: *measurable_count_space_eq2*)

lemma *simple_function_point_measure*[*simp*]:
  simple_function (*point_measure A f*) *g* ⟷ *finite* (*g* ‘ *A*)
  by (*simp add*: *point_measure_def*)

lemma *emeasure_point_measure*:
assumes *A*: *finite* {*a*∈*X*. *0* < *f a*} *X* ⊆ *A*
shows *emeasure* (*point_measure A f*) *X* = (∑ *a*|*a*∈*X* ∧ *0* < *f a*. *f a*)
proof −
have {*a*. (*a* ∈ *X* ⟶ *a* ∈ *A* ∧ *0* < *f a*) ∧ *a* ∈ *X*} = {*a*∈*X*. *0* < *f a*}
using ‹*X* ⊆ *A*› **by** *auto*
with *A* **show** *?thesis*
by (*simp add*: *emeasure_density nn_integral_count_space point_measure_def indicator_def*)
qed

lemma *emeasure_point_measure_finite*:
finite A ⟹ *X* ⊆ *A* ⟹ *emeasure* (*point_measure A f*) *X* = (∑ *a*∈*X*. *f a*)
by (*subst emeasure_point_measure*) (*auto dest*: *finite_subset intro*!: *sum.mono_neutral_left simp*: *less_le*)

lemma *emeasure_point_measure_finite2*:
X ⊆ *A* ⟹ *finite X* ⟹ *emeasure* (*point_measure A f*) *X* = (∑ *a*∈*X*. *f a*)
by (*subst emeasure_point_measure*)
(*auto dest*: *finite_subset intro*!: *sum.mono_neutral_left simp*: *less_le*)

lemma *null_sets_point_measure_iff*:
X ∈ *null_sets* (*point_measure A f*) ⟷ *X* ⊆ *A* ∧ (∀ *x*∈*X*. *f x* = *0*)
by (*auto simp*: *AE_count_space null_sets_density_iff point_measure_def*)

lemma *AE_point_measure*:
(*AE x in point_measure A f*. *P x*) ⟷ (∀ *x*∈*A*. *0* < *f x* ⟶ *P x*)
unfolding *point_measure_def*
by (*subst AE_density*) (*auto simp*: *AE_density AE_count_space point_measure_def*)

lemma *nn_integral_point_measure*:
finite {*a*∈*A*. *0* < *f a* ∧ *0* < *g a*} ⟹
integral^N (*point_measure A f*) *g* = (∑ *a*|*a*∈*A* ∧ *0* < *f a* ∧ *0* < *g a*. *f a* ∗ *g a*)
unfolding *point_measure_def*
by (*subst nn_integral_density*)
(*simp_all add*: *nn_integral_density nn_integral_count_space ennreal_zero_less_mult_iff*)

lemma *nn_integral_point_measure_finite*:
finite A ⟹ *integral*^{N} (*point_measure A f*) *g* = (∑ *a*∈*A*. *f a* ∗ *g a*)
by (*subst nn_integral_point_measure*) (*auto intro*!: *sum.mono_neutral_left simp*: *less_le*)

Uniform measure

definition *uniform_measure M A* = *density M* (λ*x*. *indicator A x* / *emeasure M A*)

lemma

shows *sets_uniform_measure[simp, measurable_cong]: sets (uniform_measure M A) = sets M*
and *space_uniform_measure[simp]: space (uniform_measure M A) = space M*
by *(auto simp: uniform_measure_def)*

lemma *emeasure_uniform_measure[simp]:*
assumes *A: A ∈ sets M* **and** *B: B ∈ sets M*
shows *emeasure (uniform_measure M A) B = emeasure M (A ∩ B) / emeasure M A*
proof −
from *A B* **have** *emeasure (uniform_measure M A) B = (∫⁺x. (1 / emeasure M A) ∗ indicator (A ∩ B) x ∂M)*
by *(auto simp add: uniform_measure_def emeasure_density divide_ennreal_def split: split_indicator intro!: nn_integral_cong)*
also have . . . *= emeasure M (A ∩ B) / emeasure M A*
using *A B*
by *(subst nn_integral_cmult_indicator) (simp_all add: sets.Int divide_ennreal_def mult.commute)*
finally show *?thesis* .
qed

lemma *measure_uniform_measure[simp]:*
assumes *A: emeasure M A ≠ 0 emeasure M A ≠ ∞* **and** *B: B ∈ sets M*
shows *measure (uniform_measure M A) B = measure M (A ∩ B) / measure M A*
using *emeasure_uniform_measure[OF emeasure_neq_0_sets[OF A(1)] B] A*
by *(cases emeasure M A emeasure M (A ∩ B) rule: ennreal2_cases) (simp_all add: measure_def divide_ennreal top_ennreal.rep_eq top_ereal_def ennreal_top_divide)*

lemma *AE_uniform_measureI:*
A ∈ sets M ⟹ (AE x in M. x ∈ A ⟶ P x) ⟹ (AE x in uniform_measure M A. P x)
unfolding *uniform_measure_def* **by** *(auto simp: AE_density divide_ennreal_def)*

lemma *emeasure_uniform_measure_1:*
emeasure M S ≠ 0 ⟹ emeasure M S ≠ ∞ ⟹ emeasure (uniform_measure M S) S = 1
by *(subst emeasure_uniform_measure) (simp_all add: emeasure_neq_0_sets emeasure_eq_ennreal_measure divide_ennreal zero_less_iff_neq_zero[symmetric])*

lemma *nn_integral_uniform_measure:*
assumes *f[measurable]: f ∈ borel_measurable M* **and** *S[measurable]: S ∈ sets M*
shows *(∫⁺x. f x ∂uniform_measure M S) = (∫⁺x. f x ∗ indicator S x ∂M) / emeasure M S*
proof −
{ **assume** *emeasure M S = ∞*

then have *?thesis*
by (*simp add*: *uniform_measure_def nn_integral_density f*) **}**
moreover
{ assume [*simp*]: *emeasure M S* = *0*
then have *ae*: *AE x in M*. $x \notin S$
using *sets.sets_into_space*[*OF S*]
by (*subst AE_iff_measurable*[*OF _ refl*]) (*simp_all add*: *subset_eq cong*: *rev_conj_cong*)
from *ae* **have** ($\int^+ x.$ *indicator S x* / *0* ∗ *f x* ∂*M*) = *0*
by (*subst nn_integral_0_iff_AE*) *auto*
moreover from *ae* **have** ($\int^+ x.$ *f x* ∗ *indicator S x* ∂*M*) = *0*
by (*subst nn_integral_0_iff_AE*) *auto*
ultimately have *?thesis*
by (*simp add*: *uniform_measure_def nn_integral_density f*) **}**
moreover have *emeasure M S* ≠ *0* ⟹ *emeasure M S* ≠ ∞ ⟹ *?thesis*
unfolding *uniform_measure_def*
by (*subst nn_integral_density*)
(*auto simp*: *ennreal_times_divide f nn_integral_divide*[*symmetric*] *mult.commute*)
ultimately show *?thesis* **by** *blast*
qed

lemma *AE_uniform_measure*:
assumes *emeasure M A* ≠ *0 emeasure M A* < ∞
shows (*AE x in uniform_measure M A*. *P x*) ⟷ (*AE x in M*. *x* ∈ *A* ⟶ *P x*)
proof −
have *A* ∈ *sets M*
using ‹*emeasure M A* ≠ *0*› **by** (*metis emeasure_notin_sets*)
moreover have ⋀*x*. *0* < *indicator A x* / *emeasure M A* ⟷ *x* ∈ *A*
using *assms*
by (*cases emeasure M A*) (*auto split*: *split_indicator simp*: *ennreal_zero_less_divide*)
ultimately show *?thesis*
unfolding *uniform_measure_def* **by** (*simp add*: *AE_density*)
qed

Null measure

lemma *null_measure_eq_density*: *null_measure M* = *density M* (λ_. *0*)
by (*intro measure_eqI*) (*simp_all add*: *emeasure_density*)

lemma *nn_integral_null_measure*[*simp*]: ($\int^+ x.$ *f x* ∂*null_measure M*) = *0*
by (*auto simp add*: *nn_integral_def simple_integral_def SUP_constant bot_ennreal_def le_fun_def*
intro!: *exI*[*of _* λ*x*. *0*])

lemma *density_null_measure*[*simp*]: *density* (*null_measure M*) *f* = *null_measure M*
proof (*intro measure_eqI*)
fix *A* **show** *emeasure* (*density* (*null_measure M*) *f*) *A* = *emeasure* (*null_measure M*) *A*
by (*simp add*: *density_def*) (*simp only*: *null_measure_def*[*symmetric*] *emeasure_null_measure*)

qed *simp*

Uniform count measure

definition *uniform_count_measure A = point_measure A (λx. 1 / card A)*

lemma
shows *space_uniform_count_measure*: *space (uniform_count_measure A) = A*
and *sets_uniform_count_measure*: *sets (uniform_count_measure A) = Pow A*
unfolding *uniform_count_measure_def* **by** (*auto simp*: *space_point_measure sets_point_measure*)

lemma *sets_uniform_count_measure_count_space*[*measurable_cong*]:
sets (uniform_count_measure A) = sets (count_space A)
by (*simp add*: *sets_uniform_count_measure*)

lemma *emeasure_uniform_count_measure*:
finite A ⟹ X ⊆ A ⟹ emeasure (uniform_count_measure A) X = card X / card A
by (*simp add*: *emeasure_point_measure_finite uniform_count_measure_def divide_inverse ennreal_mult*
ennreal_of_nat_eq_real_of_nat)

lemma *measure_uniform_count_measure*:
finite A ⟹ X ⊆ A ⟹ measure (uniform_count_measure A) X = card X / card A
by (*simp add*: *emeasure_point_measure_finite uniform_count_measure_def measure_def enn2real_mult*)

lemma *space_uniform_count_measure_empty_iff* [*simp*]:
space (uniform_count_measure X) = {} ⟷ X = {}
by(*simp add*: *space_uniform_count_measure*)

lemma *sets_uniform_count_measure_eq_UNIV* [*simp*]:
sets (uniform_count_measure UNIV) = UNIV ⟷ True
UNIV = sets (uniform_count_measure UNIV) ⟷ True
by(*simp_all add*: *sets_uniform_count_measure*)

Scaled measure

lemma *nn_integral_scale_measure*:
assumes *f*: *f ∈ borel_measurable M*
shows *nn_integral (scale_measure r M) f = r ∗ nn_integral M f*
using *f*
proof *induction*
case (*cong f g*)
thus *?case*
by(*simp add*: *cong.hyps space_scale_measure cong*: *nn_integral_cong_simp*)
next
case (*mult f c*)

thus *?case*
by(*simp add*: *nn_integral_cmult max_def mult.assoc mult.left_commute*)
next
case (*add f g*)
thus *?case*
by(*simp add*: *nn_integral_add distrib_left*)
next
case (*seq U*)
thus *?case*
by(*simp add*: *nn_integral_monotone_convergence_SUP SUP_mult_left_ennreal image_comp*)
qed *simp*

end

6.7 Binary Product Measure

theory *Binary_Product_Measure*
imports *Nonnegative_Lebesgue_Integration*
begin

lemma *Pair_vimage_times*[*simp*]: *Pair x* −' (*A* × *B*) = (*if x* ∈ *A then B else* {})
by *auto*

lemma *rev_Pair_vimage_times*[*simp*]: (λ*x*. (*x*, *y*)) −' (*A* × *B*) = (*if y* ∈ *B then A else* {})
by *auto*

6.7.1 Binary products

definition *pair_measure* (**infixr** $\bigotimes_M$ *80*) **where**
A $\bigotimes_M$ *B* = *measure_of* (*space A* × *space B*)
{*a* × *b* | *a b*. *a* ∈ *sets A* ∧ *b* ∈ *sets B*}
(λ*X*. ∫⁺*x*. (∫⁺*y*. *indicator X* (*x*,*y*) ∂*B*) ∂*A*)

lemma *pair_measure_closed*: {*a* × *b* | *a b*. *a* ∈ *sets A* ∧ *b* ∈ *sets B*} ⊆ *Pow* (*space A* × *space B*)
using *sets.space_closed*[*of A*] *sets.space_closed*[*of B*] **by** *auto*

lemma *space_pair_measure*:
space (*A* $\bigotimes_M$ *B*) = *space A* × *space B*
unfolding *pair_measure_def* **using** *pair_measure_closed*[*of A B*]
by (*rule space_measure_of*)

lemma *SIGMA_Collect_eq*: (*SIGMA x*:*space M*. {*y*∈*space N*. *P x y*}) = {*x*∈*space* (*M* $\bigotimes_M$ *N*). *P* (*fst x*) (*snd x*)}
by (*auto simp*: *space_pair_measure*)

lemma *sets_pair_measure*:

sets (*A* ⨂_{*M*} *B*) = *sigma_sets* (*space A* × *space B*) {*a* × *b* | *a b*. *a* ∈ *sets A* ∧ *b* ∈ *sets B*}
unfolding *pair_measure_def* **using** *pair_measure_closed*[*of A B*]
by (*rule sets_measure_of*)

lemma *sets_pair_measure_cong*[*measurable_cong*, *cong*]:
sets M1 = *sets M1'* ⟹ *sets M2* = *sets M2'* ⟹ *sets* (*M1* ⨂_{*M*} *M2*) = *sets* (*M1'* ⨂_{*M*} *M2'*)
unfolding *sets_pair_measure* **by** (*simp cong*: *sets_eq_imp_space_eq*)

lemma *pair_measureI*[*intro*, *simp*, *measurable*]:
x ∈ *sets A* ⟹ *y* ∈ *sets B* ⟹ *x* × *y* ∈ *sets* (*A* ⨂_{*M*} *B*)
by (*auto simp*: *sets_pair_measure*)

lemma *sets_Pair*: {*x*} ∈ *sets M1* ⟹ {*y*} ∈ *sets M2* ⟹ {(*x*, *y*)} ∈ *sets* (*M1* ⨂_{*M*} *M2*)
using *pair_measureI*[*of* {*x*} *M1* {*y*} *M2*] **by** *simp*

lemma *measurable_pair_measureI*:
assumes *1*: *f* ∈ *space M* → *space M1* × *space M2*
assumes *2*: ⋀*A B*. *A* ∈ *sets M1* ⟹ *B* ∈ *sets M2* ⟹ *f* −' (*A* × *B*) ∩ *space M* ∈ *sets M*
shows *f* ∈ *measurable M* (*M1* ⨂_{*M*} *M2*)
unfolding *pair_measure_def* **using** *1 2*
by (*intro measurable_measure_of*) (*auto dest*: *sets.sets_into_space*)

lemma *measurable_split_replace*[*measurable* (*raw*)]:
(λ*x*. *f x* (*fst* (*g x*)) (*snd* (*g x*))) ∈ *measurable M N* ⟹ (λ*x*. *case_prod* (*f x*) (*g x*)) ∈ *measurable M N*
unfolding *split_beta'* **.**

lemma *measurable_Pair*[*measurable* (*raw*)]:
assumes *f*: *f* ∈ *measurable M M1* **and** *g*: *g* ∈ *measurable M M2*
shows (λ*x*. (*f x*, *g x*)) ∈ *measurable M* (*M1* ⨂_{*M*} *M2*)
proof (*rule measurable_pair_measureI*)
show (λ*x*. (*f x*, *g x*)) ∈ *space M* → *space M1* × *space M2*
using *f g* **by** (*auto simp*: *measurable_def*)
fix *A B* **assume** ∗: *A* ∈ *sets M1 B* ∈ *sets M2*
have (λ*x*. (*f x*, *g x*)) −' (*A* × *B*) ∩ *space M* = (*f* −' *A* ∩ *space M*) ∩ (*g* −' *B* ∩ *space M*)
by *auto*
also have . . . ∈ *sets M*
by (*rule sets.Int*) (*auto intro*!: *measurable_sets* ∗ *f g*)
finally show (λ*x*. (*f x*, *g x*)) −' (*A* × *B*) ∩ *space M* ∈ *sets M* **.**
qed

lemma *measurable_fst*[*intro*!, *simp*, *measurable*]: *fst* ∈ *measurable* (*M1* ⨂_{*M*} *M2*) *M1*
by (*auto simp*: *fst_vimage_eq_Times space_pair_measure sets.sets_into_space Times_Int_Times*

measurable_def)

lemma *measurable_snd*[*intro*!, *simp*, *measurable*]: *snd* ∈ *measurable* (*M1* $\bigotimes_M$ *M2*) *M2*
by (*auto simp*: *snd_vimage_eq_Times space_pair_measure sets.sets_into_space Times_Int_Times measurable_def*)

lemma *measurable_Pair_compose_split*[*measurable_dest*]:
assumes *f*: *case_prod f* ∈ *measurable* (*M1* $\bigotimes_M$ *M2*) *N*
assumes *g*: *g* ∈ *measurable M M1* **and** *h*: *h* ∈ *measurable M M2*
shows (λ*x*. *f* (*g x*) (*h x*)) ∈ *measurable M N*
using *measurable_compose*[*OF measurable_Pair f*, *OF g h*] **by** *simp*

lemma *measurable_Pair1_compose*[*measurable_dest*]:
assumes *f*: (λ*x*. (*f x*, *g x*)) ∈ *measurable M* (*M1* $\bigotimes_M$ *M2*)
assumes [*measurable*]: *h* ∈ *measurable N M*
shows (λ*x*. *f* (*h x*)) ∈ *measurable N M1*
using *measurable_compose*[*OF f measurable_fst*] **by** *simp*

lemma *measurable_Pair2_compose*[*measurable_dest*]:
assumes *f*: (λ*x*. (*f x*, *g x*)) ∈ *measurable M* (*M1* $\bigotimes_M$ *M2*)
assumes [*measurable*]: *h* ∈ *measurable N M*
shows (λ*x*. *g* (*h x*)) ∈ *measurable N M2*
using *measurable_compose*[*OF f measurable_snd*] **by** *simp*

lemma *measurable_pair*:
assumes (*fst* ∘ *f*) ∈ *measurable M M1* (*snd* ∘ *f*) ∈ *measurable M M2*
shows *f* ∈ *measurable M* (*M1* $\bigotimes_M$ *M2*)
using *measurable_Pair*[*OF assms*] **by** *simp*

lemma
assumes *f*[*measurable*]: *f* ∈ *measurable M* (*N* $\bigotimes_M$ *P*)
shows *measurable_fst′*: (λ*x*. *fst* (*f x*)) ∈ *measurable M N*
and *measurable_snd′*: (λ*x*. *snd* (*f x*)) ∈ *measurable M P*
by *simp_all*

lemma
assumes *f*[*measurable*]: *f* ∈ *measurable M N*
shows *measurable_fst″*: (λ*x*. *f* (*fst x*)) ∈ *measurable* (*M* $\bigotimes_M$ *P*) *N*
and *measurable_snd″*: (λ*x*. *f* (*snd x*)) ∈ *measurable* (*P* $\bigotimes_M$ *M*) *N*
by *simp_all*

lemma *sets_pair_in_sets*:
assumes $\bigwedge$*a b*. *a* ∈ *sets A* ⟹ *b* ∈ *sets B* ⟹ *a* × *b* ∈ *sets N*
shows *sets* (*A* $\bigotimes_M$ *B*) ⊆ *sets N*
unfolding *sets_pair_measure*
by (*intro sets.sigma_sets_subset′*) (*auto intro*!: *assms*)

lemma *sets_pair_eq_sets_fst_snd*:

sets (*A* $\bigotimes_M$ *B*) = *sets* (*Sup* {*vimage_algebra* (*space A* × *space B*) *fst A*, *vimage_algebra* (*space A* × *space B*) *snd B*})
  (**is** *?P* = *sets* (*Sup* {*?fst*, *?snd*}))
proof −
 { **fix** *a b* **assume** *ab*: *a* ∈ *sets A b* ∈ *sets B*
  then have *a* × *b* = (*fst* −' *a* ∩ (*space A* × *space B*)) ∩ (*snd* −' *b* ∩ (*space A* × *space B*))
   by (*auto dest*: *sets.sets_into_space*)
  also have ... ∈ *sets* (*Sup* {*?fst*, *?snd*})
   apply (*rule sets.Int*)
   apply (*rule in_sets_Sup*)
   apply *auto* []
   apply (*rule insertI1*)
   apply (*auto intro*: *ab in_vimage_algebra*) []
   apply (*rule in_sets_Sup*)
   apply *auto* []
   apply (*rule insertI2*)
   apply (*auto intro*: *ab in_vimage_algebra*)
   done
  finally have *a* × *b* ∈ *sets* (*Sup* {*?fst*, *?snd*}) . }
 moreover have *sets ?fst* ⊆ *sets* (*A* $\bigotimes_M$ *B*)
  by (*rule sets_image_in_sets*) (*auto simp*: *space_pair_measure*[*symmetric*])
 moreover have *sets ?snd* ⊆ *sets* (*A* $\bigotimes_M$ *B*)
  by (*rule sets_image_in_sets*) (*auto simp*: *space_pair_measure*)
 ultimately show *?thesis*
  apply (*intro antisym*[*of sets A* **for** *A*] *sets_Sup_in_sets sets_pair_in_sets*)
  apply *simp*
  apply *simp*
  apply *simp*
  apply (*elim disjE*)
  apply (*simp add*: *space_pair_measure*)
  apply (*simp add*: *space_pair_measure*)
  apply (*auto simp add*: *space_pair_measure*)
  done
qed

lemma *measurable_pair_iff*:
 f ∈ *measurable M* (*M1* $\bigotimes_M$ *M2*) ⟷ (*fst* ∘ *f*) ∈ *measurable M M1* ∧ (*snd* ∘ *f*) ∈ *measurable M M2*
 by (*auto intro*: *measurable_pair*[*of f M M1 M2*])

lemma *measurable_split_conv*:
 (λ(*x*, *y*). *f x y*) ∈ *measurable A B* ⟷ (λ*x*. *f* (*fst x*) (*snd x*)) ∈ *measurable A B*
 by (*intro arg_cong2*[**where** *f*=(∈)]) *auto*

lemma *measurable_pair_swap'*: (λ(*x*,*y*). (*y*, *x*)) ∈ *measurable* (*M1* $\bigotimes_M$ *M2*) (*M2* $\bigotimes_M$ *M1*)
 by (*auto intro*!: *measurable_Pair simp*: *measurable_split_conv*)

lemma *measurable_pair_swap*:
assumes *f*: *f* ∈ *measurable* (*M1* $\bigotimes_M$ *M2*) *M* **shows** (λ(*x*,*y*). *f* (*y*, *x*)) ∈ *measurable* (*M2* $\bigotimes_M$ *M1*) *M*
using *measurable_comp*[*OF measurable_Pair f*] **by** (*auto simp*: *measurable_split_conv comp_def*)

lemma *measurable_pair_swap_iff*:
f ∈ *measurable* (*M2* $\bigotimes_M$ *M1*) *M* ⟷ (λ(*x*,*y*). *f* (*y*,*x*)) ∈ *measurable* (*M1* $\bigotimes_M$ *M2*) *M*
by (*auto dest*: *measurable_pair_swap*)

lemma *measurable_Pair1'*: *x* ∈ *space M1* ⟹ *Pair x* ∈ *measurable M2* (*M1* $\bigotimes_M$ *M2*)
by *simp*

lemma *sets_Pair1*[*measurable* (*raw*)]:
assumes *A*: *A* ∈ *sets* (*M1* $\bigotimes_M$ *M2*) **shows** *Pair x* −‘ *A* ∈ *sets M2*
proof −
have *Pair x* −‘ *A* = (*if x* ∈ *space M1 then Pair x* −‘ *A* ∩ *space M2 else* {})
using *A*[*THEN sets.sets_into_space*] **by** (*auto simp*: *space_pair_measure*)
also have . . . ∈ *sets M2*
using *A* **by** (*auto simp add*: *measurable_Pair1' intro*!: *measurable_sets split*: *if_split_asm*)
finally show *?thesis* **.**
qed

lemma *measurable_Pair2'*: *y* ∈ *space M2* ⟹ (λ*x*. (*x*, *y*)) ∈ *measurable M1* (*M1* $\bigotimes_M$ *M2*)
by (*auto intro*!: *measurable_Pair*)

lemma *sets_Pair2*: **assumes** *A*: *A* ∈ *sets* (*M1* $\bigotimes_M$ *M2*) **shows** (λ*x*. (*x*, *y*)) −‘ *A* ∈ *sets M1*
proof −
have (λ*x*. (*x*, *y*)) −‘ *A* = (*if y* ∈ *space M2 then* (λ*x*. (*x*, *y*)) −‘ *A* ∩ *space M1 else* {})
using *A*[*THEN sets.sets_into_space*] **by** (*auto simp*: *space_pair_measure*)
also have . . . ∈ *sets M1*
using *A* **by** (*auto simp add*: *measurable_Pair2' intro*!: *measurable_sets split*: *if_split_asm*)
finally show *?thesis* **.**
qed

lemma *measurable_Pair2*:
assumes *f*: *f* ∈ *measurable* (*M1* $\bigotimes_M$ *M2*) *M* **and** *x*: *x* ∈ *space M1*
shows (λ*y*. *f* (*x*, *y*)) ∈ *measurable M2 M*
using *measurable_comp*[*OF measurable_Pair1' f*, *OF x*]
by (*simp add*: *comp_def*)

lemma *measurable_Pair1*:

assumes *f*: *f* ∈ *measurable* (*M1* ⨂$_M$ *M2*) *M* **and** *y*: *y* ∈ *space M2*
shows (λ*x*. *f* (*x*, *y*)) ∈ *measurable M1 M*
using *measurable_comp*[*OF measurable_Pair2' f*, *OF y*]
by (*simp add*: *comp_def*)

lemma *Int_stable_pair_measure_generator*: *Int_stable* {*a* × *b* | *a b*. *a* ∈ *sets A* ∧ *b* ∈ *sets B*}
unfolding *Int_stable_def*
by *safe* (*auto simp add*: *Times_Int_Times*)

lemma (**in** *finite_measure*) *finite_measure_cut_measurable*:
assumes [*measurable*]: *Q* ∈ *sets* (*N* ⨂$_M$ *M*)
shows (λ*x*. *emeasure M* (*Pair x* −' *Q*)) ∈ *borel_measurable N*
(**is** *?s Q* ∈ _)
using *Int_stable_pair_measure_generator pair_measure_closed assms*
unfolding *sets_pair_measure*
proof (*induct rule*: *sigma_sets_induct_disjoint*)
case (*compl A*)
with *sets.sets_into_space* **have** ⋀*x*. *emeasure M* (*Pair x* −' ((*space N* × *space M*) − *A*)) =
(*if x* ∈ *space N then emeasure M* (*space M*) − *?s A x else 0*)
unfolding *sets_pair_measure*[*symmetric*]
by (*auto intro*!: *emeasure_compl simp*: *vimage_Diff sets_Pair1*)
with *compl sets.top* **show** *?case*
by (*auto intro*!: *measurable_If simp*: *space_pair_measure*)
next
case (*union F*)
then have ⋀*x*. *emeasure M* (*Pair x* −' (⋃*i*. *F i*)) = (∑*i*. *?s* (*F i*) *x*)
by (*simp add*: *suminf_emeasure disjoint_family_on_vimageI subset_eq vimage_UN sets_pair_measure*[*symmetric*])
with *union* **show** *?case*
unfolding *sets_pair_measure*[*symmetric*] **by** *simp*
qed (*auto simp add*: *if_distrib Int_def*[*symmetric*] *intro*!: *measurable_If*)

lemma (**in** *sigma_finite_measure*) *measurable_emeasure_Pair*:
assumes *Q*: *Q* ∈ *sets* (*N* ⨂$_M$ *M*) **shows** (λ*x*. *emeasure M* (*Pair x* −' *Q*)) ∈ *borel_measurable N* (**is** *?s Q* ∈ _)
proof −
from *sigma_finite_disjoint* **guess** *F* **. note** *F* = *this*
then have *F_sets*: ⋀*i*. *F i* ∈ *sets M* **by** *auto*
let *?C* = λ*x i*. *F i* ∩ *Pair x* −' *Q*
{ **fix** *i*
have [*simp*]: *space N* × *F i* ∩ *space N* × *space M* = *space N* × *F i*
using *F sets.sets_into_space* **by** *auto*
let *?R* = *density M* (*indicator* (*F i*))
have *finite_measure ?R*
using *F* **by** (*intro finite_measureI*) (*auto simp*: *emeasure_restricted subset_eq*)
then have (λ*x*. *emeasure ?R* (*Pair x* −' (*space N* × *space ?R* ∩ *Q*))) ∈ *borel_measurable N*

by (*rule finite_measure.finite_measure_cut_measurable*) (*auto intro*: *Q*)
moreover have $\bigwedge x$. *emeasure ?R* (*Pair x* −' (*space N* × *space ?R* ∩ *Q*))
= *emeasure M* (*F i* ∩ *Pair x* −' (*space N* × *space ?R* ∩ *Q*))
using *Q F_sets* **by** (*intro emeasure_restricted*) (*auto intro*: *sets_Pair1*)
moreover have $\bigwedge x$. *F i* ∩ *Pair x* −' (*space N* × *space ?R* ∩ *Q*) = *?C x i*
using *sets.sets_into_space*[*OF Q*] **by** (*auto simp*: *space_pair_measure*)
ultimately have (λx. *emeasure M* (*?C x i*)) ∈ *borel_measurable N*
by *simp* **}**
moreover
{ fix *x*
have ($\sum$ *i*. *emeasure M* (*?C x i*)) = *emeasure M* ($\bigcup$ *i*. *?C x i*)
proof (*intro suminf_emeasure*)
show *range* (*?C x*) ⊆ *sets M*
using *F* ‹*Q* ∈ *sets* (*N* $\bigotimes_M$ *M*)› **by** (*auto intro*!: *sets_Pair1*)
have *disjoint_family F* **using** *F* **by** *auto*
show *disjoint_family* (*?C x*)
by (*rule disjoint_family_on_bisimulation*[*OF* ‹*disjoint_family F*›]) *auto*
qed
also have ($\bigcup$ *i*. *?C x i*) = *Pair x* −' *Q*
using *F sets.sets_into_space*[*OF* ‹*Q* ∈ *sets* (*N* $\bigotimes_M$ *M*)›]
by (*auto simp*: *space_pair_measure*)
finally have *emeasure M* (*Pair x* −' *Q*) = ($\sum$ *i*. *emeasure M* (*?C x i*))
by *simp* **}**
ultimately show *?thesis* **using** ‹*Q* ∈ *sets* (*N* $\bigotimes_M$ *M*)› *F_sets*
by *auto*
qed

lemma (**in** *sigma_finite_measure*) *measurable_emeasure*[*measurable* (*raw*)]:
assumes *space*: $\bigwedge x$. *x* ∈ *space N* ⟹ *A x* ⊆ *space M*
assumes *A*: {*x*∈*space* (*N* $\bigotimes_M$ *M*). *snd x* ∈ *A* (*fst x*)} ∈ *sets* (*N* $\bigotimes_M$ *M*)
shows (λx. *emeasure M* (*A x*)) ∈ *borel_measurable N*
proof −
from *space* **have** $\bigwedge x$. *x* ∈ *space N* ⟹ *Pair x* −' {*x* ∈ *space* (*N* $\bigotimes_M$ *M*). *snd x* ∈ *A* (*fst x*)} = *A x*
by (*auto simp*: *space_pair_measure*)
with *measurable_emeasure_Pair*[*OF A*] **show** *?thesis*
by (*auto cong*: *measurable_cong*)
qed

lemma (**in** *sigma_finite_measure*) *emeasure_pair_measure*:
assumes *X* ∈ *sets* (*N* $\bigotimes_M$ *M*)
shows *emeasure* (*N* $\bigotimes_M$ *M*) *X* = ($\int^+$ *x*. $\int^+$ *y*. *indicator X* (*x*, *y*) ∂*M* ∂*N*) (**is** _ = *?μ X*)
proof (*rule emeasure_measure_of*[*OF pair_measure_def*])
show *positive* (*sets* (*N* $\bigotimes_M$ *M*)) *?μ*
by (*auto simp*: *positive_def*)
have *eq*[*simp*]: $\bigwedge$*A x y*. *indicator A* (*x*, *y*) = *indicator* (*Pair x* −' *A*) *y*
by (*auto simp*: *indicator_def*)
show *countably_additive* (*sets* (*N* $\bigotimes_M$ *M*)) *?μ*

proof (*rule countably_additiveI*)
fix F :: *nat* ⇒ (′*b* × ′*a*) *set* **assume** F: *range* F ⊆ *sets* ($N \bigotimes_M M$) *disjoint_family* F
from F **have** ∗: ⋀*i*. F *i* ∈ *sets* ($N \bigotimes_M M$) **by** *auto*
moreover have ⋀*x*. *disjoint_family* (λ*i*. *Pair x* −‘ F *i*)
by (*intro disjoint_family_on_bisimulation*[*OF F*(*2*)]) *auto*
moreover have ⋀*x*. *range* (λ*i*. *Pair x* −‘ F *i*) ⊆ *sets M*
using F **by** (*auto simp*: *sets_Pair1*)
ultimately show (∑ *n*. ?μ (F *n*)) = ?μ (⋃ *i*. F *i*)
by (*auto simp add*: *nn_integral_suminf*[*symmetric*] *vimage_UN suminf_emeasure intro*!: *nn_integral_cong nn_integral_indicator*[*symmetric*])
qed
show {*a* × *b* |*a b*. *a* ∈ *sets N* ∧ *b* ∈ *sets M*} ⊆ *Pow* (*space N* × *space M*)
using *sets.space_closed*[*of N*] *sets.space_closed*[*of M*] **by** *auto*
qed *fact*

lemma (**in** *sigma_finite_measure*) *emeasure_pair_measure_alt*:
assumes X: X ∈ *sets* ($N \bigotimes_M M$)
shows *emeasure* ($N \bigotimes_M M$) X = (∫⁺*x*. *emeasure M* (*Pair x* −‘ X) ∂N)
proof −
have [*simp*]: ⋀*x y*. *indicator* X (*x*, *y*) = *indicator* (*Pair x* −‘ X) *y*
by (*auto simp*: *indicator_def*)
show ?*thesis*
using X **by** (*auto intro*!: *nn_integral_cong simp*: *emeasure_pair_measure sets_Pair1*)
qed

proposition (**in** *sigma_finite_measure*) *emeasure_pair_measure_Times*:
assumes A: A ∈ *sets N* **and** B: B ∈ *sets M*
shows *emeasure* ($N \bigotimes_M M$) ($A \times B$) = *emeasure N A* ∗ *emeasure M B*
proof −
have *emeasure* ($N \bigotimes_M M$) ($A \times B$) = (∫⁺*x*. *emeasure M B* ∗ *indicator A x* ∂N)
using A B **by** (*auto intro*!: *nn_integral_cong simp*: *emeasure_pair_measure_alt*)
also have ... = *emeasure M B* ∗ *emeasure N A*
using A **by** (*simp add*: *nn_integral_cmult_indicator*)
finally show ?*thesis*
by (*simp add*: *ac_simps*)
qed

6.7.2 Binary products of σ-finite emeasure spaces

locale *pair_sigma_finite* = *M1*?: *sigma_finite_measure M1* + *M2*?: *sigma_finite_measure M2*
for *M1* :: ′*a measure* **and** *M2* :: ′*b measure*

lemma (**in** *pair_sigma_finite*) *measurable_emeasure_Pair1*:
Q ∈ *sets* ($M1 \bigotimes_M M2$) ⟹ (λ*x*. *emeasure M2* (*Pair x* −‘ Q)) ∈ *borel_measurable M1*
using *M2.measurable_emeasure_Pair* .

lemma (**in** *pair_sigma_finite*) *measurable_emeasure_Pair2*:
assumes *Q*: $Q \in sets\ (M1 \bigotimes_M M2)$ **shows** $(\lambda y.\ emeasure\ M1\ ((\lambda x.\ (x,\ y))\ -`\ Q)) \in borel\_measurable\ M2$
proof –
have $(\lambda(x,\ y).\ (y,\ x))\ -`\ Q \cap space\ (M2 \bigotimes_M M1) \in sets\ (M2 \bigotimes_M M1)$
using *Q measurable_pair_swap'* **by** (*auto intro*: *measurable_sets*)
note *M1.measurable_emeasure_Pair*[*OF this*]
moreover have $\bigwedge y.\ Pair\ y\ -`\ ((\lambda(x,\ y).\ (y,\ x))\ -`\ Q \cap space\ (M2 \bigotimes_M M1)) = (\lambda x.\ (x,\ y))\ -`\ Q$
using *Q*[*THEN sets.sets_into_space*] **by** (*auto simp*: *space_pair_measure*)
ultimately show *?thesis* **by** *simp*
qed

proposition (**in** *pair_sigma_finite*) *sigma_finite_up_in_pair_measure_generator*:
defines $E \equiv \{A \times B \mid A\ B.\ A \in sets\ M1 \wedge B \in sets\ M2\}$
shows $\exists F::nat \Rightarrow ('a \times 'b)\ set.\ range\ F \subseteq E \wedge incseq\ F \wedge (\bigcup i.\ F\ i) = space\ M1 \times space\ M2 \wedge$
$(\forall i.\ emeasure\ (M1 \bigotimes_M M2)\ (F\ i) \neq \infty)$
proof –
from *M1.sigma_finite_incseq* **guess** *F1* . **note** *F1* = *this*
from *M2.sigma_finite_incseq* **guess** *F2* . **note** *F2* = *this*
from *F1 F2* **have** *space*: $space\ M1 = (\bigcup i.\ F1\ i)\ space\ M2 = (\bigcup i.\ F2\ i)$ **by** *auto*
let $?F = \lambda i.\ F1\ i \times F2\ i$
show *?thesis*
proof (*intro exI*[*of _ ?F*] *conjI allI*)
show $range\ ?F \subseteq E$ **using** *F1 F2* **by** (*auto simp*: *E_def*) (*metis range_subsetD*)
next
have $space\ M1 \times space\ M2 \subseteq (\bigcup i.\ ?F\ i)$
proof (*intro subsetI*)
fix *x* **assume** $x \in space\ M1 \times space\ M2$
then obtain *i j* **where** $fst\ x \in F1\ i\ snd\ x \in F2\ j$
by (*auto simp*: *space*)
then have $fst\ x \in F1\ (max\ i\ j)\ snd\ x \in F2\ (max\ j\ i)$
using ‹*incseq F1*› ‹*incseq F2*› **unfolding** *incseq_def*
by (*force split*: *split_max*)+
then have $(fst\ x,\ snd\ x) \in F1\ (max\ i\ j) \times F2\ (max\ i\ j)$
by (*intro SigmaI*) (*auto simp add*: *max.commute*)
then show $x \in (\bigcup i.\ ?F\ i)$ **by** *auto*
qed
then show $(\bigcup i.\ ?F\ i) = space\ M1 \times space\ M2$
using *space* **by** (*auto simp*: *space*)
next
fix *i* **show** $incseq\ (\lambda i.\ F1\ i \times F2\ i)$
using ‹*incseq F1*› ‹*incseq F2*› **unfolding** *incseq_Suc_iff* **by** *auto*
next
fix *i*
from *F1 F2* **have** $F1\ i \in sets\ M1\ F2\ i \in sets\ M2$ **by** *auto*

with *F1 F2* **show** *emeasure* $(M1 \bigotimes_M M2)$ $(F1\ i \times F2\ i) \neq \infty$
by (*auto simp add*: *emeasure_pair_measure_Times ennreal_mult_eq_top_iff*)
qed
qed

sublocale *pair_sigma_finite* ⊆ *P?*: *sigma_finite_measure* $M1 \bigotimes_M M2$
proof
from *M1.sigma_finite_countable* **guess** *F1* **..**
moreover from *M2.sigma_finite_countable* **guess** *F2* **..**
ultimately show
$\exists A.$ *countable* $A \wedge A \subseteq$ *sets* $(M1 \bigotimes_M M2) \wedge \bigcup A =$ *space* $(M1 \bigotimes_M M2) \wedge$ $(\forall a\in A.$ *emeasure* $(M1 \bigotimes_M M2)\ a \neq \infty)$
by (*intro exI*[*of* _ $(\lambda(a, b).\ a \times b)$ ‘ $(F1 \times F2)$] *conjI*)
(*auto simp*: *M2.emeasure_pair_measure_Times space_pair_measure set_eq_iff subset_eq ennreal_mult_eq_top_iff*)
qed

lemma *sigma_finite_pair_measure*:
assumes *A*: *sigma_finite_measure A* **and** *B*: *sigma_finite_measure B*
shows *sigma_finite_measure* $(A \bigotimes_M B)$
proof −
interpret *A*: *sigma_finite_measure A* **by** *fact*
interpret *B*: *sigma_finite_measure B* **by** *fact*
interpret *AB*: *pair_sigma_finite A B* **..**
show *?thesis* **..**
qed

lemma *sets_pair_swap*:
assumes $A \in$ *sets* $(M1 \bigotimes_M M2)$
shows $(\lambda(x, y).\ (y, x))$ −‘ $A \cap$ *space* $(M2 \bigotimes_M M1) \in$ *sets* $(M2 \bigotimes_M M1)$
using *measurable_pair_swap′ assms* **by** (*rule measurable_sets*)

lemma (**in** *pair_sigma_finite*) *distr_pair_swap*:
$M1 \bigotimes_M M2 =$ *distr* $(M2 \bigotimes_M M1)$ $(M1 \bigotimes_M M2)$ $(\lambda(x, y).\ (y, x))$ (**is** *?P* = *?D*)
proof −
from *sigma_finite_up_in_pair_measure_generator* **guess** *F* :: *nat* ⇒ (′*a* × ′*b*) *set* **..** **note** *F* = *this*
let *?E* = $\{a \times b\ |a\ b.\ a \in$ *sets* $M1 \wedge b \in$ *sets* $M2\}$
show *?thesis*
proof (*rule measure_eqI_generator_eq*[*OF Int_stable_pair_measure_generator*[*of M1 M2*]])
show *?E* ⊆ *Pow* (*space ?P*)
using *sets.space_closed*[*of M1*] *sets.space_closed*[*of M2*] **by** (*auto simp*: *space_pair_measure*)
show *sets ?P* = *sigma_sets* (*space ?P*) *?E*
by (*simp add*: *sets_pair_measure space_pair_measure*)
then show *sets ?D* = *sigma_sets* (*space ?P*) *?E*
by *simp*

next
show *range F* ⊆ *?E* (⋃*i. F i*) = *space ?P* ⋀*i. emeasure ?P* (*F i*) ≠ ∞
using *F* **by** (*auto simp*: *space_pair_measure*)
next
fix *X* **assume** *X* ∈ *?E*
then obtain *A B* **where** *X*[*simp*]: *X* = *A* × *B* **and** *A*: *A* ∈ *sets M1* **and** *B*: *B* ∈ *sets M2* **by** *auto*
have (λ(*y*, *x*). (*x*, *y*)) −' *X* ∩ *space* (*M2* ⨂*M* *M1*) = *B* × *A*
using *sets.sets_into_space*[*OF A*] *sets.sets_into_space*[*OF B*] **by** (*auto simp*: *space_pair_measure*)
with *A B* **show** *emeasure* (*M1* ⨂*M* *M2*) *X* = *emeasure ?D X*
by (*simp add*: *M2.emeasure_pair_measure_Times M1.emeasure_pair_measure_Times emeasure_distr*
measurable_pair_swap' ac_simps)
qed
qed

lemma (**in** *pair_sigma_finite*) *emeasure_pair_measure_alt2*:
assumes *A*: *A* ∈ *sets* (*M1* ⨂*M* *M2*)
shows *emeasure* (*M1* ⨂*M* *M2*) *A* = (∫⁺*y*. *emeasure M1* ((λ*x*. (*x*, *y*)) −' *A*) ∂*M2*)
(**is** _ = *?ν A*)
proof −
have [*simp*]: ⋀*y*. (*Pair y* −' ((λ(*x*, *y*). (*y*, *x*)) −' *A* ∩ *space* (*M2* ⨂*M* *M1*))) = (λ*x*. (*x*, *y*)) −' *A*
using *sets.sets_into_space*[*OF A*] **by** (*auto simp*: *space_pair_measure*)
show *?thesis* **using** *A*
by (*subst distr_pair_swap*)
(*simp_all del*: *vimage_Int add*: *measurable_sets*[*OF measurable_pair_swap'*]
M1.emeasure_pair_measure_alt emeasure_distr[*OF measurable_pair_swap' A*])
qed

lemma (**in** *pair_sigma_finite*) *AE_pair*:
assumes *AE x in* (*M1* ⨂*M* *M2*). *Q x*
shows *AE x in M1*. (*AE y in M2*. *Q* (*x*, *y*))
proof −
obtain *N* **where** *N*: *N* ∈ *sets* (*M1* ⨂*M* *M2*) *emeasure* (*M1* ⨂*M* *M2*) *N* = *0* {*x*∈*space* (*M1* ⨂*M* *M2*). ¬ *Q x*} ⊆ *N*
using *assms* **unfolding** *eventually_ae_filter* **by** *auto*
show *?thesis*
proof (*rule AE_I*)
from *N measurable_emeasure_Pair1*[*OF* ‹*N* ∈ *sets* (*M1* ⨂*M* *M2*)›]
show *emeasure M1* {*x*∈*space M1*. *emeasure M2* (*Pair x* −' *N*) ≠ *0*} = *0*
by (*auto simp*: *M2.emeasure_pair_measure_alt nn_integral_0_iff*)
show {*x* ∈ *space M1*. *emeasure M2* (*Pair x* −' *N*) ≠ *0*} ∈ *sets M1*
by (*intro borel_measurable_eq measurable_emeasure_Pair1 N sets.sets_Collect_neg N*) *simp*
{ **fix** *x* **assume** *x* ∈ *space M1 emeasure M2* (*Pair x* −' *N*) = *0*

```
      have AE y in M2. Q (x, y)
      proof (rule AE_I)
        show emeasure M2 (Pair x -` N) = 0 by fact
        show Pair x -` N ∈ sets M2 using N(1) by (rule sets_Pair1)
        show {y ∈ space M2. ¬ Q (x, y)} ⊆ Pair x -` N
          using N ‹x ∈ space M1› unfolding space_pair_measure by auto
      qed }
    then show {x ∈ space M1. ¬ (AE y in M2. Q (x, y))} ⊆ {x ∈ space M1.
emeasure M2 (Pair x -` N) ≠ 0}
      by auto
  qed
qed

lemma (in pair_sigma_finite) AE_pair_measure:
  assumes {x∈space (M1 ⨂M M2). P x} ∈ sets (M1 ⨂M M2)
  assumes ae: AE x in M1. AE y in M2. P (x, y)
  shows AE x in M1 ⨂M M2. P x
proof (subst AE_iff_measurable[OF _ refl])
  show {x∈space (M1 ⨂M M2). ¬ P x} ∈ sets (M1 ⨂M M2)
    by (rule sets.sets_Collect) fact
  then have emeasure (M1 ⨂M M2) {x ∈ space (M1 ⨂M M2). ¬ P x} =
      (∫+ x. ∫+ y. indicator {x ∈ space (M1 ⨂M M2). ¬ P x} (x, y) ∂M2 ∂M1)
    by (simp add: M2.emeasure_pair_measure)
  also have ... = (∫+ x. ∫+ y. 0 ∂M2 ∂M1)
    using ae
    apply (safe intro!: nn_integral_cong_AE)
    apply (intro AE_I2)
    apply (safe intro!: nn_integral_cong_AE)
    apply auto
    done
  finally show emeasure (M1 ⨂M M2) {x ∈ space (M1 ⨂M M2). ¬ P x} = 0
by simp
qed

lemma (in pair_sigma_finite) AE_pair_iff:
  {x∈space (M1 ⨂M M2). P (fst x) (snd x)} ∈ sets (M1 ⨂M M2) ⟹
    (AE x in M1. AE y in M2. P x y) ⟷ (AE x in (M1 ⨂M M2). P (fst x)
(snd x))
  using AE_pair[of λx. P (fst x) (snd x)] AE_pair_measure[of λx. P (fst x) (snd
x)] by auto

lemma (in pair_sigma_finite) AE_commute:
  assumes P: {x∈space (M1 ⨂M M2). P (fst x) (snd x)} ∈ sets (M1 ⨂M M2)
  shows (AE x in M1. AE y in M2. P x y) ⟷ (AE y in M2. AE x in M1. P x
y)
proof -
  interpret Q: pair_sigma_finite M2 M1 ..
  have [simp]: ⋀x. (fst (case x of (x, y) ⇒ (y, x))) = snd x ⋀x. (snd (case x of
(x, y) ⇒ (y, x))) = fst x
```

    by auto
  have {x ∈ space (M2 ⨂M M1). P (snd x) (fst x)} =
    (λ(x, y). (y, x)) −‘ {x ∈ space (M1 ⨂M M2). P (fst x) (snd x)} ∩ space (M2 ⨂M M1)
    by (auto simp: space_pair_measure)
  also have ... ∈ sets (M2 ⨂M M1)
    by (intro sets_pair_swap P)
  finally show ?thesis
    apply (subst AE_pair_iff[OF P])
    apply (subst distr_pair_swap)
    apply (subst AE_distr_iff[OF measurable_pair_swap' P])
    apply (subst Q.AE_pair_iff)
    apply simp_all
    done
qed

6.7.3 Fubinis theorem

lemma measurable_compose_Pair1:
  x ∈ space M1 ⟹ g ∈ measurable (M1 ⨂M M2) L ⟹ (λy. g (x, y)) ∈ measurable M2 L
  by simp

lemma (in sigma_finite_measure) borel_measurable_nn_integral_fst:
  assumes f: f ∈ borel_measurable (M1 ⨂M M)
  shows (λx. ∫⁺ y. f (x, y) ∂M) ∈ borel_measurable M1
using f proof induct
  case (cong u v)
  then have ⋀w x. w ∈ space M1 ⟹ x ∈ space M ⟹ u (w, x) = v (w, x)
    by (auto simp: space_pair_measure)
  show ?case
    apply (subst measurable_cong)
    apply (rule nn_integral_cong)
    apply fact+
    done
next
  case (set Q)
  have [simp]: ⋀x y. indicator Q (x, y) = indicator (Pair x −‘ Q) y
    by (auto simp: indicator_def)
  have ⋀x. x ∈ space M1 ⟹ emeasure M (Pair x −‘ Q) = ∫⁺ y. indicator Q (x, y) ∂M
    by (simp add: sets_Pair1[OF set])
  from this measurable_emeasure_Pair[OF set] show ?case
    by (rule measurable_cong[THEN iffD1])
qed (simp_all add: nn_integral_add nn_integral_cmult measurable_compose_Pair1
                 nn_integral_monotone_convergence_SUP incseq_def le_fun_def image_comp
        cong: measurable_cong)

lemma (**in** *sigma_finite_measure*) *nn_integral_fst*:
assumes *f*: $f \in$ *borel_measurable* $(M1 \bigotimes_M M)$
shows $(\int^+ x. \int^+ y. f\ (x, y)\ \partial M\ \partial M1) =$ *integral*^{N} $(M1 \bigotimes_M M)$ *f* (**is** *?I f* = _)
using *f* **proof** *induct*
case (*cong u v*)
then have *?I u* = *?I v*
by (*intro nn_integral_cong*) (*auto simp*: *space_pair_measure*)
with *cong* **show** *?case*
by (*simp cong*: *nn_integral_cong*)
qed (*simp_all add*: *emeasure_pair_measure nn_integral_cmult nn_integral_add nn_integral_monotone_convergence_SUP measurable_compose_Pair1 borel_measurable_nn_integral_fst nn_integral_mono incseq_def le_fun_def image_comp*
cong: *nn_integral_cong*)

lemma (**in** *sigma_finite_measure*) *borel_measurable_nn_integral*[*measurable* (*raw*)]:
case_prod f $\in$ *borel_measurable* $(N \bigotimes_M M) \Longrightarrow (\lambda x. \int^+ y. f\ x\ y\ \partial M) \in$ *borel_measurable N*
using *borel_measurable_nn_integral_fst*[*of case_prod f N*] **by** *simp*

proposition (**in** *pair_sigma_finite*) *nn_integral_snd*:
assumes *f*[*measurable*]: $f \in$ *borel_measurable* $(M1 \bigotimes_M M2)$
shows $(\int^+ y. (\int^+ x. f\ (x, y)\ \partial M1)\ \partial M2) =$ *integral*^{N} $(M1 \bigotimes_M M2)$ *f*
proof −
note *measurable_pair_swap*[*OF f*]
from *M1.nn_integral_fst*[*OF this*]
have $(\int^+ y. (\int^+ x. f\ (x, y)\ \partial M1)\ \partial M2) = (\int^+ (x, y). f\ (y, x)\ \partial(M2 \bigotimes_M M1))$
by *simp*
also have $(\int^+ (x, y). f\ (y, x)\ \partial(M2 \bigotimes_M M1)) =$ *integral*^{N} $(M1 \bigotimes_M M2)$ *f*
by (*subst distr_pair_swap*) (*auto simp add*: *nn_integral_distr intro*!: *nn_integral_cong*)
finally show *?thesis* .
qed

theorem (**in** *pair_sigma_finite*) *Fubini*:
assumes *f*: $f \in$ *borel_measurable* $(M1 \bigotimes_M M2)$
shows $(\int^+ y. (\int^+ x. f\ (x, y)\ \partial M1)\ \partial M2) = (\int^+ x. (\int^+ y. f\ (x, y)\ \partial M2)\ \partial M1)$
unfolding *nn_integral_snd*[*OF assms*] *M2.nn_integral_fst*[*OF assms*] ..

theorem (**in** *pair_sigma_finite*) *Fubini'*:
assumes *f*: *case_prod f* $\in$ *borel_measurable* $(M1 \bigotimes_M M2)$
shows $(\int^+ y. (\int^+ x. f\ x\ y\ \partial M1)\ \partial M2) = (\int^+ x. (\int^+ y. f\ x\ y\ \partial M2)\ \partial M1)$
using *Fubini*[*OF f*] **by** *simp*

6.7.4 Products on counting spaces, densities and distributions

proposition *sigma_prod*:
  assumes *X_cover*: $\exists E \subseteq A$. *countable* $E \wedge X = \bigcup E$ **and** *A*: $A \subseteq Pow\ X$
  assumes *Y_cover*: $\exists E \subseteq B$. *countable* $E \wedge Y = \bigcup E$ **and** *B*: $B \subseteq Pow\ Y$
  shows *sigma X A* $\bigotimes_M$ *sigma Y B* = *sigma* $(X \times Y)$ $\{a \times b \mid a\ b.\ a \in A \wedge b \in B\}$
    (**is** *?P* = *?S*)
proof (*rule measure_eqI*)
  have [*simp*]: *snd* $\in X \times Y \to Y$ *fst* $\in X \times Y \to X$
    by *auto*
  let *?XY* = $\{\{fst -` a \cap X \times Y \mid a.\ a \in A\}, \{snd -` b \cap X \times Y \mid b.\ b \in B\}\}$
  have *sets ?P* = *sets* (*SUP xy*∈*?XY*. *sigma* $(X \times Y)$ *xy*)
    by (*simp add*: *vimage_algebra_sigma sets_pair_eq_sets_fst_snd A B*)
  also have ... = *sets* (*sigma* $(X \times Y)$ ($\bigcup$ *?XY*))
    by (*intro Sup_sigma arg_cong*[**where** *f*=*sets*]) *auto*
  also have ... = *sets ?S*
  proof (*intro arg_cong*[**where** *f*=*sets*] *sigma_eqI sigma_sets_eqI*)
   show $\bigcup$ *?XY* $\subseteq$ *Pow* $(X \times Y)$ $\{a \times b \mid a\ b.\ a \in A \wedge b \in B\} \subseteq$ *Pow* $(X \times Y)$
     using *A B* **by** *auto*
  next
    interpret *XY*: *sigma_algebra* $X \times Y$ *sigma_sets* $(X \times Y)$ $\{a \times b \mid a\ b.\ a \in A \wedge b \in B\}$
      using *A B* **by** (*intro sigma_algebra_sigma_sets*) *auto*
    fix *Z* **assume** $Z \in \bigcup$ *?XY*
    then show $Z \in$ *sigma_sets* $(X \times Y)$ $\{a \times b \mid a\ b.\ a \in A \wedge b \in B\}$
    proof *safe*
      fix *a* **assume** $a \in A$
      from *Y_cover* **obtain** *E* **where** *E*: $E \subseteq B$ *countable E* **and** $Y = \bigcup E$
        by *auto*
      with ‹$a \in A$› *A* **have** *eq*: *fst* $-`\ a \cap X \times Y = (\bigcup e \in E.\ a \times e)$
        by *auto*
     show *fst* $-`\ a \cap X \times Y \in$ *sigma_sets* $(X \times Y)$ $\{a \times b \mid a\ b.\ a \in A \wedge b \in B\}$
        using ‹$a \in A$› *E* **unfolding** *eq* **by** (*auto intro*!: *XY.countable_UN′*)
    next
      fix *b* **assume** $b \in B$
      from *X_cover* **obtain** *E* **where** *E*: $E \subseteq A$ *countable E* **and** $X = \bigcup E$
        by *auto*
      with ‹$b \in B$› *B* **have** *eq*: *snd* $-`\ b \cap X \times Y = (\bigcup e \in E.\ e \times b)$
        by *auto*
      show *snd* $-`\ b \cap X \times Y \in$ *sigma_sets* $(X \times Y)$ $\{a \times b \mid a\ b.\ a \in A \wedge b \in B\}$
        using ‹$b \in B$› *E* **unfolding** *eq* **by** (*auto intro*!: *XY.countable_UN′*)
    qed
  next
    fix *Z* **assume** $Z \in \{a \times b \mid a\ b.\ a \in A \wedge b \in B\}$
    then obtain *a b* **where** $Z = a \times b$ **and** *ab*: $a \in A$ $b \in B$
      by *auto*
    then have *Z*: $Z = (fst -`\ a \cap X \times Y) \cap (snd -`\ b \cap X \times Y)$

```
      using A B by auto
    interpret XY: sigma_algebra X × Y sigma_sets (X × Y) (⋃ ?XY)
      by (intro sigma_algebra_sigma_sets) auto
    show Z ∈ sigma_sets (X × Y) (⋃ ?XY)
      unfolding Z by (rule XY.Int) (blast intro: ab)+
  qed
  finally show sets ?P = sets ?S .
next
  interpret finite_measure sigma X A for X A
    proof qed (simp add: emeasure_sigma)
  fix A assume A ∈ sets ?P then show emeasure ?P A = emeasure ?S A
    by (simp add: emeasure_pair_measure_alt emeasure_sigma)
qed

lemma sigma_sets_pair_measure_generator_finite:
  assumes finite A and finite B
  shows sigma_sets (A × B) { a × b | a b. a ⊆ A ∧ b ⊆ B} = Pow (A × B)
  (is sigma_sets ?prod ?sets = _)
proof safe
  have fin: finite (A × B) using assms by (rule finite_cartesian_product)
  fix x assume subset: x ⊆ A × B
  hence finite x using fin by (rule finite_subset)
  from this subset show x ∈ sigma_sets ?prod ?sets
  proof (induct x)
    case empty show ?case by (rule sigma_sets.Empty)
  next
    case (insert a x)
    hence {a} ∈ sigma_sets ?prod ?sets by auto
    moreover have x ∈ sigma_sets ?prod ?sets using insert by auto
   ultimately show ?case unfolding insert_is_Un[of a x] by (rule sigma_sets_Un)
  qed
next
  fix x a b
  assume x ∈ sigma_sets ?prod ?sets and (a, b) ∈ x
  from sigma_sets_into_sp[OF _ this(1)] this(2)
  show a ∈ A and b ∈ B by auto
qed

proposition sets_pair_eq:
  assumes Ea: Ea ⊆ Pow (space A) sets A = sigma_sets (space A) Ea
    and Ca: countable Ca Ca ⊆ Ea ⋃ Ca = space A
    and Eb: Eb ⊆ Pow (space B) sets B = sigma_sets (space B) Eb
    and Cb: countable Cb Cb ⊆ Eb ⋃ Cb = space B
  shows sets (A ⨂M B) = sets (sigma (space A × space B) { a × b | a b. a ∈
Ea ∧ b ∈ Eb })
    (is _ = sets (sigma ?Ω ?E))
proof
  show sets (sigma ?Ω ?E) ⊆ sets (A ⨂M B)
    using Ea(1) Eb(1) by (subst sigma_le_sets) (auto simp: Ea(2) Eb(2))
```

have *?E* ⊆ *Pow ?Ω*
using *Ea*(*1*) *Eb*(*1*) **by** *auto*
then have *E*: *a* ∈ *Ea* ⟹ *b* ∈ *Eb* ⟹ *a* × *b* ∈ *sets* (*sigma ?Ω ?E*) **for** *a b*
by *auto*
have *sets* (*A* ⨂$_M$ *B*) ⊆ *sets* (*Sup* {*vimage_algebra ?Ω fst A*, *vimage_algebra ?Ω snd B*})
unfolding *sets_pair_eq_sets_fst_snd* **..**
also have *vimage_algebra ?Ω fst A* = *vimage_algebra ?Ω fst* (*sigma* (*space A*) *Ea*)
by (*intro vimage_algebra_cong*[*OF refl refl*]) (*simp add*: *Ea*)
also have ... = *sigma ?Ω* {*fst* −' *A* ∩ *?Ω* |*A*. *A* ∈ *Ea*}
by (*intro Ea vimage_algebra_sigma*) *auto*
also have *vimage_algebra ?Ω snd B* = *vimage_algebra ?Ω snd* (*sigma* (*space B*) *Eb*)
by (*intro vimage_algebra_cong*[*OF refl refl*]) (*simp add*: *Eb*)
also have ... = *sigma ?Ω* {*snd* −' *A* ∩ *?Ω* |*A*. *A* ∈ *Eb*}
by (*intro Eb vimage_algebra_sigma*) *auto*
also have {*sigma ?Ω* {*fst* −' *Aa* ∩ *?Ω* |*Aa*. *Aa* ∈ *Ea*}, *sigma ?Ω* {*snd* −' *Aa* ∩ *?Ω* |*Aa*. *Aa* ∈ *Eb*}} =
sigma ?Ω ' {{*fst* −' *Aa* ∩ *?Ω* |*Aa*. *Aa* ∈ *Ea*}, {*snd* −' *Aa* ∩ *?Ω* |*Aa*. *Aa* ∈ *Eb*}}
by *auto*
also have *sets* (*SUP S*∈{{*fst* −' *Aa* ∩ *?Ω* |*Aa*. *Aa* ∈ *Ea*}, {*snd* −' *Aa* ∩ *?Ω* |*Aa*. *Aa* ∈ *Eb*}}. *sigma ?Ω S*) =
sets (*sigma ?Ω* (⋃{{*fst* −' *Aa* ∩ *?Ω* |*Aa*. *Aa* ∈ *Ea*}, {*snd* −' *Aa* ∩ *?Ω* |*Aa*. *Aa* ∈ *Eb*}}))
using *Ea*(*1*) *Eb*(*1*) **by** (*intro sets_Sup_sigma*) *auto*
also have ... ⊆ *sets* (*sigma ?Ω ?E*)
proof (*subst sigma_le_sets*, *safe intro*!: *space_in_measure_of*)
fix *a* **assume** *a* ∈ *Ea*
then have *fst* −' *a* ∩ *?Ω* = (⋃*b*∈*Cb*. *a* × *b*)
using *Cb*(*3*)[*symmetric*] *Ea*(*1*) **by** *auto*
then show *fst* −' *a* ∩ *?Ω* ∈ *sets* (*sigma ?Ω ?E*)
using *Cb* ‹*a* ∈ *Ea*› **by** (*auto intro*!: *sets.countable_UN'* *E*)
next
fix *b* **assume** *b* ∈ *Eb*
then have *snd* −' *b* ∩ *?Ω* = (⋃*a*∈*Ca*. *a* × *b*)
using *Ca*(*3*)[*symmetric*] *Eb*(*1*) **by** *auto*
then show *snd* −' *b* ∩ *?Ω* ∈ *sets* (*sigma ?Ω ?E*)
using *Ca* ‹*b* ∈ *Eb*› **by** (*auto intro*!: *sets.countable_UN'* *E*)
qed
finally show *sets* (*A* ⨂$_M$ *B*) ⊆ *sets* (*sigma ?Ω ?E*) **.**
qed

proposition *borel_prod*:
(*borel* ⨂$_M$ *borel*) = (*borel* :: ('*a*::*second_countable_topology* × '*b*::*second_countable_topology*) *measure*)
(**is** *?P* = *?B*)
proof −

```
  have ?B = sigma UNIV {A × B | A B. open A ∧ open B}
    by (rule second_countable_borel_measurable[OF open_prod_generated])
  also have . . . = ?P
    unfolding borel_def
    by (subst sigma_prod) (auto intro!: exI[of _ {UNIV}])
  finally show ?thesis ..
qed

proposition pair_measure_count_space:
  assumes A: finite A and B: finite B
  shows count_space A ⨂M count_space B = count_space (A × B) (is ?P = ?C)
proof (rule measure_eqI)
  interpret A: finite_measure count_space A by (rule finite_measure_count_space)
fact
  interpret B: finite_measure count_space B by (rule finite_measure_count_space)
fact
  interpret P: pair_sigma_finite count_space A count_space B ..
  show eq: sets ?P = sets ?C
    by (simp add: sets_pair_measure sigma_sets_pair_measure_generator_finite A B)
  fix X assume X: X ∈ sets ?P
  with eq have X_subset: X ⊆ A × B by simp
  with A B have fin_Pair: ⋀x. finite (Pair x −‘ X)
    by (intro finite_subset[OF _ B]) auto
  have fin_X: finite X using X_subset by (rule finite_subset) (auto simp: A B)
  have card: 0 < card (Pair a −‘ X) if (a, b) ∈ X for a b
    using card_gt_0_iff fin_Pair that by auto
  then have emeasure ?P X = ∫+ x. emeasure (count_space B) (Pair x −‘ X)
          ∂count_space A
    by (simp add: B.emeasure_pair_measure_alt X)
  also have ... = emeasure ?C X
    apply (subst emeasure_count_space)
    using card X_subset A fin_Pair fin_X
    apply (auto simp add: nn_integral_count_space
                      of_nat_sum[symmetric] card_SigmaI[symmetric]
                simp del: card_SigmaI
                intro!: arg_cong[where f=card])
    done
  finally show emeasure ?P X = emeasure ?C X .
qed


lemma emeasure_prod_count_space:
  assumes A: A ∈ sets (count_space UNIV ⨂M M) (is A ∈ sets (?A ⨂M ?B))
  shows emeasure (?A ⨂M ?B) A = (∫+ x. ∫+ y. indicator A (x, y) ∂?B ∂?A)
  by (rule emeasure_measure_of[OF pair_measure_def])
     (auto simp: countably_additive_def positive_def suminf_indicator A
             nn_integral_suminf[symmetric] dest: sets.sets_into_space)

lemma emeasure_prod_count_space_single[simp]: emeasure (count_space UNIV ⨂M
```

count_space UNIV) {*x*} = *1*
proof −
  have [*simp*]: ⋀*a b x y*. *indicator* {(*a*, *b*)} (*x*, *y*) = (*indicator* {*a*} *x* ∗ *indicator* {*b*} *y*::*ennreal*)
    by (*auto split*: *split_indicator*)
  show *?thesis*
    by (*cases x*) (*auto simp*: *emeasure_prod_count_space nn_integral_cmult sets_Pair*)
qed

lemma *emeasure_count_space_prod_eq*:
  fixes *A* :: (′*a* × ′*b*) *set*
  assumes *A*: *A* ∈ *sets* (*count_space UNIV* ⨂$_M$ *count_space UNIV*) (**is** *A* ∈ *sets* (*?A* ⨂$_M$ *?B*))
  shows *emeasure* (*?A* ⨂$_M$ *?B*) *A* = *emeasure* (*count_space UNIV*) *A*
proof −
  { fix *A* :: (′*a* × ′*b*) *set* **assume** *countable A*
    then have *emeasure* (*?A* ⨂$_M$ *?B*) (⋃*a*∈*A*. {*a*}) = (∫$^+$*a*. *emeasure* (*?A* ⨂$_M$ *?B*) {*a*} ∂*count_space A*)
      by (*intro emeasure_UN_countable*) (*auto simp*: *sets_Pair disjoint_family_on_def*)
    also have ... = (∫$^+$*a*. *indicator A a* ∂*count_space UNIV*)
      by (*subst nn_integral_count_space_indicator*) *auto*
    finally have *emeasure* (*?A* ⨂$_M$ *?B*) *A* = *emeasure* (*count_space UNIV*) *A*
      by *simp* **}**
  note ∗ = *this*

  show *?thesis*
  proof *cases*
    assume *finite A* **then show** *?thesis*
      by (*intro* ∗ *countable_finite*)
  next
    assume *infinite A*
    then obtain *C* **where** *countable C* **and** *infinite C* **and** *C* ⊆ *A*
      by (*auto dest*: *infinite_countable_subset*′)
    with *A* **have** *emeasure* (*?A* ⨂$_M$ *?B*) *C* ≤ *emeasure* (*?A* ⨂$_M$ *?B*) *A*
      by (*intro emeasure_mono*) *auto*
    also have *emeasure* (*?A* ⨂$_M$ *?B*) *C* = *emeasure* (*count_space UNIV*) *C*
      using ‹*countable C*› **by** (*rule* ∗)
    finally show *?thesis*
      using ‹*infinite C*› ‹*infinite A*› **by** (*simp add*: *top_unique*)
  qed
qed

lemma *nn_integral_count_space_prod_eq*:
  nn_integral (*count_space UNIV* ⨂$_M$ *count_space UNIV*) *f* = *nn_integral* (*count_space UNIV*) *f*
    (**is** *nn_integral ?P f* = _)
proof *cases*
  assume *cntbl*: *countable* {*x*. *f x* ≠ *0*}
  have [*simp*]: ⋀*x*. *card* ({*x*} ∩ {*x*. *f x* ≠ *0*}) = (*indicator* {*x*. *f x* ≠ *0*} *x*::*ennreal*)

```
    by (auto split: split_indicator)
  have [measurable]: ⋀y. (λx. indicator {y} x) ∈ borel_measurable ?P
    by (rule measurable_discrete_difference[of λx. 0 _ borel {y} λx. indicator {y}
x for y])
       (auto intro: sets_Pair)

  have (∫⁺x. f x ∂?P) = (∫⁺x. ∫⁺x'. f x * indicator {x} x' ∂count_space {x. f
x ≠ 0} ∂?P)
   by (auto simp add: nn_integral_cmult nn_integral_indicator' intro!: nn_integral_cong
split: split_indicator)
  also have ... = (∫⁺x. ∫⁺x'. f x' * indicator {x'} x ∂count_space {x. f x ≠ 0}
∂?P)
    by (auto intro!: nn_integral_cong split: split_indicator)
  also have ... = (∫⁺x'. ∫⁺x. f x' * indicator {x'} x ∂?P ∂count_space {x. f x
≠ 0})
    by (intro nn_integral_count_space_nn_integral cntbl) auto
  also have ... = (∫⁺x'. f x' ∂count_space {x. f x ≠ 0})
    by (intro nn_integral_cong) (auto simp: nn_integral_cmult sets_Pair)
  finally show ?thesis
     by (auto simp add: nn_integral_count_space_indicator intro!: nn_integral_cong
split: split_indicator)
next
  { fix x assume f x ≠ 0
    then have (∃r≥0. 0 < r ∧ f x = ennreal r) ∨ f x = ∞
      by (cases f x rule: ennreal_cases) (auto simp: less_le)
    then have ∃n. ennreal (1 / real (Suc n)) ≤ f x
      by (auto elim!: nat_approx_posE intro!: less_imp_le) }
  note * = this

  assume cntbl: uncountable {x. f x ≠ 0}
  also have {x. f x ≠ 0} = (⋃n. {x. 1/Suc n ≤ f x})
    using * by auto
  finally obtain n where infinite {x. 1/Suc n ≤ f x}
    by (meson countableI_type countable_UN uncountable_infinite)
  then obtain C where C: C ⊆ {x. 1/Suc n ≤ f x} and countable C infinite C
    by (metis infinite_countable_subset')

  have [measurable]: C ∈ sets ?P
    using sets.countable[OF _ ‹countable C›, of ?P] by (auto simp: sets_Pair)

  have (∫⁺x. ennreal (1/Suc n) * indicator C x ∂?P) ≤ nn_integral ?P f
   using C by (intro nn_integral_mono) (auto split: split_indicator simp: zero_ereal_def[symmetric])
  moreover have (∫⁺x. ennreal (1/Suc n) * indicator C x ∂?P) = ∞
   using ‹infinite C› by (simp add: nn_integral_cmult emeasure_count_space_prod_eq
ennreal_mult_top)
  moreover have (∫⁺x. ennreal (1/Suc n) * indicator C x ∂count_space UNIV)
≤ nn_integral (count_space UNIV) f
   using C by (intro nn_integral_mono) (auto split: split_indicator simp: zero_ereal_def[symmetric])
  moreover have (∫⁺x. ennreal (1/Suc n) * indicator C x ∂count_space UNIV)
```

= ∞
  using ‹*infinite C*› **by** (*simp add*: *nn_integral_cmult ennreal_mult_top*)
 ultimately show *?thesis*
  by (*simp add*: *top_unique*)
qed

theorem *pair_measure_density*:
 assumes *f*: *f* ∈ *borel_measurable M1*
 assumes *g*: *g* ∈ *borel_measurable M2*
 assumes *sigma_finite_measure M2 sigma_finite_measure* (*density M2 g*)
 shows *density M1 f* ⨂$_M$ *density M2 g* = *density* (*M1* ⨂$_M$ *M2*) (λ(*x*,*y*). *f x* ∗ *g y*) (**is** *?L* = *?R*)
proof (*rule measure_eqI*)
 interpret *M2*: *sigma_finite_measure M2* **by** *fact*
 interpret *D2*: *sigma_finite_measure density M2 g* **by** *fact*

 fix *A* **assume** *A*: *A* ∈ *sets ?L*
 with *f g* **have** (∫ $^+$ *x*. *f x* ∗ ∫ $^+$ *y*. *g y* ∗ *indicator A* (*x*, *y*) ∂*M2* ∂*M1*) =
  (∫ $^+$ *x*. ∫ $^+$ *y*. *f x* ∗ *g y* ∗ *indicator A* (*x*, *y*) ∂*M2* ∂*M1*)
  by (*intro nn_integral_cong_AE*)
   (*auto simp add*: *nn_integral_cmult*[*symmetric*] *ac_simps*)
 with *A f g* **show** *emeasure ?L A* = *emeasure ?R A*
  by (*simp add*: *D2.emeasure_pair_measure emeasure_density nn_integral_density*
     M2.nn_integral_fst[*symmetric*]
    cong: *nn_integral_cong*)
qed *simp*

lemma *sigma_finite_measure_distr*:
 assumes *sigma_finite_measure* (*distr M N f*) **and** *f*: *f* ∈ *measurable M N*
 shows *sigma_finite_measure M*
proof −
 interpret *sigma_finite_measure distr M N f* **by** *fact*
 from *sigma_finite_countable* **guess** *A* .. **note** *A* = *this*
 show *?thesis*
 proof
  show ∃ *A*. *countable A* ∧ *A* ⊆ *sets M* ∧ ⋃ *A* = *space M* ∧ (∀ *a*∈*A*. *emeasure M a* ≠ ∞)
   using *A f*
   by (*intro exI*[*of* _ (λ*a*. *f* −‘ *a* ∩ *space M*) ‘ *A*])
    (*auto simp*: *emeasure_distr set_eq_iff subset_eq intro*: *measurable_space*)
 qed
qed

lemma *pair_measure_distr*:
 assumes *f*: *f* ∈ *measurable M S* **and** *g*: *g* ∈ *measurable N T*
 assumes *sigma_finite_measure* (*distr N T g*)
 shows *distr M S f* ⨂$_M$ *distr N T g* = *distr* (*M* ⨂$_M$ *N*) (*S* ⨂$_M$ *T*) (λ(*x*, *y*). (*f x*, *g y*)) (**is** *?P* = *?D*)
proof (*rule measure_eqI*)

```
  interpret T: sigma_finite_measure distr N T g by fact
  interpret N: sigma_finite_measure N by (rule sigma_finite_measure_distr) fact+

  fix A assume A: A ∈ sets ?P
  with f g show emeasure ?P A = emeasure ?D A
    by (auto simp add: N.emeasure_pair_measure_alt space_pair_measure emeasure_distr
                    T.emeasure_pair_measure_alt nn_integral_distr
           intro!: nn_integral_cong arg_cong[where f=emeasure N])
qed simp

lemma pair_measure_eqI:
  assumes sigma_finite_measure M1 sigma_finite_measure M2
  assumes sets: sets (M1 ⨂M M2) = sets M
  assumes emeasure: ⋀A B. A ∈ sets M1 ⟹ B ∈ sets M2 ⟹ emeasure M1 A * emeasure M2 B = emeasure M (A × B)
  shows M1 ⨂M M2 = M
proof -
  interpret M1: sigma_finite_measure M1 by fact
  interpret M2: sigma_finite_measure M2 by fact
  interpret pair_sigma_finite M1 M2 ..
  from sigma_finite_up_in_pair_measure_generator guess F :: nat ⇒ ('a × 'b) set .. note F = this
  let ?E = {a × b |a b. a ∈ sets M1 ∧ b ∈ sets M2}
  let ?P = M1 ⨂M M2
  show ?thesis
  proof (rule measure_eqI_generator_eq[OF Int_stable_pair_measure_generator[of M1 M2]])
    show ?E ⊆ Pow (space ?P)
      using sets.space_closed[of M1] sets.space_closed[of M2] by (auto simp: space_pair_measure)
    show sets ?P = sigma_sets (space ?P) ?E
      by (simp add: sets_pair_measure space_pair_measure)
    then show sets M = sigma_sets (space ?P) ?E
      using sets[symmetric] by simp
  next
    show range F ⊆ ?E (⋃i. F i) = space ?P ⋀i. emeasure ?P (F i) ≠ ∞
      using F by (auto simp: space_pair_measure)
  next
    fix X assume X ∈ ?E
    then obtain A B where X[simp]: X = A × B and A: A ∈ sets M1 and B: B ∈ sets M2 by auto
    then have emeasure ?P X = emeasure M1 A * emeasure M2 B
      by (simp add: M2.emeasure_pair_measure_Times)
    also have ... = emeasure M (A × B)
      using A B emeasure by auto
    finally show emeasure ?P X = emeasure M X
      by simp
  qed
```

qed

lemma *sets_pair_countable*:
assumes *countable S1 countable S2*
assumes *M*: *sets M = Pow S1* **and** *N*: *sets N = Pow S2*
shows *sets* ($M \bigotimes_M N$) = *Pow* (*S1* × *S2*)
proof *auto*
fix *x a b* **assume** *x*: *x* ∈ *sets* ($M \bigotimes_M N$) (*a*, *b*) ∈ *x*
from *sets.sets_into_space*[*OF x*(*1*)] *x*(*2*)
sets_eq_imp_space_eq[*of N count_space S2*] *sets_eq_imp_space_eq*[*of M count_space S1*] *M N*
show *a* ∈ *S1 b* ∈ *S2*
by (*auto simp*: *space_pair_measure*)
next
fix *X* **assume** *X*: *X* ⊆ *S1* × *S2*
then have *countable X*
by (*metis countable_subset* ‹*countable S1*› ‹*countable S2*› *countable_SIGMA*)
have *X* = (⋃(*a*, *b*)∈*X*. {*a*} × {*b*}) **by** *auto*
also have ... ∈ *sets* ($M \bigotimes_M N$)
using *X*
by (*safe intro*!: *sets.countable_UN'* ‹*countable X*› *subsetI pair_measureI*) (*auto simp*: *M N*)
finally show *X* ∈ *sets* ($M \bigotimes_M N$) **.**
qed

lemma *pair_measure_countable*:
assumes *countable S1 countable S2*
shows *count_space S1* $\bigotimes_M$ *count_space S2* = *count_space* (*S1* × *S2*)
proof (*rule pair_measure_eqI*)
show *sigma_finite_measure* (*count_space S1*) *sigma_finite_measure* (*count_space S2*)
using *assms* **by** (*auto intro*!: *sigma_finite_measure_count_space_countable*)
show *sets* (*count_space S1* $\bigotimes_M$ *count_space S2*) = *sets* (*count_space* (*S1* × *S2*))
by (*subst sets_pair_countable*[*OF assms*]) *auto*
next
fix *A B* **assume** *A* ∈ *sets* (*count_space S1*) *B* ∈ *sets* (*count_space S2*)
then show *emeasure* (*count_space S1*) *A* ∗ *emeasure* (*count_space S2*) *B* =
emeasure (*count_space* (*S1* × *S2*)) (*A* × *B*)
by (*subst* (*1 2 3*) *emeasure_count_space*) (*auto simp*: *finite_cartesian_product_iff ennreal_mult_top ennreal_top_mult*)
qed

proposition *nn_integral_fst_count_space*:
($\int^+$ *x*. $\int^+$ *y*. *f* (*x*, *y*) ∂*count_space UNIV* ∂*count_space UNIV*) = *integral*^{N} (*count_space UNIV*) *f*
(**is** *?lhs* = *?rhs*)
proof(*cases*)
assume ∗: *countable* {*xy*. *f xy* ≠ *0*}
let *?A* = *fst* ‘ {*xy*. *f xy* ≠ *0*}

let *?B* = *snd* ‘ {*xy*. *f xy* ≠ *0*}
from ∗ **have** [*simp*]: *countable ?A countable ?B* **by**(*rule countable_image*)+
have *?lhs* = (∫⁺ *x*. ∫⁺ *y*. *f* (*x*, *y*) ∂*count_space UNIV* ∂*count_space ?A*)
by(*rule nn_integral_count_space_eq*)
(*auto simp add*: *nn_integral_0_iff_AE AE_count_space not_le intro*: *rev_image_eqI*)
also have ... = (∫⁺ *x*. ∫⁺ *y*. *f* (*x*, *y*) ∂*count_space ?B* ∂*count_space ?A*)
by(*intro nn_integral_count_space_eq nn_integral_cong*)(*auto intro*: *rev_image_eqI*)
also have ... = (∫⁺ *xy*. *f xy* ∂*count_space* (*?A* × *?B*))
by(*subst sigma_finite_measure.nn_integral_fst*)
(*simp_all add*: *sigma_finite_measure_count_space_countable pair_measure_countable*)
also have ... = *?rhs*
by(*rule nn_integral_count_space_eq*)(*auto intro*: *rev_image_eqI*)
finally show *?thesis* **.**
next
{ **fix** *xy* **assume** *f xy* ≠ *0*
then have (∃ *r*≥*0*. *0* < *r* ∧ *f xy* = *ennreal r*) ∨ *f xy* = ∞
by (*cases f xy rule*: *ennreal_cases*) (*auto simp*: *less_le*)
then have ∃ *n*. *ennreal* (*1* / *real* (*Suc n*)) ≤ *f xy*
by (*auto elim*!: *nat_approx_posE intro*!: *less_imp_le*) }
note ∗ = *this*

assume *cntbl*: *uncountable* {*xy*. *f xy* ≠ *0*}
also have {*xy*. *f xy* ≠ *0*} = (⋃ *n*. {*xy*. *1*/*Suc n* ≤ *f xy*})
using ∗ **by** *auto*
finally obtain *n* **where** *infinite* {*xy*. *1*/*Suc n* ≤ *f xy*}
by (*meson countableI_type countable_UN uncountable_infinite*)
then obtain *C* **where** *C*: *C* ⊆ {*xy*. *1*/*Suc n* ≤ *f xy*} **and** *countable C infinite C*
by (*metis infinite_countable_subset'*)

have ∞ = (∫⁺ *xy*. *ennreal* (*1* / *Suc n*) ∗ *indicator C xy* ∂*count_space UNIV*)
using ‹*infinite C*› **by**(*simp add*: *nn_integral_cmult ennreal_mult_top*)
also have ... ≤ *?rhs* **using** *C*
by(*intro nn_integral_mono*)(*auto split*: *split_indicator*)
finally have *?rhs* = ∞ **by** (*simp add*: *top_unique*)
moreover have *?lhs* = ∞
proof(*cases finite* (*fst* ‘ *C*))
case *True*
then obtain *x C'* **where** *x*: *x* ∈ *fst* ‘ *C*
and *C'*: *C'* = *fst* −‘ {*x*} ∩ *C*
and *infinite C'*
using ‹*infinite C*› **by**(*auto elim*!: *inf_img_fin_domE'*)
from *x C C'* **have** ∗∗: *C'* ⊆ {*xy*. *1* / *Suc n* ≤ *f xy*} **by** *auto*

from *C'* ‹*infinite C'*› **have** *infinite* (*snd* ‘ *C'*)
by(*auto dest*!: *finite_imageD simp add*: *inj_on_def*)
then have ∞ = (∫⁺ *y*. *ennreal* (*1* / *Suc n*) ∗ *indicator* (*snd* ‘ *C'*) *y* ∂*count_space UNIV*)
by(*simp add*: *nn_integral_cmult ennreal_mult_top*)

also have ... = ($\int^+$ *y. ennreal* (*1* / *Suc n*) ∗ *indicator C′* (*x*, *y*) ∂*count_space UNIV*)
by(*rule nn_integral_cong*)(*force split*: *split_indicator intro*: *rev_image_eqI simp add*: *C′*)
also have ... = ($\int^+$ *x′*. ($\int^+$ *y. ennreal* (*1* / *Suc n*) ∗ *indicator C′* (*x*, *y*) ∂*count_space UNIV*) ∗ *indicator* {*x*} *x′* ∂*count_space UNIV*)
by(*simp add*: *one_ereal_def*[*symmetric*])
also have ... ≤ ($\int^+$ *x*. $\int^+$ *y. ennreal* (*1* / *Suc n*) ∗ *indicator C′* (*x*, *y*) ∂*count_space UNIV* ∂*count_space UNIV*)
by(*rule nn_integral_mono*)(*simp split*: *split_indicator*)
also have ... ≤ *?lhs* **using** ∗∗
by(*intro nn_integral_mono*)(*auto split*: *split_indicator*)
finally show *?thesis* **by** (*simp add*: *top_unique*)
next
case *False*
define *C′* **where** *C′* = *fst* ‘ *C*
have ∞ = $\int^+$ *x. ennreal* (*1* / *Suc n*) ∗ *indicator C′ x* ∂*count_space UNIV*
using *C′_def False* **by**(*simp add*: *nn_integral_cmult ennreal_mult_top*)
also have ... = $\int^+$ *x*. $\int^+$ *y. ennreal* (*1* / *Suc n*) ∗ *indicator C′ x* ∗ *indicator* {*SOME y*. (*x*, *y*) ∈ *C*} *y* ∂*count_space UNIV* ∂*count_space UNIV*
by(*auto simp add*: *one_ereal_def*[*symmetric*] *max_def intro*: *nn_integral_cong*)
also have ... ≤ $\int^+$ *x*. $\int^+$ *y. ennreal* (*1* / *Suc n*) ∗ *indicator C* (*x*, *y*) ∂*count_space UNIV* ∂*count_space UNIV*
by(*intro nn_integral_mono*)(*auto simp add*: *C′_def split*: *split_indicator intro*: *someI*)
also have ... ≤ *?lhs* **using** *C*
by(*intro nn_integral_mono*)(*auto split*: *split_indicator*)
finally show *?thesis* **by** (*simp add*: *top_unique*)
qed
ultimately show *?thesis* **by** *simp*
qed

proposition *nn_integral_snd_count_space*:
($\int^+$ *y*. $\int^+$ *x. f* (*x*, *y*) ∂*count_space UNIV* ∂*count_space UNIV*) = *integral*^{N} (*count_space UNIV*) *f*
(**is** *?lhs* = *?rhs*)
proof −
have *?lhs* = ($\int^+$ *y*. $\int^+$ *x*. (λ(*y*, *x*). *f* (*x*, *y*)) (*y*, *x*) ∂*count_space UNIV* ∂*count_space UNIV*)
by(*simp*)
also have ... = $\int^+$ *yx*. (λ(*y*, *x*). *f* (*x*, *y*)) *yx* ∂*count_space UNIV*
by(*rule nn_integral_fst_count_space*)
also have ... = $\int^+$ *xy*. *f xy* ∂*count_space* ((λ(*x*, *y*). (*y*, *x*)) ‘ *UNIV*)
by(*subst nn_integral_bij_count_space*[*OF inj_on_imp_bij_betw*, *symmetric*])
(*simp_all add*: *inj_on_def split_def*)
also have ... = *?rhs* **by**(*rule nn_integral_count_space_eq*) *auto*
finally show *?thesis* **.**
qed

lemma *measurable_pair_measure_countable1*:
  assumes *countable A*
  and [*measurable*]: $\bigwedge x.\ x \in A \Longrightarrow (\lambda y.\ f\ (x,\ y)) \in measurable\ N\ K$
  shows $f \in measurable\ (count\_space\ A \bigotimes_M N)\ K$
using _ _ *assms*(*1*)
by(*rule measurable_compose_countable*′[**where** $f=\lambda a\ b.\ f\ (a,\ snd\ b)$ **and** $g=fst$ **and** $I=A$, *simplified*])*simp_all*

6.7.5 Product of Borel spaces

theorem *borel_Times*:
  fixes A :: ′*a*::*topological_space set* **and** B :: ′*b*::*topological_space set*
  assumes A: $A \in sets\ borel$ **and** B: $B \in sets\ borel$
  shows $A \times B \in sets\ borel$
proof −
  have $A \times B = (A\times UNIV) \cap (UNIV \times B)$
    by *auto*
  moreover
  { **have** $A \in sigma\_sets\ UNIV\ \{S.\ open\ S\}$ **using** A **by** (*simp add*: *sets_borel*)
    then have $A\times UNIV \in sets\ borel$
    proof (*induct A*)
      case (*Basic S*) **then show** *?case*
        by (*auto intro*!: *borel_open open_Times*)
    next
      case (*Compl A*)
      moreover have ∗: $(UNIV - A) \times UNIV = UNIV - (A \times UNIV)$
        by *auto*
      ultimately show *?case*
        unfolding ∗ **by** *auto*
    next
      case (*Union A*)
       moreover have ∗: $(\bigcup(A\ `\ UNIV)) \times UNIV = \bigcup((\lambda i.\ A\ i \times UNIV)\ `\ UNIV)$
        by *auto*
      ultimately show *?case*
        unfolding ∗ **by** *auto*
    qed *simp* }
  moreover
  { **have** $B \in sigma\_sets\ UNIV\ \{S.\ open\ S\}$ **using** B **by** (*simp add*: *sets_borel*)
    then have $UNIV\times B \in sets\ borel$
    proof (*induct B*)
      case (*Basic S*) **then show** *?case*
        by (*auto intro*!: *borel_open open_Times*)
    next
      case (*Compl B*)
      moreover have ∗: $UNIV \times (UNIV - B) = UNIV - (UNIV \times B)$
        by *auto*
      ultimately show *?case*
        unfolding ∗ **by** *auto*

next
case (*Union B*)
moreover have ∗: *UNIV* × (⋃(*B* ‘ *UNIV*)) = ⋃((λ*i*. *UNIV* × *B i*) ‘ *UNIV*)
by *auto*
ultimately show *?case*
unfolding ∗ **by** *auto*
qed *simp* **}**
ultimately show *?thesis*
by *auto*
qed

lemma *finite_measure_pair_measure*:
assumes *finite_measure M finite_measure N*
shows *finite_measure* (*N* $\bigotimes_M$ *M*)
proof (*rule finite_measureI*)
interpret *M*: *finite_measure M* **by** *fact*
interpret *N*: *finite_measure N* **by** *fact*
show *emeasure* (*N* $\bigotimes_M$ *M*) (*space* (*N* $\bigotimes_M$ *M*)) ≠ ∞
by (*auto simp*: *space_pair_measure M.emeasure_pair_measure_Times ennreal_mult_eq_top_iff*)
qed

end

6.8 Finite Product Measure

theory *Finite_Product_Measure*
imports *Binary_Product_Measure Function_Topology*
begin

lemma *PiE_choice*: (∃ *f*∈Pi_E *I F*. ∀ *i*∈*I*. *P i* (*f i*)) ⟷ (∀ *i*∈*I*. ∃ *x*∈*F i*. *P i x*)
by (*auto simp*: *Bex_def PiE_iff Ball_def dest*!: *choice_iff* ′[*THEN iffD1*])
(*force intro*: *exI*[*of* _ *restrict f I* **for** *f*])

lemma *case_prod_const*: (λ(*i*, *j*). *c*) = (λ_. *c*)
by *auto*

6.8.1 More about Function restricted by *extensional*

definition
merge I J = (λ(*x*, *y*) *i*. *if i* ∈ *I then x i else if i* ∈ *J then y i else undefined*)

lemma *merge_apply*[*simp*]:
I ∩ *J* = {} ⟹ *i* ∈ *I* ⟹ *merge I J* (*x*, *y*) *i* = *x i*
I ∩ *J* = {} ⟹ *i* ∈ *J* ⟹ *merge I J* (*x*, *y*) *i* = *y i*
J ∩ *I* = {} ⟹ *i* ∈ *I* ⟹ *merge I J* (*x*, *y*) *i* = *x i*
J ∩ *I* = {} ⟹ *i* ∈ *J* ⟹ *merge I J* (*x*, *y*) *i* = *y i*
i ∉ *I* ⟹ *i* ∉ *J* ⟹ *merge I J* (*x*, *y*) *i* = *undefined*
unfolding *merge_def* **by** *auto*

lemma *merge_commute*:
$I \cap J = \{\} \Longrightarrow$ *merge* I J $(x, y) =$ *merge* J I (y, x)
by (*force simp*: *merge_def*)

lemma *Pi_cancel_merge_range*[*simp*]:
$I \cap J = \{\} \Longrightarrow x \in Pi\ I$ (*merge* I J $(A, B)) \longleftrightarrow x \in Pi\ I\ A$
$I \cap J = \{\} \Longrightarrow x \in Pi\ I$ (*merge* J I $(B, A)) \longleftrightarrow x \in Pi\ I\ A$
$J \cap I = \{\} \Longrightarrow x \in Pi\ I$ (*merge* I J $(A, B)) \longleftrightarrow x \in Pi\ I\ A$
$J \cap I = \{\} \Longrightarrow x \in Pi\ I$ (*merge* J I $(B, A)) \longleftrightarrow x \in Pi\ I\ A$
by (*auto simp*: *Pi_def*)

lemma *Pi_cancel_merge*[*simp*]:
$I \cap J = \{\} \Longrightarrow$ *merge* I J $(x, y) \in Pi\ I\ B \longleftrightarrow x \in Pi\ I\ B$
$J \cap I = \{\} \Longrightarrow$ *merge* I J $(x, y) \in Pi\ I\ B \longleftrightarrow x \in Pi\ I\ B$
$I \cap J = \{\} \Longrightarrow$ *merge* I J $(x, y) \in Pi\ J\ B \longleftrightarrow y \in Pi\ J\ B$
$J \cap I = \{\} \Longrightarrow$ *merge* I J $(x, y) \in Pi\ J\ B \longleftrightarrow y \in Pi\ J\ B$
by (*auto simp*: *Pi_def*)

lemma *extensional_merge*[*simp*]: *merge* I J $(x, y) \in$ *extensional* $(I \cup J)$
by (*auto simp*: *extensional_def*)

lemma *restrict_merge*[*simp*]:
$I \cap J = \{\} \Longrightarrow$ *restrict* (*merge* I J (x, y)) I = *restrict* x I
$I \cap J = \{\} \Longrightarrow$ *restrict* (*merge* I J (x, y)) J = *restrict* y J
$J \cap I = \{\} \Longrightarrow$ *restrict* (*merge* I J (x, y)) I = *restrict* x I
$J \cap I = \{\} \Longrightarrow$ *restrict* (*merge* I J (x, y)) J = *restrict* y J
by (*auto simp*: *restrict_def*)

lemma *split_merge*: P (*merge* I J (x,y) $i) \longleftrightarrow (i \in I \longrightarrow P\ (x\ i)) \wedge (i \in J - I \longrightarrow P\ (y\ i)) \wedge (i \notin I \cup J \longrightarrow P$ *undefined*)
unfolding *merge_def* **by** *auto*

lemma *PiE_cancel_merge*[*simp*]:
$I \cap J = \{\} \Longrightarrow$
merge I J $(x, y) \in Pi_E\ (I \cup J)\ B \longleftrightarrow x \in Pi\ I\ B \wedge y \in Pi\ J\ B$
by (*auto simp*: *PiE_def restrict_Pi_cancel*)

lemma *merge_singleton*[*simp*]: $i \notin I \Longrightarrow$ *merge* I $\{i\}$ (x,y) = *restrict* $(x(i := y\ i))$ (*insert* i I)
unfolding *merge_def* **by** (*auto simp*: *fun_eq_iff*)

lemma *extensional_merge_sub*: $I \cup J \subseteq K \Longrightarrow$ *merge* I J $(x, y) \in$ *extensional* K
unfolding *merge_def extensional_def* **by** *auto*

lemma *merge_restrict*[*simp*]:
merge I J (*restrict* x I, y) = *merge* I J (x, y)
merge I J $(x$, *restrict* y $J)$ = *merge* I J (x, y)
unfolding *merge_def* **by** *auto*

lemma *merge_x_x_eq_restrict*[*simp*]:
merge I J (*x*, *x*) = *restrict x* (*I* ∪ *J*)
unfolding *merge_def* **by** *auto*

lemma *injective_vimage_restrict*:
assumes *J*: *J* ⊆ *I*
and *sets*: *A* ⊆ (Π_E *i*∈*J*. *S i*) *B* ⊆ (Π_E *i*∈*J*. *S i*) **and** *ne*: (Π_E *i*∈*I*. *S i*) ≠ {}
and *eq*: (λ*x*. *restrict x J*) −' *A* ∩ (Π_E *i*∈*I*. *S i*) = (λ*x*. *restrict x J*) −' *B* ∩ (Π_E *i*∈*I*. *S i*)
shows *A* = *B*
proof (*intro set_eqI*)
fix *x*
from *ne* **obtain** *y* **where** *y*: ⋀*i*. *i* ∈ *I* ⟹ *y i* ∈ *S i* **by** *auto*
have *J* ∩ (*I* − *J*) = {} **by** *auto*
show *x* ∈ *A* ⟷ *x* ∈ *B*
proof *cases*
assume *x*: *x* ∈ (Π_E *i*∈*J*. *S i*)
have *x* ∈ *A* ⟷ *merge J* (*I* − *J*) (*x*,*y*) ∈ (λ*x*. *restrict x J*) −' *A* ∩ (Π_E *i*∈*I*. *S i*)
using *y x* ‹*J* ⊆ *I*› *PiE_cancel_merge*[*of J I* − *J x y S*]
by (*auto simp del*: *PiE_cancel_merge simp add*: *Un_absorb1*)
then show *x* ∈ *A* ⟷ *x* ∈ *B*
using *y x* ‹*J* ⊆ *I*› *PiE_cancel_merge*[*of J I* − *J x y S*]
by (*auto simp del*: *PiE_cancel_merge simp add*: *Un_absorb1 eq*)
qed (*insert sets*, *auto*)
qed

lemma *restrict_vimage*:
I ∩ *J* = {} ⟹
(λ*x*. (*restrict x I*, *restrict x J*)) −' (*Pi*$_E$ *I E* × *Pi*$_E$ *J F*) = *Pi* (*I* ∪ *J*) (*merge I J* (*E*, *F*))
by (*auto simp*: *restrict_Pi_cancel PiE_def*)

lemma *merge_vimage*:
I ∩ *J* = {} ⟹ *merge I J* −' *Pi*$_E$ (*I* ∪ *J*) *E* = *Pi I E* × *Pi J E*
by (*auto simp*: *restrict_Pi_cancel PiE_def*)

6.8.2 Finite product spaces

definition *prod_emb* **where**
prod_emb I M K X = (λ*x*. *restrict x K*) −' *X* ∩ (Π_E *i*∈*I*. *space* (*M i*))

lemma *prod_emb_iff*:
f ∈ *prod_emb I M K X* ⟷ *f* ∈ *extensional I* ∧ (*restrict f K* ∈ *X*) ∧ (∀ *i*∈*I*. *f i* ∈ *space* (*M i*))
unfolding *prod_emb_def PiE_def* **by** *auto*

lemma

shows *prod_emb_empty*[*simp*]: *prod_emb M L K* {} = {}
and *prod_emb_Un*[*simp*]: *prod_emb M L K* $(A \cup B)$ = *prod_emb M L K A* $\cup$ *prod_emb M L K B*
and *prod_emb_Int*: *prod_emb M L K* $(A \cap B)$ = *prod_emb M L K A* $\cap$ *prod_emb M L K B*
and *prod_emb_UN*[*simp*]: *prod_emb M L K* $(\bigcup i{\in}I.\ F\ i)$ = $(\bigcup i{\in}I.$ *prod_emb M L K* $(F\ i))$
and *prod_emb_INT*[*simp*]: $I \neq \{\} \Longrightarrow$ *prod_emb M L K* $(\bigcap i{\in}I.\ F\ i)$ = $(\bigcap i{\in}I.$ *prod_emb M L K* $(F\ i))$
and *prod_emb_Diff*[*simp*]: *prod_emb M L K* $(A - B)$ = *prod_emb M L K A* $-$ *prod_emb M L K B*
by (*auto simp*: *prod_emb_def*)

lemma *prod_emb_PiE*: $J \subseteq I \Longrightarrow (\bigwedge i.\ i \in J \Longrightarrow E\ i \subseteq$ *space* $(M\ i)) \Longrightarrow$
prod_emb I M J $(\Pi_E\ i{\in}J.\ E\ i) = (\Pi_E\ i{\in}I.$ *if* $i \in J$ *then* $E\ i$ *else space* $(M\ i))$
by (*force simp*: *prod_emb_def PiE_iff if_split_mem2*)

lemma *prod_emb_PiE_same_index*[*simp*]:
$(\bigwedge i.\ i \in I \Longrightarrow E\ i \subseteq$ *space* $(M\ i)) \Longrightarrow$ *prod_emb I M I* $(Pi_E\ I\ E) = Pi_E\ I\ E$
by (*auto simp*: *prod_emb_def PiE_iff*)

lemma *prod_emb_trans*[*simp*]:
$J \subseteq K \Longrightarrow K \subseteq L \Longrightarrow$ *prod_emb L M K* (*prod_emb K M J X*) = *prod_emb L M J X*
by (*auto simp add*: *Int_absorb1 prod_emb_def PiE_def*)

lemma *prod_emb_Pi*:
assumes $X \in (\Pi\ j{\in}J.$ *sets* $(M\ j))$ $J \subseteq K$
shows *prod_emb K M J* $(Pi_E\ J\ X) = (\Pi_E\ i{\in}K.$ *if* $i \in J$ *then* $X\ i$ *else space* $(M\ i))$
using *assms sets.space_closed*
by (*auto simp*: *prod_emb_def PiE_iff split*: *if_split_asm*) *blast*+

lemma *prod_emb_id*:
$B \subseteq (\Pi_E\ i{\in}L.$ *space* $(M\ i)) \Longrightarrow$ *prod_emb L M L B* = *B*
by (*auto simp*: *prod_emb_def subset_eq extensional_restrict*)

lemma *prod_emb_mono*:
$F \subseteq G \Longrightarrow$ *prod_emb A M B F* $\subseteq$ *prod_emb A M B G*
by (*auto simp*: *prod_emb_def*)

definition *PiM* :: *'i set* $\Rightarrow$ (*'i* $\Rightarrow$ *'a measure*) $\Rightarrow$ (*'i* $\Rightarrow$ *'a*) *measure* **where**
PiM I M = *extend_measure* $(\Pi_E\ i{\in}I.$ *space* $(M\ i))$
$\{(J, X).\ (J \neq \{\} \lor I = \{\}) \land$ *finite* $J \land J \subseteq I \land X \in (\Pi\ j{\in}J.$ *sets* $(M\ j))\}$
$(\lambda(J, X).$ *prod_emb I M J* $(\Pi_E\ j{\in}J.\ X\ j))$
$(\lambda(J, X).\ \prod j{\in}J \cup \{i{\in}I.$ *emeasure* $(M\ i)$ (*space* $(M\ i)) \neq 1\}$. *if* $j \in J$ *then* *emeasure* $(M\ j)\ (X\ j)$ *else emeasure* $(M\ j)$ (*space* $(M\ j)))$

definition *prod_algebra* :: *'i set* $\Rightarrow$ (*'i* $\Rightarrow$ *'a measure*) $\Rightarrow$ (*'i* $\Rightarrow$ *'a*) *set set* **where**

prod_algebra I M = (λ(*J*, *X*). *prod_emb I M J* (Π_E *j*∈*J*. *X j*)) ‘
{(*J*, *X*). (*J* ≠ {} ∨ *I* = {}) ∧ *finite J* ∧ *J* ⊆ *I* ∧ *X* ∈ (Π *j*∈*J*. *sets* (*M j*))}

abbreviation
Pi_M *I M* ≡ *PiM I M*

syntax
_PiM :: *pttrn* ⇒ *'i set* ⇒ *'a measure* ⇒ (*'i* => *'a*) *measure* ((*3*Π_M _∈_./ _) *10*)
translations
Π_M *x*∈*I*. *M* == *CONST PiM I* (%*x*. *M*)

lemma *extend_measure_cong*:
assumes Ω = Ω′ *I* = *I′ G* = *G′* ⋀*i*. *i* ∈ *I′* ⟹ μ *i* = μ′ *i*
shows *extend_measure* Ω *I G* μ = *extend_measure* Ω′ *I′ G′* μ′
unfolding *extend_measure_def* **by** (*auto simp add*: *assms*)

lemma *Pi_cong_sets*:
⟦*I* = *J*; ⋀*x*. *x* ∈ *I* ⟹ *M x* = *N x*⟧ ⟹ *Pi I M* = *Pi J N*
unfolding *Pi_def* **by** *auto*

lemma *PiM_cong*:
assumes *I* = *J* ⋀*x*. *x* ∈ *I* ⟹ *M x* = *N x*
shows *PiM I M* = *PiM J N*
unfolding *PiM_def*
proof (*rule extend_measure_cong*, *goal_cases*)
case *1*
show *?case* **using** *assms*
by (*subst assms*(*1*), *intro PiE_cong*[*of J* λ*i*. *space* (*M i*) λ*i*. *space* (*N i*)])
simp_all
next
case *2*
have ⋀*K*. *K* ⊆ *J* ⟹ (Π *j*∈*K*. *sets* (*M j*)) = (Π *j*∈*K*. *sets* (*N j*))
using *assms* **by** (*intro Pi_cong_sets*) *auto*
thus *?case* **by** (*auto simp*: *assms*)
next
case *3*
show *?case* **using** *assms*
by (*intro ext*) (*auto simp*: *prod_emb_def dest*: *PiE_mem*)
next
case (*4 x*)
thus *?case* **using** *assms*
by (*auto intro*!: *prod.cong split*: *if_split_asm*)
qed

lemma *prod_algebra_sets_into_space*:
prod_algebra I M ⊆ *Pow* (Π_E *i*∈*I*. *space* (*M i*))
by (*auto simp*: *prod_emb_def prod_algebra_def*)

lemma *prod_algebra_eq_finite*:
assumes *I*: *finite I*
shows *prod_algebra I M* = {(Π_E *i*∈*I*. *X i*) |*X*. *X* ∈ (Π *j*∈*I*. *sets* (*M j*))} (**is** *?L* = *?R*)
proof (*intro iffI set_eqI*)
fix *A* **assume** *A* ∈ *?L*
then obtain *J E* **where** *J*: *J* ≠ {} ∨ *I* = {} *finite J J* ⊆ *I* ∀ *i*∈*J*. *E i* ∈ *sets* (*M i*)
and *A*: *A* = *prod_emb I M J* (Π_E *j*∈*J*. *E j*)
by (*auto simp*: *prod_algebra_def*)
let *?A* = Π_E *i*∈*I*. *if i* ∈ *J then E i else space* (*M i*)
have *A*: *A* = *?A*
unfolding *A* **using** *J* **by** (*intro prod_emb_PiE sets.sets_into_space*) *auto*
show *A* ∈ *?R* **unfolding** *A* **using** *J sets.top*
by (*intro CollectI exI*[*of* _ λ*i*. *if i* ∈ *J then E i else space* (*M i*)]) *simp*
next
fix *A* **assume** *A* ∈ *?R*
then obtain *X* **where** *A*: *A* = (Π_E *i*∈*I*. *X i*) **and** *X*: *X* ∈ (Π *j*∈*I*. *sets* (*M j*)) **by** *auto*
then have *A*: *A* = *prod_emb I M I* (Π_E *i*∈*I*. *X i*)
by (*simp add*: *prod_emb_PiE_same_index*[*OF sets.sets_into_space*] *Pi_iff*)
from *X I* **show** *A* ∈ *?L* **unfolding** *A*
by (*auto simp*: *prod_algebra_def*)
qed

lemma *prod_algebraI*:
finite J ⟹ (*J* ≠ {} ∨ *I* = {}) ⟹ *J* ⊆ *I* ⟹ (⋀*i*. *i* ∈ *J* ⟹ *E i* ∈ *sets* (*M i*))
⟹ *prod_emb I M J* (Π_E *j*∈*J*. *E j*) ∈ *prod_algebra I M*
by (*auto simp*: *prod_algebra_def*)

lemma *prod_algebraI_finite*:
finite I ⟹ (∀ *i*∈*I*. *E i* ∈ *sets* (*M i*)) ⟹ (*Pi*$_E$ *I E*) ∈ *prod_algebra I M*
using *prod_algebraI*[*of I I E M*] *prod_emb_PiE_same_index*[*of I E M*, *OF sets.sets_into_space*]
by *simp*

lemma *Int_stable_PiE*: *Int_stable* {*Pi*$_E$ *J E* | *E*. ∀ *i*∈*I*. *E i* ∈ *sets* (*M i*)}
proof (*safe intro*!: *Int_stableI*)
fix *E F* **assume** ∀ *i*∈*I*. *E i* ∈ *sets* (*M i*) ∀ *i*∈*I*. *F i* ∈ *sets* (*M i*)
then show ∃ *G*. *Pi*$_E$ *J E* ∩ *Pi*$_E$ *J F* = *Pi*$_E$ *J G* ∧ (∀ *i*∈*I*. *G i* ∈ *sets* (*M i*))
by (*auto intro*!: *exI*[*of* _ λ*i*. *E i* ∩ *F i*] *simp*: *PiE_Int*)
qed

lemma *prod_algebraE*:
assumes *A*: *A* ∈ *prod_algebra I M*
obtains *J E* **where** *A* = *prod_emb I M J* (Π_E *j*∈*J*. *E j*)
finite J J ≠ {} ∨ *I* = {} *J* ⊆ *I* ⋀*i*. *i* ∈ *J* ⟹ *E i* ∈ *sets* (*M i*)
using *A* **by** (*auto simp*: *prod_algebra_def*)

lemma *prod_algebraE_all*:

assumes *A*: *A* ∈ *prod_algebra I M*
obtains *E* **where** *A* = Pi_E *I E E* ∈ (Π *i*∈*I*. *sets* (*M i*))
proof –
from *A* **obtain** *E J* **where** *A*: *A* = *prod_emb I M J* (Pi_E *J E*)
and *J*: *J* ⊆ *I* **and** *E*: *E* ∈ (Π *i*∈*J*. *sets* (*M i*))
by (*auto simp*: *prod_algebra_def*)
from *E* **have** ⋀*i*. *i* ∈ *J* ⟹ *E i* ⊆ *space* (*M i*)
using *sets.sets_into_space* **by** *auto*
then have *A* = (Π_E *i*∈*I*. *if i*∈*J then E i else space* (*M i*))
using *A J* **by** (*auto simp*: *prod_emb_PiE*)
moreover have (λ*i*. *if i*∈*J then E i else space* (*M i*)) ∈ (Π *i*∈*I*. *sets* (*M i*))
using *sets.top E* **by** *auto*
ultimately show *?thesis* **using** *that* **by** *auto*
qed

lemma *Int_stable_prod_algebra*: *Int_stable* (*prod_algebra I M*)
proof (*unfold Int_stable_def*, *safe*)
fix *A* **assume** *A* ∈ *prod_algebra I M*
from *prod_algebraE*[*OF this*] **guess** *J E* **. note** *A* = *this*
fix *B* **assume** *B* ∈ *prod_algebra I M*
from *prod_algebraE*[*OF this*] **guess** *K F* **. note** *B* = *this*
have *A* ∩ *B* = *prod_emb I M* (*J* ∪ *K*) (Π_E *i*∈*J* ∪ *K*. (*if i* ∈ *J then E i else space* (*M i*)) ∩
(*if i* ∈ *K then F i else space* (*M i*)))
unfolding *A B* **using** *A*(*2*,*3*,*4*) *A*(*5*)[*THEN sets.sets_into_space*] *B*(*2*,*3*,*4*)
B(*5*)[*THEN sets.sets_into_space*]
apply (*subst* (*1 2 3*) *prod_emb_PiE*)
apply (*simp_all add*: *subset_eq PiE_Int*)
apply *blast*
apply (*intro PiE_cong*)
apply *auto*
done
also have ... ∈ *prod_algebra I M*
using *A B* **by** (*auto intro*!: *prod_algebraI*)
finally show *A* ∩ *B* ∈ *prod_algebra I M* **.**
qed

proposition *prod_algebra_mono*:
assumes *space*: ⋀*i*. *i* ∈ *I* ⟹ *space* (*E i*) = *space* (*F i*)
assumes *sets*: ⋀*i*. *i* ∈ *I* ⟹ *sets* (*E i*) ⊆ *sets* (*F i*)
shows *prod_algebra I E* ⊆ *prod_algebra I F*
proof
fix *A* **assume** *A* ∈ *prod_algebra I E*
then obtain *J G* **where** *J*: *J* ≠ {} ∨ *I* = {} *finite J J* ⊆ *I*
and *A*: *A* = *prod_emb I E J* (Π_E *i*∈*J*. *G i*)
and *G*: ⋀*i*. *i* ∈ *J* ⟹ *G i* ∈ *sets* (*E i*)
by (*auto simp*: *prod_algebra_def*)
moreover
from *space* **have** (Π_E *i*∈*I*. *space* (*E i*)) = (Π_E *i*∈*I*. *space* (*F i*))

by (*rule PiE_cong*)
with A **have** $A = prod\_emb\ I\ F\ J\ (\Pi_E\ i{\in}J.\ G\ i)$
by (*simp add*: *prod_emb_def*)
moreover
from *sets G J* **have** $\bigwedge i.\ i \in J \Longrightarrow G\ i \in sets\ (F\ i)$
by *auto*
ultimately show $A \in prod\_algebra\ I\ F$
apply (*simp add*: *prod_algebra_def image_iff*)
apply (*intro exI*[*of* _ *J*] *exI*[*of* _ *G*] *conjI*)
apply *auto*
done
qed
proposition *prod_algebra_cong*:
assumes $I = J$ **and** *sets*: ($\bigwedge i.\ i \in I \Longrightarrow sets\ (M\ i) = sets\ (N\ i)$)
shows $prod\_algebra\ I\ M = prod\_algebra\ J\ N$
proof −
have *space*: $\bigwedge i.\ i \in I \Longrightarrow space\ (M\ i) = space\ (N\ i)$
using *sets_eq_imp_space_eq*[*OF sets*] **by** *auto*
with *sets* **show** *?thesis* **unfolding** ‹$I = J$›
by (*intro antisym prod_algebra_mono*) *auto*
qed

lemma *space_in_prod_algebra*:
$(\Pi_E\ i{\in}I.\ space\ (M\ i)) \in prod\_algebra\ I\ M$
proof *cases*
assume $I = \{\}$ **then show** *?thesis*
by (*auto simp add*: *prod_algebra_def image_iff prod_emb_def*)
next
assume $I \neq \{\}$
then obtain i **where** $i \in I$ **by** *auto*
then have $(\Pi_E\ i{\in}I.\ space\ (M\ i)) = prod\_emb\ I\ M\ \{i\}\ (\Pi_E\ i{\in}\{i\}.\ space\ (M\ i))$
by (*auto simp*: *prod_emb_def*)
also have $\ldots \in prod\_algebra\ I\ M$
using ‹$i \in I$› **by** (*intro prod_algebraI*) *auto*
finally show *?thesis* **.**
qed

lemma *space_PiM*: $space\ (\Pi_M\ i{\in}I.\ M\ i) = (\Pi_E\ i{\in}I.\ space\ (M\ i))$
using *prod_algebra_sets_into_space* **unfolding** *PiM_def prod_algebra_def* **by** (*intro space_extend_measure*) *simp*

lemma *prod_emb_subset_PiM*[*simp*]: $prod\_emb\ I\ M\ K\ X \subseteq space\ (PiM\ I\ M)$
by (*auto simp*: *prod_emb_def space_PiM*)

lemma *space_PiM_empty_iff*[*simp*]: $space\ (PiM\ I\ M) = \{\} \longleftrightarrow (\exists\, i{\in}I.\ space\ (M\ i) = \{\})$
by (*auto simp*: *space_PiM PiE_eq_empty_iff*)

lemma *undefined_in_PiM_empty*[*simp*]: $(\lambda x.\ undefined) \in space\ (PiM\ \{\}\ M)$

by (*auto simp*: *space_PiM*)

lemma *sets_PiM*: *sets* (Π_M *i*∈*I*. *M i*) = *sigma_sets* (Π_E *i*∈*I*. *space* (*M i*)) (*prod_algebra I M*)
using *prod_algebra_sets_into_space* **unfolding** *PiM_def prod_algebra_def* **by** (*intro sets_extend_measure*) *simp*

proposition *sets_PiM_single*: *sets* (*PiM I M*) =
sigma_sets (Π_E *i*∈*I*. *space* (*M i*)) {{*f*∈Π_E *i*∈*I*. *space* (*M i*). *f i* ∈ *A*} | *i A*. *i* ∈ *I* ∧ *A* ∈ *sets* (*M i*)}
(**is** _ = *sigma_sets ?Ω ?R*)
unfolding *sets_PiM*
proof (*rule sigma_sets_eqI*)
interpret *R*: *sigma_algebra ?Ω sigma_sets ?Ω ?R* **by** (*rule sigma_algebra_sigma_sets*) *auto*
fix *A* **assume** *A* ∈ *prod_algebra I M*
from *prod_algebraE*[*OF this*] **guess** *J X* **. note** *X* = *this*
show *A* ∈ *sigma_sets ?Ω ?R*
proof *cases*
assume *I* = {}
with *X* **have** *A* = {λ*x*. *undefined*} **by** (*auto simp*: *prod_emb_def*)
with ‹*I* = {}› **show** *?thesis* **by** (*auto intro*!: *sigma_sets_top*)
next
assume *I* ≠ {}
with *X* **have** *A* = (⋂*j*∈*J*. {*f*∈(Π_E *i*∈*I*. *space* (*M i*)). *f j* ∈ *X j*})
by (*auto simp*: *prod_emb_def*)
also have ... ∈ *sigma_sets ?Ω ?R*
using *X* ‹*I* ≠ {}› **by** (*intro R.finite_INT sigma_sets.Basic*) *auto*
finally show *A* ∈ *sigma_sets ?Ω ?R* **.**
qed
next
fix *A* **assume** *A* ∈ *?R*
then obtain *i B* **where** *A*: *A* = {*f*∈Π_E *i*∈*I*. *space* (*M i*). *f i* ∈ *B*} *i* ∈ *I B* ∈ *sets* (*M i*)
by *auto*
then have *A* = *prod_emb I M* {*i*} (Π_E *i*∈{*i*}. *B*)
by (*auto simp*: *prod_emb_def*)
also have ... ∈ *sigma_sets ?Ω* (*prod_algebra I M*)
using *A* **by** (*intro sigma_sets.Basic prod_algebraI*) *auto*
finally show *A* ∈ *sigma_sets ?Ω* (*prod_algebra I M*) **.**
qed

lemma *sets_PiM_eq_proj*:
I ≠ {} ⟹ *sets* (*PiM I M*) = *sets* (*SUP i*∈*I*. *vimage_algebra* (Π_E *i*∈*I*. *space* (*M i*)) (λ*x*. *x i*) (*M i*))
apply (*simp add*: *sets_PiM_single*)
apply (*subst sets_Sup_eq*[**where** *X*=Π_E *i*∈*I*. *space* (*M i*)])
apply *auto* []
apply *auto* []

```
  apply simp
  apply (subst arg_cong [of _ _ Sup, OF image_cong, OF refl])
  apply (rule sets_vimage_algebra2)
  apply (auto intro!: arg_cong2[where f=sigma_sets])
  done

lemma
  shows space_PiM_empty: space (Pi_M {} M) = {λk. undefined}
    and sets_PiM_empty: sets (Pi_M {} M) = { {}, {λk. undefined} }
 by (simp_all add: space_PiM sets_PiM_single image_constant sigma_sets_empty_eq)

proposition sets_PiM_sigma:
  assumes Ω_cover: ⋀i. i ∈ I ⟹ ∃S⊆E i. countable S ∧ Ω i = ⋃S
  assumes E: ⋀i. i ∈ I ⟹ E i ⊆ Pow (Ω i)
  assumes J: ⋀j. j ∈ J ⟹ finite j ⋃J = I
  defines P ≡ {{f∈(Π_E i∈I. Ω i). ∀i∈j. f i ∈ A i} | A j. j ∈ J ∧ A ∈ Pi j E}
  shows sets (Π_M i∈I. sigma (Ω i) (E i)) = sets (sigma (Π_E i∈I. Ω i) P)
proof cases
  assume I = {}
  with ‹⋃J = I› have P = {{λ_. undefined}} ∨ P = {}
    by (auto simp: P_def)
  with ‹I = {}› show ?thesis
    by (auto simp add: sets_PiM_empty sigma_sets_empty_eq)
next
  let ?F = λi. {(λx. x i) −‘ A ∩ Pi_E I Ω |A. A ∈ E i}
  assume I ≠ {}
  then have sets (Pi_M I (λi. sigma (Ω i) (E i))) =
      sets (SUP i∈I. vimage_algebra (Π_E i∈I. Ω i) (λx. x i) (sigma (Ω i) (E i)))
    by (subst sets_PiM_eq_proj) (auto simp: space_measure_of_conv)
  also have ... = sets (SUP i∈I. sigma (Pi_E I Ω) (?F i))
   using E by (intro sets_SUP_cong arg_cong[where f=sets] vimage_algebra_sigma)
auto
  also have ... = sets (sigma (Pi_E I Ω) (⋃i∈I. ?F i))
    using ‹I ≠ {}› by (intro arg_cong[where f=sets] SUP_sigma_sigma) auto
  also have ... = sets (sigma (Pi_E I Ω) P)
  proof (intro arg_cong[where f=sets] sigma_eqI sigma_sets_eqI)
    show (⋃i∈I. ?F i) ⊆ Pow (Pi_E I Ω) P ⊆ Pow (Pi_E I Ω)
      by (auto simp: P_def)
  next
    interpret P: sigma_algebra Π_E i∈I. Ω i sigma_sets (Π_E i∈I. Ω i) P
      by (auto intro!: sigma_algebra_sigma_sets simp: P_def)

    fix Z assume Z ∈ (⋃i∈I. ?F i)
     then obtain i A where i: i ∈ I A ∈ E i and Z_def: Z = (λx. x i) −‘ A ∩
Pi_E I Ω
      by auto
    from ‹i ∈ I› J obtain j where j: i ∈ j j ∈ J j ⊆ I finite j
      by auto
    obtain S where S: ⋀i. i ∈ j ⟹ S i ⊆ E i ⋀i. i ∈ j ⟹ countable (S i)
```

$\bigwedge i.\ i \in j \Longrightarrow \Omega\ i = \bigcup (S\ i)$
by (*metis subset_eq* Ω*_cover* ‹$j \subseteq I$›)
define A' **where** $A'\ n = n(i := A)$ **for** n
then have *A'_i*: $\bigwedge n.\ A'\ n\ i = A$
by *simp*
{ fix n **assume** $n \in Pi_E\ (j - \{i\})\ S$
then have $A'\ n \in Pi\ j\ E$
unfolding *PiE_def Pi_def* **using** *S(1)* **by** (*auto simp*: *A'_def* ‹$A \in E\ i$›)
with ‹$j \in J$› **have** $\{f \in Pi_E\ I\ \Omega.\ \forall i{\in}j.\ f\ i \in A'\ n\ i\} \in P$
by (*auto simp*: *P_def*) **}**
note *A'_in_P* = *this*

{ fix x **assume** $x\ i \in A\ x \in Pi_E\ I\ \Omega$
with *S(3)* ‹$j \subseteq I$› **have** $\forall i{\in}j.\ \exists s{\in}S\ i.\ x\ i \in s$
by (*auto simp*: *PiE_def Pi_def*)
then obtain s **where** s: $\bigwedge i.\ i \in j \Longrightarrow s\ i \in S\ i$ $\bigwedge i.\ i \in j \Longrightarrow x\ i \in s\ i$
by *metis*
with ‹$x\ i \in A$› **have** $\exists n{\in}Pi_E\ (j{-}\{i\})\ S.\ \forall i{\in}j.\ x\ i \in A'\ n\ i$
by (*intro bexI*[*of* _ *restrict* ($s(i := A)$) ($j{-}\{i\}$)]) (*auto simp*: *A'_def split*: *if_splits*) **}**
then have $Z = (\bigcup n{\in}Pi_E\ (j{-}\{i\})\ S.\ \{f{\in}(\Pi_E\ i{\in}I.\ \Omega\ i).\ \forall i{\in}j.\ f\ i \in A'\ n\ i\})$
unfolding *Z_def*
by (*auto simp add*: *set_eq_iff ball_conj_distrib* ‹$i{\in}j$› *A'_i dest*: *bspec*[*OF* _ ‹$i{\in}j$›] *cong*: *conj_cong*)
also have ... $\in$ *sigma_sets* ($\Pi_E\ i{\in}I.\ \Omega\ i$) P
using ‹*finite j*› *S(2)*
by (*intro P.countable_UN' countable_PiE*) (*simp_all add*: *image_subset_iff A'_in_P*)
finally show $Z \in$ *sigma_sets* ($\Pi_E\ i{\in}I.\ \Omega\ i$) P **.**
next
interpret F: *sigma_algebra* $\Pi_E\ i{\in}I.\ \Omega\ i$ *sigma_sets* ($\Pi_E\ i{\in}I.\ \Omega\ i$) ($\bigcup i{\in}I.$ *?F* i)
by (*auto intro*!: *sigma_algebra_sigma_sets*)

fix b **assume** $b \in P$
then obtain $A\ j$ **where** b: $b = \{f{\in}(\Pi_E\ i{\in}I.\ \Omega\ i).\ \forall i{\in}j.\ f\ i \in A\ i\}$ $j \in J$ $A \in Pi\ j\ E$
by (*auto simp*: *P_def*)
show $b \in$ *sigma_sets* ($Pi_E\ I\ \Omega$) ($\bigcup i{\in}I.$ *?F* i)
proof *cases*
assume $j = \{\}$
with b **have** $b = (\Pi_E\ i{\in}I.\ \Omega\ i)$
by *auto*
then show *?thesis*
by *blast*
next
assume $j \neq \{\}$
with J *b(2,3)* **have** *eq*: $b = (\bigcap i{\in}j.\ ((\lambda x.\ x\ i) -`\ A\ i \cap Pi_E\ I\ \Omega))$
unfolding *b(1)*

by (*auto simp*: *PiE_def Pi_def*)
show *?thesis*
unfolding *eq* **using** ‹$A \in Pi\ j\ E$› ‹$j \in J$› *J*(*2*)
by (*intro F.finite_INT J* ‹$j \in J$› ‹$j \neq \{\}$› *sigma_sets.Basic*) *blast*
qed
qed
finally show *?thesis* **.**
qed

lemma *sets_PiM_in_sets*:
assumes *space*: $space\ N = (\Pi_E\ i{\in}I.\ space\ (M\ i))$
assumes *sets*: $\bigwedge i\ A.\ i \in I \Longrightarrow A \in sets\ (M\ i) \Longrightarrow \{x{\in}space\ N.\ x\ i \in A\} \in sets\ N$
shows $sets\ (\Pi_M\ i \in I.\ M\ i) \subseteq sets\ N$
unfolding *sets_PiM_single space*[*symmetric*]
by (*intro sets.sigma_sets_subset subsetI*) (*auto intro*: *sets*)

lemma *sets_PiM_cong*[*measurable_cong*]:
assumes $I = J\ \bigwedge i.\ i \in J \Longrightarrow sets\ (M\ i) = sets\ (N\ i)$ **shows** $sets\ (PiM\ I\ M) = sets\ (PiM\ J\ N)$
using *assms sets_eq_imp_space_eq*[*OF assms*(*2*)] **by** (*simp add*: *sets_PiM_single cong*: *PiE_cong conj_cong*)

lemma *sets_PiM_I*:
assumes *finite J* $J \subseteq I\ \forall i{\in}J.\ E\ i \in sets\ (M\ i)$
shows $prod\_emb\ I\ M\ J\ (\Pi_E\ j{\in}J.\ E\ j) \in sets\ (\Pi_M\ i{\in}I.\ M\ i)$
proof *cases*
assume $J = \{\}$
then have $prod\_emb\ I\ M\ J\ (\Pi_E\ j{\in}J.\ E\ j) = (\Pi_E\ j{\in}I.\ space\ (M\ j))$
by (*auto simp*: *prod_emb_def*)
then show *?thesis*
by (*auto simp add*: *sets_PiM intro*!: *sigma_sets_top*)
next
assume $J \neq \{\}$ **with** *assms* **show** *?thesis*
by (*force simp add*: *sets_PiM prod_algebra_def*)
qed

proposition *measurable_PiM*:
assumes *space*: $f \in space\ N \to (\Pi_E\ i{\in}I.\ space\ (M\ i))$
assumes *sets*: $\bigwedge X\ J.\ J \neq \{\} \lor I = \{\} \Longrightarrow finite\ J \Longrightarrow J \subseteq I \Longrightarrow (\bigwedge i.\ i \in J \Longrightarrow X\ i \in sets\ (M\ i)) \Longrightarrow$
$f\ -`\ prod\_emb\ I\ M\ J\ (Pi_E\ J\ X) \cap space\ N \in sets\ N$
shows $f \in measurable\ N\ (PiM\ I\ M)$
using *sets_PiM prod_algebra_sets_into_space space*
proof (*rule measurable_sigma_sets*)
fix *A* **assume** $A \in prod\_algebra\ I\ M$
from *prod_algebraE*[*OF this*] **guess** *J X* **.**
with *sets*[*of J X*] **show** $f\ -`\ A \cap space\ N \in sets\ N$ **by** *auto*
qed

lemma *measurable_PiM_Collect*:
assumes *space*: $f \in space\ N \rightarrow (\Pi_E\ i{\in}I.\ space\ (M\ i))$
assumes *sets*: $\bigwedge X\ J.\ J \neq \{\} \vee I = \{\} \Longrightarrow finite\ J \Longrightarrow J \subseteq I \Longrightarrow (\bigwedge i.\ i \in J \Longrightarrow X\ i \in sets\ (M\ i)) \Longrightarrow$
$\{\omega{\in}space\ N.\ \forall\ i{\in}J.\ f\ \omega\ i \in X\ i\} \in sets\ N$
shows $f \in measurable\ N\ (PiM\ I\ M)$
using *sets_PiM prod_algebra_sets_into_space space*
proof (*rule measurable_sigma_sets*)
fix *A* **assume** $A \in prod\_algebra\ I\ M$
from *prod_algebraE*[*OF this*] **guess** *J X* **. note** $X = this$
then have $f\ -`\ A \cap space\ N = \{\omega \in space\ N.\ \forall\ i{\in}J.\ f\ \omega\ i \in X\ i\}$
using *space* **by** (*auto simp*: *prod_emb_def del*: *PiE_I*)
also have $\ldots \in sets\ N$ **using** $X(3,2,4,5)$ **by** (*rule sets*)
finally show $f\ -`\ A \cap space\ N \in sets\ N$ **.**
qed

lemma *measurable_PiM_single*:
assumes *space*: $f \in space\ N \rightarrow (\Pi_E\ i{\in}I.\ space\ (M\ i))$
assumes *sets*: $\bigwedge A\ i.\ i \in I \Longrightarrow A \in sets\ (M\ i) \Longrightarrow \{\omega \in space\ N.\ f\ \omega\ i \in A\} \in sets\ N$
shows $f \in measurable\ N\ (PiM\ I\ M)$
using *sets_PiM_single*
proof (*rule measurable_sigma_sets*)
fix *A* **assume** $A \in \{\{f \in \Pi_E\ i{\in}I.\ space\ (M\ i).\ f\ i \in A\}\ |i\ A.\ i \in I \wedge A \in sets\ (M\ i)\}$
then obtain *B i* **where** $A = \{f \in \Pi_E\ i{\in}I.\ space\ (M\ i).\ f\ i \in B\}$ **and** *B*: $i \in I$ $B \in sets\ (M\ i)$
by *auto*
with *space* **have** $f\ -`\ A \cap space\ N = \{\omega \in space\ N.\ f\ \omega\ i \in B\}$ **by** *auto*
also have $\ldots \in sets\ N$ **using** *B* **by** (*rule sets*)
finally show $f\ -`\ A \cap space\ N \in sets\ N$ **.**
qed (*auto simp*: *space*)

lemma *measurable_PiM_single'*:
assumes *f*: $\bigwedge i.\ i \in I \Longrightarrow f\ i \in measurable\ N\ (M\ i)$
and $(\lambda\omega\ i.\ f\ i\ \omega) \in space\ N \rightarrow (\Pi_E\ i{\in}I.\ space\ (M\ i))$
shows $(\lambda\omega\ i.\ f\ i\ \omega) \in measurable\ N\ (Pi_M\ I\ M)$
proof (*rule measurable_PiM_single*)
fix *A i* **assume** *A*: $i \in I$ $A \in sets\ (M\ i)$
then have $\{\omega \in space\ N.\ f\ i\ \omega \in A\} = f\ i\ -`\ A \cap space\ N$
by *auto*
then show $\{\omega \in space\ N.\ f\ i\ \omega \in A\} \in sets\ N$
using *A f* **by** (*auto intro*!: *measurable_sets*)
qed *fact*

lemma *sets_PiM_I_finite*[*measurable*]:
assumes *finite I* **and** *sets*: $(\bigwedge i.\ i \in I \Longrightarrow E\ i \in sets\ (M\ i))$
shows $(\Pi_E\ j{\in}I.\ E\ j) \in sets\ (\Pi_M\ i{\in}I.\ M\ i)$

using *sets_PiM_I[of I I E M] sets.sets_into_space[OF sets]* ‹*finite I*› *sets* **by** *auto*

lemma *measurable_component_singleton[measurable (raw)]*:
assumes $i \in I$ **shows** $(\lambda x.\ x\ i) \in$ *measurable* $(Pi_M\ I\ M)\ (M\ i)$
proof (*unfold measurable_def*, *intro CollectI conjI ballI*)
fix A **assume** $A \in$ *sets* $(M\ i)$
then have $(\lambda x.\ x\ i)$ −‘ $A \cap$ *space* $(Pi_M\ I\ M)$ = *prod_emb* $I\ M\ \{i\}\ (\Pi_E\ j{\in}\{i\}.\ A)$
using *sets.sets_into_space* ‹$i \in I$›
by (*fastforce dest*: *Pi_mem simp*: *prod_emb_def space_PiM split*: *if_split_asm*)
then show $(\lambda x.\ x\ i)$ −‘ $A \cap$ *space* $(Pi_M\ I\ M) \in$ *sets* $(Pi_M\ I\ M)$
using ‹$A \in$ *sets* $(M\ i)$› ‹$i \in I$› **by** (*auto intro*!: *sets_PiM_I*)
qed (*insert* ‹$i \in I$›, *auto simp*: *space_PiM*)

lemma *measurable_component_singleton′[measurable_dest]*:
assumes f: $f \in$ *measurable* $N\ (Pi_M\ I\ M)$
assumes g: $g \in$ *measurable* $L\ N$
assumes i: $i \in I$
shows $(\lambda x.\ (f\ (g\ x))\ i) \in$ *measurable* $L\ (M\ i)$
using *measurable_compose[OF measurable_compose[OF g f] measurable_component_singleton, OF i]* **.**

lemma *measurable_PiM_component_rev*:
$i \in I \Longrightarrow f \in$ *measurable* $(M\ i)\ N \Longrightarrow (\lambda x.\ f\ (x\ i)) \in$ *measurable* $(PiM\ I\ M)\ N$
by *simp*

lemma *measurable_case_nat[measurable (raw)]*:
assumes [*measurable (raw)*]: $i = 0 \Longrightarrow f \in$ *measurable* $M\ N$
$\bigwedge j.\ i = Suc\ j \Longrightarrow (\lambda x.\ g\ x\ j) \in$ *measurable* $M\ N$
shows $(\lambda x.\ case\_nat\ (f\ x)\ (g\ x)\ i) \in$ *measurable* $M\ N$
by (*cases i*) *simp_all*

lemma *measurable_case_nat′[measurable (raw)]*:
assumes *fg[measurable]*: $f \in$ *measurable* $N\ M$ $g \in$ *measurable* $N\ (\Pi_M\ i{\in}UNIV.\ M)$
shows $(\lambda x.\ case\_nat\ (f\ x)\ (g\ x)) \in$ *measurable* $N\ (\Pi_M\ i{\in}UNIV.\ M)$
using *fg[THEN measurable_space]*
by (*auto intro*!: *measurable_PiM_single′ simp add*: *space_PiM PiE_iff split*: *nat.split*)

lemma *measurable_add_dim[measurable]*:
$(\lambda(f, y).\ f(i := y)) \in$ *measurable* $(Pi_M\ I\ M \bigotimes_M M\ i)\ (Pi_M\ (insert\ i\ I)\ M)$
(**is** *?f* $\in$ *measurable ?P ?I*)
proof (*rule measurable_PiM_single*)
fix $j\ A$ **assume** j: $j \in$ *insert* $i\ I$ **and** A: $A \in$ *sets* $(M\ j)$
have $\{\omega \in$ *space ?P*. $(\lambda(f, x).\ fun\_upd\ f\ i\ x)\ \omega\ j \in A\}$ =
(*if* $j = i$ *then space* $(Pi_M\ I\ M) \times A$ *else* $((\lambda x.\ x\ j) \circ fst)$ −‘ $A \cap$ *space ?P*)
using *sets.sets_into_space[OF A]* **by** (*auto simp add*: *space_pair_measure space_PiM*)
also have ... $\in$ *sets ?P*
using $A\ j$

by (*auto intro*!: *measurable_sets*[*OF measurable_comp*, *OF _ measurable_component_singleton*])
finally show $\{\omega \in space\ ?P.\ case\_prod\ (\lambda f.\ fun\_upd\ f\ i)\ \omega\ j \in A\} \in sets\ ?P$ **.**
qed (*auto simp*: *space_pair_measure space_PiM PiE_def*)

proposition *measurable_fun_upd*:
assumes *I*: $I = J \cup \{i\}$
assumes *f*[*measurable*]: $f \in measurable\ N\ (PiM\ J\ M)$
assumes *h*[*measurable*]: $h \in measurable\ N\ (M\ i)$
shows $(\lambda x.\ (f\ x)\ (i := h\ x)) \in measurable\ N\ (PiM\ I\ M)$
proof (*intro measurable_PiM_single'*)
fix *j* **assume** $j \in I$ **then show** $(\lambda\omega.\ ((f\ \omega)(i := h\ \omega))\ j) \in measurable\ N\ (M\ j)$
unfolding *I* **by** (*cases* $j = i$) *auto*
next
show $(\lambda x.\ (f\ x)(i := h\ x)) \in space\ N \rightarrow (\Pi_E\ i{\in}I.\ space\ (M\ i))$
using *I f*[*THEN measurable_space*] *h*[*THEN measurable_space*]
by (*auto simp*: *space_PiM PiE_iff extensional_def*)
qed

lemma *measurable_component_update*:
$x \in space\ (Pi_M\ I\ M) \Longrightarrow i \notin I \Longrightarrow (\lambda v.\ x(i := v)) \in measurable\ (M\ i)\ (Pi_M\ (insert\ i\ I)\ M)$
by *simp*

lemma *measurable_merge*[*measurable*]:
$merge\ I\ J \in measurable\ (Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)\ (Pi_M\ (I \cup J)\ M)$
(**is** $?f \in measurable\ ?P\ ?U$)
proof (*rule measurable_PiM_single*)
fix *i A* **assume** *A*: $A \in sets\ (M\ i)\ i \in I \cup J$
then have $\{\omega \in space\ ?P.\ merge\ I\ J\ \omega\ i \in A\} =$
$(if\ i \in I\ then\ ((\lambda x.\ x\ i) \circ fst) -` A \cap space\ ?P\ else\ ((\lambda x.\ x\ i) \circ snd) -` A \cap space\ ?P)$
by (*auto simp*: *merge_def*)
also have $\ldots \in sets\ ?P$
using *A*
by (*auto intro*!: *measurable_sets*[*OF measurable_comp*, *OF _ measurable_component_singleton*])
finally show $\{\omega \in space\ ?P.\ merge\ I\ J\ \omega\ i \in A\} \in sets\ ?P$ **.**
qed (*auto simp*: *space_pair_measure space_PiM PiE_iff merge_def extensional_def*)

lemma *measurable_restrict*[*measurable* (*raw*)]:
assumes *X*: $\bigwedge i.\ i \in I \Longrightarrow X\ i \in measurable\ N\ (M\ i)$
shows $(\lambda x.\ \lambda i{\in}I.\ X\ i\ x) \in measurable\ N\ (Pi_M\ I\ M)$
proof (*rule measurable_PiM_single*)
fix *A i* **assume** *A*: $i \in I\ A \in sets\ (M\ i)$
then have $\{\omega \in space\ N.\ (\lambda i{\in}I.\ X\ i\ \omega)\ i \in A\} = X\ i -` A \cap space\ N$
by *auto*
then show $\{\omega \in space\ N.\ (\lambda i{\in}I.\ X\ i\ \omega)\ i \in A\} \in sets\ N$
using *A X* **by** (*auto intro*!: *measurable_sets*)
qed (*insert X*, *auto simp add*: *PiE_def dest*: *measurable_space*)

lemma *measurable_abs_UNIV*:
$(\bigwedge n.\ (\lambda\omega.\ f\ n\ \omega) \in measurable\ M\ (N\ n)) \Longrightarrow (\lambda\omega\ n.\ f\ n\ \omega) \in measurable\ M$ $(PiM\ UNIV\ N)$
by (*intro measurable_PiM_single*) (*auto dest*: *measurable_space*)

lemma *measurable_restrict_subset*: $J \subseteq L \Longrightarrow (\lambda f.\ restrict\ f\ J) \in measurable\ (Pi_M\ L\ M)\ (Pi_M\ J\ M)$
by (*intro measurable_restrict measurable_component_singleton*) *auto*

lemma *measurable_restrict_subset'*:
assumes $J \subseteq L\ \bigwedge x.\ x \in J \Longrightarrow sets\ (M\ x) = sets\ (N\ x)$
shows $(\lambda f.\ restrict\ f\ J) \in measurable\ (Pi_M\ L\ M)\ (Pi_M\ J\ N)$
proof−
from *assms(1)* **have** $(\lambda f.\ restrict\ f\ J) \in measurable\ (Pi_M\ L\ M)\ (Pi_M\ J\ M)$
by (*rule measurable_restrict_subset*)
also from *assms(2)* **have** $measurable\ (Pi_M\ L\ M)\ (Pi_M\ J\ M) = measurable\ (Pi_M\ L\ M)\ (Pi_M\ J\ N)$
by (*intro sets_PiM_cong measurable_cong_sets*) *simp_all*
finally show *?thesis* **.**
qed

lemma *measurable_prod_emb*[*intro*, *simp*]:
$J \subseteq L \Longrightarrow X \in sets\ (Pi_M\ J\ M) \Longrightarrow prod\_emb\ L\ M\ J\ X \in sets\ (Pi_M\ L\ M)$
unfolding *prod_emb_def space_PiM*[*symmetric*]
by (*auto intro*!: *measurable_sets measurable_restrict measurable_component_singleton*)

lemma *merge_in_prod_emb*:
assumes $y \in space\ (PiM\ I\ M)\ x \in X$ **and** X: $X \in sets\ (Pi_M\ J\ M)$ **and** $J \subseteq I$
shows $merge\ J\ I\ (x,\ y) \in prod\_emb\ I\ M\ J\ X$
using *assms sets.sets_into_space*[*OF X*]
by (*simp add*: *merge_def prod_emb_def subset_eq space_PiM PiE_def extensional_restrict Pi_iff*
cong: *if_cong restrict_cong*)
(*simp add*: *extensional_def*)

lemma *prod_emb_eq_emptyD*:
assumes J: $J \subseteq I$ **and** *ne*: $space\ (PiM\ I\ M) \neq \{\}$ **and** X: $X \in sets\ (Pi_M\ J\ M)$
and $*$: $prod\_emb\ I\ M\ J\ X = \{\}$
shows $X = \{\}$
proof *safe*
fix x **assume** $x \in X$
obtain ω **where** $\omega \in space\ (PiM\ I\ M)$
using *ne* **by** *blast*
from *merge_in_prod_emb*[*OF this* ‹$x \in X$› *X J*] $*$ **show** $x \in \{\}$ **by** *auto*
qed

lemma *sets_in_Pi_aux*:
$finite\ I \Longrightarrow (\bigwedge j.\ j \in I \Longrightarrow \{x \in space\ (M\ j).\ x \in F\ j\} \in sets\ (M\ j)) \Longrightarrow$

$\{x \in space\ (PiM\ I\ M).\ x \in Pi\ I\ F\} \in sets\ (PiM\ I\ M)$
by (*simp add*: *subset_eq Pi_iff*)

lemma *sets_in_Pi*[*measurable* (*raw*)]:
$finite\ I \Longrightarrow f \in measurable\ N\ (PiM\ I\ M) \Longrightarrow$
$(\bigwedge j.\ j \in I \Longrightarrow \{x \in space\ (M\ j).\ x \in F\ j\} \in sets\ (M\ j)) \Longrightarrow$
$Measurable.pred\ N\ (\lambda x.\ f\ x \in Pi\ I\ F)$
unfolding *pred_def*
by (*rule measurable_sets_Collect*[*of f N PiM I M*, *OF _ sets_in_Pi_aux*]) *auto*

lemma *sets_in_extensional_aux*:
$\{x \in space\ (PiM\ I\ M).\ x \in extensional\ I\} \in sets\ (PiM\ I\ M)$
proof −
have $\{x \in space\ (PiM\ I\ M).\ x \in extensional\ I\} = space\ (PiM\ I\ M)$
by (*auto simp add*: *extensional_def space_PiM*)
then show *?thesis* **by** *simp*
qed

lemma *sets_in_extensional*[*measurable* (*raw*)]:
$f \in measurable\ N\ (PiM\ I\ M) \Longrightarrow Measurable.pred\ N\ (\lambda x.\ f\ x \in extensional\ I)$
unfolding *pred_def*
by (*rule measurable_sets_Collect*[*of f N PiM I M*, *OF _ sets_in_extensional_aux*])
auto

lemma *sets_PiM_I_countable*:
assumes *I*: *countable I* **and** *E*: $\bigwedge i.\ i \in I \Longrightarrow E\ i \in sets\ (M\ i)$ **shows** $Pi_E\ I\ E$
$\in sets\ (Pi_M\ I\ M)$
proof *cases*
assume $I \neq \{\}$
then have $Pi_E\ I\ E = (\bigcap i \in I.\ prod\_emb\ I\ M\ \{i\}\ (Pi_E\ \{i\}\ E))$
using *E*[*THEN sets.sets_into_space*] **by** (*auto simp*: *PiE_iff prod_emb_def fun_eq_iff*)
also have $\ldots \in sets\ (PiM\ I\ M)$
using *I* ‹$I \neq \{\}$› **by** (*safe intro*!: *sets.countable_INT′ measurable_prod_emb*
sets_PiM_I_finite E)
finally show *?thesis* **.**
qed (*simp add*: *sets_PiM_empty*)

lemma *sets_PiM_D_countable*:
assumes *A*: $A \in PiM\ I\ M$
shows $\exists J \subseteq I.\ \exists X \in PiM\ J\ M.\ countable\ J \wedge A = prod\_emb\ I\ M\ J\ X$
using *A*[*unfolded sets_PiM_single*]
proof *induction*
case (*Basic A*)
then obtain *i X* **where** ∗: $i \in I\ X \in sets\ (M\ i)$ **and** $A = \{f \in \Pi_E\ i \in I.\ space$
$(M\ i).\ f\ i \in X\}$
by *auto*
then have *A*: $A = prod\_emb\ I\ M\ \{i\}\ (\Pi_E\ \_ \in \{i\}.\ X)$
by (*auto simp*: *prod_emb_def*)
then show *?case*

by (*intro exI*[*of* _ {*i*}] *conjI bexI*[*of* _ Π_E _∈{*i*}. *X*])
(*auto intro*: *countable_finite* ∗ *sets_PiM_I_finite*)
next
case *Empty* **then show** *?case*
by (*intro exI*[*of* _ {}] *conjI bexI*[*of* _ {}]) *auto*
next
case (*Compl A*)
then obtain *J X* **where** *J* ⊆ *I X* ∈ *sets* (Pi_M *J M*) *countable J A* = *prod_emb I M J X*
by *auto*
then show *?case*
by (*intro exI*[*of* _ *J*] *bexI*[*of* _ *space* (*PiM J M*) − *X*] *conjI*)
(*auto simp add*: *space_PiM prod_emb_PiE intro*!: *sets_PiM_I_countable*)
next
case (*Union K*)
obtain *J X* **where** *J*: ⋀*i*. *J i* ⊆ *I* ⋀*i*. *countable* (*J i*) **and** *X*: ⋀*i*. *X i* ∈ *sets* (Pi_M (*J i*) *M*)
and *K*: ⋀*i*. *K i* = *prod_emb I M* (*J i*) (*X i*)
by (*metis Union.IH*)
show *?case*
proof (*intro exI*[*of* _ ⋃*i*. *J i*] *bexI*[*of* _ ⋃*i*. *prod_emb* (⋃*i*. *J i*) *M* (*J i*) (*X i*)] *conjI*)
show (⋃*i*. *J i*) ⊆ *I countable* (⋃*i*. *J i*) **using** *J* **by** *auto*
with *J* **show** ⋃(*K* ‘ *UNIV*) = *prod_emb I M* (⋃*i*. *J i*) (⋃*i*. *prod_emb* (⋃*i*. *J i*) *M* (*J i*) (*X i*))
by (*simp add*: *K*[*abs_def*] *SUP_upper*)
qed(*auto intro*: *X*)
qed

proposition *measure_eqI_PiM_finite*:
assumes [*simp*]: *finite I sets P* = *PiM I M sets Q* = *PiM I M*
assumes *eq*: ⋀*A*. (⋀*i*. *i* ∈ *I* ⟹ *A i* ∈ *sets* (*M i*)) ⟹ *P* (Pi_E *I A*) = *Q* (Pi_E *I A*)
assumes *A*: *range A* ⊆ *prod_algebra I M* (⋃*i*. *A i*) = *space* (*PiM I M*) ⋀*i*::*nat*. *P* (*A i*) ≠ ∞
shows *P* = *Q*
proof (*rule measure_eqI_generator_eq*[*OF Int_stable_prod_algebra prod_algebra_sets_into_space*])
show *range A* ⊆ *prod_algebra I M* (⋃*i*. *A i*) = (Π_E *i*∈*I*. *space* (*M i*)) ⋀*i*. *P* (*A i*) ≠ ∞
unfolding *space_PiM*[*symmetric*] **by** *fact*+
fix *X* **assume** *X* ∈ *prod_algebra I M*
then obtain *J E* **where** *X*: *X* = *prod_emb I M J* (Π_E *j*∈*J*. *E j*)
and *J*: *finite J J* ⊆ *I* ⋀*j*. *j* ∈ *J* ⟹ *E j* ∈ *sets* (*M j*)
by (*force elim*!: *prod_algebraE*)
then show *emeasure P X* = *emeasure Q X*
unfolding *X* **by** (*subst* (*1 2*) *prod_emb_Pi*) (*auto simp*: *eq*)
qed (*simp_all add*: *sets_PiM*)

proposition *measure_eqI_PiM_infinite*:

assumes [*simp*]: *sets P* = *PiM I M sets Q* = *PiM I M*
assumes *eq*: $\bigwedge$*A J*. *finite J* $\Longrightarrow$ *J* $\subseteq$ *I* $\Longrightarrow$ ($\bigwedge$*i*. *i* $\in$ *J* $\Longrightarrow$ *A i* $\in$ *sets* (*M i*)) $\Longrightarrow$
P (*prod_emb I M J* (Pi_E *J A*)) = *Q* (*prod_emb I M J* (Pi_E *J A*))
assumes *A*: *finite_measure P*
shows *P* = *Q*
proof (*rule measure_eqI_generator_eq*[*OF Int_stable_prod_algebra prod_algebra_sets_into_space*])
interpret *finite_measure P* **by** *fact*
define *i* **where** *i* = (*SOME i*. *i* $\in$ *I*)
have *i*: *I* $\neq$ {} $\Longrightarrow$ *i* $\in$ *I*
unfolding *i_def* **by** (*rule someI_ex*) *auto*
define *A* **where** *A n* =
(*if I* = {} *then prod_emb I M* {} (Π_E *i*$\in${}. {}) *else prod_emb I M* {*i*} (Π_E *i*$\in${*i*}. *space* (*M i*)))
for *n* :: *nat*
then show *range A* $\subseteq$ *prod_algebra I M*
using *prod_algebraI*[*of* {} *I* λ*i*. *space* (*M i*) *M*] **by** (*auto intro*!: *prod_algebraI i*)
have $\bigwedge$*i*. *A i* = *space* (*PiM I M*)
by (*auto simp*: *prod_emb_def space_PiM PiE_iff A_def i ex_in_conv*[*symmetric*] *exI*)
then show ($\bigcup$*i*. *A i*) = (Π_E *i*$\in$*I*. *space* (*M i*)) $\bigwedge$*i*. *emeasure P* (*A i*) $\neq$ ∞
by (*auto simp*: *space_PiM*)
next
fix *X* **assume** *X*: *X* $\in$ *prod_algebra I M*
then obtain *J E* **where** *X*: *X* = *prod_emb I M J* (Π_E *j*$\in$*J*. *E j*)
and *J*: *finite J J* $\subseteq$ *I* $\bigwedge$*j*. *j* $\in$ *J* $\Longrightarrow$ *E j* $\in$ *sets* (*M j*)
by (*force elim*!: *prod_algebraE*)
then show *emeasure P X* = *emeasure Q X*
by (*auto intro*!: *eq*)
qed (*auto simp*: *sets_PiM*)

locale *product_sigma_finite* =
fixes *M* :: *'i* $\Rightarrow$ *'a measure*
assumes *sigma_finite_measures*: $\bigwedge$*i*. *sigma_finite_measure* (*M i*)

sublocale *product_sigma_finite* $\subseteq$ *M?*: *sigma_finite_measure M i* **for** *i*
by (*rule sigma_finite_measures*)

locale *finite_product_sigma_finite* = *product_sigma_finite M* **for** *M* :: *'i* $\Rightarrow$ *'a measure* +
fixes *I* :: *'i set*
assumes *finite_index*: *finite I*

proposition (**in** *finite_product_sigma_finite*) *sigma_finite_pairs*:
$\exists$ *F*::*'i* $\Rightarrow$ *nat* $\Rightarrow$ *'a set*.
($\forall$ *i*$\in$*I*. *range* (*F i*) $\subseteq$ *sets* (*M i*)) $\wedge$
($\forall$ *k*. $\forall$ *i*$\in$*I*. *emeasure* (*M i*) (*F i k*) $\neq$ ∞) $\wedge$ *incseq* (λ*k*. Π_E *i*$\in$*I*. *F i k*) $\wedge$
($\bigcup$ *k*. Π_E *i*$\in$*I*. *F i k*) = *space* (*PiM I M*)

```
proof -
  have ∀ i::'i. ∃ F::nat ⇒ 'a set. range F ⊆ sets (M i) ∧ incseq F ∧ (⋃ i. F i) =
space (M i) ∧ (∀ k. emeasure (M i) (F k) ≠ ∞)
    using M.sigma_finite_incseq by metis
  from choice[OF this] guess F :: 'i ⇒ nat ⇒ 'a set ..
  then have F: ⋀i. range (F i) ⊆ sets (M i) ⋀i. incseq (F i) ⋀i. (⋃ j. F i j) =
space (M i) ⋀i k. emeasure (M i) (F i k) ≠ ∞
    by auto
  let ?F = λk. Π_E i∈I. F i k
  note space_PiM[simp]
  show ?thesis
  proof (intro exI[of _ F] conjI allI incseq_SucI set_eqI iffI ballI)
    fix i show range (F i) ⊆ sets (M i) by fact
  next
    fix i k show emeasure (M i) (F i k) ≠ ∞ by fact
  next
    fix x assume x ∈ (⋃ i. ?F i) with F(1) show x ∈ space (PiM I M)
      by (auto simp: PiE_def dest!: sets.sets_into_space)
  next
    fix f assume f ∈ space (PiM I M)
    with Pi_UN[OF finite_index, of λk i. F i k] F
    show f ∈ (⋃ i. ?F i) by (auto simp: incseq_def PiE_def)
  next
    fix i show ?F i ⊆ ?F (Suc i)
      using ‹⋀i. incseq (F i)›[THEN incseq_SucD] by auto
  qed
qed

lemma emeasure_PiM_empty[simp]: emeasure (PiM {} M) {λ_. undefined} = 1
proof -
  let ?μ = λA. if A = {} then 0 else (1::ennreal)
  have emeasure (Pi_M {} M) (prod_emb {} M {} (Π_E i∈{}. {})) = 1
  proof (subst emeasure_extend_measure_Pair[OF PiM_def])
    show positive (PiM {} M) ?μ
      by (auto simp: positive_def)
    show countably_additive (PiM {} M) ?μ
      by (rule sets.countably_additiveI_finite)
        (auto simp: additive_def positive_def sets_PiM_empty space_PiM_empty intro!:
)
  qed (auto simp: prod_emb_def)
  also have (prod_emb {} M {} (Π_E i∈{}. {})) = {λ_. undefined}
    by (auto simp: prod_emb_def)
  finally show ?thesis
    by simp
qed

lemma PiM_empty: PiM {} M = count_space {λ_. undefined}
  by (rule measure_eqI) (auto simp add: sets_PiM_empty)
```

lemma (**in** *product_sigma_finite*) *emeasure_PiM*:
finite I $\Longrightarrow$ ($\bigwedge$*i*. *i*$\in$*I* $\Longrightarrow$ *A i* $\in$ *sets* (*M i*)) $\Longrightarrow$ *emeasure* (*PiM I M*) (Pi_E *I A*) = ($\prod$*i*$\in$*I*. *emeasure* (*M i*) (*A i*))
proof (*induct I arbitrary*: *A rule*: *finite_induct*)
case (*insert i I*)
interpret *finite_product_sigma_finite M I* **by** *standard fact*
have *finite* (*insert i I*) **using** ‹*finite I*› **by** *auto*
interpret *I'*: *finite_product_sigma_finite M insert i I* **by** *standard fact*
let *?h* = (λ(*f*, *y*). *f*(*i* := *y*))

let *?P* = *distr* (Pi_M *I M* $\bigotimes_M$ *M i*) (Pi_M (*insert i I*) *M*) *?h*
let *?μ* = *emeasure ?P*
let *?I* = {*j* $\in$ *insert i I*. *emeasure* (*M j*) (*space* (*M j*)) $\neq$ *1*}
let *?f* = λ*J E j*. *if j* $\in$ *J then emeasure* (*M j*) (*E j*) *else emeasure* (*M j*) (*space* (*M j*))

have *emeasure* (Pi_M (*insert i I*) *M*) (*prod_emb* (*insert i I*) *M* (*insert i I*) (Pi_E (*insert i I*) *A*)) =
($\prod$*i*$\in$*insert i I*. *emeasure* (*M i*) (*A i*))
proof (*subst emeasure_extend_measure_Pair*[*OF PiM_def*])
fix *J E* **assume** (*J* $\neq$ {} $\vee$ *insert i I* = {}) $\wedge$ *finite J* $\wedge$ *J* $\subseteq$ *insert i I* $\wedge$ *E* $\in$ (Π *j*$\in$*J*. *sets* (*M j*))
then have *J*: *J* $\neq$ {} *finite J J* $\subseteq$ *insert i I* **and** *E*: $\forall$*j*$\in$*J*. *E j* $\in$ *sets* (*M j*) **by** *auto*
let *?p* = *prod_emb* (*insert i I*) *M J* (Pi_E *J E*)
let *?p'* = *prod_emb I M* (*J* − {*i*}) (Π_E *j*$\in$*J*−{*i*}. *E j*)
have *?μ ?p* =
emeasure (Pi_M *I M* $\bigotimes_M$ (*M i*)) (*?h* −‘ *?p* $\cap$ *space* (Pi_M *I M* $\bigotimes_M$ *M i*))
by (*intro emeasure_distr measurable_add_dim sets_PiM_I*) *fact*+
also have *?h* −‘ *?p* $\cap$ *space* (Pi_M *I M* $\bigotimes_M$ *M i*) = *?p'* $\times$ (*if i* $\in$ *J then E i else space* (*M i*))
using *J E*[*rule_format*, *THEN sets.sets_into_space*]
by (*force simp*: *space_pair_measure space_PiM prod_emb_iff PiE_def Pi_iff split*: *if_split_asm*)
also have *emeasure* (Pi_M *I M* $\bigotimes_M$ (*M i*)) (*?p'* $\times$ (*if i* $\in$ *J then E i else space* (*M i*))) =
emeasure (Pi_M *I M*) *?p'* $*$ *emeasure* (*M i*) (*if i* $\in$ *J then* (*E i*) *else space* (*M i*))
using *J E* **by** (*intro M.emeasure_pair_measure_Times sets_PiM_I*) *auto*
also have *?p'* = (Π_E *j*$\in$*I*. *if j* $\in$ *J*−{*i*} *then E j else space* (*M j*))
using *J E*[*rule_format*, *THEN sets.sets_into_space*]
by (*auto simp*: *prod_emb_iff PiE_def Pi_iff split*: *if_split_asm*) *blast*+
also have *emeasure* (Pi_M *I M*) (Π_E *j*$\in$*I*. *if j* $\in$ *J*−{*i*} *then E j else space* (*M j*)) =
($\prod$ *j*$\in$*I*. *if j* $\in$ *J*−{*i*} *then emeasure* (*M j*) (*E j*) *else emeasure* (*M j*) (*space* (*M j*)))
using *E* **by** (*subst insert*) (*auto intro*!: *prod.cong*)
also have ($\prod$*j*$\in$*I*. *if j* $\in$ *J* − {*i*} *then emeasure* (*M j*) (*E j*) *else emeasure* (*M j*) (*space* (*M j*))) $*$

emeasure (*M i*) (*if i* ∈ *J then E i else space* (*M i*)) = (∏*j*∈*insert i I*. *?f J E j*)
using *insert* **by** (*auto simp*: *mult.commute intro*!: *arg_cong2*[**where** *f*=(∗)] *prod.cong*)
also have ... = (∏*j*∈*J* ∪ *?I*. *?f J E j*)
using *insert*(*1*,*2*) *J E* **by** (*intro prod.mono_neutral_right*) *auto*
finally show *?μ ?p* = ...

show *prod_emb* (*insert i I*) *M J* (Pi_E *J E*) ∈ *Pow* (Π_E *i*∈*insert i I*. *space* (*M i*))
using *J E*[*rule_format*, *THEN sets.sets_into_space*] **by** (*auto simp*: *prod_emb_iff PiE_def*)
next
show *positive* (*sets* (Pi_M (*insert i I*) *M*)) *?μ countably_additive* (*sets* (Pi_M (*insert i I*) *M*)) *?μ*
using *emeasure_positive*[*of ?P*] *emeasure_countably_additive*[*of ?P*] **by** *simp_all*
next
show (*insert i I* ≠ {} ∨ *insert i I* = {}) ∧ *finite* (*insert i I*) ∧
insert i I ⊆ *insert i I* ∧ *A* ∈ (Π *j*∈*insert i I*. *sets* (*M j*))
using *insert* **by** *auto*
qed (*auto intro*!: *prod.cong*)
with *insert* **show** *?case*
by (*subst* (*asm*) *prod_emb_PiE_same_index*) (*auto intro*!: *sets.sets_into_space*)
qed *simp*

lemma (**in** *product_sigma_finite*) *PiM_eqI*:
assumes *I*[*simp*]: *finite I* **and** *P*: *sets P* = *PiM I M*
assumes *eq*: ⋀*A*. (⋀*i*. *i* ∈ *I* ⟹ *A i* ∈ *sets* (*M i*)) ⟹ *P* (Pi_E *I A*) = (∏*i*∈*I*. *emeasure* (*M i*) (*A i*))
shows *P* = *PiM I M*
proof −
interpret *finite_product_sigma_finite M I*
proof qed *fact*
from *sigma_finite_pairs* **guess** *C* .. **note** *C* = *this*
show *?thesis*
proof (*rule measure_eqI_PiM_finite*[*OF I refl P*, *symmetric*])
show (⋀*i*. *i* ∈ *I* ⟹ *A i* ∈ *sets* (*M i*)) ⟹ (Pi_M *I M*) (Pi_E *I A*) = *P* (Pi_E *I A*) **for** *A*
by (*simp add*: *eq emeasure_PiM*)
define *A* **where** *A n* = (Π_E *i*∈*I*. *C i n*) **for** *n*
with *C* **show** *range A* ⊆ *prod_algebra I M* ⋀*i*. *emeasure* (Pi_M *I M*) (*A i*) ≠ ∞ (⋃*i*. *A i*) = *space* (*PiM I M*)
by (*auto intro*!: *prod_algebraI_finite simp*: *emeasure_PiM subset_eq ennreal_prod_eq_top*)
qed
qed

lemma (**in** *product_sigma_finite*) *sigma_finite*:
assumes *finite I*
shows *sigma_finite_measure* (*PiM I M*)

proof
interpret *finite_product_sigma_finite M I* **by** *standard fact*

obtain *F* **where** *F*: $\bigwedge j.$ *countable* (*F j*) $\bigwedge j\, f.\ f \in F\ j \Longrightarrow f \in$ *sets* (*M j*)
$\bigwedge j\, f.\ f \in F\ j \Longrightarrow$ *emeasure* (*M j*) $f \neq \infty$ **and**
in_space: $\bigwedge j.$ *space* (*M j*) $= \bigcup(F\ j)$
using *sigma_finite_countable* **by** (*metis subset_eq*)
moreover have $(\bigcup(Pi_E\ I\ `\ Pi_E\ I\ F)) =$ *space* $(Pi_M\ I\ M)$
using *in_space* **by** (*auto simp*: *space_PiM PiE_iff intro*!: *PiE_choice*[*THEN iffD2*])
ultimately show $\exists A.$ *countable* $A \wedge A \subseteq$ *sets* $(Pi_M\ I\ M) \wedge \bigcup A =$ *space* $(Pi_M\ I\ M) \wedge (\forall a{\in}A.$ *emeasure* $(Pi_M\ I\ M)\ a \neq \infty)$
by (*intro exI*[*of* _ $Pi_E\ I\ `\ Pi_E\ I\ F$])
(*auto intro*!: *countable_PiE sets_PiM_I_finite*
simp: *PiE_iff emeasure_PiM finite_index ennreal_prod_eq_top*)
qed

sublocale *finite_product_sigma_finite* $\subseteq$ *sigma_finite_measure* $Pi_M\ I\ M$
using *sigma_finite*[*OF finite_index*] **.**

lemma (**in** *finite_product_sigma_finite*) *measure_times*:
$(\bigwedge i.\ i \in I \Longrightarrow A\ i \in$ *sets* $(M\ i)) \Longrightarrow$ *emeasure* $(Pi_M\ I\ M)\ (Pi_E\ I\ A) = (\prod i{\in}I.$ *emeasure* $(M\ i)\ (A\ i))$
using *emeasure_PiM*[*OF finite_index*] **by** *auto*

lemma (**in** *product_sigma_finite*) *nn_integral_empty*:
$0 \le f\ (\lambda k.$ *undefined*$) \Longrightarrow$ *integral*^{N} $(Pi_M\ \{\}\ M)\ f = f\ (\lambda k.$ *undefined*$)$
by (*simp add*: *PiM_empty nn_integral_count_space_finite max.absorb2*)

lemma (**in** *product_sigma_finite*) *distr_merge*:
assumes *IJ*[*simp*]: $I \cap J = \{\}$ **and** *fin*: *finite I finite J*
shows *distr* $(Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)\ (Pi_M\ (I \cup J)\ M)$ (*merge I J*) $= Pi_M\ (I \cup J)\ M$
(**is** *?D* = *?P*)
proof (*rule PiM_eqI*)
interpret *I*: *finite_product_sigma_finite M I* **by** *standard fact*
interpret *J*: *finite_product_sigma_finite M J* **by** *standard fact*
fix *A* **assume** *A*: $\bigwedge i.\ i \in I \cup J \Longrightarrow A\ i \in$ *sets* $(M\ i)$
have $*$: (*merge I J* $-`\ Pi_E\ (I \cup J)\ A \cap$ *space* $(Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)) = Pi_E\ I\ A \times Pi_E\ J\ A$
using *A*[*THEN sets.sets_into_space*] **by** (*auto simp*: *space_PiM space_pair_measure*)
from *A fin* **show** *emeasure* (*distr* $(Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)\ (Pi_M\ (I \cup J)\ M)$ (*merge I J*)) $(Pi_E\ (I \cup J)\ A) =$
$(\prod i{\in}I \cup J.$ *emeasure* $(M\ i)\ (A\ i))$
by (*subst emeasure_distr*)
(*auto simp*: $*$ *J.emeasure_pair_measure_Times I.measure_times J.measure_times prod.union_disjoint*)
qed (*insert fin*, *simp_all*)

proposition (**in** *product_sigma_finite*) *product_nn_integral_fold*:
 assumes *IJ*: $I \cap J = \{\}$ *finite I finite J*
 and *f*[*measurable*]: $f \in$ *borel_measurable* (Pi_M ($I \cup J$) M)
 shows $integral^N$ (Pi_M ($I \cup J$) M) f =
 ($\int^+ x.$ ($\int^+ y.$ *f* (*merge I J* (*x*, *y*)) ∂(Pi_M *J M*)) ∂(Pi_M *I M*))
proof −
 interpret *I*: *finite_product_sigma_finite M I* **by** *standard fact*
 interpret *J*: *finite_product_sigma_finite M J* **by** *standard fact*
 interpret *P*: *pair_sigma_finite* Pi_M *I M* Pi_M *J M* **by** *standard*
 have *P_borel*: ($\lambda x.$ *f* (*merge I J x*)) $\in$ *borel_measurable* (Pi_M *I M* $\bigotimes_M$ Pi_M *J M*)
 using *measurable_comp*[*OF measurable_merge f*] **by** (*simp add*: *comp_def*)
 show *?thesis*
 apply (*subst distr_merge*[*OF IJ*, *symmetric*])
 apply (*subst nn_integral_distr*[*OF measurable_merge*])
 apply *measurable* []
 apply (*subst J.nn_integral_fst*[*symmetric*, *OF P_borel*])
 apply *simp*
 done
qed

lemma (**in** *product_sigma_finite*) *distr_singleton*:
 distr (Pi_M {*i*} *M*) (*M i*) ($\lambda x.$ *x i*) = *M i* (**is** *?D* = _)
proof (*intro measure_eqI*[*symmetric*])
 interpret *I*: *finite_product_sigma_finite M* {*i*} **by** *standard simp*
 fix *A* **assume** *A*: $A \in$ *sets* (*M i*)
 then have ($\lambda x.$ *x i*) −‘ *A* $\cap$ *space* (Pi_M {*i*} *M*) = (Π_E *i*∈{*i*}. *A*)
 using *sets.sets_into_space* **by** (*auto simp*: *space_PiM*)
 then show *emeasure* (*M i*) *A* = *emeasure ?D A*
 using *A I.measure_times*[*of* λ_. *A*]
 by (*simp add*: *emeasure_distr measurable_component_singleton*)
qed *simp*

lemma (**in** *product_sigma_finite*) *product_nn_integral_singleton*:
 assumes *f*: $f \in$ *borel_measurable* (*M i*)
 shows $integral^N$ (Pi_M {*i*} *M*) ($\lambda x.$ *f* (*x i*)) = $integral^N$ (*M i*) *f*
proof −
 interpret *I*: *finite_product_sigma_finite M* {*i*} **by** *standard simp*
 from *f* **show** *?thesis*
 apply (*subst distr_singleton*[*symmetric*])
 apply (*subst nn_integral_distr*[*OF measurable_component_singleton*])
 apply *simp_all*
 done
qed

proposition (**in** *product_sigma_finite*) *product_nn_integral_insert*:
 assumes *I*[*simp*]: *finite I* $i \notin I$
 and *f*: $f \in$ *borel_measurable* (Pi_M (*insert i I*) *M*)
 shows $integral^N$ (Pi_M (*insert i I*) *M*) *f* = ($\int^+ x.$ ($\int^+ y.$ *f* (*x*(*i* := *y*)) ∂(*M i*))

$\partial(Pi_M\ I\ M))$
proof −
  interpret *I*: *finite_product_sigma_finite M I* **by** *standard auto*
  interpret *i*: *finite_product_sigma_finite M {i}* **by** *standard auto*
  have *IJ*: $I \cap \{i\} = \{\}$ **and** *insert*: $I \cup \{i\} = insert\ i\ I$
    using *f* **by** *auto*
  show *?thesis*
   unfolding *product_nn_integral_fold*[*OF IJ*, *unfolded insert*, *OF I*(*1*) *i.finite_index f*]
  proof (*rule nn_integral_cong*, *subst product_nn_integral_singleton*[*symmetric*])
    fix *x* **assume** *x*: $x \in space\ (Pi_M\ I\ M)$
    let *?f* = $\lambda y.\ f\ (x(i := y))$
    show *?f* $\in$ *borel_measurable* (*M i*)
      using *measurable_comp*[*OF measurable_component_update f*, *OF x* ‹$i \notin I$›]
      unfolding *comp_def* **.**
    show $(\int^+ y.\ f\ (merge\ I\ \{i\}\ (x,\ y))\ \partial Pi_M\ \{i\}\ M) = (\int^+ y.\ f\ (x(i := y\ i))\ \partial Pi_M\ \{i\}\ M)$
      using *x*
      by (*auto intro*!: *nn_integral_cong arg_cong*[**where** *f*=*f*]
            simp add: *space_PiM extensional_def PiE_def*)
  qed
qed

lemma (**in** *product_sigma_finite*) *product_nn_integral_insert_rev*:
  assumes *I*[*simp*]: *finite I* $i \notin I$
    and [*measurable*]: $f \in borel\_measurable\ (Pi_M\ (insert\ i\ I)\ M)$
  shows $integral^N\ (Pi_M\ (insert\ i\ I)\ M)\ f = (\int^+ y.\ (\int^+ x.\ f\ (x(i := y))\ \partial(Pi_M\ I\ M))\ \partial(M\ i))$
  apply (*subst product_nn_integral_insert*[*OF assms*])
  apply (*rule pair_sigma_finite.Fubini'*)
  apply *intro_locales* []
  apply (*rule sigma_finite*[*OF I*(*1*)])
  apply *measurable*
  done

lemma (**in** *product_sigma_finite*) *product_nn_integral_prod*:
  assumes *finite I* $\bigwedge i.\ i \in I \Longrightarrow f\ i \in borel\_measurable\ (M\ i)$
  shows $(\int^+ x.\ (\prod i{\in}I.\ f\ i\ (x\ i))\ \partial Pi_M\ I\ M) = (\prod i{\in}I.\ integral^N\ (M\ i)\ (f\ i))$
using *assms* **proof** (*induction I*)
  case (*insert i I*)
  note *insert.prems*[*measurable*]
  note ‹*finite I*›[*intro*, *simp*]
  interpret *I*: *finite_product_sigma_finite M I* **by** *standard auto*
  have $*$: $\bigwedge x\ y.\ (\prod j{\in}I.\ f\ j\ (if\ j = i\ then\ y\ else\ x\ j)) = (\prod j{\in}I.\ f\ j\ (x\ j))$
    using *insert* **by** (*auto intro*!: *prod.cong*)
  have *prod*: $\bigwedge J.\ J \subseteq insert\ i\ I \Longrightarrow (\lambda x.\ (\prod i{\in}J.\ f\ i\ (x\ i))) \in borel\_measurable\ (Pi_M\ J\ M)$
    using *sets.sets_into_space insert*
    by (*intro borel_measurable_prod_ennreal*

measurable_comp[OF measurable_component_singleton, unfolded comp_def])
auto
then show ?case
apply (simp add: product_nn_integral_insert[OF insert(1,2)])
apply (simp add: insert(2−) ∗ nn_integral_multc)
apply (subst nn_integral_cmult)
apply (auto simp add: insert(2−))
done
qed (simp add: space_PiM)

proposition (**in** product_sigma_finite) product_nn_integral_pair:
assumes [measurable]: case_prod f ∈ borel_measurable (M x $\bigotimes_M$ M y)
assumes xy: x ≠ y
shows ($\int^+$σ. f (σ x) (σ y) ∂PiM {x, y} M) = ($\int^+$z. f (fst z) (snd z) ∂(M x $\bigotimes_M$ M y))
proof −
interpret psm: pair_sigma_finite M x M y
unfolding pair_sigma_finite_def **using** sigma_finite_measures **by** simp_all
have {x, y} = {y, x} **by** auto
also have ($\int^+$σ. f (σ x) (σ y) ∂PiM {y, x} M) = ($\int^+$y. $\int^+$σ. f (σ x) y ∂PiM {x} M ∂M y)
using xy **by** (subst product_nn_integral_insert_rev) simp_all
also have ... = ($\int^+$y. $\int^+$x. f x y ∂M x ∂M y)
by (intro nn_integral_cong, subst product_nn_integral_singleton) simp_all
also have ... = ($\int^+$z. f (fst z) (snd z) ∂(M x $\bigotimes_M$ M y))
by (subst psm.nn_integral_snd[symmetric]) simp_all
finally show ?thesis **.**
qed

lemma (**in** product_sigma_finite) distr_component:
distr (M i) (Pi_M {i} M) (λx. λi∈{i}. x) = Pi_M {i} M (**is** ?D = ?P)
proof (intro PiM_eqI)
fix A **assume** A: ⋀ia. ia ∈ {i} ⟹ A ia ∈ sets (M ia)
then have (λx. λi∈{i}. x) −‘ Pi_E {i} A ∩ space (M i) = A i
by (fastforce dest: sets.sets_into_space)
with A **show** emeasure (distr (M i) (Pi_M {i} M) (λx. λi∈{i}. x)) (Pi_E {i} A) = (∏i∈{i}. emeasure (M i) (A i))
by (subst emeasure_distr) (auto intro!: sets_PiM_I_finite measurable_restrict)
qed simp_all

lemma (**in** product_sigma_finite)
assumes IJ: I ∩ J = {} finite I finite J **and** A: A ∈ sets (Pi_M (I ∪ J) M)
shows emeasure_fold_integral:
emeasure (Pi_M (I ∪ J) M) A = ($\int^+$x. emeasure (Pi_M J M) ((λy. merge I J (x, y)) −‘ A ∩ space (Pi_M J M)) ∂Pi_M I M) (**is** ?I)
and emeasure_fold_measurable:
(λx. emeasure (Pi_M J M) ((λy. merge I J (x, y)) −‘ A ∩ space (Pi_M J M))) ∈ borel_measurable (Pi_M I M) (**is** ?B)
proof −

interpret *I*: *finite_product_sigma_finite M I* **by** *standard fact*
interpret *J*: *finite_product_sigma_finite M J* **by** *standard fact*
interpret *IJ*: *pair_sigma_finite* Pi_M *I M* Pi_M *J M* **..**
have *merge*: *merge I J* −‘ *A* ∩ *space* (Pi_M *I M* $\bigotimes_M$ Pi_M *J M*) ∈ *sets* (Pi_M *I M* $\bigotimes_M$ Pi_M *J M*)
by (*intro measurable_sets*[*OF _ A*] *measurable_merge assms*)

show *?I*
apply (*subst distr_merge*[*symmetric*, *OF IJ*])
apply (*subst emeasure_distr*[*OF measurable_merge A*])
apply (*subst J.emeasure_pair_measure_alt*[*OF merge*])
apply (*auto intro*!: *nn_integral_cong arg_cong2*[**where** *f*=*emeasure*] *simp*: *space_pair_measure*)
done

show *?B*
using *IJ.measurable_emeasure_Pair1*[*OF merge*]
by (*simp add*: *vimage_comp comp_def space_pair_measure cong*: *measurable_cong*)
qed

lemma *sets_Collect_single*:
i ∈ *I* ⟹ *A* ∈ *sets* (*M i*) ⟹ { *x* ∈ *space* (Pi_M *I M*). *x i* ∈ *A* } ∈ *sets* (Pi_M *I M*)
by *simp*

lemma *pair_measure_eq_distr_PiM*:
fixes *M1* :: *'a measure* **and** *M2* :: *'a measure*
assumes *sigma_finite_measure M1 sigma_finite_measure M2*
shows (*M1* $\bigotimes_M$ *M2*) = *distr* (Pi_M *UNIV* (*case_bool M1 M2*)) (*M1* $\bigotimes_M$ *M2*) (λ*x*. (*x True*, *x False*))
(**is** *?P* = *?D*)
proof (*rule pair_measure_eqI*[*OF assms*])
interpret *B*: *product_sigma_finite case_bool M1 M2*
unfolding *product_sigma_finite_def* **using** *assms* **by** (*auto split*: *bool.split*)
let *?B* = Pi_M *UNIV* (*case_bool M1 M2*)

have [*simp*]: *fst* ∘ (λ*x*. (*x True*, *x False*)) = (λ*x*. *x True*) *snd* ∘ (λ*x*. (*x True*, *x False*)) = (λ*x*. *x False*)
by *auto*
fix *A B* **assume** *A*: *A* ∈ *sets M1* **and** *B*: *B* ∈ *sets M2*
have *emeasure M1 A* ∗ *emeasure M2 B* = (∏ *i*∈*UNIV*. *emeasure* (*case_bool M1 M2 i*) (*case_bool A B i*))
by (*simp add*: *UNIV_bool ac_simps*)
also have ... = *emeasure ?B* (Pi_E *UNIV* (*case_bool A B*))
using *A B* **by** (*subst B.emeasure_PiM*) (*auto split*: *bool.split*)
also have Pi_E *UNIV* (*case_bool A B*) = (λ*x*. (*x True*, *x False*)) −‘ (*A* × *B*) ∩ *space ?B*
using *A*[*THEN sets.sets_into_space*] *B*[*THEN sets.sets_into_space*]
by (*auto simp*: *PiE_iff all_bool_eq space_PiM split*: *bool.split*)
finally show *emeasure M1 A* ∗ *emeasure M2 B* = *emeasure ?D* (*A* × *B*)

```
    using A B
      measurable_component_singleton[of True UNIV case_bool M1 M2]
      measurable_component_singleton[of False UNIV case_bool M1 M2]
    by (subst emeasure_distr) (auto simp: measurable_pair_iff)
qed simp

lemma infprod_in_sets[intro]:
  fixes E :: nat ⇒ 'a set assumes E: ⋀i. E i ∈ sets (M i)
  shows Pi UNIV E ∈ sets (Π_M i∈UNIV::nat set. M i)
proof –
  have Pi UNIV E = (⋂i. prod_emb UNIV M {..i} (Π_E j∈{..i}. E j))
    using E E[THEN sets.sets_into_space]
    by (auto simp: prod_emb_def Pi_iff extensional_def)
  with E show ?thesis by auto
qed
```

6.8.3 Measurability

There are two natural sigma-algebras on a product space: the borel sigma algebra, generated by open sets in the product, and the product sigma algebra, countably generated by products of measurable sets along finitely many coordinates. The second one is defined and studied in `Finite_Product_Measure.thy`.

These sigma-algebra share a lot of natural properties (measurability of coordinates, for instance), but there is a fundamental difference: open sets are generated by arbitrary unions, not only countable ones, so typically many open sets will not be measurable with respect to the product sigma algebra (while all sets in the product sigma algebra are borel). The two sigma algebras coincide only when everything is countable (i.e., the product is countable, and the borel sigma algebra in the factor is countably generated).

In this paragraph, we develop basic measurability properties for the borel sigma algebra, and compare it with the product sigma algebra as explained above.

```
lemma measurable_product_coordinates [measurable (raw)]:
  (λx. x i) ∈ measurable borel borel
by (rule borel_measurable_continuous_onI[OF continuous_on_product_coordinates])

lemma measurable_product_then_coordinatewise:
  fixes f::'a ⇒ 'b ⇒ ('c::topological_space)
  assumes [measurable]: f ∈ borel_measurable M
  shows (λx. f x i) ∈ borel_measurable M
proof –
  have (λx. f x i) = (λy. y i) o f
    unfolding comp_def by auto
  then show ?thesis by simp
qed
```

To compare the Borel sigma algebra with the product sigma algebra, we give a presentation of the product sigma algebra that is more similar to the one we used above for the product topology.

```
lemma sets_PiM_finite:
  sets (Pi_M I M) = sigma_sets (Π_E i∈I. space (M i))
       {(Π_E i∈I. X i) |X. (∀ i. X i ∈ sets (M i)) ∧ finite {i. X i ≠ space (M i)}}
proof
  have {(Π_E i∈I. X i) |X. (∀ i. X i ∈ sets (M i)) ∧ finite {i. X i ≠ space (M
i)}} ⊆ sets (Pi_M I M)
  proof (auto)
    fix X assume H: ∀ i. X i ∈ sets (M i) finite {i. X i ≠ space (M i)}
    then have ∗: X i ∈ sets (M i) for i by simp
    define J where J = {i ∈ I. X i ≠ space (M i)}
    have finite J J ⊆ I unfolding J_def using H by auto
    define Y where Y = (Π_E j∈J. X j)
    have prod_emb I M J Y ∈ sets (Pi_M I M)
      unfolding Y_def apply (rule sets_PiM_I) using ‹finite J› ‹J ⊆ I› ∗ by auto
    moreover have prod_emb I M J Y = (Π_E i∈I. X i)
      unfolding prod_emb_def Y_def J_def using H sets.sets_into_space[OF ∗]
      by (auto simp add: PiE_iff, blast)
    ultimately show Pi_E I X ∈ sets (Pi_M I M) by simp
  qed
  then show sigma_sets (Π_E i∈I. space (M i)) {(Π_E i∈I. X i) |X. (∀ i. X i ∈
sets (M i)) ∧ finite {i. X i ≠ space (M i)}}
            ⊆ sets (Pi_M I M)
    by (metis (mono_tags, lifting) sets.sigma_sets_subset' sets.top space_PiM)

  have ∗: ∃ X. {f. (∀ i∈I. f i ∈ space (M i)) ∧ f ∈ extensional I ∧ f i ∈ A} =
Pi_E I X ∧
              (∀ i. X i ∈ sets (M i)) ∧ finite {i. X i ≠ space (M i)}
    if i ∈ I A ∈ sets (M i) for i A
  proof −
    define X where X = (λj. if j = i then A else space (M j))
    have {f. (∀ i∈I. f i ∈ space (M i)) ∧ f ∈ extensional I ∧ f i ∈ A} = Pi_E I X
      unfolding X_def using sets.sets_into_space[OF ‹A ∈ sets (M i)›] ‹i ∈ I›
      by (auto simp add: PiE_iff extensional_def, metis subsetCE, metis)
    moreover have X j ∈ sets (M j) for j
      unfolding X_def using ‹A ∈ sets (M i)› by auto
    moreover have finite {j. X j ≠ space (M j)}
      unfolding X_def by simp
    ultimately show ?thesis by auto
  qed
  show sets (Pi_M I M) ⊆ sigma_sets (Π_E i∈I. space (M i)) {(Π_E i∈I. X i) |X.
(∀ i. X i ∈ sets (M i)) ∧ finite {i. X i ≠ space (M i)}}
    unfolding sets_PiM_single
    apply (rule sigma_sets_mono')
    apply (auto simp add: PiE_iff ∗)
    done
qed
```

lemma *sets_PiM_subset_borel*:
sets (Pi_M *UNIV* (λ_. *borel*)) ⊆ *sets borel*
proof –
have ∗: Pi_E *UNIV X* ∈ *sets borel* **if** [*measurable*]: ⋀*i*. *X i* ∈ *sets borel finite* {*i*. *X i* ≠ *UNIV*} **for** *X*::′*a* ⇒ ′*b set*
proof –
define *I* **where** *I* = {*i*. *X i* ≠ *UNIV*}
have *finite I* **unfolding** *I_def* **using** *that* **by** *simp*
have Pi_E *UNIV X* = (⋂*i*∈*I*. (λ*x*. *x i*)–‘(*X i*) ∩ *space borel*) ∩ *space borel*
unfolding *I_def* **by** *auto*
also have ... ∈ *sets borel*
using *that* ‹*finite I*› **by** *measurable*
finally show *?thesis* **by** *simp*
qed
then have {(Π$_E$ *i*∈*UNIV*. *X i*) |*X*::(′*a* ⇒ ′*b set*). (∀ *i*. *X i* ∈ *sets borel*) ∧ *finite* {*i*. *X i* ≠ *space borel*}} ⊆ *sets borel*
by *auto*
then show *?thesis* **unfolding** *sets_PiM_finite space_borel*
by (*simp add*: ∗ *sets.sigma_sets_subset*′)
qed

proposition *sets_PiM_equal_borel*:
sets (Pi_M *UNIV* (λ*i*::(′*a*::*countable*). *borel*::(′*b*::*second_countable_topology measure*))) = *sets borel*
proof
obtain *K*::(′*a* ⇒ ′*b*) *set set* **where** *K*: *topological_basis K countable K*
⋀*k*. *k* ∈ *K* ⟹ ∃ *X*. (*k* = Pi_E *UNIV X*) ∧ (∀ *i*. *open* (*X i*)) ∧ *finite* {*i*. *X i* ≠ *UNIV*}
using *product_topology_countable_basis* **by** *fast*
have ∗: *k* ∈ *sets* (Pi_M *UNIV* (λ_. *borel*)) **if** *k* ∈ *K* **for** *k*
proof –
obtain *X* **where** *H*: *k* = Pi_E *UNIV X* ⋀*i*. *open* (*X i*) *finite* {*i*. *X i* ≠ *UNIV*}
using *K*(*3*)[*OF* ‹*k* ∈ *K*›] **by** *blast*
show *?thesis* **unfolding** *H*(*1*) *sets_PiM_finite space_borel* **using** *borel_open*[*OF* *H*(*2*)] *H*(*3*) **by** *auto*
qed
have ∗∗: *U* ∈ *sets* (Pi_M *UNIV* (λ_. *borel*)) **if** *open U* **for** *U*::(′*a* ⇒ ′*b*) *set*
proof –
obtain *B* **where** *B* ⊆ *K U* = (⋃*B*)
using ‹*open U*› ‹*topological_basis K*› **by** (*metis topological_basis_def*)
have *countable B* **using** ‹*B* ⊆ *K*› ‹*countable K*› *countable_subset* **by** *blast*
moreover have *k* ∈ *sets* (Pi_M *UNIV* (λ_. *borel*)) **if** *k* ∈ *B* **for** *k*
using ‹*B* ⊆ *K*› ∗ *that* **by** *auto*
ultimately show *?thesis* **unfolding** ‹*U* = (⋃*B*)› **by** *auto*
qed
have *sigma_sets UNIV* (*Collect open*) ⊆ *sets* (Pi_M *UNIV* (λ*i*::′*a*. (*borel*::(′*b measure*))))
apply (*rule sets.sigma_sets_subset*′) **using** ∗∗ **by** *auto*
then show *sets* (*borel*::(′*a* ⇒ ′*b*) *measure*) ⊆ *sets* (Pi_M *UNIV* (λ_. *borel*))

unfolding *borel_def* **by** *auto*
qed (*simp add*: *sets_PiM_subset_borel*)

lemma *measurable_coordinatewise_then_product*:
fixes f::$'a \Rightarrow ('b$::*countable*$) \Rightarrow ('c$::*second_countable_topology*)
assumes [*measurable*]: $\bigwedge i.\ (\lambda x.\ f\ x\ i) \in$ *borel_measurable M*
shows $f \in$ *borel_measurable M*
proof −
have $f \in$ *measurable M* (Pi_M *UNIV* (λ_. *borel*))
by (*rule measurable_PiM_single'*, *auto simp add*: *assms*)
then show *?thesis* **using** *sets_PiM_equal_borel measurable_cong_sets* **by** *blast*
qed

end

6.9 Caratheodory Extension Theorem

theory *Caratheodory*
imports *Measure_Space*
begin

Originally from the Hurd/Coble measure theory development, translated by Lawrence Paulson.

lemma *suminf_ennreal_2dimen*:
fixes f:: *nat* $\times$ *nat* $\Rightarrow$ *ennreal*
assumes $\bigwedge m.\ g\ m = (\sum n.\ f\ (m,n))$
shows $(\sum i.\ f\ (prod\_decode\ i)) = suminf\ g$
proof −
have *g_def*: $g = (\lambda m.\ (\sum n.\ f\ (m,n)))$
using *assms* **by** (*simp add*: *fun_eq_iff*)
have *reindex*: $\bigwedge B.\ (\sum x{\in}B.\ f\ (prod\_decode\ x)) = sum\ f\ (prod\_decode\ `\ B)$
by (*simp add*: *sum.reindex*[*OF inj_prod_decode*] *comp_def*)
have $(SUP\ n.\ \sum i{<}n.\ f\ (prod\_decode\ i)) = (SUP\ p \in UNIV \times UNIV.\ \sum i{<}fst\ p.\ \sum n{<}snd\ p.\ f\ (i,\ n))$
proof (*intro SUP_eq*; *clarsimp simp*: *sum.cartesian_product reindex*)
fix n
let $?M = \lambda f.\ Suc\ (Max\ (f\ `\ prod\_decode\ `\ \{..{<}n\}))$
{ **fix** $a\ b\ x$ **assume** $x < n$ **and** [*symmetric*]: $(a,\ b) = prod\_decode\ x$
then have $a < ?M\ fst\ b < ?M\ snd$
by (*auto intro*!: *Max_ge le_imp_less_Suc image_eqI*) }
then have $sum\ f\ (prod\_decode\ `\ \{..{<}n\}) \le sum\ f\ (\{..{<}?M\ fst\} \times \{..{<}?M\ snd\})$
by (*auto intro*!: *sum_mono2*)
then show $\exists\, a\ b.\ sum\ f\ (prod\_decode\ `\ \{..{<}n\}) \le sum\ f\ (\{..{<}a\} \times \{..{<}b\})$ **by** *auto*
next
fix $a\ b$
let $?M = prod\_decode\ `\ \{..{<}Suc\ (Max\ (prod\_encode\ `\ (\{..{<}a\} \times \{..{<}b\})))\}$
{ **fix** $a'\ b'$ **assume** $a' < a\ b' < b$ **then have** $(a',\ b') \in ?M$

by (*auto intro*!: *Max_ge le_imp_less_Suc image_eqI*[**where** *x*=*prod_encode* (*a'*, *b'*)]) **}**
then have *sum f* ({..<*a*} × {..<*b*}) ≤ *sum f ?M*
by (*auto intro*!: *sum_mono2*)
then show ∃ *n*. *sum f* ({..<*a*} × {..<*b*}) ≤ *sum f* (*prod_decode* ' {..<*n*})
by *auto*
qed
also have ... = (*SUP p*. ∑ *i*<*p*. ∑ *n*. *f* (*i*, *n*))
unfolding *suminf_sum*[*OF summableI*, *symmetric*]
by (*simp add*: *suminf_eq_SUP SUP_pair sum.swap*[*of* _ {..< *fst* _}])
finally show *?thesis* **unfolding** *g_def*
by (*simp add*: *suminf_eq_SUP*)
qed

6.9.1 Characterizations of Measures

definition *outer_measure_space* **where**
outer_measure_space M f ⟷ *positive M f* ∧ *increasing M f* ∧ *countably_subadditive M f*

Lambda Systems

definition *lambda_system* :: *'a set* ⇒ *'a set set* ⇒ (*'a set* ⇒ *ennreal*) ⇒ *'a set set* **where**
lambda_system Ω *M f* = {*l* ∈ *M*. ∀ *x* ∈ *M*. *f* (*l* ∩ *x*) + *f* ((Ω − *l*) ∩ *x*) = *f x*}

lemma (**in** *algebra*) *lambda_system_eq*:
lambda_system Ω *M f* = {*l* ∈ *M*. ∀ *x* ∈ *M*. *f* (*x* ∩ *l*) + *f* (*x* − *l*) = *f x*}
proof −
have [*simp*]: ⋀*l x*. *l* ∈ *M* ⟹ *x* ∈ *M* ⟹ (Ω − *l*) ∩ *x* = *x* − *l*
by (*metis Int_Diff Int_absorb1 Int_commute sets_into_space*)
show *?thesis*
by (*auto simp add*: *lambda_system_def*) (*metis Int_commute*)+
qed

lemma (**in** *algebra*) *lambda_system_empty*: *positive M f* ⟹ {} ∈ *lambda_system* Ω *M f*
by (*auto simp add*: *positive_def lambda_system_eq*)

lemma *lambda_system_sets*: *x* ∈ *lambda_system* Ω *M f* ⟹ *x* ∈ *M*
by (*simp add*: *lambda_system_def*)

lemma (**in** *algebra*) *lambda_system_Compl*:
fixes *f*:: *'a set* ⇒ *ennreal*
assumes *x*: *x* ∈ *lambda_system* Ω *M f*
shows Ω − *x* ∈ *lambda_system* Ω *M f*
proof −
have *x* ⊆ Ω
by (*metis sets_into_space lambda_system_sets x*)
hence Ω − (Ω − *x*) = *x*

```
    by (metis double_diff equalityE)
  with x show ?thesis
    by (force simp add: lambda_system_def ac_simps)
qed

lemma (in algebra) lambda_system_Int:
  fixes f:: 'a set ⇒ ennreal
  assumes xl: x ∈ lambda_system Ω M f and yl: y ∈ lambda_system Ω M f
  shows x ∩ y ∈ lambda_system Ω M f
proof -
  from xl yl show ?thesis
  proof (auto simp add: positive_def lambda_system_eq Int)
    fix u
    assume x: x ∈ M and y: y ∈ M and u: u ∈ M
       and fx: ∀ z∈M. f (z ∩ x) + f (z - x) = f z
       and fy: ∀ z∈M. f (z ∩ y) + f (z - y) = f z
    have u - x ∩ y ∈ M
      by (metis Diff Diff_Int Un u x y)
    moreover
    have (u - (x ∩ y)) ∩ y = u ∩ y - x by blast
    moreover
    have u - x ∩ y - y = u - y by blast
    ultimately
    have ey: f (u - x ∩ y) = f (u ∩ y - x) + f (u - y) using fy
      by force
    have f (u ∩ (x ∩ y)) + f (u - x ∩ y)
          = (f (u ∩ (x ∩ y)) + f (u ∩ y - x)) + f (u - y)
      by (simp add: ey ac_simps)
    also have ... =  (f ((u ∩ y) ∩ x) + f (u ∩ y - x)) + f (u - y)
      by (simp add: Int_ac)
    also have ... = f (u ∩ y) + f (u - y)
      using fx [THEN bspec, of u ∩ y] Int y u
      by force
    also have ... = f u
      by (metis fy u)
    finally show f (u ∩ (x ∩ y)) + f (u - x ∩ y) = f u .
  qed
qed

lemma (in algebra) lambda_system_Un:
  fixes f:: 'a set ⇒ ennreal
  assumes xl: x ∈ lambda_system Ω M f and yl: y ∈ lambda_system Ω M f
  shows x ∪ y ∈ lambda_system Ω M f
proof -
  have (Ω - x) ∩ (Ω - y) ∈ M
    by (metis Diff_Un Un compl_sets lambda_system_sets xl yl)
  moreover
  have x ∪ y = Ω - ((Ω - x) ∩ (Ω - y))
    by auto (metis subsetD lambda_system_sets sets_into_space xl yl)+
```

```
  ultimately show ?thesis
    by (metis lambda_system_Compl lambda_system_Int xl yl)
qed

lemma (in algebra) lambda_system_algebra:
  positive M f ⟹ algebra Ω (lambda_system Ω M f)
  apply (auto simp add: algebra_iff_Un)
  apply (metis lambda_system_sets subsetD sets_into_space)
  apply (metis lambda_system_empty)
  apply (metis lambda_system_Compl)
  apply (metis lambda_system_Un)
  done

lemma (in algebra) lambda_system_strong_additive:
  assumes z: z ∈ M and disj: x ∩ y = {}
      and xl: x ∈ lambda_system Ω M f and yl: y ∈ lambda_system Ω M f
  shows f (z ∩ (x ∪ y)) = f (z ∩ x) + f (z ∩ y)
proof −
  have z ∩ x = (z ∩ (x ∪ y)) ∩ x using disj by blast
  moreover
  have z ∩ y = (z ∩ (x ∪ y)) − x using disj by blast
  moreover
  have (z ∩ (x ∪ y)) ∈ M
    by (metis Int Un lambda_system_sets xl yl z)
  ultimately show ?thesis using xl yl
    by (simp add: lambda_system_eq)
qed

lemma (in algebra) lambda_system_additive: additive (lambda_system Ω M f) f
proof (auto simp add: additive_def)
  fix x and y
  assume disj: x ∩ y = {}
     and xl: x ∈ lambda_system Ω M f and yl: y ∈ lambda_system Ω M f
  hence  x ∈ M y ∈ M by (blast intro: lambda_system_sets)+
  thus f (x ∪ y) = f x + f y
    using lambda_system_strong_additive [OF top disj xl yl]
    by (simp add: Un)
qed

lemma lambda_system_increasing: increasing M f ⟹ increasing (lambda_system
Ω M f) f
  by (simp add: increasing_def lambda_system_def)

lemma lambda_system_positive: positive M f ⟹ positive (lambda_system Ω M f)
f
  by (simp add: positive_def lambda_system_def)

lemma (in algebra) lambda_system_strong_sum:
  fixes A:: nat ⇒ 'a set and f :: 'a set ⇒ ennreal
```

```
  assumes f: positive M f and a: a ∈ M
      and A: range A ⊆ lambda_system Ω M f
      and disj: disjoint_family A
  shows (∑ i = 0..<n. f (a ∩A i)) = f (a ∩ (⋃ i∈{0..<n}. A i))
proof (induct n)
  case 0 show ?case using f by (simp add: positive_def)
next
  case (Suc n)
  have 2: A n ∩ ⋃ (A ‘ {0..<n}) = {} using disj
    by (force simp add: disjoint_family_on_def neq_iff)
  have 3: A n ∈ lambda_system Ω M f using A
    by blast
  interpret l: algebra Ω lambda_system Ω M f
    using f by (rule lambda_system_algebra)
  have 4: ⋃ (A ‘ {0..<n}) ∈ lambda_system Ω M f
    using A l.UNION_in_sets by simp
  from Suc.hyps show ?case
     by (simp add: atLeastLessThanSuc lambda_system_strong_additive [OF a 2 3
4])
qed

proposition (in sigma_algebra) lambda_system_caratheodory:
  assumes oms: outer_measure_space M f
      and A: range A ⊆ lambda_system Ω M f
      and disj: disjoint_family A
  shows (⋃ i. A i) ∈ lambda_system Ω M f ∧ (∑ i. f (A i)) = f (⋃ i. A i)
proof –
  have pos: positive M f and inc: increasing M f
   and csa: countably_subadditive M f
    by (metis oms outer_measure_space_def)+
  have sa: subadditive M f
    by (metis countably_subadditive_subadditive csa pos)
  have A': ⋀S. A‘S ⊆ (lambda_system Ω M f) using A
    by auto
  interpret ls: algebra Ω lambda_system Ω M f
    using pos by (rule lambda_system_algebra)
  have A'': range A ⊆ M
     by (metis A image_subset_iff lambda_system_sets)

  have U_in: (⋃ i. A i) ∈ M
    by (metis A'' countable_UN)
  have U_eq: f (⋃ i. A i) = (∑ i. f (A i))
  proof (rule antisym)
    show f (⋃ i. A i) ≤ (∑ i. f (A i))
      using csa[unfolded countably_subadditive_def] A'' disj U_in by auto
   have dis: ⋀N. disjoint_family_on A {..<N} by (intro disjoint_family_on_mono[OF
_ disj]) auto
    show (∑ i. f (A i)) ≤ f (⋃ i. A i)
     using ls.additive_sum [OF lambda_system_positive[OF pos] lambda_system_additive
```

_ A′ dis] A′′
    by (*intro suminf_le_const*[*OF summableI*]) (*auto intro*!: *increasingD*[*OF inc*] *countable_UN*)
  qed
  have $f\ (a \cap (\bigcup i.\ A\ i)) + f\ (a - (\bigcup i.\ A\ i)) = f\ a$
    if *a* [*iff*]: $a \in M$ **for** *a*
  proof (*rule antisym*)
    have *range* $(\lambda i.\ a \cap A\ i) \subseteq M$ **using** A''
      by *blast*
    moreover
    have *disjoint_family* $(\lambda i.\ a \cap A\ i)$ **using** *disj*
      by (*auto simp add*: *disjoint_family_on_def*)
    moreover
    have $a \cap (\bigcup i.\ A\ i) \in M$
      by (*metis Int U_in a*)
    ultimately
    have $f\ (a \cap (\bigcup i.\ A\ i)) \le (\sum i.\ f\ (a \cap A\ i))$
      using *csa*[*unfolded countably_subadditive_def*, *rule_format*, *of* $(\lambda i.\ a \cap A\ i)$]
      by (*simp add*: *o_def*)
    hence $f\ (a \cap (\bigcup i.\ A\ i)) + f\ (a - (\bigcup i.\ A\ i)) \le (\sum i.\ f\ (a \cap A\ i)) + f\ (a - (\bigcup i.\ A\ i))$
      by (*rule add_right_mono*)
    also have $\ldots \le f\ a$
    proof (*intro ennreal_suminf_bound_add*)
      fix *n*
      have *UNION_in*: $(\bigcup i\in\{0..<n\}.\ A\ i) \in M$
        by (*metis* A'' *UNION_in_sets*)
      have *le_fa*: $f\ (\bigcup\ (A\ `\ \{0..<n\}) \cap a) \le f\ a$ **using** A''
        by (*blast intro*: *increasingD* [*OF inc*] A'' *UNION_in_sets*)
      have *ls*: $(\bigcup i\in\{0..<n\}.\ A\ i) \in$ *lambda_system* Ω *M f*
        using *ls.UNION_in_sets* **by** (*simp add*: *A*)
      hence *eq_fa*: $f\ a = f\ (a \cap (\bigcup i\in\{0..<n\}.\ A\ i)) + f\ (a - (\bigcup i\in\{0..<n\}.\ A\ i))$
        by (*simp add*: *lambda_system_eq UNION_in*)
      have $f\ (a - (\bigcup i.\ A\ i)) \le f\ (a - (\bigcup i\in\{0..<n\}.\ A\ i))$
        by (*blast intro*: *increasingD* [*OF inc*] *UNION_in U_in*)
      thus $(\sum i<n.\ f\ (a \cap A\ i)) + f\ (a - (\bigcup i.\ A\ i)) \le f\ a$
          by (*simp add*: *lambda_system_strong_sum pos A disj eq_fa add_left_mono atLeast0LessThan*[*symmetric*])
    qed
    finally show $f\ (a \cap (\bigcup i.\ A\ i)) + f\ (a - (\bigcup i.\ A\ i)) \le f\ a$
      by *simp*
  next
    have $f\ a \le f\ (a \cap (\bigcup i.\ A\ i) \cup (a - (\bigcup i.\ A\ i)))$
      by (*blast intro*: *increasingD* [*OF inc*] *U_in*)
    also have ... $\le f\ (a \cap (\bigcup i.\ A\ i)) + f\ (a - (\bigcup i.\ A\ i))$
      by (*blast intro*: *subadditiveD* [*OF sa*] *U_in*)
    finally show $f\ a \le f\ (a \cap (\bigcup i.\ A\ i)) + f\ (a - (\bigcup i.\ A\ i))$ **.**
  qed

thus *?thesis*
by (*simp add*: *lambda_system_eq sums_iff U_eq U_in*)
qed

proposition (**in** *sigma_algebra*) *caratheodory_lemma*:
assumes *oms*: *outer_measure_space M f*
defines $L \equiv$ *lambda_system* Ω *M f*
shows *measure_space* Ω *L f*
proof −
have *pos*: *positive M f*
by (*metis oms outer_measure_space_def*)
have *alg*: *algebra* Ω *L*
using *lambda_system_algebra* [*of f*, *OF pos*]
by (*simp add*: *algebra_iff_Un L_def*)
then
have *sigma_algebra* Ω *L*
using *lambda_system_caratheodory* [*OF oms*]
by (*simp add*: *sigma_algebra_disjoint_iff L_def*)
moreover
have *countably_additive L f positive L f*
using *pos lambda_system_caratheodory* [*OF oms*]
by (*auto simp add*: *lambda_system_sets L_def countably_additive_def positive_def*)
ultimately
show *?thesis*
using *pos* **by** (*simp add*: *measure_space_def*)
qed

definition *outer_measure* :: *'a set set* ⇒ (*'a set* ⇒ *ennreal*) ⇒ *'a set* ⇒ *ennreal*
where
outer_measure M f X =
(*INF A*∈{*A*. *range A* ⊆ *M* ∧ *disjoint_family A* ∧ *X* ⊆ ($\bigcup$*i*. *A i*)}. $\sum$*i*. *f* (*A i*))

lemma (**in** *ring_of_sets*) *outer_measure_agrees*:
assumes *posf*: *positive M f* **and** *ca*: *countably_additive M f* **and** *s*: *s* ∈ *M*
shows *outer_measure M f s* = *f s*
unfolding *outer_measure_def*
proof (*safe intro*!: *antisym INF_greatest*)
fix *A* :: *nat* ⇒ *'a set* **assume** *A*: *range A* ⊆ *M* **and** *dA*: *disjoint_family A* **and** *sA*: *s* ⊆ ($\bigcup$*x*. *A x*)
have *inc*: *increasing M f*
by (*metis additive_increasing ca countably_additive_additive posf*)
have *f s* = *f* ($\bigcup$*i*. *A i* ∩ *s*)
using *sA* **by** (*auto simp*: *Int_absorb1*)
also have . . . = ($\sum$*i*. *f* (*A i* ∩ *s*))
using *sA dA A s*
by (*intro ca*[*unfolded countably_additive_def*, *rule_format*, *symmetric*])
(*auto simp*: *Int_absorb1 disjoint_family_on_def*)
also have ... ≤ ($\sum$*i*. *f* (*A i*))

using *A s* **by** (*auto intro*!: *suminf_le increasingD*[*OF inc*])
finally show *f s* ≤ ($\sum$*i*. *f* (*A i*)) .
next
have ($\sum$*i*. *f* (*if i* = *0 then s else* {})) ≤ *f s*
using *positiveD1*[*OF posf*] **by** (*subst suminf_finite*[*of* {*0*}]) *auto*
with *s* **show** (*INF A*∈{*A*. *range A* ⊆ *M* ∧ *disjoint_family A* ∧ *s* ⊆ ⋃(*A* ' *UNIV*)}. $\sum$*i*. *f* (*A i*)) ≤ *f s*
by (*intro INF_lower2*[*of* λ*i*. *if i* = *0 then s else* {}])
(*auto simp*: *disjoint_family_on_def*)
qed

lemma *outer_measure_empty*:
positive M f ⟹ {} ∈ *M* ⟹ *outer_measure M f* {} = *0*
unfolding *outer_measure_def*
by (*intro antisym INF_lower2*[*of* λ_. {}]) (*auto simp*: *disjoint_family_on_def positive_def*)

lemma (**in** *ring_of_sets*) *positive_outer_measure*:
assumes *positive M f* **shows** *positive* (*Pow* Ω) (*outer_measure M f*)
unfolding *positive_def* **by** (*auto simp*: *assms outer_measure_empty*)

lemma (**in** *ring_of_sets*) *increasing_outer_measure*: *increasing* (*Pow* Ω) (*outer_measure M f*)
by (*force simp*: *increasing_def outer_measure_def intro*!: *INF_greatest intro*: *INF_lower*)

lemma (**in** *ring_of_sets*) *outer_measure_le*:
assumes *pos*: *positive M f* **and** *inc*: *increasing M f* **and** *A*: *range A* ⊆ *M* **and** *X*: *X* ⊆ (⋃*i*. *A i*)
shows *outer_measure M f X* ≤ ($\sum$*i*. *f* (*A i*))
unfolding *outer_measure_def*
proof (*safe intro*!: *INF_lower2*[*of disjointed A*] *del*: *subsetI*)
show *dA*: *range* (*disjointed A*) ⊆ *M*
by (*auto intro*!: *A range_disjointed_sets*)
have ∀*n*. *f* (*disjointed A n*) ≤ *f* (*A n*)
by (*metis increasingD* [*OF inc*] *UNIV_I dA image_subset_iff disjointed_subset A*)
then show ($\sum$*i*. *f* (*disjointed A i*)) ≤ ($\sum$*i*. *f* (*A i*))
by (*blast intro*!: *suminf_le*)
qed (*auto simp*: *X UN_disjointed_eq disjoint_family_disjointed*)

lemma (**in** *ring_of_sets*) *outer_measure_close*:
outer_measure M f X < *e* ⟹ ∃*A*. *range A* ⊆ *M* ∧ *disjoint_family A* ∧ *X* ⊆ (⋃*i*. *A i*) ∧ ($\sum$*i*. *f* (*A i*)) < *e*
unfolding *outer_measure_def INF_less_iff* **by** *auto*

lemma (**in** *ring_of_sets*) *countably_subadditive_outer_measure*:
assumes *posf*: *positive M f* **and** *inc*: *increasing M f*
shows *countably_subadditive* (*Pow* Ω) (*outer_measure M f*)
proof (*simp add*: *countably_subadditive_def*, *safe*)

```
  fix A :: nat ⇒ _ assume A: range A ⊆ Pow (Ω) and sb: (⋃i. A i) ⊆ Ω
  let ?O = outer_measure M f
  show ?O (⋃i. A i) ≤ (∑n. ?O (A n))
  proof (rule ennreal_le_epsilon)
    fix b and e :: real assume 0 < e (∑n. outer_measure M f (A n)) < top
    then have *: ⋀n. outer_measure M f (A n) < outer_measure M f (A n) + e
* (1/2)^Suc n
      by (auto simp add: less_top dest!: ennreal_suminf_lessD)
    obtain B
      where B: ⋀n. range (B n) ⊆ M
      and sbB: ⋀n. A n ⊆ (⋃i. B n i)
      and Ble: ⋀n. (∑i. f (B n i)) ≤ ?O (A n) + e * (1/2)^(Suc n)
      by (metis less_imp_le outer_measure_close[OF *])

    define C where C = case_prod B o prod_decode
    from B have B_in_M: ⋀i j. B i j ∈ M
      by (rule range_subsetD)
    then have C: range C ⊆ M
      by (auto simp add: C_def split_def)
    have A_C: (⋃i. A i) ⊆ (⋃i. C i)
     using sbB by (auto simp add: C_def subset_eq) (metis prod.case prod_encode_inverse)

    have ?O (⋃i. A i) ≤ ?O (⋃i. C i)
     using A_C A C by (intro increasing_outer_measure[THEN increasingD]) (auto
dest!: sets_into_space)
    also have ... ≤ (∑i. f (C i))
      using C by (intro outer_measure_le[OF posf inc]) auto
    also have ... = (∑n. ∑i. f (B n i))
      using B_in_M unfolding C_def comp_def by (intro suminf_ennreal_2dimen)
auto
    also have ... ≤ (∑n. ?O (A n) + e * (1/2) ^ Suc n)
      using B_in_M by (intro suminf_le suminf_nonneg allI Ble) auto
     also have ... = (∑n. ?O (A n)) + (∑n. ennreal e * ennreal ((1/2) ^ Suc
n))
      using ‹0 < e› by (subst suminf_add[symmetric])
                        (auto simp del: ennreal_suminf_cmult simp add: en-
nreal_mult[symmetric])
    also have ... = (∑n. ?O (A n)) + e
      unfolding ennreal_suminf_cmult
      by (subst suminf_ennreal_eq[OF zero_le_power power_half_series]) auto
    finally show ?O (⋃i. A i) ≤ (∑n. ?O (A n)) + e .
  qed
qed

lemma (in ring_of_sets) outer_measure_space_outer_measure:
 positive M f ⟹ increasing M f ⟹ outer_measure_space (Pow Ω) (outer_measure
M f)
  by (simp add: outer_measure_space_def
   positive_outer_measure increasing_outer_measure countably_subadditive_outer_measure)
```

lemma (**in** *ring_of_sets*) *algebra_subset_lambda_system*:
assumes *posf*: *positive M f* **and** *inc*: *increasing M f*
and *add*: *additive M f*
shows $M \subseteq$ *lambda_system* Ω (*Pow* Ω) (*outer_measure M f*)
proof (*auto dest*: *sets_into_space*
simp add: *algebra.lambda_system_eq* [*OF algebra_Pow*])
fix *x s* **assume** *x*: $x \in M$ **and** *s*: $s \subseteq \Omega$
have [*simp*]: $\bigwedge x.\ x \in M \Longrightarrow s \cap (\Omega - x) = s - x$ **using** *s*
by *blast*
have *outer_measure M f* $(s \cap x)$ + *outer_measure M f* $(s - x) \leq$ *outer_measure M f s*
unfolding *outer_measure_def*[*of M f s*]
proof (*safe intro*!: *INF_greatest*)
fix *A* :: *nat* ⇒ *'a set* **assume** *A*: *disjoint_family A range A* ⊆ *M s* ⊆ ($\bigcup i.\ A\ i$)
have *outer_measure M f* $(s \cap x) \leq (\sum i.\ f\ (A\ i \cap x))$
unfolding *outer_measure_def*
proof (*safe intro*!: *INF_lower2*[*of* $\lambda i.\ A\ i \cap x$])
from *A*(*1*) **show** *disjoint_family* $(\lambda i.\ A\ i \cap x)$
by (*rule disjoint_family_on_bisimulation*) *auto*
qed (*insert x A*, *auto*)
moreover
have *outer_measure M f* $(s - x) \leq (\sum i.\ f\ (A\ i - x))$
unfolding *outer_measure_def*
proof (*safe intro*!: *INF_lower2*[*of* $\lambda i.\ A\ i - x$])
from *A*(*1*) **show** *disjoint_family* $(\lambda i.\ A\ i - x)$
by (*rule disjoint_family_on_bisimulation*) *auto*
qed (*insert x A*, *auto*)
ultimately have *outer_measure M f* $(s \cap x)$ + *outer_measure M f* $(s - x) \leq$
$(\sum i.\ f\ (A\ i \cap x)) + (\sum i.\ f\ (A\ i - x))$ **by** (*rule add_mono*)
also have ... $= (\sum i.\ f\ (A\ i \cap x) + f\ (A\ i - x))$
using *A*(*2*) *x posf* **by** (*subst suminf_add*) (*auto simp*: *positive_def*)
also have ... $= (\sum i.\ f\ (A\ i))$
using *A x*
by (*subst add*[*THEN additiveD*, *symmetric*])
(*auto intro*!: *arg_cong*[**where** *f*=*suminf*] *arg_cong*[**where** *f*=*f*])
finally show *outer_measure M f* $(s \cap x)$ + *outer_measure M f* $(s - x) \leq (\sum i.\ f\ (A\ i))$.
qed
moreover
have *outer_measure M f s* ≤ *outer_measure M f* $(s \cap x)$ + *outer_measure M f* $(s - x)$
proof −
have *outer_measure M f s* = *outer_measure M f* $((s \cap x) \cup (s - x))$
by (*metis Un_Diff_Int Un_commute*)
also have ... ≤ *outer_measure M f* $(s \cap x)$ + *outer_measure M f* $(s - x)$
apply (*rule subadditiveD*)
apply (*rule ring_of_sets.countably_subadditive_subadditive* [*OF ring_of_sets_Pow*])
apply (*simp add*: *positive_def outer_measure_empty*[*OF posf*])

apply (*rule countably_subadditive_outer_measure*)
using *s* **by** (*auto intro*!: *posf inc*)
finally show *?thesis* **.**
qed
ultimately
show *outer_measure M f* $(s \cap x)$ + *outer_measure M f* $(s - x)$ = *outer_measure M f s*
by (*rule order_antisym*)
qed

lemma *measure_down*: *measure_space* Ω *N* $\mu \Longrightarrow$ *sigma_algebra* Ω *M* $\Longrightarrow M \subseteq N$ $\Longrightarrow$ *measure_space* Ω *M* μ
by (*auto simp add*: *measure_space_def positive_def countably_additive_def subset_eq*)

6.9.2 Caratheodory's theorem

theorem (**in** *ring_of_sets*) *caratheodory'*:
assumes *posf*: *positive M f* **and** *ca*: *countably_additive M f*
shows $\exists\, \mu$ :: *'a set* $\Rightarrow$ *ennreal*. ($\forall\, s \in M.\ \mu\ s = f\ s$) $\wedge$ *measure_space* Ω (*sigma_sets* Ω *M*) μ
proof −
have *inc*: *increasing M f*
by (*metis additive_increasing ca countably_additive_additive posf*)
let *?O* = *outer_measure M f*
define *ls* **where** *ls* = *lambda_system* Ω (*Pow* Ω) *?O*
have *mls*: *measure_space* Ω *ls ?O*
using *sigma_algebra.caratheodory_lemma*
[*OF sigma_algebra_Pow outer_measure_space_outer_measure* [*OF posf inc*]]
by (*simp add*: *ls_def*)
hence *sls*: *sigma_algebra* Ω *ls*
by (*simp add*: *measure_space_def*)
have $M \subseteq$ *ls*
by (*simp add*: *ls_def*)
(*metis ca posf inc countably_additive_additive algebra_subset_lambda_system*)
hence *sgs_sb*: *sigma_sets* (Ω) (*M*) $\subseteq$ *ls*
using *sigma_algebra.sigma_sets_subset* [*OF sls*, *of M*]
by *simp*
have *measure_space* Ω (*sigma_sets* Ω *M*) *?O*
by (*rule measure_down* [*OF mls*], *rule sigma_algebra_sigma_sets*)
(*simp_all add*: *sgs_sb space_closed*)
thus *?thesis* **using** *outer_measure_agrees* [*OF posf ca*]
by (*intro exI*[*of _ ?O*]) *auto*
qed

lemma (**in** *ring_of_sets*) *caratheodory_empty_continuous*:
assumes *f*: *positive M f additive M f* **and** *fin*: $\bigwedge A.\ A \in M \Longrightarrow f\ A \neq \infty$
assumes *cont*: $\bigwedge A.$ *range* $A \subseteq M \Longrightarrow$ *decseq* $A \Longrightarrow (\bigcap i.\ A\ i) = \{\} \Longrightarrow (\lambda i.\ f\ (A\ i)) \longrightarrow 0$

shows $\exists\,\mu$ :: *'a set* $\Rightarrow$ *ennreal*. ($\forall\, s \in M.\ \mu\ s = f\ s$) $\wedge$ *measure_space* Ω (*sigma_sets* Ω *M*) μ
proof (*intro caratheodory' empty_continuous_imp_countably_additive f*)
show $\forall\, A{\in}M.\ f\ A \neq \infty$ **using** *fin* **by** *auto*
qed (*rule cont*)

6.9.3 Volumes

definition *volume* :: *'a set set* $\Rightarrow$ (*'a set* $\Rightarrow$ *ennreal*) $\Rightarrow$ *bool* **where**
volume M f $\longleftrightarrow$
(f {} = *0*) $\wedge$ ($\forall\, a{\in}M.\ 0 \le f\ a$) $\wedge$
($\forall\, C{\subseteq}M.$ *disjoint C* $\longrightarrow$ *finite C* $\longrightarrow \bigcup C \in M \longrightarrow f\ (\bigcup C) = (\sum c{\in}C.\ f\ c)$)

lemma *volumeI*:
assumes f {} = *0*
assumes $\bigwedge a.\ a \in M \Longrightarrow 0 \le f\ a$
assumes $\bigwedge C.\ C \subseteq M \Longrightarrow$ *disjoint C* $\Longrightarrow$ *finite C* $\Longrightarrow \bigcup C \in M \Longrightarrow f\ (\bigcup C) = (\sum c{\in}C.\ f\ c)$
shows *volume M f*
using *assms* **by** (*auto simp*: *volume_def*)

lemma *volume_positive*:
volume M f $\Longrightarrow a \in M \Longrightarrow 0 \le f\ a$
by (*auto simp*: *volume_def*)

lemma *volume_empty*:
volume M f $\Longrightarrow f$ {} = *0*
by (*auto simp*: *volume_def*)

proposition *volume_finite_additive*:
assumes *volume M f*
assumes *A*: $\bigwedge i.\ i \in I \Longrightarrow A\ i \in M$ *disjoint_family_on A I finite I* $\bigcup (A\ `\ I) \in M$
shows $f\ (\bigcup (A\ `\ I)) = (\sum i{\in}I.\ f\ (A\ i))$
proof –
have $A`I \subseteq M$ *disjoint* ($A`I$) *finite* ($A`I$) $\bigcup (A`I) \in M$
using *A* **by** (*auto simp*: *disjoint_family_on_disjoint_image*)
with ‹*volume M f*› **have** $f\ (\bigcup (A`I)) = (\sum a{\in}A`I.\ f\ a)$
unfolding *volume_def* **by** *blast*
also have ... = $(\sum i{\in}I.\ f\ (A\ i))$
proof (*subst sum.reindex_nontrivial*)
fix *i j* **assume** $i \in I\ j \in I\ i \neq j\ A\ i = A\ j$
with ‹*disjoint_family_on A I*› **have** $A\ i$ = {}
by (*auto simp*: *disjoint_family_on_def*)
then show $f\ (A\ i) = 0$
using *volume_empty*[*OF* ‹*volume M f*›] **by** *simp*
qed (*auto intro*: ‹*finite I*›)
finally show $f\ (\bigcup (A\ `\ I)) = (\sum i{\in}I.\ f\ (A\ i))$
by *simp*

qed

lemma (**in** *ring_of_sets*) *volume_additiveI*:
  assumes *pos*: $\bigwedge a.\ a \in M \Longrightarrow 0 \le \mu\ a$
  assumes [*simp*]: $\mu\ \{\} = 0$
  assumes *add*: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a \cap b = \{\} \Longrightarrow \mu\ (a \cup b) = \mu\ a + \mu\ b$
  shows *volume M* μ
proof (*unfold volume_def*, *safe*)
  fix *C* **assume** *finite C C* $\subseteq$ *M disjoint C*
  then show $\mu\ (\bigcup C) = sum\ \mu\ C$
  proof (*induct C*)
    case (*insert c C*)
    from *insert*(*1*,*2*,*4*,*5*) **have** $\mu\ (\bigcup(insert\ c\ C)) = \mu\ c + \mu\ (\bigcup C)$
      by (*auto intro*!: *add simp*: *disjoint_def*)
    with *insert* **show** *?case*
      by (*simp add*: *disjoint_def*)
  qed *simp*
qed *fact+*

proposition (**in** *semiring_of_sets*) *extend_volume*:
  assumes *volume M* μ
  shows $\exists\mu'.\ volume\ generated\_ring\ \mu' \wedge (\forall a{\in}M.\ \mu'\ a = \mu\ a)$
proof −
  let *?R* = *generated_ring*
  have $\forall a{\in}?R.\ \exists m.\ \exists C{\subseteq}M.\ a = \bigcup C \wedge finite\ C \wedge disjoint\ C \wedge m = (\sum c{\in}C.\ \mu\ c)$
    by (*auto simp*: *generated_ring_def*)
  from *bchoice*[*OF this*] **guess** μ' **..** **note** μ'*_spec* = *this*

  { fix *C* **assume** *C*: *C* $\subseteq$ *M finite C disjoint C*
    fix *D* **assume** *D*: *D* $\subseteq$ *M finite D disjoint D*
    assume $\bigcup C = \bigcup D$
    have $(\sum d{\in}D.\ \mu\ d) = (\sum d{\in}D.\ \sum c{\in}C.\ \mu\ (c \cap d))$
    proof (*intro sum.cong refl*)
      fix *d* **assume** $d \in D$
      have *Un_eq_d*: $(\bigcup c{\in}C.\ c \cap d) = d$
        using ‹$d \in D$› ‹$\bigcup C = \bigcup D$› **by** *auto*
      moreover have $\mu\ (\bigcup c{\in}C.\ c \cap d) = (\sum c{\in}C.\ \mu\ (c \cap d))$
      proof (*rule volume_finite_additive*)
        { fix *c* **assume** $c \in C$ **then show** $c \cap d \in M$
          using *C D* ‹$d \in D$› **by** *auto* **}**
        show $(\bigcup a{\in}C.\ a \cap d) \in M$
          unfolding *Un_eq_d* **using** ‹$d \in D$› *D* **by** *auto*
        show *disjoint_family_on* $(\lambda a.\ a \cap d)$ *C*
          using ‹*disjoint C*› **by** (*auto simp*: *disjoint_family_on_def disjoint_def*)
      qed *fact+*
      ultimately show $\mu\ d = (\sum c{\in}C.\ \mu\ (c \cap d))$ **by** *simp*
    qed }

```
  note split_sum = this

  { fix C assume C: C ⊆ M finite C disjoint C
    fix D assume D: D ⊆ M finite D disjoint D
    assume ⋃C = ⋃D
    with split_sum[OF C D] split_sum[OF D C]
    have (∑ d∈D. μ d) = (∑ c∈C. μ c)
      by (simp, subst sum.swap, simp add: ac_simps) }
  note sum_eq = this

  { fix C assume C: C ⊆ M finite C disjoint C
    then have ⋃C ∈ ?R by (auto simp: generated_ring_def)
    with μ'_spec[THEN bspec, of ⋃C]
    obtain D where
      D: D ⊆ M finite D disjoint D ⋃C = ⋃D and μ' (⋃C) = (∑ d∈D. μ d)
      by auto
    with sum_eq[OF C D] have μ' (⋃C) = (∑ c∈C. μ c) by simp }
  note μ' = this

  show ?thesis
  proof (intro exI conjI ring_of_sets.volume_additiveI[OF generating_ring] ballI)
    fix a assume a ∈ M with μ'[of {a}] show μ' a = μ a
      by (simp add: disjoint_def)
  next
    fix a assume a ∈ ?R then guess Ca .. note Ca = this
    with μ'[of Ca] ‹volume M μ›[THEN volume_positive]
    show 0 ≤ μ' a
      by (auto intro!: sum_nonneg)
  next
    show μ' {} = 0 using μ'[of {}] by auto
  next
    fix a assume a ∈ ?R then guess Ca .. note Ca = this
    fix b assume b ∈ ?R then guess Cb .. note Cb = this
    assume a ∩ b = {}
    with Ca Cb have Ca ∩ Cb ⊆ {{}} by auto
    then have C_Int_cases: Ca ∩ Cb = {{}} ∨ Ca ∩ Cb = {} by auto

    from ‹a ∩ b = {}› have μ' (⋃(Ca ∪ Cb)) = (∑ c∈Ca ∪ Cb. μ c)
      using Ca Cb by (intro μ') (auto intro!: disjoint_union)
    also have ... = (∑ c∈Ca ∪ Cb. μ c) + (∑ c∈Ca ∩ Cb. μ c)
      using C_Int_cases volume_empty[OF ‹volume M μ›] by (elim disjE) simp_all
    also have ... = (∑ c∈Ca. μ c) + (∑ c∈Cb. μ c)
      using Ca Cb by (simp add: sum.union_inter)
    also have ... = μ' a + μ' b
      using Ca Cb by (simp add: μ')
    finally show μ' (a ∪ b) = μ' a + μ' b
      using Ca Cb by simp
  qed
qed
```

Caratheodory on semirings

theorem (**in** *semiring_of_sets*) *caratheodory*:
  assumes *pos*: *positive M μ* **and** *ca*: *countably_additive M μ*
  shows ∃ μ′ :: ′*a set* ⇒ *ennreal*. (∀ *s* ∈ *M*. μ′ *s* = μ *s*) ∧ *measure_space* Ω (*sigma_sets* Ω *M*) μ′
proof −
  have *volume M μ*
  proof (*rule volumeI*)
    { fix *a* **assume** *a* ∈ *M* **then show** *0* ≤ μ *a*
      using *pos* **unfolding** *positive_def* **by** *auto* **}**
    note *p* = *this*

    fix *C* **assume** *sets_C*: *C* ⊆ *M* ⋃ *C* ∈ *M* **and** *disjoint C finite C*
    have ∃ *F′*. *bij_betw F′* {..<*card C*} *C*
      by (*rule finite_same_card_bij*[*OF* _ ‹*finite C*›]) *auto*
    then guess *F′* **.. note** *F′* = *this*
    then have *F′*: *C* = *F′* ‘ {..< *card C*} *inj_on F′* {..< *card C*}
      by (*auto simp*: *bij_betw_def*)
    { fix *i j* **assume** ∗: *i* < *card C j* < *card C i* ≠ *j*
      with *F′* **have** *F′ i* ∈ *C F′ j* ∈ *C F′ i* ≠ *F′ j*
        unfolding *inj_on_def* **by** *auto*
      with ‹*disjoint C*›[*THEN disjointD*]
      have *F′ i* ∩ *F′ j* = {}
        by *auto* **}**
    note *F′_disj* = *this*
    define *F* **where** *F i* = (*if i* < *card C then F′ i else* {}) **for** *i*
    then have *disjoint_family F*
      using *F′_disj* **by** (*auto simp*: *disjoint_family_on_def*)
    moreover from *F′* **have** (⋃ *i*. *F i*) = ⋃ *C*
      by (*auto simp add*: *F_def split*: *if_split_asm cong del*: *SUP_cong*)
    moreover have *sets_F*: ⋀*i*. *F i* ∈ *M*
      using *F′ sets_C* **by** (*auto simp*: *F_def*)
    moreover note *sets_C*
    ultimately have μ (⋃ *C*) = (∑ *i*. μ (*F i*))
      using *ca*[*unfolded countably_additive_def*, *THEN spec*, *of F*] **by** *auto*
    also have ... = (∑ *i*<*card C*. μ (*F′ i*))
    proof −
     have (λ*i*. *if i* ∈ {..< *card C*} *then* μ (*F′ i*) *else 0*) *sums* (∑ *i*<*card C*. μ (*F′ i*))
        by (*rule sums_If_finite_set*) *auto*
      also have (λ*i*. *if i* ∈ {..< *card C*} *then* μ (*F′ i*) *else 0*) = (λ*i*. μ (*F i*))
        using *pos* **by** (*auto simp*: *positive_def F_def*)
      finally show (∑ *i*. μ (*F i*)) = (∑ *i*<*card C*. μ (*F′ i*))
        by (*simp add*: *sums_iff*)
    qed
    also have ... = (∑ *c*∈*C*. μ *c*)
      using *F′*(*2*) **by** (*subst* (*2*) *F′*) (*simp add*: *sum.reindex*)
    finally show μ (⋃ *C*) = (∑ *c*∈*C*. μ *c*) **.**
  next

```
    show μ {} = 0
      using ‹positive M μ› by (rule positiveD1)
  qed
  from extend_volume[OF this] obtain μ_r where
    V: volume generated_ring μ_r ⋀a. a ∈ M ⟹ μ a = μ_r a
    by auto

  interpret G: ring_of_sets Ω generated_ring
    by (rule generating_ring)

  have pos: positive generated_ring μ_r
   using V unfolding positive_def by (auto simp: positive_def intro!: volume_positive
volume_empty)

  have countably_additive generated_ring μ_r
  proof (rule countably_additiveI)
    fix A' :: nat ⇒ 'a set assume A': range A' ⊆ generated_ring disjoint_family A'
      and Un_A: (⋃i. A' i) ∈ generated_ring

    from generated_ringE[OF Un_A] guess C' . note C' = this

    { fix c assume c ∈ C'
      moreover define A where [abs_def]: A i = A' i ∩ c for i
      ultimately have A: range A ⊆ generated_ring disjoint_family A
        and Un_A: (⋃i. A i) ∈ generated_ring
        using A' C'
          by (auto intro!: G.Int G.finite_Union intro: generated_ringI_Basic simp:
disjoint_family_on_def)
      from A C' ‹c ∈ C'› have UN_eq: (⋃i. A i) = c
        by (auto simp: A_def)

      have ∀i::nat. ∃f::nat ⇒ 'a set. μ_r (A i) = (∑j. μ_r (f j)) ∧ disjoint_family
f ∧ ⋃(range f) = A i ∧ (∀j. f j ∈ M)
        (is ∀i. ?P i)
      proof
        fix i
        from A have Ai: A i ∈ generated_ring by auto
        from generated_ringE[OF this] guess C . note C = this

        have ∃F'. bij_betw F' {..<card C} C
          by (rule finite_same_card_bij[OF _ ‹finite C›]) auto
        then guess F .. note F = this
        define f where [abs_def]: f i = (if i < card C then F i else {}) for i
        then have f: bij_betw f {..< card C} C
          by (intro bij_betw_cong[THEN iffD1, OF _ F]) auto
        with C have ∀j. f j ∈ M
          by (auto simp: Pi_iff f_def dest!: bij_betw_imp_funcset)
        moreover
        from f C have d_f: disjoint_family_on f {..<card C}
```

```
      by (intro disjoint_image_disjoint_family_on) (auto simp: bij_betw_def)
    then have disjoint_family f
      by (auto simp: disjoint_family_on_def f_def)
    moreover
    have Ai_eq: A i = (⋃x<card C. f x)
      using f C Ai unfolding bij_betw_def by auto
    then have ⋃(range f) = A i
      using f by (auto simp add: f_def)
    moreover
    { have (∑j. μ_r (f j)) = (∑j. if j ∈ {..< card C} then μ_r (f j) else 0)
     using volume_empty[OF V(1)] by (auto intro!: arg_cong[where f=suminf]
simp: f_def)
      also have ... = (∑j<card C. μ_r (f j))
        by (rule sums_If_finite_set[THEN sums_unique, symmetric]) simp
      also have ... = μ_r (A i)
        using C f[THEN bij_betw_imp_funcset] unfolding Ai_eq
        by (intro volume_finite_additive[OF V(1) _ d_f, symmetric])
          (auto simp: Pi_iff Ai_eq intro: generated_ringI_Basic)
      finally have μ_r (A i) = (∑j. μ_r (f j)) .. }
    ultimately show ?P i
      by blast
  qed
  from choice[OF this] guess f .. note f = this
  then have UN_f_eq: (⋃i. case_prod f (prod_decode i)) = (⋃i. A i)
    unfolding UN_extend_simps surj_prod_decode by (auto simp: set_eq_iff)

  have d: disjoint_family (λi. case_prod f (prod_decode i))
    unfolding disjoint_family_on_def
  proof (intro ballI impI)
    fix m n :: nat assume m ≠ n
    then have neq: prod_decode m ≠ prod_decode n
      using inj_prod_decode[of UNIV] by (auto simp: inj_on_def)
    show case_prod f (prod_decode m) ∩ case_prod f (prod_decode n) = {}
    proof cases
      assume fst (prod_decode m) = fst (prod_decode n)
      then show ?thesis
        using neq f by (fastforce simp: disjoint_family_on_def)
    next
      assume neq: fst (prod_decode m) ≠ fst (prod_decode n)
      have case_prod f (prod_decode m) ⊆ A (fst (prod_decode m))
        case_prod f (prod_decode n) ⊆ A (fst (prod_decode n))
        using f[THEN spec, of fst (prod_decode m)]
        using f[THEN spec, of fst (prod_decode n)]
        by (auto simp: set_eq_iff)
      with f A neq show ?thesis
        by (fastforce simp: disjoint_family_on_def subset_eq set_eq_iff)
    qed
  qed
  from f have (∑n. μ_r (A n)) = (∑n. μ_r (case_prod f (prod_decode n)))
```

by (*intro suminf_ennreal_2dimen*[*symmetric*] *generated_ringI_Basic*)
(*auto split*: *prod.split*)
also have ... = ($\sum$ *n*. *μ* (*case_prod f* (*prod_decode n*)))
using *f V*(*2*) **by** (*auto intro*!: *arg_cong*[**where** *f*=*suminf*] *split*: *prod.split*)
also have ... = *μ* ($\bigcup$ *i*. *case_prod f* (*prod_decode i*))
using *f* ‹*c* ∈ *C'*› *C'*
by (*intro ca*[*unfolded countably_additive_def*, *rule_format*])
(*auto split*: *prod.split simp*: *UN_f_eq d UN_eq*)
finally have ($\sum$ *n*. *μ_r* (*A' n* ∩ *c*)) = *μ c*
using *UN_f_eq UN_eq* **by** (*simp add*: *A_def*) **}**
note *eq* = *this*

have ($\sum$ *n*. *μ_r* (*A' n*)) = ($\sum$ *n*. $\sum$ *c*∈*C'*. *μ_r* (*A' n* ∩ *c*))
using *C' A'*
by (*subst volume_finite_additive*[*symmetric*, *OF V*(*1*)])
(*auto simp*: *disjoint_def disjoint_family_on_def*
intro!: *G.Int G.finite_Union arg_cong*[**where** *f*=*λX*. *suminf* (*λi*. *μ_r* (*X i*))] *ext*
intro: *generated_ringI_Basic*)
also have ... = ($\sum$ *c*∈*C'*. $\sum$ *n*. *μ_r* (*A' n* ∩ *c*))
using *C' A'*
by (*intro suminf_sum G.Int G.finite_Union*) (*auto intro*: *generated_ringI_Basic*)
also have ... = ($\sum$ *c*∈*C'*. *μ_r c*)
using *eq V C'* **by** (*auto intro*!: *sum.cong*)
also have ... = *μ_r* ($\bigcup$ *C'*)
using *C' Un_A*
by (*subst volume_finite_additive*[*symmetric*, *OF V*(*1*)])
(*auto simp*: *disjoint_family_on_def disjoint_def*
intro: *generated_ringI_Basic*)
finally show ($\sum$ *n*. *μ_r* (*A' n*)) = *μ_r* ($\bigcup$ *i*. *A' i*)
using *C'* **by** *simp*
qed
from *G.caratheodory'*[*OF* ‹*positive generated_ring μ_r*› ‹*countably_additive generated_ring μ_r*›]
guess *μ'* ..
with *V* **show** *?thesis*
unfolding *sigma_sets_generated_ring_eq*
by (*intro exI*[*of* _ *μ'*]) (*auto intro*: *generated_ringI_Basic*)
qed

lemma *extend_measure_caratheodory*:
fixes *G* :: *'i* ⇒ *'a set*
assumes *M*: *M* = *extend_measure* Ω *I G μ*
assumes *i* ∈ *I*
assumes *semiring_of_sets* Ω (*G* ‘ *I*)
assumes *empty*: ⋀*i*. *i* ∈ *I* ⟹ *G i* = {} ⟹ *μ i* = *0*
assumes *inj*: ⋀*i j*. *i* ∈ *I* ⟹ *j* ∈ *I* ⟹ *G i* = *G j* ⟹ *μ i* = *μ j*
assumes *nonneg*: ⋀*i*. *i* ∈ *I* ⟹ *0* ≤ *μ i*
assumes *add*: ⋀*A*::*nat* ⇒ *'i*. ⋀*j*. *A* ∈ *UNIV* → *I* ⟹ *j* ∈ *I* ⟹ *disjoint_family*

$(G \circ A) \Longrightarrow$
  $(\bigcup i.\ G\ (A\ i)) = G\ j \Longrightarrow (\sum n.\ \mu\ (A\ n)) = \mu\ j$
  shows *emeasure M* $(G\ i) = \mu\ i$

proof −
  interpret *semiring_of_sets* Ω *G ‘ I*
    by *fact*
  have $\forall g{\in}G`I.\ \exists i{\in}I.\ g = G\ i$
    by *auto*
  then obtain *sel* **where** *sel*: $\bigwedge g.\ g \in G\ `\ I \Longrightarrow sel\ g \in I\ \bigwedge g.\ g \in G\ `\ I \Longrightarrow G$ $(sel\ g) = g$
    by *metis*

  have $\exists \mu'.\ (\forall s{\in}G\ `\ I.\ \mu'\ s = \mu\ (sel\ s)) \wedge$ *measure_space* $\Omega$ (*sigma_sets* $\Omega$ $(G\ `\ I))\ \mu'$
  proof (*rule caratheodory*)
    show *positive* $(G\ `\ I)\ (\lambda s.\ \mu\ (sel\ s))$
      by (*auto simp*: *positive_def intro*!: *empty sel nonneg*)
    show *countably_additive* $(G\ `\ I)\ (\lambda s.\ \mu\ (sel\ s))$
    proof (*rule countably_additiveI*)
      fix $A :: nat \Rightarrow 'a\ set$ **assume** *range* $A \subseteq G\ `\ I$ *disjoint_family* $A\ (\bigcup i.\ A\ i) \in G\ `\ I$
      then show $(\sum i.\ \mu\ (sel\ (A\ i))) = \mu\ (sel\ (\bigcup i.\ A\ i))$
        by (*intro add*) (*auto simp*: *sel image_subset_iff_funcset comp_def Pi_iff intro*!: *sel*)
    qed
  qed
  then obtain μ' **where** μ': $\forall s{\in}G\ `\ I.\ \mu'\ s = \mu\ (sel\ s)$ *measure_space* $\Omega$ (*sigma_sets* $\Omega\ (G\ `\ I))\ \mu'$
    by *metis*

  show *?thesis*
  proof (*rule emeasure_extend_measure*[*OF M*])
    { fix *i* **assume** $i \in I$ **then show** $\mu'\ (G\ i) = \mu\ i$
      using μ' **by** (*auto intro*!: *inj sel*) **}**
    show $G\ `\ I \subseteq Pow\ \Omega$
      by (*rule space_closed*)
    then show *positive* (*sets M*) μ' *countably_additive* (*sets M*) μ'
      using μ' **by** (*simp_all add*: *M sets_extend_measure measure_space_def*)
  qed *fact*
qed

proposition *extend_measure_caratheodory_pair*:
  fixes $G :: 'i \Rightarrow 'j \Rightarrow 'a\ set$
  assumes *M*: $M =$ *extend_measure* $\Omega\ \{(a, b).\ P\ a\ b\}\ (\lambda(a, b).\ G\ a\ b)\ (\lambda(a, b).\ \mu\ a\ b)$
  assumes $P\ i\ j$
  assumes *semiring*: *semiring_of_sets* $\Omega\ \{G\ a\ b \mid a\ b.\ P\ a\ b\}$
  assumes *empty*: $\bigwedge i\ j.\ P\ i\ j \Longrightarrow G\ i\ j = \{\} \Longrightarrow \mu\ i\ j = 0$

assumes *inj*: $\bigwedge i\ j\ k\ l.\ P\ i\ j \Longrightarrow P\ k\ l \Longrightarrow G\ i\ j = G\ k\ l \Longrightarrow \mu\ i\ j = \mu\ k\ l$
assumes *nonneg*: $\bigwedge i\ j.\ P\ i\ j \Longrightarrow 0 \le \mu\ i\ j$
assumes *add*: $\bigwedge A{::}nat \Rightarrow 'i.\ \bigwedge B{::}nat \Rightarrow 'j.\ \bigwedge j\ k.$
$(\bigwedge n.\ P\ (A\ n)\ (B\ n)) \Longrightarrow P\ j\ k \Longrightarrow disjoint\_family\ (\lambda n.\ G\ (A\ n)\ (B\ n)) \Longrightarrow$
$(\bigcup i.\ G\ (A\ i)\ (B\ i)) = G\ j\ k \Longrightarrow (\sum n.\ \mu\ (A\ n)\ (B\ n)) = \mu\ j\ k$
shows *emeasure M* $(G\ i\ j) = \mu\ i\ j$
proof −
have *emeasure M* $((\lambda(a, b).\ G\ a\ b)\ (i, j)) = (\lambda(a, b).\ \mu\ a\ b)\ (i, j)$
proof (*rule extend_measure_caratheodory*[*OF M*])
show *semiring_of_sets* Ω $((\lambda(a, b).\ G\ a\ b)$ ' $\{(a, b).\ P\ a\ b\})$
using *semiring* **by** (*simp add*: *image_def conj_commute*)
next
fix $A :: nat \Rightarrow ('i \times 'j)$ **and** j **assume** $A \in UNIV \to \{(a, b).\ P\ a\ b\}\ j \in \{(a, b).\ P\ a\ b\}$
disjoint_family $((\lambda(a, b).\ G\ a\ b) \circ A)$
$(\bigcup i.\ case\ A\ i\ of\ (a, b) \Rightarrow G\ a\ b) = (case\ j\ of\ (a, b) \Rightarrow G\ a\ b)$
then show $(\sum n.\ case\ A\ n\ of\ (a, b) \Rightarrow \mu\ a\ b) = (case\ j\ of\ (a, b) \Rightarrow \mu\ a\ b)$
using *add*[*of* $\lambda i.\ fst\ (A\ i)\ \lambda i.\ snd\ (A\ i)\ fst\ j\ snd\ j$]
by (*simp add*: *split_beta' comp_def Pi_iff*)
qed (*auto split*: *prod.splits intro*: *assms*)
then show *?thesis* **by** *simp*
qed

end

6.10 Bochner Integration for Vector-Valued Functions

theory *Bochner_Integration*
imports *Finite_Product_Measure*
begin

In the following development of the Bochner integral we use second countable topologies instead of separable spaces. A second countable topology is also separable.

proposition *borel_measurable_implies_sequence_metric*:
fixes $f :: 'a \Rightarrow 'b :: \{metric\_space, second\_countable\_topology\}$
assumes [*measurable*]: $f \in borel\_measurable\ M$
shows $\exists F.\ (\forall i.\ simple\_function\ M\ (F\ i)) \wedge (\forall x{\in}space\ M.\ (\lambda i.\ F\ i\ x) \longrightarrow f\ x) \wedge$
$(\forall i.\ \forall x{\in}space\ M.\ dist\ (F\ i\ x)\ z \le 2 * dist\ (f\ x)\ z)$
proof −
obtain $D :: 'b\ set$ **where** *countable D* **and** D: $\bigwedge X.\ open\ X \Longrightarrow X \neq \{\} \Longrightarrow \exists d{\in}D.\ d \in X$
by (*erule countable_dense_setE*)

define e **where** $e = from\_nat\_into\ D$
{ **fix** $n\ x$

obtain *d* **where** *d* ∈ *D* **and** *d*: *d* ∈ *ball x* (*1* / *Suc n*)
  using *D*[*of ball x* (*1* / *Suc n*)] **by** *auto*
from ‹*d* ∈ *D*› *D*[*of UNIV*] ‹*countable D*› **obtain** *i* **where** *d* = *e i*
  unfolding *e_def* **by** (*auto dest*: *from_nat_into_surj*)
with *d* **have** ∃ *i*. *dist x* (*e i*) < *1* / *Suc n*
  by *auto* **}**
note *e* = *this*

define *A* **where** [*abs_def*]: *A m n* =
{*x*∈*space M*. *dist* (*f x*) (*e n*) < *1* / (*Suc m*) ∧ *1* / (*Suc m*) ≤ *dist* (*f x*) *z*} **for** *m n*
define *B* **where** [*abs_def*]: *B m* = *disjointed* (*A m*) **for** *m*

define *m* **where** [*abs_def*]: *m N x* = *Max* {*m*. *m* ≤ *N* ∧ *x* ∈ (⋃*n*≤*N*. *B m n*)} **for** *N x*
define *F* **where** [*abs_def*]: *F N x* =
(*if* (∃ *m*≤*N*. *x* ∈ (⋃*n*≤*N*. *B m n*)) ∧ (∃ *n*≤*N*. *x* ∈ *B* (*m N x*) *n*)
then e (*LEAST n*. *x* ∈ *B* (*m N x*) *n*) *else z*) **for** *N x*

have *B_imp_A*[*intro*, *simp*]: ⋀*x m n*. *x* ∈ *B m n* ⟹ *x* ∈ *A m n*
  using *disjointed_subset*[*of A m* **for** *m*] **unfolding** *B_def* **by** *auto*

{ fix *m*
  have ⋀*n*. *A m n* ∈ *sets M*
    by (*auto simp*: *A_def*)
  then have ⋀*n*. *B m n* ∈ *sets M*
    using *sets.range_disjointed_sets*[*of A m M*] **by** (*auto simp*: *B_def*) **}**
note *this*[*measurable*]

{ fix *N i x* **assume** ∃ *m*≤*N*. *x* ∈ (⋃*n*≤*N*. *B m n*)
  then have *m N x* ∈ {*m*::*nat*. *m* ≤ *N* ∧ *x* ∈ (⋃*n*≤*N*. *B m n*)}
    unfolding *m_def* **by** (*intro Max_in*) *auto*
  then have *m N x* ≤ *N* ∃ *n*≤*N*. *x* ∈ *B* (*m N x*) *n*
    by *auto* **}**
note *m* = *this*

{ fix *j N i x* **assume** *j* ≤ *N i* ≤ *N x* ∈ *B j i*
  then have *j* ≤ *m N x*
    unfolding *m_def* **by** (*intro Max_ge*) *auto* **}**
note *m_upper* = *this*

show *?thesis*
  unfolding *simple_function_def*
proof (*safe intro*!: *exI*[*of _ F*])
  have [*measurable*]: ⋀*i*. *F i* ∈ *borel_measurable M*
    unfolding *F_def m_def* **by** *measurable*
  show ⋀*x i*. *F i* −‘ {*x*} ∩ *space M* ∈ *sets M*
    by *measurable*

```
{ fix i
  { fix n x assume x ∈ B (m i x) n
    then have (LEAST n. x ∈ B (m i x) n) ≤ n
      by (intro Least_le)
    also assume n ≤ i
    finally have (LEAST n. x ∈ B (m i x) n) ≤ i . }
  then have F i ` space M ⊆ {z} ∪ e ` {.. i}
    by (auto simp: F_def)
  then show finite (F i ` space M)
    by (rule finite_subset) auto }

{ fix N i n x assume i ≤ N n ≤ N x ∈ B i n
  then have 1: ∃ m≤N. x ∈ (⋃ n≤N. B m n) by auto
  from m[OF this] obtain n where n: m N x ≤ N n ≤ N x ∈ B (m N x) n
by auto
  moreover
  define L where L = (LEAST n. x ∈ B (m N x) n)
  have dist (f x) (e L) < 1 / Suc (m N x)
  proof -
    have x ∈ B (m N x) L
      using n(3) unfolding L_def by (rule LeastI)
    then have x ∈ A (m N x) L
      by auto
    then show ?thesis
      unfolding A_def by simp
  qed
  ultimately have dist (f x) (F N x) < 1 / Suc (m N x)
    by (auto simp add: F_def L_def) }
note * = this

fix x assume x ∈ space M
show (λi. F i x) ⟶ f x
proof cases
  assume f x = z
  then have ⋀i n. x ∉ A i n
    unfolding A_def by auto
  then have ⋀i. F i x = z
    by (auto simp: F_def)
  then show ?thesis
    using ‹f x = z› by auto
next
  assume f x ≠ z

  show ?thesis
  proof (rule tendstoI)
    fix e :: real assume 0 < e
    with ‹f x ≠ z› obtain n where 1 / Suc n < e 1 / Suc n < dist (f x) z
      by (metis dist_nz order_less_trans neq_iff nat_approx_posE)
    with ‹x∈space M› ‹f x ≠ z› have x ∈ (⋃ i. B n i)
```

```
        unfolding A_def B_def UN_disjointed_eq using e by auto
      then obtain i where i: x ∈ B n i by auto

      show eventually (λi. dist (F i x) (f x) < e) sequentially
        using eventually_ge_at_top[of max n i]
      proof eventually_elim
        fix j assume j: max n i ≤ j
        with i have dist (f x) (F j x) < 1 / Suc (m j x)
          by (intro *[OF _ _ i]) auto
        also have ... ≤ 1 / Suc n
          using j m_upper[OF _ _ i]
          by (auto simp: field_simps)
        also note ‹1 / Suc n < e›
        finally show dist (F j x) (f x) < e
          by (simp add: less_imp_le dist_commute)
      qed
    qed
  qed
  fix i
  { fix n m assume x ∈ A n m
    then have dist (e m) (f x) + dist (f x) z ≤ 2 ∗ dist (f x) z
      unfolding A_def by (auto simp: dist_commute)
    also have dist (e m) z ≤ dist (e m) (f x) + dist (f x) z
      by (rule dist_triangle)
    finally (xtrans) have dist (e m) z ≤ 2 ∗ dist (f x) z . }
  then show dist (F i x) z ≤ 2 ∗ dist (f x) z
    unfolding F_def
    apply auto
    apply (rule LeastI2)
    apply auto
    done
 qed
qed

lemma
  fixes f :: 'a ⇒ 'b::semiring_1 assumes finite A
  shows sum_mult_indicator[simp]: (∑ x ∈ A. f x ∗ indicator (B x) (g x)) =
(∑ x∈{x∈A. g x ∈ B x}. f x)
  and sum_indicator_mult[simp]: (∑ x ∈ A. indicator (B x) (g x) ∗ f x) = (∑ x∈{x∈A.
g x ∈ B x}. f x)
  unfolding indicator_def
  using assms by (auto intro!: sum.mono_neutral_cong_right split: if_split_asm)

lemma borel_measurable_induct_real[consumes 2, case_names set mult add seq]:
  fixes P :: ('a ⇒ real) ⇒ bool
  assumes u: u ∈ borel_measurable M ⋀x. 0 ≤ u x
  assumes set: ⋀A. A ∈ sets M ⟹ P (indicator A)
  assumes mult: ⋀u c. 0 ≤ c ⟹ u ∈ borel_measurable M ⟹ (⋀x. 0 ≤ u x)
⟹ P u ⟹ P (λx. c ∗ u x)
```

assumes *add*: $\bigwedge u\ v.\ u \in borel\_measurable\ M \Longrightarrow (\bigwedge x.\ 0 \leq u\ x) \Longrightarrow P\ u \Longrightarrow v \in borel\_measurable\ M \Longrightarrow (\bigwedge x.\ 0 \leq v\ x) \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow u\ x = 0 \vee v\ x = 0) \Longrightarrow P\ v \Longrightarrow P\ (\lambda x.\ v\ x + u\ x)$
assumes *seq*: $\bigwedge U.\ (\bigwedge i.\ U\ i \in borel\_measurable\ M) \Longrightarrow (\bigwedge i\ x.\ 0 \leq U\ i\ x) \Longrightarrow (\bigwedge i.\ P\ (U\ i)) \Longrightarrow incseq\ U \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow (\lambda i.\ U\ i\ x) \longrightarrow u\ x) \Longrightarrow P\ u$
shows $P\ u$
proof –
have $(\lambda x.\ ennreal\ (u\ x)) \in borel\_measurable\ M$ **using** u **by** *auto*
from *borel_measurable_implies_simple_function_sequence'[OF this]*
obtain U **where** U: $\bigwedge i.\ simple\_function\ M\ (U\ i)\ incseq\ U\ \bigwedge i\ x.\ U\ i\ x < top$ **and**
sup: $\bigwedge x.\ (SUP\ i.\ U\ i\ x) = ennreal\ (u\ x)$
by *blast*

define U' **where** *[abs_def]*: $U'\ i\ x = indicator\ (space\ M)\ x * enn2real\ (U\ i\ x)$ **for** $i\ x$
then have *U'_sf[measurable]*: $\bigwedge i.\ simple\_function\ M\ (U'\ i)$
using U **by** (*auto intro*!: *simple_function_compose1*[**where** *g=enn2real*])

show $P\ u$
proof (*rule seq*)
show U': $U'\ i \in borel\_measurable\ M\ \bigwedge x.\ 0 \leq U'\ i\ x$ **for** i
using U **by** (*auto*
intro: *borel_measurable_simple_function*
intro!: *borel_measurable_enn2real borel_measurable_times*
simp: *U'_def zero_le_mult_iff*)
show *incseq* U'
using $U(2,3)$
by (*auto simp*: *incseq_def le_fun_def image_iff eq_commute U'_def indicator_def enn2real_mono*)

fix x **assume** x: $x \in space\ M$
have $(\lambda i.\ U\ i\ x) \longrightarrow (SUP\ i.\ U\ i\ x)$
using $U(2)$ **by** (*intro LIMSEQ_SUP*) (*auto simp*: *incseq_def le_fun_def*)
moreover have $(\lambda i.\ U\ i\ x) = (\lambda i.\ ennreal\ (U'\ i\ x))$
using $x\ U(3)$ **by** (*auto simp*: *fun_eq_iff U'_def image_iff eq_commute*)
moreover have $(SUP\ i.\ U\ i\ x) = ennreal\ (u\ x)$
using *sup* $u(2)$ **by** (*simp add*: *max_def*)
ultimately show $(\lambda i.\ U'\ i\ x) \longrightarrow u\ x$
using $u\ U'$ **by** *simp*
next
fix i
have $U'\ i\ `\ space\ M \subseteq enn2real\ `\ (U\ i\ `\ space\ M)\ finite\ (U\ i\ `\ space\ M)$
unfolding *U'_def* **using** $U(1)$ **by** (*auto dest*: *simple_functionD*)
then have *fin*: $finite\ (U'\ i\ `\ space\ M)$
by (*metis finite_subset finite_imageI*)
moreover have $\bigwedge z.\ \{y.\ U'\ i\ z = y \wedge y \in U'\ i\ `\ space\ M \wedge z \in space\ M\} = (if\ z \in space\ M\ then\ \{U'\ i\ z\}\ else\ \{\})$

by *auto*
ultimately have U': $(\lambda z. \sum y{\in}U'\ i`space\ M.\ y * indicator\ \{x{\in}space\ M.\ U'\ i\ x = y\}\ z) = U'\ i$
by (*simp add*: *U'_def fun_eq_iff*)
have $\bigwedge x.\ x \in U'\ i\ `\ space\ M \Longrightarrow 0 \leq x$
by (*auto simp*: *U'_def*)
with *fin* **have** $P\ (\lambda z. \sum y{\in}U'\ i`space\ M.\ y * indicator\ \{x{\in}space\ M.\ U'\ i\ x = y\}\ z)$
proof *induct*
case *empty* **from** *set*[*of* {}] **show** *?case*
by (*simp add*: *indicator_def*[*abs_def*])
next
case (*insert x F*)
from *insert.prems* **have** *nonneg*: $x \geq 0\ \bigwedge y.\ y \in F \Longrightarrow y \geq 0$
by *simp_all*
hence ∗: $P\ (\lambda xa.\ x * indicat\_real\ \{x' \in space\ M.\ U'\ i\ x' = x\}\ xa)$
by (*intro mult set*) *auto*
have $P\ (\lambda z.\ x * indicat\_real\ \{x' \in space\ M.\ U'\ i\ x' = x\}\ z +$
$(\sum y{\in}F.\ y * indicat\_real\ \{x \in space\ M.\ U'\ i\ x = y\}\ z))$
using *insert*(*1−3*)
by (*intro add* ∗ *sum_nonneg mult_nonneg_nonneg*)
(*auto simp*: *nonneg indicator_def sum_nonneg_eq_0_iff*)
thus *?case*
using *insert.hyps* **by** (*subst sum.insert*) *auto*
qed
with U' **show** $P\ (U'\ i)$ **by** *simp*
qed
qed

lemma *scaleR_cong_right*:
fixes x :: *'a* :: *real_vector*
shows $(x \neq 0 \Longrightarrow r = p) \Longrightarrow r *_R x = p *_R x$
by (*cases* $x = 0$) *auto*

inductive *simple_bochner_integrable* :: *'a measure* ⇒ (*'a* ⇒ *'b*::*real_vector*) ⇒ *bool*
for *M f* **where**
simple_function M f ⟹ *emeasure M* $\{y{\in}space\ M.\ f\ y \neq 0\} \neq \infty \Longrightarrow$
simple_bochner_integrable M f

lemma *simple_bochner_integrable_compose2*:
assumes *p_0*: $p\ 0\ 0 = 0$
shows *simple_bochner_integrable M f* ⟹ *simple_bochner_integrable M g* ⟹
simple_bochner_integrable M $(\lambda x.\ p\ (f\ x)\ (g\ x))$
proof (*safe intro*!: *simple_bochner_integrable.intros elim*!: *simple_bochner_integrable.cases del*: *notI*)
assume *sf*: *simple_function M f simple_function M g*
then show *simple_function M* $(\lambda x.\ p\ (f\ x)\ (g\ x))$
by (*rule simple_function_compose2*)

```
  from sf have [measurable]:
      f ∈ measurable M (count_space UNIV)
      g ∈ measurable M (count_space UNIV)
    by (auto intro: measurable_simple_function)

  assume fin: emeasure M {y ∈ space M. f y ≠ 0} ≠ ∞ emeasure M {y ∈ space
M. g y ≠ 0} ≠ ∞

  have emeasure M {x∈space M. p (f x) (g x) ≠ 0} ≤
      emeasure M ({x∈space M. f x ≠ 0} ∪ {x∈space M. g x ≠ 0})
    by (intro emeasure_mono) (auto simp: p_0)
  also have ... ≤ emeasure M {x∈space M. f x ≠ 0} + emeasure M {x∈space
M. g x ≠ 0}
    by (intro emeasure_subadditive) auto
  finally show emeasure M {y ∈ space M. p (f y) (g y) ≠ 0} ≠ ∞
    using fin by (auto simp: top_unique)
qed

lemma simple_function_finite_support:
  assumes f: simple_function M f and fin: (∫+x. f x ∂M) < ∞ and nn: ⋀x. 0
≤ f x
  shows emeasure M {x∈space M. f x ≠ 0} ≠ ∞
proof cases
  from f have meas[measurable]: f ∈ borel_measurable M
    by (rule borel_measurable_simple_function)

  assume non_empty: ∃x∈space M. f x ≠ 0

  define m where m = Min (f`space M − {0})
  have m ∈ f`space M − {0}
      unfolding m_def using f non_empty by (intro Min_in) (auto simp: sim-
ple_function_def)
  then have m: 0 < m
    using nn by (auto simp: less_le)

  from m have m ∗ emeasure M {x∈space M. 0 ≠ f x} =
    (∫+x. m ∗ indicator {x∈space M. 0 ≠ f x} x ∂M)
    using f by (intro nn_integral_cmult_indicator[symmetric]) auto
  also have ... ≤ (∫+x. f x ∂M)
    using AE_space
  proof (intro nn_integral_mono_AE, eventually_elim)
    fix x assume x ∈ space M
    with nn show m ∗ indicator {x ∈ space M. 0 ≠ f x} x ≤ f x
      using f by (auto split: split_indicator simp: simple_function_def m_def)
  qed
  also note ‹... < ∞›
  finally show ?thesis
    using m by (auto simp: ennreal_mult_less_top)
next
```

assume ¬ (∃ *x*∈*space M*. *f x* ≠ *0*)
with *nn* **have** ∗: {*x*∈*space M*. *f x* ≠ *0*} = {}
by *auto*
show *?thesis* **unfolding** ∗ **by** *simp*
qed

lemma *simple_bochner_integrableI_bounded*:
assumes *f*: *simple_function M f* **and** *fin*: (∫ ⁺*x*. *norm* (*f x*) ∂*M*) < ∞
shows *simple_bochner_integrable M f*
proof
have *emeasure M* {*y* ∈ *space M*. *ennreal* (*norm* (*f y*)) ≠ *0*} ≠ ∞
proof (*rule simple_function_finite_support*)
show *simple_function M* (λ*x*. *ennreal* (*norm* (*f x*)))
using *f* **by** (*rule simple_function_compose1*)
show (∫ ⁺ *y*. *ennreal* (*norm* (*f y*)) ∂*M*) < ∞ **by** *fact*
qed *simp*
then show *emeasure M* {*y* ∈ *space M*. *f y* ≠ *0*} ≠ ∞ **by** *simp*
qed *fact*

definition *simple_bochner_integral* :: ′*a measure* ⇒ (′*a* ⇒ ′*b*::*real_vector*) ⇒ ′*b*
where
simple_bochner_integral M f = (∑ *y*∈*f*'*space M*. *measure M* {*x*∈*space M*. *f x* = *y*} ∗*R* *y*)

proposition *simple_bochner_integral_partition*:
assumes *f*: *simple_bochner_integrable M f* **and** *g*: *simple_function M g*
assumes *sub*: ⋀*x y*. *x* ∈ *space M* ⟹ *y* ∈ *space M* ⟹ *g x* = *g y* ⟹ *f x* = *f y*
assumes *v*: ⋀*x*. *x* ∈ *space M* ⟹ *f x* = *v* (*g x*)
shows *simple_bochner_integral M f* = (∑ *y*∈*g* ' *space M*. *measure M* {*x*∈*space M*. *g x* = *y*} ∗*R* *v y*)
(**is** _ = *?r*)
proof −
from *f g* **have** [*simp*]: *finite* (*f*'*space M*) *finite* (*g*'*space M*)
by (*auto simp*: *simple_function_def elim*: *simple_bochner_integrable.cases*)

from *f* **have** [*measurable*]: *f* ∈ *measurable M* (*count_space UNIV*)
by (*auto intro*: *measurable_simple_function elim*: *simple_bochner_integrable.cases*)

from *g* **have** [*measurable*]: *g* ∈ *measurable M* (*count_space UNIV*)
by (*auto intro*: *measurable_simple_function elim*: *simple_bochner_integrable.cases*)

{ fix *y* **assume** *y* ∈ *space M*
then have *f* ' *space M* ∩ {*i*. ∃ *x*∈*space M*. *i* = *f x* ∧ *g y* = *g x*} = {*v* (*g y*)}
by (*auto cong*: *sub simp*: *v*[*symmetric*]) **}**
note *eq* = *this*

have *simple_bochner_integral M f* =
(∑ *y*∈*f*'*space M*. (∑ *z*∈*g*'*space M*.
if ∃ *x*∈*space M*. *y* = *f x* ∧ *z* = *g x then measure M* {*x*∈*space M*. *g x* = *z*}

else 0) $*_R$ *y*)
  unfolding *simple_bochner_integral_def*
 proof (*safe intro*!: *sum.cong scaleR_cong_right*)
  fix *y* **assume** *y*: *y* ∈ *space M f y* ≠ *0*
  have [*simp*]: *g* ‘ *space M* ∩ {*z*. ∃ *x*∈*space M*. *f y* = *f x* ∧ *z* = *g x*} =
    {*z*. ∃ *x*∈*space M*. *f y* = *f x* ∧ *z* = *g x*}
   by *auto*
  have *eq*:{*x* ∈ *space M*. *f x* = *f y*} =
    (⋃ *i*∈{*z*. ∃ *x*∈*space M*. *f y* = *f x* ∧ *z* = *g x*}. {*x* ∈ *space M*. *g x* = *i*})
   by (*auto simp*: *eq_commute cong*: *sub rev_conj_cong*)
  have *finite* (*g*‘*space M*) **by** *simp*
  then have *finite* {*z*. ∃ *x*∈*space M*. *f y* = *f x* ∧ *z* = *g x*}
   by (*rule rev_finite_subset*) *auto*
  moreover
  { **fix** *x* **assume** *x* ∈ *space M f x* = *f y*
   then have *x* ∈ *space M f x* ≠ *0*
    using *y* **by** *auto*
   then have *emeasure M* {*y* ∈ *space M*. *g y* = *g x*} ≤ *emeasure M* {*y* ∈ *space M*. *f y* ≠ *0*}
    by (*auto intro*!: *emeasure_mono cong*: *sub*)
   then have *emeasure M* {*xa* ∈ *space M*. *g xa* = *g x*} < ∞
    using *f* **by** (*auto simp*: *simple_bochner_integrable.simps less_top*) }
  ultimately
  show *measure M* {*x* ∈ *space M*. *f x* = *f y*} =
   (∑ *z*∈*g* ‘ *space M*. *if* ∃ *x*∈*space M*. *f y* = *f x* ∧ *z* = *g x then measure M* {*x* ∈ *space M*. *g x* = *z*} *else 0*)
   apply (*simp add*: *sum.If_cases eq*)
   apply (*subst measure_finite_Union*[*symmetric*])
   apply (*auto simp*: *disjoint_family_on_def less_top*)
   done
 qed
 also have . . . = (∑ *y*∈*f*‘*space M*. (∑ *z*∈*g*‘*space M*.
   if ∃ *x*∈*space M*. *y* = *f x* ∧ *z* = *g x then measure M* {*x*∈*space M*. *g x* = *z*} $*_R$ *y else 0*))
  by (*auto intro*!: *sum.cong simp*: *scaleR_sum_left*)
 also have . . . = *?r*
  by (*subst sum.swap*)
   (*auto intro*!: *sum.cong simp*: *sum.If_cases scaleR_sum_right*[*symmetric*] *eq*)
 finally show *simple_bochner_integral M f* = *?r* **.**
qed

lemma *simple_bochner_integral_add*:
 assumes *f*: *simple_bochner_integrable M f* **and** *g*: *simple_bochner_integrable M g*
 shows *simple_bochner_integral M* (λ*x*. *f x* + *g x*) =
  simple_bochner_integral M f + *simple_bochner_integral M g*
proof −
 from *f g* **have** *simple_bochner_integral M* (λ*x*. *f x* + *g x*) =
  (∑ *y*∈(λ*x*. (*f x*, *g x*)) ‘ *space M*. *measure M* {*x* ∈ *space M*. (*f x*, *g x*) = *y*} $*_R$ (*fst y* + *snd y*))

by (*intro simple_bochner_integral_partition*)
(*auto simp*: *simple_bochner_integrable_compose2 elim*: *simple_bochner_integrable.cases*)
moreover from *f g* **have** *simple_bochner_integral M f* =
($\sum$ *y*∈(λ*x*. (*f x*, *g x*)) ‘ *space M*. *measure M* {*x* ∈ *space M*. (*f x*, *g x*) = *y*} $*_R$
fst y)
by (*intro simple_bochner_integral_partition*)
(*auto simp*: *simple_bochner_integrable_compose2 elim*: *simple_bochner_integrable.cases*)
moreover from *f g* **have** *simple_bochner_integral M g* =
($\sum$ *y*∈(λ*x*. (*f x*, *g x*)) ‘ *space M*. *measure M* {*x* ∈ *space M*. (*f x*, *g x*) = *y*} $*_R$
snd y)
by (*intro simple_bochner_integral_partition*)
(*auto simp*: *simple_bochner_integrable_compose2 elim*: *simple_bochner_integrable.cases*)
ultimately show *?thesis*
by (*simp add*: *sum.distrib*[*symmetric*] *scaleR_add_right*)
qed

lemma *simple_bochner_integral_linear*:
assumes *linear f*
assumes *g*: *simple_bochner_integrable M g*
shows *simple_bochner_integral M* (λ*x*. *f* (*g x*)) = *f* (*simple_bochner_integral M*
g)
proof −
interpret *linear f* **by** *fact*
from *g* **have** *simple_bochner_integral M* (λ*x*. *f* (*g x*)) =
($\sum$ *y*∈*g* ‘ *space M*. *measure M* {*x* ∈ *space M*. *g x* = *y*} $*_R$ *f y*)
by (*intro simple_bochner_integral_partition*)
(*auto simp*: *simple_bochner_integrable_compose2*[**where** *p*=λ*x y*. *f x*]
elim: *simple_bochner_integrable.cases*)
also have . . . = *f* (*simple_bochner_integral M g*)
by (*simp add*: *simple_bochner_integral_def sum scale*)
finally show *?thesis* **.**
qed

lemma *simple_bochner_integral_minus*:
assumes *f*: *simple_bochner_integrable M f*
shows *simple_bochner_integral M* (λ*x*. − *f x*) = − *simple_bochner_integral M f*
proof −
from *linear_uminus f* **show** *?thesis*
by (*rule simple_bochner_integral_linear*)
qed

lemma *simple_bochner_integral_diff*:
assumes *f*: *simple_bochner_integrable M f* **and** *g*: *simple_bochner_integrable M g*
shows *simple_bochner_integral M* (λ*x*. *f x* − *g x*) =
simple_bochner_integral M f − *simple_bochner_integral M g*
unfolding *diff_conv_add_uminus* **using** *f g*
by (*subst simple_bochner_integral_add*)
(*auto simp*: *simple_bochner_integral_minus simple_bochner_integrable_compose2*[**where**
p=λ*x y*. − *y*])

lemma *simple_bochner_integral_norm_bound*:
assumes *f*: *simple_bochner_integrable M f*
shows *norm* (*simple_bochner_integral M f*) ≤ *simple_bochner_integral M* (λ*x*. *norm* (*f x*))
proof −
have *norm* (*simple_bochner_integral M f*) ≤
(∑ *y*∈*f* ' *space M*. *norm* (*measure M* {*x* ∈ *space M*. *f x* = *y*} $*_R$ *y*))
unfolding *simple_bochner_integral_def* **by** (*rule norm_sum*)
also have ... = (∑ *y*∈*f* ' *space M*. *measure M* {*x* ∈ *space M*. *f x* = *y*} $*_R$ *norm y*)
by *simp*
also have ... = *simple_bochner_integral M* (λ*x*. *norm* (*f x*))
using *f*
by (*intro simple_bochner_integral_partition*[*symmetric*])
(*auto intro*: *f simple_bochner_integrable_compose2 elim*: *simple_bochner_integrable.cases*)
finally show *?thesis* .
qed

lemma *simple_bochner_integral_nonneg*[*simp*]:
fixes *f* :: *'a* ⇒ *real*
shows (⋀*x*. *0* ≤ *f x*) ⟹ *0* ≤ *simple_bochner_integral M f*
by (*force simp add*: *simple_bochner_integral_def intro*: *sum_nonneg*)

lemma *simple_bochner_integral_eq_nn_integral*:
assumes *f*: *simple_bochner_integrable M f* ⋀*x*. *0* ≤ *f x*
shows *simple_bochner_integral M f* = (∫⁺*x*. *f x* ∂*M*)
proof −
{ fix *x y z* **have** (*x* ≠ *0* ⟹ *y* = *z*) ⟹ *ennreal x* ∗ *y* = *ennreal x* ∗ *z*
by (*cases x* = *0*) (*auto simp*: *zero_ennreal_def*[*symmetric*]) **}**
note *ennreal_cong_mult* = *this*

have [*measurable*]: *f* ∈ *borel_measurable M*
using *f*(*1*) **by** (*auto intro*: *borel_measurable_simple_function elim*: *simple_bochner_integrable.cases*)

{ fix *y* **assume** *y*: *y* ∈ *space M f y* ≠ *0*
have *ennreal* (*measure M* {*x* ∈ *space M*. *f x* = *f y*}) = *emeasure M* {*x* ∈ *space M*. *f x* = *f y*}
proof (*rule emeasure_eq_ennreal_measure*[*symmetric*])
have *emeasure M* {*x* ∈ *space M*. *f x* = *f y*} ≤ *emeasure M* {*x* ∈ *space M*. *f x* ≠ *0*}
using *y* **by** (*intro emeasure_mono*) *auto*
with *f* **show** *emeasure M* {*x* ∈ *space M*. *f x* = *f y*} ≠ *top*
by (*auto simp*: *simple_bochner_integrable.simps top_unique*)
qed
moreover have {*x* ∈ *space M*. *f x* = *f y*} = (λ*x*. *ennreal* (*f x*)) −' {*ennreal* (*f y*)} ∩ *space M*
using *f* **by** *auto*
ultimately have *ennreal* (*measure M* {*x* ∈ *space M*. *f x* = *f y*}) =

emeasure M ((*λx. ennreal* (*f x*)) −‘ {*ennreal* (*f y*)} ∩ *space M*) **by** *simp* **}**
with *f* **have** *simple_bochner_integral M f* = (∫ *^S^x. f x ∂M*)
unfolding *simple_integral_def*
by (*subst simple_bochner_integral_partition*[*OF f*(*1*), **where** *g*=*λx. ennreal* (*f x*) **and** *v*=*enn2real*])
(*auto intro*: *f simple_function_compose1 elim*: *simple_bochner_integrable.cases*
intro!: *sum.cong ennreal_cong_mult*
simp: *ac_simps ennreal_mult*
simp flip: *sum_ennreal*)
also have ... = (∫ *^+^x. f x ∂M*)
using *f*
by (*intro nn_integral_eq_simple_integral*[*symmetric*])
(*auto simp*: *simple_function_compose1 simple_bochner_integrable.simps*)
finally show *?thesis* **.**
qed

lemma *simple_bochner_integral_bounded*:
fixes *f* :: *'a* ⇒ *'b*::{*real_normed_vector*, *second_countable_topology*}
assumes *f*[*measurable*]: *f* ∈ *borel_measurable M*
assumes *s*: *simple_bochner_integrable M s* **and** *t*: *simple_bochner_integrable M t*
shows *ennreal* (*norm* (*simple_bochner_integral M s* − *simple_bochner_integral M t*)) ≤
(∫ *^+^ x. norm* (*f x* − *s x*) *∂M*) + (∫ *^+^ x. norm* (*f x* − *t x*) *∂M*)
(**is** *ennreal* (*norm* (*?s* − *?t*)) ≤ *?S* + *?T*)
proof −
have [*measurable*]: *s* ∈ *borel_measurable M t* ∈ *borel_measurable M*
using *s t* **by** (*auto intro*: *borel_measurable_simple_function elim*: *simple_bochner_integrable.cases*)

have *ennreal* (*norm* (*?s* − *?t*)) = *norm* (*simple_bochner_integral M* (*λx. s x* − *t x*))
using *s t* **by** (*subst simple_bochner_integral_diff*) *auto*
also have ... ≤ *simple_bochner_integral M* (*λx. norm* (*s x* − *t x*))
using *simple_bochner_integrable_compose2*[*of* (−) *M s t*] *s t*
by (*auto intro*!: *simple_bochner_integral_norm_bound*)
also have ... = (∫ *^+^x. norm* (*s x* − *t x*) *∂M*)
using *simple_bochner_integrable_compose2*[*of λx y. norm* (*x* − *y*) *M s t*] *s t*
by (*auto intro*!: *simple_bochner_integral_eq_nn_integral*)
also have ... ≤ (∫ *^+^x. ennreal* (*norm* (*f x* − *s x*)) + *ennreal* (*norm* (*f x* − *t x*)) *∂M*)
by (*auto intro*!: *nn_integral_mono simp flip*: *ennreal_plus*)
(*metis* (*erased*, *hide_lams*) *add_diff_cancel_left add_diff_eq diff_add_eq order_trans*
norm_minus_commute norm_triangle_ineq4 order_refl)
also have ... = *?S* + *?T*
by (*rule nn_integral_add*) *auto*
finally show *?thesis* **.**
qed

inductive *has_bochner_integral* :: *'a measure* ⇒ (*'a* ⇒ *'b*) ⇒ *'b*::{*real_normed_vector*,

second_countable_topology} ⇒ *bool*
for *M f x* **where**
f ∈ *borel_measurable M* ⟹
(⋀*i*. *simple_bochner_integrable M* (*s i*)) ⟹
(λ*i*. ∫⁺*x*. *norm* (*f x* − *s i x*) ∂*M*) ⟶ *0* ⟹
(λ*i*. *simple_bochner_integral M* (*s i*)) ⟶ *x* ⟹
has_bochner_integral M f x

lemma *has_bochner_integral_cong*:
assumes *M* = *N* ⋀*x*. *x* ∈ *space N* ⟹ *f x* = *g x x* = *y*
shows *has_bochner_integral M f x* ⟷ *has_bochner_integral N g y*
unfolding *has_bochner_integral.simps assms*(*1*,*3*)
using *assms*(*2*) **by** (*simp cong*: *measurable_cong_simp nn_integral_cong_simp*)

lemma *has_bochner_integral_cong_AE*:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ (*AE x in M*. *f x* = *g x*) ⟹
has_bochner_integral M f x ⟷ *has_bochner_integral M g x*
unfolding *has_bochner_integral.simps*
by (*intro arg_cong*[**where** *f*=*Ex*] *ext conj_cong rev_conj_cong refl arg_cong*[**where** *f*=λ*x*. *x* ⟶ *0*]
nn_integral_cong_AE)
auto

lemma *borel_measurable_has_bochner_integral*:
has_bochner_integral M f x ⟹ *f* ∈ *borel_measurable M*
by (*rule has_bochner_integral.cases*)

lemma *borel_measurable_has_bochner_integral'*[*measurable_dest*]:
has_bochner_integral M f x ⟹ *g* ∈ *measurable N M* ⟹ (λ*x*. *f* (*g x*)) ∈ *borel_measurable N*
using *borel_measurable_has_bochner_integral*[*measurable*] **by** *measurable*

lemma *has_bochner_integral_simple_bochner_integrable*:
simple_bochner_integrable M f ⟹ *has_bochner_integral M f* (*simple_bochner_integral M f*)
by (*rule has_bochner_integral.intros*[**where** *s*=λ_. *f*])
(*auto intro*: *borel_measurable_simple_function*
elim: *simple_bochner_integrable.cases*
simp: *zero_ennreal_def*[*symmetric*])

lemma *has_bochner_integral_real_indicator*:
assumes [*measurable*]: *A* ∈ *sets M* **and** *A*: *emeasure M A* < ∞
shows *has_bochner_integral M* (*indicator A*) (*measure M A*)
proof −
have *sbi*: *simple_bochner_integrable M* (*indicator A*::'*a* ⇒ *real*)
proof
have {*y* ∈ *space M*. (*indicator A y*::*real*) ≠ *0*} = *A*
using *sets.sets_into_space*[*OF* ‹*A*∈*sets M*›] **by** (*auto split*: *split_indicator*)

then show emeasure M {y ∈ space M. (indicator A y::real) ≠ 0} ≠ ∞
using A **by** auto
qed (rule simple_function_indicator assms)+
moreover have simple_bochner_integral M (indicator A) = measure M A
using simple_bochner_integral_eq_nn_integral[OF sbi] A
by (simp add: ennreal_indicator emeasure_eq_ennreal_measure)
ultimately show ?thesis
by (metis has_bochner_integral_simple_bochner_integrable)
qed

lemma has_bochner_integral_add[intro]:
has_bochner_integral M f x ⟹ has_bochner_integral M g y ⟹
has_bochner_integral M (λx. f x + g x) (x + y)
proof (safe intro!: has_bochner_integral.intros elim!: has_bochner_integral.cases)
fix sf sg
assume f_sf: (λi. ∫⁺ x. norm (f x − sf i x) ∂M) ⟶ 0
assume g_sg: (λi. ∫⁺ x. norm (g x − sg i x) ∂M) ⟶ 0

assume sf: ∀ i. simple_bochner_integrable M (sf i)
and sg: ∀ i. simple_bochner_integrable M (sg i)
then have [measurable]: ⋀i. sf i ∈ borel_measurable M ⋀i. sg i ∈ borel_measurable M
by (auto intro: borel_measurable_simple_function elim: simple_bochner_integrable.cases)
assume [measurable]: f ∈ borel_measurable M g ∈ borel_measurable M

show ⋀i. simple_bochner_integrable M (λx. sf i x + sg i x)
using sf sg **by** (simp add: simple_bochner_integrable_compose2)

show (λi. ∫⁺ x. (norm (f x + g x − (sf i x + sg i x))) ∂M) ⟶ 0
(**is** ?f ⟶ 0)
proof (rule tendsto_sandwich)
show eventually (λn. 0 ≤ ?f n) sequentially (λ_. 0) ⟶ 0
by auto
show eventually (λi. ?f i ≤ (∫⁺ x. (norm (f x − sf i x)) ∂M) + ∫⁺ x. (norm (g x − sg i x)) ∂M) sequentially
(**is** eventually (λi. ?f i ≤ ?g i) sequentially)
proof (intro always_eventually allI)
fix i **have** ?f i ≤ (∫⁺ x. (norm (f x − sf i x)) + ennreal (norm (g x − sg i x)) ∂M)
by (auto intro!: nn_integral_mono norm_diff_triangle_ineq
simp flip: ennreal_plus)
also have … = ?g i
by (intro nn_integral_add) auto
finally show ?f i ≤ ?g i **.**
qed
show ?g ⟶ 0
using tendsto_add[OF f_sf g_sg] **by** simp
qed
qed (auto simp: simple_bochner_integral_add tendsto_add)

```
lemma has_bochner_integral_bounded_linear:
  assumes bounded_linear T
  shows has_bochner_integral M f x ⟹ has_bochner_integral M (λx. T (f x)) (T
x)
proof (safe intro!: has_bochner_integral.intros elim!: has_bochner_integral.cases)
  interpret T: bounded_linear T by fact
  have [measurable]: T ∈ borel_measurable borel
    by (intro borel_measurable_continuous_onI T.continuous_on continuous_on_id)
  assume [measurable]: f ∈ borel_measurable M
  then show (λx. T (f x)) ∈ borel_measurable M
    by auto

  fix s assume f_s: (λi. ∫+ x. norm (f x − s i x) ∂M) ⟶ 0
  assume s: ∀ i. simple_bochner_integrable M (s i)
  then show ⋀i. simple_bochner_integrable M (λx. T (s i x))
    by (auto intro: simple_bochner_integrable_compose2 T.zero)

  have [measurable]: ⋀i. s i ∈ borel_measurable M
   using s by (auto intro: borel_measurable_simple_function elim: simple_bochner_integrable.cases)

  obtain K where K: K > 0 ⋀x i. norm (T (f x) − T (s i x)) ≤ norm (f x −
s i x) ∗ K
    using T.pos_bounded by (auto simp: T.diff[symmetric])

  show (λi. ∫+ x. norm (T (f x) − T (s i x)) ∂M) ⟶ 0
    (is ?f ⟶ 0)
  proof (rule tendsto_sandwich)
    show eventually (λn. 0 ≤ ?f n) sequentially (λ_. 0) ⟶ 0
      by auto

    show eventually (λi. ?f i ≤ K ∗ (∫+ x. norm (f x − s i x) ∂M)) sequentially
      (is eventually (λi. ?f i ≤ ?g i) sequentially)
    proof (intro always_eventually allI)
      fix i have ?f i ≤ (∫+ x. ennreal K ∗ norm (f x − s i x) ∂M)
      using K by (intro nn_integral_mono) (auto simp: ac_simps ennreal_mult[symmetric])
      also have ... = ?g i
        using K by (intro nn_integral_cmult) auto
      finally show ?f i ≤ ?g i .
    qed
    show ?g ⟶ 0
      using ennreal_tendsto_cmult[OF _ f_s] by simp
  qed

  assume (λi. simple_bochner_integral M (s i)) ⟶ x
  with s show (λi. simple_bochner_integral M (λx. T (s i x))) ⟶ T x
   by (auto intro!: T.tendsto simp: simple_bochner_integral_linear T.linear_axioms)
qed
```

lemma *has_bochner_integral_zero*[*intro*]: *has_bochner_integral M* (λ*x*. *0*) *0*
by (*auto intro*!: *has_bochner_integral.intros*[**where** *s*=λ_ _. *0*]
simp: *zero_ennreal_def*[*symmetric*] *simple_bochner_integrable.simps*
simple_bochner_integral_def image_constant_conv)

lemma *has_bochner_integral_scaleR_left*[*intro*]:
(*c* ≠ *0* ⟹ *has_bochner_integral M f x*) ⟹ *has_bochner_integral M* (λ*x*. *f x* $*_R$ *c*) (*x* $*_R$ *c*)
by (*cases c* = *0*) (*auto simp add*: *has_bochner_integral_bounded_linear*[*OF bounded_linear_scaleR_left*])

lemma *has_bochner_integral_scaleR_right*[*intro*]:
(*c* ≠ *0* ⟹ *has_bochner_integral M f x*) ⟹ *has_bochner_integral M* (λ*x*. *c* $*_R$ *f x*) (*c* $*_R$ *x*)
by (*cases c* = *0*) (*auto simp add*: *has_bochner_integral_bounded_linear*[*OF bounded_linear_scaleR_right*])

lemma *has_bochner_integral_mult_left*[*intro*]:
fixes *c* :: _::{*real_normed_algebra*,*second_countable_topology*}
shows (*c* ≠ *0* ⟹ *has_bochner_integral M f x*) ⟹ *has_bochner_integral M* (λ*x*. *f x* ∗ *c*) (*x* ∗ *c*)
by (*cases c* = *0*) (*auto simp add*: *has_bochner_integral_bounded_linear*[*OF bounded_linear_mult_left*])

lemma *has_bochner_integral_mult_right*[*intro*]:
fixes *c* :: _::{*real_normed_algebra*,*second_countable_topology*}
shows (*c* ≠ *0* ⟹ *has_bochner_integral M f x*) ⟹ *has_bochner_integral M* (λ*x*. *c* ∗ *f x*) (*c* ∗ *x*)
by (*cases c* = *0*) (*auto simp add*: *has_bochner_integral_bounded_linear*[*OF bounded_linear_mult_right*])

lemmas *has_bochner_integral_divide* =
has_bochner_integral_bounded_linear[*OF bounded_linear_divide*]

lemma *has_bochner_integral_divide_zero*[*intro*]:
fixes *c* :: _::{*real_normed_field*, *field*, *second_countable_topology*}
shows (*c* ≠ *0* ⟹ *has_bochner_integral M f x*) ⟹ *has_bochner_integral M* (λ*x*. *f x* / *c*) (*x* / *c*)
using *has_bochner_integral_divide* **by** (*cases c* = *0*) *auto*

lemma *has_bochner_integral_inner_left*[*intro*]:
(*c* ≠ *0* ⟹ *has_bochner_integral M f x*) ⟹ *has_bochner_integral M* (λ*x*. *f x* · *c*) (*x* · *c*)
by (*cases c* = *0*) (*auto simp add*: *has_bochner_integral_bounded_linear*[*OF bounded_linear_inner_left*])

lemma *has_bochner_integral_inner_right*[*intro*]:
(*c* ≠ *0* ⟹ *has_bochner_integral M f x*) ⟹ *has_bochner_integral M* (λ*x*. *c* · *f x*) (*c* · *x*)
by (*cases c* = *0*) (*auto simp add*: *has_bochner_integral_bounded_linear*[*OF bounded_linear_inner_right*])

lemmas *has_bochner_integral_minus* =
has_bochner_integral_bounded_linear[*OF bounded_linear_minus*[*OF bounded_linear_ident*]]
lemmas *has_bochner_integral_Re* =

has_bochner_integral_bounded_linear[*OF bounded_linear_Re*]
lemmas *has_bochner_integral_Im* =
has_bochner_integral_bounded_linear[*OF bounded_linear_Im*]
lemmas *has_bochner_integral_cnj* =
has_bochner_integral_bounded_linear[*OF bounded_linear_cnj*]
lemmas *has_bochner_integral_of_real* =
has_bochner_integral_bounded_linear[*OF bounded_linear_of_real*]
lemmas *has_bochner_integral_fst* =
has_bochner_integral_bounded_linear[*OF bounded_linear_fst*]
lemmas *has_bochner_integral_snd* =
has_bochner_integral_bounded_linear[*OF bounded_linear_snd*]

lemma *has_bochner_integral_indicator*:
$A \in sets\ M \Longrightarrow emeasure\ M\ A < \infty \Longrightarrow$
has_bochner_integral $M\ (\lambda x.\ indicator\ A\ x *_R c)\ (measure\ M\ A *_R c)$
by (*intro has_bochner_integral_scaleR_left has_bochner_integral_real_indicator*)

lemma *has_bochner_integral_diff*:
has_bochner_integral $M\ f\ x \Longrightarrow$ *has_bochner_integral* $M\ g\ y \Longrightarrow$
has_bochner_integral $M\ (\lambda x.\ f\ x - g\ x)\ (x - y)$
unfolding *diff_conv_add_uminus*
by (*intro has_bochner_integral_add has_bochner_integral_minus*)

lemma *has_bochner_integral_sum*:
$(\bigwedge i.\ i \in I \Longrightarrow$ *has_bochner_integral* $M\ (f\ i)\ (x\ i)) \Longrightarrow$
has_bochner_integral $M\ (\lambda x.\ \sum i{\in}I.\ f\ i\ x)\ (\sum i{\in}I.\ x\ i)$
by (*induct I rule*: *infinite_finite_induct*) *auto*

proposition *has_bochner_integral_implies_finite_norm*:
has_bochner_integral $M\ f\ x \Longrightarrow (\int^+ x.\ norm\ (f\ x)\ \partial M) < \infty$
proof (*elim has_bochner_integral.cases*)
fix *s v*
assume [*measurable*]: $f \in$ *borel_measurable* M **and** *s*: $\bigwedge i.$ *simple_bochner_integrable*
$M\ (s\ i)$ **and**
lim_0: $(\lambda i.\ \int^+ x.\ ennreal\ (norm\ (f\ x - s\ i\ x))\ \partial M) \longrightarrow 0$
from *order_tendstoD*[*OF lim_0*, *of* ∞]
obtain *i* **where** *f_s_fin*: $(\int^+ x.\ ennreal\ (norm\ (f\ x - s\ i\ x))\ \partial M) < \infty$
by (*auto simp*: *eventually_sequentially*)

have [*measurable*]: $\bigwedge i.\ s\ i \in$ *borel_measurable* M
using *s* **by** (*auto intro*: *borel_measurable_simple_function elim*: *simple_bochner_integrable.cases*)

define *m* **where** $m = (if\ space\ M = \{\}\ then\ 0\ else\ Max\ ((\lambda x.\ norm\ (s\ i\ x))\ `space$
$M))$
have *finite* $(s\ i\ `\ space\ M)$
using *s* **by** (*auto simp*: *simple_function_def simple_bochner_integrable.simps*)
then have *finite* $(norm\ `\ s\ i\ `\ space\ M)$
by (*rule finite_imageI*)
then have $\bigwedge x.\ x \in space\ M \Longrightarrow norm\ (s\ i\ x) \le m\ 0 \le m$

by (auto simp: m_def image_comp comp_def Max_ge_iff)
then have (∫⁺x. norm (s i x) ∂M) ≤ (∫⁺x. ennreal m ∗ indicator {x∈space M. s i x ≠ 0} x ∂M)
by (auto split: split_indicator intro!: Max_ge nn_integral_mono simp:)
also have . . . < ∞
using s **by** (subst nn_integral_cmult_indicator) (auto simp: ‹0 ≤ m› simple_bochner_integrable.simps ennreal_mult_less_top less_top)
finally have s_fin: (∫⁺x. norm (s i x) ∂M) < ∞ **.**

have (∫⁺ x. norm (f x) ∂M) ≤ (∫⁺ x. ennreal (norm (f x − s i x)) + ennreal (norm (s i x)) ∂M)
by (auto intro!: nn_integral_mono simp flip: ennreal_plus)
(metis add.commute norm_triangle_sub)
also have . . . = (∫⁺x. norm (f x − s i x) ∂M) + (∫⁺x. norm (s i x) ∂M)
by (rule nn_integral_add) auto
also have . . . < ∞
using s_fin f_s_fin **by** auto
finally show (∫⁺ x. ennreal (norm (f x)) ∂M) < ∞ **.**
qed

proposition has_bochner_integral_norm_bound:
assumes i: has_bochner_integral M f x
shows norm x ≤ (∫⁺x. norm (f x) ∂M)
using assms **proof**
fix s **assume**
x: (λi. simple_bochner_integral M (s i)) ⟶ x (**is** ?s ⟶ x) **and**
s[simp]: ⋀i. simple_bochner_integrable M (s i) **and**
lim: (λi. ∫⁺ x. ennreal (norm (f x − s i x)) ∂M) ⟶ 0 **and**
f[measurable]: f ∈ borel_measurable M

have [measurable]: ⋀i. s i ∈ borel_measurable M
using s **by** (auto simp: simple_bochner_integrable.simps intro: borel_measurable_simple_function)

show norm x ≤ (∫⁺x. norm (f x) ∂M)
proof (rule LIMSEQ_le)
show (λi. ennreal (norm (?s i))) ⟶ norm x
using x **by** (auto simp: tendsto_ennreal_iff intro: tendsto_intros)
show ∃N. ∀n≥N. norm (?s n) ≤ (∫⁺x. norm (f x − s n x) ∂M) + (∫⁺x. norm (f x) ∂M)
(**is** ∃N. ∀n≥N. _ ≤ ?t n)
proof (intro exI allI impI)
fix n
have ennreal (norm (?s n)) ≤ simple_bochner_integral M (λx. norm (s n x))
by (auto intro!: simple_bochner_integral_norm_bound)
also have . . . = (∫⁺x. norm (s n x) ∂M)
by (intro simple_bochner_integral_eq_nn_integral)
(auto intro: s simple_bochner_integrable_compose2)
also have . . . ≤ (∫⁺x. ennreal (norm (f x − s n x)) + norm (f x) ∂M)
by (auto intro!: nn_integral_mono simp flip: ennreal_plus)

```
        (metis add.commute norm_minus_commute norm_triangle_sub)
      also have ... = ?t n
        by (rule nn_integral_add) auto
      finally show norm (?s n) ≤ ?t n .
    qed
    have ?t ⟶ 0 + (∫+ x. ennreal (norm (f x)) ∂M)
      using has_bochner_integral_implies_finite_norm[OF i]
      by (intro tendsto_add tendsto_const lim)
    then show ?t ⟶ ∫+ x. ennreal (norm (f x)) ∂M
      by simp
  qed
qed

lemma has_bochner_integral_eq:
  has_bochner_integral M f x ⟹ has_bochner_integral M f y ⟹ x = y
proof (elim has_bochner_integral.cases)
  assume f[measurable]: f ∈ borel_measurable M

  fix s t
  assume (λi. ∫+ x. norm (f x − s i x) ∂M) ⟶ 0 (is ?S ⟶ 0)
  assume (λi. ∫+ x. norm (f x − t i x) ∂M) ⟶ 0 (is ?T ⟶ 0)
  assume s: ⋀i. simple_bochner_integrable M (s i)
  assume t: ⋀i. simple_bochner_integrable M (t i)

  have [measurable]: ⋀i. s i ∈ borel_measurable M ⋀i. t i ∈ borel_measurable M
   using s t by (auto intro: borel_measurable_simple_function elim: simple_bochner_integrable.cases)

  let ?s = λi. simple_bochner_integral M (s i)
  let ?t = λi. simple_bochner_integral M (t i)
  assume ?s ⟶ x ?t ⟶ y
  then have (λi. norm (?s i − ?t i)) ⟶ norm (x − y)
    by (intro tendsto_intros)
  moreover
  have (λi. ennreal (norm (?s i − ?t i))) ⟶ ennreal 0
  proof (rule tendsto_sandwich)
     show eventually (λi. 0 ≤ ennreal (norm (?s i − ?t i))) sequentially (λ_. 0)
⟶ ennreal 0
      by auto

    show eventually (λi. norm (?s i − ?t i) ≤ ?S i + ?T i) sequentially
      by (intro always_eventually allI simple_bochner_integral_bounded s t f)
    show (λi. ?S i + ?T i) ⟶ ennreal 0
      using tendsto_add[OF ‹?S ⟶ 0› ‹?T ⟶ 0›] by simp
  qed
  then have (λi. norm (?s i − ?t i)) ⟶ 0
    by (simp flip: ennreal_0)
  ultimately have norm (x − y) = 0
    by (rule LIMSEQ_unique)
  then show x = y by simp
```

qed

lemma *has_bochner_integralI_AE*:
assumes *f*: *has_bochner_integral M f x*
and *g*: *g* ∈ *borel_measurable M*
and *ae*: *AE x in M. f x = g x*
shows *has_bochner_integral M g x*
using *f*
proof (*safe intro!*: *has_bochner_integral.intros elim!*: *has_bochner_integral.cases*)
fix *s* **assume** ($\lambda i.\ \int^+ x.\ ennreal\ (norm\ (f\ x - s\ i\ x))\ \partial M) \longrightarrow 0$
also have ($\lambda i.\ \int^+ x.\ ennreal\ (norm\ (f\ x - s\ i\ x))\ \partial M) = (\lambda i.\ \int^+ x.\ ennreal\ (norm\ (g\ x - s\ i\ x))\ \partial M)$
using *ae*
by (*intro ext nn_integral_cong_AE*, *eventually_elim*) *simp*
finally show ($\lambda i.\ \int^+ x.\ ennreal\ (norm\ (g\ x - s\ i\ x))\ \partial M) \longrightarrow 0$.
qed (*auto intro*: *g*)

lemma *has_bochner_integral_eq_AE*:
assumes *f*: *has_bochner_integral M f x*
and *g*: *has_bochner_integral M g y*
and *ae*: *AE x in M. f x = g x*
shows *x = y*
proof −
from *assms* **have** *has_bochner_integral M g x*
by (*auto intro*: *has_bochner_integralI_AE*)
from *this g* **show** *x = y*
by (*rule has_bochner_integral_eq*)
qed

lemma *simple_bochner_integrable_restrict_space*:
fixes *f* :: _ ⇒ *'b::real_normed_vector*
assumes Ω: Ω ∩ *space M* ∈ *sets M*
shows *simple_bochner_integrable* (*restrict_space M* Ω) *f* ⟷
simple_bochner_integrable M (λ*x*. *indicator* Ω *x* $*_R$ *f x*)
by (*simp add*: *simple_bochner_integrable.simps space_restrict_space*
simple_function_restrict_space[*OF* Ω] *emeasure_restrict_space*[*OF* Ω] *Collect_restrict*
indicator_eq_0_iff conj_left_commute)

lemma *simple_bochner_integral_restrict_space*:
fixes *f* :: _ ⇒ *'b::real_normed_vector*
assumes Ω: Ω ∩ *space M* ∈ *sets M*
assumes *f*: *simple_bochner_integrable* (*restrict_space M* Ω) *f*
shows *simple_bochner_integral* (*restrict_space M* Ω) *f* =
simple_bochner_integral M (λ*x*. *indicator* Ω *x* $*_R$ *f x*)
proof −
have *finite* ((λ*x*. *indicator* Ω *x* $*_R$ *f x*)‘*space M*)
using *f simple_bochner_integrable_restrict_space*[*OF* Ω, *of f*]
by (*simp add*: *simple_bochner_integrable.simps simple_function_def*)
then show *?thesis*

```
  by (auto simp: space_restrict_space measure_restrict_space[OF Ω(1)] le_infI2
          simple_bochner_integral_def Collect_restrict
        split: split_indicator split_indicator_asm
        intro!: sum.mono_neutral_cong_left arg_cong2[where f=measure])
qed

context
  notes [[inductive_internals]]
begin

inductive integrable for M f where
  has_bochner_integral M f x ⟹ integrable M f

end

definition lebesgue_integral (integral^L) where
  integral^L M f = (if ∃x. has_bochner_integral M f x then THE x. has_bochner_integral
M f x else 0)

syntax
  _lebesgue_integral :: pttrn ⇒ real ⇒ 'a measure ⇒ real (∫ ((2 _./ _)/ ∂_) [60,61]
110)

translations
  ∫ x. f ∂M == CONST lebesgue_integral M (λx. f)

syntax
  _ascii_lebesgue_integral :: pttrn ⇒ 'a measure ⇒ real ⇒ real ((3LINT (1_)/|(_)./
_) [0,110,60] 60)

translations
  LINT x|M. f == CONST lebesgue_integral M (λx. f)

lemma has_bochner_integral_integral_eq: has_bochner_integral M f x ⟹ integral^L
M f = x
  by (metis the_equality has_bochner_integral_eq lebesgue_integral_def)

lemma has_bochner_integral_integrable:
  integrable M f ⟹ has_bochner_integral M f (integral^L M f)
  by (auto simp: has_bochner_integral_integral_eq integrable.simps)

lemma has_bochner_integral_iff:
  has_bochner_integral M f x ⟷ integrable M f ∧ integral^L M f = x
  by (metis has_bochner_integral_integrable has_bochner_integral_integral_eq integrable.intros)

lemma simple_bochner_integrable_eq_integral:
  simple_bochner_integrable M f ⟹ simple_bochner_integral M f = integral^L M f
  using has_bochner_integral_simple_bochner_integrable[of M f]
  by (simp add: has_bochner_integral_integral_eq)
```

lemma *not_integrable_integral_eq*: ¬ *integrable M f* ⟹ *integral*^{L} *M f* = *0*
unfolding *integrable.simps lebesgue_integral_def* **by** (*auto intro*!: *arg_cong*[**where** *f*=*The*])

lemma *integral_eq_cases*:
integrable M f ⟷ *integrable N g* ⟹
(*integrable M f* ⟹ *integrable N g* ⟹ *integral*^{L} *M f* = *integral*^{L} *N g*) ⟹
integral^L *M f* = *integral*^{L} *N g*
by (*metis not_integrable_integral_eq*)

lemma *borel_measurable_integrable*[*measurable_dest*]: *integrable M f* ⟹ *f* ∈ *borel_measurable M*
by (*auto elim*: *integrable.cases has_bochner_integral.cases*)

lemma *borel_measurable_integrable'*[*measurable_dest*]:
integrable M f ⟹ *g* ∈ *measurable N M* ⟹ (λ*x*. *f* (*g x*)) ∈ *borel_measurable N*
using *borel_measurable_integrable*[*measurable*] **by** *measurable*

lemma *integrable_cong*:
M = *N* ⟹ (⋀*x*. *x* ∈ *space N* ⟹ *f x* = *g x*) ⟹ *integrable M f* ⟷ *integrable N g*
by (*simp cong*: *has_bochner_integral_cong add*: *integrable.simps*)

lemma *integrable_cong_AE*:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ *AE x in M*. *f x* = *g x* ⟹
integrable M f ⟷ *integrable M g*
unfolding *integrable.simps*
by (*intro has_bochner_integral_cong_AE arg_cong*[**where** *f*=*Ex*] *ext*)

lemma *integrable_cong_AE_imp*:
integrable M g ⟹ *f* ∈ *borel_measurable M* ⟹ (*AE x in M*. *g x* = *f x*) ⟹ *integrable M f*
using *integrable_cong_AE*[*of f M g*] **by** (*auto simp*: *eq_commute*)

lemma *integral_cong*:
M = *N* ⟹ (⋀*x*. *x* ∈ *space N* ⟹ *f x* = *g x*) ⟹ *integral*^{L} *M f* = *integral*^{L} *N g*
by (*simp cong*: *has_bochner_integral_cong cong del*: *if_weak_cong add*: *lebesgue_integral_def*)

lemma *integral_cong_AE*:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ *AE x in M*. *f x* = *g x* ⟹
integral^L *M f* = *integral*^{L} *M g*
unfolding *lebesgue_integral_def*
by (*rule arg_cong*[**where** *x*=*has_bochner_integral M f*]) (*intro has_bochner_integral_cong_AE ext*)

lemma *integrable_add*[*simp*, *intro*]: *integrable M f* $\Longrightarrow$ *integrable M g* $\Longrightarrow$ *integrable M* (λx. *f x* + *g x*)
  by (*auto simp*: *integrable.simps*)

lemma *integrable_zero*[*simp*, *intro*]: *integrable M* (λx. *0*)
  by (*metis has_bochner_integral_zero integrable.simps*)

lemma *integrable_sum*[*simp*, *intro*]: ($\bigwedge i$. *i* $\in$ *I* $\Longrightarrow$ *integrable M* (*f i*)) $\Longrightarrow$ *integrable M* (λx. $\sum$*i*$\in$*I*. *f i x*)
  by (*metis has_bochner_integral_sum integrable.simps*)

lemma *integrable_indicator*[*simp*, *intro*]: *A* $\in$ *sets M* $\Longrightarrow$ *emeasure M A* $< \infty \Longrightarrow$
  integrable M (λx. *indicator A x* $*_R$ *c*)
  by (*metis has_bochner_integral_indicator integrable.simps*)

lemma *integrable_real_indicator*[*simp*, *intro*]: *A* $\in$ *sets M* $\Longrightarrow$ *emeasure M A* $< \infty$
$\Longrightarrow$
  integrable M (*indicator A* :: *'a* $\Rightarrow$ *real*)
  by (*metis has_bochner_integral_real_indicator integrable.simps*)

lemma *integrable_diff*[*simp*, *intro*]: *integrable M f* $\Longrightarrow$ *integrable M g* $\Longrightarrow$ *integrable M* (λx. *f x* $-$ *g x*)
  by (*auto simp*: *integrable.simps intro*: *has_bochner_integral_diff*)

lemma *integrable_bounded_linear*: *bounded_linear T* $\Longrightarrow$ *integrable M f* $\Longrightarrow$ *integrable M* (λx. *T* (*f x*))
  by (*auto simp*: *integrable.simps intro*: *has_bochner_integral_bounded_linear*)

lemma *integrable_scaleR_left*[*simp*, *intro*]: (*c* $\neq$ *0* $\Longrightarrow$ *integrable M f*) $\Longrightarrow$ *integrable M* (λx. *f x* $*_R$ *c*)
  unfolding *integrable.simps* **by** *fastforce*

lemma *integrable_scaleR_right*[*simp*, *intro*]: (*c* $\neq$ *0* $\Longrightarrow$ *integrable M f*) $\Longrightarrow$ *integrable M* (λx. *c* $*_R$ *f x*)
  unfolding *integrable.simps* **by** *fastforce*

lemma *integrable_mult_left*[*simp*, *intro*]:
  fixes *c* :: _::{*real_normed_algebra*,*second_countable_topology*}
  shows (*c* $\neq$ *0* $\Longrightarrow$ *integrable M f*) $\Longrightarrow$ *integrable M* (λx. *f x* $*$ *c*)
  unfolding *integrable.simps* **by** *fastforce*

lemma *integrable_mult_right*[*simp*, *intro*]:
  fixes *c* :: _::{*real_normed_algebra*,*second_countable_topology*}
  shows (*c* $\neq$ *0* $\Longrightarrow$ *integrable M f*) $\Longrightarrow$ *integrable M* (λx. *c* $*$ *f x*)
  unfolding *integrable.simps* **by** *fastforce*

lemma *integrable_divide_zero*[*simp*, *intro*]:
  fixes *c* :: _::{*real_normed_field*, *field*, *second_countable_topology*}
  shows (*c* $\neq$ *0* $\Longrightarrow$ *integrable M f*) $\Longrightarrow$ *integrable M* (λx. *f x* / *c*)

unfolding *integrable.simps* **by** *fastforce*

lemma *integrable_inner_left*[*simp*, *intro*]:
($c \neq 0 \Longrightarrow$ *integrable* $M\ f$) $\Longrightarrow$ *integrable* M ($\lambda x.\ f\ x \cdot c$)
unfolding *integrable.simps* **by** *fastforce*

lemma *integrable_inner_right*[*simp*, *intro*]:
($c \neq 0 \Longrightarrow$ *integrable* $M\ f$) $\Longrightarrow$ *integrable* M ($\lambda x.\ c \cdot f\ x$)
unfolding *integrable.simps* **by** *fastforce*

lemmas *integrable_minus*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_minus*[*OF bounded_linear_ident*]]
lemmas *integrable_divide*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_divide*]
lemmas *integrable_Re*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_Re*]
lemmas *integrable_Im*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_Im*]
lemmas *integrable_cnj*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_cnj*]
lemmas *integrable_of_real*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_of_real*]
lemmas *integrable_fst*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_fst*]
lemmas *integrable_snd*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_snd*]

lemma *integral_zero*[*simp*]: *integral*^{L} M ($\lambda x.\ 0$) = 0
by (*intro has_bochner_integral_integral_eq has_bochner_integral_zero*)

lemma *integral_add*[*simp*]: *integrable* $M\ f \Longrightarrow$ *integrable* $M\ g \Longrightarrow$
integral^L M ($\lambda x.\ f\ x + g\ x$) = *integral*^{L} $M\ f$ + *integral*^{L} $M\ g$
by (*intro has_bochner_integral_integral_eq has_bochner_integral_add has_bochner_integral_integrable*)

lemma *integral_diff*[*simp*]: *integrable* $M\ f \Longrightarrow$ *integrable* $M\ g \Longrightarrow$
integral^L M ($\lambda x.\ f\ x - g\ x$) = *integral*^{L} $M\ f$ − *integral*^{L} $M\ g$
by (*intro has_bochner_integral_integral_eq has_bochner_integral_diff has_bochner_integral_integrable*)

lemma *integral_sum*: ($\bigwedge i.\ i \in I \Longrightarrow$ *integrable* M ($f\ i$)) $\Longrightarrow$
integral^L M ($\lambda x.\ \sum i{\in}I.\ f\ i\ x$) = ($\sum i{\in}I.$ *integral*^{L} M ($f\ i$))
by (*intro has_bochner_integral_integral_eq has_bochner_integral_sum has_bochner_integral_integrable*)

lemma *integral_sum'*[*simp*]: ($\bigwedge i.\ i \in I$ =*simp*=> *integrable* M ($f\ i$)) $\Longrightarrow$
integral^L M ($\lambda x.\ \sum i{\in}I.\ f\ i\ x$) = ($\sum i{\in}I.$ *integral*^{L} M ($f\ i$))
unfolding *simp_implies_def* **by** (*rule integral_sum*)

lemma *integral_bounded_linear*: *bounded_linear* $T \Longrightarrow$ *integrable* $M\ f \Longrightarrow$
integral^L M ($\lambda x.\ T$ ($f\ x$)) = T (*integral*^{L} $M\ f$)
by (*metis has_bochner_integral_bounded_linear has_bochner_integral_integrable has_bochner_integral_integr

```
lemma integral_bounded_linear':
  assumes T: bounded_linear T and T': bounded_linear T'
  assumes *: ¬ (∀x. T x = 0) ⟹ (∀x. T' (T x) = x)
  shows integral^L M (λx. T (f x)) = T (integral^L M f)
proof cases
  assume (∀x. T x = 0) then show ?thesis
    by simp
next
  assume **: ¬ (∀x. T x = 0)
  show ?thesis
  proof cases
    assume integrable M f with T show ?thesis
      by (rule integral_bounded_linear)
  next
    assume not: ¬ integrable M f
    moreover have ¬ integrable M (λx. T (f x))
    proof
      assume integrable M (λx. T (f x))
      from integrable_bounded_linear[OF T' this] not *[OF **]
      show False
        by auto
    qed
    ultimately show ?thesis
      using T by (simp add: not_integrable_integral_eq linear_simps)
  qed
qed

lemma integral_scaleR_left[simp]: (c ≠ 0 ⟹ integrable M f) ⟹ (∫ x. f x *R c
∂M) = integral^L M f *R c
  by (intro has_bochner_integral_integral_eq has_bochner_integral_integrable has_bochner_integral_scaleR_left)

lemma integral_scaleR_right[simp]: (∫ x. c *R f x ∂M) = c *R integral^L M f
  by (rule integral_bounded_linear'[OF bounded_linear_scaleR_right bounded_linear_scaleR_right[of
1 / c]]) simp

lemma integral_mult_left[simp]:
  fixes c :: _::{real_normed_algebra,second_countable_topology}
  shows (c ≠ 0 ⟹ integrable M f) ⟹ (∫ x. f x * c ∂M) = integral^L M f * c
  by (intro has_bochner_integral_integral_eq has_bochner_integral_integrable has_bochner_integral_mult_left)

lemma integral_mult_right[simp]:
  fixes c :: _::{real_normed_algebra,second_countable_topology}
  shows (c ≠ 0 ⟹ integrable M f) ⟹ (∫ x. c * f x ∂M) = c * integral^L M f
  by (intro has_bochner_integral_integral_eq has_bochner_integral_integrable has_bochner_integral_mult_right)

lemma integral_mult_left_zero[simp]:
  fixes c :: _::{real_normed_field,second_countable_topology}
  shows (∫ x. f x * c ∂M) = integral^L M f * c
```

by (*rule integral_bounded_linear'*[*OF bounded_linear_mult_left bounded_linear_mult_left*[*of 1 / c*]]) *simp*

lemma *integral_mult_right_zero*[*simp*]:
fixes c :: _::{*real_normed_field*,*second_countable_topology*}
shows ($\int$ x. $c * f\ x\ \partial M$) = $c *$ *integral*^{L} $M\ f$
by (*rule integral_bounded_linear'*[*OF bounded_linear_mult_right bounded_linear_mult_right*[*of 1 / c*]]) *simp*

lemma *integral_inner_left*[*simp*]: ($c \neq 0 \Longrightarrow$ *integrable* $M\ f$) $\Longrightarrow$ ($\int$ x. $f\ x \cdot c\ \partial M$) = *integral*^{L} $M\ f \cdot c$
by (*intro has_bochner_integral_integral_eq has_bochner_integral_integrable has_bochner_integral_inner_left*)

lemma *integral_inner_right*[*simp*]: ($c \neq 0 \Longrightarrow$ *integrable* $M\ f$) $\Longrightarrow$ ($\int$ x. $c \cdot f\ x$ ∂M) = $c \cdot$ *integral*^{L} $M\ f$
by (*intro has_bochner_integral_integral_eq has_bochner_integral_integrable has_bochner_integral_inner_right*

lemma *integral_divide_zero*[*simp*]:
fixes c :: _::{*real_normed_field*, *field*, *second_countable_topology*}
shows *integral*^{L} M (λx. $f\ x$ / c) = *integral*^{L} $M\ f$ / c
by (*rule integral_bounded_linear'*[*OF bounded_linear_divide bounded_linear_mult_left*[*of c*]]) *simp*

lemma *integral_minus*[*simp*]: *integral*^{L} M (λx. $-$ $f\ x$) = $-$ *integral*^{L} $M\ f$
by (*rule integral_bounded_linear'*[*OF bounded_linear_minus*[*OF bounded_linear_ident*] *bounded_linear_minus*[*OF bounded_linear_ident*]]) *simp*

lemma *integral_complex_of_real*[*simp*]: *integral*^{L} M (λx. *complex_of_real* ($f\ x$)) = *of_real* (*integral*^{L} $M\ f$)
by (*rule integral_bounded_linear'*[*OF bounded_linear_of_real bounded_linear_Re*]) *simp*

lemma *integral_cnj*[*simp*]: *integral*^{L} M (λx. *cnj* ($f\ x$)) = *cnj* (*integral*^{L} $M\ f$)
by (*rule integral_bounded_linear'*[*OF bounded_linear_cnj bounded_linear_cnj*]) *simp*

lemmas *integral_divide*[*simp*] =
integral_bounded_linear[*OF bounded_linear_divide*]
lemmas *integral_Re*[*simp*] =
integral_bounded_linear[*OF bounded_linear_Re*]
lemmas *integral_Im*[*simp*] =
integral_bounded_linear[*OF bounded_linear_Im*]
lemmas *integral_of_real*[*simp*] =
integral_bounded_linear[*OF bounded_linear_of_real*]
lemmas *integral_fst*[*simp*] =
integral_bounded_linear[*OF bounded_linear_fst*]
lemmas *integral_snd*[*simp*] =
integral_bounded_linear[*OF bounded_linear_snd*]

lemma *integral_norm_bound_ennreal*:

```
  integrable M f ⟹ norm (integral^L M f) ≤ (∫+x. norm (f x) ∂M)
  by (metis has_bochner_integral_integrable has_bochner_integral_norm_bound)

lemma integrableI_sequence:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  assumes f[measurable]: f ∈ borel_measurable M
  assumes s: ⋀i. simple_bochner_integrable M (s i)
  assumes lim: (λi. ∫+x. norm (f x − s i x) ∂M) ⟶ 0 (is ?S ⟶ 0)
  shows integrable M f
proof −
  let ?s = λn. simple_bochner_integral M (s n)

  have ∃x. ?s ⟶ x
    unfolding convergent_eq_Cauchy
  proof (rule metric_CauchyI)
    fix e :: real assume 0 < e
    then have 0 < ennreal (e / 2) by auto
    from order_tendstoD(2)[OF lim this]
    obtain M where M: ⋀n. M ≤ n ⟹ ?S n < e / 2
      by (auto simp: eventually_sequentially)
    show ∃M. ∀m≥M. ∀n≥M. dist (?s m) (?s n) < e
    proof (intro exI allI impI)
      fix m n assume m: M ≤ m and n: M ≤ n
      have ?S n ≠ ∞
        using M[OF n] by auto
      have norm (?s n − ?s m) ≤ ?S n + ?S m
        by (intro simple_bochner_integral_bounded s f)
      also have ... < ennreal (e / 2) + e / 2
        by (intro add_strict_mono M n m)
      also have ... = e using ‹0<e› by (simp flip: ennreal_plus)
      finally show dist (?s n) (?s m) < e
        using ‹0<e› by (simp add: dist_norm ennreal_less_iff)
    qed
  qed
  then obtain x where ?s ⟶ x ..
  show ?thesis
    by (rule, rule) fact+
qed

proposition nn_integral_dominated_convergence_norm:
  fixes u' :: _ ⇒ _::{real_normed_vector, second_countable_topology}
  assumes [measurable]:
      ⋀i. u i ∈ borel_measurable M u' ∈ borel_measurable M w ∈ borel_measurable
M
    and bound: ⋀j. AE x in M. norm (u j x) ≤ w x
    and w: (∫+x. w x ∂M) < ∞
    and u': AE x in M. (λi. u i x) ⟶ u' x
  shows (λi. (∫+x. norm (u' x − u i x) ∂M)) ⟶ 0
proof −
```

have *AE x in M*. $\forall j$. *norm* (*u j x*) $\leq$ *w x*
unfolding *AE_all_countable* **by** *rule fact*
with *u′* **have** *bnd*: *AE x in M*. $\forall j$. *norm* (*u′ x* − *u j x*) $\leq$ *2* ∗ *w x*
proof (*eventually_elim*, *intro allI*)
fix *i x* **assume** (λi. *u i x*) $\longrightarrow$ *u′ x* $\forall j$. *norm* (*u j x*) $\leq$ *w x* $\forall j$. *norm* (*u j x*) $\leq$ *w x*
then have *norm* (*u′ x*) $\leq$ *w x norm* (*u i x*) $\leq$ *w x*
by (*auto intro*: *LIMSEQ_le_const2 tendsto_norm*)
then have *norm* (*u′ x*) + *norm* (*u i x*) $\leq$ *2* ∗ *w x*
by *simp*
also have *norm* (*u′ x* − *u i x*) $\leq$ *norm* (*u′ x*) + *norm* (*u i x*)
by (*rule norm_triangle_ineq4*)
finally (*xtrans*) **show** *norm* (*u′ x* − *u i x*) $\leq$ *2* ∗ *w x* **.**
qed
have *w_nonneg*: *AE x in M*. *0* $\leq$ *w x*
using *bound*[*of 0*] **by** (*auto intro*: *order_trans*[*OF norm_ge_zero*])

have (λi. ($\int^+ x$. *norm* (*u′ x* − *u i x*) ∂M)) $\longrightarrow$ ($\int^+ x$. *0* ∂M)
proof (*rule nn_integral_dominated_convergence*)
show ($\int^+ x$. *2* ∗ *w x* ∂M) < ∞
by (*rule nn_integral_mult_bounded_inf*[*OF _ w*, *of 2*]) (*insert w_nonneg*, *auto simp*: *ennreal_mult*)
show *AE x in M*. (λi. *ennreal* (*norm* (*u′ x* − *u i x*))) $\longrightarrow$ *0*
using *u′*
proof *eventually_elim*
fix *x* **assume** (λi. *u i x*) $\longrightarrow$ *u′ x*
from *tendsto_diff*[*OF tendsto_const*[*of u′ x*] *this*]
show (λi. *ennreal* (*norm* (*u′ x* − *u i x*))) $\longrightarrow$ *0*
by (*simp add*: *tendsto_norm_zero_iff flip*: *ennreal_0*)
qed
qed (*insert bnd w_nonneg*, *auto*)
then show *?thesis* **by** *simp*
qed

proposition *integrableI_bounded*:
fixes *f* :: *′a* ⇒ *′b*::{*banach*, *second_countable_topology*}
assumes *f*[*measurable*]: *f* ∈ *borel_measurable M* **and** *fin*: ($\int^+ x$. *norm* (*f x*) ∂M) < ∞
shows *integrable M f*
proof −
from *borel_measurable_implies_sequence_metric*[*OF f*, *of 0*] **obtain** *s* **where**
s: $\bigwedge i$. *simple_function M* (*s i*) **and**
pointwise: $\bigwedge x$. *x* ∈ *space M* $\Longrightarrow$ (λi. *s i x*) $\longrightarrow$ *f x* **and**
bound: $\bigwedge i\ x$. *x* ∈ *space M* $\Longrightarrow$ *norm* (*s i x*) $\leq$ *2* ∗ *norm* (*f x*)
by *simp metis*

show *?thesis*
proof (*rule integrableI_sequence*)
{ **fix** *i*

```
      have (∫+x. norm (s i x) ∂M) ≤ (∫+x. ennreal (2 * norm (f x)) ∂M)
        by (intro nn_integral_mono) (simp add: bound)
      also have ... = 2 * (∫+x. ennreal (norm (f x)) ∂M)
        by (simp add: ennreal_mult nn_integral_cmult)
      also have ... < top
        using fin by (simp add: ennreal_mult_less_top)
      finally have (∫+x. norm (s i x) ∂M) < ∞
        by simp }
    note fin_s = this

    show ⋀i. simple_bochner_integrable M (s i)
      by (rule simple_bochner_integrableI_bounded) fact+

    show (λi. ∫+ x. ennreal (norm (f x − s i x)) ∂M) ⟶ 0
    proof (rule nn_integral_dominated_convergence_norm)
      show ⋀j. AE x in M. norm (s j x) ≤ 2 * norm (f x)
        using bound by auto
      show ⋀i. s i ∈ borel_measurable M (λx. 2 * norm (f x)) ∈ borel_measurable
M
        using s by (auto intro: borel_measurable_simple_function)
      show (∫+ x. ennreal (2 * norm (f x)) ∂M) < ∞
       using fin by (simp add: nn_integral_cmult ennreal_mult ennreal_mult_less_top)
      show AE x in M. (λi. s i x) ⟶ f x
        using pointwise by auto
    qed fact
  qed fact
qed

lemma integrableI_bounded_set:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  assumes [measurable]: A ∈ sets M f ∈ borel_measurable M
  assumes finite: emeasure M A < ∞
    and bnd: AE x in M. x ∈ A ⟶ norm (f x) ≤ B
    and null: AE x in M. x ∉ A ⟶ f x = 0
  shows integrable M f
proof (rule integrableI_bounded)
  { fix x :: 'b have norm x ≤ B ⟹ 0 ≤ B
      using norm_ge_zero[of x] by arith }
  with bnd null have (∫+ x. ennreal (norm (f x)) ∂M) ≤ (∫+ x. ennreal (max
0 B) * indicator A x ∂M)
    by (intro nn_integral_mono_AE) (auto split: split_indicator split_max)
  also have ... < ∞
   using finite by (subst nn_integral_cmult_indicator) (auto simp: ennreal_mult_less_top)
  finally show (∫+ x. ennreal (norm (f x)) ∂M) < ∞ .
qed simp

lemma integrableI_bounded_set_indicator:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  shows A ∈ sets M ⟹ f ∈ borel_measurable M ⟹
```

emeasure M A $< \infty \Longrightarrow$ (*AE x in M*. $x \in A \longrightarrow$ *norm* (*f x*) $\leq B$) $\Longrightarrow$
integrable M (λx. *indicator A x* $*_R$ *f x*)
by (*rule integrableI_bounded_set*[**where** *A*=*A*]) *auto*

lemma *integrableI_nonneg*:
fixes *f* :: *'a* ⇒ *real*
assumes $f \in$ *borel_measurable M AE x in M*. $0 \leq f\,x$ ($\int^+ x.\ f\,x\ \partial M$) $< \infty$
shows *integrable M f*
proof –
have ($\int^+ x.$ *norm* (*f x*) ∂M) = ($\int^+ x.\ f\,x\ \partial M$)
using *assms* **by** (*intro nn_integral_cong_AE*) *auto*
then show *?thesis*
using *assms* **by** (*intro integrableI_bounded*) *auto*
qed

lemma *integrable_iff_bounded*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
shows *integrable M f* ⟷ $f \in$ *borel_measurable M* ∧ ($\int^+ x.$ *norm* (*f x*) ∂M) <
∞
using *integrableI_bounded*[*of f M*] *has_bochner_integral_implies_finite_norm*[*of M*
f]
unfolding *integrable.simps has_bochner_integral.simps*[*abs_def*] **by** *auto*

lemma *integrable_bound*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
and *g* :: *'a* ⇒ *'c*::{*banach*, *second_countable_topology*}
shows *integrable M f* ⟹ $g \in$ *borel_measurable M* ⟹ (*AE x in M*. *norm* (*g x*)
≤ *norm* (*f x*)) ⟹
integrable M g
unfolding *integrable_iff_bounded*
proof *safe*
assume $f \in$ *borel_measurable M* $g \in$ *borel_measurable M*
assume *AE x in M*. *norm* (*g x*) ≤ *norm* (*f x*)
then have ($\int^+ x.$ *ennreal* (*norm* (*g x*)) ∂M) ≤ ($\int^+ x.$ *ennreal* (*norm* (*f x*))
∂M)
by (*intro nn_integral_mono_AE*) *auto*
also assume ($\int^+ x.$ *ennreal* (*norm* (*f x*)) ∂M) < ∞
finally show ($\int^+ x.$ *ennreal* (*norm* (*g x*)) ∂M) < ∞ **.**
qed

lemma *integrable_mult_indicator*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
shows $A \in$ *sets M* ⟹ *integrable M f* ⟹ *integrable M* (λx. *indicator A x* $*_R$ *f*
x)
by (*rule integrable_bound*[*of M f*]) (*auto split*: *split_indicator*)

lemma *integrable_real_mult_indicator*:
fixes *f* :: *'a* ⇒ *real*
shows $A \in$ *sets M* ⟹ *integrable M f* ⟹ *integrable M* (λx. *f x* ∗ *indicator A*

x)
  using *integrable_mult_indicator*[*of A M f*] **by** (*simp add*: *mult_ac*)

lemma *integrable_abs*[*simp*, *intro*]:
  fixes *f* :: *'a* ⇒ *real*
  assumes [*measurable*]: *integrable M f* **shows** *integrable M* (λ*x*. |*f x*|)
  using *assms* **by** (*rule integrable_bound*) *auto*

lemma *integrable_norm*[*simp*, *intro*]:
  fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
  assumes [*measurable*]: *integrable M f* **shows** *integrable M* (λ*x*. *norm* (*f x*))
  using *assms* **by** (*rule integrable_bound*) *auto*

lemma *integrable_norm_cancel*:
  fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
  assumes [*measurable*]: *integrable M* (λ*x*. *norm* (*f x*)) *f* ∈ *borel_measurable M*
shows *integrable M f*
  using *assms* **by** (*rule integrable_bound*) *auto*

lemma *integrable_norm_iff*:
  fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
  shows *f* ∈ *borel_measurable M* ⟹ *integrable M* (λ*x*. *norm* (*f x*)) ⟷ *integrable M f*
  by (*auto intro*: *integrable_norm_cancel*)

lemma *integrable_abs_cancel*:
  fixes *f* :: *'a* ⇒ *real*
  assumes [*measurable*]: *integrable M* (λ*x*. |*f x*|) *f* ∈ *borel_measurable M* **shows** *integrable M f*
  using *assms* **by** (*rule integrable_bound*) *auto*

lemma *integrable_abs_iff*:
  fixes *f* :: *'a* ⇒ *real*
  shows *f* ∈ *borel_measurable M* ⟹ *integrable M* (λ*x*. |*f x*|) ⟷ *integrable M f*
  by (*auto intro*: *integrable_abs_cancel*)

lemma *integrable_max*[*simp*, *intro*]:
  fixes *f* :: *'a* ⇒ *real*
  assumes *fg*[*measurable*]: *integrable M f integrable M g*
  shows *integrable M* (λ*x*. *max* (*f x*) (*g x*))
  using *integrable_add*[*OF integrable_norm*[*OF fg*(*1*)] *integrable_norm*[*OF fg*(*2*)]]
  by (*rule integrable_bound*) *auto*

lemma *integrable_min*[*simp*, *intro*]:
  fixes *f* :: *'a* ⇒ *real*
  assumes *fg*[*measurable*]: *integrable M f integrable M g*
  shows *integrable M* (λ*x*. *min* (*f x*) (*g x*))
  using *integrable_add*[*OF integrable_norm*[*OF fg*(*1*)] *integrable_norm*[*OF fg*(*2*)]]
  by (*rule integrable_bound*) *auto*

lemma *integral_minus_iff* [*simp*]:
integrable M ($\lambda x.$ − *f x* ::′*a*::{*banach*, *second_countable_topology*}) ⟷ *integrable M f*
unfolding *integrable_iff_bounded*
by (*auto*)

lemma *integrable_indicator_iff*:
integrable M (*indicator A*::_ ⇒ *real*) ⟷ *A* ∩ *space M* ∈ *sets M* ∧ *emeasure M* (*A* ∩ *space M*) < ∞
by (*simp add*: *integrable_iff_bounded borel_measurable_indicator_iff ennreal_indicator nn_integral_indicator′*
cong: *conj_cong*)

lemma *integral_indicator*[*simp*]: *integral*^{L} *M* (*indicator A*) = *measure M* (*A* ∩ *space M*)
proof *cases*
assume ∗: *A* ∩ *space M* ∈ *sets M* ∧ *emeasure M* (*A* ∩ *space M*) < ∞
have *integral*^{L} *M* (*indicator A*) = *integral*^{L} *M* (*indicator* (*A* ∩ *space M*))
by (*intro integral_cong*) (*auto split*: *split_indicator*)
also have ... = *measure M* (*A* ∩ *space M*)
using ∗ **by** (*intro has_bochner_integral_integral_eq has_bochner_integral_real_indicator*) *auto*
finally show *?thesis* **.**
next
assume ∗: ¬ (*A* ∩ *space M* ∈ *sets M* ∧ *emeasure M* (*A* ∩ *space M*) < ∞)
have *integral*^{L} *M* (*indicator A*) = *integral*^{L} *M* (*indicator* (*A* ∩ *space M*) :: _ ⇒ *real*)
by (*intro integral_cong*) (*auto split*: *split_indicator*)
also have ... = *0*
using ∗ **by** (*subst not_integrable_integral_eq*) (*auto simp*: *integrable_indicator_iff*)
also have ... = *measure M* (*A* ∩ *space M*)
using ∗ **by** (*auto simp*: *measure_def emeasure_notin_sets not_less top_unique*)
finally show *?thesis* **.**
qed

lemma *integrable_discrete_difference*:
fixes *f* :: ′*a* ⇒ ′*b*::{*banach*, *second_countable_topology*}
assumes *X*: *countable X*
assumes *null*: ⋀*x*. *x* ∈ *X* ⟹ *emeasure M* {*x*} = *0*
assumes *sets*: ⋀*x*. *x* ∈ *X* ⟹ {*x*} ∈ *sets M*
assumes *eq*: ⋀*x*. *x* ∈ *space M* ⟹ *x* ∉ *X* ⟹ *f x* = *g x*
shows *integrable M f* ⟷ *integrable M g*
unfolding *integrable_iff_bounded*
proof (*rule conj_cong*)
{ assume *f* ∈ *borel_measurable M* **then have** *g* ∈ *borel_measurable M*
by (*rule measurable_discrete_difference*[**where** *X*=*X*]) (*auto simp*: *assms*) **}**
moreover
{ assume *g* ∈ *borel_measurable M* **then have** *f* ∈ *borel_measurable M*

```
      by (rule measurable_discrete_difference[where X=X]) (auto simp: assms) }
  ultimately show f ∈ borel_measurable M ⟷ g ∈ borel_measurable M ..
next
  have AE x in M. x ∉ X
    by (rule AE_discrete_difference) fact+
  then have (∫+ x. norm (f x) ∂M) = (∫+ x. norm (g x) ∂M)
    by (intro nn_integral_cong_AE) (auto simp: eq)
  then show (∫+ x. norm (f x) ∂M) < ∞ ⟷ (∫+ x. norm (g x) ∂M) < ∞
    by simp
qed

lemma integral_discrete_difference:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  assumes X: countable X
  assumes null: ⋀x. x ∈ X ⟹ emeasure M {x} = 0
  assumes sets: ⋀x. x ∈ X ⟹ {x} ∈ sets M
  assumes eq: ⋀x. x ∈ space M ⟹ x ∉ X ⟹ f x = g x
  shows integral^L M f = integral^L M g
proof (rule integral_eq_cases)
  show eq: integrable M f ⟷ integrable M g
    by (rule integrable_discrete_difference[where X=X]) fact+

  assume f: integrable M f
  show integral^L M f = integral^L M g
  proof (rule integral_cong_AE)
    show f ∈ borel_measurable M g ∈ borel_measurable M
      using f eq by (auto intro: borel_measurable_integrable)

    have AE x in M. x ∉ X
      by (rule AE_discrete_difference) fact+
    with AE_space show AE x in M. f x = g x
      by eventually_elim fact
  qed
qed

lemma has_bochner_integral_discrete_difference:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  assumes X: countable X
  assumes null: ⋀x. x ∈ X ⟹ emeasure M {x} = 0
  assumes sets: ⋀x. x ∈ X ⟹ {x} ∈ sets M
  assumes eq: ⋀x. x ∈ space M ⟹ x ∉ X ⟹ f x = g x
  shows has_bochner_integral M f x ⟷ has_bochner_integral M g x
  using integrable_discrete_difference[of X M f g, OF assms]
  using integral_discrete_difference[of X M f g, OF assms]
  by (metis has_bochner_integral_iff)

lemma
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology} and w :: 'a ⇒ real
  assumes f ∈ borel_measurable M ⋀i. s i ∈ borel_measurable M integrable M w
```

```
  assumes lim: AE x in M. (λi. s i x) ⟶ f x
  assumes bound: ⋀i. AE x in M. norm (s i x) ≤ w x
  shows integrable_dominated_convergence: integrable M f
    and integrable_dominated_convergence2: ⋀i. integrable M (s i)
    and integral_dominated_convergence: (λi. integral^L M (s i)) ⟶ integral^L
M f
proof -
  have w_nonneg: AE x in M. 0 ≤ w x
    using bound[of 0] by eventually_elim (auto intro: norm_ge_zero order_trans)
  then have (∫+x. w x ∂M) = (∫+x. norm (w x) ∂M)
    by (intro nn_integral_cong_AE) auto
  with ‹integrable M w› have w: w ∈ borel_measurable M (∫+x. w x ∂M) < ∞
    unfolding integrable_iff_bounded by auto

  show int_s: ⋀i. integrable M (s i)
    unfolding integrable_iff_bounded
  proof
    fix i
    have (∫+ x. ennreal (norm (s i x)) ∂M) ≤ (∫+x. w x ∂M)
      using bound[of i] w_nonneg by (intro nn_integral_mono_AE) auto
    with w show (∫+ x. ennreal (norm (s i x)) ∂M) < ∞ by auto
  qed fact

  have all_bound: AE x in M. ∀i. norm (s i x) ≤ w x
    using bound unfolding AE_all_countable by auto

  show int_f: integrable M f
    unfolding integrable_iff_bounded
  proof
    have (∫+ x. ennreal (norm (f x)) ∂M) ≤ (∫+x. w x ∂M)
      using all_bound lim w_nonneg
    proof (intro nn_integral_mono_AE, eventually_elim)
      fix x assume ∀i. norm (s i x) ≤ w x (λi. s i x) ⟶ f x 0 ≤ w x
      then show ennreal (norm (f x)) ≤ ennreal (w x)
        by (intro LIMSEQ_le_const2[where X=λi. ennreal (norm (s i x))]) (auto
intro: tendsto_intros)
    qed
    with w show (∫+ x. ennreal (norm (f x)) ∂M) < ∞ by auto
  qed fact

  have (λn. ennreal (norm (integral^L M (s n) - integral^L M f))) ⟶ ennreal
0 (is ?d ⟶ ennreal 0)
  proof (rule tendsto_sandwich)
    show eventually (λn. ennreal 0 ≤ ?d n) sequentially (λ_. ennreal 0) ⟶
ennreal 0 by auto
    show eventually (λn. ?d n ≤ (∫+x. norm (s n x - f x) ∂M)) sequentially
    proof (intro always_eventually allI)
      fix n
      have ?d n = norm (integral^L M (λx. s n x - f x))
```

```
      using int_f int_s by simp
    also have ... ≤ (∫⁺x. norm (s n x − f x) ∂M)
      by (intro int_f int_s integrable_diff integral_norm_bound_ennreal)
    finally show ?d n ≤ (∫⁺x. norm (s n x − f x) ∂M) .
  qed
  show (λn. ∫⁺x. norm (s n x − f x) ∂M) ⟶ ennreal 0
    unfolding ennreal_0
    apply (subst norm_minus_commute)
  proof (rule nn_integral_dominated_convergence_norm[where w=w])
    show ⋀n. s n ∈ borel_measurable M
      using int_s unfolding integrable_iff_bounded by auto
  qed fact+
qed
then have (λn. integral^L M (s n) − integral^L M f) ⟶ 0
  by (simp add: tendsto_norm_zero_iff del: ennreal_0)
from tendsto_add[OF this tendsto_const[of integral^L M f]]
show (λi. integral^L M (s i)) ⟶ integral^L M f by simp
qed

context
  fixes s :: real ⇒ 'a ⇒ 'b::{banach, second_countable_topology} and w :: 'a ⇒
real
    and f :: 'a ⇒ 'b and M
  assumes f ∈ borel_measurable M ⋀t. s t ∈ borel_measurable M integrable M w
  assumes lim: AE x in M. ((λi. s i x) ⟶ f x) at_top
  assumes bound: ∀F i in at_top. AE x in M. norm (s i x) ≤ w x
begin

lemma integral_dominated_convergence_at_top: ((λt. integral^L M (s t)) ⟶ in-
tegral^L M f) at_top
proof (rule tendsto_at_topI_sequentially)
  fix X :: nat ⇒ real assume X: filterlim X at_top sequentially
  from filterlim_iff[THEN iffD1, OF this, rule_format, OF bound]
  obtain N where w: ⋀n. N ≤ n ⟹ AE x in M. norm (s (X n) x) ≤ w x
    by (auto simp: eventually_sequentially)

  show (λn. integral^L M (s (X n))) ⟶ integral^L M f
  proof (rule LIMSEQ_offset, rule integral_dominated_convergence)
    show AE x in M. norm (s (X (n + N)) x) ≤ w x for n
      by (rule w) auto
    show AE x in M. (λn. s (X (n + N)) x) ⟶ f x
      using lim
    proof eventually_elim
      fix x assume ((λi. s i x) ⟶ f x) at_top
      then show (λn. s (X (n + N)) x) ⟶ f x
        by (intro LIMSEQ_ignore_initial_segment filterlim_compose[OF _ X])
    qed
  qed fact+
qed
```

lemma *integrable_dominated_convergence_at_top*: *integrable M f*
proof –
  from *bound* **obtain** *N* **where** *w*: $\bigwedge n.\ N \leq n \Longrightarrow$ *AE x in M. norm* (*s n x*) $\leq$ *w x*
    by (*auto simp*: *eventually_at_top_linorder*)
  show *?thesis*
  proof (*rule integrable_dominated_convergence*)
    show *AE x in M. norm* (*s* (*N* + *i*) *x*) $\leq$ *w x* **for** *i* :: *nat*
      by (*intro w*) *auto*
    show *AE x in M.* ($\lambda i.$ *s* (*N* + *real i*) *x*) $\longrightarrow$ *f x*
      using *lim*
    proof *eventually_elim*
      fix *x* **assume** (($\lambda i.$ *s i x*) $\longrightarrow$ *f x*) *at_top*
      then show ($\lambda n.$ *s* (*N* + *n*) *x*) $\longrightarrow$ *f x*
        by (*rule filterlim_compose*)
          (*auto intro*!: *filterlim_tendsto_add_at_top filterlim_real_sequentially*)
    qed
  qed *fact*+
qed

end

lemma *integrable_mult_left_iff* [*simp*]:
  fixes *f* :: *'a* $\Rightarrow$ *real*
  shows *integrable M* ($\lambda x.$ *c* $*$ *f x*) $\longleftrightarrow$ *c* = *0* $\vee$ *integrable M f*
  using *integrable_mult_left*[*of c M f*] *integrable_mult_left*[*of 1 / c M* $\lambda x.$ *c* $*$ *f x*]
  by (*cases c* = *0*) *auto*

lemma *integrable_mult_right_iff* [*simp*]:
  fixes *f* :: *'a* $\Rightarrow$ *real*
  shows *integrable M* ($\lambda x.$ *f x* $*$ *c*) $\longleftrightarrow$ *c* = *0* $\vee$ *integrable M f*
  using *integrable_mult_left_iff* [*of M c f*] **by** (*simp add*: *mult.commute*)

lemma *integrableI_nn_integral_finite*:
  assumes [*measurable*]: *f* $\in$ *borel_measurable M*
    and *nonneg*: *AE x in M. 0* $\leq$ *f x*
    and *finite*: ($\int^+ x.$ *f x* ∂M) = *ennreal x*
  shows *integrable M f*
proof (*rule integrableI_bounded*)
  have ($\int^+ x.$ *ennreal* (*norm* (*f x*)) ∂M) = ($\int^+ x.$ *ennreal* (*f x*) ∂M)
    using *nonneg* **by** (*intro nn_integral_cong_AE*) *auto*
  with *finite* **show** ($\int^+ x.$ *ennreal* (*norm* (*f x*)) ∂M) < ∞
    by *auto*
qed *simp*

lemma *integral_nonneg_AE*:
  fixes *f* :: *'a* $\Rightarrow$ *real*
  assumes *nonneg*: *AE x in M. 0* $\leq$ *f x*

shows $0 \le integral^L\ M\ f$
proof *cases*
assume f: *integrable M f*
then have [*measurable*]: $f \in M \rightarrow_M borel$
by *auto*
have $(\lambda x.\ max\ 0\ (f\ x)) \in M \rightarrow_M borel\ \bigwedge x.\ 0 \le max\ 0\ (f\ x)\ integrable\ M\ (\lambda x.\ max\ 0\ (f\ x))$
using f **by** *auto*
from *this* **have** $0 \le integral^L\ M\ (\lambda x.\ max\ 0\ (f\ x))$
proof (*induction rule*: *borel_measurable_induct_real*)
case (*add f g*)
then have *integrable M f integrable M g*
by (*auto intro*!: *integrable_bound*[*OF add.prems*])
with *add* **show** *?case*
by (*simp add*: *nn_integral_add*)
next
case (*seq U*)
show *?case*
proof (*rule LIMSEQ_le_const*)
have *U_le*: $x \in space\ M \Longrightarrow U\ i\ x \le max\ 0\ (f\ x)$ **for** $x\ i$
using *seq* **by** (*intro incseq_le*) (*auto simp*: *incseq_def le_fun_def*)
with *seq nonneg* **show** $(\lambda i.\ integral^L\ M\ (U\ i)) \longrightarrow LINT\ x|M.\ max\ 0\ (f\ x)$
by (*intro integral_dominated_convergence*) *auto*
have *integrable M* (*U i*) **for** *i*
using *seq.prems* **by** (*rule integrable_bound*) (*insert U_le seq*, *auto*)
with *seq* **show** $\exists N.\ \forall n \ge N.\ 0 \le integral^L\ M\ (U\ n)$
by *auto*
qed
qed (*auto*)
also have $\ldots = integral^L\ M\ f$
using *nonneg* **by** (*auto intro*!: *integral_cong_AE*)
finally show *?thesis* **.**
qed (*simp add*: *not_integrable_integral_eq*)

lemma *integral_nonneg*[*simp*]:
fixes $f :: 'a \Rightarrow real$
shows $(\bigwedge x.\ x \in space\ M \Longrightarrow 0 \le f\ x) \Longrightarrow 0 \le integral^L\ M\ f$
by (*intro integral_nonneg_AE*) *auto*

proposition *nn_integral_eq_integral*:
assumes f: *integrable M f*
assumes *nonneg*: $AE\ x\ in\ M.\ 0 \le f\ x$
shows $(\int^+ x.\ f\ x\ \partial M) = integral^L\ M\ f$
proof −
{ **fix** $f :: 'a \Rightarrow real$ **assume** f: $f \in borel\_measurable\ M\ \bigwedge x.\ 0 \le f\ x\ integrable\ M\ f$
then have $(\int^+ x.\ f\ x\ \partial M) = integral^L\ M\ f$
proof (*induct rule*: *borel_measurable_induct_real*)

case (*set A*) **then show** *?case*
by (*simp add*: *integrable_indicator_iff ennreal_indicator emeasure_eq_ennreal_measure*)
next
case (*mult f c*) **then show** *?case*
by (*auto simp add*: *nn_integral_cmult ennreal_mult integral_nonneg_AE*)
next
case (*add g f*)
then have *integrable M f integrable M g*
by (*auto intro*!: *integrable_bound*[*OF add.prems*])
with *add* **show** *?case*
by (*simp add*: *nn_integral_add integral_nonneg_AE*)
next
case (*seq U*)
show *?case*
proof (*rule LIMSEQ_unique*)
have *U_le_f*: $x \in space\ M \Longrightarrow U\ i\ x \le f\ x$ **for** *x i*
using *seq* **by** (*intro incseq_le*) (*auto simp*: *incseq_def le_fun_def*)
have *int_U*: $\bigwedge i.\ integrable\ M\ (U\ i)$
using *seq f U_le_f* **by** (*intro integrable_bound*[*OF f*(*3*)]) *auto*
from *U_le_f seq* **have** $(\lambda i.\ integral^L\ M\ (U\ i)) \longrightarrow integral^L\ M\ f$
by (*intro integral_dominated_convergence*) *auto*
then show $(\lambda i.\ ennreal\ (integral^L\ M\ (U\ i))) \longrightarrow ennreal\ (integral^L\ M\ f)$
using *seq f int_U* **by** (*simp add*: *f integral_nonneg_AE*)
have $(\lambda i.\ \int^+ x.\ U\ i\ x\ \partial M) \longrightarrow \int^+ x.\ f\ x\ \partial M$
using *seq U_le_f f*
by (*intro nn_integral_dominated_convergence*[**where** *w=f*]) (*auto simp*: *integrable_iff_bounded*)
then show $(\lambda i.\ \int x.\ U\ i\ x\ \partial M) \longrightarrow \int^+ x.\ f\ x\ \partial M$
using *seq int_U* **by** *simp*
qed
qed **}**
from *this*[*of* $\lambda x.\ max\ 0\ (f\ x)$] *assms* **have** $(\int^+ x.\ max\ 0\ (f\ x)\ \partial M) = integral^L\ M\ (\lambda x.\ max\ 0\ (f\ x))$
by *simp*
also have $\ldots = integral^L\ M\ f$
using *assms* **by** (*auto intro*!: *integral_cong_AE simp*: *integral_nonneg_AE*)
also have $(\int^+ x.\ max\ 0\ (f\ x)\ \partial M) = (\int^+ x.\ f\ x\ \partial M)$
using *assms* **by** (*auto intro*!: *nn_integral_cong_AE simp*: *max_def*)
finally show *?thesis* **.**
qed

lemma *nn_integral_eq_integrable*:
assumes *f*: $f \in M \rightarrow_M borel\ AE\ x\ in\ M.\ 0 \le f\ x$ **and** $0 \le x$
shows $(\int^+ x.\ f\ x\ \partial M) = ennreal\ x \longleftrightarrow (integrable\ M\ f \wedge integral^L\ M\ f = x)$
proof (*safe intro*!: *nn_integral_eq_integral assms*)
assume $*$: $(\int^+ x.\ f\ x\ \partial M) = ennreal\ x$
with *integrableI_nn_integral_finite*[*OF f this*] *nn_integral_eq_integral*[*of M f*, *OF _ f*(*2*)]

```
  show integrable M f integral^L M f = x
    by (simp_all add: * assms integral_nonneg_AE)
qed

lemma
  fixes f :: _ ⇒ _ ⇒ 'a :: {banach, second_countable_topology}
  assumes integrable[measurable]: ⋀i. integrable M (f i)
  and summable: AE x in M. summable (λi. norm (f i x))
  and sums: summable (λi. (∫ x. norm (f i x) ∂M))
  shows integrable_suminf: integrable M (λx. (∑ i. f i x)) (is integrable M ?S)
    and sums_integral: (λi. integral^L M (f i)) sums (∫ x. (∑ i. f i x) ∂M) (is ?f
sums ?x)
    and integral_suminf: (∫ x. (∑ i. f i x) ∂M) = (∑ i. integral^L M (f i))
    and summable_integral: summable (λi. integral^L M (f i))
proof -
  have 1: integrable M (λx. ∑ i. norm (f i x))
  proof (rule integrableI_bounded)
    have (∫+ x. ennreal (norm (∑ i. norm (f i x))) ∂M) = (∫+ x. (∑ i. ennreal
(norm (f i x))) ∂M)
      apply (intro nn_integral_cong_AE)
      using summable
      apply eventually_elim
      apply (simp add: suminf_nonneg ennreal_suminf_neq_top)
      done
    also have ... = (∑ i. ∫+ x. norm (f i x) ∂M)
      by (intro nn_integral_suminf) auto
    also have ... = (∑ i. ennreal (∫ x. norm (f i x) ∂M))
     by (intro arg_cong[where f=suminf] ext nn_integral_eq_integral integrable_norm
integrable) auto
    finally show (∫+ x. ennreal (norm (∑ i. norm (f i x))) ∂M) < ∞
     by (simp add: sums ennreal_suminf_neq_top less_top[symmetric] integral_nonneg_AE)
  qed simp

  have 2: AE x in M. (λn. ∑ i<n. f i x) ⟶ (∑ i. f i x)
   using summable by eventually_elim (auto intro: summable_LIMSEQ summable_norm_cancel)

  have 3: ⋀j. AE x in M. norm (∑ i<j. f i x) ≤ (∑ i. norm (f i x))
    using summable
  proof eventually_elim
    fix j x assume [simp]: summable (λi. norm (f i x))
    have norm (∑ i<j. f i x) ≤ (∑ i<j. norm (f i x)) by (rule norm_sum)
    also have ... ≤ (∑ i. norm (f i x))
      using sum_le_suminf[of λi. norm (f i x)] unfolding sums_iff by auto
    finally show norm (∑ i<j. f i x) ≤ (∑ i. norm (f i x)) by simp
  qed

  note ibl = integrable_dominated_convergence[OF _ _ 1 2 3]
  note int = integral_dominated_convergence[OF _ _ 1 2 3]
```

show *integrable M ?S*
by (*rule ibl*) *measurable*

show *?f sums ?x* **unfolding** *sums_def*
using *int* **by** (*simp add*: *integrable*)
then show *?x* = *suminf ?f summable ?f*
unfolding *sums_iff* **by** *auto*
qed

proposition *integral_norm_bound* [*simp*]:
fixes *f* :: _ ⇒ *'a* :: {*banach*, *second_countable_topology*}
shows *norm* (*integral*^{L} *M f*) ≤ (∫ *x*. *norm* (*f x*) ∂*M*)
proof (*cases integrable M f*)
case *True* **then show** *?thesis*
using *nn_integral_eq_integral*[*of M λx*. *norm* (*f x*)] *integral_norm_bound_ennreal*[*of M f*]
by (*simp add*: *integral_nonneg_AE*)
next
case *False*
then have *norm* (*integral*^{L} *M f*) = *0* **by** (*simp add*: *not_integrable_integral_eq*)
moreover have (∫ *x*. *norm* (*f x*) ∂*M*) ≥ *0* **by** *auto*
ultimately show *?thesis* **by** *simp*
qed

proposition *integral_abs_bound* [*simp*]:
fixes *f* :: *'a* ⇒ *real* **shows** *abs* (∫ *x*. *f x* ∂*M*) ≤ (∫ *x*. |*f x*| ∂*M*)
using *integral_norm_bound*[*of M f*] **by** *auto*

lemma *integral_eq_nn_integral*:
assumes [*measurable*]: *f* ∈ *borel_measurable M*
assumes *nonneg*: *AE x in M*. *0* ≤ *f x*
shows *integral*^{L} *M f* = *enn2real* (∫$^+$ *x*. *ennreal* (*f x*) ∂*M*)
proof *cases*
assume ∗: (∫$^+$ *x*. *ennreal* (*f x*) ∂*M*) = ∞
also have (∫$^+$ *x*. *ennreal* (*f x*) ∂*M*) = (∫$^+$ *x*. *ennreal* (*norm* (*f x*)) ∂*M*)
using *nonneg* **by** (*intro nn_integral_cong_AE*) *auto*
finally have ¬ *integrable M f*
by (*auto simp*: *integrable_iff_bounded*)
then show *?thesis*
by (*simp add*: ∗ *not_integrable_integral_eq*)
next
assume (∫$^+$ *x*. *ennreal* (*f x*) ∂*M*) ≠ ∞
then have *integrable M f*
by (*cases* ∫$^+$ *x*. *ennreal* (*f x*) ∂*M rule*: *ennreal_cases*)
(*auto intro*!: *integrableI_nn_integral_finite assms*)
from *nn_integral_eq_integral*[*OF this*] *nonneg* **show** *?thesis*
by (*simp add*: *integral_nonneg_AE*)
qed

```
lemma enn2real_nn_integral_eq_integral:
  assumes eq: AE x in M. f x = ennreal (g x) and nn: AE x in M. 0 ≤ g x
    and fin: (∫+x. f x ∂M) < top
    and [measurable]: g ∈ M →M borel
  shows enn2real (∫+x. f x ∂M) = (∫ x. g x ∂M)
proof –
  have ennreal (enn2real (∫+x. f x ∂M)) = (∫+x. f x ∂M)
    using fin by (intro ennreal_enn2real) auto
  also have ... = (∫+x. g x ∂M)
    using eq by (rule nn_integral_cong_AE)
  also have ... = (∫ x. g x ∂M)
  proof (rule nn_integral_eq_integral)
    show integrable M g
    proof (rule integrableI_bounded)
      have (∫+ x. ennreal (norm (g x)) ∂M) = (∫+ x. f x ∂M)
        using eq nn by (auto intro!: nn_integral_cong_AE elim!: eventually_elim2)
      also note fin
      finally show (∫+ x. ennreal (norm (g x)) ∂M) < ∞
        by simp
    qed simp
  qed fact
  finally show ?thesis
    using nn by (simp add: integral_nonneg_AE)
qed

lemma has_bochner_integral_nn_integral:
  assumes f ∈ borel_measurable M AE x in M. 0 ≤ f x 0 ≤ x
  assumes (∫+x. f x ∂M) = ennreal x
  shows has_bochner_integral M f x
  unfolding has_bochner_integral_iff
 using assms by (auto simp: assms integral_eq_nn_integral intro: integrableI_nn_integral_finite)

lemma integrableI_simple_bochner_integrable:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  shows simple_bochner_integrable M f ⟹ integrable M f
  by (intro integrableI_sequence[where s=λ_. f] borel_measurable_simple_function)
     (auto simp: zero_ennreal_def[symmetric] simple_bochner_integrable.simps)

proposition integrable_induct[consumes 1, case_names base add lim, induct pred:
integrable]:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  assumes integrable M f
  assumes base: ⋀A c. A ∈ sets M ⟹ emeasure M A < ∞ ⟹ P (λx. indicator
A x *R c)
  assumes add: ⋀f g. integrable M f ⟹ P f ⟹ integrable M g ⟹ P g ⟹ P
(λx. f x + g x)
  assumes lim: ⋀f s. (⋀i. integrable M (s i)) ⟹ (⋀i. P (s i)) ⟹
   (⋀x. x ∈ space M ⟹ (λi. s i x) ⟶ f x) ⟹
   (⋀i x. x ∈ space M ⟹ norm (s i x) ≤ 2 * norm (f x)) ⟹ integrable M f ⟹
```

P f
  shows *P f*
proof –
  from ⟨*integrable M f*⟩ **have** *f*: $f \in$ *borel_measurable M* ($\int^+ x$. *norm* (*f x*) ∂M) $< \infty$
    unfolding *integrable_iff_bounded* **by** *auto*
  from *borel_measurable_implies_sequence_metric*[*OF f*(*1*)]
  obtain *s* **where** *s*: $\bigwedge i$. *simple_function M* (*s i*) $\bigwedge x$. $x \in$ *space M* $\Longrightarrow$ (λi. *s i x*) $\longrightarrow$ *f x*
    $\bigwedge i\ x$. $x \in$ *space M* $\Longrightarrow$ *norm* (*s i x*) $\leq 2 *$ *norm* (*f x*)
    unfolding *norm_conv_dist* **by** *metis*

  { fix *f A*
    have [*simp*]: *P* (λx. *0*)
      using *base*[*of* {} *undefined*] **by** *simp*
    have ($\bigwedge i$::*'b*. $i \in A \Longrightarrow$ *integrable M* (*f i*::*'a* $\Rightarrow$ *'b*)) $\Longrightarrow$
    ($\bigwedge i$. $i \in A \Longrightarrow$ *P* (*f i*)) $\Longrightarrow$ *P* (λx. $\sum i{\in}A$. *f i x*)
    by (*induct A rule*: *infinite_finite_induct*) (*auto intro*!: *add*) **}**
  note *sum* = *this*

  define *s'* **where** [*abs_def*]: *s' i z* = *indicator* (*space M*) *z* $*_R$ *s i z* **for** *i z*
  then have *s'_eq_s*: $\bigwedge i\ x$. $x \in$ *space M* $\Longrightarrow$ *s' i x* = *s i x*
    by *simp*

  have *sf*[*measurable*]: $\bigwedge i$. *simple_function M* (*s' i*)
    unfolding *s'_def* **using** *s*(*1*)
    by (*intro simple_function_compose2*[**where** *h*=($*_R$)] *simple_function_indicator*) *auto*

  { fix *i*
    have $\bigwedge z$. {*y*. *s' i z* = *y* $\wedge$ *y* $\in$ *s' i* ‘ *space M* $\wedge$ *y* $\neq$ *0* $\wedge$ *z* $\in$ *space M*} =
        (*if* *z* $\in$ *space M* $\wedge$ *s' i z* $\neq$ *0* *then* {*s' i z*} *else* {})
      by (*auto simp add*: *s'_def split*: *split_indicator*)
    then have $\bigwedge z$. *s' i* = (λz. $\sum y{\in}$*s' i*‘*space M* − {*0*}. *indicator* {*x*$\in$*space M*. *s' i x* = *y*} *z* $*_R$ *y*)
      using *sf* **by** (*auto simp*: *fun_eq_iff simple_function_def s'_def*) **}**
  note *s'_eq* = *this*

  show *P f*
  proof (*rule lim*)
    fix *i*

    have ($\int^+ x$. *norm* (*s' i x*) ∂M) $\leq$ ($\int^+ x$. *ennreal* (*2* $*$ *norm* (*f x*)) ∂M)
      using *s* **by** (*intro nn_integral_mono*) (*auto simp*: *s'_eq_s*)
    also have ... $< \infty$
      using *f* **by** (*simp add*: *nn_integral_cmult ennreal_mult_less_top ennreal_mult*)
    finally have *sbi*: *simple_bochner_integrable M* (*s' i*)
      using *sf* **by** (*intro simple_bochner_integrableI_bounded*) *auto*
    then show *integrable M* (*s' i*)

```
      by (rule integrableI_simple_bochner_integrable)

    { fix x assumex ∈ space M s' i x ≠ 0
      then have emeasure M {y ∈ space M. s' i y = s' i x} ≤ emeasure M {y ∈
space M. s' i y ≠ 0}
        by (intro emeasure_mono) auto
      also have ... < ∞
        using sbi by (auto elim: simple_bochner_integrable.cases simp: less_top)
      finally have emeasure M {y ∈ space M. s' i y = s' i x} ≠ ∞ by simp }
    then show P (s' i)
      by (subst s'_eq) (auto intro!: sum base simp: less_top)

    fix x assume x ∈ space M with s show (λi. s' i x) ⟶ f x
      by (simp add: s'_eq_s)
    show norm (s' i x) ≤ 2 * norm (f x)
      using ‹x ∈ space M› s by (simp add: s'_eq_s)
  qed fact
qed

lemma integral_eq_zero_AE:
  (AE x in M. f x = 0) ⟹ integral^L M f = 0
  using integral_cong_AE[of f M λ_. 0]
  by (cases integrable M f) (simp_all add: not_integrable_integral_eq)

lemma integral_nonneg_eq_0_iff_AE:
  fixes f :: _ ⇒ real
  assumes f[measurable]: integrable M f and nonneg: AE x in M. 0 ≤ f x
  shows integral^L M f = 0 ⟷ (AE x in M. f x = 0)
proof
  assume integral^L M f = 0
  then have integral^N M f = 0
    using nn_integral_eq_integral[OF f nonneg] by simp
  then have AE x in M. ennreal (f x) ≤ 0
    by (simp add: nn_integral_0_iff_AE)
  with nonneg show AE x in M. f x = 0
    by auto
qed (auto simp add: integral_eq_zero_AE)

lemma integral_mono_AE:
  fixes f :: 'a ⇒ real
  assumes integrable M f integrable M g AE x in M. f x ≤ g x
  shows integral^L M f ≤ integral^L M g
proof -
  have 0 ≤ integral^L M (λx. g x - f x)
    using assms by (intro integral_nonneg_AE integrable_diff assms) auto
  also have ... = integral^L M g - integral^L M f
    by (intro integral_diff assms)
  finally show ?thesis by simp
qed
```

lemma *integral_mono*:
fixes f :: $'a \Rightarrow real$
shows *integrable* M f $\Longrightarrow$ *integrable* M g $\Longrightarrow$ $(\bigwedge x.\ x \in space\ M \Longrightarrow f\ x \leq g\ x)$ $\Longrightarrow$
$integral^L\ M\ f \leq integral^L\ M\ g$
by (*intro integral_mono_AE*) *auto*

lemma *integral_norm_bound_integral*:
fixes f :: $'a::euclidean\_space \Rightarrow 'b::\{banach,second\_countable\_topology\}$
assumes *integrable* M f *integrable* M g $\bigwedge x.\ x \in space\ M \Longrightarrow norm(f\ x) \leq g\ x$
shows $norm\ (\int x.\ f\ x\ \partial M) \leq (\int x.\ g\ x\ \partial M)$
proof −
have $norm\ (\int x.\ f\ x\ \partial M) \leq (\int x.\ norm\ (f\ x)\ \partial M)$
by (*rule integral_norm_bound*)
also have ... $\leq (\int x.\ g\ x\ \partial M)$
using *assms integrable_norm integral_mono* **by** *blast*
finally show *?thesis* **.**
qed

lemma *integral_abs_bound_integral*:
fixes f :: $'a::euclidean\_space \Rightarrow real$
assumes *integrable* M f *integrable* M g $\bigwedge x.\ x \in space\ M \Longrightarrow |f\ x| \leq g\ x$
shows $|\int x.\ f\ x\ \partial M| \leq (\int x.\ g\ x\ \partial M)$
by (*metis integral_norm_bound_integral assms real_norm_def*)

The next two statements are useful to bound Lebesgue integrals, as they avoid one integrability assumption. The price to pay is that the upper function has to be nonnegative, but this is often true and easy to check in computations.

lemma *integral_mono_AE'*:
fixes f::_ $\Rightarrow$ *real*
assumes *integrable* M f *AE* x *in* M. $g\ x \leq f\ x$ *AE* x *in* M. $0 \leq f\ x$
shows $(\int x.\ g\ x\ \partial M) \leq (\int x.\ f\ x\ \partial M)$
proof (*cases integrable M g*)
case *True*
show *?thesis* **by** (*rule integral_mono_AE*, *auto simp add*: *assms True*)
next
case *False*
then have $(\int x.\ g\ x\ \partial M) = 0$ **by** (*simp add*: *not_integrable_integral_eq*)
also have ... $\leq (\int x.\ f\ x\ \partial M)$ **by** (*simp add*: *integral_nonneg_AE*[*OF assms*(*3*)])
finally show *?thesis* **by** *simp*
qed

lemma *integral_mono'*:
fixes f::_ $\Rightarrow$ *real*
assumes *integrable* M f $\bigwedge x.\ x \in space\ M \Longrightarrow g\ x \leq f\ x$ $\bigwedge x.\ x \in space\ M \Longrightarrow 0 \leq f\ x$
shows $(\int x.\ g\ x\ \partial M) \leq (\int x.\ f\ x\ \partial M)$

by (*rule integral_mono_AE'*, *insert assms*, *auto*)

lemma (**in** *finite_measure*) *integrable_measure*:
assumes *I*: *disjoint_family_on X I countable I*
shows *integrable* (*count_space I*) (λi. *measure M* (*X i*))
proof –
have ($\int^+ i$. *measure M* (*X i*) ∂*count_space I*) = ($\int^+ i$. *measure M* (*if X i* $\in$ *sets M then X i else* {}) ∂*count_space I*)
by (*auto intro*!: *nn_integral_cong measure_notin_sets*)
also have ... = *measure M* ($\bigcup i \in I$. *if X i* $\in$ *sets M then X i else* {})
using *I* **unfolding** *emeasure_eq_measure*[*symmetric*]
by (*subst emeasure_UN_countable*) (*auto simp*: *disjoint_family_on_def*)
finally show *?thesis*
by (*auto intro*!: *integrableI_bounded*)
qed

lemma *integrableI_real_bounded*:
assumes *f*: *f* $\in$ *borel_measurable M* **and** *ae*: *AE x in M*. *0* $\le$ *f x* **and** *fin*: *integral*^{N} *M f* < ∞
shows *integrable M f*
proof (*rule integrableI_bounded*)
have ($\int^+$ *x*. *ennreal* (*norm* (*f x*)) ∂M) = $\int^+$ *x*. *ennreal* (*f x*) ∂M
using *ae* **by** (*auto intro*: *nn_integral_cong_AE*)
also note *fin*
finally show ($\int^+$ *x*. *ennreal* (*norm* (*f x*)) ∂M) < ∞ .
qed *fact*

lemma *nn_integral_nonneg_infinite*:
fixes *f*::'*a* $\Rightarrow$ *real*
assumes *f* $\in$ *borel_measurable M* $\neg$ *integrable M f AE x in M*. *f x* $\ge$ *0*
shows ($\int^+ x$. *f x* ∂M) = ∞
using *assms integrableI_real_bounded less_top* **by** *auto*

lemma *integral_real_bounded*:
assumes *0* $\le$ *r integral*^{N} *M f* $\le$ *ennreal r*
shows *integral*^{L} *M f* $\le$ *r*
proof *cases*
assume [*simp*]: *integrable M f*

have *integral*^{L} *M* (λx. *max 0* (*f x*)) = *integral*^{N} *M* (λx. *max 0* (*f x*))
by (*intro nn_integral_eq_integral*[*symmetric*]) *auto*
also have ... = *integral*^{N} *M f*
by (*intro nn_integral_cong*) (*simp add*: *max_def ennreal_neg*)
also have ... $\le$ *r*
by *fact*
finally have *integral*^{L} *M* (λx. *max 0* (*f x*)) $\le$ *r*
using ‹*0* $\le$ *r*› **by** *simp*

moreover have *integral*^{L} *M f* $\le$ *integral*^{L} *M* (λx. *max 0* (*f x*))

by (*rule integral_mono_AE*) *auto*
ultimately show *?thesis*
by *simp*
next
assume ¬ *integrable M f* **then show** *?thesis*
using ‹*0* ≤ *r*› **by** (*simp add*: *not_integrable_integral_eq*)
qed

lemma *integrable_MIN*:
fixes *f*:: _ ⇒ _ ⇒ *real*
shows ⟦ *finite I*; *I* ≠ {}; ⋀*i*. *i* ∈ *I* ⟹ *integrable M* (*f i*) ⟧
⟹ *integrable M* (λ*x*. *MIN i*∈*I*. *f i x*)
by (*induct rule*: *finite_ne_induct*) *simp*+

lemma *integrable_MAX*:
fixes *f*:: _ ⇒ _ ⇒ *real*
shows ⟦ *finite I*; *I* ≠ {}; ⋀*i*. *i* ∈ *I* ⟹ *integrable M* (*f i*) ⟧
⟹ *integrable M* (λ*x*. *MAX i*∈*I*. *f i x*)
by (*induct rule*: *finite_ne_induct*) *simp*+

theorem *integral_Markov_inequality*:
assumes [*measurable*]: *integrable M u* **and** *AE x in M*. *0* ≤ *u x 0* < (*c*::*real*)
shows (*emeasure M*) {*x*∈*space M*. *u x* ≥ *c*} ≤ (*1*/*c*) ∗ (∫ *x*. *u x* ∂*M*)
proof −
have (∫ ⁺ *x*. *ennreal*(*u x*) ∗ *indicator* (*space M*) *x* ∂*M*) ≤ (∫ ⁺ *x*. *u x* ∂*M*)
by (*rule nn_integral_mono_AE*, *auto simp add*: ‹*c*>*0*› *less_eq_real_def*)
also have ... = (∫ *x*. *u x* ∂*M*)
by (*rule nn_integral_eq_integral*, *auto simp add*: *assms*)
finally have ∗: (∫ ⁺ *x*. *ennreal*(*u x*) ∗ *indicator* (*space M*) *x* ∂*M*) ≤ (∫ *x*. *u x* ∂*M*)
by *simp*

have {*x* ∈ *space M*. *u x* ≥ *c*} = {*x* ∈ *space M*. *ennreal*(*1*/*c*) ∗ *u x* ≥ *1*} ∩ (*space M*)
using ‹*c*>*0*› **by** (*auto simp*: *ennreal_mult'*[*symmetric*])
then have *emeasure M* {*x* ∈ *space M*. *u x* ≥ *c*} = *emeasure M* ({*x* ∈ *space M*. *ennreal*(*1*/*c*) ∗ *u x* ≥ *1*} ∩ (*space M*))
by *simp*
also have ... ≤ *ennreal*(*1*/*c*) ∗ (∫ ⁺ *x*. *ennreal*(*u x*) ∗ *indicator* (*space M*) *x* ∂*M*)
by (*rule nn_integral_Markov_inequality*) (*auto simp add*: *assms*)
also have ... ≤ *ennreal*(*1*/*c*) ∗ (∫ *x*. *u x* ∂*M*)
apply (*rule mult_left_mono*) **using** ∗ ‹*c*>*0*› **by** *auto*
finally show *?thesis*
using ‹*0*<*c*› **by** (*simp add*: *ennreal_mult'*[*symmetric*])
qed

lemma *integral_ineq_eq_0_then_AE*:
fixes *f*::_ ⇒ *real*

```
  assumes AE x in M. f x ≤ g x integrable M f integrable M g
        (∫ x. f x ∂M) = (∫ x. g x ∂M)
  shows AE x in M. f x = g x
proof −
  define h where h = (λx. g x − f x)
  have AE x in M. h x = 0
    apply (subst integral_nonneg_eq_0_iff_AE[symmetric])
    unfolding h_def using assms by auto
  then show ?thesis unfolding h_def by auto
qed

lemma not_AE_zero_int_E:
  fixes f::'a ⇒ real
  assumes AE x in M. f x ≥ 0 (∫ x. f x ∂M) > 0
      and [measurable]: f ∈ borel_measurable M
  shows ∃ A e. A ∈ sets M ∧ e>0 ∧ emeasure M A > 0 ∧ (∀ x ∈ A. f x ≥ e)
proof (rule not_AE_zero_E, auto simp add: assms)
  assume ∗: AE x in M. f x = 0
  have (∫ x. f x ∂M) = (∫ x. 0 ∂M) by (rule integral_cong_AE, auto simp add:
∗)
  then have (∫ x. f x ∂M) = 0 by simp
  then show False using assms(2) by simp
qed

proposition tendsto_L1_int:
  fixes u :: _ ⇒ _ ⇒ 'b::{banach, second_countable_topology}
  assumes [measurable]: ⋀n. integrable M (u n) integrable M f
        and ((λn. (∫+x. norm(u n x − f x) ∂M)) ⟶ 0) F
  shows ((λn. (∫ x. u n x ∂M)) ⟶ (∫ x. f x ∂M)) F
proof −
  have ((λn. norm((∫ x. u n x ∂M) − (∫ x. f x ∂M))) ⟶ (0::ennreal)) F
  proof (rule tendsto_sandwich[of λ_. 0, where ?h = λn. (∫+x. norm(u n x − f
x) ∂M)], auto simp add: assms)
    {
      fix n
      have (∫ x. u n x ∂M) − (∫ x. f x ∂M) = (∫ x. u n x − f x ∂M)
        apply (rule Bochner_Integration.integral_diff[symmetric]) using assms by
auto
      then have norm((∫ x. u n x ∂M) − (∫ x. f x ∂M)) = norm (∫ x. u n x − f
x ∂M)
        by auto
      also have ... ≤ (∫ x. norm(u n x − f x) ∂M)
        by (rule integral_norm_bound)
      finally have ennreal(norm((∫ x. u n x ∂M) − (∫ x. f x ∂M))) ≤ (∫ x. norm(u
n x − f x) ∂M)
        by simp
      also have ... = (∫+x. norm(u n x − f x) ∂M)
        apply (rule nn_integral_eq_integral[symmetric]) using assms by auto
      finally have norm((∫ x. u n x ∂M) − (∫ x. f x ∂M)) ≤ (∫+x. norm(u n x
```

− *f x*) ∂*M*) **by** *simp*
}
then show *eventually* (λ*n*. *norm*((∫ *x*. *u n x* ∂*M*) − (∫ *x*. *f x* ∂*M*)) ≤ (∫⁺*x*. *norm*(*u n x* − *f x*) ∂*M*)) *F*
by *auto*
qed
then have ((λ*n*. *norm*((∫ *x*. *u n x* ∂*M*) − (∫ *x*. *f x* ∂*M*))) ⟶ *0*) *F*
by (*simp flip*: *ennreal_0*)
then have ((λ*n*. ((∫ *x*. *u n x* ∂*M*) − (∫ *x*. *f x* ∂*M*))) ⟶ *0*) *F* **using** *tendsto_norm_zero_iff* **by** *blast*
then show *?thesis* **using** *Lim_null* **by** *auto*
qed

The next lemma asserts that, if a sequence of functions converges in L^1, then it admits a subsequence that converges almost everywhere.

proposition *tendsto_L1_AE_subseq*:
fixes *u* :: *nat* ⇒ *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes [*measurable*]: ⋀*n*. *integrable M* (*u n*)
and (λ*n*. (∫ *x*. *norm*(*u n x*) ∂*M*)) ⟶ *0*
shows ∃ *r*::*nat*⇒*nat*. *strict_mono r* ∧ (*AE x in M*. (λ*n*. *u* (*r n*) *x*) ⟶ *0*)
proof −
{
fix *k*
have *eventually* (λ*n*. (∫ *x*. *norm*(*u n x*) ∂*M*) < (*1*/*2*) ^*k*) *sequentially*
using *order_tendstoD*(*2*)[*OF assms*(*2*)] **by** *auto*
with *eventually_elim2*[*OF eventually_gt_at_top*[*of k*] *this*]
have ∃ *n*>*k*. (∫ *x*. *norm*(*u n x*) ∂*M*) < (*1*/*2*) ^*k*
by (*metis eventually_False_sequentially*)
}
then have ∃ *r*. ∀ *n*. *True* ∧ (*r* (*Suc n*) > *r n* ∧ (∫ *x*. *norm*(*u* (*r* (*Suc n*)) *x*) ∂*M*) < (*1*/*2*) ^(*r n*))
by (*intro dependent_nat_choice*, *auto*)
then obtain *r0* **where** *r0*: *strict_mono r0* ⋀*n*. (∫ *x*. *norm*(*u* (*r0* (*Suc n*)) *x*) ∂*M*) < (*1*/*2*) ^(*r0 n*)
by (*auto simp*: *strict_mono_Suc_iff*)
define *r* **where** *r* = (λ*n*. *r0*(*n*+*1*))
have *strict_mono r* **unfolding** *r_def* **using** *r0*(*1*) **by** (*simp add*: *strict_mono_Suc_iff*)
have *I*: (∫⁺*x*. *norm*(*u* (*r n*) *x*) ∂*M*) < *ennreal*((*1*/*2*) ^*n*) **for** *n*
proof −
have *r0 n* ≥ *n* **using** ‹*strict_mono r0*› **by** (*simp add*: *seq_suble*)
have (*1*/*2*::*real*) ^(*r0 n*) ≤ (*1*/*2*) ^*n* **by** (*rule power_decreasing*[*OF* ‹*r0 n* ≥ *n*›], *auto*)
then have (∫ *x*. *norm*(*u* (*r0* (*Suc n*)) *x*) ∂*M*) < (*1*/*2*) ^*n*
using *r0*(*2*) *less_le_trans* **by** *blast*
then have (∫ *x*. *norm*(*u* (*r n*) *x*) ∂*M*) < (*1*/*2*) ^*n*
unfolding *r_def* **by** *auto*
moreover have (∫⁺*x*. *norm*(*u* (*r n*) *x*) ∂*M*) = (∫ *x*. *norm*(*u* (*r n*) *x*) ∂*M*)
by (*rule nn_integral_eq_integral*, *auto simp add*: *integrable_norm*[*OF assms*(*1*)[*of r n*]])

```
    ultimately show ?thesis by (auto intro: ennreal_lessI)
  qed

  have AE x in M. limsup (λn. ennreal (norm(u (r n) x))) ≤ 0
  proof (rule AE_upper_bound_inf_ennreal)
    fix e::real assume e > 0
    define A where A = (λn. {x ∈ space M. norm(u (r n) x) ≥ e})
    have A_meas [measurable]: ⋀n. A n ∈ sets M unfolding A_def by auto
    have A_bound: emeasure M (A n) < (1/e) ∗ ennreal((1/2)^n) for n
    proof −
      have ∗: indicator (A n) x ≤ (1/e) ∗ ennreal(norm(u (r n) x)) for x
         apply (cases x ∈ A n) unfolding A_def using ‹0 < e› by (auto simp:
ennreal_mult[symmetric])
      have emeasure M (A n) = (∫+x. indicator (A n) x ∂M) by auto
      also have ... ≤ (∫+x. (1/e) ∗ ennreal(norm(u (r n) x)) ∂M)
        apply (rule nn_integral_mono) using ∗ by auto
      also have ... = (1/e) ∗ (∫+x. norm(u (r n) x) ∂M)
        apply (rule nn_integral_cmult) using ‹e > 0› by auto
      also have ... < (1/e) ∗ ennreal((1/2)^n)
        using I[of n] ‹e > 0› by (intro ennreal_mult_strict_left_mono) auto
      finally show ?thesis by simp
    qed
    have A_fin: emeasure M (A n) < ∞ for n
      using ‹e > 0› A_bound[of n]
      by (auto simp add: ennreal_mult less_top[symmetric])

    have A_sum: summable (λn. measure M (A n))
    proof (rule summable_comparison_test'[of λn. (1/e) ∗ (1/2)^n 0])
      have summable (λn. (1/(2::real))^n) by (simp add: summable_geometric)
      then show summable (λn. (1/e) ∗ (1/2)^n) using summable_mult by blast
      fix n::nat assume n ≥ 0
      have norm(measure M (A n)) = measure M (A n) by simp
     also have ... = enn2real(emeasure M (A n)) unfolding measure_def by simp
      also have ... < enn2real((1/e) ∗ (1/2)^n)
        using A_bound[of n] ‹emeasure M (A n) < ∞› ‹0 < e›
         by (auto simp: emeasure_eq_ennreal_measure ennreal_mult[symmetric] en-
nreal_less_iff)
      also have ... = (1/e) ∗ (1/2)^n
        using ‹0<e› by auto
      finally show norm(measure M (A n)) ≤ (1/e) ∗ (1/2)^n by simp
    qed

    have AE x in M. eventually (λn. x ∈ space M − A n) sequentially
      by (rule borel_cantelli_AE1[OF A_meas A_fin A_sum])
    moreover
    {
      fix x assume eventually (λn. x ∈ space M − A n) sequentially
      moreover have norm(u (r n) x) ≤ ennreal e if x ∈ space M − A n for n
        using that unfolding A_def by (auto intro: ennreal_leI)
```

ultimately have *eventually* (λn. *norm*(*u* (*r n*) *x*) ≤ *ennreal e*) *sequentially*
by (*simp add*: *eventually_mono*)
then have *limsup* (λn. *ennreal* (*norm*(*u* (*r n*) *x*))) ≤ *e*
by (*simp add*: *Limsup_bounded*)
}
ultimately show *AE x in M*. *limsup* (λn. *ennreal* (*norm*(*u* (*r n*) *x*))) ≤ *0* + *ennreal e* **by** *auto*
qed
moreover
{
fix *x* **assume** *limsup* (λn. *ennreal* (*norm*(*u* (*r n*) *x*))) ≤ *0*
moreover then have *liminf* (λn. *ennreal* (*norm*(*u* (*r n*) *x*))) ≤ *0*
by (*rule order_trans*[*rotated*]) (*auto intro*: *Liminf_le_Limsup*)
ultimately have (λn. *ennreal* (*norm*(*u* (*r n*) *x*))) ⟶ *0*
using *tendsto_Limsup*[*of sequentially* λn. *ennreal* (*norm*(*u* (*r n*) *x*))] **by** *auto*
then have (λn. *norm*(*u* (*r n*) *x*)) ⟶ *0*
by (*simp flip*: *ennreal_0*)
then have (λn. *u* (*r n*) *x*) ⟶ *0*
by (*simp add*: *tendsto_norm_zero_iff*)
}
ultimately have *AE x in M*. (λn. *u* (*r n*) *x*) ⟶ *0* **by** *auto*
then show *?thesis* **using** ‹*strict_mono r*› **by** *auto*
qed

6.10.1 Restricted measure spaces

lemma *integrable_restrict_space*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes Ω[*simp*]: Ω ∩ *space M* ∈ *sets M*
shows *integrable* (*restrict_space M* Ω) *f* ⟷ *integrable M* (λx. *indicator* Ω *x* $*_R$ *f x*)
unfolding *integrable_iff_bounded*
borel_measurable_restrict_space_iff [*OF* Ω]
nn_integral_restrict_space[*OF* Ω]
by (*simp add*: *ac_simps ennreal_indicator ennreal_mult*)

lemma *integral_restrict_space*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes Ω[*simp*]: Ω ∩ *space M* ∈ *sets M*
shows $integral^L$ (*restrict_space M* Ω) *f* = $integral^L$ *M* (λx. *indicator* Ω *x* $*_R$ *f x*)
proof (*rule integral_eq_cases*)
assume *integrable* (*restrict_space M* Ω) *f*
then show *?thesis*
proof *induct*
case (*base A c*) **then show** *?case*
by (*simp add*: *indicator_inter_arith*[*symmetric*] *sets_restrict_space_iff*
emeasure_restrict_space Int_absorb1 measure_restrict_space)
next

```
    case (add g f) then show ?case
      by (simp add: scaleR_add_right integrable_restrict_space)
  next
    case (lim f s)
    show ?case
    proof (rule LIMSEQ_unique)
     show (λi. integral^L (restrict_space M Ω) (s i)) ⟶ integral^L (restrict_space
M Ω) f
        using lim by (intro integral_dominated_convergence[where w=λx. 2 * norm
(f x)]) simp_all

       show (λi. integral^L (restrict_space M Ω) (s i)) ⟶ (∫ x. indicator Ω x
*R f x ∂M)
         unfolding lim
         using lim
       by (intro integral_dominated_convergence[where w=λx. 2 * norm (indicator
Ω x *R f x)])
             (auto simp add: space_restrict_space integrable_restrict_space simp del:
norm_scaleR
                split: split_indicator)
    qed
  qed
qed (simp add: integrable_restrict_space)

lemma integral_empty:
  assumes space M = {}
  shows integral^L M f = 0
proof -
  have (∫ x. f x ∂M) = (∫ x. 0 ∂M)
    by(rule integral_cong)(simp_all add: assms)
  thus ?thesis by simp
qed
```

6.10.2 Measure spaces with an associated density

```
lemma integrable_density:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology} and g :: 'a ⇒ real
  assumes [measurable]: f ∈ borel_measurable M g ∈ borel_measurable M
    and nn: AE x in M. 0 ≤ g x
  shows integrable (density M g) f ⟷ integrable M (λx. g x *R f x)
  unfolding integrable_iff_bounded using nn
  apply (simp add: nn_integral_density less_top[symmetric])
  apply (intro arg_cong2[where f=(=)] refl nn_integral_cong_AE)
  apply (auto simp: ennreal_mult)
  done

lemma integral_density:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology} and g :: 'a ⇒ real
  assumes f: f ∈ borel_measurable M
```

and *g*[*measurable*]: *g* ∈ *borel_measurable M AE x in M*. *0* ≤ *g x*
shows $integral^L$ (*density M g*) *f* = $integral^L$ *M* (λ*x*. *g x* $*_R$ *f x*)
proof (*rule integral_eq_cases*)
assume *integrable* (*density M g*) *f*
then show *?thesis*
proof *induct*
case (*base A c*)
then have [*measurable*]: *A* ∈ *sets M* **by** *auto*

have *int*: *integrable M* (λ*x*. *g x* ∗ *indicator A x*)
using *g base integrable_density*[*of indicator A* :: ′*a* ⇒ *real M g*] **by** *simp*
then have $integral^L$ *M* (λ*x*. *g x* ∗ *indicator A x*) = (∫ + *x*. *ennreal* (*g x* ∗ *indicator A x*) ∂*M*)
using *g* **by** (*subst nn_integral_eq_integral*) *auto*
also have ... = (∫ + *x*. *ennreal* (*g x*) ∗ *indicator A x* ∂*M*)
by (*intro nn_integral_cong*) (*auto split*: *split_indicator*)
also have ... = *emeasure* (*density M g*) *A*
by (*rule emeasure_density*[*symmetric*]) *auto*
also have ... = *ennreal* (*measure* (*density M g*) *A*)
using *base* **by** (*auto intro*: *emeasure_eq_ennreal_measure*)
also have ... = $integral^L$ (*density M g*) (*indicator A*)
using *base* **by** *simp*
finally show *?case*
using *base g*
apply (*simp add*: *int integral_nonneg_AE*)
apply (*subst* (*asm*) *ennreal_inj*)
apply (*auto intro*!: *integral_nonneg_AE*)
done
next
case (*add f h*)
then have [*measurable*]: *f* ∈ *borel_measurable M h* ∈ *borel_measurable M*
by (*auto dest*!: *borel_measurable_integrable*)
from *add g* **show** *?case*
by (*simp add*: *scaleR_add_right integrable_density*)
next
case (*lim f s*)
have [*measurable*]: *f* ∈ *borel_measurable M* ⋀*i*. *s i* ∈ *borel_measurable M*
using *lim*(*1*,*5*)[*THEN borel_measurable_integrable*] **by** *auto*

show *?case*
proof (*rule LIMSEQ_unique*)
show (λ*i*. $integral^L$ *M* (λ*x*. *g x* $*_R$ *s i x*)) ⟶ $integral^L$ *M* (λ*x*. *g x* $*_R$ *f x*)
proof (*rule integral_dominated_convergence*)
show *integrable M* (λ*x*. *2* ∗ *norm* (*g x* $*_R$ *f x*))
by (*intro integrable_mult_right integrable_norm integrable_density*[*THEN iffD1*] *lim g*) *auto*
show *AE x in M*. (λ*i*. *g x* $*_R$ *s i x*) ⟶ *g x* $*_R$ *f x*
using *lim*(*3*) **by** (*auto intro*!: *tendsto_scaleR AE_I2*[*of M*])

```
        show ⋀i. AE x in M. norm (g x *R s i x) ≤ 2 * norm (g x *R f x)
            using lim(4) g by (auto intro!: AE_I2[of M] mult_left_mono simp:
field_simps)
      qed auto
      show (λi. integral^L M (λx. g x *R s i x)) ⟶ integral^L (density M g) f
        unfolding lim(2)[symmetric]
        by (rule integral_dominated_convergence[where w=λx. 2 * norm (f x)])
          (insert lim(3-5), auto)
    qed
  qed
qed (simp add: f g integrable_density)

lemma
  fixes g :: 'a ⇒ real
  assumes f ∈ borel_measurable M AE x in M. 0 ≤ f x g ∈ borel_measurable M
  shows integral_real_density: integral^L (density M f) g = (∫ x. f x * g x ∂M)
    and integrable_real_density: integrable (density M f) g ⟷ integrable M (λx. f
x * g x)
  using assms integral_density[of g M f] integrable_density[of g M f] by auto

lemma has_bochner_integral_density:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology} and g :: 'a ⇒ real
  shows f ∈ borel_measurable M ⟹ g ∈ borel_measurable M ⟹ (AE x in M. 0
≤ g x) ⟹
    has_bochner_integral M (λx. g x *R f x) x ⟹ has_bochner_integral (density M
g) f x
  by (simp add: has_bochner_integral_iff integrable_density integral_density)
```

6.10.3 Distributions

```
lemma integrable_distr_eq:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  assumes [measurable]: g ∈ measurable M N f ∈ borel_measurable N
  shows integrable (distr M N g) f ⟷ integrable M (λx. f (g x))
  unfolding integrable_iff_bounded by (simp_all add: nn_integral_distr)

lemma integrable_distr:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  shows T ∈ measurable M M' ⟹ integrable (distr M M' T) f ⟹ integrable M
(λx. f (T x))
  by (subst integrable_distr_eq[symmetric, where g=T])
    (auto dest: borel_measurable_integrable)

lemma integral_distr:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  assumes g[measurable]: g ∈ measurable M N and f: f ∈ borel_measurable N
  shows integral^L (distr M N g) f = integral^L M (λx. f (g x))
proof (rule integral_eq_cases)
  assume integrable (distr M N g) f
```

then show *?thesis*
proof *induct*
case (*base A c*)
then have [*measurable*]: $A \in$ *sets N* **by** *auto*
from *base* **have** *int*: *integrable* (*distr M N g*) ($\lambda a.$ *indicator A a* $*_R$ *c*)
by (*intro integrable_indicator*)

have *integral*^{L} (*distr M N g*) ($\lambda a.$ *indicator A a* $*_R$ *c*) = *measure* (*distr M N g*) *A* $*_R$ *c*
using *base* **by** *auto*
also have ... = *measure M* (*g* −‘ *A* ∩ *space M*) $*_R$ *c*
by (*subst measure_distr*) *auto*
also have ... = *integral*^{L} *M* ($\lambda a.$ *indicator* (*g* −‘ *A* ∩ *space M*) *a* $*_R$ *c*)
using *base* **by** (*auto simp*: *emeasure_distr*)
also have ... = *integral*^{L} *M* ($\lambda a.$ *indicator A* (*g a*) $*_R$ *c*)
using *int base* **by** (*intro integral_cong_AE*) (*auto simp*: *emeasure_distr split*: *split_indicator*)
finally show *?case* **.**
next
case (*add f h*)
then have [*measurable*]: $f \in$ *borel_measurable N h* $\in$ *borel_measurable N*
by (*auto dest*!: *borel_measurable_integrable*)
from *add g* **show** *?case*
by (*simp add*: *scaleR_add_right integrable_distr_eq*)
next
case (*lim f s*)
have [*measurable*]: $f \in$ *borel_measurable N* $\bigwedge i.$ *s i* $\in$ *borel_measurable N*
using *lim*(*1*,*5*)[*THEN borel_measurable_integrable*] **by** *auto*

show *?case*
proof (*rule LIMSEQ_unique*)
show ($\lambda i.$ *integral*^{L} *M* ($\lambda x.$ *s i* (*g x*))) ⟶ *integral*^{L} *M* ($\lambda x.$ *f* (*g x*))
proof (*rule integral_dominated_convergence*)
show *integrable M* ($\lambda x.$ *2* ∗ *norm* (*f* (*g x*)))
using *lim* **by** (*auto simp*: *integrable_distr_eq*)
show *AE x in M*. ($\lambda i.$ *s i* (*g x*)) ⟶ *f* (*g x*)
using *lim*(*3*) *g*[*THEN measurable_space*] **by** *auto*
show $\bigwedge i.$ *AE x in M*. *norm* (*s i* (*g x*)) ≤ *2* ∗ *norm* (*f* (*g x*))
using *lim*(*4*) *g*[*THEN measurable_space*] **by** *auto*
qed *auto*
show ($\lambda i.$ *integral*^{L} *M* ($\lambda x.$ *s i* (*g x*))) ⟶ *integral*^{L} (*distr M N g*) *f*
unfolding *lim*(*2*)[*symmetric*]
by (*rule integral_dominated_convergence*[**where** *w*=$\lambda x.$ *2* ∗ *norm* (*f x*)])
(*insert lim*(*3*−*5*), *auto*)
qed
qed
qed (*simp add*: *f g integrable_distr_eq*)

lemma *has_bochner_integral_distr*:

fixes f :: $'a \Rightarrow 'b$::{*banach*, *second_countable_topology*}
shows $f \in$ *borel_measurable* $N \Longrightarrow g \in$ *measurable* M $N \Longrightarrow$
has_bochner_integral M (λx. f (g x)) $x \Longrightarrow$ *has_bochner_integral* (*distr* M N g) f x
by (*simp add*: *has_bochner_integral_iff integrable_distr_eq integral_distr*)

6.10.4 Lebesgue integration on *count_space*

lemma *integrable_count_space*:
fixes f :: $'a \Rightarrow 'b$::{*banach*,*second_countable_topology*}
shows *finite* $X \Longrightarrow$ *integrable* (*count_space* X) f
by (*auto simp*: *nn_integral_count_space integrable_iff_bounded*)

lemma *measure_count_space*[*simp*]:
$B \subseteq A \Longrightarrow$ *finite* $B \Longrightarrow$ *measure* (*count_space* A) B = *card* B
unfolding *measure_def* **by** (*subst emeasure_count_space*) *auto*

lemma *lebesgue_integral_count_space_finite_support*:
assumes f: *finite* $\{a \in A.\ f\ a \neq 0\}$
shows $(\int x.\ f\ x\ \partial count\_space\ A) = (\sum a \mid a \in A \wedge f\ a \neq 0.\ f\ a)$
proof –
have *eq*: $\bigwedge x.\ x \in A \Longrightarrow (\sum a \mid x = a \wedge a \in A \wedge f\ a \neq 0.\ f\ a) = (\sum x \in \{x\}.\ f\ x)$
by (*intro sum.mono_neutral_cong_left*) *auto*

have $(\int x.\ f\ x\ \partial count\_space\ A) = (\int x.\ (\sum a \mid a \in A \wedge f\ a \neq 0.\ indicator\ \{a\}\ x *_R f\ a)\ \partial count\_space\ A)$
by (*intro integral_cong refl*) (*simp add*: *f eq*)
also have ... $= (\sum a \mid a \in A \wedge f\ a \neq 0.\ measure\ (count\_space\ A)\ \{a\} *_R f\ a)$
by (*subst integral_sum*) (*auto intro*!: *sum.cong*)
finally show *?thesis*
by *auto*
qed

lemma *lebesgue_integral_count_space_finite*: *finite* $A \Longrightarrow (\int x.\ f\ x\ \partial count\_space\ A) = (\sum a \in A.\ f\ a)$
by (*subst lebesgue_integral_count_space_finite_support*)
(*auto intro*!: *sum.mono_neutral_cong_left*)

lemma *integrable_count_space_nat_iff*:
fixes f :: *nat* $\Rightarrow$ _::{*banach*,*second_countable_topology*}
shows *integrable* (*count_space UNIV*) $f \longleftrightarrow$ *summable* (λx. *norm* (f x))
by (*auto simp add*: *integrable_iff_bounded nn_integral_count_space_nat ennreal_suminf_neq_top*
intro: *summable_suminf_not_top*)

lemma *sums_integral_count_space_nat*:
fixes f :: *nat* $\Rightarrow$ _::{*banach*,*second_countable_topology*}
assumes $*$: *integrable* (*count_space UNIV*) f
shows f *sums* (*integral*^{L} (*count_space UNIV*) f)

proof −
  let *?f* = $\lambda n\ i.$ *indicator* {*n*} *i* $*_R$ *f i*
  have *f'*: $\bigwedge n\ i.$ *?f n i* = *indicator* {*n*} *i* $*_R$ *f n*
    by (*auto simp*: *fun_eq_iff split*: *split_indicator*)

  have ($\lambda i.$ $\int$ *n*. *?f i n* ∂*count_space UNIV*) *sums* $\int$ *n*. ($\sum$ *i*. *?f i n*) ∂*count_space UNIV*
  proof (*rule sums_integral*)
    show $\bigwedge i.$ *integrable* (*count_space UNIV*) (*?f i*)
      using ∗ **by** (*intro integrable_mult_indicator*) *auto*
    show *AE n in count_space UNIV*. *summable* ($\lambda i.$ *norm* (*?f i n*))
      using *summable_finite*[*of* {*n*} $\lambda i.$ *norm* (*?f i n*) **for** *n*] **by** *simp*
    show *summable* ($\lambda i.$ $\int$ *n*. *norm* (*?f i n*) ∂*count_space UNIV*)
      using ∗ **by** (*subst f'*) (*simp add*: *integrable_count_space_nat_iff*)
  qed
  also have ($\int$ *n*. ($\sum$ *i*. *?f i n*) ∂*count_space UNIV*) = ($\int$ *n*. *f n* ∂*count_space UNIV*)
    using *suminf_finite*[*of* {*n*} $\lambda i.$ *?f i n* **for** *n*] **by** (*auto intro*!: *integral_cong*)
  also have ($\lambda i.$ $\int$ *n*. *?f i n* ∂*count_space UNIV*) = *f*
    by (*subst f'*) *simp*
  finally show *?thesis* **.**
qed

lemma *integral_count_space_nat*:
  fixes *f* :: *nat* ⇒ _::{*banach*,*second_countable_topology*}
  shows *integrable* (*count_space UNIV*) *f* ⟹ *integral*^{L} (*count_space UNIV*) *f* = ($\sum$ *x*. *f x*)
  using *sums_integral_count_space_nat* **by** (*rule sums_unique*)

lemma *integrable_bij_count_space*:
  fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
  assumes *g*: *bij_betw g A B*
  shows *integrable* (*count_space A*) ($\lambda x.$ *f* (*g x*)) ⟷ *integrable* (*count_space B*) *f*
  unfolding *integrable_iff_bounded* **by** (*subst nn_integral_bij_count_space*[*OF g*]) *auto*

lemma *integral_bij_count_space*:
  fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
  assumes *g*: *bij_betw g A B*
  shows *integral*^{L} (*count_space A*) ($\lambda x.$ *f* (*g x*)) = *integral*^{L} (*count_space B*) *f*
  using *g*[*THEN bij_betw_imp_funcset*]
  apply (*subst distr_bij_count_space*[*OF g*, *symmetric*])
  apply (*intro integral_distr*[*symmetric*])
  apply *auto*
  done

lemma *has_bochner_integral_count_space_nat*:
  fixes *f* :: *nat* ⇒ _::{*banach*,*second_countable_topology*}
  shows *has_bochner_integral* (*count_space UNIV*) *f x* ⟹ *f sums x*

unfolding *has_bochner_integral_iff* **by** (*auto intro*!: *sums_integral_count_space_nat*)

6.10.5 Point measure

lemma *lebesgue_integral_point_measure_finite*:
fixes g :: $'a \Rightarrow 'b$::{*banach*, *second_countable_topology*}
shows *finite A* $\Longrightarrow$ ($\bigwedge a.\ a \in A \Longrightarrow 0 \le f\ a$) $\Longrightarrow$
integral^L (*point_measure A f*) g = ($\sum a{\in}A.\ f\ a *_R g\ a$)
by (*simp add*: *lebesgue_integral_count_space_finite AE_count_space integral_density point_measure_def*)

proposition *integrable_point_measure_finite*:
fixes g :: $'a \Rightarrow 'b$::{*banach*, *second_countable_topology*} **and** f :: $'a \Rightarrow$ *real*
shows *finite A* $\Longrightarrow$ *integrable* (*point_measure A f*) g
unfolding *point_measure_def*
apply (*subst density_cong*[**where** $f'=\lambda x.$ *ennreal* (*max 0* ($f\ x$))])
apply (*auto split*: *split_max simp*: *ennreal_neg*)
apply (*subst integrable_density*)
apply (*auto simp*: *AE_count_space integrable_count_space*)
done

6.10.6 Lebesgue integration on *null_measure*

lemma *has_bochner_integral_null_measure_iff*[*iff*]:
has_bochner_integral (*null_measure M*) *f 0* $\longleftrightarrow$ $f \in$ *borel_measurable M*
by (*auto simp add*: *has_bochner_integral.simps simple_bochner_integral_def*[*abs_def*]
intro!: *exI*[*of _* $\lambda n\ x.\ 0$] *simple_bochner_integrable.intros*)

lemma *integrable_null_measure_iff*[*iff*]: *integrable* (*null_measure M*) $f \longleftrightarrow f \in$ *borel_measurable M*
by (*auto simp add*: *integrable.simps*)

lemma *integral_null_measure*[*simp*]: *integral*^{L} (*null_measure M*) $f = 0$
by (*cases integrable* (*null_measure M*) *f*)
(*auto simp add*: *not_integrable_integral_eq has_bochner_integral_integral_eq*)

6.10.7 Legacy lemmas for the real-valued Lebesgue integral

theorem *real_lebesgue_integral_def*:
assumes *f*[*measurable*]: *integrable M f*
shows *integral*^{L} $M\ f$ = *enn2real* ($\int^+ x.\ f\ x\ \partial M$) − *enn2real* ($\int^+ x.$ *ennreal* ($-\ f\ x$) ∂M)
proof −
have *integral*^{L} $M\ f$ = *integral*^{L} M ($\lambda x.$ *max 0* ($f\ x$) − *max 0* ($-\ f\ x$))
by (*auto intro*!: *arg_cong*[**where** f=*integral*^{L} M])
also have ... = *integral*^{L} M ($\lambda x.$ *max 0* ($f\ x$)) − *integral*^{L} M ($\lambda x.$ *max 0* ($-\ f\ x$))
by (*intro integral_diff integrable_max integrable_minus integrable_zero f*)
also have *integral*^{L} M ($\lambda x.$ *max 0* ($f\ x$)) = *enn2real* ($\int^+ x.$ *ennreal* ($f\ x$) ∂M)

by (*subst integral_eq_nn_integral*) (*auto intro*!: *arg_cong*[**where** *f*=*enn2real*] *nn_integral_cong simp*: *max_def ennreal_neg*)
also have $integral^L$ *M* (λ*x*. *max 0* (− *f x*)) = *enn2real* (∫ $^+$*x*. *ennreal* (− *f x*) ∂*M*)
by (*subst integral_eq_nn_integral*) (*auto intro*!: *arg_cong*[**where** *f*=*enn2real*] *nn_integral_cong simp*: *max_def ennreal_neg*)
finally show *?thesis* **.**
qed

theorem *real_integrable_def*:
integrable M f ⟷ *f* ∈ *borel_measurable M* ∧
(∫ $^+$ *x*. *ennreal* (*f x*) ∂*M*) ≠ ∞ ∧ (∫ $^+$ *x*. *ennreal* (− *f x*) ∂*M*) ≠ ∞
unfolding *integrable_iff_bounded*
proof (*safe del*: *notI*)
assume ∗: (∫ $^+$ *x*. *ennreal* (*norm* (*f x*)) ∂*M*) < ∞
have (∫ $^+$ *x*. *ennreal* (*f x*) ∂*M*) ≤ (∫ $^+$ *x*. *ennreal* (*norm* (*f x*)) ∂*M*)
by (*intro nn_integral_mono*) *auto*
also note ∗
finally show (∫ $^+$ *x*. *ennreal* (*f x*) ∂*M*) ≠ ∞
by *simp*
have (∫ $^+$ *x*. *ennreal* (− *f x*) ∂*M*) ≤ (∫ $^+$ *x*. *ennreal* (*norm* (*f x*)) ∂*M*)
by (*intro nn_integral_mono*) *auto*
also note ∗
finally show (∫ $^+$ *x*. *ennreal* (− *f x*) ∂*M*) ≠ ∞
by *simp*
next
assume [*measurable*]: *f* ∈ *borel_measurable M*
assume *fin*: (∫ $^+$ *x*. *ennreal* (*f x*) ∂*M*) ≠ ∞ (∫ $^+$ *x*. *ennreal* (− *f x*) ∂*M*) ≠ ∞
have (∫ $^+$ *x*. *norm* (*f x*) ∂*M*) = (∫ $^+$ *x*. *ennreal* (*f x*) + *ennreal* (− *f x*) ∂*M*)
by (*intro nn_integral_cong*) (*auto simp*: *abs_real_def ennreal_neg*)
also have. . . = (∫ $^+$ *x*. *ennreal* (*f x*) ∂*M*) + (∫ $^+$ *x*. *ennreal* (− *f x*) ∂*M*)
by (*intro nn_integral_add*) *auto*
also have . . . < ∞
using *fin* **by** (*auto simp*: *less_top*)
finally show (∫ $^+$ *x*. *norm* (*f x*) ∂*M*) < ∞ **.**
qed

lemma *integrableD*[*dest*]:
assumes *integrable M f*
shows *f* ∈ *borel_measurable M* (∫ $^+$ *x*. *ennreal* (*f x*) ∂*M*) ≠ ∞ (∫ $^+$ *x*. *ennreal* (− *f x*) ∂*M*) ≠ ∞
using *assms* **unfolding** *real_integrable_def* **by** *auto*

lemma *integrableE*:
assumes *integrable M f*
obtains *r q* **where** *0* ≤ *r 0* ≤ *q*
(∫ $^+$*x*. *ennreal* (*f x*)∂*M*) = *ennreal r*
(∫ $^+$*x*. *ennreal* (−*f x*)∂*M*) = *ennreal q*
f ∈ *borel_measurable M* $integral^L$ *M f* = *r* − *q*

using *assms* **unfolding** *real_integrable_def real_lebesgue_integral_def*[*OF assms*]
by (*cases rule*: *ennreal2_cases*[*of* ($\int^+ x$. *ennreal* ($-f\ x$)∂M) ($\int^+ x$. *ennreal* (*f* *x*)∂M)]) *auto*

lemma *integral_monotone_convergence_nonneg*:
fixes *f* :: *nat* ⇒ *'a* ⇒ *real*
assumes *i*: ⋀*i*. *integrable M* (*f i*) **and** *mono*: *AE x in M*. *mono* (λ*n*. *f n x*)
and *pos*: ⋀*i*. *AE x in M*. *0* ≤ *f i x*
and *lim*: *AE x in M*. (λ*i*. *f i x*) ⟶ *u x*
and *ilim*: (λ*i*. *integral*^{L} *M* (*f i*)) ⟶ *x*
and *u*: *u* ∈ *borel_measurable M*
shows *integrable M u*
and *integral*^{L} *M u* = *x*
proof −
have *nn*: *AE x in M*. ∀ *i*. *0* ≤ *f i x*
using *pos* **unfolding** *AE_all_countable* **by** *auto*
with *lim* **have** *u_nn*: *AE x in M*. *0* ≤ *u x*
by *eventually_elim* (*auto intro*: *LIMSEQ_le_const*)
have [*simp*]: *0* ≤ *x*
by (*intro LIMSEQ_le_const*[*OF ilim*] *allI exI impI integral_nonneg_AE pos*)
have ($\int^+ x$. *ennreal* (*u x*) ∂M) = (*SUP n*. ($\int^+ x$. *ennreal* (*f n x*) ∂M))
proof (*subst nn_integral_monotone_convergence_SUP_AE*[*symmetric*])
fix *i*
from *mono nn* **show** *AE x in M*. *ennreal* (*f i x*) ≤ *ennreal* (*f* (*Suc i*) *x*)
by *eventually_elim* (*auto simp*: *mono_def*)
show (λ*x*. *ennreal* (*f i x*)) ∈ *borel_measurable M*
using *i* **by** *auto*
next
show ($\int^+ x$. *ennreal* (*u x*) ∂M) = $\int^+ x$. (*SUP i*. *ennreal* (*f i x*)) ∂M
apply (*rule nn_integral_cong_AE*)
using *lim mono nn u_nn*
apply *eventually_elim*
apply (*simp add*: *LIMSEQ_unique*[*OF _ LIMSEQ_SUP*] *incseq_def*)
done
qed
also have ... = *ennreal x*
using *mono i nn* **unfolding** *nn_integral_eq_integral*[*OF i pos*]
by (*subst LIMSEQ_unique*[*OF LIMSEQ_SUP*]) (*auto simp*: *mono_def integral_nonneg_AE pos intro*!: *integral_mono_AE ilim*)
finally have ($\int^+ x$. *ennreal* (*u x*) ∂M) = *ennreal x* **.**
moreover have ($\int^+ x$. *ennreal* (− *u x*) ∂M) = *0*
using *u u_nn* **by** (*subst nn_integral_0_iff_AE*) (*auto simp add*: *ennreal_neg*)
ultimately show *integrable M u integral*^{L} *M u* = *x*
by (*auto simp*: *real_integrable_def real_lebesgue_integral_def u*)
qed

lemma
fixes *f* :: *nat* ⇒ *'a* ⇒ *real*
assumes *f*: ⋀*i*. *integrable M* (*f i*) **and** *mono*: *AE x in M*. *mono* (λ*n*. *f n x*)

and *lim*: $AE\ x\ in\ M.\ (\lambda i.\ f\ i\ x) \longrightarrow u\ x$
and *ilim*: $(\lambda i.\ integral^L\ M\ (f\ i)) \longrightarrow x$
and *u*: $u \in borel\_measurable\ M$
shows *integrable_monotone_convergence*: *integrable M u*
and *integral_monotone_convergence*: $integral^L\ M\ u = x$
and *has_bochner_integral_monotone_convergence*: *has_bochner_integral M u x*
proof −
have *1*: $\bigwedge i.\ integrable\ M\ (\lambda x.\ f\ i\ x - f\ 0\ x)$
using *f* **by** *auto*
have *2*: $AE\ x\ in\ M.\ mono\ (\lambda n.\ f\ n\ x - f\ 0\ x)$
using *mono* **by** (*auto simp*: *mono_def le_fun_def*)
have *3*: $\bigwedge n.\ AE\ x\ in\ M.\ 0 \le f\ n\ x - f\ 0\ x$
using *mono* **by** (*auto simp*: *field_simps mono_def le_fun_def*)
have *4*: $AE\ x\ in\ M.\ (\lambda i.\ f\ i\ x - f\ 0\ x) \longrightarrow u\ x - f\ 0\ x$
using *lim* **by** (*auto intro*!: *tendsto_diff*)
have *5*: $(\lambda i.\ (\int x.\ f\ i\ x - f\ 0\ x\ \partial M)) \longrightarrow x - integral^L\ M\ (f\ 0)$
using *f ilim* **by** (*auto intro*!: *tendsto_diff*)
have *6*: $(\lambda x.\ u\ x - f\ 0\ x) \in borel\_measurable\ M$
using *f*[*of 0*] *u* **by** *auto*
note *diff* = *integral_monotone_convergence_nonneg*[*OF 1 2 3 4 5 6*]
have *integrable M* $(\lambda x.\ (u\ x - f\ 0\ x) + f\ 0\ x)$
using *diff*(*1*) *f* **by** (*rule integrable_add*)
with *diff*(*2*) *f* **show** *integrable M u* $integral^L\ M\ u = x$
by *auto*
then show *has_bochner_integral M u x*
by (*metis has_bochner_integral_integrable*)
qed

lemma *integral_norm_eq_0_iff*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes *f*[*measurable*]: *integrable M f*
shows $(\int x.\ norm\ (f\ x)\ \partial M) = 0 \longleftrightarrow emeasure\ M\ \{x{\in}space\ M.\ f\ x \neq 0\} = 0$
proof −
have $(\int^+ x.\ norm\ (f\ x)\ \partial M) = (\int x.\ norm\ (f\ x)\ \partial M)$
using *f* **by** (*intro nn_integral_eq_integral integrable_norm*) *auto*
then have $(\int x.\ norm\ (f\ x)\ \partial M) = 0 \longleftrightarrow (\int^+ x.\ norm\ (f\ x)\ \partial M) = 0$
by *simp*
also have $\ldots \longleftrightarrow emeasure\ M\ \{x{\in}space\ M.\ ennreal\ (norm\ (f\ x)) \neq 0\} = 0$
by (*intro nn_integral_0_iff*) *auto*
finally show *?thesis*
by *simp*
qed

lemma *integral_0_iff*:
fixes *f* :: *'a* ⇒ *real*
shows $integrable\ M\ f \Longrightarrow (\int x.\ |f\ x|\ \partial M) = 0 \longleftrightarrow emeasure\ M\ \{x{\in}space\ M.\ f\ x \neq 0\} = 0$
using *integral_norm_eq_0_iff*[*of M f*] **by** *simp*

lemma (**in** *finite_measure*) *integrable_const*[*intro*!, *simp*]: *integrable M* ($\lambda x.\ a$)
using *integrable_indicator*[*of space M M a*] **by** (*simp cong*: *integrable_cong add*: *less_top*[*symmetric*])

lemma *lebesgue_integral_const*[*simp*]:
fixes *a* :: *'a* :: {*banach*, *second_countable_topology*}
shows ($\int x.\ a\ \partial M$) = *measure M* (*space M*) $*_R$ *a*
proof –
{ **assume** *emeasure M* (*space M*) = ∞ *a* ≠ *0*
then have *?thesis*
by (*auto simp add*: *not_integrable_integral_eq ennreal_mult_less_top measure_def integrable_iff_bounded*) }
moreover
{ **assume** *a* = *0* **then have** *?thesis* **by** *simp* }
moreover
{ **assume** *emeasure M* (*space M*) ≠ ∞
interpret *finite_measure M*
proof qed *fact*
have ($\int x.\ a\ \partial M$) = ($\int x.$ *indicator* (*space M*) *x* $*_R$ *a* ∂M)
by (*intro integral_cong*) *auto*
also have ... = *measure M* (*space M*) $*_R$ *a*
by (*simp add*: *less_top*[*symmetric*])
finally have *?thesis* **.** }
ultimately show *?thesis* **by** *blast*
qed

lemma (**in** *finite_measure*) *integrable_const_bound*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*,*second_countable_topology*}
shows *AE x in M*. *norm* (*f x*) ≤ *B* ⟹ *f* ∈ *borel_measurable M* ⟹ *integrable M f*
apply (*rule integrable_bound*[*OF integrable_const*[*of B*], *of f*])
apply *assumption*
apply (*cases 0* ≤ *B*)
apply *auto*
done

lemma (**in** *finite_measure*) *integral_bounded_eq_bound_then_AE*:
assumes *AE x in M*. *f x* ≤ (*c*::*real*)
integrable M f ($\int x.\ f\ x\ \partial M$) = *c* ∗ *measure M* (*space M*)
shows *AE x in M*. *f x* = *c*
apply (*rule integral_ineq_eq_0_then_AE*) **using** *assms* **by** *auto*

lemma *integral_indicator_finite_real*:
fixes *f* :: *'a* ⇒ *real*
assumes [*simp*]: *finite A*
assumes [*measurable*]: ⋀*a*. *a* ∈ *A* ⟹ {*a*} ∈ *sets M*
assumes *finite*: ⋀*a*. *a* ∈ *A* ⟹ *emeasure M* {*a*} < ∞
shows ($\int x.\ f\ x$ ∗ *indicator A x* ∂M) = ($\sum$ *a*∈*A*. *f a* ∗ *measure M* {*a*})
proof –

have ($\int x.\ f\,x * indicator\ A\ x\ \partial M$) = ($\int x.$ ($\sum a{\in}A.\ f\ a * indicator\ \{a\}\ x$) ∂M)
proof (*intro integral_cong refl*)
fix *x* **show** $f\,x * indicator\ A\ x = (\sum a{\in}A.\ f\ a * indicator\ \{a\}\ x)$
by (*auto split*: *split_indicator simp*: *eq_commute*[*of x*] *cong*: *conj_cong*)
qed
also have ... = ($\sum a{\in}A.\ f\ a * measure\ M\ \{a\}$)
using *finite* **by** (*subst integral_sum*) (*auto*)
finally show *?thesis* **.**
qed

lemma (**in** *finite_measure*) *ennreal_integral_real*:
assumes [*measurable*]: $f \in borel\_measurable\ M$
assumes *ae*: *AE x in M*. $f\,x \le ennreal\ B\ 0 \le B$
shows *ennreal* ($\int x.\ enn2real\ (f\,x)\ \partial M$) = ($\int^+ x.\ f\ x\ \partial M$)
proof (*subst nn_integral_eq_integral*[*symmetric*])
show *integrable M* ($\lambda x.\ enn2real\ (f\,x)$)
using *ae* **by** (*intro integrable_const_bound*[**where** *B*=*B*]) (*auto simp*: *enn2real_leI*)
show ($\int^+ x.\ ennreal\ (enn2real\ (f\,x))\ \partial M$) = $integral^N\ M\ f$
using *ae* **by** (*intro nn_integral_cong_AE*) (*auto simp*: *le_less_trans*[*OF _ ennreal_less_top*])
qed *auto*

lemma (**in** *finite_measure*) *integral_less_AE*:
fixes $X\ Y :: 'a \Rightarrow real$
assumes *int*: *integrable M X integrable M Y*
assumes *A*: (*emeasure M*) $A \neq 0\ A \in sets\ M$ *AE x in M*. $x \in A \longrightarrow X\,x \neq Y\,x$
assumes *gt*: *AE x in M*. $X\,x \le Y\,x$
shows $integral^L\ M\ X < integral^L\ M\ Y$
proof −
have $integral^L\ M\ X \le integral^L\ M\ Y$
using *gt int* **by** (*intro integral_mono_AE*) *auto*
moreover
have $integral^L\ M\ X \neq integral^L\ M\ Y$
proof
assume *eq*: $integral^L\ M\ X = integral^L\ M\ Y$
have $integral^L\ M\ (\lambda x.\ |Y\,x - X\,x|) = integral^L\ M\ (\lambda x.\ Y\,x - X\,x)$
using *gt int* **by** (*intro integral_cong_AE*) *auto*
also have ... = *0*
using *eq int* **by** *simp*
finally have (*emeasure M*) $\{x \in space\ M.\ Y\,x - X\,x \neq 0\} = 0$
using *int* **by** (*simp add*: *integral_0_iff*)
moreover
have ($\int^+ x.\ indicator\ A\ x\ \partial M$) $\le$ ($\int^+ x.\ indicator\ \{x \in space\ M.\ Y\,x - X\,x \neq 0\}\ x\ \partial M$)
using *A* **by** (*intro nn_integral_mono_AE*) *auto*
then have (*emeasure M*) $A \le$ (*emeasure M*) $\{x \in space\ M.\ Y\,x - X\,x \neq 0\}$
using *int A* **by** (*simp add*: *integrable_def*)
ultimately have *emeasure M A* = *0*
by *simp*

```
    with ‹(emeasure M) A ≠ 0› show False by auto
  qed
  ultimately show ?thesis by auto
qed

lemma (in finite_measure) integral_less_AE_space:
  fixes X Y :: 'a ⇒ real
  assumes int: integrable M X integrable M Y
  assumes gt: AE x in M. X x < Y x emeasure M (space M) ≠ 0
  shows integral^L M X < integral^L M Y
  using gt by (intro integral_less_AE[OF int, where A=space M]) auto

lemma tendsto_integral_at_top:
  fixes f :: real ⇒ 'a::{banach, second_countable_topology}
  assumes [measurable_cong]: sets M = sets borel and f[measurable]: integrable M
f
  shows ((λy. ∫ x. indicator {.. y} x *R f x ∂M) ⟶ ∫ x. f x ∂M) at_top
proof (rule tendsto_at_topI_sequentially)
  fix X :: nat ⇒ real assume filterlim X at_top sequentially
  show (λn. ∫x. indicator {..X n} x *R f x ∂M) ⟶ integral^L M f
  proof (rule integral_dominated_convergence)
    show integrable M (λx. norm (f x))
      by (rule integrable_norm) fact
    show AE x in M. (λn. indicator {..X n} x *R f x) ⟶ f x
    proof
      fix x
      from ‹filterlim X at_top sequentially›
      have eventually (λn. x ≤ X n) sequentially
        unfolding filterlim_at_top_ge[where c=x] by auto
      then show (λn. indicator {..X n} x *R f x) ⟶ f x
          by (intro tendsto_eventually) (auto split: split_indicator elim!: eventu-
ally_mono)
    qed
    fix n show AE x in M. norm (indicator {..X n} x *R f x) ≤ norm (f x)
      by (auto split: split_indicator)
  qed auto
qed

lemma
  fixes f :: real ⇒ real
  assumes M: sets M = sets borel
  assumes nonneg: AE x in M. 0 ≤ f x
  assumes borel: f ∈ borel_measurable borel
  assumes int: ⋀y. integrable M (λx. f x * indicator {.. y} x)
  assumes conv: ((λy. ∫ x. f x * indicator {.. y} x ∂M) ⟶ x) at_top
  shows has_bochner_integral_monotone_convergence_at_top: has_bochner_integral M
f x
    and integrable_monotone_convergence_at_top: integrable M f
    and integral_monotone_convergence_at_top:integral^L M f = x
```

proof −
  from *nonneg* **have** $AE\ x\ in\ M.\ mono\ (\lambda n{::}nat.\ f\ x * indicator\ \{..real\ n\}\ x)$
    by (*auto split*: *split_indicator intro*!: *monoI*)
  { **fix** x **have** $eventually\ (\lambda n.\ f\ x * indicator\ \{..real\ n\}\ x = f\ x)\ sequentially$
      by (*rule eventually_sequentiallyI*[*of nat* $\lceil x\rceil$])
        (*auto split*: *split_indicator simp*: *nat_le_iff ceiling_le_iff*) }
  from *filterlim_cong*[*OF refl refl this*]
  have $AE\ x\ in\ M.\ (\lambda i.\ f\ x * indicator\ \{..real\ i\}\ x) \longrightarrow f\ x$
    by *simp*
  have $(\lambda i.\ \int\ x.\ f\ x * indicator\ \{..real\ i\}\ x\ \partial M) \longrightarrow x$
    using *conv filterlim_real_sequentially* **by** (*rule filterlim_compose*)
  have *M_measure*[*simp*]: *borel_measurable M* = *borel_measurable borel*
    using *M* **by** (*simp add*: *sets_eq_imp_space_eq measurable_def*)
  have $f \in$ *borel_measurable M*
    using *borel* **by** *simp*
  show *has_bochner_integral M f x*
    by (*rule has_bochner_integral_monotone_convergence*) *fact*+
  then show *integrable M f integral*^{L} $M\ f = x$
    by (*auto simp*: *_has_bochner_integral_iff*)
qed

6.10.8 Product measure

lemma (**in** *sigma_finite_measure*) *borel_measurable_lebesgue_integrable*[*measurable* (*raw*)]:
  fixes f :: _ ⇒ _ ⇒ _::{*banach*, *second_countable_topology*}
  assumes [*measurable*]: *case_prod f* $\in$ *borel_measurable* $(N \bigotimes_M M)$
  shows *Measurable.pred N* $(\lambda x.\ integrable\ M\ (f\ x))$
proof −
  have [*simp*]: $\bigwedge x.\ x \in space\ N \Longrightarrow integrable\ M\ (f\ x) \longleftrightarrow (\int^+ y.\ norm\ (f\ x\ y)\ \partial M) < \infty$
    unfolding *integrable_iff_bounded* **by** *simp*
  show *?thesis*
    by (*simp cong*: *measurable_cong*)
qed

lemma (**in** *sigma_finite_measure*) *measurable_measure*[*measurable* (*raw*)]:
  $(\bigwedge x.\ x \in space\ N \Longrightarrow A\ x \subseteq space\ M) \Longrightarrow$
    $\{x \in space\ (N \bigotimes_M M).\ snd\ x \in A\ (fst\ x)\} \in sets\ (N \bigotimes_M M) \Longrightarrow$
    $(\lambda x.\ measure\ M\ (A\ x)) \in$ *borel_measurable N*
 unfolding *measure_def* **by** (*intro measurable_emeasure borel_measurable_enn2real*) *auto*

proposition (**in** *sigma_finite_measure*) *borel_measurable_lebesgue_integral*[*measurable* (*raw*)]:
  fixes f :: _ ⇒ _ ⇒ _::{*banach*, *second_countable_topology*}
  assumes f[*measurable*]: *case_prod f* $\in$ *borel_measurable* $(N \bigotimes_M M)$
  shows $(\lambda x.\ \int\ y.\ f\ x\ y\ \partial M) \in$ *borel_measurable N*
proof −

from *borel_measurable_implies_sequence_metric*[*OF f*, *of 0*] **guess** *s* **..**
then have *s*: ⋀*i*. *simple_function* (*N* ⨂$_M$ *M*) (*s i*)
⋀*x y*. *x* ∈ *space N* ⟹ *y* ∈ *space M* ⟹ (λ*i*. *s i* (*x*, *y*)) ⟶ *f x y*
⋀*i x y*. *x* ∈ *space N* ⟹ *y* ∈ *space M* ⟹ *norm* (*s i* (*x*, *y*)) ≤ *2* ∗ *norm* (*f x y*)
by (*auto simp*: *space_pair_measure*)

have [*measurable*]: ⋀*i*. *s i* ∈ *borel_measurable* (*N* ⨂$_M$ *M*)
by (*rule borel_measurable_simple_function*) *fact*

have ⋀*i*. *s i* ∈ *measurable* (*N* ⨂$_M$ *M*) (*count_space UNIV*)
by (*rule measurable_simple_function*) *fact*

define *f'* **where** [*abs_def*]: *f' i x* =
(*if integrable M* (*f x*) *then simple_bochner_integral M* (λ*y*. *s i* (*x*, *y*)) *else 0*)
for *i x*

{ **fix** *i x* **assume** *x* ∈ *space N*
then have *simple_bochner_integral M* (λ*y*. *s i* (*x*, *y*)) =
(∑*z*∈*s i* ' (*space N* × *space M*). *measure M* {*y* ∈ *space M*. *s i* (*x*, *y*) = *z*} ∗$_R$ *z*)
using *s*(*1*)[*THEN simple_functionD*(*1*)]
unfolding *simple_bochner_integral_def*
by (*intro sum.mono_neutral_cong_left*)
(*auto simp*: *eq_commute space_pair_measure image_iff cong*: *conj_cong*) }
note *eq* = *this*

show *?thesis*
proof (*rule borel_measurable_LIMSEQ_metric*)
fix *i* **show** *f' i* ∈ *borel_measurable N*
unfolding *f'_def* **by** (*simp_all add*: *eq cong*: *measurable_cong if_cong*)
next
fix *x* **assume** *x*: *x* ∈ *space N*
{ **assume** *int_f*: *integrable M* (*f x*)
have *int_2f*: *integrable M* (λ*y*. *2* ∗ *norm* (*f x y*))
by (*intro integrable_norm integrable_mult_right int_f*)
have (λ*i*. *integral*^{L} *M* (λ*y*. *s i* (*x*, *y*))) ⟶ *integral*^{L} *M* (*f x*)
proof (*rule integral_dominated_convergence*)
from *int_f* **show** *f x* ∈ *borel_measurable M* **by** *auto*
show ⋀*i*. (λ*y*. *s i* (*x*, *y*)) ∈ *borel_measurable M*
using *x* **by** *simp*
show *AE xa in M*. (λ*i*. *s i* (*x*, *xa*)) ⟶ *f x xa*
using *x s*(*2*) **by** *auto*
show ⋀*i*. *AE xa in M*. *norm* (*s i* (*x*, *xa*)) ≤ *2* ∗ *norm* (*f x xa*)
using *x s*(*3*) **by** *auto*
qed *fact*
moreover
{ **fix** *i*
have *simple_bochner_integrable M* (λ*y*. *s i* (*x*, *y*))

proof (*rule simple_bochner_integrableI_bounded*)
have (λ*y*. *s i* (*x*, *y*)) ' *space M* ⊆ *s i* ' (*space N* × *space M*)
using *x* **by** *auto*
then show *simple_function M* (λ*y*. *s i* (*x*, *y*))
using *simple_functionD*(*1*)[*OF s*(*1*), *of i*] *x*
by (*intro simple_function_borel_measurable*)
(*auto simp*: *space_pair_measure dest*: *finite_subset*)
have (∫ + *y*. *ennreal* (*norm* (*s i* (*x*, *y*))) ∂*M*) ≤ (∫ + *y*. *2* ∗ *norm* (*f x y*) ∂*M*)
using *x s* **by** (*intro nn_integral_mono*) *auto*
also have (∫ + *y*. *2* ∗ *norm* (*f x y*) ∂*M*) < ∞
using *int_2f* **unfolding** *integrable_iff_bounded* **by** *simp*
finally show (∫ + *xa*. *ennreal* (*norm* (*s i* (*x*, *xa*))) ∂*M*) < ∞ **.**
qed
then have *integral*^{L} *M* (λ*y*. *s i* (*x*, *y*)) = *simple_bochner_integral M* (λ*y*. *s i* (*x*, *y*))
by (*rule simple_bochner_integrable_eq_integral*[*symmetric*]) **}**
ultimately have (λ*i*. *simple_bochner_integral M* (λ*y*. *s i* (*x*, *y*))) ⟶ *integral*^{L} *M* (*f x*)
by *simp* **}**
then
show (λ*i*. *f ′ i x*) ⟶ *integral*^{L} *M* (*f x*)
unfolding *f ′_def*
by (*cases integrable M* (*f x*)) (*simp_all add*: *not_integrable_integral_eq*)
qed
qed

lemma (**in** *pair_sigma_finite*) *integrable_product_swap*:
fixes *f* :: _ ⇒ _::{*banach*, *second_countable_topology*}
assumes *integrable* (*M1* ⨂$_M$ *M2*) *f*
shows *integrable* (*M2* ⨂$_M$ *M1*) (λ(*x*,*y*). *f* (*y*,*x*))
proof −
interpret *Q*: *pair_sigma_finite M2 M1* **..**
have ∗: (λ(*x*,*y*). *f* (*y*,*x*)) = (λ*x*. *f* (*case x of* (*x*,*y*)⇒(*y*,*x*))) **by** (*auto simp*: *fun_eq_iff*)
show *?thesis* **unfolding** ∗
by (*rule integrable_distr*[*OF measurable_pair_swap'*])
(*simp add*: *distr_pair_swap*[*symmetric*] *assms*)
qed

lemma (**in** *pair_sigma_finite*) *integrable_product_swap_iff*:
fixes *f* :: _ ⇒ _::{*banach*, *second_countable_topology*}
shows *integrable* (*M2* ⨂$_M$ *M1*) (λ(*x*,*y*). *f* (*y*,*x*)) ⟷ *integrable* (*M1* ⨂$_M$ *M2*) *f*
proof −
interpret *Q*: *pair_sigma_finite M2 M1* **..**
from *Q*.*integrable_product_swap*[*of* λ(*x*,*y*). *f* (*y*,*x*)] *integrable_product_swap*[*of f*]
show *?thesis* **by** *auto*
qed

lemma (**in** *pair_sigma_finite*) *integral_product_swap*:
fixes *f* :: _ ⇒ _::{*banach*, *second_countable_topology*}
assumes *f*: *f* ∈ *borel_measurable* (*M1* ⨂$_M$ *M2*)
shows (∫ (*x*,*y*). *f* (*y*,*x*) ∂(*M2* ⨂$_M$ *M1*)) = *integral*^{L} (*M1* ⨂$_M$ *M2*) *f*
proof –
have ∗: (λ(*x*,*y*). *f* (*y*,*x*)) = (λ*x*. *f* (*case x of* (*x*,*y*)⇒(*y*,*x*))) **by** (*auto simp*: *fun_eq_iff*)
show *?thesis* **unfolding** ∗
by (*simp add*: *integral_distr*[*symmetric*, *OF measurable_pair_swap′ f*] *distr_pair_swap*[*symmetric*])
qed

theorem (**in** *pair_sigma_finite*) *Fubini_integrable*:
fixes *f* :: _ ⇒ _::{*banach*, *second_countable_topology*}
assumes *f*[*measurable*]: *f* ∈ *borel_measurable* (*M1* ⨂$_M$ *M2*)
and *integ1*: *integrable M1* (λ*x*. ∫ *y*. *norm* (*f* (*x*, *y*)) ∂*M2*)
and *integ2*: *AE x in M1*. *integrable M2* (λ*y*. *f* (*x*, *y*))
shows *integrable* (*M1* ⨂$_M$ *M2*) *f*
proof (*rule integrableI_bounded*)
have (∫$^+$ *p*. *norm* (*f p*) ∂(*M1* ⨂$_M$ *M2*)) = (∫$^+$ *x*. (∫$^+$ *y*. *norm* (*f* (*x*, *y*)) ∂*M2*) ∂*M1*)
by (*simp add*: *M2*.*nn_integral_fst* [*symmetric*])
also have ... = (∫$^+$ *x*. |∫ *y*. *norm* (*f* (*x*, *y*)) ∂*M2*| ∂*M1*)
apply (*intro nn_integral_cong_AE*)
using *integ2*
proof *eventually_elim*
fix *x* **assume** *integrable M2* (λ*y*. *f* (*x*, *y*))
then have *f*: *integrable M2* (λ*y*. *norm* (*f* (*x*, *y*)))
by *simp*
then have (∫$^+$*y*. *ennreal* (*norm* (*f* (*x*, *y*))) ∂*M2*) = *ennreal* (*LINT y*|*M2*. *norm* (*f* (*x*, *y*)))
by (*rule nn_integral_eq_integral*) *simp*
also have ... = *ennreal* |*LINT y*|*M2*. *norm* (*f* (*x*, *y*))|
using *f* **by** *simp*
finally show (∫$^+$*y*. *ennreal* (*norm* (*f* (*x*, *y*))) ∂*M2*) = *ennreal* |*LINT y*|*M2*. *norm* (*f* (*x*, *y*))| .
qed
also have ... < ∞
using *integ1* **by** (*simp add*: *integrable_iff_bounded integral_nonneg_AE*)
finally show (∫$^+$ *p*. *norm* (*f p*) ∂(*M1* ⨂$_M$ *M2*)) < ∞ .
qed *fact*

lemma (**in** *pair_sigma_finite*) *emeasure_pair_measure_finite*:
assumes *A*: *A* ∈ *sets* (*M1* ⨂$_M$ *M2*) **and** *finite*: *emeasure* (*M1* ⨂$_M$ *M2*) *A* < ∞
shows *AE x in M1*. *emeasure M2* {*y*∈*space M2*. (*x*, *y*) ∈ *A*} < ∞
proof –
from *M2*.*emeasure_pair_measure_alt*[*OF A*] *finite*
have (∫$^+$ *x*. *emeasure M2* (*Pair x* –‘ *A*) ∂*M1*) ≠ ∞

by *simp*
then have *AE x in M1. emeasure M2 (Pair x −' A)* ≠ ∞
by (*rule nn_integral_PInf_AE*[*rotated*]) (*intro M2.measurable_emeasure_Pair A*)
moreover have ⋀*x. x* ∈ *space M1* ⟹ *Pair x −' A* = {*y*∈*space M2.* (*x, y*) ∈ *A*}
using *sets.sets_into_space*[*OF A*] **by** (*auto simp*: *space_pair_measure*)
ultimately show *?thesis* **by** (*auto simp*: *less_top*)
qed

lemma (**in** *pair_sigma_finite*) *AE_integrable_fst'*:
fixes *f* :: _ ⇒ _::{*banach, second_countable_topology*}
assumes *f*[*measurable*]: *integrable* (*M1* ⨂_M *M2*) *f*
shows *AE x in M1. integrable M2* (λ*y. f* (*x, y*))
proof −
have (∫⁺*x.* (∫⁺*y. norm* (*f* (*x, y*)) ∂*M2*) ∂*M1*) = (∫⁺*x. norm* (*f x*) ∂(*M1* ⨂_M *M2*))
by (*rule M2.nn_integral_fst*) *simp*
also have (∫⁺*x. norm* (*f x*) ∂(*M1* ⨂_M *M2*)) ≠ ∞
using *f* **unfolding** *integrable_iff_bounded* **by** *simp*
finally have *AE x in M1.* (∫⁺*y. norm* (*f* (*x, y*)) ∂*M2*) ≠ ∞
by (*intro nn_integral_PInf_AE M2.borel_measurable_nn_integral*)
(*auto simp*: *measurable_split_conv*)
with *AE_space* **show** *?thesis*
by *eventually_elim*
(*auto simp*: *integrable_iff_bounded measurable_compose*[*OF _ borel_measurable_integrable*[*OF f*]] *less_top*)
qed

lemma (**in** *pair_sigma_finite*) *integrable_fst'*:
fixes *f* :: _ ⇒ _::{*banach, second_countable_topology*}
assumes *f*[*measurable*]: *integrable* (*M1* ⨂_M *M2*) *f*
shows *integrable M1* (λ*x.* ∫ *y. f* (*x, y*) ∂*M2*)
unfolding *integrable_iff_bounded*
proof
show (λ*x.* ∫ *y. f* (*x, y*) ∂*M2*) ∈ *borel_measurable M1*
by (*rule M2.borel_measurable_lebesgue_integral*) *simp*
have (∫⁺ *x. ennreal* (*norm* (∫ *y. f* (*x, y*) ∂*M2*)) ∂*M1*) ≤ (∫⁺*x.* (∫⁺*y. norm* (*f* (*x, y*)) ∂*M2*) ∂*M1*)
using *AE_integrable_fst'*[*OF f*] **by** (*auto intro*!: *nn_integral_mono_AE integral_norm_bound_ennreal*)
also have (∫⁺*x.* (∫⁺*y. norm* (*f* (*x, y*)) ∂*M2*) ∂*M1*) = (∫⁺*x. norm* (*f x*) ∂(*M1* ⨂_M *M2*))
by (*rule M2.nn_integral_fst*) *simp*
also have (∫⁺*x. norm* (*f x*) ∂(*M1* ⨂_M *M2*)) < ∞
using *f* **unfolding** *integrable_iff_bounded* **by** *simp*
finally show (∫⁺ *x. ennreal* (*norm* (∫ *y. f* (*x, y*) ∂*M2*)) ∂*M1*) < ∞ **.**
qed

proposition (**in** *pair_sigma_finite*) *integral_fst'*:

```
  fixes f :: _ ⇒ _::{banach, second_countable_topology}
  assumes f: integrable (M1 ⨂M M2) f
  shows (∫ x. (∫ y. f (x, y) ∂M2) ∂M1) = integralL (M1 ⨂M M2) f
using f proof induct
  case (base A c)
  have A[measurable]: A ∈ sets (M1 ⨂M M2) by fact

  have eq: ⋀x y. x ∈ space M1 ⟹ indicator A (x, y) = indicator {y∈space M2.
(x, y) ∈ A} y
   using sets.sets_into_space[OF A] by (auto split: split_indicator simp: space_pair_measure)

  have int_A: integrable (M1 ⨂M M2) (indicator A :: _ ⇒ real)
    using base by (rule integrable_real_indicator)

   have (∫ x. ∫ y. indicator A (x, y) *R c ∂M2 ∂M1) = (∫ x. measure M2
{y∈space M2. (x, y) ∈ A} *R c ∂M1)
  proof (intro integral_cong_AE, simp, simp)
    from AE_integrable_fst'[OF int_A] AE_space
   show AE x in M1. (∫ y. indicator A (x, y) *R c ∂M2) = measure M2 {y∈space
M2. (x, y) ∈ A} *R c
      by eventually_elim (simp add: eq integrable_indicator_iff)
  qed
  also have ... = measure (M1 ⨂M M2) A *R c
  proof (subst integral_scaleR_left)
    have (∫+x. ennreal (measure M2 {y ∈ space M2. (x, y) ∈ A}) ∂M1) =
      (∫+x. emeasure M2 {y ∈ space M2. (x, y) ∈ A} ∂M1)
      using emeasure_pair_measure_finite[OF base]
     by (intro nn_integral_cong_AE, eventually_elim) (simp add: emeasure_eq_ennreal_measure)
    also have ... = emeasure (M1 ⨂M M2) A
      using sets.sets_into_space[OF A]
      by (subst M2.emeasure_pair_measure_alt)
            (auto intro!: nn_integral_cong arg_cong[where f=emeasure M2] simp:
space_pair_measure)
     finally have *: (∫+x. ennreal (measure M2 {y ∈ space M2. (x, y) ∈ A})
∂M1) = emeasure (M1 ⨂M M2) A .

     from base * show integrable M1 (λx. measure M2 {y ∈ space M2. (x, y) ∈
A})
      by (simp add: integrable_iff_bounded)
    then have (∫ x. measure M2 {y ∈ space M2. (x, y) ∈ A} ∂M1) =
      (∫+x. ennreal (measure M2 {y ∈ space M2. (x, y) ∈ A}) ∂M1)
      by (rule nn_integral_eq_integral[symmetric]) simp
    also note *
     finally show (∫ x. measure M2 {y ∈ space M2. (x, y) ∈ A} ∂M1) *R c =
measure (M1 ⨂M M2) A *R c
      using base by (simp add: emeasure_eq_ennreal_measure)
  qed
  also have ... = (∫ a. indicator A a *R c ∂(M1 ⨂M M2))
    using base by simp
```

```
  finally show ?case .
next
  case (add f g)
 then have [measurable]: f ∈ borel_measurable (M1 ⨂M M2) g ∈ borel_measurable
(M1 ⨂M M2)
    by auto
  have (∫ x. ∫ y. f (x, y) + g (x, y) ∂M2 ∂M1) =
    (∫ x. (∫ y. f (x, y) ∂M2) + (∫ y. g (x, y) ∂M2) ∂M1)
    apply (rule integral_cong_AE)
    apply simp_all
    using AE_integrable_fst'[OF add(1)] AE_integrable_fst'[OF add(3)]
    apply eventually_elim
    apply simp
    done
  also have ... = (∫ x. f x ∂(M1 ⨂M M2)) + (∫ x. g x ∂(M1 ⨂M M2))
    using integrable_fst'[OF add(1)] integrable_fst'[OF add(3)] add(2,4) by simp
  finally show ?case
    using add by simp
next
  case (lim f s)
 then have [measurable]: f ∈ borel_measurable (M1 ⨂M M2) ⋀i. s i ∈ borel_measurable
(M1 ⨂M M2)
    by auto

  show ?case
  proof (rule LIMSEQ_unique)
    show (λi. integral^L (M1 ⨂M M2) (s i)) ⟶ integral^L (M1 ⨂M M2) f
    proof (rule integral_dominated_convergence)
      show integrable (M1 ⨂M M2) (λx. 2 * norm (f x))
        using lim(5) by auto
    qed (insert lim, auto)
     have (λi. ∫ x. ∫ y. s i (x, y) ∂M2 ∂M1) ⟶ ∫ x. ∫ y. f (x, y) ∂M2
∂M1
    proof (rule integral_dominated_convergence)
      have AE x in M1. ∀ i. integrable M2 (λy. s i (x, y))
        unfolding AE_all_countable using AE_integrable_fst'[OF lim(1)] ..
      with AE_space AE_integrable_fst'[OF lim(5)]
      show AE x in M1. (λi. ∫ y. s i (x, y) ∂M2) ⟶ ∫ y. f (x, y) ∂M2
      proof eventually_elim
        fix x assume x: x ∈ space M1 and
          s: ∀ i. integrable M2 (λy. s i (x, y)) and f: integrable M2 (λy. f (x, y))
        show (λi. ∫ y. s i (x, y) ∂M2) ⟶ ∫ y. f (x, y) ∂M2
        proof (rule integral_dominated_convergence)
          show integrable M2 (λy. 2 * norm (f (x, y)))
             using f by auto
          show AE xa in M2. (λi. s i (x, xa)) ⟶ f (x, xa)
            using x lim(3) by (auto simp: space_pair_measure)
          show ⋀i. AE xa in M2. norm (s i (x, xa)) ≤ 2 * norm (f (x, xa))
            using x lim(4) by (auto simp: space_pair_measure)
```

qed (*insert x*, *measurable*)
qed
show *integrable M1* (λ*x*. (∫ *y*. *2* ∗ *norm* (*f* (*x*, *y*)) ∂*M2*))
by (*intro integrable_mult_right integrable_norm integrable_fst′ lim*)
fix *i* **show** *AE x in M1*. *norm* (∫ *y*. *s i* (*x*, *y*) ∂*M2*) ≤ (∫ *y*. *2* ∗ *norm* (*f* (*x*, *y*)) ∂*M2*)
using *AE_space AE_integrable_fst′*[*OF lim*(*1*), *of i*] *AE_integrable_fst′*[*OF lim*(*5*)]
proof *eventually_elim*
fix *x* **assume** *x*: *x* ∈ *space M1*
and *s*: *integrable M2* (λ*y*. *s i* (*x*, *y*)) **and** *f*: *integrable M2* (λ*y*. *f* (*x*, *y*))
from *s* **have** *norm* (∫ *y*. *s i* (*x*, *y*) ∂*M2*) ≤ (∫⁺*y*. *norm* (*s i* (*x*, *y*)) ∂*M2*)
by (*rule integral_norm_bound_ennreal*)
also have ... ≤ (∫⁺*y*. *2* ∗ *norm* (*f* (*x*, *y*)) ∂*M2*)
using *x lim* **by** (*auto intro*!: *nn_integral_mono simp*: *space_pair_measure*)
also have ... = (∫ *y*. *2* ∗ *norm* (*f* (*x*, *y*)) ∂*M2*)
using *f* **by** (*intro nn_integral_eq_integral*) *auto*
finally show *norm* (∫ *y*. *s i* (*x*, *y*) ∂*M2*) ≤ (∫ *y*. *2* ∗ *norm* (*f* (*x*, *y*)) ∂*M2*)
by *simp*
qed
qed *simp_all*
then show (λ*i*. *integral*^{L} (*M1* ⨂$_M$ *M2*) (*s i*)) ⟶ ∫ *x*. ∫ *y*. *f* (*x*, *y*) ∂*M2* ∂*M1*
using *lim* **by** *simp*
qed
qed

lemma (**in** *pair_sigma_finite*)
fixes *f* :: _ ⇒ _ ⇒ _::{*banach*, *second_countable_topology*}
assumes *f*: *integrable* (*M1* ⨂$_M$ *M2*) (*case_prod f*)
shows *AE_integrable_fst*: *AE x in M1*. *integrable M2* (λ*y*. *f x y*) (**is** *?AE*)
and *integrable_fst*: *integrable M1* (λ*x*. ∫ *y*. *f x y* ∂*M2*) (**is** *?INT*)
and *integral_fst*: (∫ *x*. (∫ *y*. *f x y* ∂*M2*) ∂*M1*) = *integral*^{L} (*M1* ⨂$_M$ *M2*) (λ(*x*, *y*). *f x y*) (**is** *?EQ*)
using *AE_integrable_fst′*[*OF f*] *integrable_fst′*[*OF f*] *integral_fst′*[*OF f*] **by** *auto*

lemma (**in** *pair_sigma_finite*)
fixes *f* :: _ ⇒ _ ⇒ _::{*banach*, *second_countable_topology*}
assumes *f*[*measurable*]: *integrable* (*M1* ⨂$_M$ *M2*) (*case_prod f*)
shows *AE_integrable_snd*: *AE y in M2*. *integrable M1* (λ*x*. *f x y*) (**is** *?AE*)
and *integrable_snd*: *integrable M2* (λ*y*. ∫ *x*. *f x y* ∂*M1*) (**is** *?INT*)
and *integral_snd*: (∫ *y*. (∫ *x*. *f x y* ∂*M1*) ∂*M2*) = *integral*^{L} (*M1* ⨂$_M$ *M2*) (*case_prod f*) (**is** *?EQ*)
proof −
interpret *Q*: *pair_sigma_finite M2 M1* ..
have *Q_int*: *integrable* (*M2* ⨂$_M$ *M1*) (λ(*x*, *y*). *f y x*)
using *f* **unfolding** *integrable_product_swap_iff*[*symmetric*] **by** *simp*
show *?AE* **using** *Q*.*AE_integrable_fst′*[*OF Q_int*] **by** *simp*

show *?INT* **using** *Q.integrable_fst'*[*OF Q_int*] **by** *simp*
show *?EQ* **using** *Q.integral_fst'*[*OF Q_int*]
using *integral_product_swap*[*of case_prod f*] **by** *simp*
qed

proposition (**in** *pair_sigma_finite*) *Fubini_integral*:
fixes *f* :: _ ⇒ _ ⇒ _ :: {*banach*, *second_countable_topology*}
assumes *f*: *integrable* (*M1* $\bigotimes_M$ *M2*) (*case_prod f*)
shows (∫ *y*. (∫ *x*. *f x y* ∂*M1*) ∂*M2*) = (∫ *x*. (∫ *y*. *f x y* ∂*M2*) ∂*M1*)
unfolding *integral_snd*[*OF assms*] *integral_fst*[*OF assms*] **..**

lemma (**in** *product_sigma_finite*) *product_integral_singleton*:
fixes *f* :: _ ⇒ _::{*banach*, *second_countable_topology*}
shows *f* ∈ *borel_measurable* (*M i*) ⟹ (∫ *x*. *f* (*x i*) ∂Pi_M {*i*} *M*) = $integral^L$ (*M i*) *f*
apply (*subst distr_singleton*[*symmetric*])
apply (*subst integral_distr*)
apply *simp_all*
done

lemma (**in** *product_sigma_finite*) *product_integral_fold*:
fixes *f* :: _ ⇒ _::{*banach*, *second_countable_topology*}
assumes *IJ*[*simp*]: *I* ∩ *J* = {} **and** *fin*: *finite I finite J*
and *f*: *integrable* (Pi_M (*I* ∪ *J*) *M*) *f*
shows $integral^L$ (Pi_M (*I* ∪ *J*) *M*) *f* = (∫ *x*. (∫ *y*. *f* (*merge I J* (*x*, *y*)) ∂Pi_M *J* *M*) ∂Pi_M *I M*)
proof −
interpret *I*: *finite_product_sigma_finite M I* **by** *standard fact*
interpret *J*: *finite_product_sigma_finite M J* **by** *standard fact*
have *finite* (*I* ∪ *J*) **using** *fin* **by** *auto*
interpret *IJ*: *finite_product_sigma_finite M I* ∪ *J* **by** *standard fact*
interpret *P*: *pair_sigma_finite* Pi_M *I M* Pi_M *J M* **..**
let *?M* = *merge I J*
let *?f* = λ*x*. *f* (*?M x*)
from *f* **have** *f_borel*: *f* ∈ *borel_measurable* (Pi_M (*I* ∪ *J*) *M*)
by *auto*
have *P_borel*: (λ*x*. *f* (*merge I J x*)) ∈ *borel_measurable* (Pi_M *I M* $\bigotimes_M$ Pi_M *J M*)
using *measurable_comp*[*OF measurable_merge f_borel*] **by** (*simp add*: *comp_def*)
have *f_int*: *integrable* (Pi_M *I M* $\bigotimes_M$ Pi_M *J M*) *?f*
by (*rule integrable_distr*[*OF measurable_merge*]) (*simp add*: *distr_merge*[*OF IJ fin*] *f*)
show *?thesis*
apply (*subst distr_merge*[*symmetric*, *OF IJ fin*])
apply (*subst integral_distr*[*OF measurable_merge f_borel*])
apply (*subst P.integral_fst'*[*symmetric*, *OF f_int*])
apply *simp*
done
qed

lemma (**in** *product_sigma_finite*) *product_integral_insert*:
  fixes f :: _ ⇒ _::{*banach*, *second_countable_topology*}
  assumes I: *finite I* $i \notin I$
    and f: *integrable* (Pi_M (*insert i I*) M) f
  shows $integral^L$ (Pi_M (*insert i I*) M) f = (∫ x. (∫ y. f (x(i:=y)) ∂M i) ∂Pi_M I M)
proof −
  have $integral^L$ (Pi_M (*insert i I*) M) f = $integral^L$ (Pi_M ($I \cup \{i\}$) M) f
    by *simp*
  also have ... = (∫ x. (∫ y. f (*merge* I {i} (x,y)) ∂Pi_M {i} M) ∂Pi_M I M)
    using f I **by** (*intro product_integral_fold*) *auto*
  also have ... = (∫ x. (∫ y. f (x(i := y)) ∂M i) ∂Pi_M I M)
  proof (*rule integral_cong*[*OF refl*], *subst product_integral_singleton*[*symmetric*])
    fix x **assume** x: $x \in$ *space* (Pi_M I M)
    have *f_borel*: $f \in$ *borel_measurable* (Pi_M (*insert i I*) M)
      using f **by** *auto*
    show (λy. f (x(i := y))) ∈ *borel_measurable* (M i)
      using *measurable_comp*[*OF measurable_component_update f_borel*, *OF x* ‹$i \notin I$›]
      unfolding *comp_def* .
    from x I **show** (∫ y. f (*merge* I {i} (x,y)) ∂Pi_M {i} M) = (∫ xa. f (x(i := xa i)) ∂Pi_M {i} M)
      by (*auto intro*!: *integral_cong arg_cong*[**where** f=f] *simp*: *merge_def space_PiM extensional_def PiE_def*)
  qed
  finally show *?thesis* .
qed

lemma (**in** *product_sigma_finite*) *product_integrable_prod*:
  fixes f :: $'i$ ⇒ $'a$ ⇒ _::{*real_normed_field*,*banach*,*second_countable_topology*}
  assumes [*simp*]: *finite I* **and** *integrable*: ⋀i. $i \in I \Longrightarrow$ *integrable* (M i) (f i)
  shows *integrable* (Pi_M I M) (λx. (∏$i \in I$. f i (x i))) (**is** *integrable* _ *?f*)
proof (*unfold integrable_iff_bounded*, *intro conjI*)
  interpret *finite_product_sigma_finite M I* **by** *standard fact*

  show *?f* ∈ *borel_measurable* (Pi_M I M)
    using *assms* **by** *simp*
  have ($\int^+$ x. *ennreal* (*norm* (∏$i \in I$. f i (x i))) ∂Pi_M I M) =
      ($\int^+$ x. (∏$i \in I$. *ennreal* (*norm* (f i (x i)))) ∂Pi_M I M)
    by (*simp add*: *prod_norm prod_ennreal*)
  also have ... = (∏$i \in I$. $\int^+$ x. *ennreal* (*norm* (f i x)) ∂M i)
    using *assms* **by** (*intro product_nn_integral_prod*) *auto*
  also have ... < ∞
    using *integrable* **by** (*simp add*: *less_top*[*symmetric*] *ennreal_prod_eq_top integrable_iff_bounded*)
  finally show ($\int^+$ x. *ennreal* (*norm* (∏$i \in I$. f i (x i))) ∂Pi_M I M) < ∞ .
qed

lemma (**in** *product_sigma_finite*) *product_integral_prod*:
  fixes f :: $'i \Rightarrow 'a \Rightarrow$ _::{*real_normed_field*,*banach*,*second_countable_topology*}
  assumes *finite I* **and** *integrable*: $\bigwedge i.\ i \in I \Longrightarrow$ *integrable* (*M i*) (*f i*)
  shows ($\int x.\ (\prod i{\in}I.\ f\ i\ (x\ i))\ \partial Pi_M\ I\ M) = (\prod i{\in}I.\ integral^L\ (M\ i)\ (f\ i))$
using *assms* **proof** *induct*
  case *empty*
  interpret *finite_measure* Pi_M {} *M*
    by *rule* (*simp add*: *space_PiM*)
  show *?case* **by** (*simp add*: *space_PiM measure_def*)
next
  case (*insert i I*)
  then have *iI*: *finite* (*insert i I*) **by** *auto*
  then have *prod*: $\bigwedge J.\ J \subseteq$ *insert i I* $\Longrightarrow$
    integrable (Pi_M *J M*) ($\lambda x.\ (\prod i{\in}J.\ f\ i\ (x\ i))$)
    by (*intro product_integrable_prod insert*(*4*)) (*auto intro*: *finite_subset*)
  interpret *I*: *finite_product_sigma_finite M I* **by** *standard fact*
  have ∗: $\bigwedge x\ y.\ (\prod j{\in}I.\ f\ j\ (\textit{if}\ j = i\ \textit{then}\ y\ \textit{else}\ x\ j)) = (\prod j{\in}I.\ f\ j\ (x\ j))$
    using ‹$i \notin I$› **by** (*auto intro*!: *prod.cong*)
  show *?case*
    unfolding *product_integral_insert*[*OF insert*(*1*,*2*) *prod*[*OF subset_refl*]]
    by (*simp add*: ∗ *insert prod subset_insertI*)
qed

lemma *integrable_subalgebra*:
  fixes f :: $'a \Rightarrow 'b$::{*banach*, *second_countable_topology*}
  assumes *borel*: $f \in$ *borel_measurable N*
  and *N*: *sets N* $\subseteq$ *sets M space N* = *space M* $\bigwedge A.\ A \in$ *sets N* $\Longrightarrow$ *emeasure N A* = *emeasure M A*
  shows *integrable N f* $\longleftrightarrow$ *integrable M f* (**is** *?P*)
proof −
  have $f \in$ *borel_measurable M*
    using *assms* **by** (*auto simp*: *measurable_def*)
  with *assms* **show** *?thesis*
    using *assms* **by** (*auto simp*: *integrable_iff_bounded nn_integral_subalgebra*)
qed

lemma *integral_subalgebra*:
  fixes f :: $'a \Rightarrow 'b$::{*banach*, *second_countable_topology*}
  assumes *borel*: $f \in$ *borel_measurable N*
  and *N*: *sets N* $\subseteq$ *sets M space N* = *space M* $\bigwedge A.\ A \in$ *sets N* $\Longrightarrow$ *emeasure N A* = *emeasure M A*
  shows $integral^L\ N\ f = integral^L\ M\ f$
proof *cases*
  assume *integrable N f*
  then show *?thesis*
  proof *induct*
    case *base* **with** *assms* **show** *?case* **by** (*auto simp*: *subset_eq measure_def*)
  next
    case (*add f g*)

```
    then have (∫ a. f a + g a ∂N) = integral^L M f + integral^L M g
      by simp
    also have ... = (∫ a. f a + g a ∂M)
      using add integrable_subalgebra[OF _ N, of f] integrable_subalgebra[OF _ N,
of g] by simp
    finally show ?case .
  next
    case (lim f s)
    then have M: ⋀i. integrable M (s i) integrable M f
      using integrable_subalgebra[OF _ N, of f] integrable_subalgebra[OF _ N, of s i
for i] by simp_all
    show ?case
    proof (intro LIMSEQ_unique)
      show (λi. integral^L N (s i)) ⟶ integral^L N f
        apply (rule integral_dominated_convergence[where w=λx. 2 * norm (f x)])
        using lim
        apply auto
        done
      show (λi. integral^L N (s i)) ⟶ integral^L M f
        unfolding lim
        apply (rule integral_dominated_convergence[where w=λx. 2 * norm (f x)])
        using lim M N(2)
        apply auto
        done
    qed
  qed
qed (simp add: not_integrable_integral_eq integrable_subalgebra[OF assms])

hide_const (open) simple_bochner_integral
hide_const (open) simple_bochner_integrable

end
```

6.11 Complete Measures

```
theory Complete_Measure
  imports Bochner_Integration
begin

locale complete_measure =
  fixes M :: 'a measure
  assumes complete: ⋀A B. B ⊆ A ⟹ A ∈ null_sets M ⟹ B ∈ sets M

definition
  split_completion M A p = (if A ∈ sets M then p = (A, {}) else
   ∃N'. A = fst p ∪ snd p ∧ fst p ∩ snd p = {} ∧ fst p ∈ sets M ∧ snd p ⊆ N'
∧ N' ∈ null_sets M)

definition
```

main_part M A = fst (Eps (split_completion M A))

definition
null_part M A = snd (Eps (split_completion M A))

definition *completion* :: *'a measure ⇒ 'a measure* **where**
completion M = measure_of (space M) { S ∪ N |S N N'. S ∈ sets M ∧ N' ∈ null_sets M ∧ N ⊆ N' }
(emeasure M ∘ main_part M)

lemma *completion_into_space*:
{ S ∪ N |S N N'. S ∈ sets M ∧ N' ∈ null_sets M ∧ N ⊆ N' } ⊆ Pow (space M)
using *sets.sets_into_space* **by** *auto*

lemma *space_completion*[*simp*]: *space (completion M) = space M*
unfolding *completion_def* **using** *space_measure_of*[*OF completion_into_space*] **by** *simp*

lemma *completionI*:
assumes *A = S ∪ N N ⊆ N' N' ∈ null_sets M S ∈ sets M*
shows *A ∈ { S ∪ N |S N N'. S ∈ sets M ∧ N' ∈ null_sets M ∧ N ⊆ N' }*
using *assms* **by** *auto*

lemma *completionE*:
assumes *A ∈ { S ∪ N |S N N'. S ∈ sets M ∧ N' ∈ null_sets M ∧ N ⊆ N' }*
obtains *S N N'* **where** *A = S ∪ N N ⊆ N' N' ∈ null_sets M S ∈ sets M*
using *assms* **by** *auto*

lemma *sigma_algebra_completion*:
sigma_algebra (space M) { S ∪ N |S N N'. S ∈ sets M ∧ N' ∈ null_sets M ∧ N ⊆ N' }
(**is** *sigma_algebra _ ?A*)
unfolding *sigma_algebra_iff2*
proof (*intro conjI ballI allI impI*)
show *?A ⊆ Pow (space M)*
using *sets.sets_into_space* **by** *auto*
next
show {} ∈ *?A* **by** *auto*
next
let *?C = space M*
fix *A* **assume** *A ∈ ?A* **from** *completionE*[*OF this*] **guess** *S N N'* **.**
then show *space M − A ∈ ?A*
by (*intro completionI*[*of _ (?C − S) ∩ (?C − N') (?C − S) ∩ N' ∩ (?C − N)*]) *auto*
next
fix *A* :: *nat ⇒ 'a set* **assume** *A*: *range A ⊆ ?A*
then have *∀ n. ∃ S N N'. A n = S ∪ N ∧ S ∈ sets M ∧ N' ∈ null_sets M ∧ N ⊆ N'*

by (*auto simp*: *image_subset_iff*)
from *choice*[*OF this*] **guess** *S* ..
from *choice*[*OF this*] **guess** *N* ..
from *choice*[*OF this*] **guess** *N'* ..
then show $\bigcup(A$ ‘ *UNIV*) ∈ *?A*
using *null_sets_UN*[*of N'*]
by (*intro completionI*[*of* _ $\bigcup(S$ ‘ *UNIV*) $\bigcup(N$ ‘ *UNIV*) $\bigcup(N'$ ‘ *UNIV*)]) *auto*
qed

lemma *sets_completion*:
sets (*completion M*) = { *S* ∪ *N* |*S N N'*. *S* ∈ *sets M* ∧ *N'* ∈ *null_sets M* ∧ *N* ⊆ *N'* }
using *sigma_algebra.sets_measure_of_eq*[*OF sigma_algebra_completion*]
by (*simp add*: *completion_def*)

lemma *sets_completionE*:
assumes *A* ∈ *sets* (*completion M*)
obtains *S N N'* **where** *A* = *S* ∪ *N N* ⊆ *N' N'* ∈ *null_sets M S* ∈ *sets M*
using *assms* **unfolding** *sets_completion* **by** *auto*

lemma *sets_completionI*:
assumes *A* = *S* ∪ *N N* ⊆ *N' N'* ∈ *null_sets M S* ∈ *sets M*
shows *A* ∈ *sets* (*completion M*)
using *assms* **unfolding** *sets_completion* **by** *auto*

lemma *sets_completionI_sets*[*intro*, *simp*]:
A ∈ *sets M* ⟹ *A* ∈ *sets* (*completion M*)
unfolding *sets_completion* **by** *force*

lemma *measurable_completion*: *f* ∈ *M* $\rightarrow_M$ *N* ⟹ *f* ∈ *completion M* $\rightarrow_M$ *N*
by (*auto simp*: *measurable_def*)

lemma *null_sets_completion*:
assumes *N'* ∈ *null_sets M N* ⊆ *N'* **shows** *N* ∈ *sets* (*completion M*)
using *assms* **by** (*intro sets_completionI*[*of N* {} *N N'*]) *auto*

lemma *split_completion*:
assumes *A* ∈ *sets* (*completion M*)
shows *split_completion M A* (*main_part M A*, *null_part M A*)
proof *cases*
assume *A* ∈ *sets M* **then show** *?thesis*
by (*simp add*: *split_completion_def*[*abs_def*] *main_part_def null_part_def*)
next
assume *nA*: *A* ∉ *sets M*
show *?thesis*
unfolding *main_part_def null_part_def if_not_P*[*OF nA*]
proof (*rule someI2_ex*)
from *assms*[*THEN sets_completionE*] **guess** *S N N'* . **note** *A* = *this*
let *?P* = (*S*, *N* − *S*)

show ∃ *p*. *split_completion M A p*
unfolding *split_completion_def if_not_P*[*OF nA*] **using** *A*
proof (*intro exI conjI*)
show *A* = *fst ?P* ∪ *snd ?P* **using** *A* **by** *auto*
show *snd ?P* ⊆ *N′* **using** *A* **by** *auto*
qed *auto*
qed *auto*
qed

lemma *sets_restrict_space_subset*:
assumes *S*: *S* ∈ *sets* (*completion M*)
shows *sets* (*restrict_space* (*completion M*) *S*) ⊆ *sets* (*completion M*)
by (*metis assms sets.Int_space_eq2 sets_restrict_space_iff subsetI*)

lemma
assumes *S* ∈ *sets* (*completion M*)
shows *main_part_sets*[*intro*, *simp*]: *main_part M S* ∈ *sets M*
and *main_part_null_part_Un*[*simp*]: *main_part M S* ∪ *null_part M S* = *S*
and *main_part_null_part_Int*[*simp*]: *main_part M S* ∩ *null_part M S* = {}
using *split_completion*[*OF assms*]
by (*auto simp*: *split_completion_def split*: *if_split_asm*)

lemma *main_part*[*simp*]: *S* ∈ *sets M* ⟹ *main_part M S* = *S*
using *split_completion*[*of S M*]
by (*auto simp*: *split_completion_def split*: *if_split_asm*)

lemma *null_part*:
assumes *S* ∈ *sets* (*completion M*) **shows** ∃ *N*. *N*∈*null_sets M* ∧ *null_part M S*
⊆ *N*
using *split_completion*[*OF assms*] **by** (*auto simp*: *split_completion_def split*: *if_split_asm*)

lemma *null_part_sets*[*intro*, *simp*]:
assumes *S* ∈ *sets M* **shows** *null_part M S* ∈ *sets M emeasure M* (*null_part M*
S) = *0*
proof −
have *S*: *S* ∈ *sets* (*completion M*) **using** *assms* **by** *auto*
have *S* − *main_part M S* ∈ *sets M* **using** *assms* **by** *auto*
moreover
from *main_part_null_part_Un*[*OF S*] *main_part_null_part_Int*[*OF S*]
have *S* − *main_part M S* = *null_part M S* **by** *auto*
ultimately show *sets*: *null_part M S* ∈ *sets M* **by** *auto*
from *null_part*[*OF S*] **guess** *N* **..**
with *emeasure_eq_0*[*of N _ null_part M S*] *sets*
show *emeasure M* (*null_part M S*) = *0* **by** *auto*
qed

lemma *emeasure_main_part_UN*:
fixes *S* :: *nat* ⇒ ′*a set*
assumes *range S* ⊆ *sets* (*completion M*)

shows *emeasure M* (*main_part M* ($\bigcup$*i*. (*S i*))) = *emeasure M* ($\bigcup$*i*. *main_part M* (*S i*))
proof −
have *S*: $\bigwedge$*i*. *S i* ∈ *sets* (*completion M*) **using** *assms* **by** *auto*
then have *UN*: ($\bigcup$*i*. *S i*) ∈ *sets* (*completion M*) **by** *auto*
have ∀ *i*. ∃ *N*. *N* ∈ *null_sets M* ∧ *null_part M* (*S i*) ⊆ *N*
using *null_part*[*OF S*] **by** *auto*
from *choice*[*OF this*] **guess** *N* **..** **note** *N* = *this*
then have *UN_N*: ($\bigcup$*i*. *N i*) ∈ *null_sets M* **by** (*intro null_sets_UN*) *auto*
have ($\bigcup$*i*. *S i*) ∈ *sets* (*completion M*) **using** *S* **by** *auto*
from *null_part*[*OF this*] **guess** *N'* **..** **note** *N'* = *this*
let *?N* = ($\bigcup$*i*. *N i*) ∪ *N'*
have *null_set*: *?N* ∈ *null_sets M* **using** *N' UN_N* **by** (*intro null_sets.Un*) *auto*
have *main_part M* ($\bigcup$*i*. *S i*) ∪ *?N* = (*main_part M* ($\bigcup$*i*. *S i*) ∪ *null_part M* ($\bigcup$*i*. *S i*)) ∪ *?N*
using *N'* **by** *auto*
also have . . . = ($\bigcup$*i*. *main_part M* (*S i*) ∪ *null_part M* (*S i*)) ∪ *?N*
unfolding *main_part_null_part_Un*[*OF S*] *main_part_null_part_Un*[*OF UN*] **by** *auto*
also have . . . = ($\bigcup$*i*. *main_part M* (*S i*)) ∪ *?N*
using *N* **by** *auto*
finally have ∗: *main_part M* ($\bigcup$*i*. *S i*) ∪ *?N* = ($\bigcup$*i*. *main_part M* (*S i*)) ∪ *?N* **.**
have *emeasure M* (*main_part M* ($\bigcup$*i*. *S i*)) = *emeasure M* (*main_part M* ($\bigcup$*i*. *S i*) ∪ *?N*)
using *null_set UN* **by** (*intro emeasure_Un_null_set*[*symmetric*]) *auto*
also have . . . = *emeasure M* (($\bigcup$*i*. *main_part M* (*S i*)) ∪ *?N*)
unfolding ∗ **..**
also have . . . = *emeasure M* ($\bigcup$*i*. *main_part M* (*S i*))
using *null_set S* **by** (*intro emeasure_Un_null_set*) *auto*
finally show *?thesis* **.**
qed

lemma *emeasure_completion*[*simp*]:
assumes *S*: *S* ∈ *sets* (*completion M*)
shows *emeasure* (*completion M*) *S* = *emeasure M* (*main_part M S*)
proof (*subst emeasure_measure_of*[*OF completion_def completion_into_space*])
let *?μ* = *emeasure M* ∘ *main_part M*
show *S* ∈ *sets* (*completion M*) *?μ S* = *emeasure M* (*main_part M S*) **using** *S* **by** *simp_all*
show *positive* (*sets* (*completion M*)) *?μ*
by (*simp add*: *positive_def*)
show *countably_additive* (*sets* (*completion M*)) *?μ*
proof (*intro countably_additiveI*)
fix *A* :: *nat* ⇒ *'a set* **assume** *A*: *range A* ⊆ *sets* (*completion M*) *disjoint_family A*
have *disjoint_family* (λ*i*. *main_part M* (*A i*))
proof (*intro disjoint_family_on_bisimulation*[*OF A*(*2*)])
fix *n m* **assume** *A n* ∩ *A m* = {}

then have (*main_part M* (*A n*) ∪ *null_part M* (*A n*)) ∩ (*main_part M* (*A m*) ∪ *null_part M* (*A m*)) = {}
using *A* **by** (*subst* (*1 2*) *main_part_null_part_Un*) *auto*
then show *main_part M* (*A n*) ∩ *main_part M* (*A m*) = {} **by** *auto*
qed
then have (∑ *n. emeasure M* (*main_part M* (*A n*))) = *emeasure M* (⋃ *i. main_part M* (*A i*))
using *A* **by** (*auto intro*!: *suminf_emeasure*)
then show (∑ *n. ?μ* (*A n*)) = *?μ* (⋃(*A* ' *UNIV*))
by (*simp add*: *completion_def emeasure_main_part_UN*[*OF A*(*1*)])
qed
qed

lemma *measure_completion*[*simp*]: *S* ∈ *sets M* ⟹ *measure* (*completion M*) *S* = *measure M S*
unfolding *measure_def* **by** *auto*

lemma *emeasure_completion_UN*:
range S ⊆ *sets* (*completion M*) ⟹
emeasure (*completion M*) (⋃ *i::nat.* (*S i*)) = *emeasure M* (⋃ *i. main_part M* (*S i*))
by (*subst emeasure_completion*) (*auto simp add*: *emeasure_main_part_UN*)

lemma *emeasure_completion_Un*:
assumes *S*: *S* ∈ *sets* (*completion M*) **and** *T*: *T* ∈ *sets* (*completion M*)
shows *emeasure* (*completion M*) (*S* ∪ *T*) = *emeasure M* (*main_part M S* ∪ *main_part M T*)
proof (*subst emeasure_completion*)
have *UN*: (⋃ *i. binary* (*main_part M S*) (*main_part M T*) *i*) = (⋃ *i. main_part M* (*binary S T i*))
unfolding *binary_def* **by** (*auto split*: *if_split_asm*)
show *emeasure M* (*main_part M* (*S* ∪ *T*)) = *emeasure M* (*main_part M S* ∪ *main_part M T*)
using *emeasure_main_part_UN*[*of binary S T M*] *assms*
by (*simp add*: *range_binary_eq*, *simp add*: *Un_range_binary UN*)
qed (*auto intro*: *S T*)

lemma *sets_completionI_sub*:
assumes *N*: *N'* ∈ *null_sets M N* ⊆ *N'*
shows *N* ∈ *sets* (*completion M*)
using *assms* **by** (*intro sets_completionI*[*of _* {} *N N'*]) *auto*

lemma *completion_ex_simple_function*:
assumes *f*: *simple_function* (*completion M*) *f*
shows ∃*f'*. *simple_function M f'* ∧ (*AE x in M. f x* = *f' x*)
proof –
let *?F* = λ*x. f* –' {*x*} ∩ *space M*
have *F*: ⋀*x. ?F x* ∈ *sets* (*completion M*) **and** *fin*: *finite* (*f'space M*)
using *simple_functionD*[*OF f*] *simple_functionD*[*OF f*] **by** *simp_all*

```
  have ∀x. ∃N. N ∈ null_sets M ∧ null_part M (?F x) ⊆ N
    using F null_part by auto
  from choice[OF this] obtain N where
    N: ⋀x. null_part M (?F x) ⊆ N x ⋀x. N x ∈ null_sets M by auto
  let ?N = ⋃x∈f`space M. N x
  let ?f' = λx. if x ∈ ?N then undefined else f x
  have sets: ?N ∈ null_sets M using N fin by (intro null_sets.finite_UN) auto
  show ?thesis unfolding simple_function_def
  proof (safe intro!: exI[of _ ?f'])
    have ?f' ` space M ⊆ f`space M ∪ {undefined} by auto
    from finite_subset[OF this] simple_functionD(1)[OF f]
    show finite (?f' ` space M) by auto
  next
    fix x assume x ∈ space M
    have ?f' -` {?f' x} ∩ space M =
      (if x ∈ ?N then ?F undefined ∪ ?N
       else if f x = undefined then ?F (f x) ∪ ?N
       else ?F (f x) - ?N)
     using N(2) sets.sets_into_space by (auto split: if_split_asm simp: null_sets_def)
    moreover { fix y have ?F y ∪ ?N ∈ sets M
      proof cases
        assume y: y ∈ f`space M
        have ?F y ∪ ?N = (main_part M (?F y) ∪ null_part M (?F y)) ∪ ?N
          using main_part_null_part_Un[OF F] by auto
        also have ... = main_part M (?F y) ∪ ?N
          using y N by auto
        finally show ?thesis
          using F sets by auto
      next
        assume y ∉ f`space M then have ?F y = {} by auto
        then show ?thesis using sets by auto
      qed }
    moreover {
      have ?F (f x) - ?N = main_part M (?F (f x)) ∪ null_part M (?F (f x)) -
?N
        using main_part_null_part_Un[OF F] by auto
      also have ... = main_part M (?F (f x)) - ?N
        using N ‹x ∈ space M› by auto
      finally have ?F (f x) - ?N ∈ sets M
        using F sets by auto }
    ultimately show ?f' -` {?f' x} ∩ space M ∈ sets M by auto
  next
    show AE x in M. f x = ?f' x
      by (rule AE_I', rule sets) auto
  qed
qed

lemma completion_ex_borel_measurable:
  fixes g :: 'a ⇒ ennreal
```

assumes *g*: *g* ∈ *borel_measurable* (*completion M*)
shows ∃ *g'*∈*borel_measurable M*. (*AE x in M*. *g x* = *g' x*)
proof −
from *g*[*THEN borel_measurable_implies_simple_function_sequence'*] **guess** *f* . **note** *f* = *this*
from *this*(*1*)[*THEN completion_ex_simple_function*]
have ∀ *i*. ∃ *f'*. *simple_function M f'* ∧ (*AE x in M*. *f i x* = *f' x*) ..
from *this*[*THEN choice*] **obtain** *f'* **where**
sf: ⋀*i*. *simple_function M* (*f' i*) **and**
AE: ∀ *i*. *AE x in M*. *f i x* = *f' i x* **by** *auto*
show *?thesis*
proof (*intro bexI*)
from *AE*[*unfolded AE_all_countable*[*symmetric*]]
show *AE x in M*. *g x* = (*SUP i*. *f' i x*) (**is** *AE x in M*. *g x* = *?f x*)
proof (*elim AE_mp*, *safe intro*!: *AE_I2*)
fix *x* **assume** *eq*: ∀ *i*. *f i x* = *f' i x*
moreover have *g x* = (*SUP i*. *f i x*)
unfolding *f* **by** (*auto split*: *split_max*)
ultimately show *g x* = *?f x* **by** *auto*
qed
show *?f* ∈ *borel_measurable M*
using *sf*[*THEN borel_measurable_simple_function*] **by** *auto*
qed
qed

lemma *null_sets_completionI*: *N* ∈ *null_sets M* ⟹ *N* ∈ *null_sets* (*completion M*)
by (*auto simp*: *null_sets_def*)

lemma *AE_completion*: (*AE x in M*. *P x*) ⟹ (*AE x in completion M*. *P x*)
unfolding *eventually_ae_filter* **by** (*auto intro*: *null_sets_completionI*)

lemma *null_sets_completion_iff*: *N* ∈ *sets M* ⟹ *N* ∈ *null_sets* (*completion M*) ⟷ *N* ∈ *null_sets M*
by (*auto simp*: *null_sets_def*)

lemma *sets_completion_AE*: (*AE x in M*. ¬ *P x*) ⟹ *Measurable.pred* (*completion M*) *P*
unfolding *pred_def sets_completion eventually_ae_filter*
by *auto*

lemma *null_sets_completion_iff2*:
A ∈ *null_sets* (*completion M*) ⟷ (∃ *N'*∈*null_sets M*. *A* ⊆ *N'*)
proof *safe*
assume *A* ∈ *null_sets* (*completion M*)
then have *A*: *A* ∈ *sets* (*completion M*) **and** *main_part M A* ∈ *null_sets M*
by (*auto simp*: *null_sets_def*)
moreover obtain *N* **where** *N* ∈ *null_sets M null_part M A* ⊆ *N*
using *null_part*[*OF A*] **by** *auto*
ultimately show ∃ *N'*∈*null_sets M*. *A* ⊆ *N'*

proof (*intro bexI*)
show *A* ⊆ *N* ∪ *main_part M A*
using ‹*null_part M A* ⊆ *N*› **by** (*subst main_part_null_part_Un*[*OF A*, *symmetric*]) *auto*
qed *auto*
next
fix *N* **assume** *N* ∈ *null_sets M A* ⊆ *N*
then have *A* ∈ *sets* (*completion M*) **and** *N*: *N* ∈ *sets M A* ⊆ *N emeasure M N* = *0*
by (*auto intro*: *null_sets_completion*)
moreover have *emeasure* (*completion M*) *A* = *0*
using *N* **by** (*intro emeasure_eq_0*[*of N _ A*]) *auto*
ultimately show *A* ∈ *null_sets* (*completion M*)
by *auto*
qed

lemma *null_sets_completion_subset*:
B ⊆ *A* ⟹ *A* ∈ *null_sets* (*completion M*) ⟹ *B* ∈ *null_sets* (*completion M*)
unfolding *null_sets_completion_iff2* **by** *auto*

interpretation *completion*: *complete_measure completion M* **for** *M*
proof
show *B* ⊆ *A* ⟹ *A* ∈ *null_sets* (*completion M*) ⟹ *B* ∈ *sets* (*completion M*) **for** *B A*
using *null_sets_completion_subset*[*of B A M*] **by** (*simp add*: *null_sets_def*)
qed

lemma *null_sets_restrict_space*:
Ω ∈ *sets M* ⟹ *A* ∈ *null_sets* (*restrict_space M* Ω) ⟷ *A* ⊆ Ω ∧ *A* ∈ *null_sets M*
by (*auto simp*: *null_sets_def emeasure_restrict_space sets_restrict_space*)

lemma *completion_ex_borel_measurable_real*:
fixes *g* :: '*a* ⇒ *real*
assumes *g*: *g* ∈ *borel_measurable* (*completion M*)
shows ∃ *g*'∈*borel_measurable M*. (*AE x in M*. *g x* = *g*' *x*)
proof −
have (λ*x*. *ennreal* (*g x*)) ∈ *completion M* $\to_M$ *borel* (λ*x*. *ennreal* (− *g x*)) ∈ *completion M* $\to_M$ *borel*
using *g* **by** *auto*
from *this*[*THEN completion_ex_borel_measurable*]
obtain *pf nf* :: '*a* ⇒ *ennreal*
where [*measurable*]: *nf* ∈ *M* $\to_M$ *borel pf* ∈ *M* $\to_M$ *borel*
and *ae*: *AE x in M*. *pf x* = *ennreal* (*g x*) *AE x in M*. *nf x* = *ennreal* (− *g x*)
by (*auto simp*: *eq_commute*)
then have *AE x in M*. *pf x* = *ennreal* (*g x*) ∧ *nf x* = *ennreal* (− *g x*)
by *auto*
then obtain *N* **where** *N* ∈ *null_sets M* {*x*∈*space M*. *pf x* ≠ *ennreal* (*g x*) ∧ *nf x* ≠ *ennreal* (− *g x*)} ⊆ *N*

    by (auto elim!: AE_E)
  show ?thesis
  proof
    let ?F = λx. indicator (space M − N) x ∗ (enn2real (pf x) − enn2real (nf x))
    show ?F ∈ M →$_M$ borel
      using ‹N ∈ null_sets M› by auto
    show AE x in M. g x = ?F x
      using ‹N ∈ null_sets M›[THEN AE_not_in] ae AE_space
      apply eventually_elim
      subgoal for x
        by (cases 0::real g x rule: linorder_le_cases) (auto simp: ennreal_neg)
      done
  qed
qed

lemma *simple_function_completion*: *simple_function M f* ⟹ *simple_function* (*completion M*) *f*
  by (*simp add*: *simple_function_def*)

lemma *simple_integral_completion*:
  simple_function M f ⟹ *simple_integral* (*completion M*) *f* = *simple_integral M f*
  unfolding *simple_integral_def* **by** *simp*

lemma *nn_integral_completion*: *nn_integral* (*completion M*) *f* = *nn_integral M f*
  unfolding *nn_integral_def*
proof (*safe intro*!: *SUP_eq*)
  fix *s* **assume** *s*: *simple_function* (*completion M*) *s* **and** *s* ≤ *f*
  then obtain *s′* **where** *s′*: *simple_function M s′ AE x in M. s x = s′ x*
    by (*auto dest*: *completion_ex_simple_function*)
  then obtain *N* **where** *N*: *N* ∈ *null_sets M* {*x*∈*space M*. *s x* ≠ *s′ x*} ⊆ *N*
    by (*auto elim*!: *AE_E*)
  then have *ae_N*: *AE x in M*. (*s x* ≠ *s′ x* ⟶ *x* ∈ *N*) ∧ *x* ∉ *N*
    by (*auto dest*: *AE_not_in*)
  define *s′′* **where** *s′′ x* = (*if x* ∈ *N then 0 else s x*) **for** *x*
  then have *ae_s_eq_s′′*: *AE x in completion M*. *s x* = *s′′ x*
    using *s′ ae_N* **by** (*intro AE_completion*) *auto*
  have *s′′*: *simple_function M s′′*
  proof (*subst simple_function_cong*)
    show *t* ∈ *space M* ⟹ *s′′ t* = (*if t* ∈ *N then 0 else s′ t*) **for** *t*
      using *N* **by** (*auto simp*: *s′′_def dest*: *sets.sets_into_space*)
    show *simple_function M* (λ*t*. *if t* ∈ *N then 0 else s′ t*)
      unfolding *s′′_def*[*abs_def*] **using** *N* **by** (*auto intro*!: *simple_function_If s′*)
  qed

  show ∃*j*∈{*g*. *simple_function M g* ∧ *g* ≤ *f*}. *integral^S* (*completion M*) *s* ≤ *integral^S M j*
  proof (*safe intro*!: *bexI*[*of _ s′′*])
    have *integral^S* (*completion M*) *s* = *integral^S* (*completion M*) *s′′*
      by (*intro simple_integral_cong_AE s simple_function_completion s′′ ae_s_eq_s′′*)

```
    then show integral^S (completion M) s ≤ integral^S M s''
      using s'' by (simp add: simple_integral_completion)
    from ‹s ≤ f› show s'' ≤ f
      unfolding s''_def le_fun_def by auto
  qed fact
next
  fix s assume simple_function M s s ≤ f
  then show ∃j∈{g. simple_function (completion M) g ∧ g ≤ f}. integral^S M s
≤ integral^S (completion M) j
   by (intro bexI[of _ s]) (auto simp: simple_integral_completion simple_function_completion)
qed

lemma integrable_completion:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  shows f ∈ M →M borel ⟹ integrable (completion M) f ⟷ integrable M f
  by (rule integrable_subalgebra[symmetric]) auto

lemma integral_completion:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  assumes f: f ∈ M →M borel shows integral^L (completion M) f = integral^L M
f
  by (rule integral_subalgebra[symmetric]) (auto intro: f)

locale semifinite_measure =
  fixes M :: 'a measure
  assumes semifinite:
    ⋀A. A ∈ sets M ⟹ emeasure M A = ∞ ⟹ ∃B∈sets M. B ⊆ A ∧ emeasure
M B < ∞

locale locally_determined_measure = semifinite_measure +
  assumes locally_determined:
    ⋀A. A ⊆ space M ⟹ (⋀B. B ∈ sets M ⟹ emeasure M B < ∞ ⟹ A ∩ B
∈ sets M) ⟹ A ∈ sets M

locale cld_measure =
  complete_measure M + locally_determined_measure M for M :: 'a measure

definition outer_measure_of :: 'a measure ⇒ 'a set ⇒ ennreal
  where outer_measure_of M A = (INF B ∈ {B∈sets M. A ⊆ B}. emeasure M B)

lemma outer_measure_of_eq[simp]: A ∈ sets M ⟹ outer_measure_of M A = emea-
sure M A
  by (auto simp: outer_measure_of_def intro!: INF_eqI emeasure_mono)

lemma outer_measure_of_mono: A ⊆ B ⟹ outer_measure_of M A ≤ outer_measure_of
M B
  unfolding outer_measure_of_def by (intro INF_superset_mono) auto

lemma outer_measure_of_attain:
```

assumes $A \subseteq$ *space M*
shows $\exists E \in$*sets M*. $A \subseteq E \wedge$ *outer_measure_of M A* = *emeasure M E*
proof –
have *emeasure M* ‘ $\{B \in$ *sets M*. $A \subseteq B\} \neq \{\}$
using ‹$A \subseteq$ *space M*› **by** *auto*
from *ennreal_Inf_countable_INF*[*OF this*]
obtain *f*
where *f*: *range f* $\subseteq$ *emeasure M* ‘ $\{B \in$ *sets M*. $A \subseteq B\}$ *decseq f*
and *outer_measure_of M A* = (*INF i. f i*)
unfolding *outer_measure_of_def* **by** *auto*
have $\exists E. \forall n.$ ($E\ n \in$ *sets M* $\wedge A \subseteq E\ n \wedge$ *emeasure M* ($E\ n$) $\leq f\ n$) $\wedge E$ (*Suc n*) $\subseteq E\ n$
proof (*rule dependent_nat_choice*)
show $\exists x. x \in$ *sets M* $\wedge A \subseteq x \wedge$ *emeasure M x* $\leq f\ 0$
using *f*(*1*) **by** (*fastforce simp*: *image_subset_iff image_iff intro*: *eq_refl*[*OF sym*])
next
fix *E n* **assume** $E \in$ *sets M* $\wedge A \subseteq E \wedge$ *emeasure M E* $\leq f\ n$
moreover obtain *F* **where** $F \in$ *sets M* $A \subseteq F$ *f* (*Suc n*) = *emeasure M F*
using *f*(*1*) **by** (*auto simp*: *image_subset_iff image_iff*)
ultimately show $\exists y.$ ($y \in$ *sets M* $\wedge A \subseteq y \wedge$ *emeasure M y* $\leq f$ (*Suc n*)) $\wedge y \subseteq E$
by (*auto intro*!: *exI*[*of _ F* $\cap$ *E*] *emeasure_mono*)
qed
then obtain *E*
where [*simp*]: $\bigwedge n.$ $E\ n \in$ *sets M*
and $\bigwedge n.$ $A \subseteq E\ n$
and *le_f*: $\bigwedge n.$ *emeasure M* ($E\ n$) $\leq f\ n$
and *decseq E*
by (*auto simp*: *decseq_Suc_iff*)
show *?thesis*
proof *cases*
assume *fin*: $\exists i.$ *emeasure M* ($E\ i$) $< \infty$
show *?thesis*
proof (*intro bexI*[*of _* $\bigcap i.$ *E i*] *conjI*)
show $A \subseteq (\bigcap i.\ E\ i)$ $(\bigcap i.\ E\ i) \in$ *sets M*
using ‹$\bigwedge n.$ $A \subseteq E\ n$› **by** *auto*

have (*INF i. emeasure M* ($E\ i$)) $\leq$ *outer_measure_of M A*
unfolding ‹*outer_measure_of M A* = (*INF n. f n*)›
by (*intro INF_superset_mono le_f*) *auto*
moreover have *outer_measure_of M A* $\leq$ (*INF i. outer_measure_of M* ($E\ i$))
by (*intro INF_greatest outer_measure_of_mono* ‹$\bigwedge n.$ $A \subseteq E\ n$›)
ultimately have *outer_measure_of M A* = (*INF i. emeasure M* ($E\ i$))
by *auto*
also have ... = *emeasure M* ($\bigcap i.\ E\ i$)
using *fin* **by** (*intro INF_emeasure_decseq'* ‹*decseq E*›) (*auto simp*: *less_top*)
finally show *outer_measure_of M A* = *emeasure M* ($\bigcap i.\ E\ i$) **.**
qed

```
  next
    assume ∄i. emeasure M (E i) < ∞
    then have f n = ∞ for n
      using le_f by (auto simp: not_less top_unique)
    moreover have ∃E∈sets M. A ⊆ E ∧ f 0 = emeasure M E
      using f by auto
    ultimately show ?thesis
      unfolding ‹outer_measure_of M A = (INF n. f n)› by simp
  qed
qed

lemma SUP_outer_measure_of_incseq:
  assumes A: ⋀n. A n ⊆ space M and incseq A
  shows (SUP n. outer_measure_of M (A n)) = outer_measure_of M (⋃i. A i)
proof (rule antisym)
  obtain E
    where E: ⋀n. E n ∈ sets M ⋀n. A n ⊆ E n ⋀n. outer_measure_of M (A n)
= emeasure M (E n)
    using outer_measure_of_attain[OF A] by metis

  define F where F n = (⋂i∈{n ..}. E i) for n
  with E have F: incseq F ⋀n. F n ∈ sets M
    by (auto simp: incseq_def)
  have A n ⊆ F n for n
    using incseqD[OF ‹incseq A›, of n] ‹⋀n. A n ⊆ E n› by (auto simp: F_def)

  have eq: outer_measure_of M (A n) = outer_measure_of M (F n) for n
  proof (intro antisym)
    have outer_measure_of M (F n) ≤ outer_measure_of M (E n)
      by (intro outer_measure_of_mono) (auto simp add: F_def)
    with E show outer_measure_of M (F n) ≤ outer_measure_of M (A n)
      by auto
    show outer_measure_of M (A n) ≤ outer_measure_of M (F n)
      by (intro outer_measure_of_mono ‹A n ⊆ F n›)
  qed

  have outer_measure_of M (⋃n. A n) ≤ outer_measure_of M (⋃n. F n)
    using ‹⋀n. A n ⊆ F n› by (intro outer_measure_of_mono) auto
  also have ... = (SUP n. emeasure M (F n))
    using F by (simp add: SUP_emeasure_incseq subset_eq)
  finally show outer_measure_of M (⋃n. A n) ≤ (SUP n. outer_measure_of M (A
n))
    by (simp add: eq F)
qed (auto intro: SUP_least outer_measure_of_mono)

definition measurable_envelope :: 'a measure ⇒ 'a set ⇒ 'a set ⇒ bool
  where measurable_envelope M A E ⟷
   (A ⊆ E ∧ E ∈ sets M ∧ (∀F∈sets M. emeasure M (F ∩ E) = outer_measure_of
M (F ∩ A)))
```

lemma *measurable_envelopeD*:
assumes *measurable_envelope M A E*
shows $A \subseteq E$
and $E \in$ *sets M*
and $\bigwedge F$. $F \in$ *sets M* $\Longrightarrow$ *emeasure M* $(F \cap E) =$ *outer_measure_of M* $(F \cap A)$
and $A \subseteq$ *space M*
using *assms sets.sets_into_space*[*of E*] **by** (*auto simp*: *measurable_envelope_def*)

lemma *measurable_envelopeD1*:
assumes *E*: *measurable_envelope M A E* **and** *F*: $F \in$ *sets M* $F \subseteq E - A$
shows *emeasure M F* = *0*
proof –
have *emeasure M F* = *emeasure M* $(F \cap E)$
using *F* **by** (*intro arg_cong2*[**where** *f*=*emeasure*]) *auto*
also have ... = *outer_measure_of M* $(F \cap A)$
using *measurable_envelopeD*[*OF E*] ‹$F \in$ *sets M*› **by** (*auto simp*: *measurable_envelope_def*)
also have ... = *outer_measure_of M* {}
using ‹$F \subseteq E - A$› **by** (*intro arg_cong2*[**where** *f*=*outer_measure_of*]) *auto*
finally show *emeasure M F* = *0*
by *simp*
qed

lemma *measurable_envelope_eq1*:
assumes $A \subseteq E$ $E \in$ *sets M*
shows *measurable_envelope M A E* $\longleftrightarrow$ ($\forall F \in$*sets M*. $F \subseteq E - A \longrightarrow$ *emeasure M F* = *0*)
proof *safe*
assume ∗: $\forall F \in$*sets M*. $F \subseteq E - A \longrightarrow$ *emeasure M F* = *0*
show *measurable_envelope M A E*
unfolding *measurable_envelope_def*
proof (*rule ccontr*, *auto simp add*: ‹$E \in$ *sets M*› ‹$A \subseteq E$›)
fix *F* **assume** $F \in$ *sets M* *emeasure M* $(F \cap E) \neq$ *outer_measure_of M* $(F \cap A)$
then have *outer_measure_of M* $(F \cap A) <$ *emeasure M* $(F \cap E)$
using *outer_measure_of_mono*[*of* $F \cap A$ $F \cap E$ *M*] ‹$A \subseteq E$› ‹$E \in$ *sets M*› **by** (*auto simp*: *less_le*)
then obtain *G* **where** *G*: $G \in$ *sets M* $F \cap A \subseteq G$ **and** *less*: *emeasure M G* < *emeasure M* $(E \cap F)$
unfolding *outer_measure_of_def INF_less_iff* **by** (*auto simp*: *ac_simps*)
have *le*: *emeasure M* $(G \cap E \cap F) \leq$ *emeasure M G*
using ‹$E \in$ *sets M*› ‹$G \in$ *sets M*› ‹$F \in$ *sets M*› **by** (*auto intro*!: *emeasure_mono*)

from *G* **have** $E \cap F - G \in$ *sets M* $E \cap F - G \subseteq E - A$
using ‹$F \in$ *sets M*› ‹$E \in$ *sets M*› **by** *auto*
with ∗ **have** *0* = *emeasure M* $(E \cap F - G)$
by *auto*

also have $E \cap F - G = E \cap F - (G \cap E \cap F)$
by *auto*
also have *emeasure M* $(E \cap F - (G \cap E \cap F))$ = *emeasure M* $(E \cap F)$ − *emeasure M* $(G \cap E \cap F)$
using ‹$E \in$ *sets M*› ‹$F \in$ *sets M*› *le less G* **by** (*intro emeasure_Diff*) (*auto simp*: *top_unique*)
also have ... > *0*
using *le less* **by** (*intro diff_gr0_ennreal*) *auto*
finally show *False* **by** *auto*
qed
qed (*rule measurable_envelopeD1*)

lemma *measurable_envelopeD2*:
assumes *E*: *measurable_envelope M A E* **shows** *emeasure M E* = *outer_measure_of M A*
proof −
from ‹*measurable_envelope M A E*› **have** *emeasure M* $(E \cap E)$ = *outer_measure_of M* $(E \cap A)$
by (*auto simp*: *measurable_envelope_def*)
with *measurable_envelopeD*[*OF E*] **show** *emeasure M E* = *outer_measure_of M A*
by (*auto simp*: *Int_absorb1*)
qed

lemma *measurable_envelope_eq2*:
assumes $A \subseteq E$ $E \in$ *sets M emeasure M E* < ∞
shows *measurable_envelope M A E* ⟷ (*emeasure M E* = *outer_measure_of M A*)
proof *safe*
assume ∗: *emeasure M E* = *outer_measure_of M A*
show *measurable_envelope M A E*
unfolding *measurable_envelope_eq1*[*OF* ‹$A \subseteq E$› ‹$E \in$ *sets M*›]
proof (*intro conjI ballI impI assms*)
fix *F* **assume** *F*: $F \in$ *sets M* $F \subseteq E - A$
with ‹$E \in$ *sets M*› **have** *le*: *emeasure M F* ≤ *emeasure M E*
by (*intro emeasure_mono*) *auto*
from *F* ‹$A \subseteq E$› **have** *outer_measure_of M A* ≤ *outer_measure_of M* $(E - F)$
by (*intro outer_measure_of_mono*) *auto*
then have *emeasure M E* − *0* ≤ *emeasure M* $(E - F)$
using ∗ ‹$E \in$ *sets M*› ‹$F \in$ *sets M*› **by** *simp*
also have ... = *emeasure M E* − *emeasure M F*
using ‹$E \in$ *sets M*› ‹*emeasure M E* < ∞› *F le* **by** (*intro emeasure_Diff*) (*auto simp*: *top_unique*)
finally show *emeasure M F* = *0*
using *ennreal_mono_minus_cancel*[*of emeasure M E 0 emeasure M F*] *le assms* **by** *auto*
qed
qed (*auto intro*: *measurable_envelopeD2*)

lemma *measurable_envelopeI_countable*:
  fixes A :: $nat \Rightarrow 'a\ set$
  assumes E: $\bigwedge n.$ *measurable_envelope* M $(A\ n)$ $(E\ n)$
  shows *measurable_envelope* M $(\bigcup n.\ A\ n)$ $(\bigcup n.\ E\ n)$
proof (*subst measurable_envelope_eq1*)
  show $(\bigcup n.\ A\ n) \subseteq (\bigcup n.\ E\ n)$ $(\bigcup n.\ E\ n) \in$ *sets* M
    using *measurable_envelopeD*(*1*,*2*)[*OF E*] **by** *auto*
  show $\forall F{\in}sets\ M.\ F \subseteq (\bigcup n.\ E\ n) - (\bigcup n.\ A\ n) \longrightarrow$ *emeasure* M $F = 0$
  proof *safe*
    fix F **assume** F: $F \in$ *sets* M $F \subseteq (\bigcup n.\ E\ n) - (\bigcup n.\ A\ n)$
    then have $F \cap E\ n \in$ *sets* M $F \cap E\ n \subseteq E\ n - A\ n$ $F \subseteq (\bigcup n.\ E\ n)$ **for** n
      using *measurable_envelopeD*(*1*,*2*)[*OF E*] **by** *auto*
    then have *emeasure* M $(\bigcup n.\ F \cap E\ n) = 0$
      by (*intro emeasure_UN_eq_0 measurable_envelopeD1*[*OF E*]) *auto*
    then show *emeasure* M $F = 0$
      using ‹$F \subseteq (\bigcup n.\ E\ n)$› **by** (*auto simp*: *Int_absorb2*)
  qed
qed

lemma *measurable_envelopeI_countable_cover*:
  fixes A **and** C :: $nat \Rightarrow 'a\ set$
  assumes C: $A \subseteq (\bigcup n.\ C\ n)$ $\bigwedge n.\ C\ n \in$ *sets* M $\bigwedge n.$ *emeasure* M $(C\ n) < \infty$
  shows $\exists E{\subseteq}(\bigcup n.\ C\ n).$ *measurable_envelope* M A E
proof −
  have $A \cap C\ n \subseteq$ *space* M **for** n
    using ‹$C\ n \in$ *sets* M› **by** (*auto dest*: *sets.sets_into_space*)
  then have $\forall n.\ \exists E{\in}sets\ M.\ A \cap C\ n \subseteq E \wedge$ *outer_measure_of* M $(A \cap C\ n) =$ *emeasure* M E
    using *outer_measure_of_attain*[*of* $A \cap C\ n$ M **for** n] **by** *auto*
  then obtain E
    where E: $\bigwedge n.\ E\ n \in$ *sets* M $\bigwedge n.\ A \cap C\ n \subseteq E\ n$
      and *eq*: $\bigwedge n.$ *outer_measure_of* M $(A \cap C\ n) =$ *emeasure* M $(E\ n)$
    by *metis*

  have *outer_measure_of* M $(A \cap C\ n) \leq$ *outer_measure_of* M $(E\ n \cap C\ n)$ **for** n
    using E **by** (*intro outer_measure_of_mono*) *auto*
  moreover have *outer_measure_of* M $(E\ n \cap C\ n) \leq$ *outer_measure_of* M $(E\ n)$ **for** n
    by (*intro outer_measure_of_mono*) *auto*
  ultimately have *eq*: *outer_measure_of* M $(A \cap C\ n) =$ *emeasure* M $(E\ n \cap C\ n)$ **for** n
    using E C **by** (*intro antisym*) (*auto simp*: *eq*)

  { **fix** n
    have *outer_measure_of* M $(A \cap C\ n) \leq$ *outer_measure_of* M $(C\ n)$
      by (*intro outer_measure_of_mono*) *simp*
    also have $\ldots < \infty$
      using *assms* **by** *auto*
    finally have *emeasure* M $(E\ n \cap C\ n) < \infty$

```
      using eq by simp }
  then have measurable_envelope M (⋃n. A ∩ C n) (⋃n. E n ∩ C n)
   using E C by (intro measurable_envelopeI_countable measurable_envelope_eq2[THEN
iffD2]) (auto simp: eq)
  with ‹A ⊆ (⋃n. C n)› show ?thesis
    by (intro exI[of _ (⋃n. E n ∩ C n)]) (auto simp add: Int_absorb2)
qed

lemma (in complete_measure) complete_sets_sandwich:
  assumes [measurable]: A ∈ sets M C ∈ sets M and subset: A ⊆ B B ⊆ C
    and measure: emeasure M A = emeasure M C emeasure M A < ∞
  shows B ∈ sets M
proof –
  have B − A ∈ sets M
  proof (rule complete)
    show B − A ⊆ C − A
      using subset by auto
    show C − A ∈ null_sets M
      using measure subset by(simp add: emeasure_Diff null_setsI)
  qed
  then have A ∪ (B − A) ∈ sets M
    by measurable
  also have A ∪ (B − A) = B
    using ‹A ⊆ B› by auto
  finally show ?thesis .
qed

lemma (in complete_measure) complete_sets_sandwich_fmeasurable:
  assumes [measurable]: A ∈ fmeasurable M C ∈ fmeasurable M and subset: A ⊆
B B ⊆ C
    and measure: measure M A = measure M C
  shows B ∈ fmeasurable M
proof (rule fmeasurableI2)
  show B ⊆ C C ∈ fmeasurable M by fact+
  show B ∈ sets M
  proof (rule complete_sets_sandwich)
    show A ∈ sets M C ∈ sets M A ⊆ B B ⊆ C
      using assms by auto
    show emeasure M A < ∞
      using ‹A ∈ fmeasurable M› by (auto simp: fmeasurable_def)
    show emeasure M A = emeasure M C
      using assms by (simp add: emeasure_eq_measure2)
  qed
qed

lemma AE_completion_iff: (AE x in completion M. P x) ⟷ (AE x in M. P x)
proof
  assume AE x in completion M. P x
  then obtain N where N ∈ null_sets (completion M) and P: {x∈space M. ¬ P
```

x} ⊆ *N*
    unfolding *eventually_ae_filter* **by** *auto*
  then obtain *N'* **where** *N'*: *N'* ∈ *null_sets M* **and** *N* ⊆ *N'*
    unfolding *null_sets_completion_iff2* **by** *auto*
  from *P* ‹*N* ⊆ *N'*› **have** {*x*∈*space M*. ¬ *P x*} ⊆ *N'*
    by *auto*
  with *N'* **show** *AE x in M*. *P x*
    unfolding *eventually_ae_filter* **by** *auto*
qed (*rule AE_completion*)

lemma *null_part_null_sets*: *S* ∈ *completion M* ⟹ *null_part M S* ∈ *null_sets* (*completion M*)
  by (*auto dest*!: *null_part intro*: *null_sets_completionI null_sets_completion_subset*)

lemma *AE_notin_null_part*: *S* ∈ *completion M* ⟹ (*AE x in M*. *x* ∉ *null_part M S*)
  by (*auto dest*!: *null_part_null_sets AE_not_in simp*: *AE_completion_iff*)

lemma *completion_upper*:
  assumes *A*: *A* ∈ *sets* (*completion M*)
  shows ∃ *A'*∈*sets M*. *A* ⊆ *A'* ∧ *emeasure* (*completion M*) *A* = *emeasure M A'*
proof −
  from *AE_notin_null_part*[*OF A*] **obtain** *N* **where** *N*: *N* ∈ *null_sets M null_part M A* ⊆ *N*
    unfolding *eventually_ae_filter* **using** *null_part_null_sets*[*OF A*, *THEN null_setsD2*, *THEN sets.sets_into_space*] **by** *auto*
  show *?thesis*
  proof (*intro bexI conjI*)
    show *A* ⊆ *main_part M A* ∪ *N*
      using ‹*null_part M A* ⊆ *N*› **by** (*subst main_part_null_part_Un*[*symmetric*, *OF A*]) *auto*
    show *emeasure* (*completion M*) *A* = *emeasure M* (*main_part M A* ∪ *N*)
      using *A* ‹*N* ∈ *null_sets M*› **by** (*simp add*: *emeasure_Un_null_set*)
  qed (*use A N* **in** *auto*)
qed

lemma *AE_in_main_part*:
  assumes *A*: *A* ∈ *completion M* **shows** *AE x in M*. *x* ∈ *main_part M A* ⟷ *x* ∈ *A*
  using *AE_notin_null_part*[*OF A*]
  by (*subst* (*2*) *main_part_null_part_Un*[*symmetric*, *OF A*]) *auto*

lemma *completion_density_eq*:
  assumes *ae*: *AE x in M*. *0* < *f x* **and** [*measurable*]: *f* ∈ *M* $\rightarrow_M$ *borel*
  shows *completion* (*density M f*) = *density* (*completion M*) *f*
proof (*intro measure_eqI*)
  have *N'* ∈ *sets M* ∧ (*AE x*∈*N' in M*. *f x* = *0*) ⟷ *N'* ∈ *null_sets M* **for** *N'*
  proof *safe*
    assume *N'*: *N'* ∈ *sets M* **and** *ae_N'*: *AE x*∈*N' in M*. *f x* = *0*

```
    from ae_N' ae have AE x in M. x ∉ N'
      by eventually_elim auto
    then show N' ∈ null_sets M
      using N' by (simp add: AE_iff_null_sets)
  next
    assume N': N' ∈ null_sets M then show N' ∈ sets M AE x∈N' in M. f x =
0
      using ae AE_not_in[OF N'] by (auto simp: less_le)
  qed
  then show sets_eq: sets (completion (density M f)) = sets (density (completion
M) f)
    by (simp add: sets_completion null_sets_density_iff)

  fix A assume A: ‹A ∈ completion (density M f)›
  moreover
  have A ∈ completion M
    using A unfolding sets_eq by simp
  moreover
  have main_part (density M f) A ∈ M
    using A main_part_sets[of A density M f] unfolding sets_density sets_eq by
simp
  moreover have AE x in M. x ∈ main_part (density M f) A ⟷ x ∈ A
    using AE_in_main_part[OF ‹A ∈ completion (density M f)›] ae by (auto simp
add: AE_density)
  ultimately show emeasure (completion (density M f)) A = emeasure (density
(completion M) f) A
    by (auto simp add: emeasure_density measurable_completion nn_integral_completion
intro!: nn_integral_cong_AE)
qed

lemma null_sets_subset: B ∈ null_sets M ⟹ A ∈ sets M ⟹ A ⊆ B ⟹ A ∈
null_sets M
  using emeasure_mono[of A B M] by (simp add: null_sets_def)

lemma (in complete_measure) complete2: A ⊆ B ⟹ B ∈ null_sets M ⟹ A ∈
null_sets M
  using complete[of A B] null_sets_subset[of B M A] by simp

lemma (in complete_measure) AE_iff_null_sets: (AE x in M. P x) ⟷ {x∈space
M. ¬ P x} ∈ null_sets M
  unfolding eventually_ae_filter by (auto intro: complete2)

lemma (in complete_measure) null_sets_iff_AE: A ∈ null_sets M ⟷ ((AE x in
M. x ∉ A) ∧ A ⊆ space M)
  unfolding AE_iff_null_sets by (auto cong: rev_conj_cong dest: sets.sets_into_space
simp: subset_eq)

lemma (in complete_measure) in_sets_AE:
  assumes ae: AE x in M. x ∈ A ⟷ x ∈ B and A: A ∈ sets M and B: B ⊆
```

space M
  shows *B* ∈ *sets M*
proof −
  have (*AE x in M*. *x* ∉ *B* − *A* ∧ *x* ∉ *A* − *B*)
    using *ae* **by** *eventually_elim auto*
  then have *B* − *A* ∈ *null_sets M A* − *B* ∈ *null_sets M*
    using *A B* **unfolding** *null_sets_iff_AE* **by** (*auto dest*: *sets.sets_into_space*)
  then have *A* ∪ (*B* − *A*) − (*A* − *B*) ∈ *sets M*
    using *A* **by** *blast*
  also have *A* ∪ (*B* − *A*) − (*A* − *B*) = *B*
    by *auto*
  finally show *B* ∈ *sets M* **.**
qed

lemma (**in** *complete_measure*) *vimage_null_part_null_sets*:
  assumes *f*: *f* ∈ *M* $\rightarrow_M$ *N* **and** *eq*: *null_sets N* ⊆ *null_sets* (*distr M N f*)
    and *A*: *A* ∈ *completion N*
  shows *f* −‘ *null_part N A* ∩ *space M* ∈ *null_sets M*
proof −
  obtain *N′* **where** *N′* ∈ *null_sets N null_part N A* ⊆ *N′*
    using *null_part*[*OF A*] **by** *auto*
  then have *N′*: *N′* ∈ *null_sets* (*distr M N f*)
    using *eq* **by** *auto*
  show *?thesis*
  proof (*rule complete2*)
    show *f* −‘ *null_part N A* ∩ *space M* ⊆ *f* −‘ *N′* ∩ *space M*
      using ‹*null_part N A* ⊆ *N′*› **by** *auto*
    show *f* −‘ *N′* ∩ *space M* ∈ *null_sets M*
      using *f N′* **by** (*auto simp*: *null_sets_def emeasure_distr*)
  qed
qed

lemma (**in** *complete_measure*) *vimage_null_part_sets*:
  f ∈ *M* $\rightarrow_M$ *N* ⟹ *null_sets N* ⊆ *null_sets* (*distr M N f*) ⟹ *A* ∈ *completion N* ⟹
  f −‘ *null_part N A* ∩ *space M* ∈ *sets M*
  using *vimage_null_part_null_sets*[*of f N A*] **by** *auto*

lemma (**in** *complete_measure*) *measurable_completion2*:
  assumes *f*: *f* ∈ *M* $\rightarrow_M$ *N* **and** *null_sets*: *null_sets N* ⊆ *null_sets* (*distr M N f*)
  shows *f* ∈ *M* $\rightarrow_M$ *completion N*
proof (*rule measurableI*)
  show *x* ∈ *space M* ⟹ *f x* ∈ *space* (*completion N*) **for** *x*
    using *f*[*THEN measurable_space*] **by** *auto*
  fix *A* **assume** *A*: *A* ∈ *sets* (*completion N*)
  have *f* −‘ *A* ∩ *space M* = (*f* −‘ *main_part N A* ∩ *space M*) ∪ (*f* −‘ *null_part N A* ∩ *space M*)
    using *main_part_null_part_Un*[*OF A*] **by** *auto*
  then show *f* −‘ *A* ∩ *space M* ∈ *sets M*

using *f A null_sets* **by** (*auto intro*: *vimage_null_part_sets measurable_sets*)
qed

lemma (**in** *complete_measure*) *completion_distr_eq*:
assumes *X*: $X \in M \rightarrow_M N$ **and** *null_sets*: *null_sets* (*distr M N X*) = *null_sets N*
shows *completion* (*distr M N X*) = *distr M* (*completion N*) *X*
proof (*rule measure_eqI*)
show *eq*: *sets* (*completion* (*distr M N X*)) = *sets* (*distr M* (*completion N*) *X*)
by (*simp add*: *sets_completion null_sets*)

fix *A* **assume** *A*: *A* ∈ *completion* (*distr M N X*)
moreover have *A'*: *A* ∈ *completion N*
using *A* **by** (*simp add*: *eq*)
moreover have *main_part* (*distr M N X*) *A* ∈ *sets N*
using *main_part_sets*[*OF A*] **by** *simp*
moreover have *X* −' *A* ∩ *space M* = (*X* −' *main_part* (*distr M N X*) *A* ∩ *space M*) ∪ (*X* −' *null_part* (*distr M N X*) *A* ∩ *space M*)
using *main_part_null_part_Un*[*OF A*] **by** *auto*
moreover have *X* −' *null_part* (*distr M N X*) *A* ∩ *space M* ∈ *null_sets M*
using *X A* **by** (*intro vimage_null_part_null_sets*) (*auto cong*: *distr_cong*)
ultimately show *emeasure* (*completion* (*distr M N X*)) *A* = *emeasure* (*distr M* (*completion N*) *X*) *A*
using *X* **by** (*auto simp*: *emeasure_distr measurable_completion null_sets measurable_completion2*
intro!: *emeasure_Un_null_set*[*symmetric*])
qed

lemma *distr_completion*: $X \in M \rightarrow_M N \Longrightarrow$ *distr* (*completion M*) *N X* = *distr M N X*
by (*intro measure_eqI*) (*auto simp*: *emeasure_distr measurable_completion*)

proposition (**in** *complete_measure*) *fmeasurable_inner_outer*:
S ∈ *fmeasurable M* ⟷
(∀ *e*>*0*. ∃ *T*∈*fmeasurable M*. ∃ *U*∈*fmeasurable M*. *T* ⊆ *S* ∧ *S* ⊆ *U* ∧ |*measure M T* − *measure M U*| < *e*)
(**is** _ ⟷ *?approx*)
proof *safe*
let *?μ* = *measure M* **let** *?D* = λ*T U* . |*?μ T* − *?μ U*|
assume *?approx*
have ∃ *A*. ∀ *n*. (*fst* (*A n*) ∈ *fmeasurable M* ∧ *snd* (*A n*) ∈ *fmeasurable M* ∧ *fst* (*A n*) ⊆ *S* ∧ *S* ⊆ *snd* (*A n*) ∧
?D (*fst* (*A n*)) (*snd* (*A n*)) < *1*/*Suc n*) ∧ (*fst* (*A n*) ⊆ *fst* (*A* (*Suc n*)) ∧ *snd* (*A* (*Suc n*)) ⊆ *snd* (*A n*))
(**is** ∃ *A*. ∀ *n*. *?P n* (*A n*) ∧ *?Q* (*A n*) (*A* (*Suc n*)))
proof (*intro dependent_nat_choice*)
show ∃ *A*. *?P 0 A*
using ‹*?approx*›[*THEN spec*, *of 1*] **by** *auto*
next

fix *A n* **assume** *?P n A*
moreover
from ‹*?approx*›[*THEN spec*, *of 1/Suc* (*Suc n*)]
obtain *T U* **where** ∗: *T* ∈ *fmeasurable M U* ∈ *fmeasurable M T* ⊆ *S S* ⊆ *U ?D T U* < *1* / *Suc* (*Suc n*)
by *auto*
ultimately have *?μ T* ≤ *?μ* (*T* ∪ *fst A*) *?μ* (*U* ∩ *snd A*) ≤ *?μ U*
?μ T ≤ *?μ U ?μ* (*T* ∪ *fst A*) ≤ *?μ* (*U* ∩ *snd A*)
by (*auto intro*!: *measure_mono_fmeasurable*)
then have *?D* (*T* ∪ *fst A*) (*U* ∩ *snd A*) ≤ *?D T U*
by *auto*
also have *?D T U* < *1/Suc* (*Suc n*) **by** *fact*
finally show ∃*B*. *?P* (*Suc n*) *B* ∧ *?Q A B*
using ‹*?P n A*› ∗
by (*intro exI*[*of* _ (*T* ∪ *fst A*, *U* ∩ *snd A*)] *conjI*) *auto*
qed
then obtain *A*
where *lm*: ⋀*n*. *fst* (*A n*) ∈ *fmeasurable M* ⋀*n*. *snd* (*A n*) ∈ *fmeasurable M*
and *set_bound*: ⋀*n*. *fst* (*A n*) ⊆ *S* ⋀*n*. *S* ⊆ *snd* (*A n*)
and *mono*: ⋀*n*. *fst* (*A n*) ⊆ *fst* (*A* (*Suc n*)) ⋀*n*. *snd* (*A* (*Suc n*)) ⊆ *snd* (*A n*)
and *bound*: ⋀*n*. *?D* (*fst* (*A n*)) (*snd* (*A n*)) < *1/Suc n*
by *metis*

have *INT_sA*: (⋂*n*. *snd* (*A n*)) ∈ *fmeasurable M*
using *lm* **by** (*intro fmeasurable_INT*[*of* _ *0*]) *auto*
have *UN_fA*: (⋃*n*. *fst* (*A n*)) ∈ *fmeasurable M*
using *lm order_trans*[*OF set_bound*(*1*) *set_bound*(*2*)[*of 0*]] **by** (*intro fmeasurable_UN*[*of* _ _ *snd* (*A 0*)]) *auto*

have (λ*n*. *?μ* (*fst* (*A n*)) − *?μ* (*snd* (*A n*))) ⟶ *0*
using *bound*
by (*subst tendsto_rabs_zero_iff*[*symmetric*])
(*intro tendsto_sandwich*[*OF* _ _ *tendsto_const LIMSEQ_inverse_real_of_nat*];
auto intro!: *always_eventually less_imp_le simp*: *divide_inverse*)
moreover
have (λ*n*. *?μ* (*fst* (*A n*)) − *?μ* (*snd* (*A n*))) ⟶ *?μ* (⋃*n*. *fst* (*A n*)) − *?μ* (⋂*n*. *snd* (*A n*))
proof (*intro tendsto_diff Lim_measure_incseq Lim_measure_decseq*)
show *range* (λ*i*. *fst* (*A i*)) ⊆ *sets M range* (λ*i*. *snd* (*A i*)) ⊆ *sets M*
incseq (λ*i*. *fst* (*A i*)) *decseq* (λ*n*. *snd* (*A n*))
using *mono lm* **by** (*auto simp*: *incseq_Suc_iff decseq_Suc_iff intro*!: *measure_mono_fmeasurable*)
show *emeasure M* (⋃*x*. *fst* (*A x*)) ≠ ∞ *emeasure M* (*snd* (*A n*)) ≠ ∞ **for** *n*
using *lm*(*2*)[*of n*] *UN_fA* **by** (*auto simp*: *fmeasurable_def*)
qed
ultimately have *eq*: *0* = *?μ* (⋃*n*. *fst* (*A n*)) − *?μ* (⋂*n*. *snd* (*A n*))
by (*rule LIMSEQ_unique*)

```
  show S ∈ fmeasurable M
    using UN_fA INT_sA
  proof (rule complete_sets_sandwich_fmeasurable)
    show (⋃n. fst (A n)) ⊆ S S ⊆ (⋂n. snd (A n))
      using set_bound by auto
    show ?μ (⋃n. fst (A n)) = ?μ (⋂n. snd (A n))
      using eq by auto
  qed
qed (auto intro!: bexI[of _ S])

lemma (in complete_measure) fmeasurable_measure_inner_outer:
   (S ∈ fmeasurable M ∧ m = measure M S) ⟷
      (∀e>0. ∃T∈fmeasurable M. T ⊆ S ∧ m − e < measure M T) ∧
      (∀e>0. ∃U∈fmeasurable M. S ⊆ U ∧ measure M U < m + e)
    (is ?lhs = ?rhs)
proof
  assume RHS: ?rhs
  then have T: ⋀e. 0 < e ⟶ (∃T∈fmeasurable M. T ⊆ S ∧ m − e < measure
M T)
      and U: ⋀e. 0 < e ⟶ (∃U∈fmeasurable M. S ⊆ U ∧ measure M U < m
+ e)
    by auto
  have S ∈ fmeasurable M
  proof (subst fmeasurable_inner_outer, safe)
    fix e::real assume 0 < e
    with RHS obtain T U where T: T ∈ fmeasurable M T ⊆ S m − e/2 <
measure M T
                        and U: U ∈ fmeasurable M S ⊆ U measure M U < m + e/2
      by (meson half_gt_zero)+
    moreover have measure M U − measure M T < (m + e/2) − (m − e/2)
      by (intro diff_strict_mono) fact+
    moreover have measure M T ≤ measure M U
      using T U by (intro measure_mono_fmeasurable) auto
    ultimately show ∃T∈fmeasurable M. ∃U∈fmeasurable M. T ⊆ S ∧ S ⊆ U
∧ |measure M T − measure M U| < e
      apply (rule_tac bexI[OF _ ‹T ∈ fmeasurable M›])
      apply (rule_tac bexI[OF _ ‹U ∈ fmeasurable M›])
      by auto
  qed
  moreover have m = measure M S
    using ‹S ∈ fmeasurable M› U[of measure M S − m] T[of m − measure M S]
    by (cases m measure M S rule: linorder_cases)
      (auto simp: not_le[symmetric] measure_mono_fmeasurable)
  ultimately show ?lhs
    by simp
qed (auto intro!: bexI[of _ S])

lemma (in complete_measure) null_sets_outer:
  S ∈ null_sets M ⟷ (∀e>0. ∃T∈fmeasurable M. S ⊆ T ∧ measure M T < e)
```

proof −
have $S \in null\_sets\ M \longleftrightarrow (S \in fmeasurable\ M \wedge 0 = measure\ M\ S)$
by (*auto simp*: *null_sets_def emeasure_eq_measure2 intro*: *fmeasurableI*) (*simp add*: *measure_def*)
also have ... $= (\forall e>0.\ \exists T\in fmeasurable\ M.\ S \subseteq T \wedge measure\ M\ T < e)$
unfolding *fmeasurable_measure_inner_outer* **by** *auto*
finally show *?thesis* **.**
qed

lemma (**in** *complete_measure*) *fmeasurable_measure_inner_outer_le*:
$(S \in fmeasurable\ M \wedge m = measure\ M\ S) \longleftrightarrow$
$(\forall e>0.\ \exists T\in fmeasurable\ M.\ T \subseteq S \wedge m - e \leq measure\ M\ T) \wedge$
$(\forall e>0.\ \exists U\in fmeasurable\ M.\ S \subseteq U \wedge measure\ M\ U \leq m + e)$ (**is** *?T1*)
and *null_sets_outer_le*:
$S \in null\_sets\ M \longleftrightarrow (\forall e>0.\ \exists T\in fmeasurable\ M.\ S \subseteq T \wedge measure\ M\ T \leq e)$ (**is** *?T2*)
proof −
have $e > 0 \wedge m - e/2 \leq t \Longrightarrow m - e < t$
$e > 0 \wedge t \leq m + e/2 \Longrightarrow t < m + e$
$e > 0 \longleftrightarrow e/2 > 0$
for *e t*
by *auto*
then show *?T1 ?T2*
unfolding *fmeasurable_measure_inner_outer null_sets_outer*
by (*meson dense le_less_trans less_imp_le*)+
qed

lemma (**in** *cld_measure*) *notin_sets_outer_measure_of_cover*:
assumes E: $E \subseteq space\ M\ E \notin sets\ M$
shows $\exists B\in sets\ M.\ 0 < emeasure\ M\ B \wedge emeasure\ M\ B < \infty \wedge$
$outer\_measure\_of\ M\ (B \cap E) = emeasure\ M\ B \wedge outer\_measure\_of\ M\ (B - E) = emeasure\ M\ B$
proof −
from *locally_determined*[*OF* ‹$E \subseteq space\ M$›] ‹$E \notin sets\ M$›
obtain *F*
where [*measurable*]: $F \in sets\ M$ **and** $emeasure\ M\ F < \infty\ E \cap F \notin sets\ M$
by *blast*
then obtain $H\ H'$
where H: $measurable\_envelope\ M\ (F \cap E)\ H$ **and** H': $measurable\_envelope\ M\ (F - E)\ H'$
using *measurable_envelopeI_countable_cover*[*of* $F \cap E\ \lambda\_.\ F\ M$]
measurable_envelopeI_countable_cover[*of* $F - E\ \lambda\_.\ F\ M$]
by *auto*
note *measurable_envelopeD*(*2*)[*OF* H', *measurable*] *measurable_envelopeD*(*2*)[*OF* H, *measurable*]

from *measurable_envelopeD*(*1*)[*OF* H'] *measurable_envelopeD*(*1*)[*OF* H]
have *subset*: $F - H' \subseteq F \cap E\ F \cap E \subseteq F \cap H$
by *auto*

moreover define *G* **where** *G* = (*F* ∩ *H*) − (*F* − *H'*)
ultimately have *G*: *G* = *F* ∩ *H* ∩ *H'*
by *auto*
have *emeasure M* (*F* ∩ *H*) ≠ *0*
proof
assume *emeasure M* (*F* ∩ *H*) = *0*
then have *F* ∩ *H* ∈ *null_sets M*
by *auto*
with ‹*E* ∩ *F* ∉ *sets M*› **show** *False*
using *complete*[*OF* ‹*F* ∩ *E* ⊆ *F* ∩ *H*›] **by** (*auto simp*: *Int_commute*)
qed
moreover
have *emeasure M* (*F* − *H'*) ≠ *emeasure M* (*F* ∩ *H*)
proof
assume *emeasure M* (*F* − *H'*) = *emeasure M* (*F* ∩ *H*)
with ‹*E* ∩ *F* ∉ *sets M*› *emeasure_mono*[*of F* ∩ *H F M*] ‹*emeasure M F* < ∞›
have *F* ∩ *E* ∈ *sets M*
by (*intro complete_sets_sandwich*[*OF* _ _ *subset*]) *auto*
with ‹*E* ∩ *F* ∉ *sets M*› **show** *False*
by (*simp add*: *Int_commute*)
qed
moreover have *emeasure M* (*F* − *H'*) ≤ *emeasure M* (*F* ∩ *H*)
using *subset* **by** (*intro emeasure_mono*) *auto*
ultimately have *emeasure M G* ≠ *0*
unfolding *G_def* **using** *subset*
by (*subst emeasure_Diff*) (*auto simp*: *top_unique diff_eq_0_iff_ennreal*)
show *?thesis*
proof (*intro bexI conjI*)
have *emeasure M G* ≤ *emeasure M F*
unfolding *G* **by** (*auto intro*!: *emeasure_mono*)
with ‹*emeasure M F* < ∞› **show** *0* < *emeasure M G emeasure M G* < ∞
using ‹*emeasure M G* ≠ *0*› **by** (*auto simp*: *zero_less_iff_neq_zero*)
show [*measurable*]: *G* ∈ *sets M*
unfolding *G* **by** *auto*

have *emeasure M G* = *outer_measure_of M* (*F* ∩ *H'* ∩ (*F* ∩ *E*))
using *measurable_envelopeD*(*3*)[*OF H*, *of F* ∩ *H'*] **unfolding** *G* **by** (*simp add*: *ac_simps*)
also have . . . ≤ *outer_measure_of M* (*G* ∩ *E*)
using *measurable_envelopeD*(*1*)[*OF H*] **by** (*intro outer_measure_of_mono*) (*auto simp*: *G*)
finally show *outer_measure_of M* (*G* ∩ *E*) = *emeasure M G*
using *outer_measure_of_mono*[*of G* ∩ *E G M*] **by** *auto*

have *emeasure M G* = *outer_measure_of M* (*F* ∩ *H* ∩ (*F* − *E*))
using *measurable_envelopeD*(*3*)[*OF H'*, *of F* ∩ *H*] **unfolding** *G* **by** (*simp add*: *ac_simps*)
also have . . . ≤ *outer_measure_of M* (*G* − *E*)
using *measurable_envelopeD*(*1*)[*OF H'*] **by** (*intro outer_measure_of_mono*)

(*auto simp*: *G*)
  finally show *outer_measure_of M* (*G* − *E*) = *emeasure M G*
    using *outer_measure_of_mono*[*of G* − *E G M*] **by** *auto*
  qed
qed

The following theorem is a specialization of D.H. Fremlin, Measure Theory vol 4I (413G). We only show one direction and do not use a inner regular family *K*.

lemma (**in** *cld_measure*) *borel_measurable_cld*:
  fixes *f* :: *'a* ⇒ *real*
  assumes ⋀*A a b*. *A* ∈ *sets M* ⟹ *0* < *emeasure M A* ⟹ *emeasure M A* < ∞ ⟹ *a* < *b* ⟹
    min (*outer_measure_of M* {*x*∈*A*. *f x* ≤ *a*}) (*outer_measure_of M* {*x*∈*A*. *b* ≤ *f x*}) < *emeasure M A*
  shows *f* ∈ *M* →$_M$ *borel*
proof (*rule ccontr*)
  let *?E* = λ*a*. {*x*∈*space M*. *f x* ≤ *a*} **and** *?F* = λ*a*. {*x*∈*space M*. *a* ≤ *f x*}

  assume *f* ∉ *M* →$_M$ *borel*
  then obtain *a* **where** *?E a* ∉ *sets M*
    unfolding *borel_measurable_iff_le* **by** *blast*
  from *notin_sets_outer_measure_of_cover*[*OF _ this*]
  obtain *K*
    where *K*: *K* ∈ *sets M 0* < *emeasure M K emeasure M K* < ∞
      and *eq1*: *outer_measure_of M* (*K* ∩ *?E a*) = *emeasure M K*
      and *eq2*: *outer_measure_of M* (*K* − *?E a*) = *emeasure M K*
    by *auto*
  then have *me_K*: *measurable_envelope M* (*K* ∩ *?E a*) *K*
    by (*subst measurable_envelope_eq2*) *auto*

  define *b* **where** *b n* = *a* + *inverse* (*real* (*Suc n*)) **for** *n*
  have (*SUP n*. *outer_measure_of M* (*K* ∩ *?F* (*b n*))) = *outer_measure_of M* (⋃*n*. *K* ∩ *?F* (*b n*))
  proof (*intro SUP_outer_measure_of_incseq*)
    have *x* ≤ *y* ⟹ *b y* ≤ *b x* **for** *x y*
      by (*auto simp*: *b_def field_simps*)
    then show *incseq* (λ*n*. *K* ∩ {*x* ∈ *space M*. *b n* ≤ *f x*})
      by (*auto simp*: *incseq_def intro*: *order_trans*)
  qed *auto*
  also have (⋃*n*. *K* ∩ *?F* (*b n*)) = *K* − *?E a*
  proof −
    have *b* ⟶ *a*
      unfolding *b_def* **by** (*rule LIMSEQ_inverse_real_of_nat_add*)
    then have ∀ *n*. ¬ *b n* ≤ *f x* ⟹ *f x* ≤ *a* **for** *x*
      by (*rule LIMSEQ_le_const*) (*auto intro*: *less_imp_le simp*: *not_le*)
    moreover have ¬ *b n* ≤ *a* **for** *n*
      by (*auto simp*: *b_def*)
    ultimately show *?thesis*

using ‹*K* ∈ *sets M*›[*THEN sets.sets_into_space*] **by** (*auto simp*: *subset_eq intro*: *order_trans*)
qed
finally have *0* < (*SUP n. outer_measure_of M* (*K* ∩ *?F* (*b n*)))
using *K* **by** (*simp add*: *eq2*)
then obtain *n* **where** *pos_b*: *0* < *outer_measure_of M* (*K* ∩ *?F* (*b n*)) **and** *a* < *b n*
unfolding *less_SUP_iff* **by** (*auto simp*: *b_def*)
from *measurable_envelopeI_countable_cover*[*of K* ∩ *?F* (*b n*) λ_. *K M*] *K*
obtain *K'* **where** *K'* ⊆ *K* **and** *me_K'*: *measurable_envelope M* (*K* ∩ *?F* (*b n*)) *K'*
by *auto*
then have *K'_le_K*: *emeasure M K'* ≤ *emeasure M K*
by (*intro emeasure_mono K*)
have *K'* ∈ *sets M*
using *me_K'* **by** (*rule measurable_envelopeD*)

have *min* (*outer_measure_of M* {*x*∈*K'*. *f x* ≤ *a*}) (*outer_measure_of M* {*x*∈*K'*. *b n* ≤ *f x*}) < *emeasure M K'*
proof (*rule assms*)
show *0* < *emeasure M K'* *emeasure M K'* < ∞
using *measurable_envelopeD2*[*OF me_K'*] *pos_b K K'_le_K* **by** *auto*
qed *fact*+
also have {*x*∈*K'*. *f x* ≤ *a*} = *K'* ∩ (*K* ∩ *?E a*)
using ‹*K'* ∈ *sets M*›[*THEN sets.sets_into_space*] ‹*K'* ⊆ *K*› **by** *auto*
also have {*x*∈*K'*. *b n* ≤ *f x*} = *K'* ∩ (*K* ∩ *?F* (*b n*))
using ‹*K'* ∈ *sets M*›[*THEN sets.sets_into_space*] ‹*K'* ⊆ *K*› **by** *auto*
finally have *min* (*emeasure M K*) (*emeasure M K'*) < *emeasure M K'*
unfolding
measurable_envelopeD(*3*)[*OF me_K* ‹*K'* ∈ *sets M*›, *symmetric*]
measurable_envelopeD(*3*)[*OF me_K'* ‹*K'* ∈ *sets M*›, *symmetric*]
using ‹*K'* ⊆ *K*› **by** (*simp add*: *Int_absorb1 Int_absorb2*)
with *K'_le_K* **show** *False*
by (*auto simp*: *min_def split*: *if_split_asm*)
qed

end

6.12 Regularity of Measures

theory *Regularity*
imports *Measure_Space Borel_Space*
begin

theorem
fixes *M*::'*a*::{*second_countable_topology*, *complete_space*} *measure*
assumes *sb*: *sets M* = *sets borel*
assumes *emeasure M* (*space M*) ≠ ∞
assumes *B* ∈ *sets borel*

shows *inner_regular*: *emeasure M B* =
(*SUP K* ∈ {*K*. *K* ⊆ *B* ∧ *compact K*}. *emeasure M K*) (**is** *?inner B*)
and *outer_regular*: *emeasure M B* =
(*INF U* ∈ {*U*. *B* ⊆ *U* ∧ *open U*}. *emeasure M U*) (**is** *?outer B*)
proof −
have *Us*: *UNIV* = *space M* **by** (*metis assms*(*1*) *sets_eq_imp_space_eq space_borel*)
hence *sU*: *space M* = *UNIV* **by** *simp*
interpret *finite_measure M* **by** *rule fact*
have *approx_inner*: ⋀*A*. *A* ∈ *sets M* ⟹
(⋀*e*. *e* > *0* ⟹ ∃*K*. *K* ⊆ *A* ∧ *compact K* ∧ *emeasure M A* ≤ *emeasure M K* + *ennreal e*) ⟹ *?inner A*
by (*rule ennreal_approx_SUP*)
(*force intro*!: *emeasure_mono simp*: *compact_imp_closed emeasure_eq_measure*)+
have *approx_outer*: ⋀*A*. *A* ∈ *sets M* ⟹
(⋀*e*. *e* > *0* ⟹ ∃*B*. *A* ⊆ *B* ∧ *open B* ∧ *emeasure M B* ≤ *emeasure M A* + *ennreal e*) ⟹ *?outer A*
by (*rule ennreal_approx_INF*)
(*force intro*!: *emeasure_mono simp*: *emeasure_eq_measure sb*)+
from *countable_dense_setE* **guess** *X*::'*a set* **. note** *X* = *this*
{
fix *r*::*real* **assume** *r* > *0* **hence** ⋀*y*. *open* (*ball y r*) ⋀*y*. *ball y r* ≠ {} **by** *auto*
with *X*(*2*)[*OF this*]
have *x*: *space M* = (⋃*x*∈*X*. *cball x r*)
by (*auto simp add*: *sU*) (*metis dist_commute order_less_imp_le*)
let *?U* = ⋃*k*. (⋃*n*∈{*0*..*k*}. *cball* (*from_nat_into X n*) *r*)
have (λ*k*. *emeasure M* (⋃*n*∈{*0*..*k*}. *cball* (*from_nat_into X n*) *r*)) ⟶ *M ?U*
by (*rule Lim_emeasure_incseq*) (*auto intro*!: *borel_closed bexI simp*: *incseq_def Us sb*)
also have *?U* = *space M*
proof *safe*
fix *x* **from** *X*(*2*)[*OF open_ball*[*of x r*]] ‹*r* > *0*› **obtain** *d* **where** *d*: *d*∈*X d* ∈ *ball x r* **by** *auto*
show *x* ∈ *?U*
using *X*(*1*) *d*
by *simp* (*auto intro*!: *exI* [**where** *x* = *to_nat_on X d*] *simp*: *dist_commute Bex_def*)
qed (*simp add*: *sU*)
finally have (λ*k*. *M* (⋃*n*∈{*0*..*k*}. *cball* (*from_nat_into X n*) *r*)) ⟶ *M* (*space M*) **.**
} **note** *M_space* = *this*
{
fix *e* ::*real* **and** *n* :: *nat* **assume** *e* > *0 n* > *0*
hence *1*/*n* > *0 e* ∗ *2 powr* − *n* > *0* **by** (*auto*)
from *M_space*[*OF* ‹*1*/*n*>*0*›]
have (λ*k*. *measure M* (⋃*i*∈{*0*..*k*}. *cball* (*from_nat_into X i*) (*1*/*real n*))) ⟶ *measure M* (*space M*)
unfolding *emeasure_eq_measure* **by** (*auto*)
from *metric_LIMSEQ_D*[*OF this* ‹*0* < *e* ∗ *2 powr* −*n*›]

obtain *k* **where** *dist* (*measure M* (⋃*i*∈{*0*..*k*}. *cball* (*from_nat_into X i*) (*1*/*real* *n*))) (*measure M* (*space M*)) <
e ∗ *2 powr* −*n*
by *auto*
hence *measure M* (⋃*i*∈{*0*..*k*}. *cball* (*from_nat_into X i*) (*1*/*real n*)) ≥
measure M (*space M*) − *e* ∗ *2 powr* −*real n*
by (*auto simp*: *dist_real_def*)
hence ∃ *k*. *measure M* (⋃*i*∈{*0*..*k*}. *cball* (*from_nat_into X i*) (*1*/*real n*)) ≥
measure M (*space M*) − *e* ∗ *2 powr* − *real n* **..**
} **note** *k*=*this*
hence ∀ *e*∈{*0*<..}. ∀ (*n*::*nat*)∈{*0*<..}. ∃ *k*.
measure M (⋃*i*∈{*0*..*k*}. *cball* (*from_nat_into X i*) (*1*/*real n*)) ≥ *measure M* (*space M*) − *e* ∗ *2 powr* − *real n*
by *blast*
then obtain *k* **where** *k*: ∀ *e*∈{*0*<..}. ∀ *n*∈{*0*<..}. *measure M* (*space M*) − *e* ∗ *2 powr* − *real* (*n*::*nat*)
≤ *measure M* (⋃*i*∈{*0*..*k e n*}. *cball* (*from_nat_into X i*) (*1* / *n*))
by *metis*
hence *k*: ⋀*e n*. *e* > *0* ⟹ *n* > *0* ⟹ *measure M* (*space M*) − *e* ∗ *2 powr* − *n*
≤ *measure M* (⋃*i*∈{*0*..*k e n*}. *cball* (*from_nat_into X i*) (*1* / *n*))
unfolding *Ball_def* **by** *blast*
have *approx_space*:
∃ *K* ∈ {*K*. *K* ⊆ *space M* ∧ *compact K*}. *emeasure M* (*space M*) ≤ *emeasure* *M K* + *ennreal e*
(**is** *?thesis e*) **if** *0* < *e* **for** *e* :: *real*
proof −
define *B* **where** [*abs_def*]:
B n = (⋃*i*∈{*0*..*k e* (*Suc n*)}. *cball* (*from_nat_into X i*) (*1* / *Suc n*)) **for** *n*
have ⋀*n*. *closed* (*B n*) **by** (*auto simp*: *B_def*)
hence [*simp*]: ⋀*n*. *B n* ∈ *sets M* **by** (*simp add*: *sb*)
from *k*[*OF* ‹*e* > *0*› *zero_less_Suc*]
have ⋀*n*. *measure M* (*space M*) − *measure M* (*B n*) ≤ *e* ∗ *2 powr* − *real* (*Suc* *n*)
by (*simp add*: *algebra_simps B_def finite_measure_compl*)
hence *B_compl_le*: ⋀*n*::*nat*. *measure M* (*space M* − *B n*) ≤ *e* ∗ *2 powr* − *real* (*Suc n*)
by (*simp add*: *finite_measure_compl*)
define *K* **where** *K* = (⋂*n*. *B n*)
from ‹*closed* (*B* _)› **have** *closed K* **by** (*auto simp*: *K_def*)
hence [*simp*]: *K* ∈ *sets M* **by** (*simp add*: *sb*)
have *measure M* (*space M*) − *measure M K* = *measure M* (*space M* − *K*)
by (*simp add*: *finite_measure_compl*)
also have ... = *emeasure M* (⋃*n*. *space M* − *B n*) **by** (*auto simp*: *K_def* *emeasure_eq_measure*)
also have ... ≤ (∑*n*. *emeasure M* (*space M* − *B n*))
by (*rule emeasure_subadditive_countably*) (*auto simp*: *summable_def*)
also have ... ≤ (∑*n*. *ennreal* (*e*∗*2 powr* − *real* (*Suc n*)))
using *B_compl_le* **by** (*intro suminf_le*) (*simp_all add*: *emeasure_eq_measure* *ennreal_leI*)

also have ... ≤ (∑ *n*. *ennreal* (*e* ∗ (*1* / *2*) ˆ *Suc n*))
by (*simp add*: *powr_minus powr_realpow field_simps del*: *of_nat_Suc*)
also have ... = *ennreal e* ∗ (∑ *n*. *ennreal* ((*1* / *2*) ˆ *Suc n*))
unfolding *ennreal_power*[*symmetric*]
using ‹*0* < *e*›
by (*simp add*: *ac_simps ennreal_mult′ divide_ennreal*[*symmetric*] *divide_ennreal_def ennreal_power*[*symmetric*])
also have ... = *e*
by (*subst suminf_ennreal_eq*[*OF zero_le_power power_half_series*]) *auto*
finally have *measure M* (*space M*) ≤ *measure M K* + *e*
using ‹*0* < *e*› **by** *simp*
hence *emeasure M* (*space M*) ≤ *emeasure M K* + *e*
using ‹*0* < *e*› **by** (*simp add*: *emeasure_eq_measure flip*: *ennreal_plus*)
moreover have *compact K*
unfolding *compact_eq_totally_bounded*
proof *safe*
show *complete K* **using** ‹*closed K*› **by** (*simp add*: *complete_eq_closed*)
fix *e′*::*real* **assume** *0* < *e′*
from *nat_approx_posE*[*OF this*] **guess** *n* **. note** *n* = *this*
let *?k* = *from_nat_into X* ‘ {*0*..*k e* (*Suc n*)}
have *finite ?k* **by** *simp*
moreover have *K* ⊆ (⋃*x*∈*?k*. *ball x e′*) **unfolding** *K_def B_def* **using** *n* **by** *force*
ultimately show ∃ *k*. *finite k* ∧ *K* ⊆ (⋃*x*∈*k*. *ball x e′*) **by** *blast*
qed
ultimately
show *?thesis* **by** (*auto simp*: *sU*)
qed
{ **fix** *A*::′*a set* **assume** *closed A* **hence** *A* ∈ *sets borel* **by** (*simp add*: *compact_imp_closed*)
hence [*simp*]: *A* ∈ *sets M* **by** (*simp add*: *sb*)
have *?inner A*
proof (*rule approx_inner*)
fix *e*::*real* **assume** *e* > *0*
from *approx_space*[*OF this*] **obtain** *K* **where**
K: *K* ⊆ *space M compact K emeasure M* (*space M*) ≤ *emeasure M K* + *e*
by (*auto simp*: *emeasure_eq_measure*)
hence [*simp*]: *K* ∈ *sets M* **by** (*simp add*: *sb compact_imp_closed*)
have *measure M A* − *measure M* (*A* ∩ *K*) = *measure M* (*A* − *A* ∩ *K*)
by (*subst finite_measure_Diff*) *auto*
also have *A* − *A* ∩ *K* = *A* ∪ *K* − *K* **by** *auto*
also have *measure M* ... = *measure M* (*A* ∪ *K*) − *measure M K*
by (*subst finite_measure_Diff*) *auto*
also have ... ≤ *measure M* (*space M*) − *measure M K*
by (*simp add*: *emeasure_eq_measure sU sb finite_measure_mono*)
also have ... ≤ *e*
using *K* ‹*0* < *e*› **by** (*simp add*: *emeasure_eq_measure flip*: *ennreal_plus*)
finally have *emeasure M A* ≤ *emeasure M* (*A* ∩ *K*) + *ennreal e*
using ‹*0*<*e*› **by** (*simp add*: *emeasure_eq_measure algebra_simps flip*: *en-*

nreal_plus)
moreover have *A* ∩ *K* ⊆ *A compact* (*A* ∩ *K*) **using** ‹*closed A*› ‹*compact K*› **by** *auto*
ultimately show ∃ *K* ⊆ *A*. *compact K* ∧ *emeasure M A* ≤ *emeasure M K* + *ennreal e*
by *blast*
qed *simp*
have *?outer A*
proof *cases*
assume *A* ≠ {}
let *?G* = λ*d*. {*x*. *infdist x A* < *d*}
{
fix *d*
have *?G d* = (λ*x*. *infdist x A*) −' {..<*d*} **by** *auto*
also have *open* ...
by (*intro continuous_open_vimage*) (*auto intro*!: *continuous_infdist continuous_ident*)
finally have *open* (*?G d*) **.**
} **note** *open_G* = *this*
from *in_closed_iff_infdist_zero*[*OF* ‹*closed A*› ‹*A* ≠ {}›]
have *A* = {*x*. *infdist x A* = *0*} **by** *auto*
also have ... = (⋂ *i*. *?G* (*1*/*real* (*Suc i*)))
proof (*auto simp del*: *of_nat_Suc*, *rule ccontr*)
fix *x*
assume *infdist x A* ≠ *0*
hence *pos*: *infdist x A* > *0* **using** *infdist_nonneg*[*of x A*] **by** *simp*
from *nat_approx_posE*[*OF this*] **guess** *n* **.**
moreover
assume ∀ *i*. *infdist x A* < *1* / *real* (*Suc i*)
hence *infdist x A* < *1* / *real* (*Suc n*) **by** *auto*
ultimately show *False* **by** *simp*
qed
also have *M* ... = (*INF n*. *emeasure M* (*?G* (*1* / *real* (*Suc n*))))
proof (*rule INF_emeasure_decseq*[*symmetric*], *safe*)
fix *i*::*nat*
from *open_G*[*of 1* / *real* (*Suc i*)]
show *?G* (*1* / *real* (*Suc i*)) ∈ *sets M* **by** (*simp add*: *sb borel_open*)
next
show *decseq* (λ*i*. {*x*. *infdist x A* < *1* / *real* (*Suc i*)})
by (*auto intro*: *less_trans intro*!: *divide_strict_left_mono*
simp: *decseq_def le_eq_less_or_eq*)
qed *simp*
finally
have *emeasure M A* = (*INF n*. *emeasure M* {*x*. *infdist x A* < *1* / *real* (*Suc n*)}) **.**
moreover
have ... ≥ (*INF U*∈{*U*. *A* ⊆ *U* ∧ *open U*}. *emeasure M U*)
proof (*intro INF_mono*)
fix *m*

have *?G (1 / real (Suc m))* ∈ {*U*. *A* ⊆ *U* ∧ *open U*} **using** *open_G* **by** *auto*
moreover have *M (?G (1 / real (Suc m)))* ≤ *M (?G (1 / real (Suc m)))* **by** *simp*
ultimately show ∃ *U*∈{*U*. *A* ⊆ *U* ∧ *open U*}.
emeasure M U ≤ *emeasure M* {*x*. *infdist x A* < *1* / *real (Suc m)*}
by *blast*
qed
moreover
have *emeasure M A* ≤ (*INF U*∈{*U*. *A* ⊆ *U* ∧ *open U*}. *emeasure M U*)
by (*rule INF_greatest*) (*auto intro*!: *emeasure_mono simp*: *sb*)
ultimately show *?thesis* **by** *simp*
qed (*auto intro*!: *INF_eqI*)
note ‹*?inner A*› ‹*?outer A*› **}**
note *closed_in_D* = *this*
from ‹*B* ∈ *sets borel*›
have *Int_stable* (*Collect closed*) *Collect closed* ⊆ *Pow UNIV B* ∈ *sigma_sets UNIV* (*Collect closed*)
by (*auto simp*: *Int_stable_def borel_eq_closed*)
then show *?inner B ?outer B*
proof (*induct B rule*: *sigma_sets_induct_disjoint*)
case *empty*
{ case *1* **show** *?case* **by** (*intro SUP_eqI*[*symmetric*]) *auto* **}**
{ case *2* **show** *?case* **by** (*intro INF_eqI*[*symmetric*]) (*auto elim*!: *meta_allE*[*of* _ {}]) **}**
next
case (*basic B*)
{ case *1* **from** *basic closed_in_D* **show** *?case* **by** *auto* **}**
{ case *2* **from** *basic closed_in_D* **show** *?case* **by** *auto* **}**
next
case (*compl B*)
note *inner* = *compl*(*2*) **and** *outer* = *compl*(*3*)
from *compl* **have** [*simp*]: *B* ∈ *sets M* **by** (*auto simp*: *sb borel_eq_closed*)
case *2*
have *M* (*space M* − *B*) = *M* (*space M*) − *emeasure M B* **by** (*auto simp*: *emeasure_compl*)
also have . . . = (*INF K*∈{*K*. *K* ⊆ *B* ∧ *compact K*}. *M* (*space M*) − *M K*)
by (*subst ennreal_SUP_const_minus*) (*auto simp*: *less_top*[*symmetric*] *inner*)
also have . . . = (*INF U*∈{*U*. *U* ⊆ *B* ∧ *compact U*}. *M* (*space M* − *U*))
by (*auto simp add*: *emeasure_compl sb compact_imp_closed*)
also have . . . ≥ (*INF U*∈{*U*. *U* ⊆ *B* ∧ *closed U*}. *M* (*space M* − *U*))
by (*rule INF_superset_mono*) (*auto simp add*: *compact_imp_closed*)
also have (*INF U*∈{*U*. *U* ⊆ *B* ∧ *closed U*}. *M* (*space M* − *U*)) =
(*INF U*∈{*U*. *space M* − *B* ⊆ *U* ∧ *open U*}. *emeasure M U*)
apply (*rule arg_cong* [*of* _ _ *Inf*])
using *sU*
apply (*auto simp add*: *image_iff*)
apply (*rule exI* [*of* _ *UNIV* − *y* **for** *y*])
apply *safe*

apply (*auto simp add*: *double_diff*)
done
finally have
($INF\ U{\in}\{U.\ space\ M - B \subseteq U \wedge open\ U\}.\ emeasure\ M\ U) \leq emeasure\ M\ (space\ M - B)$ **.**
moreover have
($INF\ U{\in}\{U.\ space\ M - B \subseteq U \wedge open\ U\}.\ emeasure\ M\ U) \geq emeasure\ M\ (space\ M - B)$
by (*auto simp*: *sb sU intro*!: *INF_greatest emeasure_mono*)
ultimately show *?case* **by** (*auto intro*!: *antisym simp*: *sets_eq_imp_space_eq*[*OF sb*])

case *1*
have $M\ (space\ M - B) = M\ (space\ M) - emeasure\ M\ B$ **by** (*auto simp*: *emeasure_compl*)
also have $\ldots = (SUP\ U{\in}\ \{U.\ B \subseteq U \wedge open\ U\}.\ M\ (space\ M) -\ M\ U)$
unfolding *outer* **by** (*subst ennreal_INF_const_minus*) *auto*
also have $\ldots = (SUP\ U{\in}\{U.\ B \subseteq U \wedge open\ U\}.\ M\ (space\ M - U))$
by (*auto simp add*: *emeasure_compl sb compact_imp_closed*)
also have $\ldots = (SUP\ K{\in}\{K.\ K \subseteq space\ M - B \wedge closed\ K\}.\ emeasure\ M\ K)$
unfolding *SUP_image* [*of* _ $\lambda u.\ space\ M - u$ _, *symmetric*, *unfolded comp_def*]
apply (*rule arg_cong* [*of* _ _ *Sup*])
using *sU* **apply** (*auto intro*!: *imageI*)
done
also have $\ldots = (SUP\ K{\in}\{K.\ K \subseteq space\ M - B \wedge compact\ K\}.\ emeasure\ M\ K)$
proof (*safe intro*!: *antisym SUP_least*)
fix *K* **assume** *closed K* $K \subseteq space\ M - B$
from *closed_in_D*[*OF* ‹*closed K*›]
have *K_inner*: $emeasure\ M\ K = (SUP\ K{\in}\{Ka.\ Ka \subseteq K \wedge compact\ Ka\}.\ emeasure\ M\ K)$ **by** *simp*
show $emeasure\ M\ K \leq (SUP\ K{\in}\{K.\ K \subseteq space\ M - B \wedge compact\ K\}.\ emeasure\ M\ K)$
unfolding *K_inner* **using** ‹$K \subseteq space\ M - B$›
by (*auto intro*!: *SUP_upper SUP_least*)
qed (*fastforce intro*!: *SUP_least SUP_upper simp*: *compact_imp_closed*)
finally show *?case* **by** (*auto intro*!: *antisym simp*: *sets_eq_imp_space_eq*[*OF sb*])
next
case (*union D*)
then have $range\ D \subseteq sets\ M$ **by** (*auto simp*: *sb borel_eq_closed*)
with *union* **have** *M*[*symmetric*]: $(\sum i.\ M\ (D\ i)) = M\ (\bigcup i.\ D\ i)$ **by** (*intro suminf_emeasure*)
also have $(\lambda n.\ \sum i{<}n.\ M\ (D\ i)) \longrightarrow (\sum i.\ M\ (D\ i))$
by (*intro summable_LIMSEQ*) *auto*
finally have *measure_LIMSEQ*: $(\lambda n.\ \sum i{<}n.\ measure\ M\ (D\ i)) \longrightarrow measure\ M\ (\bigcup i.\ D\ i)$
by (*simp add*: *emeasure_eq_measure sum_nonneg*)
have $(\bigcup i.\ D\ i) \in sets\ M$ **using** ‹$range\ D \subseteq sets\ M$› **by** *auto*

case *1*
show *?case*
proof (*rule approx_inner*)
fix *e*::*real* **assume** *e* > *0*
with *measure_LIMSEQ*
have ∃ *no*. ∀ *n*≥*no*. |(∑ *i*<*n*. *measure M* (*D i*)) −*measure M* (⋃ *x*. *D x*)| < *e*/*2*
by (*auto simp*: *lim_sequentially dist_real_def simp del*: *less_divide_eq_numeral1*)
hence ∃ *n0*. |(∑ *i*<*n0*. *measure M* (*D i*)) − *measure M* (⋃ *x*. *D x*)| < *e*/*2* **by** *auto*
then obtain *n0* **where** *n0*: |(∑ *i*<*n0*. *measure M* (*D i*)) − *measure M* (⋃ *i*. *D i*)| < *e*/*2*
unfolding *choice_iff* **by** *blast*
have *ennreal* (∑ *i*<*n0*. *measure M* (*D i*)) = (∑ *i*<*n0*. *M* (*D i*))
by (*auto simp add*: *emeasure_eq_measure*)
also have ... ≤ (∑ *i*. *M* (*D i*)) **by** (*rule sum_le_suminf*) *auto*
also have ... = *M* (⋃ *i*. *D i*) **by** (*simp add*: *M*)
also have ... = *measure M* (⋃ *i*. *D i*) **by** (*simp add*: *emeasure_eq_measure*)
finally have *n0*: *measure M* (⋃ *i*. *D i*) − (∑ *i*<*n0*. *measure M* (*D i*)) < *e*/*2*
using *n0* **by** (*auto simp*: *sum_nonneg*)
have ∀ *i*. ∃ *K*. *K* ⊆ *D i* ∧ *compact K* ∧ *emeasure M* (*D i*) ≤ *emeasure M K* + *e*/(*2*∗*Suc n0*)
proof
fix *i*
from ‹*0* < *e*› **have** *0* < *e*/(*2*∗*Suc n0*) **by** *simp*
have *emeasure M* (*D i*) = (*SUP K*∈{*K*. *K* ⊆ (*D i*) ∧ *compact K*}. *emeasure M K*)
using *union* **by** *blast*
from *SUP_approx_ennreal*[*OF* ‹*0* < *e*/(*2*∗*Suc n0*)› _ *this*]
show ∃ *K*. *K* ⊆ *D i* ∧ *compact K* ∧ *emeasure M* (*D i*) ≤ *emeasure M K* + *e*/(*2*∗*Suc n0*)
by (*auto simp*: *emeasure_eq_measure intro*: *less_imp_le compact_empty*)
qed
then obtain *K* **where** *K*: ⋀*i*. *K i* ⊆ *D i* ⋀*i*. *compact* (*K i*)
⋀*i*. *emeasure M* (*D i*) ≤ *emeasure M* (*K i*) + *e*/(*2*∗*Suc n0*)
unfolding *choice_iff* **by** *blast*
let *?K* = ⋃ *i*∈{..<*n0*}. *K i*
have *disjoint_family_on K* {..<*n0*} **using** *K* ‹*disjoint_family D*›
unfolding *disjoint_family_on_def* **by** *blast*
hence *mK*: *measure M ?K* = (∑ *i*<*n0*. *measure M* (*K i*)) **using** *K*
by (*intro finite_measure_finite_Union*) (*auto simp*: *sb compact_imp_closed*)
have *measure M* (⋃ *i*. *D i*) < (∑ *i*<*n0*. *measure M* (*D i*)) + *e*/*2* **using** *n0* **by** *simp*
also have (∑ *i*<*n0*. *measure M* (*D i*)) ≤ (∑ *i*<*n0*. *measure M* (*K i*) + *e*/(*2*∗*Suc n0*))
using *K* ‹*0* < *e*›
by (*auto intro*: *sum_mono simp*: *emeasure_eq_measure simp flip*: *ennreal_plus*)
also have ... = (∑ *i*<*n0*. *measure M* (*K i*)) + (∑ *i*<*n0*. *e*/(*2*∗*Suc n0*))

```
        by (simp add: sum.distrib)
      also have ... ≤ (∑ i<n0. measure M (K i)) +  e / 2 using ‹0 < e›
        by (auto simp: field_simps intro!: mult_left_mono)
      finally
      have measure M (⋃ i. D i) < (∑ i<n0. measure M (K i)) + e / 2 + e / 2
        by auto
      hence M (⋃ i. D i) < M ?K + e
          using ‹0<e› by (auto simp: mK emeasure_eq_measure sum_nonneg en-
nreal_less_iff simp flip: ennreal_plus)
      moreover
      have ?K ⊆ (⋃ i. D i) using K by auto
      moreover
      have compact ?K using K by auto
      ultimately
      have ?K⊆(⋃ i. D i) ∧ compact ?K ∧ emeasure M (⋃ i. D i) ≤ emeasure M
?K + ennreal e by simp
       thus ∃ K⊆⋃ i. D i. compact K ∧ emeasure M (⋃ i. D i) ≤ emeasure M K
+ ennreal e ..
    qed fact
    case 2
    show ?case
    proof (rule approx_outer[OF ‹(⋃ i. D i) ∈ sets M›])
      fix e::real assume e > 0
      have ∀ i::nat. ∃ U. D i ⊆ U ∧ open U ∧ e/(2 powr Suc i) > emeasure M U
− emeasure M (D i)
      proof
        fix i::nat
        from ‹0 < e› have 0 < e/(2 powr Suc i) by simp
         have emeasure M (D i) = (INF U∈{U. (D i) ⊆ U ∧ open U}. emeasure
M U)
          using union by blast
        from INF_approx_ennreal[OF ‹0 < e/(2 powr Suc i)› this]
         show ∃ U. D i ⊆ U ∧ open U ∧ e/(2 powr Suc i) > emeasure M U −
emeasure M (D i)
          using ‹0<e›
              by (auto simp: emeasure_eq_measure sum_nonneg ennreal_less_iff en-
nreal_minus
                          finite_measure_mono sb
                    simp flip: ennreal_plus)
      qed
      then obtain U where U: ⋀i. D i ⊆ U i ⋀i. open (U i)
        ⋀i. e/(2 powr Suc i) > emeasure M (U i) − emeasure M (D i)
        unfolding choice_iff by blast
      let ?U = ⋃ i. U i
      have ennreal (measure M ?U − measure M (⋃ i. D i)) = M ?U − M (⋃ i.
D i)
        using U(1,2)
        by (subst ennreal_minus[symmetric])
           (auto intro!: finite_measure_mono simp: sb emeasure_eq_measure)
```

also have ... = *M* (*?U* − (⋃*i*. *D i*)) **using** *U* ‹(⋃*i*. *D i*) ∈ *sets M*›
by (*subst emeasure_Diff*) (*auto simp*: *sb*)
also have ... ≤ *M* (⋃*i*. *U i* − *D i*) **using** *U* ‹*range D* ⊆ *sets M*›
by (*intro emeasure_mono*) (*auto simp*: *sb intro*!: *sets.countable_nat_UN sets.Diff*)
also have ... ≤ (∑*i*. *M* (*U i* − *D i*)) **using** *U* ‹*range D* ⊆ *sets M*›
by (*intro emeasure_subadditive_countably*) (*auto intro*!: *sets.Diff simp*: *sb*)
also have ... ≤ (∑*i*. *ennreal e*/(*2 powr Suc i*)) **using** *U* ‹*range D* ⊆ *sets M*›
using ‹*0*<*e*›
by (*intro suminf_le*, *subst emeasure_Diff*)
(*auto simp*: *emeasure_Diff emeasure_eq_measure sb ennreal_minus*
finite_measure_mono divide_ennreal ennreal_less_iff
intro: *less_imp_le*)
also have ... ≤ (∑*n*. *ennreal* (*e* ∗ (*1* / *2*) ^ *Suc n*))
using ‹*0*<*e*›
by (*simp add*: *powr_minus powr_realpow field_simps divide_ennreal del*: *of_nat_Suc*)
also have ... = *ennreal e* ∗ (∑*n*. *ennreal* ((*1* / *2*) ^ *Suc n*))
unfolding *ennreal_power*[*symmetric*]
using ‹*0* < *e*›
by (*simp add*: *ac_simps ennreal_mult′ divide_ennreal*[*symmetric*] *divide_ennreal_def*
ennreal_power[*symmetric*])
also have ... = *ennreal e*
by (*subst suminf_ennreal_eq*[*OF zero_le_power power_half_series*]) *auto*
finally have *emeasure M ?U* ≤ *emeasure M* (⋃*i*. *D i*) + *ennreal e*
using ‹*0*<*e*› **by** (*simp add*: *emeasure_eq_measure flip*: *ennreal_plus*)
moreover
have (⋃*i*. *D i*) ⊆ *?U* **using** *U* **by** *auto*
moreover
have *open ?U* **using** *U* **by** *auto*
ultimately
have (⋃*i*. *D i*) ⊆ *?U* ∧ *open ?U* ∧ *emeasure M ?U* ≤ *emeasure M* (⋃*i*. *D i*) + *ennreal e* **by** *simp*
thus ∃*B*. (⋃*i*. *D i*) ⊆ *B* ∧ *open B* ∧ *emeasure M B* ≤ *emeasure M* (⋃*i*. *D i*) + *ennreal e* **..**
qed
qed
qed

end

6.13 Lebesgue Measure

theory *Lebesgue_Measure*
imports
Finite_Product_Measure
Caratheodory
Complete_Measure

Summation_Tests
Regularity
begin

lemma *measure_eqI_lessThan*:
fixes *M N* :: *real measure*
assumes *sets*: *sets M* = *sets borel sets N* = *sets borel*
assumes *fin*: $\bigwedge x$. *emeasure M* {*x* <..} < ∞
assumes $\bigwedge x$. *emeasure M* {*x* <..} = *emeasure N* {*x* <..}
shows *M* = *N*
proof (*rule measure_eqI_generator_eq_countable*)
let *?LT* = λ*a*::*real*. {*a* <..} **let** *?E* = *range ?LT*
show *Int_stable ?E*
by (*auto simp*: *Int_stable_def lessThan_Int_lessThan*)

show *?E* ⊆ *Pow UNIV sets M* = *sigma_sets UNIV ?E sets N* = *sigma_sets UNIV ?E*
unfolding *sets borel_Ioi* **by** *auto*

show *?LT‘Rats* ⊆ *?E* (⋃*i*∈*Rats*. *?LT i*) = *UNIV* ⋀*a*. *a* ∈ *?LT‘Rats* ⟹ *emeasure M a* ≠ ∞
using *fin* **by** (*auto intro*: *Rats_no_bot_less simp*: *less_top*)
qed (*auto intro*: *assms countable_rat*)

6.13.1 Measures defined by monotonous functions

Every right-continuous and nondecreasing function gives rise to a measure on the reals:

definition *interval_measure* :: (*real* ⇒ *real*) ⇒ *real measure* **where**
interval_measure F =
extend_measure UNIV {(*a*, *b*). *a* ≤ *b*} (λ(*a*, *b*). {*a*<..*b*}) (λ(*a*, *b*). *ennreal* (*F b* − *F a*))

lemma *emeasure_interval_measure_Ioc*:
assumes *a* ≤ *b*
assumes *mono_F*: ⋀*x y*. *x* ≤ *y* ⟹ *F x* ≤ *F y*
assumes *right_cont_F* : ⋀*a*. *continuous* (*at_right a*) *F*
shows *emeasure* (*interval_measure F*) {*a*<..*b*} = *F b* − *F a*
proof (*rule extend_measure_caratheodory_pair*[*OF interval_measure_def* ‹*a* ≤ *b*›])
show *semiring_of_sets UNIV* {{*a*<..*b*} |*a b* :: *real*. *a* ≤ *b*}
proof (*unfold_locales*, *safe*)
fix *a b c d* :: *real* **assume** ∗: *a* ≤ *b c* ≤ *d*
then show ∃ *C*⊆{{*a*<..*b*} |*a b*. *a* ≤ *b*}. *finite C* ∧ *disjoint C* ∧ {*a*<..*b*} − {*c*<..*d*} = ⋃ *C*
proof *cases*
let *?C* = {{*a*<..*b*}}
assume *b* < *c* ∨ *d* ≤ *a* ∨ *d* ≤ *c*
with ∗ **have** *?C* ⊆ {{*a*<..*b*} |*a b*. *a* ≤ *b*} ∧ *finite ?C* ∧ *disjoint ?C* ∧ {*a*<..*b*} − {*c*<..*d*} = ⋃ *?C*

by (*auto simp add*: *disjoint_def*)
thus *?thesis* **..**
next
let *?C* = {{*a*<..*c*}, {*d*<..*b*}}
assume ¬ (*b* < *c* ∨ *d* ≤ *a* ∨ *d* ≤ *c*)
with ∗ **have** *?C* ⊆ {{*a*<..*b*} |*a b*. *a* ≤ *b*} ∧ *finite ?C* ∧ *disjoint ?C* ∧ {*a*<..*b*} − {*c*<..*d*} = ⋃ *?C*
by (*auto simp add*: *disjoint_def Ioc_inj*) (*metis linear*)+
thus *?thesis* **..**
qed
qed (*auto simp*: *Ioc_inj*, *metis linear*)
next
fix *l r* :: *nat* ⇒ *real* **and** *a b* :: *real*
assume *l_r*[*simp*]: ⋀*n*. *l n* ≤ *r n* **and** *a* ≤ *b* **and** *disj*: *disjoint_family* (λ*n*. {*l n*<..*r n*})
assume *lr_eq_ab*: (⋃ *i*. {*l i*<..*r i*}) = {*a*<..*b*}

have [*intro*, *simp*]: ⋀*a b*. *a* ≤ *b* ⟹ *F a* ≤ *F b*
by (*auto intro*!: *l_r mono_F*)

{ fix *S* :: *nat set* **assume** *finite S*
moreover note ‹*a* ≤ *b*›
moreover have ⋀*i*. *i* ∈ *S* ⟹ {*l i* <.. *r i*} ⊆ {*a* <.. *b*}
unfolding *lr_eq_ab*[*symmetric*] **by** *auto*
ultimately have (∑ *i*∈*S*. *F* (*r i*) − *F* (*l i*)) ≤ *F b* − *F a*
proof (*induction S arbitrary*: *a rule*: *finite_psubset_induct*)
case (*psubset S*)
show *?case*
proof *cases*
assume ∃ *i*∈*S*. *l i* < *r i*
with ‹*finite S*› **have** *Min* (*l* ‘ {*i*∈*S*. *l i* < *r i*}) ∈ *l* ‘ {*i*∈*S*. *l i* < *r i*}
by (*intro Min_in*) *auto*
then obtain *m* **where** *m*: *m* ∈ *S l m* < *r m l m* = *Min* (*l* ‘ {*i*∈*S*. *l i* < *r i*})
by *fastforce*

have (∑ *i*∈*S*. *F* (*r i*) − *F* (*l i*)) = (*F* (*r m*) − *F* (*l m*)) + (∑ *i*∈*S* − {*m*}. *F* (*r i*) − *F* (*l i*))
using *m psubset* **by** (*intro sum.remove*) *auto*
also have (∑ *i*∈*S* − {*m*}. *F* (*r i*) − *F* (*l i*)) ≤ *F b* − *F* (*r m*)
proof (*intro psubset.IH*)
show *S* − {*m*} ⊂ *S*
using ‹*m*∈*S*› **by** *auto*
show *r m* ≤ *b*
using *psubset.prems*(*2*)[*OF* ‹*m*∈*S*›] ‹*l m* < *r m*› **by** *auto*
next
fix *i* **assume** *i* ∈ *S* − {*m*}
then have *i*: *i* ∈ *S i* ≠ *m* **by** *auto*
{ assume *i'*: *l i* < *r i l i* < *r m*

```
        with ‹finite S› i m have l m ≤ l i
          by auto
        with i' have {l i <.. r i} ∩ {l m <.. r m} ≠ {}
          by auto
        then have False
          using disjoint_family_onD[OF disj, of i m] i by auto }
      then have l i ≠ r i ⟹ r m ≤ l i
        unfolding not_less[symmetric] using l_r[of i] by auto
      then show {l i <.. r i} ⊆ {r m <.. b}
        using psubset.prems(2)[OF ‹i∈S›] by auto
    qed
    also have F (r m) − F (l m) ≤ F (r m) − F a
      using psubset.prems(2)[OF ‹m ∈ S›] ‹l m < r m›
      by (auto simp add: Ioc_subset_iff intro!: mono_F)
    finally show ?case
      by (auto intro: add_mono)
  qed (auto simp add: ‹a ≤ b› less_le)
qed }
note claim1 = this


{ fix S u v and l r :: nat ⇒ real
  assume finite S ⋀i. i∈S ⟹ l i < r i {u..v} ⊆ (⋃i∈S. {l i<..< r i})
  then have F v − F u ≤ (∑i∈S. F (r i) − F (l i))
  proof (induction arbitrary: v u rule: finite_psubset_induct)
    case (psubset S)
    show ?case
    proof cases
      assume S = {} then show ?case
        using psubset by (simp add: mono_F)
    next
      assume S ≠ {}
      then obtain j where j ∈ S
        by auto

      let ?R = r j < u ∨ l j > v ∨ (∃i∈S−{j}. l i ≤ l j ∧ r j ≤ r i)
      show ?case
      proof cases
        assume ?R
        with ‹j ∈ S› psubset.prems have {u..v} ⊆ (⋃i∈S−{j}. {l i<..< r i})
          apply (auto simp: subset_eq Ball_def)
          apply (metis Diff_iff less_le_trans leD linear singletonD)
          apply (metis Diff_iff less_le_trans leD linear singletonD)
          apply (metis order_trans less_le_not_le linear)
          done
        with ‹j ∈ S› have F v − F u ≤ (∑i∈S − {j}. F (r i) − F (l i))
          by (intro psubset) auto
        also have ... ≤ (∑i∈S. F (r i) − F (l i))
```

using *psubset.prems*
by (*intro sum_mono2 psubset*) (*auto intro*: *less_imp_le*)
finally show *?thesis* **.**
next
assume ¬ *?R*
then have *j*: $u \le r\ j\ l\ j \le v\ \bigwedge i.\ i \in S - \{j\} \Longrightarrow r\ i < r\ j \lor l\ i > l\ j$
by (*auto simp*: *not_less*)
let $?S1 = \{i \in S.\ l\ i < l\ j\}$
let $?S2 = \{i \in S.\ r\ i > r\ j\}$

have $(\sum i{\in}S.\ F\ (r\ i) - F\ (l\ i)) \ge (\sum i{\in}?S1 \cup ?S2 \cup \{j\}.\ F\ (r\ i) - F\ (l\ i))$
using ‹$j \in S$› ‹*finite S*› *psubset.prems j*
by (*intro sum_mono2*) (*auto intro*: *less_imp_le*)
also have $(\sum i{\in}?S1 \cup ?S2 \cup \{j\}.\ F\ (r\ i) - F\ (l\ i)) =$
$(\sum i{\in}?S1.\ F\ (r\ i) - F\ (l\ i)) + (\sum i{\in}?S2\ .\ F\ (r\ i) - F\ (l\ i)) + (F\ (r\ j) - F\ (l\ j))$
using *psubset(1) psubset.prems(1) j*
apply (*subst sum.union_disjoint*)
apply *simp_all*
apply (*subst sum.union_disjoint*)
apply *auto*
apply (*metis less_le_not_le*)
done
also (*xtrans*) **have** $(\sum i{\in}?S1.\ F\ (r\ i) - F\ (l\ i)) \ge F\ (l\ j) - F\ u$
using ‹$j \in S$› ‹*finite S*› *psubset.prems j*
apply (*intro psubset.IH psubset*)
apply (*auto simp*: *subset_eq Ball_def*)
apply (*metis less_le_trans not_le*)
done
also (*xtrans*) **have** $(\sum i{\in}?S2.\ F\ (r\ i) - F\ (l\ i)) \ge F\ v - F\ (r\ j)$
using ‹$j \in S$› ‹*finite S*› *psubset.prems j*
apply (*intro psubset.IH psubset*)
apply (*auto simp*: *subset_eq Ball_def*)
apply (*metis le_less_trans not_le*)
done
finally (*xtrans*) **show** *?case*
by (*auto simp*: *add_mono*)
qed
qed
qed }
note *claim2* = *this*

have *ennreal* $(F\ b - F\ a) \le (\sum i.\ ennreal\ (F\ (r\ i) - F\ (l\ i)))$
proof (*rule ennreal_le_epsilon*)
fix *epsilon* :: *real* **assume** *egt0*: $epsilon > 0$
have $\forall i.\ \exists d{>}0.\ F\ (r\ i + d) < F\ (r\ i) + epsilon\ /\ 2\hat{}(i{+}2)$
proof

```
    fix i
    note right_cont_F [of r i]
    thus ∃ d>0. F (r i + d) < F (r i) + epsilon / 2^(i+2)
      apply -
      apply (subst (asm) continuous_at_right_real_increasing)
      apply (rule mono_F, assumption)
      apply (drule_tac x = epsilon / 2 ^ (i + 2) in spec)
      apply (erule impE)
      using egt0 by (auto simp add: field_simps)
  qed
  then obtain delta where
      deltai_gt0: ⋀i. delta i > 0 and
      deltai_prop: ⋀i. F (r i + delta i) < F (r i) + epsilon / 2^(i+2)
    by metis
  have ∃ a' > a. F a' - F a < epsilon / 2
    apply (insert right_cont_F [of a])
    apply (subst (asm) continuous_at_right_real_increasing)
    using mono_F apply force
    apply (drule_tac x = epsilon / 2 in spec)
    using egt0 unfolding mult.commute [of 2] by force
  then obtain a' where a'lea [arith]: a' > a and
    a_prop: F a' - F a < epsilon / 2
    by auto
  define S' where S' = {i. l i < r i}
  obtain S :: nat set where
    S ⊆ S' and finS: finite S and
    Sprop: {a'..b} ⊆ (⋃ i ∈ S. {l i<..<r i + delta i})
  proof (rule compactE_image)
    show compact {a'..b}
      by (rule compact_Icc)
    show ⋀i. i ∈ S' ⟹ open ({l i<..<r i + delta i}) by auto
    have {a'..b} ⊆ {a <.. b}
      by auto
    also have {a <.. b} = (⋃ i∈S'. {l i<..r i})
    unfolding lr_eq_ab[symmetric] by (fastforce simp add: S'_def intro: less_le_trans)
    also have ... ⊆ (⋃ i ∈ S'. {l i<..<r i + delta i})
      apply (intro UN_mono)
      apply (auto simp: S'_def)
      apply (cut_tac i=i in deltai_gt0)
      apply simp
      done
    finally show {a'..b} ⊆ (⋃ i ∈ S'. {l i<..<r i + delta i}) .
  qed
  with S'_def have Sprop2: ⋀i. i ∈ S ⟹ l i < r i by auto
  from finS have ∃ n. ∀ i ∈ S. i ≤ n
    by (subst finite_nat_set_iff_bounded_le [symmetric])
  then obtain n where Sbound [rule_format]: ∀ i ∈ S. i ≤ n ..
  have F b - F a' ≤ (∑ i∈S. F (r i + delta i) - F (l i))
    apply (rule claim2 [rule_format])
```

using *finS Sprop* **apply** *auto*
apply (*frule Sprop2*)
apply (*subgoal_tac delta i > 0*)
apply *arith*
by (*rule deltai_gt0*)
also have ... $\leq$ ($\sum$ *i* $\in$ *S*. *F*(*r i*) $-$ *F*(*l i*) + *epsilon* / *2*ˆ(*i*+*2*))
apply (*rule sum_mono*)
apply *simp*
apply (*rule order_trans*)
apply (*rule less_imp_le*)
apply (*rule deltai_prop*)
by *auto*
also have ... = ($\sum$ *i* $\in$ *S*. *F*(*r i*) $-$ *F*(*l i*)) +
(*epsilon* / *4*) $*$ ($\sum$ *i* $\in$ *S*. (*1* / *2*)ˆ*i*) (**is** _ = *?t* + _)
by (*subst sum.distrib*) (*simp add*: *field_simps sum_distrib_left*)
also have ... $\leq$ *?t* + (*epsilon* / *4*) $*$ ($\sum$ *i* < *Suc n*. (*1* / *2*)ˆ*i*)
apply (*rule add_left_mono*)
apply (*rule mult_left_mono*)
apply (*rule sum_mono2*)
using *egt0* **apply** *auto*
by (*frule Sbound*, *auto*)
also have ... $\leq$ *?t* + (*epsilon* / *2*)
apply (*rule add_left_mono*)
apply (*subst geometric_sum*)
apply *auto*
apply (*rule mult_left_mono*)
using *egt0* **apply** *auto*
done
finally have *aux2*: *F b* $-$ *F a*′ $\leq$ ($\sum$ *i*$\in$*S*. *F* (*r i*) $-$ *F* (*l i*)) + *epsilon* / *2*
by *simp*

have *F b* $-$ *F a* = (*F b* $-$ *F a*′) + (*F a*′ $-$ *F a*)
by *auto*
also have ... $\leq$ (*F b* $-$ *F a*′) + *epsilon* / *2*
using *a_prop* **by** (*intro add_left_mono*) *simp*
also have ... $\leq$ ($\sum$ *i*$\in$*S*. *F* (*r i*) $-$ *F* (*l i*)) + *epsilon* / *2* + *epsilon* / *2*
apply (*intro add_right_mono*)
apply (*rule aux2*)
done
also have ... = ($\sum$ *i*$\in$*S*. *F* (*r i*) $-$ *F* (*l i*)) + *epsilon*
by *auto*
also have ... $\leq$ ($\sum$ *i*$\leq$*n*. *F* (*r i*) $-$ *F* (*l i*)) + *epsilon*
using *finS Sbound Sprop* **by** (*auto intro*!: *add_right_mono sum_mono2*)
finally have *ennreal* (*F b* $-$ *F a*) $\leq$ ($\sum$ *i*$\leq$*n*. *ennreal* (*F* (*r i*) $-$ *F* (*l i*))) + *epsilon*
using *egt0* **by** (*simp add*: *sum_nonneg flip*: *ennreal_plus*)
then show *ennreal* (*F b* $-$ *F a*) $\leq$ ($\sum$ *i*. *ennreal* (*F* (*r i*) $-$ *F* (*l i*))) + (*epsilon* :: *real*)
by (*rule order_trans*) (*auto intro*!: *add_mono sum_le_suminf simp del*: *sum_ennreal*)

qed
moreover have $(\sum i.\ ennreal\ (F\ (r\ i) - F\ (l\ i))) \leq ennreal\ (F\ b - F\ a)$
using ‹$a \leq b$› **by** (*auto intro*!: *suminf_le_const ennreal_le_iff*[*THEN iffD2*] *claim1*)
ultimately show $(\sum n.\ ennreal\ (F\ (r\ n) - F\ (l\ n))) = ennreal\ (F\ b - F\ a)$
by (*rule antisym*[*rotated*])
qed (*auto simp*: *Ioc_inj mono_F*)

lemma *measure_interval_measure_Ioc*:
assumes $a \leq b$ **and** $\bigwedge x\ y.\ x \leq y \Longrightarrow F\ x \leq F\ y$ **and** $\bigwedge a.$ *continuous* (*at_right a*) *F*
shows *measure* (*interval_measure F*) $\{a <..\ b\} = F\ b - F\ a$
unfolding *measure_def*
by (*simp add*: *assms emeasure_interval_measure_Ioc*)

lemma *emeasure_interval_measure_Ioc_eq*:
$(\bigwedge x\ y.\ x \leq y \Longrightarrow F\ x \leq F\ y) \Longrightarrow (\bigwedge a.$ *continuous* (*at_right a*) $F) \Longrightarrow$
emeasure (*interval_measure F*) $\{a <..\ b\} = ($*if* $a \leq b$ *then* $F\ b - F\ a$ *else 0*)
using *emeasure_interval_measure_Ioc*[*of a b F*] **by** *auto*

lemma *sets_interval_measure* [*simp*, *measurable_cong*]:
sets (*interval_measure F*) = *sets borel*
apply (*simp add*: *sets_extend_measure interval_measure_def borel_sigma_sets_Ioc*)
apply (*rule sigma_sets_eqI*)
apply *auto*
apply (*case_tac* $a \leq ba$)
apply (*auto intro*: *sigma_sets.Empty*)
done

lemma *space_interval_measure* [*simp*]: *space* (*interval_measure F*) = *UNIV*
by (*simp add*: *interval_measure_def space_extend_measure*)

lemma *emeasure_interval_measure_Icc*:
assumes $a \leq b$
assumes *mono_F*: $\bigwedge x\ y.\ x \leq y \Longrightarrow F\ x \leq F\ y$
assumes *cont_F* : *continuous_on UNIV F*
shows *emeasure* (*interval_measure F*) $\{a ..\ b\} = F\ b - F\ a$
proof (*rule tendsto_unique*)
{ fix *a b* :: *real* **assume** $a \leq b$ **then have** *emeasure* (*interval_measure F*) $\{a <..\ b\} = F\ b - F\ a$
using *cont_F*
by (*subst emeasure_interval_measure_Ioc*)
(*auto intro*: *mono_F continuous_within_subset simp*: *continuous_on_eq_continuous_within*)
}
note ∗ = *this*

let *?F* = *interval_measure F*
show $((\lambda a.\ F\ b - F\ a) \longrightarrow$ *emeasure ?F* $\{a..b\})$ (*at_left a*)
proof (*rule tendsto_at_left_sequentially*)

show $a - 1 < a$ **by** *simp*
fix X **assume** $\bigwedge n.\ X\ n < a$ *incseq X X* $\longrightarrow a$
with ‹$a \le b$› **have** $(\lambda n.$ *emeasure ?F* $\{X\ n{<}..b\}) \longrightarrow$ *emeasure ?F* $(\bigcap n.$ $\{X\ n <..b\})$
apply (*intro Lim_emeasure_decseq*)
apply (*auto simp*: *decseq_def incseq_def emeasure_interval_measure_Ioc* ∗)
apply *force*
apply (*subst* (*asm*) ∗)
apply (*auto intro*: *less_le_trans less_imp_le*)
done
also have $(\bigcap n.\ \{X\ n <..b\}) = \{a..b\}$
using ‹$\bigwedge n.\ X\ n < a$›
apply *auto*
apply (*rule LIMSEQ_le_const2*[*OF* ‹$X \longrightarrow a$›])
apply (*auto intro*: *less_imp_le*)
apply (*auto intro*: *less_le_trans*)
done
also have $(\lambda n.$ *emeasure ?F* $\{X\ n{<}..b\}) = (\lambda n.\ F\ b - F\ (X\ n))$
using ‹$\bigwedge n.\ X\ n < a$› ‹$a \le b$› **by** (*subst* ∗) (*auto intro*: *less_imp_le less_le_trans*)
finally show $(\lambda n.\ F\ b - F\ (X\ n)) \longrightarrow$ *emeasure ?F* $\{a..b\}$ **.**
qed
show $((\lambda a.$ *ennreal* $(F\ b - F\ a)) \longrightarrow F\ b - F\ a)$ (*at_left a*)
by (*rule continuous_on_tendsto_compose*[**where** $g=\lambda x.\ x$ **and** $s=UNIV$])
(*auto simp*: *continuous_on_ennreal continuous_on_diff cont_F*)
qed (*rule trivial_limit_at_left_real*)

lemma *sigma_finite_interval_measure*:
assumes *mono_F*: $\bigwedge x\ y.\ x \le y \Longrightarrow F\ x \le F\ y$
assumes *right_cont_F* : $\bigwedge a.$ *continuous* (*at_right a*) F
shows *sigma_finite_measure* (*interval_measure F*)
apply *unfold_locales*
apply (*intro exI*[*of* _ $(\lambda(a, b).\ \{a <..\ b\})$ ‘ $(\mathbb{Q} \times \mathbb{Q})$])
apply (*auto intro*!: *Rats_no_top_le Rats_no_bot_less countable_rat simp*: *emeasure_interval_measure_Ioc_eq*[*OF assms*])
done

6.13.2 Lebesgue-Borel measure

definition *lborel* :: ('*a* :: *euclidean_space*) *measure* **where**
lborel = *distr* $(\Pi_M\ b{\in}Basis.$ *interval_measure* $(\lambda x.\ x))$ *borel* $(\lambda f.\ \sum b{\in}Basis.\ f\ b *_R b)$

abbreviation *lebesgue* :: '*a*::*euclidean_space measure*
where *lebesgue* ≡ *completion lborel*

abbreviation *lebesgue_on* :: '*a set* ⇒ '*a*::*euclidean_space measure*
where *lebesgue_on* Ω ≡ *restrict_space* (*completion lborel*) Ω

lemma *lebesgue_on_mono*:

assumes *major*: *AE x in lebesgue_on S*. *P x* **and** *minor*: $\bigwedge x.[\![P\ x;\ x \in S]\!] \Longrightarrow$ *Q x*
shows *AE x in lebesgue_on S*. *Q x*
proof –
have *AE a in lebesgue_on S*. *P a* ⟶ *Q a*
using *minor space_restrict_space* **by** *fastforce*
then show *?thesis*
using *major* **by** *auto*
qed

lemma *integral_eq_zero_null_sets*:
assumes *S* ∈ *null_sets lebesgue*
shows $integral^L$ (*lebesgue_on S*) *f* = *0*
proof (*rule integral_eq_zero_AE*)
show *AE x in lebesgue_on S*. *f x* = *0*
by (*metis* (*no_types*, *lifting*) *assms AE_not_in lebesgue_on_mono null_setsD2 null_sets_restrict_space order_refl*)
qed

lemma
shows *sets_lborel*[*simp*, *measurable_cong*]: *sets lborel* = *sets borel*
and *space_lborel*[*simp*]: *space lborel* = *space borel*
and *measurable_lborel1*[*simp*]: *measurable M lborel* = *measurable M borel*
and *measurable_lborel2*[*simp*]: *measurable lborel M* = *measurable borel M*
by (*simp_all add*: *lborel_def*)

lemma *space_lebesgue_on* [*simp*]: *space* (*lebesgue_on S*) = *S*
by (*simp add*: *space_restrict_space*)

lemma *sets_lebesgue_on_refl* [*iff*]: *S* ∈ *sets* (*lebesgue_on S*)
by (*metis inf_top.right_neutral sets.top space_borel space_completion space_lborel space_restrict_space*)

lemma *Compl_in_sets_lebesgue*: −*A* ∈ *sets lebesgue* ⟷ *A* ∈ *sets lebesgue*
by (*metis Compl_eq_Diff_UNIV double_compl space_borel space_completion space_lborel Sigma_Algebra.sets.compl_sets*)

lemma *measurable_lebesgue_cong*:
assumes $\bigwedge x.\ x \in S \Longrightarrow f\ x = g\ x$
shows *f* ∈ *measurable* (*lebesgue_on S*) *M* ⟷ *g* ∈ *measurable* (*lebesgue_on S*) *M*
by (*metis* (*mono_tags*, *lifting*) *IntD1 assms measurable_cong_simp space_restrict_space*)

lemma *lebesgue_on_UNIV_eq*: *lebesgue_on UNIV* = *lebesgue*
proof –
have *measure_of UNIV* (*sets lebesgue*) (*emeasure lebesgue*) = *lebesgue*
by (*metis measure_of_of_measure space_borel space_completion space_lborel*)
then show *?thesis*
by (*auto simp*: *restrict_space_def*)

qed

lemma *integral_restrict_Int*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *S* ∈ *sets lebesgue T* ∈ *sets lebesgue*
shows *integral^L* (*lebesgue_on T*) (*λx. if x* ∈ *S then f x else 0*) = *integral^L* (*lebesgue_on* (*S* ∩ *T*)) *f*
proof −
have (*λx. indicat_real T x* $*_R$ (*if x* ∈ *S then f x else 0*)) = (*λx. indicat_real* (*S* ∩ *T*) *x* $*_R$ *f x*)
by (*force simp*: *indicator_def*)
then show *?thesis*
by (*simp add*: *assms sets.Int Bochner_Integration.integral_restrict_space*)
qed

lemma *integral_restrict*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *S* ⊆ *T S* ∈ *sets lebesgue T* ∈ *sets lebesgue*
shows *integral^L* (*lebesgue_on T*) (*λx. if x* ∈ *S then f x else 0*) = *integral^L* (*lebesgue_on S*) *f*
using *integral_restrict_Int* [*of S T f*] *assms*
by (*simp add*: *Int_absorb2*)

lemma *integral_restrict_UNIV*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *S* ∈ *sets lebesgue*
shows *integral^L lebesgue* (*λx. if x* ∈ *S then f x else 0*) = *integral^L* (*lebesgue_on S*) *f*
using *integral_restrict_Int* [*of S UNIV f*] *assms*
by (*simp add*: *lebesgue_on_UNIV_eq*)

lemma *integrable_lebesgue_on_empty* [*iff*]:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::*{*second_countable_topology*,*banach*}
shows *integrable* (*lebesgue_on* {}) *f*
by (*simp add*: *integrable_restrict_space*)

lemma *integral_lebesgue_on_empty* [*simp*]:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::*{*second_countable_topology*,*banach*}
shows *integral^L* (*lebesgue_on* {}) *f* = *0*
by (*simp add*: *Bochner_Integration.integral_empty*)
lemma *has_bochner_integral_restrict_space*:
fixes *f* :: *'a* ⇒ *'b::*{*banach*, *second_countable_topology*}
assumes Ω: Ω ∩ *space M* ∈ *sets M*
shows *has_bochner_integral* (*restrict_space M* Ω) *f i*
⟷ *has_bochner_integral M* (*λx. indicator* Ω *x* $*_R$ *f x*) *i*
by (*simp add*: *integrable_restrict_space* [*OF assms*] *integral_restrict_space* [*OF assms*] *has_bochner_integral_iff*)

lemma *integrable_restrict_UNIV*:

fixes f :: $'a$::*euclidean_space* ⇒ $'b$::{*banach*, *second_countable_topology*}
assumes S: $S \in$ *sets lebesgue*
shows *integrable lebesgue* (λx. *if* $x \in S$ *then f x else 0*) ⟷ *integrable* (*lebesgue_on S*) *f*
using *has_bochner_integral_restrict_space* [*of S lebesgue f*] *assms*
by (*simp add*: *integrable.simps indicator_scaleR_eq_if*)

lemma *integral_mono_lebesgue_on_AE*:
fixes *f*::_ ⇒ *real*
assumes *f*: *integrable* (*lebesgue_on T*) *f*
and *gf*: *AE x in* (*lebesgue_on S*). $g\ x \le f\ x$
and *f0*: *AE x in* (*lebesgue_on T*). $0 \le f\ x$
and $S \subseteq T$ **and** S: $S \in$ *sets lebesgue* **and** T: $T \in$ *sets lebesgue*
shows (∫ *x*. *g x* ∂(*lebesgue_on S*)) ≤ (∫ *x*. *f x* ∂(*lebesgue_on T*))
proof –
have (∫ *x*. *g x* ∂(*lebesgue_on S*)) = (∫ *x*. (*if* $x \in S$ *then g x else 0*) ∂*lebesgue*)
by (*simp add*: *Lebesgue_Measure.integral_restrict_UNIV S*)
also have ... ≤ (∫ *x*. (*if* $x \in T$ *then f x else 0*) ∂*lebesgue*)
proof (*rule Bochner_Integration.integral_mono_AE'*)
show *integrable lebesgue* (λx. *if* $x \in T$ *then f x else 0*)
by (*simp add*: *integrable_restrict_UNIV T f*)
show *AE x in lebesgue*. (*if* $x \in S$ *then g x else 0*) ≤ (*if* $x \in T$ *then f x else 0*)
using *assms* **by** (*auto simp*: *AE_restrict_space_iff*)
show *AE x in lebesgue*. 0 ≤ (*if* $x \in T$ *then f x else 0*)
using *f0* **by** (*simp add*: *AE_restrict_space_iff T*)
qed
also have ... = (∫ *x*. *f x* ∂(*lebesgue_on T*))
using *Lebesgue_Measure.integral_restrict_UNIV T* **by** *blast*
finally show *?thesis* **.**
qed

6.13.3 Borel measurability

lemma *borel_measurable_if_I*:
fixes f :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
assumes *f*: $f \in$ *borel_measurable* (*lebesgue_on S*) **and** S: $S \in$ *sets lebesgue*
shows (λx. *if* $x \in S$ *then f x else 0*) ∈ *borel_measurable lebesgue*
proof –
have *eq*: {x. x ∉ S} ∪ {x. f x ∈ Y} = {x. x ∉ S} ∪ {x. f x ∈ Y} ∩ S **for** *Y*
by *blast*
show *?thesis*
using *f S*
apply (*simp add*: *vimage_def in_borel_measurable_borel Ball_def*)
apply (*elim all_forward imp_forward asm_rl*)
apply (*simp only*: *Collect_conj_eq Collect_disj_eq imp_conv_disj eq*)
apply (*auto simp*: *Compl_eq* [*symmetric*] *Compl_in_sets_lebesgue sets_restrict_space_iff*)
done
qed

lemma *borel_measurable_if_D*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes (*λx. if x* ∈ *S then f x else 0*) ∈ *borel_measurable lebesgue*
shows *f* ∈ *borel_measurable* (*lebesgue_on S*)
using *assms*
apply (*simp add*: *in_borel_measurable_borel Ball_def*)
apply (*elim all_forward imp_forward asm_rl*)
apply (*force simp*: *space_restrict_space sets_restrict_space image_iff intro*: *rev_bexI*)
done

lemma *borel_measurable_if*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *S* ∈ *sets lebesgue*
shows (*λx. if x* ∈ *S then f x else 0*) ∈ *borel_measurable lebesgue* ⟷ *f* ∈ *borel_measurable* (*lebesgue_on S*)
using *assms borel_measurable_if_D borel_measurable_if_I* **by** *blast*

lemma *borel_measurable_if_lebesgue_on*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *S* ∈ *sets lebesgue T* ∈ *sets lebesgue S* ⊆ *T*
shows (*λx. if x* ∈ *S then f x else 0*) ∈ *borel_measurable* (*lebesgue_on T*) ⟷ *f* ∈ *borel_measurable* (*lebesgue_on S*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
using *measurable_restrict_mono* [*OF* _ ‹*S* ⊆ *T*›]
by (*subst measurable_lebesgue_cong* [**where** *g* = (*λx. if x* ∈ *S then f x else 0*)]) *auto*
next
assume *?rhs* **then show** *?lhs*
by (*simp add*: ‹*S* ∈ *sets lebesgue*› *borel_measurable_if_I measurable_restrict_space1*)
qed

lemma *borel_measurable_vimage_halfspace_component_lt*:
f ∈ *borel_measurable* (*lebesgue_on S*) ⟷
(∀ *a i. i* ∈ *Basis* ⟶ {*x* ∈ *S. f x* · *i* < *a*} ∈ *sets* (*lebesgue_on S*))
apply (*rule trans* [*OF borel_measurable_iff_halfspace_less*])
apply (*fastforce simp add*: *space_restrict_space*)
done

lemma *borel_measurable_vimage_halfspace_component_ge*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
shows *f* ∈ *borel_measurable* (*lebesgue_on S*) ⟷
(∀ *a i. i* ∈ *Basis* ⟶ {*x* ∈ *S. f x* · *i* ≥ *a*} ∈ *sets* (*lebesgue_on S*))
apply (*rule trans* [*OF borel_measurable_iff_halfspace_ge*])
apply (*fastforce simp add*: *space_restrict_space*)
done

lemma *borel_measurable_vimage_halfspace_component_gt*:

fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
shows *f* ∈ *borel_measurable* (*lebesgue_on S*) ⟷
(∀ *a i*. *i* ∈ *Basis* ⟶ {*x* ∈ *S*. *f x* · *i* > *a*} ∈ *sets* (*lebesgue_on S*))
apply (*rule trans* [*OF borel_measurable_iff_halfspace_greater*])
apply (*fastforce simp add*: *space_restrict_space*)
done

lemma *borel_measurable_vimage_halfspace_component_le*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
shows *f* ∈ *borel_measurable* (*lebesgue_on S*) ⟷
(∀ *a i*. *i* ∈ *Basis* ⟶ {*x* ∈ *S*. *f x* · *i* ≤ *a*} ∈ *sets* (*lebesgue_on S*))
apply (*rule trans* [*OF borel_measurable_iff_halfspace_le*])
apply (*fastforce simp add*: *space_restrict_space*)
done

lemma
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
shows *borel_measurable_vimage_open_interval*:
f ∈ *borel_measurable* (*lebesgue_on S*) ⟷
(∀ *a b*. {*x* ∈ *S*. *f x* ∈ *box a b*} ∈ *sets* (*lebesgue_on S*)) (**is** *?thesis1*)
and *borel_measurable_vimage_open*:
f ∈ *borel_measurable* (*lebesgue_on S*) ⟷
(∀ *T*. *open T* ⟶ {*x* ∈ *S*. *f x* ∈ *T*} ∈ *sets* (*lebesgue_on S*)) (**is** *?thesis2*)
proof −
have {*x* ∈ *S*. *f x* ∈ *box a b*} ∈ *sets* (*lebesgue_on S*) **if** *f* ∈ *borel_measurable* (*lebesgue_on S*) **for** *a b*
proof −
have *S* = *S* ∩ *space lebesgue*
by *simp*
then have *S* ∩ (*f* −' *box a b*) ∈ *sets* (*lebesgue_on S*)
by (*metis* (*no_types*) *box_borel in_borel_measurable_borel inf_sup_aci*(*1*) *space_restrict_space that*)
then show *?thesis*
by (*simp add*: *Collect_conj_eq vimage_def*)
qed
moreover
have {*x* ∈ *S*. *f x* ∈ *T*} ∈ *sets* (*lebesgue_on S*)
if *T*: ⋀*a b*. {*x* ∈ *S*. *f x* ∈ *box a b*} ∈ *sets* (*lebesgue_on S*) *open T* **for** *T*
proof −
obtain 𝒟 **where** *countable* 𝒟 **and** 𝒟: ⋀*X*. *X* ∈ 𝒟 ⟹ ∃ *a b*. *X* = *box a b* ⋃𝒟 = *T*
using *open_countable_Union_open_box that* ‹*open T*› **by** *metis*
then have *eq*: {*x* ∈ *S*. *f x* ∈ *T*} = (⋃ *U* ∈ 𝒟. {*x* ∈ *S*. *f x* ∈ *U*})
by *blast*
have {*x* ∈ *S*. *f x* ∈ *U*} ∈ *sets* (*lebesgue_on S*) **if** *U* ∈ 𝒟 **for** *U*
using *that T* 𝒟 **by** *blast*
then show *?thesis*
by (*auto simp*: *eq intro*: *Sigma_Algebra.sets.countable_UN'* [*OF* ‹*countable* 𝒟›])

```
  qed
  moreover
  have eq: {x ∈ S. f x · i < a} = {x ∈ S. f x ∈ {y. y · i < a}} for i a
    by auto
  have f ∈ borel_measurable (lebesgue_on S)
    if ⋀T. open T ⟹ {x ∈ S. f x ∈ T} ∈ sets (lebesgue_on S)
   by (metis (no_types) eq borel_measurable_vimage_halfspace_component_lt open_halfspace_component_lt
that)
  ultimately show ?thesis1 ?thesis2
    by blast+
qed

lemma borel_measurable_vimage_closed:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  shows f ∈ borel_measurable (lebesgue_on S) ⟷
        (∀ T. closed T ⟶ {x ∈ S. f x ∈ T} ∈ sets (lebesgue_on S))
       (is ?lhs = ?rhs)
proof −
  have eq: {x ∈ S. f x ∈ T} = S − {x ∈ S. f x ∈ (− T)} for T
    by auto
  show ?thesis
    apply (simp add: borel_measurable_vimage_open, safe)
     apply (simp_all (no_asm) add: eq)
     apply (intro sets.Diff sets_lebesgue_on_refl, force simp: closed_open)
    apply (intro sets.Diff sets_lebesgue_on_refl, force simp: open_closed)
    done
qed

lemma borel_measurable_vimage_closed_interval:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  shows f ∈ borel_measurable (lebesgue_on S) ⟷
        (∀ a b. {x ∈ S. f x ∈ cbox a b} ∈ sets (lebesgue_on S))
       (is ?lhs = ?rhs)
proof
  assume ?lhs then show ?rhs
    using borel_measurable_vimage_closed by blast
next
  assume RHS: ?rhs
  have {x ∈ S. f x ∈ T} ∈ sets (lebesgue_on S) if open T for T
  proof −
    obtain 𝒟 where countable 𝒟 and 𝒟: 𝒟 ⊆ Pow T ⋀X. X ∈ 𝒟 ⟹ ∃ a b. X
= cbox a b ⋃𝒟 = T
      using open_countable_Union_open_cbox that ‹open T› by metis
    then have eq: {x ∈ S. f x ∈ T} = (⋃ U ∈ 𝒟. {x ∈ S. f x ∈ U})
      by blast
    have {x ∈ S. f x ∈ U} ∈ sets (lebesgue_on S) if U ∈ 𝒟 for U
      using that 𝒟 by (metis RHS)
    then show ?thesis
       by (auto simp: eq intro: Sigma_Algebra.sets.countable_UN' [OF ‹countable
```

```
𝒟›])
  qed
  then show ?lhs
    by (simp add: borel_measurable_vimage_open)
qed

lemma borel_measurable_vimage_borel:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  shows f ∈ borel_measurable (lebesgue_on S) ⟷
       (∀ T. T ∈ sets borel ⟶ {x ∈ S. f x ∈ T} ∈ sets (lebesgue_on S))
      (is ?lhs = ?rhs)
proof
  assume f: ?lhs
  then show ?rhs
    using measurable_sets [OF f]
      by (simp add: Collect_conj_eq inf_sup_aci(1) space_restrict_space vimage_def)
qed (simp add: borel_measurable_vimage_open_interval)

lemma lebesgue_measurable_vimage_borel:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes f ∈ borel_measurable lebesgue T ∈ sets borel
  shows {x. f x ∈ T} ∈ sets lebesgue
  using assms borel_measurable_vimage_borel [of f UNIV] by auto

lemma borel_measurable_lebesgue_preimage_borel:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  shows f ∈ borel_measurable lebesgue ⟷
       (∀ T. T ∈ sets borel ⟶ {x. f x ∈ T} ∈ sets lebesgue)
  apply (intro iffI allI impI lebesgue_measurable_vimage_borel)
   apply (auto simp: in_borel_measurable_borel vimage_def)
  done
```

6.13.4 Measurability of continuous functions

```
lemma continuous_imp_measurable_on_sets_lebesgue:
  assumes f: continuous_on S f and S: S ∈ sets lebesgue
  shows f ∈ borel_measurable (lebesgue_on S)
proof −
  have sets (restrict_space borel S) ⊆ sets (lebesgue_on S)
    by (simp add: mono_restrict_space subsetI)
  then show ?thesis
   by (simp add: borel_measurable_continuous_on_restrict [OF f] borel_measurable_subalgebra

                 space_restrict_space)
qed

lemma id_borel_measurable_lebesgue [iff]: id ∈ borel_measurable lebesgue
  by (simp add: measurable_completion)
```

lemma *id_borel_measurable_lebesgue_on* [*iff*]: *id* ∈ *borel_measurable* (*lebesgue_on S*)
by (*simp add*: *measurable_completion measurable_restrict_space1*)

context
begin

interpretation *sigma_finite_measure interval_measure* (λ*x*. *x*)
by (*rule sigma_finite_interval_measure*) *auto*
interpretation *finite_product_sigma_finite* λ_. *interval_measure* (λ*x*. *x*) *Basis*
proof qed *simp*

lemma *lborel_eq_real*: *lborel* = *interval_measure* (λ*x*. *x*)
unfolding *lborel_def Basis_real_def*
using *distr_id*[*of interval_measure* (λ*x*. *x*)]
by (*subst distr_component*[*symmetric*])
(*simp_all add*: *distr_distr comp_def del*: *distr_id cong*: *distr_cong*)

lemma *lborel_eq*: *lborel* = *distr* (Π_M *b*∈*Basis*. *lborel*) *borel* (λ*f*. ∑*b*∈*Basis*. *f b* $*_R$ *b*)
by (*subst lborel_def*) (*simp add*: *lborel_eq_real*)

lemma *nn_integral_lborel_prod*:
assumes [*measurable*]: ⋀*b*. *b* ∈ *Basis* ⟹ *f b* ∈ *borel_measurable borel*
assumes *nn*[*simp*]: ⋀*b x*. *b* ∈ *Basis* ⟹ *0* ≤ *f b x*
shows (∫ $^+$*x*. (∏*b*∈*Basis*. *f b* (*x* · *b*)) ∂*lborel*) = (∏*b*∈*Basis*. (∫ $^+$*x*. *f b x* ∂*lborel*))
by (*simp add*: *lborel_def nn_integral_distr product_nn_integral_prod*
product_nn_integral_singleton)

lemma *emeasure_lborel_Icc*[*simp*]:
fixes *l u* :: *real*
assumes [*simp*]: *l* ≤ *u*
shows *emeasure lborel* {*l* .. *u*} = *u* − *l*
proof −
have ((λ*f*. *f 1*) −‘ {*l*..*u*} ∩ *space* (Pi_M {*1*} (λ*b*. *interval_measure* (λ*x*. *x*)))) = {*1*::*real*} $\to_E$ {*l*..*u*}
by (*auto simp*: *space_PiM*)
then show *?thesis*
by (*simp add*: *lborel_def emeasure_distr emeasure_PiM emeasure_interval_measure_Icc*)
qed

lemma *emeasure_lborel_Icc_eq*: *emeasure lborel* {*l* .. *u*} = *ennreal* (*if l* ≤ *u then u* − *l else 0*)
by *simp*

lemma *emeasure_lborel_cbox*[*simp*]:
assumes [*simp*]: ⋀*b*. *b* ∈ *Basis* ⟹ *l* · *b* ≤ *u* · *b*
shows *emeasure lborel* (*cbox l u*) = (∏*b*∈*Basis*. (*u* − *l*) · *b*)
proof −

have ($\lambda x. \prod b{\in}Basis.\ indicator\ \{l{\bullet}b\ ..\ u{\bullet}b\}\ (x \cdot b) :: ennreal) = indicator\ (cbox\ l\ u)$
by (*auto simp*: *fun_eq_iff cbox_def split*: *split_indicator*)
then have *emeasure lborel* (*cbox l u*) = ($\int^+ x. (\prod b{\in}Basis.\ indicator\ \{l{\bullet}b\ ..\ u{\bullet}b\}\ (x \cdot b))\ \partial lborel$)
by *simp*
also have ... = ($\prod b{\in}Basis.\ (u - l) \cdot b$)
by (*subst nn_integral_lborel_prod*) (*simp_all add*: *prod_ennreal inner_diff_left*)
finally show *?thesis* **.**
qed

lemma *AE_lborel_singleton*: *AE x in lborel::'a::euclidean_space measure.* $x \neq c$
using *SOME_Basis AE_discrete_difference* [*of* {*c*} *lborel*] *emeasure_lborel_cbox* [*of c c*]
by (*auto simp add*: *power_0_left*)

lemma *emeasure_lborel_Ioo*[*simp*]:
assumes [*simp*]: $l \leq u$
shows *emeasure lborel* {*l* <..< *u*} = *ennreal* (*u* − *l*)
proof −
have *emeasure lborel* {*l* <..< *u*} = *emeasure lborel* {*l* .. *u*}
using *AE_lborel_singleton*[*of u*] *AE_lborel_singleton*[*of l*] **by** (*intro emeasure_eq_AE*) *auto*
then show *?thesis*
by *simp*
qed

lemma *emeasure_lborel_Ioc*[*simp*]:
assumes [*simp*]: $l \leq u$
shows *emeasure lborel* {*l* <.. *u*} = *ennreal* (*u* − *l*)
proof −
have *emeasure lborel* {*l* <.. *u*} = *emeasure lborel* {*l* .. *u*}
using *AE_lborel_singleton*[*of u*] *AE_lborel_singleton*[*of l*] **by** (*intro emeasure_eq_AE*) *auto*
then show *?thesis*
by *simp*
qed

lemma *emeasure_lborel_Ico*[*simp*]:
assumes [*simp*]: $l \leq u$
shows *emeasure lborel* {*l* ..< *u*} = *ennreal* (*u* − *l*)
proof −
have *emeasure lborel* {*l* ..< *u*} = *emeasure lborel* {*l* .. *u*}
using *AE_lborel_singleton*[*of u*] *AE_lborel_singleton*[*of l*] **by** (*intro emeasure_eq_AE*) *auto*
then show *?thesis*
by *simp*
qed

lemma *emeasure_lborel_box*[*simp*]:
assumes [*simp*]: $\bigwedge b.\ b \in Basis \Longrightarrow l \cdot b \leq u \cdot b$
shows *emeasure lborel* (*box l u*) = ($\prod b \in Basis.\ (u - l) \cdot b$)
proof −
have ($\lambda x.\ \prod b \in Basis.$ *indicator* {$l \cdot b$ <..< $u \cdot b$} ($x \cdot b$) :: *ennreal*) = *indicator* (*box l u*)
by (*auto simp*: *fun_eq_iff box_def split*: *split_indicator*)
then have *emeasure lborel* (*box l u*) = ($\int^+ x.$ ($\prod b \in Basis.$ *indicator* {$l \cdot b$ <..< $u \cdot b$} ($x \cdot b$)) ∂*lborel*)
by *simp*
also have ... = ($\prod b \in Basis.\ (u - l) \cdot b$)
by (*subst nn_integral_lborel_prod*) (*simp_all add*: *prod_ennreal inner_diff_left*)
finally show *?thesis* **.**
qed

lemma *emeasure_lborel_cbox_eq*:
emeasure lborel (*cbox l u*) = (*if* $\forall b \in Basis.\ l \cdot b \leq u \cdot b$ *then* $\prod b \in Basis.\ (u - l) \cdot b$ *else 0*)
using *box_eq_empty*(*2*)[*THEN iffD2*, *of u l*] **by** (*auto simp*: *not_le*)

lemma *emeasure_lborel_box_eq*:
emeasure lborel (*box l u*) = (*if* $\forall b \in Basis.\ l \cdot b \leq u \cdot b$ *then* $\prod b \in Basis.\ (u - l) \cdot b$ *else 0*)
using *box_eq_empty*(*1*)[*THEN iffD2*, *of u l*] **by** (*auto simp*: *not_le dest*!: *less_imp_le*) *force*

lemma *emeasure_lborel_singleton*[*simp*]: *emeasure lborel* {*x*} = *0*
using *emeasure_lborel_cbox*[*of x x*] *nonempty_Basis*
by (*auto simp del*: *emeasure_lborel_cbox nonempty_Basis*)

lemma *emeasure_lborel_cbox_finite*: *emeasure lborel* (*cbox a b*) < ∞
by (*auto simp*: *emeasure_lborel_cbox_eq*)

lemma *emeasure_lborel_box_finite*: *emeasure lborel* (*box a b*) < ∞
by (*auto simp*: *emeasure_lborel_box_eq*)

lemma *emeasure_lborel_ball_finite*: *emeasure lborel* (*ball c r*) < ∞
proof −
have *bounded* (*ball c r*) **by** *simp*
from *bounded_subset_cbox_symmetric*[*OF this*] **obtain** *a* **where** *a*: *ball c r* ⊆ *cbox* (−*a*) *a*
by *auto*
hence *emeasure lborel* (*ball c r*) ≤ *emeasure lborel* (*cbox* (−*a*) *a*)
by (*intro emeasure_mono*) *auto*
also have ... < ∞ **by** (*simp add*: *emeasure_lborel_cbox_eq*)
finally show *?thesis* **.**
qed

lemma *emeasure_lborel_cball_finite*: *emeasure lborel* (*cball c r*) < ∞

proof –
have *bounded* (*cball c r*) **by** *simp*
from *bounded_subset_cbox_symmetric*[*OF this*] **obtain** *a* **where** *a*: *cball c r* ⊆ *cbox* (−*a*) *a*
by *auto*
hence *emeasure lborel* (*cball c r*) ≤ *emeasure lborel* (*cbox* (−*a*) *a*)
by (*intro emeasure_mono*) *auto*
also have . . . < ∞ **by** (*simp add*: *emeasure_lborel_cbox_eq*)
finally show *?thesis* **.**
qed

lemma *fmeasurable_cbox* [*iff*]: *cbox a b* ∈ *fmeasurable lborel*
and *fmeasurable_box* [*iff*]: *box a b* ∈ *fmeasurable lborel*
by (*auto simp*: *fmeasurable_def emeasure_lborel_box_eq emeasure_lborel_cbox_eq*)

lemma
fixes *l u* :: *real*
assumes [*simp*]: *l* ≤ *u*
shows *measure_lborel_Icc*[*simp*]: *measure lborel* {*l* .. *u*} = *u* − *l*
and *measure_lborel_Ico*[*simp*]: *measure lborel* {*l* ..< *u*} = *u* − *l*
and *measure_lborel_Ioc*[*simp*]: *measure lborel* {*l* <.. *u*} = *u* − *l*
and *measure_lborel_Ioo*[*simp*]: *measure lborel* {*l* <..< *u*} = *u* − *l*
by (*simp_all add*: *measure_def*)

lemma
assumes [*simp*]: ⋀*b*. *b* ∈ *Basis* ⟹ *l* · *b* ≤ *u* · *b*
shows *measure_lborel_box*[*simp*]: *measure lborel* (*box l u*) = (∏ *b*∈*Basis*. (*u* − *l*) · *b*)
and *measure_lborel_cbox*[*simp*]: *measure lborel* (*cbox l u*) = (∏ *b*∈*Basis*. (*u* − *l*) · *b*)
by (*simp_all add*: *measure_def inner_diff_left prod_nonneg*)

lemma *measure_lborel_cbox_eq*:
measure lborel (*cbox l u*) = (*if* ∀ *b*∈*Basis*. *l* · *b* ≤ *u* · *b then* ∏ *b*∈*Basis*. (*u* − *l*) · *b else 0*)
using *box_eq_empty*(*2*)[*THEN iffD2*, *of u l*] **by** (*auto simp*: *not_le*)

lemma *measure_lborel_box_eq*:
measure lborel (*box l u*) = (*if* ∀ *b*∈*Basis*. *l* · *b* ≤ *u* · *b then* ∏ *b*∈*Basis*. (*u* − *l*) · *b else 0*)
using *box_eq_empty*(*1*)[*THEN iffD2*, *of u l*] **by** (*auto simp*: *not_le dest*!: *less_imp_le*) *force*

lemma *measure_lborel_singleton*[*simp*]: *measure lborel* {*x*} = *0*
by (*simp add*: *measure_def*)

lemma *sigma_finite_lborel*: *sigma_finite_measure lborel*
proof
show ∃ *A*::'*a set set*. *countable A* ∧ *A* ⊆ *sets lborel* ∧ ⋃ *A* = *space lborel* ∧

(∀ *a*∈*A*. *emeasure lborel a* ≠ ∞)
  by (*intro exI*[*of* _ *range* (λ*n*::*nat*. *box* (− *real n* $*_R$ *One*) (*real n* $*_R$ *One*))])
    (*auto simp*: *emeasure_lborel_cbox_eq UN_box_eq_UNIV*)
qed

end

lemma *emeasure_lborel_UNIV* [*simp*]: *emeasure lborel* (*UNIV*::′*a*::*euclidean_space set*) = ∞
proof −
 { **fix** *n*::*nat*
  let *?Ba* = *Basis* :: ′*a set*
  have *real n* ≤ (*2*::*real*) ˆ *card ?Ba* ∗ *real n*
   by (*simp add*: *mult_le_cancel_right1*)
  also
  have ... ≤ (*2*::*real*) ˆ *card ?Ba* ∗ *real* (*Suc n*) ˆ *card ?Ba*
   apply (*rule mult_left_mono*)
   apply (*metis DIM_positive One_nat_def less_eq_Suc_le less_imp_le of_nat_le_iff of_nat_power self_le_power zero_less_Suc*)
   apply (*simp*)
   done
  finally have *real n* ≤ (*2*::*real*) ˆ *card ?Ba* ∗ *real* (*Suc n*) ˆ *card ?Ba* **.**
 } **note** [*intro*!] = *this*
 show *?thesis*
  unfolding *UN_box_eq_UNIV*[*symmetric*]
  apply (*subst SUP_emeasure_incseq*[*symmetric*])
  apply (*auto simp*: *incseq_def subset_box inner_add_left*
   simp del: *Sup_eq_top_iff SUP_eq_top_iff*
   intro!: *ennreal_SUP_eq_top*)
  done
qed

lemma *emeasure_lborel_countable*:
 fixes *A* :: ′*a*::*euclidean_space set*
 assumes *countable A*
 shows *emeasure lborel A* = *0*
proof −
 have *A* ⊆ (⋃ *i*. {*from_nat_into A i*}) **using** *from_nat_into_surj assms* **by** *force*
 then have *emeasure lborel A* ≤ *emeasure lborel* (⋃ *i*. {*from_nat_into A i*})
  by (*intro emeasure_mono*) *auto*
 also have *emeasure lborel* (⋃ *i*. {*from_nat_into A i*}) = *0*
  by (*rule emeasure_UN_eq_0*) *auto*
 finally show *?thesis*
  by (*auto simp add*:)
qed

lemma *countable_imp_null_set_lborel*: *countable A* ⟹ *A* ∈ *null_sets lborel*
 by (*simp add*: *null_sets_def emeasure_lborel_countable sets.countable*)

```
lemma finite_imp_null_set_lborel: finite A ⟹ A ∈ null_sets lborel
  by (intro countable_imp_null_set_lborel countable_finite)

lemma insert_null_sets_iff [simp]: insert a N ∈ null_sets lebesgue ⟷ N ∈ null_sets
lebesgue
    (is ?lhs = ?rhs)
proof
  assume ?lhs then show ?rhs
    by (meson completion.complete2 subset_insertI)
next
  assume ?rhs then show ?lhs
  by (simp add: null_sets.insert_in_sets null_setsI)
qed

lemma insert_null_sets_lebesgue_on_iff [simp]:
  assumes a ∈ S S ∈ sets lebesgue
  shows insert a N ∈ null_sets (lebesgue_on S) ⟷ N ∈ null_sets (lebesgue_on S)

  by (simp add: assms null_sets_restrict_space)

lemma lborel_neq_count_space[simp]: lborel ≠ count_space (A::('a::ordered_euclidean_space)
set)
proof
  assume asm: lborel = count_space A
  have space lborel = UNIV by simp
  hence [simp]: A = UNIV by (subst (asm) asm) (simp only: space_count_space)
  have emeasure lborel {undefined::'a} = 1
      by (subst asm, subst emeasure_count_space_finite) auto
  moreover have emeasure lborel {undefined} ≠ 1 by simp
  ultimately show False by contradiction
qed

lemma mem_closed_if_AE_lebesgue_open:
  assumes open S closed C
  assumes AE x ∈ S in lebesgue. x ∈ C
  assumes x ∈ S
  shows x ∈ C
proof (rule ccontr)
  assume xC: x ∉ C
  with openE[of S − C] assms
  obtain e where e: 0 < e ball x e ⊆ S − C
    by blast
  then obtain a b where box: x ∈ box a b box a b ⊆ S − C
    by (metis rational_boxes order_trans)
  then have 0 < emeasure lebesgue (box a b)
    by (auto simp: emeasure_lborel_box_eq mem_box algebra_simps intro!: prod_pos)
  also have ... ≤ emeasure lebesgue (S − C)
    using assms box
    by (auto intro!: emeasure_mono)
```

also have ... = *0*
using *assms*
by (*auto simp*: *eventually_ae_filter completion.complete2 set_diff_eq null_setsD1*)
finally show *False* **by** *simp*
qed

lemma *mem_closed_if_AE_lebesgue*: *closed C* ⟹ (*AE x in lebesgue*. *x* ∈ *C*) ⟹ *x* ∈ *C*
using *mem_closed_if_AE_lebesgue_open*[*OF open_UNIV*] **by** *simp*

6.13.5 Affine transformation on the Lebesgue-Borel

lemma *lborel_eqI*:
fixes *M* :: *'a::euclidean_space measure*
assumes *emeasure_eq*: ⋀*l u*. (⋀*b*. *b* ∈ *Basis* ⟹ *l* · *b* ≤ *u* · *b*) ⟹ *emeasure M* (*box l u*) = (∏ *b*∈*Basis*. (*u* − *l*) · *b*)
assumes *sets_eq*: *sets M* = *sets borel*
shows *lborel* = *M*
proof (*rule measure_eqI_generator_eq*)
let *?E* = *range* (λ(*a*, *b*). *box a b*::*'a set*)
show *Int_stable ?E*
by (*auto simp*: *Int_stable_def box_Int_box*)

show *?E* ⊆ *Pow UNIV sets lborel* = *sigma_sets UNIV ?E sets M* = *sigma_sets UNIV ?E*
by (*simp_all add*: *borel_eq_box sets_eq*)

let *?A* = λ*n*::*nat*. *box* (− (*real n* $*_R$ *One*)) (*real n* $*_R$ *One*) :: *'a set*
show *range ?A* ⊆ *?E* (⋃ *i*. *?A i*) = *UNIV*
unfolding *UN_box_eq_UNIV* **by** *auto*

{ fix *i* **show** *emeasure lborel* (*?A i*) ≠ ∞ **by** *auto* **}**
{ fix *X* **assume** *X* ∈ *?E* **then show** *emeasure lborel X* = *emeasure M X*
apply (*auto simp*: *emeasure_eq emeasure_lborel_box_eq*)
apply (*subst box_eq_empty*(*1*)[*THEN iffD2*])
apply (*auto intro*: *less_imp_le simp*: *not_le*)
done }
qed

lemma *lborel_affine_euclidean*:
fixes *c* :: *'a::euclidean_space* ⇒ *real* **and** *t*
defines *T x* ≡ *t* + (∑ *j*∈*Basis*. (*c j* ∗ (*x* · *j*)) $*_R$ *j*)
assumes *c*: ⋀*j*. *j* ∈ *Basis* ⟹ *c j* ≠ *0*
shows *lborel* = *density* (*distr lborel borel T*) (λ_. (∏ *j*∈*Basis*. |*c j*|)) (**is** _ = *?D*)
proof (*rule lborel_eqI*)
let *?B* = *Basis* :: *'a set*
fix *l u* **assume** *le*: ⋀*b*. *b* ∈ *?B* ⟹ *l* · *b* ≤ *u* · *b*
have [*measurable*]: *T* ∈ *borel* $\to_M$ *borel*
by (*simp add*: *T_def*[*abs_def*])

have *eq*: $T -` box\ l\ u = box$
($\sum j{\in}Basis.\ (((if\ 0 < c\ j\ then\ l - t\ else\ u - t) \cdot j) / c\ j) *_R j)$
($\sum j{\in}Basis.\ (((if\ 0 < c\ j\ then\ u - t\ else\ l - t) \cdot j) / c\ j) *_R j)$
using *c* **by** (*auto simp*: *box_def T_def field_simps inner_simps divide_less_eq*)
with *le c* **show** *emeasure ?D* (*box l u*) = ($\prod b{\in}?B.\ (u - l) \cdot b$)
by (*auto simp*: *emeasure_density emeasure_distr nn_integral_multc emeasure_lborel_box_eq inner_simps*
field_split_simps ennreal_mult'[*symmetric*] *prod_nonneg prod.distrib*[*symmetric*]
intro!: *prod.cong*)
qed *simp*

lemma *lborel_affine*:
fixes *t* :: *'a::euclidean_space*
shows $c \neq 0 \Longrightarrow lborel = density\ (distr\ lborel\ borel\ (\lambda x.\ t + c *_R x))\ (\lambda\_. |c|\,\hat{}\,DIM('a))$
using *lborel_affine_euclidean*[**where** *c*=λ_::'*a*. *c* **and** *t*=*t*]
unfolding *scaleR_scaleR*[*symmetric*] *scaleR_sum_right*[*symmetric*] *euclidean_representation prod_constant* **by** *simp*

lemma *lborel_real_affine*:
$c \neq 0 \Longrightarrow lborel = density\ (distr\ lborel\ borel\ (\lambda x.\ t + c * x))\ (\lambda\_.\ ennreal\ (abs\ c))$
using *lborel_affine*[*of c t*] **by** *simp*

lemma *AE_borel_affine*:
fixes *P* :: *real* ⇒ *bool*
shows $c \neq 0 \Longrightarrow Measurable.pred\ borel\ P \Longrightarrow AE\ x\ in\ lborel.\ P\ x \Longrightarrow AE\ x\ in\ lborel.\ P\ (t + c * x)$
by (*subst lborel_real_affine*[**where** *t*=− *t* / *c* **and** *c*=*1* / *c*])
(*simp_all add*: *AE_density AE_distr_iff field_simps*)

lemma *nn_integral_real_affine*:
fixes *c* :: *real* **assumes** [*measurable*]: $f \in borel\_measurable\ borel$ **and** *c*: $c \neq 0$
shows $(\int^+ x.\ f\ x\ \partial lborel) = |c| * (\int^+ x.\ f\ (t + c * x)\ \partial lborel)$
by (*subst lborel_real_affine*[*OF c*, *of t*])
(*simp add*: *nn_integral_density nn_integral_distr nn_integral_cmult*)

lemma *lborel_integrable_real_affine*:
fixes *f* :: *real* ⇒ '*a* :: {*banach*, *second_countable_topology*}
assumes *f*: *integrable lborel f*
shows $c \neq 0 \Longrightarrow integrable\ lborel\ (\lambda x.\ f\ (t + c * x))$
using *f f*[*THEN borel_measurable_integrable*] **unfolding** *integrable_iff_bounded*
by (*subst* (*asm*) *nn_integral_real_affine*[**where** *c*=*c* **and** *t*=*t*]) (*auto simp*: *ennreal_mult_less_top*)

lemma *lborel_integrable_real_affine_iff*:
fixes *f* :: *real* ⇒ '*a* :: {*banach*, *second_countable_topology*}
shows $c \neq 0 \Longrightarrow integrable\ lborel\ (\lambda x.\ f\ (t + c * x)) \longleftrightarrow integrable\ lborel\ f$
using

lborel_integrable_real_affine[*of f c t*]
lborel_integrable_real_affine[*of* $\lambda x.$ *f* $(t + c * x)$ *1*/*c* $-t/c$]
by (*auto simp add*: *field_simps*)

lemma *lborel_integral_real_affine*:
fixes *f* :: *real* ⇒ *'a* :: {*banach*, *second_countable_topology*} **and** *c* :: *real*
assumes *c*: $c \neq 0$ **shows** ($\int x.$ *f x* ∂ *lborel*) = $|c| *_R$ ($\int x.$ *f* $(t + c * x)$ ∂*lborel*)
proof *cases*
assume *f*[*measurable*]: *integrable lborel f* **then show** *?thesis*
using *c f f*[*THEN borel_measurable_integrable*] *f*[*THEN lborel_integrable_real_affine*, *of c t*]
by (*subst lborel_real_affine*[*OF c*, *of t*])
(*simp add*: *integral_density integral_distr*)
next
assume ¬ *integrable lborel f* **with** *c* **show** *?thesis*
by (*simp add*: *lborel_integrable_real_affine_iff not_integrable_integral_eq*)
qed

lemma
fixes *c* :: *'a*::*euclidean_space* ⇒ *real* **and** *t*
assumes *c*: $\bigwedge j.\ j \in$ *Basis* $\Longrightarrow$ *c j* $\neq 0$
defines *T* == ($\lambda x.$ *t* + ($\sum j{\in}$*Basis*. (*c j* ∗ (*x* • *j*)) $*_R$ *j*))
shows *lebesgue_affine_euclidean*: *lebesgue* = *density* (*distr lebesgue lebesgue T*) (λ_. ($\prod j{\in}$*Basis*. |*c j*|)) (**is** _ = *?D*)
and *lebesgue_affine_measurable*: *T* ∈ *lebesgue* $\rightarrow_M$ *lebesgue*
proof −
have *T_borel*[*measurable*]: *T* ∈ *borel* $\rightarrow_M$ *borel*
by (*auto simp*: *T_def*[*abs_def*])
{ fix *A* :: *'a set* **assume** *A*: *A* ∈ *sets borel*
then have *emeasure lborel A* = *0* ⟷ *emeasure* (*density* (*distr lborel borel T*) (λ_. ($\prod j{\in}$*Basis*. |*c j*|))) *A* = *0*
unfolding *T_def* **using** *c* **by** (*subst lborel_affine_euclidean*[*symmetric*]) *auto*
also have … ⟷ *emeasure* (*distr lebesgue lborel T*) *A* = *0*
using *A c* **by** (*simp add*: *distr_completion emeasure_density nn_integral_cmult prod_nonneg cong*: *distr_cong*)
finally have *emeasure lborel A* = *0* ⟷ *emeasure* (*distr lebesgue lborel T*) *A* = *0* **. }**
then have *eq*: *null_sets lborel* = *null_sets* (*distr lebesgue lborel T*)
by (*auto simp*: *null_sets_def*)

show *T* ∈ *lebesgue* $\rightarrow_M$ *lebesgue*
by (*rule completion.measurable_completion2*) (*auto simp*: *eq measurable_completion*)

have *lebesgue* = *completion* (*density* (*distr lborel borel T*) (λ_. ($\prod j{\in}$*Basis*. |*c j*|)))
using *c* **by** (*subst lborel_affine_euclidean*[*of c t*]) (*simp_all add*: *T_def*[*abs_def*])
also have … = *density* (*completion* (*distr lebesgue lborel T*)) (λ_. ($\prod j{\in}$*Basis*. |*c j*|))
using *c* **by** (*auto intro*!: *always_eventually prod_pos completion_density_eq simp*:

distr_completion cong: *distr_cong*)
also have ... = *density* (*distr lebesgue lebesgue T*) (λ_. ($\prod$*j*$\in$*Basis*. |*c j*|))
by (*subst completion.completion_distr_eq*) (*auto simp*: *eq measurable_completion*)
finally show *lebesgue* = *density* (*distr lebesgue lebesgue T*) (λ_. ($\prod$*j*$\in$*Basis*. |*c j*|)) .
qed

corollary *lebesgue_real_affine*:
c $\neq$ *0* $\Longrightarrow$ *lebesgue* = *density* (*distr lebesgue lebesgue* (λ*x*. *t* + *c* $*$ *x*)) (λ_. *ennreal* (*abs c*))
using *lebesgue_affine_euclidean* [**where** *c*= λ*x*::*real*. *c*] **by** *simp*

lemma *nn_integral_real_affine_lebesgue*:
fixes *c* :: *real* **assumes** *f*[*measurable*]: *f* $\in$ *borel_measurable lebesgue* **and** *c*: *c* $\neq$ *0*
shows ($\int^+$*x*. *f x* ∂*lebesgue*) = *ennreal*|*c*| $*$ ($\int^+$*x*. *f*(*t* + *c* $*$ *x*) ∂*lebesgue*)
proof −
have ($\int^+$*x*. *f x* ∂*lebesgue*) = ($\int^+$*x*. *f x* ∂*density* (*distr lebesgue lebesgue* (λ*x*. *t* + *c* $*$ *x*)) (λ*x*. *ennreal* |*c*|))
using *lebesgue_real_affine c* **by** *auto*
also have ... = $\int^+$ *x*. *ennreal* |*c*| $*$ *f x* ∂*distr lebesgue lebesgue* (λ*x*. *t* + *c* $*$ *x*)
by (*subst nn_integral_density*) *auto*
also have ... = *ennreal* |*c*| $*$ *integral*^{N} (*distr lebesgue lebesgue* (λ*x*. *t* + *c* $*$ *x*)) *f*
using *f measurable_distr_eq1 nn_integral_cmult* **by** *blast*
also have ... = |*c*| $*$ ($\int^+$*x*. *f*(*t* + *c* $*$ *x*) ∂*lebesgue*)
using *lebesgue_affine_measurable*[**where** *c*= λ*x*::*real*. *c*]
by (*subst nn_integral_distr*) (*force*+)
finally show *?thesis* .
qed

lemma *lebesgue_measurable_scaling*[*measurable*]: ($*_R$) *x* $\in$ *lebesgue* $\rightarrow_M$ *lebesgue*
proof *cases*
assume *x* = *0*
then have ($*_R$) *x* = (λ*x*. *0*::'*a*)
by (*auto simp*: *fun_eq_iff*)
then show *?thesis* **by** *auto*
next
assume *x* $\neq$ *0* **then show** *?thesis*
using *lebesgue_affine_measurable*[*of* λ_. *x 0*]
unfolding *scaleR_scaleR*[*symmetric*] *scaleR_sum_right*[*symmetric*] *euclidean_representation*
by (*auto simp add*: *ac_simps*)
qed

lemma
fixes *m* :: *real* **and** δ :: '*a*::*euclidean_space*
defines *T r d x* $\equiv$ *r* $*_R$ *x* + *d*
shows *emeasure_lebesgue_affine*: *emeasure lebesgue* (*T m* δ ' *S*) = |*m*| ^ *DIM*('*a*) $*$ *emeasure lebesgue S* (**is** *?e*)

```
  and measure_lebesgue_affine: measure lebesgue (T m δ ` S) = |m| ^ DIM('a) *
measure lebesgue S (is ?m)
proof -
 show ?e
 proof cases
  assume m = 0 then show ?thesis
   by (simp add: image_constant_conv T_def[abs_def])
 next
  let ?T = T m δ and ?T' = T (1 / m) (- ((1/m) *R δ))
  assume m ≠ 0
  then have s_comp_s: ?T' ∘ ?T = id ?T ∘ ?T' = id
   by (auto simp: T_def[abs_def] fun_eq_iff scaleR_add_right scaleR_diff_right)
  then have inv ?T' = ?T bij ?T'
   by (auto intro: inv_unique_comp o_bij)
  then have eq: T m δ ` S = T (1 / m) ((-1/m) *R δ) -` S ∩ space lebesgue
   using bij_vimage_eq_inv_image[OF ‹bij ?T'›, of S] by auto

  have trans_eq_T: (λx. δ + (∑j∈Basis. (m * (x · j)) *R j)) = T m δ for m δ
   unfolding T_def[abs_def] scaleR_scaleR[symmetric] scaleR_sum_right[symmetric]
   by (auto simp add: euclidean_representation ac_simps)

  have T[measurable]: T r d ∈ lebesgue →M lebesgue for r d
   using lebesgue_affine_measurable[of λ_. r d]
   by (cases r = 0) (auto simp: trans_eq_T T_def[abs_def])

  show ?thesis
  proof cases
   assume S ∈ sets lebesgue with ‹m ≠ 0› show ?thesis
    unfolding eq
    apply (subst lebesgue_affine_euclidean[of λ_. m δ])
     apply (simp_all add: emeasure_density trans_eq_T nn_integral_cmult emea-
sure_distr
              del: space_completion emeasure_completion)
    apply (simp add: vimage_comp s_comp_s)
    done
  next
   assume S ∉ sets lebesgue
   moreover have ?T ` S ∉ sets lebesgue
   proof
    assume ?T ` S ∈ sets lebesgue
    then have ?T -` (?T ` S) ∩ space lebesgue ∈ sets lebesgue
     by (rule measurable_sets[OF T])
    also have ?T -` (?T ` S) ∩ space lebesgue = S
     by (simp add: vimage_comp s_comp_s eq)
    finally show False using ‹S ∉ sets lebesgue› by auto
   qed
   ultimately show ?thesis
    by (simp add: emeasure_notin_sets)
  qed
```

qed
show *?m*
unfolding *measure_def* ‹*?e*› **by** (*simp add*: *enn2real_mult prod_nonneg*)
qed

lemma *lebesgue_real_scale*:
assumes $c \neq 0$
shows *lebesgue* = *density* (*distr lebesgue lebesgue* ($\lambda x.\ c * x$)) ($\lambda x.$ *ennreal* $|c|$)
using *assms* **by** (*subst lebesgue_affine_euclidean*[*of* $\lambda\_.\ c\ 0$]) *simp_all*

lemma *divideR_right*:
fixes *x y* :: *'a*::*real_normed_vector*
shows $r \neq 0 \Longrightarrow y = x /_R r \longleftrightarrow r *_R y = x$
using *scaleR_cancel_left*[*of r y x* $/_R$ *r*] **by** *simp*

lemma *lborel_has_bochner_integral_real_affine_iff*:
fixes *x* :: *'a* :: {*banach*, *second_countable_topology*}
shows $c \neq 0 \Longrightarrow$
has_bochner_integral lborel f x ⟷
has_bochner_integral lborel ($\lambda x.\ f\ (t + c * x)$) ($x /_R |c|$)
unfolding *has_bochner_integral_iff lborel_integrable_real_affine_iff*
by (*simp_all add*: *lborel_integral_real_affine*[*symmetric*] *divideR_right cong*: *conj_cong*)

lemma *lborel_distr_uminus*: *distr lborel borel uminus* = (*lborel* :: *real measure*)
by (*subst lborel_real_affine*[*of* -1 *0*])
(*auto simp*: *density_1 one_ennreal_def*[*symmetric*])

lemma *lborel_distr_mult*:
assumes (*c*::*real*) $\neq 0$
shows *distr lborel borel* ((∗) *c*) = *density lborel* ($\lambda\_.$ *inverse* $|c|$)
proof−
have *distr lborel borel* ((∗) *c*) = *distr lborel lborel* ((∗) *c*) **by** (*simp cong*: *distr_cong*)
also from *assms* **have** ... = *density lborel* ($\lambda\_.$ *inverse* $|c|$)
by (*subst lborel_real_affine*[*of inverse c 0*]) (*auto simp*: *o_def distr_density_distr*)
finally show *?thesis* **.**
qed

lemma *lborel_distr_mult'*:
assumes (*c*::*real*) $\neq 0$
shows *lborel* = *density* (*distr lborel borel* ((∗) *c*)) ($\lambda\_.\ |c|$)
proof−
have *lborel* = *density lborel* ($\lambda\_.$ *1*) **by** (*rule density_1*[*symmetric*])
also from *assms* **have** ($\lambda\_.$ *1* :: *ennreal*) = ($\lambda\_.$ *inverse* $|c| * |c|$) **by** (*intro ext*) *simp*
also have *density lborel* ... = *density* (*density lborel* ($\lambda\_.$ *inverse* $|c|$)) ($\lambda\_.\ |c|$)
by (*subst density_density_eq*) (*auto simp*: *ennreal_mult*)
also from *assms* **have** *density lborel* ($\lambda\_.$ *inverse* $|c|$) = *distr lborel borel* ((∗) *c*)
by (*rule lborel_distr_mult*[*symmetric*])
finally show *?thesis* **.**

qed

lemma *lborel_distr_plus*:
  fixes *c* :: *'a::euclidean_space*
  shows *distr lborel borel* ((+) *c*) = *lborel*
by (*subst lborel_affine*[*of 1 c*], *auto simp*: *density_1*)

interpretation *lborel*: *sigma_finite_measure lborel*
  by (*rule sigma_finite_lborel*)

interpretation *lborel_pair*: *pair_sigma_finite lborel lborel* **..**

lemma *lborel_prod*:
  lborel $\bigotimes_M$ *lborel* = (*lborel* :: (*'a::euclidean_space* × *'b::euclidean_space*) *measure*)
proof (*rule lborel_eqI*[*symmetric*], *clarify*)
  fix *la ua* :: *'a* **and** *lb ub* :: *'b*
  assume *lu*: $\bigwedge$*a b*. (*a*, *b*) ∈ *Basis* ⟹ (*la*, *lb*) · (*a*, *b*) ≤ (*ua*, *ub*) · (*a*, *b*)
  have [*simp*]:
    $\bigwedge$*b*. *b* ∈ *Basis* ⟹ *la* · *b* ≤ *ua* · *b*
    $\bigwedge$*b*. *b* ∈ *Basis* ⟹ *lb* · *b* ≤ *ub* · *b*
    inj_on (λ*u*. (*u*, *0*)) *Basis inj_on* (λ*u*. (*0*, *u*)) *Basis*
    (λ*u*. (*u*, *0*)) ‘ *Basis* ∩ (λ*u*. (*0*, *u*)) ‘ *Basis* = {}
    box (*la*, *lb*) (*ua*, *ub*) = *box la ua* × *box lb ub*
   using *lu*[*of _ 0*] *lu*[*of 0*] **by** (*auto intro*!: *inj_onI simp add*: *Basis_prod_def ball_Un box_def*)
  show *emeasure* (*lborel* $\bigotimes_M$ *lborel*) (*box* (*la*, *lb*) (*ua*, *ub*)) =
      ennreal (*prod* ((·) ((*ua*, *ub*) − (*la*, *lb*))) *Basis*)
   by (*simp add*: *lborel.emeasure_pair_measure_Times Basis_prod_def prod.union_disjoint prod.reindex ennreal_mult inner_diff_left prod_nonneg*)
qed (*simp add*: *borel_prod*[*symmetric*])

lemma *lborelD_Collect*[*measurable* (*raw*)]: {*x*∈*space borel*. *P x*} ∈ *sets borel* ⟹ {*x*∈*space lborel*. *P x*} ∈ *sets lborel*
  by *simp*

lemma *lborelD*[*measurable* (*raw*)]: *A* ∈ *sets borel* ⟹ *A* ∈ *sets lborel*
  by *simp*

lemma *emeasure_bounded_finite*:
  assumes *bounded A* **shows** *emeasure lborel A* < ∞
proof −
  obtain *a b* **where** *A* ⊆ *cbox a b*
    by (*meson bounded_subset_cbox_symmetric* ‹*bounded A*›)
  then have *emeasure lborel A* ≤ *emeasure lborel* (*cbox a b*)
    by (*intro emeasure_mono*) *auto*
  then show *?thesis*
   by (*auto simp*: *emeasure_lborel_cbox_eq prod_nonneg less_top*[*symmetric*] *top_unique split*: *if_split_asm*)

qed

lemma *emeasure_compact_finite*: *compact A* $\Longrightarrow$ *emeasure lborel A* $< \infty$
  using *emeasure_bounded_finite*[*of A*] **by** (*auto intro*: *compact_imp_bounded*)

lemma *borel_integrable_compact*:
  fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::{*banach*, *second_countable_topology*}
  assumes *compact S continuous_on S f*
  shows *integrable lborel* (λ*x*. *indicator S x* $*_R$ *f x*)
proof *cases*
  assume *S* ≠ {}
  have *continuous_on S* (λ*x*. *norm* (*f x*))
    using *assms* **by** (*intro continuous_intros*)
  from *continuous_attains_sup*[*OF* ‹*compact S*› ‹*S* ≠ {}› *this*]
  obtain *M* **where** *M*: ⋀*x*. *x* ∈ *S* ⟹ *norm* (*f x*) ≤ *M*
    by *auto*
  show *?thesis*
  proof (*rule integrable_bound*)
    show *integrable lborel* (λ*x*. *indicator S x* ∗ *M*)
        using *assms* **by** (*auto intro*!: *emeasure_compact_finite borel_compact integrable_mult_left*)
    show (λ*x*. *indicator S x* $*_R$ *f x*) ∈ *borel_measurable lborel*
     using *assms* **by** (*auto intro*!: *borel_measurable_continuous_on_indicator borel_compact*)
    show *AE x in lborel*. *norm* (*indicator S x* $*_R$ *f x*) ≤ *norm* (*indicator S x* ∗ *M*)
      by (*auto split*: *split_indicator simp*: *abs_real_def dest*!: *M*)
  qed
qed *simp*

lemma *borel_integrable_atLeastAtMost*:
  fixes *f* :: *real* ⇒ *real*
  assumes *f*: ⋀*x*. *a* ≤ *x* ⟹ *x* ≤ *b* ⟹ *isCont f x*
  shows *integrable lborel* (λ*x*. *f x* ∗ *indicator* {*a* .. *b*} *x*) (**is** *integrable* _ *?f*)
proof −
  have *integrable lborel* (λ*x*. *indicator* {*a* .. *b*} *x* $*_R$ *f x*)
  proof (*rule borel_integrable_compact*)
    from *f* **show** *continuous_on* {*a*..*b*} *f*
      by (*auto intro*: *continuous_at_imp_continuous_on*)
  qed *simp*
  then show *?thesis*
    by (*auto simp*: *mult.commute*)
qed

6.13.6 Lebesgue measurable sets

abbreviation *lmeasurable* :: *'a*::*euclidean_space set set*
where
  lmeasurable ≡ *fmeasurable lebesgue*

lemma *not_measurable_UNIV* [*simp*]: *UNIV* ∉ *lmeasurable*

by (*simp add*: *fmeasurable_def*)

lemma *lmeasurable_iff_integrable*:
S ∈ *lmeasurable* ⟷ *integrable lebesgue* (*indicator S* :: *'a*::*euclidean_space* ⇒ *real*)
by (*auto simp*: *fmeasurable_def integrable_iff_bounded borel_measurable_indicator_iff ennreal_indicator*)

lemma *lmeasurable_cbox* [*iff*]: *cbox a b* ∈ *lmeasurable*
and *lmeasurable_box* [*iff*]: *box a b* ∈ *lmeasurable*
by (*auto simp*: *fmeasurable_def emeasure_lborel_box_eq emeasure_lborel_cbox_eq*)

lemma
fixes *a*::*real*
shows *lmeasurable_interval* [*iff*]: {*a*..*b*} ∈ *lmeasurable* {*a*<..<*b*} ∈ *lmeasurable*
apply (*metis box_real*(*2*) *lmeasurable_cbox*)
by (*metis box_real*(*1*) *lmeasurable_box*)

lemma *fmeasurable_compact*: *compact S* ⟹ *S* ∈ *fmeasurable lborel*
using *emeasure_compact_finite*[*of S*] **by** (*intro fmeasurableI*) (*auto simp*: *borel_compact*)

lemma *lmeasurable_compact*: *compact S* ⟹ *S* ∈ *lmeasurable*
using *fmeasurable_compact* **by** (*force simp*: *fmeasurable_def*)

lemma *measure_frontier*:
bounded S ⟹ *measure lebesgue* (*frontier S*) = *measure lebesgue* (*closure S*) − *measure lebesgue* (*interior S*)
using *closure_subset interior_subset*
by (*auto simp*: *frontier_def fmeasurable_compact intro*!: *measurable_measure_Diff*)

lemma *lmeasurable_closure*:
bounded S ⟹ *closure S* ∈ *lmeasurable*
by (*simp add*: *lmeasurable_compact*)

lemma *lmeasurable_frontier*:
bounded S ⟹ *frontier S* ∈ *lmeasurable*
by (*simp add*: *compact_frontier_bounded lmeasurable_compact*)

lemma *lmeasurable_open*: *bounded S* ⟹ *open S* ⟹ *S* ∈ *lmeasurable*
using *emeasure_bounded_finite*[*of S*] **by** (*intro fmeasurableI*) (*auto simp*: *borel_open*)

lemma *lmeasurable_ball* [*simp*]: *ball a r* ∈ *lmeasurable*
by (*simp add*: *lmeasurable_open*)

lemma *lmeasurable_cball* [*simp*]: *cball a r* ∈ *lmeasurable*
by (*simp add*: *lmeasurable_compact*)

lemma *lmeasurable_interior*: *bounded S* ⟹ *interior S* ∈ *lmeasurable*
by (*simp add*: *bounded_interior lmeasurable_open*)

lemma *null_sets_cbox_Diff_box*: *cbox a b* − *box a b* ∈ *null_sets lborel*
proof −
  have *emeasure lborel* (*cbox a b* − *box a b*) = *0*
   by (*subst emeasure_Diff*) (*auto simp*: *emeasure_lborel_cbox_eq emeasure_lborel_box_eq box_subset_cbox*)
  then have *cbox a b* − *box a b* ∈ *null_sets lborel*
    by (*auto simp*: *null_sets_def*)
  then show *?thesis*
    by (*auto dest*!: *AE_not_in*)
qed

lemma *bounded_set_imp_lmeasurable*:
  assumes *bounded S S* ∈ *sets lebesgue* **shows** *S* ∈ *lmeasurable*
  by (*metis assms bounded_Un emeasure_bounded_finite emeasure_completion fmeasurableI main_part_null_part_Un*)

lemma *finite_measure_lebesgue_on*: *S* ∈ *lmeasurable* ⟹ *finite_measure* (*lebesgue_on S*)
  by (*auto simp*: *finite_measureI fmeasurable_def emeasure_restrict_space*)

lemma *integrable_const_ivl* [*iff*]:
  fixes *a*::′*a*::*ordered_euclidean_space*
  shows *integrable* (*lebesgue_on* {*a*..*b*}) (λ*x*. *c*)
  by (*metis cbox_interval finite_measure.integrable_const finite_measure_lebesgue_on lmeasurable_cbox*)

6.13.7 Translation preserves Lebesgue measure

lemma *sigma_sets_image*:
  assumes *S*: *S* ∈ *sigma_sets* Ω *M* **and** *M* ⊆ *Pow* Ω *f* ‘ Ω = Ω *inj_on f* Ω
    and *M*: ⋀*y*. *y* ∈ *M* ⟹ *f* ‘ *y* ∈ *M*
  shows (*f* ‘ *S*) ∈ *sigma_sets* Ω *M*
  using *S*
proof (*induct S rule*: *sigma_sets.induct*)
  case (*Basic a*) **then show** *?case*
    by (*simp add*: *M*)
next
  case *Empty* **then show** *?case*
    by (*simp add*: *sigma_sets.Empty*)
next
  case (*Compl a*)
  then have Ω − *a* ⊆ Ω *a* ⊆ Ω
    by (*auto simp*: *sigma_sets_into_sp* [*OF* ‹*M* ⊆ *Pow* Ω›])
  then show *?case*
    by (*auto simp*: *inj_on_image_set_diff* [*OF* ‹*inj_on f* Ω›] *assms intro*: *Compl sigma_sets.Compl*)
next
  case (*Union a*) **then show** *?case*

by (*metis image_UN sigma_sets.simps*)
qed

lemma *null_sets_translation*:
assumes $N \in$ *null_sets lborel* **shows** $\{x.\ x - a \in N\} \in$ *null_sets lborel*
proof −
have [*simp*]: $(\lambda x.\ x + a)$ ‘ $N = \{x.\ x - a \in N\}$
by *force*
show *?thesis*
using *assms emeasure_lebesgue_affine* [*of 1 a N*] **by** (*auto simp*: *null_sets_def*)
qed

lemma *lebesgue_sets_translation*:
fixes f :: $'a \Rightarrow 'a$::*euclidean_space*
assumes S: $S \in$ *sets lebesgue*
shows $((\lambda x.\ a + x)$ ‘ $S) \in$ *sets lebesgue*
proof −
have *im_eq*: $(+)\ a$ ‘ $A = \{x.\ x - a \in A\}$ **for** A
by *force*
have $((\lambda x.\ a + x)$ ‘ $S) = ((\lambda x.\ -a + x)$ −‘ $S) \cap$ (*space lebesgue*)
using *image_iff* **by** *fastforce*
also have … $\in$ *sets lebesgue*
proof (*rule measurable_sets* [*OF measurableI assms*])
fix A :: $'b$ *set*
assume A: $A \in$ *sets lebesgue*
have *vim_eq*: $(\lambda x.\ x - a)$ −‘ $A = (+)\ a$ ‘ A **for** A
by *force*
have $\exists s\ n\ N'.\ (+)\ a$ ‘ $(S \cup N) = s \cup n \wedge s \in$ *sets borel* $\wedge N' \in$ *null_sets lborel* $\wedge n \subseteq N'$
if $S \in$ *sets borel* **and** $N' \in$ *null_sets lborel* **and** $N \subseteq N'$ **for** $S\ N\ N'$
proof (*intro exI conjI*)
show $(+)\ a$ ‘ $(S \cup N) = (\lambda x.\ a + x)$ ‘ $S \cup (\lambda x.\ a + x)$ ‘ N
by *auto*
show $(\lambda x.\ a + x)$ ‘ $N' \in$ *null_sets lborel*
using *that* **by** (*auto simp*: *null_sets_translation im_eq*)
qed (*use that im_eq* **in** *auto*)
with A **have** $(\lambda x.\ x - a)$ −‘ $A \in$ *sets lebesgue*
by (*force simp*: *vim_eq completion_def intro*!: *sigma_sets_image*)
then show $(+)\ (-\ a)$ −‘ $A \cap$ *space lebesgue* $\in$ *sets lebesgue*
by (*auto simp*: *vimage_def im_eq*)
qed *auto*
finally show *?thesis* **.**
qed

lemma *measurable_translation*:
$S \in$ *lmeasurable* $\Longrightarrow ((+)\ a$ ‘ $S) \in$ *lmeasurable*
using *emeasure_lebesgue_affine* [*of 1 a S*]
apply (*auto intro*: *lebesgue_sets_translation simp add*: *fmeasurable_def cong*: *image_cong_simp*)

apply (*simp add*: *ac_simps*)
done

lemma *measurable_translation_subtract*:
$S \in lmeasurable \Longrightarrow ((\lambda x.\ x - a)\ `\ S) \in lmeasurable$
using *measurable_translation* [*of S* − *a*] **by** (*simp cong*: *image_cong_simp*)

lemma *measure_translation*:
measure lebesgue ((+) *a* ‘ *S*) = *measure lebesgue S*
using *measure_lebesgue_affine* [*of 1 a S*] **by** (*simp add*: *ac_simps cong*: *image_cong_simp*)

lemma *measure_translation_subtract*:
measure lebesgue ((λ*x*. *x* − *a*) ‘ *S*) = *measure lebesgue S*
using *measure_translation* [*of* − *a*] **by** (*simp cong*: *image_cong_simp*)

6.13.8 A nice lemma for negligibility proofs

lemma *summable_iff_suminf_neq_top*: $(\bigwedge n.\ f\ n \geq 0) \Longrightarrow \neg$ *summable f* $\Longrightarrow (\sum i.$ *ennreal* (*f i*)) = *top*
by (*metis summable_suminf_not_top*)

proposition *starlike_negligible_bounded_gmeasurable*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *S*: *S* ∈ *sets lebesgue* **and** *bounded S*
and *eq1*: $\bigwedge c\ x.\ [\![(c *_R x) \in S;\ 0 \leq c;\ x \in S]\!] \Longrightarrow c = 1$
shows *S* ∈ *null_sets lebesgue*
proof −
obtain *M* **where** $0 < M$ $S \subseteq$ *ball 0 M*
using ‹*bounded S*› **by** (*auto dest*: *bounded_subset_ballD*)

let *?f* = λ*n*. *root DIM*(*'a*) (*Suc n*)

have *vimage_eq_image*: $(*_R)$ (*?f n*) −‘ *S* = $(*_R)$ (*1* / *?f n*) ‘ *S* **for** *n*
apply *safe*
subgoal for *x* **by** (*rule image_eqI*[*of _ _ ?f n* $*_R$ *x*]) *auto*
subgoal by *auto*
done

have *eq*: (*1* / *?f n*) ^ *DIM*(*'a*) = *1* / *Suc n* **for** *n*
by (*simp add*: *field_simps*)

{ **fix** *n x* **assume** *x*: *root DIM*(*'a*) (*1* + *real n*) $*_R$ *x* ∈ *S*
have *1* ∗ *norm x* ≤ *root DIM*(*'a*) (*1* + *real n*) ∗ *norm x*
by (*rule mult_mono*) *auto*
also have . . . < *M*
using *x* ‹*S* ⊆ *ball 0 M*› **by** *auto*
finally have *norm x* < *M* **by** *simp* }
note *less_M* = *this*

have $(\sum n.\ ennreal\ (1\ /\ Suc\ n)) = top$
using *not_summable_harmonic*[**where** *'a=real*] *summable_Suc_iff* [**where** *f=λn. 1 / (real n)*]
by (*intro summable_iff_suminf_neq_top*) (*auto simp add*: *inverse_eq_divide*)
then have $top * emeasure\ lebesgue\ S = (\sum n.\ (1\ /\ ?f\ n)\hat{}DIM('a) * emeasure\ lebesgue\ S)$
unfolding *ennreal_suminf_multc eq* **by** *simp*
also have $\ldots = (\sum n.\ emeasure\ lebesgue\ ((*_R)\ (?f\ n) -` S))$
unfolding *vimage_eq_image* **using** *emeasure_lebesgue_affine*[*of 1 / ?f n 0 S* **for** *n*] **by** *simp*
also have $\ldots = emeasure\ lebesgue\ (\bigcup n.\ (*_R)\ (?f\ n) -` S)$
proof (*intro suminf_emeasure*)
show *disjoint_family* $(\lambda n.\ (*_R)\ (?f\ n) -` S)$
unfolding *disjoint_family_on_def*
proof *safe*
fix *m n* :: *nat* **and** *x* **assume** $m \neq n$ $?f\ m *_R x \in S$ $?f\ n *_R x \in S$
with *eq1*[*of* $?f\ m\ /\ ?f\ n$ $?f\ n *_R x$] **show** $x \in \{\}$
by *auto*
qed
have $(*_R)\ (?f\ i) -` S \in sets\ lebesgue$ **for** *i*
using *measurable_sets*[*OF lebesgue_measurable_scaling*[*of ?f i*] *S*] **by** *auto*
then show *range* $(\lambda i.\ (*_R)\ (?f\ i) -` S) \subseteq sets\ lebesgue$
by *auto*
qed
also have $\ldots \leq emeasure\ lebesgue\ (ball\ 0\ M :: 'a\ set)$
using *less_M* **by** (*intro emeasure_mono*) *auto*
also have $\ldots < top$
using *lmeasurable_ball* **by** (*auto simp*: *fmeasurable_def*)
finally have $emeasure\ lebesgue\ S = 0$
by (*simp add*: *ennreal_top_mult split*: *if_split_asm*)
then show $S \in null\_sets\ lebesgue$
unfolding *null_sets_def* **using** ‹$S \in sets\ lebesgue$› **by** *auto*
qed

corollary *starlike_negligible_compact*:
$compact\ S \Longrightarrow (\bigwedge c\ x.\ [\![(c *_R x) \in S;\ 0 \leq c;\ x \in S]\!] \Longrightarrow c = 1) \Longrightarrow S \in null\_sets\ lebesgue$
using *starlike_negligible_bounded_gmeasurable*[*of S*] **by** (*auto simp*: *compact_eq_bounded_closed*)

proposition *outer_regular_lborel_le*:
assumes *B*[*measurable*]: $B \in sets\ borel$ **and** $0 < (e::real)$
obtains *U* **where** *open U* $B \subseteq U$ **and** $emeasure\ lborel\ (U - B) \leq e$
proof −
let $?\mu = emeasure\ lborel$
let $?B = \lambda n::nat.\ ball\ 0\ n :: 'a\ set$
let $?e = \lambda n.\ e*((1/2)\hat{}Suc\ n)$
have $\forall n.\ \exists U.\ open\ U \wedge ?B\ n \cap B \subseteq U \wedge ?\mu\ (U - B) < ?e\ n$
proof

```
    fix n :: nat
    let ?A = density lborel (indicator (?B n))
    have emeasure_A: X ∈ sets borel ⟹ emeasure ?A X = ?μ (?B n ∩ X) for X
     by (auto simp: emeasure_density borel_measurable_indicator indicator_inter_arith[symmetric])

    have finite_A: emeasure ?A (space ?A) ≠ ∞
      using emeasure_bounded_finite[of ?B n] by (auto simp: emeasure_A)
    interpret A: finite_measure ?A
      by rule fact
   have emeasure ?A B + ?e n > (INF U∈{U. B ⊆ U ∧ open U}. emeasure ?A
U)
      using ‹0<e› by (auto simp: outer_regular[OF _ finite_A B, symmetric])
    then obtain U where U: B ⊆ U open U and muU: ?μ (?B n ∩ B) + ?e n
> ?μ (?B n ∩ U)
      unfolding INF_less_iff by (auto simp: emeasure_A)
    moreover
    { have ?μ ((?B n ∩ U) − B) = ?μ ((?B n ∩ U) − (?B n ∩ B))
        using U by (intro arg_cong[where f=?μ]) auto
      also have ... = ?μ (?B n ∩ U) − ?μ (?B n ∩ B)
        using U A.emeasure_finite[of B]
      by (intro emeasure_Diff) (auto simp del: A.emeasure_finite simp: emeasure_A)
      also have ... < ?e n
        using U muU A.emeasure_finite[of B]
        by (subst minus_less_iff_ennreal)
         (auto simp del: A.emeasure_finite simp: emeasure_A less_top ac_simps intro!:
emeasure_mono)
      finally have ?μ ((?B n ∩ U) − B) < ?e n . }
    ultimately show ∃U. open U ∧ ?B n ∩ B ⊆ U ∧ ?μ (U − B) < ?e n
      by (intro exI[of _ ?B n ∩ U]) auto
  qed
  then obtain U
    where U: ⋀n. open (U n) ⋀n. ?B n ∩ B ⊆ U n ⋀n. ?μ (U n − B) < ?e n
    by metis
  show ?thesis
  proof
    { fix x assume x ∈ B
      moreover
      obtain n where norm x < real n
        using reals_Archimedean2 by blast
      ultimately have x ∈ (⋃n. U n)
        using U(2)[of n] by auto }
    note * = this
    then show open (⋃n. U n) B ⊆ (⋃n. U n)
      using U by auto
    have ?μ (⋃n. U n − B) ≤ (∑n. ?μ (U n − B))
      using U(1) by (intro emeasure_subadditive_countably) auto
    also have ... ≤ (∑n. ennreal (?e n))
      using U(3) by (intro suminf_le) (auto intro: less_imp_le)
    also have ... = ennreal (e * 1)
```

```
    using ‹0<e› by (intro suminf_ennreal_eq sums_mult power_half_series) auto
  finally show emeasure lborel ((⋃n. U n) − B) ≤ ennreal e
    by simp
 qed
qed

lemma outer_regular_lborel:
 assumes B: B ∈ sets borel and 0 < (e::real)
 obtains U where open U B ⊆ U emeasure lborel (U − B) < e
proof −
 obtain U where U: open U B ⊆ U and emeasure lborel (U−B) ≤ e/2
   using outer_regular_lborel_le [OF B, of e/2] ‹e > 0›
   by force
 moreover have ennreal (e/2) < ennreal e
   using ‹e > 0› by (simp add: ennreal_lessI)
 ultimately have emeasure lborel (U−B) < e
   by auto
 with U show ?thesis
   using that by auto
qed

lemma completion_upper:
 assumes A: A ∈ sets (completion M)
 obtains A' where A ⊆ A' A' ∈ sets M A' − A ∈ null_sets (completion M)
                emeasure (completion M) A = emeasure M A'
proof −
 from AE_notin_null_part[OF A] obtain N where N: N ∈ null_sets M null_part
M A ⊆ N
  unfolding eventually_ae_filter using null_part_null_sets[OF A, THEN null_setsD2,
THEN sets.sets_into_space] by auto
 let ?A' = main_part M A ∪ N
 show ?thesis
 proof
   show A ⊆ ?A'
     using ‹null_part M A ⊆ N› by (subst main_part_null_part_Un[symmetric, OF
A]) auto
   have main_part M A ⊆ A
     using assms main_part_null_part_Un by auto
   then have ?A' − A ⊆ N
     by blast
   with N show ?A' − A ∈ null_sets (completion M)
       by (blast intro: null_sets_completionI completion.complete_measure_axioms
complete_measure.complete2)
   show emeasure (completion M) A = emeasure M (main_part M A ∪ N)
     using A ‹N ∈ null_sets M› by (simp add: emeasure_Un_null_set)
 qed (use A N in auto)
qed

lemma sets_lebesgue_outer_open:
```

fixes *e*::*real*
assumes *S*: *S* ∈ *sets lebesgue* **and** *e* > *0*
obtains *T* **where** *open T S* ⊆ *T* (*T* − *S*) ∈ *lmeasurable emeasure lebesgue* (*T* − *S*) < *ennreal e*
proof −
obtain *S'* **where** *S'*: *S* ⊆ *S' S'* ∈ *sets borel*
and *null*: *S'* − *S* ∈ *null_sets lebesgue*
and *em*: *emeasure lebesgue S* = *emeasure lborel S'*
using *completion_upper*[*of S lborel*] *S* **by** *auto*
then have *f_S'*: *S'* ∈ *sets borel*
using *S* **by** (*auto simp*: *fmeasurable_def*)
with *outer_regular_lborel*[*OF _ ‹0<e›*]
obtain *U* **where** *U*: *open U S'* ⊆ *U emeasure lborel* (*U* − *S'*) < *e*
by *blast*
show *thesis*
proof
show *open U S* ⊆ *U*
using *f_S' U S'* **by** *auto*
have (*U* − *S*) = (*U* − *S'*) ∪ (*S'* − *S*)
using *S' U* **by** *auto*
then have *eq*: *emeasure lebesgue* (*U* − *S*) = *emeasure lborel* (*U* − *S'*)
using *null* **by** (*simp add*: *U*(*1*) *emeasure_Un_null_set f_S' sets.Diff*)
have (*U* − *S*) ∈ *sets lebesgue*
by (*simp add*: *S U*(*1*) *sets.Diff*)
then show (*U* − *S*) ∈ *lmeasurable*
unfolding *fmeasurable_def* **using** *U*(*3*) *eq less_le_trans* **by** *fastforce*
with *eq U* **show** *emeasure lebesgue* (*U* − *S*) < *e*
by (*simp add*: *eq*)
qed
qed

lemma *sets_lebesgue_inner_closed*:
fixes *e*::*real*
assumes *S* ∈ *sets lebesgue e* > *0*
obtains *T* **where** *closed T T* ⊆ *S S*−*T* ∈ *lmeasurable emeasure lebesgue* (*S* − *T*) < *ennreal e*
proof −
have −*S* ∈ *sets lebesgue*
using *assms* **by** (*simp add*: *Compl_in_sets_lebesgue*)
then obtain *T* **where** *open T* −*S* ⊆ *T*
and *T*: (*T* − −*S*) ∈ *lmeasurable emeasure lebesgue* (*T* − −*S*) < *e*
using *sets_lebesgue_outer_open assms* **by** *blast*
show *thesis*
proof
show *closed* (−*T*)
using ‹*open T*› **by** *blast*
show −*T* ⊆ *S*
using ‹− *S* ⊆ *T*› **by** *auto*
show *S* − (−*T*) ∈ *lmeasurable emeasure lebesgue* (*S* − (− *T*)) < *e*

using *T* **by** (*auto simp*: *Int_commute*)
qed
qed

lemma *lebesgue_openin*:
⟦*openin* (*top_of_set S*) *T*; *S* ∈ *sets lebesgue*⟧ ⟹ *T* ∈ *sets lebesgue*
by (*metis borel_open openin_open sets.Int sets_completionI_sets sets_lborel*)

lemma *lebesgue_closedin*:
⟦*closedin* (*top_of_set S*) *T*; *S* ∈ *sets lebesgue*⟧ ⟹ *T* ∈ *sets lebesgue*
by (*metis borel_closed closedin_closed sets.Int sets_completionI_sets sets_lborel*)

6.13.9 *F_sigma* and *G_delta* sets.

— https://en.wikipedia.org/wiki/F-sigma_set
inductive *fsigma* :: *'a::topological_space set* ⇒ *bool* **where**
(⋀*n::nat*. *closed* (*F n*)) ⟹ *fsigma* (⋃(*F* ' *UNIV*))

inductive *gdelta* :: *'a::topological_space set* ⇒ *bool* **where**
(⋀*n::nat*. *open* (*F n*)) ⟹ *gdelta* (⋂(*F* ' *UNIV*))

lemma *fsigma_Union_compact*:
fixes *S* :: *'a::{real_normed_vector,heine_borel} set*
shows *fsigma S* ⟷ (∃ *F::nat* ⇒ *'a set*. *range F* ⊆ *Collect compact* ∧ *S* = ⋃(*F* ' *UNIV*))
proof *safe*
assume *fsigma S*
then obtain *F* :: *nat* ⇒ *'a set* **where** *F*: *range F* ⊆ *Collect closed S* = ⋃(*F* ' *UNIV*)
by (*meson fsigma.cases image_subsetI mem_Collect_eq*)
then have ∃ *D::nat* ⇒ *'a set*. *range D* ⊆ *Collect compact* ∧ ⋃(*D* ' *UNIV*) = *F i* **for** *i*
using *closed_Union_compact_subsets* [*of F i*]
by (*metis image_subsetI mem_Collect_eq range_subsetD*)
then obtain *D* :: *nat* ⇒ *nat* ⇒ *'a set*
where *D*: ⋀*i*. *range* (*D i*) ⊆ *Collect compact* ∧ ⋃((*D i*) ' *UNIV*) = *F i*
by *metis*
let *?DD* = λ*n*. (λ(*i*,*j*). *D i j*) (*prod_decode n*)
show ∃ *F::nat* ⇒ *'a set*. *range F* ⊆ *Collect compact* ∧ *S* = ⋃(*F* ' *UNIV*)
proof (*intro exI conjI*)
show *range ?DD* ⊆ *Collect compact*
using *D* **by** *clarsimp* (*metis mem_Collect_eq rangeI split_conv subsetCE surj_pair*)
show *S* = ⋃ (*range ?DD*)
proof
show *S* ⊆ ⋃ (*range ?DD*)
using *D F*
by *clarsimp* (*metis UN_iff old.prod.case prod_decode_inverse prod_encode_eq*)
show ⋃ (*range ?DD*) ⊆ *S*

```
      using D F  by fastforce
    qed
  qed
next
  fix F :: nat ⇒ 'a set
  assume range F ⊆ Collect compact and S = ⋃(F ` UNIV)
  then show fsigma (⋃(F ` UNIV))
    by (simp add: compact_imp_closed fsigma.intros image_subset_iff)
qed

lemma gdelta_imp_fsigma: gdelta S ⟹ fsigma (− S)
proof (induction rule: gdelta.induct)
  case (1 F)
  have − ⋂(F ` UNIV) = (⋃i. −(F i))
    by auto
  then show ?case
    by (simp add: fsigma.intros closed_Compl 1)
qed

lemma fsigma_imp_gdelta: fsigma S ⟹ gdelta (− S)
proof (induction rule: fsigma.induct)
  case (1 F)
  have − ⋃(F ` UNIV) = (⋂i. −(F i))
    by auto
  then show ?case
    by (simp add: 1 gdelta.intros open_closed)
qed

lemma gdelta_complement: gdelta(− S) ⟷ fsigma S
  using fsigma_imp_gdelta gdelta_imp_fsigma by force

lemma lebesgue_set_almost_fsigma:
  assumes S ∈ sets lebesgue
  obtains C T where fsigma C T ∈ null_sets lebesgue C ∪ T = S disjnt C T
proof −
  { fix n::nat
    obtain T where closed T T ⊆ S S−T ∈ lmeasurable emeasure lebesgue (S −
T) < ennreal (1 / Suc n)
      using sets_lebesgue_inner_closed [OF assms]
      by (metis of_nat_0_less_iff zero_less_Suc zero_less_divide_1_iff)
    then have ∃ T. closed T ∧ T ⊆ S ∧ S − T ∈ lmeasurable ∧ measure lebesgue
(S−T) < 1 / Suc n
      by (metis emeasure_eq_measure2 ennreal_leI not_le)
  }
  then obtain F where F: ⋀n::nat. closed (F n) ∧ F n ⊆ S ∧ S − F n ∈
lmeasurable ∧ measure lebesgue (S − F n) < 1 / Suc n
    by metis
  let ?C = ⋃(F ` UNIV)
  show thesis
```

```
  proof
    show fsigma ?C
      using F by (simp add: fsigma.intros)
    show (S − ?C) ∈ null_sets lebesgue
    proof (clarsimp simp add: completion.null_sets_outer_le)
      fix e :: real
      assume 0 < e
      then obtain n where n: 1 / Suc n < e
        using nat_approx_posE by metis
      show ∃ T ∈ lmeasurable. S − (⋃x. F x) ⊆ T ∧ measure lebesgue T ≤ e
      proof (intro bexI conjI)
        show measure lebesgue (S − F n) ≤ e
          by (meson F n less_trans not_le order.asym)
      qed (use F in auto)
    qed
    show ?C ∪ (S − ?C) = S
      using F by blast
    show disjnt ?C (S − ?C)
      by (auto simp: disjnt_def)
  qed
qed

lemma lebesgue_set_almost_gdelta:
  assumes S ∈ sets lebesgue
  obtains C T where gdelta C T ∈ null_sets lebesgue S ∪ T = C disjnt S T
proof −
  have −S ∈ sets lebesgue
    using assms Compl_in_sets_lebesgue by blast
  then obtain C T where C: fsigma C T ∈ null_sets lebesgue C ∪ T = −S disjnt
C T
    using lebesgue_set_almost_fsigma by metis
  show thesis
  proof
    show gdelta (−C)
      by (simp add: ‹fsigma C› fsigma_imp_gdelta)
    show S ∪ T = −C disjnt S T
      using C by (auto simp: disjnt_def)
  qed (use C in auto)
qed

end
```

6.14 Tagged Divisions for Henstock-Kurzweil Integration

```
theory Tagged_Division
  imports Topology_Euclidean_Space
begin
```

lemma *sum_Sigma_product*:
  assumes *finite S*
    and $\bigwedge i.\ i \in S \Longrightarrow finite\ (T\ i)$
  shows $(\sum i{\in}S.\ sum\ (x\ i)\ (T\ i)) = (\sum (i, j){\in}Sigma\ S\ T.\ x\ i\ j)$
  using *assms*
proof *induct*
  case *empty*
  then show *?case*
    by *simp*
next
  case (*insert a S*)
  show *?case*
    by (*simp add*: *insert.hyps*(*1*) *insert.prems sum.Sigma*)
qed

lemmas *scaleR_simps* = *scaleR_zero_left scaleR_minus_left scaleR_left_diff_distrib scaleR_zero_right scaleR_minus_right scaleR_right_diff_distrib scaleR_eq_0_iff scaleR_cancel_left scaleR_cancel_right scaleR_add_right scaleR_add_left real_vector_class.scaleR_one*

6.14.1 Sundries

A transitive relation is well-founded if all initial segments are finite.

lemma *wf_finite_segments*:
  assumes *irrefl r* **and** *trans r* **and** $\bigwedge x.\ finite\ \{y.\ (y, x) \in r\}$
  shows *wf* (*r*)
  apply (*simp add*: *trans_wf_iff wf_iff_acyclic_if_finite converse_def assms*)
  using *acyclic_def assms irrefl_def trans_Restr* **by** *fastforce*

For creating values between u and v.

lemma *scaling_mono*:
  fixes *u*::*'a*::*linordered_field*
  assumes $u \le v$ $0 \le r$ $r \le s$
    shows $u + r * (v - u) / s \le v$
proof −
  have $r/s \le 1$ **using** *assms*
    using *divide_le_eq_1* **by** *fastforce*
  then have $(r/s) * (v - u) \le 1 * (v - u)$
    by (*meson diff_ge_0_iff_ge mult_right_mono* ‹$u \le v$›)
  then show *?thesis*
    by (*simp add*: *field_simps*)
qed

6.14.2 Some useful lemmas about intervals

lemma *interior_subset_union_intervals*:
  assumes $i = cbox\ a\ b$
    and $j = cbox\ c\ d$
    and $interior\ j \neq \{\}$

and *i* ⊆ *j* ∪ *S*
and *interior i* ∩ *interior j* = {}
shows *interior i* ⊆ *interior S*
proof –
have *box a b* ∩ *cbox c d* = {}
using *Int_interval_mixed_eq_empty*[*of c d a b*] *assms*
unfolding *interior_cbox* **by** *auto*
moreover
have *box a b* ⊆ *cbox c d* ∪ *S*
apply (*rule order_trans*,*rule box_subset_cbox*)
using *assms* **by** *auto*
ultimately
show *?thesis*
unfolding *assms interior_cbox*
by *auto* (*metis IntI UnE empty_iff interior_maximal open_box subsetCE subsetI*)
qed

lemma *interior_Union_subset_cbox*:
assumes *finite f*
assumes *f*: ⋀*s*. *s* ∈ *f* ⟹ ∃ *a b*. *s* = *cbox a b* ⋀*s*. *s* ∈ *f* ⟹ *interior s* ⊆ *t*
and *t*: *closed t*
shows *interior* (⋃*f*) ⊆ *t*
proof –
have [*simp*]: *s* ∈ *f* ⟹ *closed s* **for** *s*
using *f* **by** *auto*
define *E* **where** *E* = {*s*∈*f*. *interior s* = {}}
then have *finite E E* ⊆ {*s*∈*f*. *interior s* = {}}
using ‹*finite f*› **by** *auto*
then have *interior* (⋃*f*) = *interior* (⋃(*f* – *E*))
proof (*induction E rule*: *finite_subset_induct'*)
case (*insert s f'*)
have *interior* (⋃(*f* – *insert s f'*) ∪ *s*) = *interior* (⋃(*f* – *insert s f'*))
using *insert.hyps* ‹*finite f*› **by** (*intro interior_closed_Un_empty_interior*) *auto*
also have ⋃(*f* – *insert s f'*) ∪ *s* = ⋃(*f* – *f'*)
using *insert.hyps* **by** *auto*
finally show *?case*
by (*simp add*: *insert.IH*)
qed *simp*
also have . . . ⊆ ⋃(*f* – *E*)
by (*rule interior_subset*)
also have . . . ⊆ *t*
proof (*rule Union_least*)
fix *s* **assume** *s* ∈ *f* – *E*
with *f*[*of s*] **obtain** *a b* **where** *s*: *s* ∈ *f s* = *cbox a b box a b* ≠ {}
by (*fastforce simp*: *E_def*)
have *closure* (*interior s*) ⊆ *closure t*
by (*intro closure_mono f* ‹*s* ∈ *f*›)
with *s* ‹*closed t*› **show** *s* ⊆ *t*
by *simp*

qed
finally show *?thesis* **.**
qed

lemma *Int_interior_Union_intervals*:
$[\![$*finite* $\mathcal{F}$; *open* S; $\bigwedge T.\ T\in\mathcal{F} \Longrightarrow \exists\, a\ b.\ T = cbox\ a\ b$; $\bigwedge T.\ T\in\mathcal{F} \Longrightarrow S \cap (interior\ T) = \{\}]\!]$
$\Longrightarrow S \cap interior\ (\bigcup\mathcal{F}) = \{\}$
using *interior_Union_subset_cbox*[*of* $\mathcal{F}$ *UNIV* − *S*] **by** *auto*

lemma *interval_split*:
fixes *a* :: *'a::euclidean_space*
assumes $k \in Basis$
shows
$cbox\ a\ b \cap \{x.\ x\cdot k \le c\} = cbox\ a\ (\sum i\in Basis.\ (if\ i = k\ then\ min\ (b\cdot k)\ c\ else\ b\cdot i) *_R\ i)$
$cbox\ a\ b \cap \{x.\ x\cdot k \ge c\} = cbox\ (\sum i\in Basis.\ (if\ i = k\ then\ max\ (a\cdot k)\ c\ else\ a\cdot i) *_R\ i)\ b$
using *assms* **by** (*rule_tac set_eqI*; *auto simp*: *mem_box*)+

lemma *interval_not_empty*: $\forall\, i\in Basis.\ a\cdot i \le b\cdot i \Longrightarrow cbox\ a\ b \neq \{\}$
by (*simp add*: *box_ne_empty*)

6.14.3 Bounds on intervals where they exist

definition *interval_upperbound* :: (*'a::euclidean_space*) *set* ⇒ *'a*
where $interval\_upperbound\ s = (\sum i\in Basis.\ (SUP\ x\in s.\ x\cdot i) *_R\ i)$

definition *interval_lowerbound* :: (*'a::euclidean_space*) *set* ⇒ *'a*
where $interval\_lowerbound\ s = (\sum i\in Basis.\ (INF\ x\in s.\ x\cdot i) *_R\ i)$

lemma *interval_upperbound*[*simp*]:
$\forall\, i\in Basis.\ a\cdot i \le b\cdot i \Longrightarrow$
interval_upperbound (*cbox a b*) = (*b::'a::euclidean_space*)
unfolding *interval_upperbound_def euclidean_representation_sum cbox_def*
by (*safe intro*!: *cSup_eq*) *auto*

lemma *interval_lowerbound*[*simp*]:
$\forall\, i\in Basis.\ a\cdot i \le b\cdot i \Longrightarrow$
interval_lowerbound (*cbox a b*) = (*a::'a::euclidean_space*)
unfolding *interval_lowerbound_def euclidean_representation_sum cbox_def*
by (*safe intro*!: *cInf_eq*) *auto*

lemmas *interval_bounds* = *interval_upperbound interval_lowerbound*

lemma
fixes *X::real set*
shows *interval_upperbound_real*[*simp*]: *interval_upperbound X* = *Sup X*
and *interval_lowerbound_real*[*simp*]: *interval_lowerbound X* = *Inf X*

by (*auto simp*: *interval_upperbound_def interval_lowerbound_def*)

lemma *interval_bounds'*[*simp*]:
assumes *cbox a b* ≠ {}
shows *interval_upperbound* (*cbox a b*) = *b*
and *interval_lowerbound* (*cbox a b*) = *a*
using *assms* **unfolding** *box_ne_empty* **by** *auto*

lemma *interval_upperbound_Times*:
assumes *A* ≠ {} **and** *B* ≠ {}
shows *interval_upperbound* (*A* × *B*) = (*interval_upperbound A*, *interval_upperbound B*)
proof−
from *assms* **have** *fst_image_times'*: *A* = *fst* ‘ (*A* × *B*) **by** *simp*
have (∑ *i*∈*Basis*. (*SUP x*∈*A* × *B*. *x* · (*i*, *0*)) $*_R$ *i*) = (∑ *i*∈*Basis*. (*SUP x*∈*A*. *x* · *i*) $*_R$ *i*)
by (*subst* (*2*) *fst_image_times'*) (*simp del*: *fst_image_times add*: *image_comp inner_Pair_0*)
moreover from *assms* **have** *snd_image_times'*: *B* = *snd* ‘ (*A* × *B*) **by** *simp*
have (∑ *i*∈*Basis*. (*SUP x*∈*A* × *B*. *x* · (*0*, *i*)) $*_R$ *i*) = (∑ *i*∈*Basis*. (*SUP x*∈*B*. *x* · *i*) $*_R$ *i*)
by (*subst* (*2*) *snd_image_times'*) (*simp del*: *snd_image_times add*: *image_comp inner_Pair_0*)
ultimately show *?thesis* **unfolding** *interval_upperbound_def*
by (*subst sum_Basis_prod_eq*) (*auto simp add*: *sum_prod*)
qed

lemma *interval_lowerbound_Times*:
assumes *A* ≠ {} **and** *B* ≠ {}
shows *interval_lowerbound* (*A* × *B*) = (*interval_lowerbound A*, *interval_lowerbound B*)
proof−
from *assms* **have** *fst_image_times'*: *A* = *fst* ‘ (*A* × *B*) **by** *simp*
have (∑ *i*∈*Basis*. (*INF x*∈*A* × *B*. *x* · (*i*, *0*)) $*_R$ *i*) = (∑ *i*∈*Basis*. (*INF x*∈*A*. *x* · *i*) $*_R$ *i*)
by (*subst* (*2*) *fst_image_times'*) (*simp del*: *fst_image_times add*: *image_comp inner_Pair_0*)
moreover from *assms* **have** *snd_image_times'*: *B* = *snd* ‘ (*A* × *B*) **by** *simp*
have (∑ *i*∈*Basis*. (*INF x*∈*A* × *B*. *x* · (*0*, *i*)) $*_R$ *i*) = (∑ *i*∈*Basis*. (*INF x*∈*B*. *x* · *i*) $*_R$ *i*)
by (*subst* (*2*) *snd_image_times'*) (*simp del*: *snd_image_times add*: *image_comp inner_Pair_0*)
ultimately show *?thesis* **unfolding** *interval_lowerbound_def*
by (*subst sum_Basis_prod_eq*) (*auto simp add*: *sum_prod*)
qed

6.14.4 The notion of a gauge — simply an open set containing the point

definition *gauge* $\gamma \longleftrightarrow (\forall x.\ x \in \gamma\ x \wedge open\ (\gamma\ x))$

lemma *gaugeI*:
  assumes $\bigwedge x.\ x \in \gamma\ x$
    and $\bigwedge x.\ open\ (\gamma\ x)$
  shows *gauge* γ
  using *assms* **unfolding** *gauge_def* **by** *auto*

lemma *gaugeD*[*dest*]:
  assumes *gauge* γ
  shows $x \in \gamma\ x$
    and $open\ (\gamma\ x)$
  using *assms* **unfolding** *gauge_def* **by** *auto*

lemma *gauge_ball_dependent*: $\forall x.\ 0 < e\ x \Longrightarrow gauge\ (\lambda x.\ ball\ x\ (e\ x))$
  unfolding *gauge_def* **by** *auto*

lemma *gauge_ball*[*intro*]: $0 < e \Longrightarrow gauge\ (\lambda x.\ ball\ x\ e)$
  unfolding *gauge_def* **by** *auto*

lemma *gauge_trivial*[*intro*!]: $gauge\ (\lambda x.\ ball\ x\ 1)$
  by (*rule gauge_ball*) *auto*

lemma *gauge_Int*[*intro*]: $gauge\ \gamma 1 \Longrightarrow gauge\ \gamma 2 \Longrightarrow gauge\ (\lambda x.\ \gamma 1\ x \cap \gamma 2\ x)$
  unfolding *gauge_def* **by** *auto*

lemma *gauge_reflect*:
  fixes γ :: *'a::euclidean_space* $\Rightarrow$ *'a set*
  shows $gauge\ \gamma \Longrightarrow gauge\ (\lambda x.\ uminus\ `\ \gamma\ (-\ x))$
  using *equation_minus_iff*
  by (*auto simp*: *gauge_def surj_def intro*!: *open_surjective_linear_image linear_uminus*)

lemma *gauge_Inter*:
  assumes *finite S*
    and $\bigwedge u.\ u{\in}S \Longrightarrow gauge\ (f\ u)$
  shows $gauge\ (\lambda x.\ \bigcap\{f\ u\ x \mid u.\ u \in S\})$
proof –
  have $*$: $\bigwedge x.\ \{f\ u\ x\ |u.\ u \in S\} = (\lambda u.\ f\ u\ x)\ `\ S$
    by *auto*
  show *?thesis*
    unfolding *gauge_def* **unfolding** $*$
    using *assms* **unfolding** *Ball_def Inter_iff mem_Collect_eq gauge_def* **by** *auto*
qed

lemma *gauge_existence_lemma*:
  $(\forall x.\ \exists d :: real.\ p\ x \longrightarrow 0 < d \wedge q\ d\ x) \longleftrightarrow (\forall x.\ \exists d{>}0.\ p\ x \longrightarrow q\ d\ x)$
  by (*metis zero_less_one*)

6.14.5 Attempt a systematic general set of "offset" results for components

lemma *gauge_modify*:
assumes $(\forall S.\ open\ S \longrightarrow open\ \{x.\ f(x) \in S\})$ *gauge d*
shows *gauge* $(\lambda x.\ \{y.\ f\ y \in d\ (f\ x)\})$
using *assms* **unfolding** *gauge_def* **by** *force*

6.14.6 Divisions

definition *division_of* (**infixl** *division′_of 40*)
where
s division_of i $\longleftrightarrow$
finite s $\wedge$
$(\forall K{\in}s.\ K \subseteq i \wedge K \neq \{\} \wedge (\exists\, a\ b.\ K = cbox\ a\ b)) \wedge$
$(\forall K1{\in}s.\ \forall K2{\in}s.\ K1 \neq K2 \longrightarrow interior(K1) \cap interior(K2) = \{\}) \wedge$
$(\bigcup s = i)$

lemma *division_ofD*[*dest*]:
assumes *s division_of i*
shows *finite s*
and $\bigwedge K.\ K \in s \Longrightarrow K \subseteq i$
and $\bigwedge K.\ K \in s \Longrightarrow K \neq \{\}$
and $\bigwedge K.\ K \in s \Longrightarrow \exists\, a\ b.\ K = cbox\ a\ b$
and $\bigwedge K1\ K2.\ K1 \in s \Longrightarrow K2 \in s \Longrightarrow K1 \neq K2 \Longrightarrow interior(K1) \cap interior(K2) = \{\}$
and $\bigcup s = i$
using *assms* **unfolding** *division_of_def* **by** *auto*

lemma *division_ofI*:
assumes *finite s*
and $\bigwedge K.\ K \in s \Longrightarrow K \subseteq i$
and $\bigwedge K.\ K \in s \Longrightarrow K \neq \{\}$
and $\bigwedge K.\ K \in s \Longrightarrow \exists\, a\ b.\ K = cbox\ a\ b$
and $\bigwedge K1\ K2.\ K1 \in s \Longrightarrow K2 \in s \Longrightarrow K1 \neq K2 \Longrightarrow interior\ K1 \cap interior\ K2 = \{\}$
and $\bigcup s = i$
shows *s division_of i*
using *assms* **unfolding** *division_of_def* **by** *auto*

lemma *division_of_finite*: *s division_of i* $\Longrightarrow$ *finite s*
by *auto*

lemma *division_of_self*[*intro*]: $cbox\ a\ b \neq \{\} \Longrightarrow \{cbox\ a\ b\}$ *division_of* $(cbox\ a\ b)$
unfolding *division_of_def* **by** *auto*

lemma *division_of_trivial*[*simp*]: *s division_of* $\{\} \longleftrightarrow s = \{\}$
unfolding *division_of_def* **by** *auto*

lemma *division_of_sing*[*simp*]:

```
  s division_of cbox a (a::'a::euclidean_space) ⟷ s = {cbox a a}
  (is ?l = ?r)
proof
  assume ?r
  moreover
  { fix K
    assume s = {{a}} K∈s
    then have ∃x y. K = cbox x y
      apply (rule_tac x=a in exI)+
      apply force
      done
  }
  ultimately show ?l
    unfolding division_of_def cbox_sing by auto
next
  assume ?l
  have x = {a} if x ∈ s for x
     by (metis ‹s division_of cbox a a› cbox_sing division_ofD(2) division_ofD(3)
subset_singletonD that)
  moreover have s ≠ {}
    using ‹s division_of cbox a a› by auto
  ultimately show ?r
    unfolding cbox_sing by auto
qed

lemma elementary_empty: obtains p where p division_of {}
  unfolding division_of_trivial by auto

lemma elementary_interval: obtains p where p division_of (cbox a b)
  by (metis division_of_trivial division_of_self)

lemma division_contains: s division_of i ⟹ ∀x∈i. ∃k∈s. x ∈ k
  unfolding division_of_def by auto

lemma forall_in_division:
  d division_of i ⟹ (∀x∈d. P x) ⟷ (∀a b. cbox a b ∈ d ⟶ P (cbox a b))
  unfolding division_of_def by fastforce

lemma cbox_division_memE:
  assumes 𝒟: K ∈ 𝒟 𝒟 division_of S
  obtains a b where K = cbox a b K ≠ {} K ⊆ S
  using assms unfolding division_of_def by metis

lemma division_of_subset:
  assumes p division_of (⋃p)
    and q ⊆ p
  shows q division_of (⋃q)
proof (rule division_ofI)
  note ∗ = division_ofD[OF assms(1)]
```

```
  show finite q
    using *(1) assms(2) infinite_super by auto
  {
    fix k
    assume k ∈ q
    then have kp: k ∈ p
      using assms(2) by auto
    show k ⊆ ⋃q
      using ‹k ∈ q› by auto
    show ∃ a b. k = cbox a b
      using *(4)[OF kp] by auto
    show k ≠ {}
      using *(3)[OF kp] by auto
  }
  fix k1 k2
  assume k1 ∈ q k2 ∈ q k1 ≠ k2
  then have **: k1 ∈ p k2 ∈ p k1 ≠ k2
    using assms(2) by auto
  show interior k1 ∩ interior k2 = {}
    using *(5)[OF **] by auto
qed auto

lemma division_of_union_self[intro]: p division_of s ⟹ p division_of (⋃p)
  unfolding division_of_def by auto

lemma division_inter:
  fixes s1 s2 :: 'a::euclidean_space set
  assumes p1 division_of s1
    and p2 division_of s2
  shows {k1 ∩ k2 | k1 k2. k1 ∈ p1 ∧ k2 ∈ p2 ∧ k1 ∩ k2 ≠ {}} division_of (s1
∩ s2)
  (is ?A' division_of _)
proof –
  let ?A = {s. s ∈ (λ(k1,k2). k1 ∩ k2) ‘ (p1 × p2) ∧ s ≠ {}}
  have *: ?A' = ?A by auto
  show ?thesis
    unfolding *
  proof (rule division_ofI)
    have ?A ⊆ (λ(x, y). x ∩ y) ‘ (p1 × p2)
      by auto
    moreover have finite (p1 × p2)
      using assms unfolding division_of_def by auto
    ultimately show finite ?A by auto
    have *: ⋀s. ⋃{x∈s. x ≠ {}} = ⋃s
      by auto
    show ⋃?A = s1 ∩ s2
      unfolding *
      using division_ofD(6)[OF assms(1)] and division_ofD(6)[OF assms(2)] by
auto
```

```
    {
      fix k
      assume k ∈ ?A
      then obtain k1 k2 where k: k = k1 ∩ k2 k1 ∈ p1 k2 ∈ p2 k ≠ {}
        by auto
      then show k ≠ {}
        by auto
      show k ⊆ s1 ∩ s2
       using division_ofD(2)[OF assms(1) k(2)] and division_ofD(2)[OF assms(2)
k(3)]
        unfolding k by auto
      obtain a1 b1 where k1: k1 = cbox a1 b1
        using division_ofD(4)[OF assms(1) k(2)] by blast
      obtain a2 b2 where k2: k2 = cbox a2 b2
        using division_ofD(4)[OF assms(2) k(3)] by blast
      show ∃ a b. k = cbox a b
        unfolding k k1 k2 unfolding Int_interval by auto
    }
    fix k1 k2
    assume k1 ∈ ?A
    then obtain x1 y1 where k1: k1 = x1 ∩ y1 x1 ∈ p1 y1 ∈ p2 k1 ≠ {}
      by auto
    assume k2 ∈ ?A
    then obtain x2 y2 where k2: k2 = x2 ∩ y2 x2 ∈ p1 y2 ∈ p2 k2 ≠ {}
      by auto
    assume k1 ≠ k2
    then have th: x1 ≠ x2 ∨ y1 ≠ y2
      unfolding k1 k2 by auto
    have ∗: interior x1 ∩ interior x2 = {} ∨ interior y1 ∩ interior y2 = {} ⟹
      interior (x1 ∩ y1) ⊆ interior x1 ⟹ interior (x1 ∩ y1) ⊆ interior y1 ⟹
      interior (x2 ∩ y2) ⊆ interior x2 ⟹ interior (x2 ∩ y2) ⊆ interior y2 ⟹
      interior (x1 ∩ y1) ∩ interior (x2 ∩ y2) = {} by auto
    show interior k1 ∩ interior k2 = {}
      unfolding k1 k2
      apply (rule ∗)
      using assms division_ofD(5) k1 k2(2) k2(3) th apply auto
      done
  qed
qed

lemma division_inter_1:
  assumes d division_of i
    and cbox a (b::'a::euclidean_space) ⊆ i
  shows {cbox a b ∩ k | k. k ∈ d ∧ cbox a b ∩ k ≠ {}} division_of (cbox a b)
proof (cases cbox a b = {})
  case True
  show ?thesis
    unfolding True and division_of_trivial by auto
next
```

```
  case False
  have *: cbox a b ∩ i = cbox a b using assms(2) by auto
  show ?thesis
    using division_inter[OF division_of_self[OF False] assms(1)]
    unfolding * by auto
qed

lemma elementary_Int:
  fixes s t :: 'a::euclidean_space set
  assumes p1 division_of s
    and p2 division_of t
  shows ∃ p. p division_of (s ∩ t)
using assms division_inter by blast

lemma elementary_Inter:
  assumes finite f
    and f ≠ {}
    and ∀ s∈f. ∃ p. p division_of (s::('a::euclidean_space) set)
  shows ∃ p. p division_of (⋂ f)
  using assms
proof (induct f rule: finite_induct)
  case (insert x f)
  show ?case
  proof (cases f = {})
    case True
    then show ?thesis
      unfolding True using insert by auto
  next
    case False
    obtain p where p division_of ⋂ f
      using insert(3)[OF False insert(5)[unfolded ball_simps,THEN conjunct2]] ..
    moreover obtain px where px division_of x
      using insert(5)[rule_format,OF insertI1] ..
    ultimately show ?thesis
      by (simp add: elementary_Int Inter_insert)
  qed
qed auto

lemma division_disjoint_union:
  assumes p1 division_of s1
    and p2 division_of s2
    and interior s1 ∩ interior s2 = {}
  shows (p1 ∪ p2) division_of (s1 ∪ s2)
proof (rule division_ofI)
  note d1 = division_ofD[OF assms(1)]
  note d2 = division_ofD[OF assms(2)]
  show finite (p1 ∪ p2)
    using d1(1) d2(1) by auto
  show ⋃ (p1 ∪ p2) = s1 ∪ s2
```

```
    using d1(6) d2(6) by auto
  {
    fix k1 k2
    assume as: k1 ∈ p1 ∪ p2 k2 ∈ p1 ∪ p2 k1 ≠ k2
    moreover
    let ?g=interior k1 ∩ interior k2 = {}
    {
      assume as: k1∈p1 k2∈p2
      have ?g
          using interior_mono[OF d1(2)[OF as(1)]] interior_mono[OF d2(2)[OF
as(2)]]
        using assms(3) by blast
    }
    moreover
    {
      assume as: k1∈p2 k2∈p1
      have ?g
          using interior_mono[OF d1(2)[OF as(2)]] interior_mono[OF d2(2)[OF
as(1)]]
        using assms(3) by blast
    }
    ultimately show ?g
      using d1(5)[OF _ _ as(3)] and d2(5)[OF _ _ as(3)] by auto
  }
  fix k
  assume k: k ∈ p1 ∪ p2
  show k ⊆ s1 ∪ s2
    using k d1(2) d2(2) by auto
  show k ≠ {}
    using k d1(3) d2(3) by auto
  show ∃ a b. k = cbox a b
    using k d1(4) d2(4) by auto
qed

lemma partial_division_extend_1:
  fixes a b c d :: 'a::euclidean_space
  assumes incl: cbox c d ⊆ cbox a b
    and nonempty: cbox c d ≠ {}
  obtains p where p division_of (cbox a b) cbox c d ∈ p
proof
  let ?B = λf::'a⇒'a × 'a.
    cbox (∑ i∈Basis. (fst (f i) · i) *R i) (∑ i∈Basis. (snd (f i) · i) *R i)
  define p where p = ?B ` (Basis →E {(a, c), (c, d), (d, b)})

  show cbox c d ∈ p
    unfolding p_def
   by (auto simp add: box_eq_empty cbox_def intro!: image_eqI[where x=λ(i::'a)∈Basis.
(c, d)])
  have ord: a · i ≤ c · i c · i ≤ d · i d · i ≤ b · i if i ∈ Basis for i
```

```
    using incl nonempty that
    unfolding box_eq_empty subset_box by (auto simp: not_le)

  show p division_of (cbox a b)
  proof (rule division_ofI)
    show finite p
      unfolding p_def by (auto intro!: finite_PiE)
    {
      fix k
      assume k ∈ p
      then obtain f where f: f ∈ Basis →E {(a, c), (c, d), (d, b)} and k: k =
?B f
        by (auto simp: p_def)
      then show ∃ a b. k = cbox a b
        by auto
      have k ⊆ cbox a b ∧ k ≠ {}
      proof (simp add: k box_eq_empty subset_box not_less, safe)
        fix i :: 'a
        assume i: i ∈ Basis
        with f have f i = (a, c) ∨ f i = (c, d) ∨ f i = (d, b)
          by (auto simp: PiE_iff)
        with i ord[of i]
        show a · i ≤ fst (f i) · i snd (f i) · i ≤ b · i fst (f i) · i ≤ snd (f i) · i
          by auto
      qed
      then show k ≠ {} k ⊆ cbox a b
        by auto
      {
        fix l
        assume l ∈ p
        then obtain g where g: g ∈ Basis →E {(a, c), (c, d), (d, b)} and l: l =
?B g
          by (auto simp: p_def)
        assume l ≠ k
        have ∃ i∈Basis. f i ≠ g i
        proof (rule ccontr)
          assume ¬ ?thesis
          with f g have f = g
            by (auto simp: PiE_iff extensional_def fun_eq_iff)
          with ‹l ≠ k› show False
            by (simp add: l k)
        qed
        then obtain i where *: i ∈ Basis f i ≠ g i ..
        then have f i = (a, c) ∨ f i = (c, d) ∨ f i = (d, b)
                  g i = (a, c) ∨ g i = (c, d) ∨ g i = (d, b)
          using f g by (auto simp: PiE_iff)
        with * ord[of i] show interior l ∩ interior k = {}
          by (auto simp add: l k disjoint_interval intro!: bexI[of _ i])
      }
```

```
      note ‹k ⊆ cbox a b›
    }
    moreover
    {
      fix x assume x: x ∈ cbox a b
      have ∀i∈Basis. ∃l. x · i ∈ {fst l · i .. snd l · i} ∧ l ∈ {(a, c), (c, d), (d, b)}
      proof
        fix i :: 'a
        assume i ∈ Basis
        with x ord[of i]
        have (a · i ≤ x · i ∧ x · i ≤ c · i) ∨ (c · i ≤ x · i ∧ x · i ≤ d · i) ∨
            (d · i ≤ x · i ∧ x · i ≤ b · i)
          by (auto simp: cbox_def)
        then show ∃l. x · i ∈ {fst l · i .. snd l · i} ∧ l ∈ {(a, c), (c, d), (d, b)}
          by auto
      qed
      then obtain f where
        f: ∀i∈Basis. x · i ∈ {fst (f i) · i..snd (f i) · i} ∧ f i ∈ {(a, c), (c, d), (d,
b)}
        unfolding bchoice_iff ..
      moreover from f have restrict f Basis ∈ Basis →E {(a, c), (c, d), (d, b)}
        by auto
      moreover from f have x ∈ ?B (restrict f Basis)
        by (auto simp: mem_box)
      ultimately have ∃k∈p. x ∈ k
        unfolding p_def by blast
    }
    ultimately show ⋃p = cbox a b
      by auto
  qed
qed

proposition partial_division_extend_interval:
  assumes p division_of (⋃p) (⋃p) ⊆ cbox a b
  obtains q where p ⊆ q q division_of cbox a (b::'a::euclidean_space)
proof (cases p = {})
  case True
  obtain q where q division_of (cbox a b)
    by (rule elementary_interval)
  then show ?thesis
    using True that by blast
next
  case False
  note p = division_ofD[OF assms(1)]
  have div_cbox: ∀k∈p. ∃q. q division_of cbox a b ∧ k ∈ q
  proof
    fix k
    assume kp: k ∈ p
    obtain c d where k: k = cbox c d
```

using *p*(*4*)[*OF kp*] **by** *blast*
have ∗: *cbox c d* ⊆ *cbox a b cbox c d* ≠ {}
using *p*(*2*,*3*)[*OF kp*, *unfolded k*] **using** *assms*(*2*)
by (*blast intro*: *order.trans*)+
obtain *q* **where** *q division_of cbox a b cbox c d* ∈ *q*
by (*rule partial_division_extend_1*[*OF* ∗])
then show ∃ *q*. *q division_of cbox a b* ∧ *k* ∈ *q*
unfolding *k* **by** *auto*
qed
obtain *q* **where** *q*: ⋀*x*. *x* ∈ *p* ⟹ *q x division_of cbox a b* ⋀*x*. *x* ∈ *p* ⟹ *x* ∈ *q x*
using *bchoice*[*OF div_cbox*] **by** *blast*
have *q x division_of* ⋃(*q x*) **if** *x*: *x* ∈ *p* **for** *x*
apply (*rule division_ofI*)
using *division_ofD*[*OF q*(*1*)[*OF x*]] **by** *auto*
then have *di*: ⋀*x*. *x* ∈ *p* ⟹ ∃ *d*. *d division_of* ⋃(*q x* − {*x*})
by (*meson Diff_subset division_of_subset*)
have ∃ *d*. *d division_of* ⋂((λ*i*. ⋃(*q i* − {*i*})) ‘ *p*)
apply (*rule elementary_Inter* [*OF finite_imageI*[*OF p*(*1*)]])
apply (*auto dest*: *di simp*: *False elementary_Inter* [*OF finite_imageI*[*OF p*(*1*)]])
done
then obtain *d* **where** *d*: *d division_of* ⋂((λ*i*. ⋃(*q i* − {*i*})) ‘ *p*) **..**
have *d* ∪ *p division_of cbox a b*
proof −
have *te*: ⋀*S f T*. *S* ≠ {} ⟹ ∀ *i*∈*S*. *f i* ∪ *i* = *T* ⟹ *T* = ⋂(*f* ‘ *S*) ∪ ⋃*S* **by** *auto*
have *cbox_eq*: *cbox a b* = ⋂((λ*i*. ⋃(*q i* − {*i*})) ‘ *p*) ∪ ⋃*p*
proof (*rule te*[*OF False*], *clarify*)
fix *i*
assume *i*: *i* ∈ *p*
show ⋃(*q i* − {*i*}) ∪ *i* = *cbox a b*
using *division_ofD*(*6*)[*OF q*(*1*)[*OF i*]] **using** *q*(*2*)[*OF i*] **by** *auto*
qed
{ fix *K*
assume *K*: *K* ∈ *p*
note *qk*=*division_ofD*[*OF q*(*1*)[*OF K*]]
have ∗: ⋀*u T S*. *T* ∩ *S* = {} ⟹ *u* ⊆ *S* ⟹ *u* ∩ *T* = {}
by *auto*
have *interior* (⋂*i*∈*p*. ⋃(*q i* − {*i*})) ∩ *interior K* = {}
proof (*rule* ∗[*OF Int_interior_Union_intervals*])
show ⋀*T*. *T*∈*q K* − {*K*} ⟹ *interior K* ∩ *interior T* = {}
using *qk*(*5*) **using** *q*(*2*)[*OF K*] **by** *auto*
show *interior* (⋂*i*∈*p*. ⋃(*q i* − {*i*})) ⊆ *interior* (⋃(*q K* − {*K*}))
apply (*rule interior_mono*)+
using *K* **by** *auto*
qed (*use qk* **in** *auto*)**}** **note** [*simp*] = *this*
show *d* ∪ *p division_of* (*cbox a b*)
unfolding *cbox_eq*
apply (*rule division_disjoint_union*[*OF d assms*(*1*)])

apply (*rule Int_interior_Union_intervals*)
apply (*rule p open_interior ballI*)+
apply *simp_all*
done
qed
then show *?thesis*
by (*meson Un_upper2 that*)
qed

lemma *elementary_bounded*[*dest*]:
fixes *S* :: *'a*::*euclidean_space set*
shows *p division_of S* $\Longrightarrow$ *bounded S*
unfolding *division_of_def* **by** (*metis bounded_Union bounded_cbox*)

lemma *elementary_subset_cbox*:
p division_of S $\Longrightarrow$ $\exists$ *a b*. *S* $\subseteq$ *cbox a* (*b*::*'a*::*euclidean_space*)
by (*meson bounded_subset_cbox_symmetric elementary_bounded*)

proposition *division_union_intervals_exists*:
fixes *a b* :: *'a*::*euclidean_space*
assumes *cbox a b* $\neq$ {}
obtains *p* **where** (*insert* (*cbox a b*) *p*) *division_of* (*cbox a b* $\cup$ *cbox c d*)
proof (*cases cbox c d* = {})
case *True*
with *assms that* **show** *?thesis* **by** *force*
next
case *False*
show *?thesis*
proof (*cases cbox a b* $\cap$ *cbox c d* = {})
case *True*
then show *?thesis*
by (*metis that False assms division_disjoint_union division_of_self insert_is_Un interior_Int interior_empty*)
next
case *False*
obtain *u v* **where** *uv*: *cbox a b* $\cap$ *cbox c d* = *cbox u v*
unfolding *Int_interval* **by** *auto*
have *uv_sub*: *cbox u v* $\subseteq$ *cbox c d* **using** *uv* **by** *auto*
obtain *p* **where** *pd*: *p division_of cbox c d* **and** *cbox u v* $\in$ *p*
by (*rule partial_division_extend_1*[*OF uv_sub False*[*unfolded uv*]])
note *p* = *this division_ofD*[*OF pd*]
have *interior* (*cbox a b* $\cap$ $\bigcup$(*p* − {*cbox u v*})) = *interior*(*cbox u v* $\cap$ $\bigcup$(*p* − {*cbox u v*}))
apply (*rule arg_cong*[*of _ _ interior*])
using *p*(*8*) *uv* **by** *auto*
also have ... = {}
unfolding *interior_Int*
apply (*rule Int_interior_Union_intervals*)
using *p*(*6*) *p*(*7*)[*OF p*(*2*)] ‹*finite p*›

apply *auto*
done
finally have [*simp*]: *interior* (*cbox a b*) ∩ *interior* (⋃(*p* − {*cbox u v*})) = {}
by *simp*
have *cbe*: *cbox a b* ∪ *cbox c d* = *cbox a b* ∪ ⋃(*p* − {*cbox u v*})
using *p*(*8*) **unfolding** *uv*[*symmetric*] **by** *auto*
have *insert* (*cbox a b*) (*p* − {*cbox u v*}) *division_of cbox a b* ∪ ⋃(*p* − {*cbox u v*})
proof −
have {*cbox a b*} *division_of cbox a b*
by (*simp add*: *assms division_of_self*)
then show *insert* (*cbox a b*) (*p* − {*cbox u v*}) *division_of cbox a b* ∪ ⋃(*p* − {*cbox u v*})
by (*metis* (*no_types*) *Diff_subset* ‹*interior* (*cbox a b*) ∩ *interior* (⋃(*p* − {*cbox u v*})) = {}› *division_disjoint_union division_of_subset insert_is_Un p*(*1*) *p*(*8*))
qed
with *that*[*of p* − {*cbox u v*}] **show** *?thesis* **by** (*simp add*: *cbe*)
qed
qed

lemma *division_of_Union*:
assumes *finite f*
and ⋀*p*. *p* ∈ *f* ⟹ *p division_of* (⋃*p*)
and ⋀*k1 k2*. *k1* ∈ ⋃*f* ⟹ *k2* ∈ ⋃*f* ⟹ *k1* ≠ *k2* ⟹ *interior k1* ∩ *interior k2* = {}
shows ⋃*f division_of* ⋃(⋃*f*)
using *assms* **by** (*auto intro*!: *division_ofI*)

lemma *elementary_union_interval*:
fixes *a b* :: ′*a*::*euclidean_space*
assumes *p division_of* ⋃*p*
obtains *q* **where** *q division_of* (*cbox a b* ∪ ⋃*p*)
proof (*cases p* = {} ∨ *cbox a b* = {})
case *True*
obtain *p* **where** *p division_of* (*cbox a b*)
by (*rule elementary_interval*)
then show *thesis*
using *True assms that* **by** *auto*
next
case *False*
then have *p* ≠ {} *cbox a b* ≠ {}
by *auto*
note *pdiv* = *division_ofD*[*OF assms*]
show *?thesis*
proof (*cases interior* (*cbox a b*) = {})
case *True*
show *?thesis*
apply (*rule that* [*of insert* (*cbox a b*) *p*, *OF division_ofI*])
using *pdiv*(*1*−*4*) *True False* **apply** *auto*

```
      apply (metis IntI equals0D pdiv(5))
     by (metis disjoint_iff_not_equal pdiv(5))
 next
   case False
   have ∀K∈p. ∃q. (insert (cbox a b) q) division_of (cbox a b ∪ K)
   proof
     fix K
     assume kp: K ∈ p
     from pdiv(4)[OF kp] obtain c d where K = cbox c d by blast
     then show ∃q. (insert (cbox a b) q) division_of (cbox a b ∪ K)
       by (meson ‹cbox a b ≠ {}› division_union_intervals_exists)
   qed
  from bchoice[OF this] obtain q where ∀x∈p. insert (cbox a b) (q x) division_of
(cbox a b) ∪ x ..
   note q = division_ofD[OF this[rule_format]]
   let ?D = ⋃{insert (cbox a b) (q K) | K. K ∈ p}
   show thesis
   proof (rule that[OF division_ofI])
    have *: {insert (cbox a b) (q K) |K. K ∈ p} = (λK. insert (cbox a b) (q K))
` p
       by auto
     show finite ?D
       using * pdiv(1) q(1) by auto
     have ⋃ ?D = (⋃x∈p. ⋃(insert (cbox a b) (q x)))
       by auto
     also have ... = ⋃{cbox a b ∪ t |t. t ∈ p}
       using q(6) by auto
     also have ... = cbox a b ∪ ⋃p
       using ‹p ≠ {}› by auto
     finally show ⋃ ?D = cbox a b ∪ ⋃p .
     show K ⊆ cbox a b ∪ ⋃p K ≠ {} if K ∈ ?D for K
       using q that by blast+
     show ∃a b. K = cbox a b if K ∈ ?D for K
       using q(4) that by auto
   next
     fix K' K
     assume K: K ∈ ?D and K': K' ∈ ?D K ≠ K'
     obtain x where x: K ∈ insert (cbox a b) (q x) x∈p
       using K by auto
     obtain x' where x': K'∈insert (cbox a b) (q x') x'∈p
       using K' by auto
     show interior K ∩ interior K' = {}
     proof (cases x = x' ∨ K  = cbox a b ∨ K' = cbox a b)
       case True
       then show ?thesis
         using True K' q(5) x' x by auto
     next
       case False
       then have as': K ≠ cbox a b K' ≠ cbox a b
```

```
        by auto
      obtain c d where K: K = cbox c d
        using q(4) x by blast
      have interior K ∩ interior (cbox a b) = {}
        using as' K'(2) q(5) x by blast
      then have interior K ⊆ interior x
      by (metis ‹interior (cbox a b) ≠ {}› K q(2) x interior_subset_union_intervals)
      moreover
      obtain c d where c_d: K' = cbox c d
        using q(4)[OF x'(2,1)] by blast
      have interior K' ∩ interior (cbox a b) = {}
        using as'(2) q(5) x' by blast
      then have interior K' ⊆ interior x'
          by (metis ‹interior (cbox a b) ≠ {}› c_d interior_subset_union_intervals
q(2) x'(1) x'(2))
      moreover have interior x ∩ interior x' = {}
        by (meson False assms division_ofD(5) x'(2) x(2))
      ultimately show ?thesis
        using ‹interior K ⊆ interior x› ‹interior K' ⊆ interior x'› by auto
    qed
  qed
 qed
qed


lemma elementary_unions_intervals:
 assumes fin: finite f
   and ⋀s. s ∈ f ⟹ ∃ a b. s = cbox a (b::'a::euclidean_space)
 obtains p where p division_of (⋃f)
proof −
 have ∃p. p division_of (⋃f)
 proof (induct_tac f rule:finite_subset_induct)
   show ∃p. p division_of ⋃{} using elementary_empty by auto
 next
   fix x F
   assume as: finite F x ∉ F ∃p. p division_of ⋃F x∈f
   from this(3) obtain p where p: p division_of ⋃F ..
   from assms(2)[OF as(4)] obtain a b where x: x = cbox a b by blast
   have *: ⋃F = ⋃p
     using division_ofD[OF p] by auto
   show ∃p. p division_of ⋃(insert x F)
     using elementary_union_interval[OF p[unfolded *], of a b]
     unfolding Union_insert x * by metis
 qed (insert assms, auto)
 then show ?thesis
   using that by auto
qed
```

```
lemma elementary_union:
  fixes S T :: 'a::euclidean_space set
  assumes ps division_of S pt division_of T
  obtains p where p division_of (S ∪ T)
proof –
  have *: S ∪ T = ⋃ps ∪ ⋃pt
    using assms unfolding division_of_def by auto
  show ?thesis
    apply (rule elementary_unions_intervals[of ps ∪ pt])
    using assms apply auto
    by (simp add: * that)
qed

lemma partial_division_extend:
  fixes T :: 'a::euclidean_space set
  assumes p division_of S
    and q division_of T
    and S ⊆ T
  obtains r where p ⊆ r and r division_of T
proof –
  note divp = division_ofD[OF assms(1)] and divq = division_ofD[OF assms(2)]
  obtain a b where ab: T ⊆ cbox a b
    using elementary_subset_cbox[OF assms(2)] by auto
  obtain r1 where p ⊆ r1 r1 division_of (cbox a b)
    using assms
    by (metis ab dual_order.trans partial_division_extend_interval divp(6))
  note r1 = this division_ofD[OF this(2)]
  obtain p' where p' division_of ⋃(r1 – p)
    apply (rule elementary_unions_intervals[of r1 – p])
    using r1(3,6)
      apply auto
    done
  then obtain r2 where r2: r2 division_of (⋃(r1 – p)) ∩ (⋃q)
    by (metis assms(2) divq(6) elementary_Int)
  {
    fix x
    assume x: x ∈ T x ∉ S
    then obtain R where r: R ∈ r1 x ∈ R
      unfolding r1 using ab
      by (meson division_contains r1(2) subsetCE)
    moreover have R ∉ p
    proof
      assume R ∈ p
      then have x ∈ S using divp(2) r by auto
      then show False using x by auto
    qed
    ultimately have x∈⋃(r1 – p) by auto
  }
  then have Teq: T = ⋃p ∪ (⋃(r1 – p) ∩ ⋃q)
```

unfolding *divp divq* **using** *assms*(*3*) **by** *auto*
have *interior S* ∩ *interior* (⋃(*r1*−*p*)) = {}
proof (*rule Int_interior_Union_intervals*)
have ∗: ⋀*S*. (⋀*x*. *x* ∈ *S* ⟹ *False*) ⟹ *S* = {}
by *auto*
show *interior S* ∩ *interior m* = {} **if** *m* ∈ *r1* − *p* **for** *m*
proof −
have *interior m* ∩ *interior* (⋃*p*) = {}
proof (*rule Int_interior_Union_intervals*)
show ⋀*T*. *T*∈*p* ⟹ *interior m* ∩ *interior T* = {}
by (*metis DiffD1 DiffD2 that r1*(*1*) *r1*(*7*) *rev_subsetD*)
qed (*use divp* **in** *auto*)
then show *interior S* ∩ *interior m* = {}
unfolding *divp* **by** *auto*
qed
qed (*use r1* **in** *auto*)
then have *interior S* ∩ *interior* (⋃(*r1*−*p*) ∩ (⋃*q*)) = {}
using *interior_subset* **by** *auto*
then have *div*: *p* ∪ *r2 division_of* ⋃*p* ∪ ⋃(*r1* − *p*) ∩ ⋃*q*
by (*simp add*: *assms*(*1*) *division_disjoint_union divp*(*6*) *r2*)
show *?thesis*
apply (*rule that*[*of p* ∪ *r2*])
apply (*auto simp*: *div Teq*)
done
qed

lemma *division_split*:
fixes *a* :: *'a*::*euclidean_space*
assumes *p division_of* (*cbox a b*)
and *k*: *k*∈*Basis*
shows {*l* ∩ {*x*. *x*•*k* ≤ *c*} | *l*. *l* ∈ *p* ∧ *l* ∩ {*x*. *x*•*k* ≤ *c*} ≠ {}} *division_of*(*cbox a b* ∩ {*x*. *x*•*k* ≤ *c*})
(**is** *?p1 division_of ?I1*)
and {*l* ∩ {*x*. *x*•*k* ≥ *c*} | *l*. *l* ∈ *p* ∧ *l* ∩ {*x*. *x*•*k* ≥ *c*} ≠ {}} *division_of* (*cbox a b* ∩ {*x*. *x*•*k* ≥ *c*})
(**is** *?p2 division_of ?I2*)
proof (*rule_tac*[!] *division_ofI*)
note *p* = *division_ofD*[*OF assms*(*1*)]
show *finite ?p1 finite ?p2*
using *p*(*1*) **by** *auto*
show ⋃ *?p1* = *?I1* ⋃ *?p2* = *?I2*
unfolding *p*(*6*)[*symmetric*] **by** *auto*
{
fix *K*
assume *K* ∈ *?p1*
then obtain *l* **where** *l*: *K* = *l* ∩ {*x*. *x* • *k* ≤ *c*} *l* ∈ *p l* ∩ {*x*. *x* • *k* ≤ *c*} ≠ {}
by *blast*
obtain *u v* **where** *uv*: *l* = *cbox u v*

```
      using assms(1) l(2) by blast
    show K ⊆ ?I1
      using l p(2) uv by force
    show  K ≠ {}
      by (simp add: l)
    show  ∃ a b. K = cbox a b
      apply (simp add: l uv p(2−3)[OF l(2)])
      apply (subst interval_split[OF k])
      apply (auto intro: order.trans)
      done
    fix K'
    assume K' ∈ ?p1
    then obtain l' where l': K' = l' ∩ {x. x · k ≤ c} l' ∈ p l' ∩ {x. x · k ≤ c}
≠ {}
      by blast
    assume K ≠ K'
    then show interior K ∩ interior K' = {}
      unfolding l l' using p(5)[OF l(2) l'(2)] by auto
  }
  {
    fix K
    assume K ∈ ?p2
    then obtain l where l: K = l ∩ {x. c ≤ x · k} l ∈ p l ∩ {x. c ≤ x · k} ≠ {}
      by blast
    obtain u v where uv: l = cbox u v
      using l(2) p(4) by blast
    show K ⊆ ?I2
      using l p(2) uv by force
    show  K ≠ {}
      by (simp add: l)
    show  ∃ a b. K = cbox a b
      apply (simp add: l uv p(2−3)[OF l(2)])
      apply (subst interval_split[OF k])
      apply (auto intro: order.trans)
      done
    fix K'
    assume K' ∈ ?p2
    then obtain l' where l': K' = l' ∩ {x. c ≤ x · k} l' ∈ p l' ∩ {x. c ≤ x · k}
≠ {}
      by blast
    assume K ≠ K'
    then show interior K ∩ interior K' = {}
      unfolding l l' using p(5)[OF l(2) l'(2)] by auto
  }
qed
```

6.14.7 Tagged (partial) divisions

definition *tagged_partial_division_of* (**infixr** *tagged'_partial'_division'_of 40*)

where *s tagged_partial_division_of i* $\longleftrightarrow$
finite s $\wedge$
$(\forall x\ K.\ (x,\ K) \in s \longrightarrow x \in K \wedge K \subseteq i \wedge (\exists a\ b.\ K = cbox\ a\ b)) \wedge$
$(\forall x1\ K1\ x2\ K2.\ (x1,\ K1) \in s \wedge (x2,\ K2) \in s \wedge (x1,\ K1) \neq (x2,\ K2) \longrightarrow$
$interior\ K1 \cap interior\ K2 = \{\})$

lemma *tagged_partial_division_ofD*:
assumes *s tagged_partial_division_of i*
shows *finite s*
and $\bigwedge x\ K.\ (x,K) \in s \Longrightarrow x \in K$
and $\bigwedge x\ K.\ (x,K) \in s \Longrightarrow K \subseteq i$
and $\bigwedge x\ K.\ (x,K) \in s \Longrightarrow \exists a\ b.\ K = cbox\ a\ b$
and $\bigwedge x1\ K1\ x2\ K2.\ (x1,K1) \in s \Longrightarrow$
$(x2,\ K2) \in s \Longrightarrow (x1,\ K1) \neq (x2,\ K2) \Longrightarrow interior\ K1 \cap interior\ K2 = \{\}$
using *assms* **unfolding** *tagged_partial_division_of_def* **by** *blast+*

definition *tagged_division_of* (**infixr** *tagged'_division'_of 40*)
where *s tagged_division_of i* $\longleftrightarrow$ *s tagged_partial_division_of i* $\wedge$ $(\bigcup\{K.\ \exists x.\ (x,K) \in s\} = i)$

lemma *tagged_division_of_finite*: *s tagged_division_of i* $\Longrightarrow$ *finite s*
unfolding *tagged_division_of_def tagged_partial_division_of_def* **by** *auto*

lemma *tagged_division_of*:
s tagged_division_of i $\longleftrightarrow$
finite s $\wedge$
$(\forall x\ K.\ (x,\ K) \in s \longrightarrow x \in K \wedge K \subseteq i \wedge (\exists a\ b.\ K = cbox\ a\ b)) \wedge$
$(\forall x1\ K1\ x2\ K2.\ (x1,\ K1) \in s \wedge (x2,\ K2) \in s \wedge (x1,\ K1) \neq (x2,\ K2) \longrightarrow$
$interior\ K1 \cap interior\ K2 = \{\}) \wedge$
$(\bigcup\{K.\ \exists x.\ (x,K) \in s\} = i)$
unfolding *tagged_division_of_def tagged_partial_division_of_def* **by** *auto*

lemma *tagged_division_ofI*:
assumes *finite s*
and $\bigwedge x\ K.\ (x,K) \in s \Longrightarrow x \in K$
and $\bigwedge x\ K.\ (x,K) \in s \Longrightarrow K \subseteq i$
and $\bigwedge x\ K.\ (x,K) \in s \Longrightarrow \exists a\ b.\ K = cbox\ a\ b$
and $\bigwedge x1\ K1\ x2\ K2.\ (x1,K1) \in s \Longrightarrow (x2,\ K2) \in s \Longrightarrow (x1,\ K1) \neq (x2,\ K2) \Longrightarrow$
$interior\ K1 \cap interior\ K2 = \{\}$
and $(\bigcup\{K.\ \exists x.\ (x,K) \in s\} = i)$
shows *s tagged_division_of i*
unfolding *tagged_division_of*
using *assms* **by** *fastforce*

lemma *tagged_division_ofD*[*dest*]:
assumes *s tagged_division_of i*
shows *finite s*
and $\bigwedge x\ K.\ (x,K) \in s \Longrightarrow x \in K$

```
    and ⋀x K. (x,K) ∈ s ⟹ K ⊆ i
    and ⋀x K. (x,K) ∈ s ⟹ ∃ a b. K = cbox a b
    and ⋀x1 K1 x2 K2. (x1, K1) ∈ s ⟹ (x2, K2) ∈ s ⟹ (x1, K1) ≠ (x2, K2)
⟹
      interior K1 ∩ interior K2 = {}
    and (⋃{K. ∃x. (x,K) ∈ s} = i)
  using assms unfolding tagged_division_of by blast+

lemma division_of_tagged_division:
  assumes s tagged_division_of i
  shows (snd ` s) division_of i
proof (rule division_ofI)
  note assm = tagged_division_ofD[OF assms]
  show ⋃(snd ` s) = i finite (snd ` s)
    using assm by auto
  fix k
  assume k: k ∈ snd ` s
  then obtain xk where xk: (xk, k) ∈ s
    by auto
  then show k ⊆ i k ≠ {} ∃ a b. k = cbox a b
    using assm by fastforce+
  fix k'
  assume k': k' ∈ snd ` s k ≠ k'
  from this(1) obtain xk' where xk': (xk', k') ∈ s
    by auto
  then show interior k ∩ interior k' = {}
    using assm(5) k'(2) xk by blast
qed

lemma partial_division_of_tagged_division:
  assumes s tagged_partial_division_of i
  shows (snd ` s) division_of ⋃(snd ` s)
proof (rule division_ofI)
  note assm = tagged_partial_division_ofD[OF assms]
  show finite (snd ` s) ⋃(snd ` s) = ⋃(snd ` s)
    using assm by auto
  fix k
  assume k: k ∈ snd ` s
  then obtain xk where xk: (xk, k) ∈ s
    by auto
  then show k ≠ {} ∃ a b. k = cbox a b k ⊆ ⋃(snd ` s)
    using assm by auto
  fix k'
  assume k': k' ∈ snd ` s k ≠ k'
  from this(1) obtain xk' where xk': (xk', k') ∈ s
    by auto
  then show interior k ∩ interior k' = {}
    using assm(5) k'(2) xk by auto
qed
```

lemma *tagged_partial_division_subset*:
  assumes *s tagged_partial_division_of i*
    and $t \subseteq s$
  shows *t tagged_partial_division_of i*
  using *assms finite_subset*[*OF assms*(*2*)]
  unfolding *tagged_partial_division_of_def*
  by *blast*

lemma *tag_in_interval*: *p tagged_division_of i* $\Longrightarrow$ $(x, k) \in p \Longrightarrow x \in i$
  by *auto*

lemma *tagged_division_of_empty*: {} *tagged_division_of* {}
  unfolding *tagged_division_of* **by** *auto*

lemma *tagged_partial_division_of_trivial*[*simp*]: *p tagged_partial_division_of* {} $\longleftrightarrow$ $p = \{\}$
  unfolding *tagged_partial_division_of_def* **by** *auto*

lemma *tagged_division_of_trivial*[*simp*]: *p tagged_division_of* {} $\longleftrightarrow$ $p = \{\}$
  unfolding *tagged_division_of* **by** *auto*

lemma *tagged_division_of_self*: $x \in$ *cbox a b* $\Longrightarrow$ {(*x*,*cbox a b*)} *tagged_division_of* (*cbox a b*)
  by (*rule tagged_division_ofI*) *auto*

lemma *tagged_division_of_self_real*: $x \in \{a .. b::real\} \Longrightarrow \{(x,\{a .. b\})\}$ *tagged_division_of* $\{a .. b\}$
  unfolding *box_real*[*symmetric*]
  by (*rule tagged_division_of_self*)

lemma *tagged_division_Un*:
  assumes *p1 tagged_division_of s1*
    and *p2 tagged_division_of s2*
    and *interior s1* $\cap$ *interior s2* = {}
  shows (*p1* $\cup$ *p2*) *tagged_division_of* (*s1* $\cup$ *s2*)
proof (*rule tagged_division_ofI*)
  note *p1* = *tagged_division_ofD*[*OF assms*(*1*)]
  note *p2* = *tagged_division_ofD*[*OF assms*(*2*)]
  show *finite* (*p1* $\cup$ *p2*)
    using *p1*(*1*) *p2*(*1*) **by** *auto*
  show $\bigcup\{k.\ \exists x.\ (x, k) \in p1 \cup p2\} = s1 \cup s2$
    using *p1*(*6*) *p2*(*6*) **by** *blast*
  fix *x k*
  assume *xk*: $(x, k) \in p1 \cup p2$
  show $x \in k$ $\exists a\ b.\ k = cbox\ a\ b$
    using *xk p1*(*2*,*4*) *p2*(*2*,*4*) **by** *auto*
  show $k \subseteq s1 \cup s2$
    using *xk p1*(*3*) *p2*(*3*) **by** *blast*

fix x' k'
assume *xk'*: $(x', k') \in p1 \cup p2$ $(x, k) \neq (x', k')$
have ∗: $\bigwedge a\ b.\ a \subseteq s1 \Longrightarrow b \subseteq s2 \Longrightarrow$ *interior* $a \cap$ *interior* $b = \{\}$
using *assms*(*3*) *interior_mono* **by** *blast*
show *interior* $k \cap$ *interior* $k' = \{\}$
apply (*cases* $(x, k) \in p1$)
apply (*meson* ∗ *UnE assms*(*1*) *assms*(*2*) *p1*(*5*) *tagged_division_ofD*(*3*) *xk'*(*1*) *xk'*(*2*))
by (*metis* ∗ *UnE assms*(*1*) *assms*(*2*) *inf_sup_aci*(*1*) *p2*(*5*) *tagged_division_ofD*(*3*) *xk xk'*(*1*) *xk'*(*2*))
qed

lemma *tagged_division_Union*:
assumes *finite I*
and *tag*: $\bigwedge i.\ i \in I \Longrightarrow$ *pfn i tagged_division_of i*
and *disj*: $\bigwedge i1\ i2.\ [\![i1 \in I;\ i2 \in I;\ i1 \neq i2]\!] \Longrightarrow$ *interior*(*i1*) $\cap$ *interior*(*i2*) = {}
shows $\bigcup$(*pfn* ‘ *I*) *tagged_division_of* ($\bigcup$*I*)
proof (*rule tagged_division_ofI*)
note *assm* = *tagged_division_ofD*[*OF tag*]
show *finite* ($\bigcup$(*pfn* ‘ *I*))
using *assms* **by** *auto*
have $\bigcup\{k.\ \exists x.\ (x, k) \in \bigcup(pfn\ `\ I)\} = \bigcup((\lambda i.\ \bigcup\{k.\ \exists x.\ (x, k) \in pfn\ i\})\ `\ I)$
by *blast*
also have ... = $\bigcup I$
using *assm*(*6*) **by** *auto*
finally show $\bigcup\{k.\ \exists x.\ (x, k) \in \bigcup(pfn\ `\ I)\} = \bigcup I$ **.**
fix *x k*
assume *xk*: $(x, k) \in \bigcup(pfn\ `\ I)$
then obtain *i* **where** *i*: $i \in I$ $(x, k) \in pfn\ i$
by *auto*
show $x \in k$ $\exists a\ b.\ k = cbox\ a\ b$ $k \subseteq \bigcup I$
using *assm*(*2*−*4*)[*OF i*] **using** *i*(*1*) **by** *auto*
fix x' k'
assume *xk'*: $(x', k') \in \bigcup(pfn\ `\ I)$ $(x, k) \neq (x', k')$
then obtain i' **where** i': $i' \in I$ $(x', k') \in pfn\ i'$
by *auto*
have ∗: $\bigwedge a\ b.\ i \neq i' \Longrightarrow a \subseteq i \Longrightarrow b \subseteq i' \Longrightarrow$ *interior* $a \cap$ *interior* $b = \{\}$
using *i*(*1*) *i'*(*1*) *disj interior_mono* **by** *blast*
show *interior* $k \cap$ *interior* $k' = \{\}$
proof (*cases* $i = i'$)
case *True* **then show** *?thesis*
using *assm*(*5*) *i' i xk'*(*2*) **by** *blast*
next
case *False* **then show** *?thesis*
using ∗ *assm*(*3*) *i' i* **by** *auto*
qed
qed

```
lemma tagged_partial_division_of_Union_self:
  assumes p tagged_partial_division_of s
  shows p tagged_division_of (⋃(snd ` p))
  apply (rule tagged_division_ofI)
  using tagged_partial_division_ofD[OF assms]
  apply auto
  done

lemma tagged_division_of_union_self:
  assumes p tagged_division_of s
  shows p tagged_division_of (⋃(snd ` p))
  apply (rule tagged_division_ofI)
  using tagged_division_ofD[OF assms]
  apply auto
  done

lemma tagged_division_Un_interval:
  fixes a :: 'a::euclidean_space
  assumes p1 tagged_division_of (cbox a b ∩ {x. x•k ≤ (c::real)})
    and p2 tagged_division_of (cbox a b ∩ {x. x•k ≥ c})
    and k: k ∈ Basis
  shows (p1 ∪ p2) tagged_division_of (cbox a b)
proof −
  have ∗: cbox a b = (cbox a b ∩ {x. x•k ≤ c}) ∪ (cbox a b ∩ {x. x•k ≥ c})
    by auto
  show ?thesis
    apply (subst ∗)
    apply (rule tagged_division_Un[OF assms(1−2)])
    unfolding interval_split[OF k] interior_cbox
    using k
    apply (auto simp add: box_def elim!: ballE[where x=k])
    done
qed

lemma tagged_division_Un_interval_real:
  fixes a :: real
  assumes p1 tagged_division_of ({a .. b} ∩ {x. x•k ≤ (c::real)})
    and p2 tagged_division_of ({a .. b} ∩ {x. x•k ≥ c})
    and k: k ∈ Basis
  shows (p1 ∪ p2) tagged_division_of {a .. b}
  using assms
  unfolding box_real[symmetric]
  by (rule tagged_division_Un_interval)

lemma tagged_division_split_left_inj:
  assumes d: d tagged_division_of i
  and tags: (x1, K1) ∈ d (x2, K2) ∈ d
  and K1 ≠ K2
  and eq: K1 ∩ {x. x · k ≤ c} = K2 ∩ {x. x · k ≤ c}
```

shows *interior* (*K1* ∩ {*x*. *x*•*k* ≤ *c*}) = {}
proof −
have *interior* (*K1* ∩ *K2*) = {} ∨ (*x2*, *K2*) = (*x1*, *K1*)
using *tags d* **by** (*metis* (*no_types*) *interior_Int tagged_division_ofD*(*5*))
then show *?thesis*
using *eq* ‹*K1* ≠ *K2*› **by** (*metis* (*no_types*) *inf_assoc inf_bot_left inf_left_idem interior_Int old.prod.inject*)
qed

lemma *tagged_division_split_right_inj*:
assumes *d*: *d tagged_division_of i*
and *tags*: (*x1*, *K1*) ∈ *d* (*x2*, *K2*) ∈ *d*
and *K1* ≠ *K2*
and *eq*: *K1* ∩ {*x*. *x*•*k* ≥ *c*} = *K2* ∩ {*x*. *x*•*k* ≥ *c*}
shows *interior* (*K1* ∩ {*x*. *x*•*k* ≥ *c*}) = {}
proof −
have *interior* (*K1* ∩ *K2*) = {} ∨ (*x2*, *K2*) = (*x1*, *K1*)
using *tags d* **by** (*metis* (*no_types*) *interior_Int tagged_division_ofD*(*5*))
then show *?thesis*
using *eq* ‹*K1* ≠ *K2*› **by** (*metis* (*no_types*) *inf_assoc inf_bot_left inf_left_idem interior_Int old.prod.inject*)
qed

lemma (**in** *comm_monoid_set*) *over_tagged_division_lemma*:
assumes *p tagged_division_of i*
and ⋀*u v*. *box u v* = {} ⟹ *d* (*cbox u v*) = **1**
shows *F* (λ(_, *k*). *d k*) *p* = *F d* (*snd* ‘ *p*)
proof −
have ∗: (λ(_ ,*k*). *d k*) = *d* ∘ *snd*
by (*simp add*: *fun_eq_iff*)
note *assm* = *tagged_division_ofD*[*OF assms*(*1*)]
show *?thesis*
unfolding ∗
proof (*rule reindex_nontrivial*[*symmetric*])
show *finite p*
using *assm* **by** *auto*
fix *x y*
assume *x*∈*p y*∈*p x*≠*y snd x* = *snd y*
obtain *a b* **where** *ab*: *snd x* = *cbox a b*
using *assm*(*4*)[*of fst x snd x*] ‹*x*∈*p*› **by** *auto*
have (*fst x*, *snd y*) ∈ *p* (*fst x*, *snd y*) ≠ *y*
using ‹*x* ∈ *p*› ‹*x* ≠ *y*› ‹*snd x* = *snd y*› [*symmetric*] **by** *auto*
with ‹*x*∈*p*› ‹*y*∈*p*› **have** *interior* (*snd x*) ∩ *interior* (*snd y*) = {}
by (*intro assm*(*5*)[*of fst x* _ *fst y*]) *auto*
then have *box a b* = {}
unfolding ‹*snd x* = *snd y*›[*symmetric*] *ab* **by** *auto*
then have *d* (*cbox a b*) = **1**
using *assm*(*2*)[*of fst x snd x*] ‹*x*∈*p*› *ab*[*symmetric*] **by** (*intro assms*(*2*)) *auto*
then show *d* (*snd x*) = **1**

```
    unfolding ab by auto
  qed
qed
```

6.14.8 Functions closed on boxes: morphisms from boxes to monoids

This auxiliary structure is used to sum up over the elements of a division. Main theorem is *operative_division*. Instances for the monoid are *'a option*, *real*, and *bool*.

Using additivity of lifted function to encode definedness.

```
definition lift_option :: ('a ⇒ 'b ⇒ 'c) ⇒ 'a option ⇒ 'b option ⇒ 'c option
where
  lift_option f a' b' = Option.bind a' (λa. Option.bind b' (λb. Some (f a b)))

lemma lift_option_simps[simp]:
  lift_option f (Some a) (Some b) = Some (f a b)
  lift_option f None b' = None
  lift_option f a' None = None
  by (auto simp: lift_option_def)

lemma comm_monoid_lift_option:
  assumes comm_monoid f z
  shows comm_monoid (lift_option f) (Some z)
proof −
  from assms interpret comm_monoid f z .
  show ?thesis
    by standard (auto simp: lift_option_def ac_simps split: bind_split)
qed

lemma comm_monoid_and: comm_monoid HOL.conj True
  by standard auto

lemma comm_monoid_set_and: comm_monoid_set HOL.conj True
  by (rule comm_monoid_set.intro) (fact comm_monoid_and)
```

Misc

```
lemma interval_real_split:
  {a .. b::real} ∩ {x. x ≤ c} = {a .. min b c}
  {a .. b} ∩ {x. c ≤ x} = {max a c .. b}
  apply (metis Int_atLeastAtMostL1 atMost_def)
  apply (metis Int_atLeastAtMostL2 atLeast_def)
  done

lemma bgauge_existence_lemma: (∀x∈s. ∃d::real. 0 < d ∧ q d x) ⟷ (∀x. ∃d>0.
x∈s ⟶ q d x)
  by (meson zero_less_one)
```

Division points **definition** *division_points* (*k*::(*'a*::*euclidean_space*) *set*) *d* =
{(*j*,*x*). *j* ∈ *Basis* ∧ (*interval_lowerbound k*)•*j* < *x* ∧ *x* < (*interval_upperbound k*)•*j* ∧
(∃ *i*∈*d*. (*interval_lowerbound i*)•*j* = *x* ∨ (*interval_upperbound i*)•*j* = *x*)}

lemma *division_points_finite*:
fixes *i* :: *'a*::*euclidean_space set*
assumes *d division_of i*
shows *finite* (*division_points i d*)
proof −
note *assm* = *division_ofD*[*OF assms*]
let *?M* = λ*j*. {(*j*,*x*)|*x*. (*interval_lowerbound i*)•*j* < *x* ∧ *x* < (*interval_upperbound i*)•*j* ∧
(∃ *i*∈*d*. (*interval_lowerbound i*)•*j* = *x* ∨ (*interval_upperbound i*)•*j* = *x*)}
have ∗: *division_points i d* = ⋃(*?M* ‘ *Basis*)
unfolding *division_points_def* **by** *auto*
show *?thesis*
unfolding ∗ **using** *assm* **by** *auto*
qed

lemma *division_points_subset*:
fixes *a* :: *'a*::*euclidean_space*
assumes *d division_of* (*cbox a b*)
and ∀ *i*∈*Basis*. *a*•*i* < *b*•*i* *a*•*k* < *c c* < *b*•*k*
and *k*: *k* ∈ *Basis*
shows *division_points* (*cbox a b* ∩ {*x*. *x*•*k* ≤ *c*}) {*l* ∩ {*x*. *x*•*k* ≤ *c*} | *l* . *l* ∈ *d* ∧ *l* ∩ {*x*. *x*•*k* ≤ *c*} ≠ {}} ⊆
division_points (*cbox a b*) *d* (**is** *?t1*)
and *division_points* (*cbox a b* ∩ {*x*. *x*•*k* ≥ *c*}) {*l* ∩ {*x*. *x*•*k* ≥ *c*} | *l* . *l* ∈ *d* ∧ ¬(*l* ∩ {*x*. *x*•*k* ≥ *c*} = {})} ⊆
division_points (*cbox a b*) *d* (**is** *?t2*)
proof −
note *assm* = *division_ofD*[*OF assms*(*1*)]
have ∗: ∀ *i*∈*Basis*. *a*•*i* ≤ *b*•*i*
∀ *i*∈*Basis*. *a*•*i* ≤ (∑ *i*∈*Basis*. (*if i* = *k then min* (*b* · *k*) *c else b* · *i*) ∗*R* *i*) · *i*
∀ *i*∈*Basis*. (∑ *i*∈*Basis*. (*if i* = *k then max* (*a* · *k*) *c else a* · *i*) ∗*R* *i*) · *i* ≤ *b*•*i*
min (*b* · *k*) *c* = *c max* (*a* · *k*) *c* = *c*
using *assms* **using** *less_imp_le* **by** *auto*
have ∃ *i*∈*d*. *interval_lowerbound i* · *x* = *y* ∨ *interval_upperbound i* · *x* = *y*
if *a* · *x* < *y y* < (*if x* = *k then c else b* · *x*)
interval_lowerbound i · *x* = *y* ∨ *interval_upperbound i* · *x* = *y*
i = *l* ∩ {*x*. *x* · *k* ≤ *c*} *l* ∈ *d l* ∩ {*x*. *x* · *k* ≤ *c*} ≠ {}
x ∈ *Basis* **for** *i l x y*
proof −
obtain *u v* **where** *l*: *l* = *cbox u v*
using ‹*l* ∈ *d*› *assms*(*1*) **by** *blast*
have ∗: ∀ *i*∈*Basis*. *u* · *i* ≤ (∑ *i*∈*Basis*. (*if i* = *k then min* (*v* · *k*) *c else v* · *i*) ∗*R* *i*) · *i*
using *that*(*6*) **unfolding** *l interval_split*[*OF k*] *box_ne_empty that* **.**

have ∗∗: ∀ *i*∈*Basis*. *u*·*i* ≤ *v*·*i*
using *l* **using** *that*(*6*) **unfolding** *box_ne_empty*[*symmetric*] **by** *auto*
show *?thesis*
apply (*rule bexI*[*OF* _ ‹*l* ∈ *d*›])
using *that*(*1*−*3*,*5*) ‹*x* ∈ *Basis*›
unfolding *l interval_bounds*[*OF* ∗∗] *interval_bounds*[*OF* ∗] *interval_split*[*OF* *k*] *that*
apply (*auto split*: *if_split_asm*)
done
qed
moreover have ⋀*x y*. ⟦*y* < (*if x* = *k then c else b* · *x*)⟧ ⟹ *y* < *b* · *x*
using ‹*c* < *b*·*k*› **by** (*auto split*: *if_split_asm*)
ultimately show *?t1*
unfolding *division_points_def interval_split*[*OF k*, *of a b*]
unfolding *interval_bounds*[*OF* ∗(*1*)] *interval_bounds*[*OF* ∗(*2*)] *interval_bounds*[*OF* ∗(*3*)]
unfolding ∗ **by** *force*
have ⋀*x y i l*. (*if x* = *k then c else a* · *x*) < *y* ⟹ *a* · *x* < *y*
using ‹*a*·*k* < *c*› **by** (*auto split*: *if_split_asm*)
moreover have ∃ *i*∈*d*. *interval_lowerbound i* · *x* = *y* ∨
interval_upperbound i · *x* = *y*
if (*if x* = *k then c else a* · *x*) < *y y* < *b* · *x*
interval_lowerbound i · *x* = *y* ∨ *interval_upperbound i* · *x* = *y*
i = *l* ∩ {*x*. *c* ≤ *x* · *k*} *l* ∈ *d l* ∩ {*x*. *c* ≤ *x* · *k*} ≠ {}
x ∈ *Basis* **for** *x y i l*
proof −
obtain *u v* **where** *l*: *l* = *cbox u v*
using ‹*l* ∈ *d*› *assm*(*4*) **by** *blast*
have ∗: ∀ *i*∈*Basis*. (∑ *i*∈*Basis*. (*if i* = *k then max* (*u* · *k*) *c else u* · *i*) ∗*R* *i*) · *i* ≤ *v* · *i*
using *that*(*6*) **unfolding** *l interval_split*[*OF k*] *box_ne_empty that* **.**
have ∗∗: ∀ *i*∈*Basis*. *u*·*i* ≤ *v*·*i*
using *l* **using** *that*(*6*) **unfolding** *box_ne_empty*[*symmetric*] **by** *auto*
show ∃ *i*∈*d*. *interval_lowerbound i* · *x* = *y* ∨ *interval_upperbound i* · *x* = *y*
apply (*rule bexI*[*OF* _ ‹*l* ∈ *d*›])
using *that*(*1*−*3*,*5*) ‹*x* ∈ *Basis*›
unfolding *l interval_bounds*[*OF* ∗∗] *interval_bounds*[*OF* ∗] *interval_split*[*OF* *k*] *that*
apply (*auto split*: *if_split_asm*)
done
qed
ultimately show *?t2*
unfolding *division_points_def interval_split*[*OF k*, *of a b*]
unfolding *interval_bounds*[*OF* ∗(*1*)] *interval_bounds*[*OF* ∗(*2*)] *interval_bounds*[*OF* ∗(*3*)]
unfolding ∗
by *force*
qed

lemma *division_points_psubset*:
  fixes *a* :: *'a::euclidean_space*
  assumes *d*: *d division_of* (*cbox a b*)
    and *altb*: ∀ *i*∈*Basis*. *a*•*i* < *b*•*i* *a*•*k* < *c c* < *b*•*k*
    and *l* ∈ *d*
    and *disj*: *interval_lowerbound l*•*k* = *c* ∨ *interval_upperbound l*•*k* = *c*
    and *k*: *k* ∈ *Basis*
  shows *division_points* (*cbox a b* ∩ {*x*. *x*•*k* ≤ *c*}) {*l* ∩ {*x*. *x*•*k* ≤ *c*} | *l*. *l*∈*d* ∧ *l* ∩ {*x*. *x*•*k* ≤ *c*} ≠ {}} ⊂
      division_points (*cbox a b*) *d* (**is** *?D1* ⊂ *?D*)
   and *division_points* (*cbox a b* ∩ {*x*. *x*•*k* ≥ *c*}) {*l* ∩ {*x*. *x*•*k* ≥ *c*} | *l*. *l*∈*d* ∧ *l* ∩ {*x*. *x*•*k* ≥ *c*} ≠ {}} ⊂
      division_points (*cbox a b*) *d* (**is** *?D2* ⊂ *?D*)
proof −
  have *ab*: ∀ *i*∈*Basis*. *a*•*i* ≤ *b*•*i*
    using *altb* **by** (*auto intro*!:*less_imp_le*)
  obtain *u v* **where** *l*: *l* = *cbox u v*
    using *d* ‹*l* ∈ *d*› **by** *blast*
  have *uv*: ∀ *i*∈*Basis*. *u*•*i* ≤ *v*•*i* ∀ *i*∈*Basis*. *a*•*i* ≤ *u*•*i* ∧ *v*•*i* ≤ *b*•*i*
    apply (*metis assms*(*5*) *box_ne_empty*(*1*) *cbox_division_memE d l*)
    by (*metis assms*(*5*) *box_ne_empty*(*1*) *cbox_division_memE d l subset_box*(*1*))
  have ∗: *interval_upperbound* (*cbox a b* ∩ {*x*. *x* · *k* ≤ *interval_upperbound l* · *k*}) · *k* = *interval_upperbound l* · *k*
      interval_upperbound (*cbox a b* ∩ {*x*. *x* · *k* ≤ *interval_lowerbound l* · *k*}) · *k* = *interval_lowerbound l* · *k*
    unfolding *l interval_split*[*OF k*] *interval_bounds*[*OF uv*(*1*)]
    using *uv*[*rule_format*, *of k*] *ab k*
    by *auto*
  have ∃ *x*. *x* ∈ *?D* − *?D1*
    using *assms*(*3*−)
    unfolding *division_points_def interval_bounds*[*OF ab*]
    by (*force simp add*: ∗)
  moreover have *?D1* ⊆ *?D*
    by (*auto simp add*: *assms division_points_subset*)
  ultimately show *?D1* ⊂ *?D*
    by *blast*
  have ∗: *interval_lowerbound* (*cbox a b* ∩ {*x*. *x* · *k* ≥ *interval_lowerbound l* · *k*}) · *k* = *interval_lowerbound l* · *k*
    interval_lowerbound (*cbox a b* ∩ {*x*. *x* · *k* ≥ *interval_upperbound l* · *k*}) · *k* = *interval_upperbound l* · *k*
    unfolding *l interval_split*[*OF k*] *interval_bounds*[*OF uv*(*1*)]
    using *uv*[*rule_format*, *of k*] *ab k*
    by *auto*
  have ∃ *x*. *x* ∈ *?D* − *?D2*
    using *assms*(*3*−)
    unfolding *division_points_def interval_bounds*[*OF ab*]
    by (*force simp add*: ∗)
  moreover have *?D2* ⊆ *?D*
    by (*auto simp add*: *assms division_points_subset*)

ultimately show *?D2* ⊂ *?D*
  by *blast*
qed

lemma *division_split_left_inj*:
  fixes *S* :: *'a::euclidean_space set*
  assumes *div*: *𝒟 division_of S*
    and *eq*: *K1* ∩ {*x::'a. x•k* ≤ *c*} = *K2* ∩ {*x. x•k* ≤ *c*}
    and *K1* ∈ *𝒟 K2* ∈ *𝒟 K1* ≠ *K2*
  shows *interior* (*K1* ∩ {*x. x•k* ≤ *c*}) = {}
proof −
  have *interior K2* ∩ *interior* {*a. a* • *k* ≤ *c*} = *interior K1* ∩ *interior* {*a. a* • *k* ≤ *c*}
    by (*metis* (*no_types*) *eq interior_Int*)
  moreover have ⋀*A. interior A* ∩ *interior K2* = {} ∨ *A* = *K2* ∨ *A* ∉ *𝒟*
    by (*meson div* ‹*K2* ∈ *𝒟*› *division_of_def*)
  ultimately show *?thesis*
    using ‹*K1* ∈ *𝒟*› ‹*K1* ≠ *K2*› **by** *auto*
qed

lemma *division_split_right_inj*:
  fixes *S* :: *'a::euclidean_space set*
  assumes *div*: *𝒟 division_of S*
    and *eq*: *K1* ∩ {*x::'a. x•k* ≥ *c*} = *K2* ∩ {*x. x•k* ≥ *c*}
    and *K1* ∈ *𝒟 K2* ∈ *𝒟 K1* ≠ *K2*
  shows *interior* (*K1* ∩ {*x. x•k* ≥ *c*}) = {}
proof −
  have *interior K2* ∩ *interior* {*a. a* • *k* ≥ *c*} = *interior K1* ∩ *interior* {*a. a* • *k* ≥ *c*}
    by (*metis* (*no_types*) *eq interior_Int*)
  moreover have ⋀*A. interior A* ∩ *interior K2* = {} ∨ *A* = *K2* ∨ *A* ∉ *𝒟*
    by (*meson div* ‹*K2* ∈ *𝒟*› *division_of_def*)
  ultimately show *?thesis*
    using ‹*K1* ∈ *𝒟*› ‹*K1* ≠ *K2*› **by** *auto*
qed

lemma *interval_doublesplit*:
  fixes *a* :: *'a::euclidean_space*
  assumes *k* ∈ *Basis*
  shows *cbox a b* ∩ {*x* . |*x•k* − *c*| ≤ (*e::real*)} =
    cbox (∑*i*∈*Basis*. (*if i* = *k then max* (*a•k*) (*c* − *e*) *else a•i*) $*_R$ *i*)
     (∑*i*∈*Basis*. (*if i* = *k then min* (*b•k*) (*c* + *e*) *else b•i*) $*_R$ *i*)
proof −
  have ∗: ⋀*x c e::real*. |*x* − *c*| ≤ *e* ⟷ *x* ≥ *c* − *e* ∧ *x* ≤ *c* + *e*
    by *auto*
  have ∗∗: ⋀*s P Q. s* ∩ {*x. P x* ∧ *Q x*} = (*s* ∩ {*x. Q x*}) ∩ {*x. P x*}
    by *blast*
  show *?thesis*
    unfolding ∗ ∗∗ *interval_split*[*OF assms*] **by** (*rule refl*)

qed

lemma *division_doublesplit*:
  fixes a :: $'a$::*euclidean_space*
  assumes p *division_of* (*cbox* a b)
    and k: $k \in$ *Basis*
  shows $(\lambda l.\ l \cap \{x.\ |x{\cdot}k - c| \le e\})$ ‘ $\{l{\in}p.\ l \cap \{x.\ |x{\cdot}k - c| \le e\} \ne \{\}\}$
      division_of (*cbox* a b $\cap$ $\{x.\ |x{\cdot}k - c| \le e\}$)
proof −
  have $*$: $\bigwedge x\ c.\ |x - c| \le e \longleftrightarrow x \ge c - e \wedge x \le c + e$
    by *auto*
  have $**$: $\bigwedge p\ q\ p'\ q'.\ p$ *division_of* $q \Longrightarrow p = p' \Longrightarrow q = q' \Longrightarrow p'$ *division_of* q'
    by *auto*
  note *division_split*(*1*)[*OF assms*, **where** c=c+e,*unfolded interval_split*[*OF k*]]
  note *division_split*(*2*)[*OF this*, **where** c=c−e **and** k=k,*OF k*]
  then show *?thesis*
    apply (*rule* $**$)
    subgoal
      apply (*simp add*: *abs_diff_le_iff field_simps Collect_conj_eq setcompr_eq_image*
[*symmetric*] *cong*: *image_cong_simp*)
      apply (*rule equalityI*)
      apply *blast*
      apply *clarsimp*
      apply (*rule_tac* x=$xa \cap \{x.\ c + e \ge x \cdot k\}$ **in** *exI*)
      apply *auto*
      done
    by (*simp add*: *interval_split k interval_doublesplit*)
qed

Operative

locale *operative* = *comm_monoid_set* +
  fixes g :: $'b$::*euclidean_space set* $\Rightarrow$ $'a$
  assumes *box_empty_imp*: $\bigwedge a\ b.$ *box* a b = {} $\Longrightarrow$ g (*cbox* a b) = **1**
    and *Basis_imp*: $\bigwedge a\ b\ c\ k.\ k \in$ *Basis* $\Longrightarrow g$ (*cbox* a b) = g (*cbox* a b $\cap$ $\{x.\ x{\cdot}k$
$\le c\}$) $*$ g (*cbox* a b $\cap$ $\{x.\ x{\cdot}k \ge c\}$)
begin

lemma *empty* [*simp*]:
  g {} = **1**
proof −
  have $*$: *cbox One* (−*One*) = ({}::$'b$ *set*)
   by (*auto simp*: *box_eq_empty inner_sum_left inner_Basis sum.If_cases ex_in_conv*)
  moreover have *box One* (−*One*) = ({}::$'b$ *set*)
    using *box_subset_cbox*[*of One* −*One*] **by** (*auto simp*: $*$)
  ultimately show *?thesis*
    using *box_empty_imp* [*of One* −*One*] **by** *simp*
qed

lemma *division*:
  F g d = g (*cbox* a b) **if** d *division_of* (*cbox* a b)

```
proof –
  define C where [abs_def]: C = card (division_points (cbox a b) d)
  then show ?thesis
  using that proof (induction C arbitrary: a b d rule: less_induct)
    case (less a b d)
    show ?case
    proof cases
      assume box a b = {}
      { fix k assume k∈d
        then obtain a' b' where k: k = cbox a' b'
          using division_ofD(4)[OF less.prems] by blast
        with ‹k∈d› division_ofD(2)[OF less.prems] have cbox a' b' ⊆ cbox a b
          by auto
        then have box a' b' ⊆ box a b
          unfolding subset_box by auto
        then have g k = 1
          using box_empty_imp [of a' b'] k by (simp add: ‹box a b = {}›) }
      then show box a b = {} ⟹ F g d = g (cbox a b)
        by (auto intro!: neutral simp: box_empty_imp)
    next
      assume box a b ≠ {}
      then have ab: ∀i∈Basis. a·i < b·i and ab': ∀i∈Basis. a·i ≤ b·i
        by (auto simp: box_ne_empty)
      show F g d = g (cbox a b)
      proof (cases division_points (cbox a b) d = {})
        case True
        { fix u v and j :: 'b
          assume j: j ∈ Basis and as: cbox u v ∈ d
          then have cbox u v ≠ {}
            using less.prems by blast
          then have uv: ∀i∈Basis. u·i ≤ v·i u·j ≤ v·j
            using j unfolding box_ne_empty by auto
         have *: ⋀p r Q. ¬ j∈Basis ∨ p ∨ r ∨ (∀x∈d. Q x) ⟹ p ∨ r ∨ Q (cbox
u v)
            using as j by auto
          have (j, u·j) ∉ division_points (cbox a b) d
               (j, v·j) ∉ division_points (cbox a b) d using True by auto
              note this[unfolded de_Morgan_conj division_points_def mem_Collect_eq
split_conv interval_bounds[OF ab'] bex_simps]
           note *[OF this(1)] *[OF this(2)] note this[unfolded interval_bounds[OF
uv(1)]]
          moreover
          have a·j ≤ u·j v·j ≤ b·j
            using division_ofD(2,2,3)[OF ‹d division_of cbox a b› as]
            apply (metis j subset_box(1) uv(1))
            by (metis ‹cbox u v ⊆ cbox a b› j subset_box(1) uv(1))
         ultimately have u·j = a·j ∧ v·j = a·j ∨ u·j = b·j ∧ v·j = b·j ∨ u·j =
a·j ∧ v·j = b·j
              unfolding not_less de_Morgan_disj using ab[rule_format,of j] uv(2) j
```

```
by force }
        then have d': ∀i∈d. ∃u v. i = cbox u v ∧
          (∀j∈Basis. u•j = a•j ∧ v•j = a•j ∨ u•j = b•j ∧ v•j = b•j ∨ u•j = a•j ∧
v•j = b•j)
          unfolding forall_in_division[OF less.prems] by blast
        have (1/2) *R (a+b) ∈ cbox a b
          unfolding mem_box using ab by (auto simp: inner_simps)
         note this[unfolded division_ofD(6)[OF ‹d division_of cbox a b›,symmetric]
Union_iff]
        then obtain i where i: i ∈ d (1 / 2) *R (a + b) ∈ i ..
        obtain u v where uv: i = cbox u v
                    ∀j∈Basis. u • j = a • j ∧ v • j = a • j ∨
                              u • j = b • j ∧ v • j = b • j ∨
                              u • j = a • j ∧ v • j = b • j
          using d' i(1) by auto
        have cbox a b ∈ d
        proof –
          have u = a v = b
            unfolding euclidean_eq_iff[where 'a='b]
          proof safe
            fix j :: 'b
            assume j: j ∈ Basis
            note i(2)[unfolded uv mem_box,rule_format,of j]
            then show u • j = a • j and v • j = b • j
              using uv(2)[rule_format,of j] j by (auto simp: inner_simps)
          qed
          then have i = cbox a b using uv by auto
          then show ?thesis using i by auto
        qed
        then have deq: d = insert (cbox a b) (d – {cbox a b})
          by auto
        have F g (d – {cbox a b}) = 1
        proof (intro neutral ballI)
          fix x
          assume x: x ∈ d – {cbox a b}
          then have x∈d
            by auto note d'[rule_format,OF this]
          then obtain u v where uv: x = cbox u v
                    ∀j∈Basis. u • j = a • j ∧ v • j = a • j ∨
                              u • j = b • j ∧ v • j = b • j ∨
                              u • j = a • j ∧ v • j = b • j
            by blast
          have u ≠ a ∨ v ≠ b
            using x[unfolded uv] by auto
          then obtain j where u•j ≠ a•j ∨ v•j ≠ b•j and j: j ∈ Basis
            unfolding euclidean_eq_iff[where 'a='b] by auto
          then have u•j = v•j
            using uv(2)[rule_format,OF j] by auto
          then have box u v = {}
```

```
          using j unfolding box_eq_empty by (auto intro!: bexI[of _ j])
        then show g x = 1
          unfolding uv(1) by (rule box_empty_imp)
      qed
      then show F g d = g (cbox a b)
        using division_ofD[OF less.prems]
        apply (subst deq)
        apply (subst insert)
        apply auto
        done
    next
      case False
      then have ∃x. x ∈ division_points (cbox a b) d
        by auto
      then obtain k c
        where k ∈ Basis interval_lowerbound (cbox a b) · k < c
              c < interval_upperbound (cbox a b) · k
              ∃i∈d. interval_lowerbound i · k = c ∨ interval_upperbound i · k = c
        unfolding division_points_def by auto
      then obtain j where a · k < c c < b · k
          and j ∈ d and j: interval_lowerbound j · k = c ∨ interval_upperbound
j · k = c
        by (metis division_of_trivial empty_iff interval_bounds' less.prems)
      let ?lec = {x. x·k ≤ c} let ?gec = {x. x·k ≥ c}
      define d1 where d1 = {l ∩ ?lec | l. l ∈ d ∧ l ∩ ?lec ≠ {}}
      define d2 where d2 = {l ∩ ?gec | l. l ∈ d ∧ l ∩ ?gec ≠ {}}
      define cb where cb = (∑ i∈Basis. (if i = k then c else b·i) *R i)
      define ca where ca = (∑ i∈Basis. (if i = k then c else a·i) *R i)
      have division_points (cbox a b ∩ ?lec) {l ∩ ?lec |l. l ∈ d ∧ l ∩ ?lec ≠ {}}
            ⊂ division_points (cbox a b) d
        by (rule division_points_psubset[OF ‹d division_of cbox a b› ab ‹a · k < c›
‹c < b · k› ‹j ∈ d› j ‹k ∈ Basis›])
      with division_points_finite[OF ‹d division_of cbox a b›]
      have card
        (division_points (cbox a b ∩ ?lec) {l ∩ ?lec |l. l ∈ d ∧ l ∩ ?lec ≠ {}})
          < card (division_points (cbox a b) d)
        by (rule psubset_card_mono)
      moreover have division_points (cbox a b ∩ {x. c ≤ x · k}) {l ∩ {x. c ≤ x
· k} |l. l ∈ d ∧ l ∩ {x. c ≤ x · k} ≠ {}}
            ⊂ division_points (cbox a b) d
        by (rule division_points_psubset[OF ‹d division_of cbox a b› ab ‹a · k < c›
‹c < b · k› ‹j ∈ d› j ‹k ∈ Basis›])
      with division_points_finite[OF ‹d division_of cbox a b›]
      have card (division_points (cbox a b ∩ ?gec) {l ∩ ?gec |l. l ∈ d ∧ l ∩ ?gec
≠ {}})
            < card (division_points (cbox a b) d)
        by (rule psubset_card_mono)
      ultimately have *: F g d1 = g (cbox a b ∩ ?lec) F g d2 = g (cbox a b ∩
?gec)
```

```
        unfolding interval_split[OF ‹k ∈ Basis›]
        apply (rule_tac[!] less.hyps)
        using division_split[OF ‹d division_of cbox a b›, where k=k and c=c] ‹k
∈ Basis›
        by (simp_all add: interval_split d1_def d2_def division_points_finite[OF ‹d
division_of cbox a b›])
      have fxk_le: g (l ∩ ?lec) = 1
        if l ∈ d y ∈ d l ∩ ?lec = y ∩ ?lec l ≠ y for l y
      proof –
        obtain u v where leq: l = cbox u v
          using ‹l ∈ d› less.prems by auto
        have interior (cbox u v ∩ ?lec) = {}
          using that division_split_left_inj leq less.prems by blast
        then show ?thesis
          unfolding leq interval_split [OF ‹k ∈ Basis›]
          by (auto intro: box_empty_imp)
      qed
      have fxk_ge: g (l ∩ {x. x · k ≥ c}) = 1
        if l ∈ d y ∈ d l ∩ ?gec = y ∩ ?gec l ≠ y for l y
      proof –
        obtain u v where leq: l = cbox u v
          using ‹l ∈ d› less.prems by auto
        have interior (cbox u v ∩ ?gec) = {}
          using that division_split_right_inj leq less.prems by blast
        then show ?thesis
          unfolding leq interval_split[OF ‹k ∈ Basis›]
          by (auto intro: box_empty_imp)
      qed
      have d1_alt: d1 = (λl. l ∩ ?lec) ` {l ∈ d. l ∩ ?lec ≠ {}}
        using d1_def by auto
      have d2_alt: d2 = (λl. l ∩ ?gec) ` {l ∈ d. l ∩ ?gec ≠ {}}
        using d2_def by auto
      have g (cbox a b) = F g d1 ∗ F g d2 (is _ = ?prev)
        unfolding ∗ using ‹k ∈ Basis›
        by (auto dest: Basis_imp)
      also have F g d1 = F (λl. g (l ∩ ?lec)) d
        unfolding d1_alt using division_of_finite[OF less.prems] fxk_le
        by (subst reindex_nontrivial) (auto intro!: mono_neutral_cong_left)
      also have F g d2 = F (λl. g (l ∩ ?gec)) d
        unfolding d2_alt using division_of_finite[OF less.prems] fxk_ge
        by (subst reindex_nontrivial) (auto intro!: mono_neutral_cong_left)
      also have ∗: ∀x∈d. g x = g (x ∩ ?lec) ∗ g (x ∩ ?gec)
        unfolding forall_in_division[OF ‹d division_of cbox a b›]
        using ‹k ∈ Basis›
        by (auto dest: Basis_imp)
      have F (λl. g (l ∩ ?lec)) d ∗ F (λl. g (l ∩ ?gec)) d = F g d
        using ∗ by (simp add: distrib)
      finally show ?thesis by auto
    qed
```

```
    qed
  qed
qed

proposition tagged_division:
  assumes d tagged_division_of (cbox a b)
  shows F (λ(_, l). g l) d = g (cbox a b)
proof −
  have F (λ(_, k). g k) d = F g (snd ‘ d)
    using assms box_empty_imp by (rule over_tagged_division_lemma)
  then show ?thesis
    unfolding assms [THEN division_of_tagged_division, THEN division] .
  qed

end

locale operative_real = comm_monoid_set +
  fixes g :: real set ⇒ 'a
  assumes neutral: b ≤ a =⇒ g {a..b} = 1
  assumes coalesce_less: a < c =⇒ c < b =⇒ g {a..c} ∗ g {c..b} = g {a..b}
begin

sublocale operative where g = g
  rewrites box = (greaterThanLessThan :: real ⇒ _)
    and cbox = (atLeastAtMost :: real ⇒ _)
    and ⋀x::real. x ∈ Basis ←→ x = 1
proof −
  show operative f z g
  proof
    show g (cbox a b) = 1 if box a b = {} for a b
      using that by (simp add: neutral)
    show g (cbox a b) = g (cbox a b ∩ {x. x · k ≤ c}) ∗ g (cbox a b ∩ {x. c ≤ x
· k})
      if k ∈ Basis for a b c k
    proof −
      from that have [simp]: k = 1
        by simp
      from neutral [of 0 1] neutral [of a a for a] coalesce_less
  have [simp]: g {} = 1 ⋀a. g {a} = 1
    ⋀a b c. a < c =⇒ c < b =⇒ g {a..c} ∗ g {c..b} = g {a..b}
    by auto
      have g {a..b} = g {a..min b c} ∗ g {max a c..b}
    by (auto simp: min_def max_def le_less)
      then show g (cbox a b) = g (cbox a b ∩ {x. x · k ≤ c}) ∗ g (cbox a b ∩ {x.
c ≤ x · k})
        by (simp add: atMost_def [symmetric] atLeast_def [symmetric])
qed
  qed
  show box = (greaterThanLessThan :: real ⇒ _)
```

```
    and cbox = (atLeastAtMost :: real ⇒ _)
    and ⋀x::real. x ∈ Basis ⟷ x = 1
    by (simp_all add: fun_eq_iff)
qed

lemma coalesce_less_eq:
  g {a..c} ∗ g {c..b} = g {a..b} if a ≤ c c ≤ b
  proof (cases c = a ∨ c = b)
    case False
  with that have a < c c < b
    by auto
    then show ?thesis
    by (rule coalesce_less)
  next
    case True
  with that box_empty_imp [of a a] box_empty_imp [of b b] show ?thesis
    by safe simp_all
    qed

end

lemma operative_realI:
  operative_real f z g if operative f z g
proof −
  interpret operative f z g
    using that .
  show ?thesis
  proof
    show g {a..b} = z if b ≤ a for a b
      using that box_empty_imp by simp
    show f (g {a..c}) (g {c..b}) = g {a..b} if a < c c < b for a b c
      using that
    using Basis_imp [of 1 a b c]
      by (simp_all add: atMost_def [symmetric] atLeast_def [symmetric] max_def
min_def)
qed
qed
```

6.14.9 Special case of additivity we need for the FTC

```
lemma additive_tagged_division_1:
  fixes f :: real ⇒ 'a::real_normed_vector
  assumes a ≤ b
    and p tagged_division_of {a..b}
  shows sum (λ(x,k). f(Sup k) − f(Inf k)) p = f b − f a
proof −
  let ?f = (λk::(real) set. if k = {} then 0 else f(interval_upperbound k) −
f(interval_lowerbound k))
  interpret operative_real plus 0 ?f
```

rewrites *comm_monoid_set.F* (+) *0* = *sum*
by *standard*[*1*] (*auto simp add*: *sum_def*)
have *p_td*: *p tagged_division_of cbox a b*
using *assms*(*2*) *box_real*(*2*) **by** *presburger*
have ∗∗: *cbox a b* ≠ {}
using *assms*(*1*) **by** *auto*
then have *f b* − *f a* = (∑(*x*, *l*)∈*p*. *if l* = {} *then 0 else f* (*interval_upperbound l*) − *f* (*interval_lowerbound l*))
proof −
have (*if cbox a b* = {} *then 0 else f* (*interval_upperbound* (*cbox a b*)) − *f* (*interval_lowerbound* (*cbox a b*))) = *f b* − *f a*
using *assms* **by** *auto*
then show *?thesis*
using *p_td assms* **by** (*simp add*: *tagged_division*)
qed
then show *?thesis*
using *assms* **by** (*auto intro*!: *sum.cong*)
qed

6.14.10 Fine-ness of a partition w.r.t. a gauge

definition *fine* (**infixr** *fine 46*)
where *d fine s* ⟷ (∀(*x*,*k*) ∈ *s*. *k* ⊆ *d x*)

lemma *fineI*:
assumes ⋀*x k*. (*x*, *k*) ∈ *s* ⟹ *k* ⊆ *d x*
shows *d fine s*
using *assms* **unfolding** *fine_def* **by** *auto*

lemma *fineD*[*dest*]:
assumes *d fine s*
shows ⋀*x k*. (*x*,*k*) ∈ *s* ⟹ *k* ⊆ *d x*
using *assms* **unfolding** *fine_def* **by** *auto*

lemma *fine_Int*: (λ*x*. *d1 x* ∩ *d2 x*) *fine p* ⟷ *d1 fine p* ∧ *d2 fine p*
unfolding *fine_def* **by** *auto*

lemma *fine_Inter*:
(λ*x*. ⋂{*f d x* | *d*. *d* ∈ *s*}) *fine p* ⟷ (∀ *d*∈*s*. (*f d*) *fine p*)
unfolding *fine_def* **by** *blast*

lemma *fine_Un*: *d fine p1* ⟹ *d fine p2* ⟹ *d fine* (*p1* ∪ *p2*)
unfolding *fine_def* **by** *blast*

lemma *fine_Union*: (⋀*p*. *p* ∈ *ps* ⟹ *d fine p*) ⟹ *d fine* (⋃*ps*)
unfolding *fine_def* **by** *auto*

lemma *fine_subset*: *p* ⊆ *q* ⟹ *d fine q* ⟹ *d fine p*
unfolding *fine_def* **by** *blast*

6.14.11 Some basic combining lemmas

```
lemma tagged_division_Union_exists:
  assumes finite I
    and ∀ i∈I. ∃ p. p tagged_division_of i ∧ d fine p
    and ∀ i1∈I. ∀ i2∈I. i1 ≠ i2 ⟶ interior i1 ∩ interior i2 = {}
    and ⋃ I = i
   obtains p where p tagged_division_of i and d fine p
proof −
  obtain pfn where pfn:
    ⋀x. x ∈ I ⟹ pfn x tagged_division_of x
    ⋀x. x ∈ I ⟹ d fine pfn x
    using bchoice[OF assms(2)] by auto
  show thesis
    apply (rule_tac p=⋃(pfn ‘ I) in that)
    using assms(1) assms(3) assms(4) pfn(1) tagged_division_Union apply force
    by (metis (mono_tags, lifting) fine_Union imageE pfn(2))
qed
```

6.14.12 The set we're concerned with must be closed

```
lemma division_of_closed:
  fixes i :: 'n::euclidean_space set
  shows s division_of i ⟹ closed i
  unfolding division_of_def by fastforce
```

6.14.13 General bisection principle for intervals; might be useful elsewhere

```
lemma interval_bisection_step:
  fixes type :: 'a::euclidean_space
  assumes emp: P {}
    and Un: ⋀S T. ⟦P S; P T; interior(S) ∩ interior(T) = {}⟧ ⟹ P (S ∪ T)
    and non: ¬ P (cbox a (b::'a))
  obtains c d where ¬ P (cbox c d)
    and ⋀i. i ∈ Basis ⟹ a∙i ≤ c∙i ∧ c∙i ≤ d∙i ∧ d∙i ≤ b∙i ∧ 2 ∗ (d∙i − c∙i) ≤
b∙i − a∙i
proof −
  have cbox a b ≠ {}
    using emp non by metis
  then have ab: ⋀i. i∈Basis ⟹ a ∙ i ≤ b ∙ i
    by (force simp: mem_box)
  have UN_cases: ⟦finite ℱ;
          ⋀S. S∈ℱ ⟹ P S;
          ⋀S. S∈ℱ ⟹ ∃ a b. S = cbox a b;
         ⋀S T. S∈ℱ ⟹ T∈ℱ ⟹ S ≠ T ⟹ interior S ∩ interior T = {}⟧ ⟹
P (⋃ℱ) for ℱ
  proof (induct ℱ rule: finite_induct)
    case empty show ?case
      using emp by auto
```

next
case (*insert x f*)
then show *?case*
unfolding *Union_insert* **by** (*metis Int_interior_Union_intervals Un insert_iff open_interior*)
qed
let *?ab* = $\lambda i.$ ($a{\bullet}i$ + $b{\bullet}i$) / *2*
let *?A* = {*cbox c d* | *c d*::*'a*. $\forall$ *i*$\in$*Basis*. ($c{\bullet}i$ = $a{\bullet}i$) $\wedge$ ($d{\bullet}i$ = *?ab i*) $\vee$
($c{\bullet}i$ = *?ab i*) $\wedge$ ($d{\bullet}i$ = $b{\bullet}i$)}
have *P* ($\bigcup$ *?A*)
if $\bigwedge$*c d*. $\forall$ *i*$\in$*Basis*. $a{\bullet}i \le c{\bullet}i \wedge c{\bullet}i \le d{\bullet}i \wedge d{\bullet}i \le b{\bullet}i \wedge 2 * (d{\bullet}i - c{\bullet}i) \le b{\bullet}i - a{\bullet}i \Longrightarrow$ *P* (*cbox c d*)
proof (*rule UN_cases*)
let *?B* = (λS. *cbox* ($\sum$ *i*$\in$*Basis*. (*if i* $\in$ *S then* $a{\bullet}i$ *else ?ab i*) $*_R$ *i*::*'a*)
($\sum$ *i*$\in$*Basis*. (*if i* $\in$ *S then ?ab i else* $b{\bullet}i$) $*_R$ *i*)) ‘ {*s*. *s* $\subseteq$ *Basis*}
have *?A* $\subseteq$ *?B*
proof
fix *x*
assume *x* $\in$ *?A*
then obtain *c d*
where *x*: *x* = *cbox c d*
$\bigwedge$*i*. *i* $\in$ *Basis* $\Longrightarrow$
$c \bullet i = a \bullet i \wedge d \bullet i =$ *?ab i* $\vee$ $c \bullet i =$ *?ab i* $\wedge$ $d \bullet i = b \bullet i$
by *blast*
have *c* = ($\sum$ *i*$\in$*Basis*. (*if* $c \bullet i = a \bullet i$ *then* $a \bullet i$ *else ?ab i*) $*_R$ *i*)
d = ($\sum$ *i*$\in$*Basis*. (*if* $c \bullet i = a \bullet i$ *then ?ab i else* $b \bullet i$) $*_R$ *i*)
using *x*(*2*) *ab* **by** (*fastforce simp add*: *euclidean_eq_iff* [**where** *'a*=*'a*])+
then show *x* $\in$ *?B*
unfolding *x* **by** (*rule_tac x*={*i*. *i*$\in$*Basis* $\wedge$ $c{\bullet}i$ = $a{\bullet}i$} **in** *image_eqI*) *auto*
qed
then show *finite ?A*
by (*rule finite_subset*) *auto*
next
fix *S*
assume *S* $\in$ *?A*
then obtain *c d*
where *s*: *S* = *cbox c d*
$\bigwedge$*i*. *i* $\in$ *Basis* $\Longrightarrow$ $c \bullet i = a \bullet i \wedge d \bullet i =$ *?ab i* $\vee$ $c \bullet i =$ *?ab i* $\wedge$ $d \bullet i = b \bullet i$
by *blast*
show *P S*
unfolding *s* **using** *ab s*(*2*) **by** (*fastforce intro*!: *that*)
show $\exists$ *a b*. *S* = *cbox a b*
unfolding *s* **by** *auto*
fix *T*
assume *T* $\in$ *?A*
then obtain *e f* **where** *t*:
T = *cbox e f*
$\bigwedge$*i*. *i* $\in$ *Basis* $\Longrightarrow$ $e \bullet i = a \bullet i \wedge f \bullet i =$ *?ab i* $\vee$ $e \bullet i =$ *?ab i* $\wedge$ $f \bullet i = b \bullet i$

```
      by blast
    assume S ≠ T
    then have ¬ (c = e ∧ d = f)
      unfolding s t by auto
    then obtain i where c·i ≠ e·i ∨ d·i ≠ f·i and i': i ∈ Basis
      unfolding euclidean_eq_iff[where 'a='a] by auto
    then have i: c·i ≠ e·i d·i ≠ f·i
      using s(2) t(2) apply fastforce
      using t(2)[OF i'] ‹c · i ≠ e · i ∨ d · i ≠ f · i› i' s(2) t(2) by fastforce
    have *: ⋀s t. (⋀a. a ∈ s ⟹ a ∈ t ⟹ False) ⟹ s ∩ t = {}
      by auto
    show interior S ∩ interior T = {}
      unfolding s t interior_cbox
    proof (rule *)
      fix x
      assume x ∈ box c d x ∈ box e f
      then have x: c·i < d·i e·i < f·i c·i < f·i e·i < d·i
        unfolding mem_box using i'  by force+
      show False  using s(2)[OF i'] t(2)[OF i'] and i x
        by auto
    qed
  qed
  also have ⋃ ?A = cbox a b
  proof (rule set_eqI,rule)
    fix x
    assume x ∈ ⋃ ?A
    then obtain c d where x:
      x ∈ cbox c d
      ⋀i. i ∈ Basis ⟹ c · i = a · i ∧ d · i = ?ab i ∨ c · i = ?ab i ∧ d · i = b · i
      by blast
    then show x∈cbox a b
      unfolding mem_box by force
  next
    fix x
    assume x: x ∈ cbox a b
    then have ∀i∈Basis. ∃c d. (c = a·i ∧ d = ?ab i ∨ c = ?ab i ∧ d = b·i) ∧
c≤x·i ∧ x·i ≤ d
      (is ∀i∈Basis. ∃c d. ?P i c d)
      unfolding mem_box by (metis linear)
    then obtain α β where ∀i∈Basis. (α · i = a · i ∧ β · i = ?ab i ∨
        α · i = ?ab i ∧ β · i = b · i) ∧ α · i ≤ x · i ∧ x · i ≤ β · i
      by (auto simp: choice_Basis_iff)
    then show x∈⋃ ?A
      by (force simp add: mem_box)
  qed
  finally show thesis
      by (metis (no_types, lifting) assms(3) that)
qed
```

lemma *interval_bisection*:
 fixes *type* :: *'a::euclidean_space*
 assumes *P* {}
  and *Un*: ⋀*S T*. ⟦*P S*; *P T*; *interior*(*S*) ∩ *interior*(*T*) = {}⟧ ⟹ *P* (*S* ∪ *T*)
  and ¬ *P* (*cbox a* (*b::'a*))
 obtains *x* **where** *x* ∈ *cbox a b*
  and ∀ *e*>*0*. ∃ *c d*. *x* ∈ *cbox c d* ∧ *cbox c d* ⊆ *ball x e* ∧ *cbox c d* ⊆ *cbox a b* ∧ ¬ *P* (*cbox c d*)
proof −
 have ∀ *x*. ∃ *y*. ¬ *P* (*cbox* (*fst x*) (*snd x*)) ⟶ (¬ *P* (*cbox* (*fst y*) (*snd y*)) ∧
  (∀ *i*∈*Basis*. *fst x*·*i* ≤ *fst y*·*i* ∧ *fst y*·*i* ≤ *snd y*·*i* ∧ *snd y*·*i* ≤ *snd x*·*i* ∧
   2 ∗ (*snd y*·*i* − *fst y*·*i*) ≤ *snd x*·*i* − *fst x*·*i*)) (**is** ∀ *x*. *?P x*)
 proof
  show *?P x* **for** *x*
  proof (*cases P* (*cbox* (*fst x*) (*snd x*)))
   case *True*
   then show *?thesis* **by** *auto*
  next
   case *False*
   obtain *c d* **where** ¬ *P* (*cbox c d*)
    ⋀*i*. *i* ∈ *Basis* ⟹
     fst x · *i* ≤ *c* · *i* ∧
     c · *i* ≤ *d* · *i* ∧
     d · *i* ≤ *snd x* · *i* ∧
     2 ∗ (*d* · *i* − *c* · *i*) ≤ *snd x* · *i* − *fst x* · *i*
    by (*blast intro*: *interval_bisection_step*[*of P*, *OF assms*(*1*−*2*) *False*])
   then show *?thesis*
    by (*rule_tac x*=(*c*,*d*) **in** *exI*) *auto*
  qed
 qed
 then obtain *f* **where** *f*:
  ∀ *x*.
   ¬ *P* (*cbox* (*fst x*) (*snd x*)) ⟶
   ¬ *P* (*cbox* (*fst* (*f x*)) (*snd* (*f x*))) ∧
    (∀ *i*∈*Basis*.
      fst x · *i* ≤ *fst* (*f x*) · *i* ∧
      fst (*f x*) · *i* ≤ *snd* (*f x*) · *i* ∧
      snd (*f x*) · *i* ≤ *snd x* · *i* ∧
      2 ∗ (*snd* (*f x*) · *i* − *fst* (*f x*) · *i*) ≤ *snd x* · *i* − *fst x* · *i*) **by** *metis*
 define *AB A B* **where** *ab_def*: *AB n* = (*f* ^^ *n*) (*a*,*b*) *A n* = *fst*(*AB n*) *B n* = *snd*(*AB n*) **for** *n*
 have [*simp*]: *A 0* = *a B 0* = *b* **and** *ABRAW*: ⋀*n*. ¬ *P* (*cbox* (*A*(*Suc n*)) (*B*(*Suc n*))) ∧
  (∀ *i*∈*Basis*. *A*(*n*)·*i* ≤ *A*(*Suc n*)·*i* ∧ *A*(*Suc n*)·*i* ≤ *B*(*Suc n*)·*i* ∧ *B*(*Suc n*)·*i* ≤ *B*(*n*)·*i* ∧
  2 ∗ (*B*(*Suc n*)·*i* − *A*(*Suc n*)·*i*) ≤ *B*(*n*)·*i* − *A*(*n*)·*i*) (**is** ⋀*n*. *?P n*)
 proof −
  show *A 0* = *a B 0* = *b*
   unfolding *ab_def* **by** *auto*

```
    note S = ab_def funpow.simps o_def id_apply
    show ?P n for n
    proof (induct n)
      case 0
      then show ?case
        unfolding S using ‹¬ P (cbox a b)› f by auto
    next
      case (Suc n)
      show ?case
        unfolding S
        apply (rule f[rule_format])
        using Suc
        unfolding S
        apply auto
        done
    qed
  qed
  then have AB: A(n)•i ≤ A(Suc n)•i A(Suc n)•i ≤ B(Suc n)•i
                B(Suc n)•i ≤ B(n)•i 2 ∗ (B(Suc n)•i − A(Suc n)•i) ≤ B(n)•i −
A(n)•i
              if i∈Basis for i n
    using that by blast+
  have notPAB: ¬ P (cbox (A(Suc n)) (B(Suc n))) for n
    using ABRAW by blast
  have interv: ∃ n. ∀ x∈cbox (A n) (B n). ∀ y∈cbox (A n) (B n). dist x y < e
    if e: 0 < e for e
  proof −
    obtain n where n: (∑ i∈Basis. b • i − a • i) / e < 2 ^ n
      using real_arch_pow[of 2 (sum (λi. b•i − a•i) Basis) / e] by auto
    show ?thesis
    proof (rule exI [where x=n], clarify)
      fix x y
      assume xy: x∈cbox (A n) (B n) y∈cbox (A n) (B n)
      have dist x y ≤ sum (λi. |(x − y)•i|) Basis
        unfolding dist_norm by(rule norm_le_l1)
      also have . . . ≤ sum (λi. B n•i − A n•i) Basis
      proof (rule sum_mono)
        fix i :: 'a
        assume i: i ∈ Basis
        show |(x − y) • i| ≤ B n • i − A n • i
          using xy[unfolded mem_box,THEN bspec, OF i]
          by (auto simp: inner_diff_left)
      qed
      also have . . . ≤ sum (λi. b•i − a•i) Basis / 2^n
        unfolding sum_divide_distrib
      proof (rule sum_mono)
        show B n • i − A n • i ≤ (b • i − a • i) / 2 ^ n if i: i ∈ Basis for i
        proof (induct n)
          case 0
```

```
        then show ?case
          unfolding AB by auto
      next
        case (Suc n)
        have B (Suc n) · i − A (Suc n) · i ≤ (B n · i − A n · i) / 2
          using AB(3) that AB(4)[of i n] using i by auto
        also have . . . ≤ (b · i − a · i) / 2 ˆ Suc n
          using Suc by (auto simp add: field_simps)
        finally show ?case .
      qed
    qed
    also have . . . < e
      using n using e by (auto simp add: field_simps)
    finally show dist x y < e .
  qed
qed
{
  fix n m :: nat
  assume m ≤ n then have cbox (A n) (B n) ⊆ cbox (A m) (B m)
  proof (induction rule: inc_induct)
    case (step i)
    show ?case
      using AB by (intro order_trans[OF step.IH] subset_box_imp) auto
  qed simp
} note ABsubset = this
have ⋀n. cbox (A n) (B n) ≠ {}
  by (meson AB dual_order.trans interval_not_empty)
then obtain x0 where x0: ⋀n. x0 ∈ cbox (A n) (B n)
  using decreasing_closed_nest [OF closed_cbox] ABsubset interv by blast
show thesis
proof (rule that[rule_format, of x0])
  show x0∈cbox a b
    using ‹A 0 = a› ‹B 0 = b› x0 by blast
  fix e :: real
  assume e > 0
  from interv[OF this] obtain n
    where n: ∀ x∈cbox (A n) (B n). ∀ y∈cbox (A n) (B n). dist x y < e ..
  have ¬ P (cbox (A n) (B n))
  proof (cases 0 < n)
    case True then show ?thesis
      by (metis Suc_pred' notPAB)
  next
    case False then show ?thesis
      using ‹A 0 = a› ‹B 0 = b› ‹¬ P (cbox a b)› by blast
  qed
  moreover have cbox (A n) (B n) ⊆ ball x0 e
    using n using x0[of n] by auto
  moreover have cbox (A n) (B n) ⊆ cbox a b
    using ABsubset ‹A 0 = a› ‹B 0 = b› by blast
```

```
    ultimately show ∃c d. x0 ∈ cbox c d ∧ cbox c d ⊆ ball x0 e ∧ cbox c d ⊆
cbox a b ∧ ¬ P (cbox c d)
      apply (rule_tac x=A n in exI)
      apply (rule_tac x=B n in exI)
      apply (auto simp: x0)
      done
  qed
qed
```

6.14.14 Cousin's lemma

```
lemma fine_division_exists:
  fixes a b :: 'a::euclidean_space
  assumes gauge g
  obtains p where p tagged_division_of (cbox a b) g fine p
proof (cases ∃p. p tagged_division_of (cbox a b) ∧ g fine p)
  case True
  then show ?thesis
    using that by auto
next
  case False
  assume ¬ (∃p. p tagged_division_of (cbox a b) ∧ g fine p)
  obtain x where x:
      x ∈ (cbox a b)
      ⋀e. 0 < e ⟹
        ∃c d.
          x ∈ cbox c d ∧
          cbox c d ⊆ ball x e ∧
          cbox c d ⊆ (cbox a b) ∧
          ¬ (∃p. p tagged_division_of cbox c d ∧ g fine p)
    apply (rule interval_bisection[of λs. ∃p. p tagged_division_of s ∧ g fine p, OF
_ _ False])
    apply (simp add: fine_def)
    apply (metis tagged_division_Un fine_Un)
    apply auto
    done
  obtain e where e: e > 0 ball x e ⊆ g x
    using gaugeD[OF assms, of x] unfolding open_contains_ball by auto
  from x(2)[OF e(1)]
  obtain c d where c_d: x ∈ cbox c d
                        cbox c d ⊆ ball x e
                        cbox c d ⊆ cbox a b
                        ¬ (∃p. p tagged_division_of cbox c d ∧ g fine p)
    by blast
  have g fine {(x, cbox c d)}
    unfolding fine_def using e using c_d(2) by auto
  then show ?thesis
    using tagged_division_of_self[OF c_d(1)] using c_d by auto
qed
```

lemma *fine_division_exists_real*:
  fixes *a b* :: *real*
  assumes *gauge g*
  obtains *p* **where** *p tagged_division_of* {*a* .. *b*} *g fine p*
  by (*metis assms box_real*(*2*) *fine_division_exists*)

6.14.15 A technical lemma about "refinement" of division

lemma *tagged_division_finer*:
  fixes *p* :: (*'a*::*euclidean_space* × (*'a*::*euclidean_space set*)) *set*
  assumes *ptag*: *p tagged_division_of* (*cbox a b*)
    and *gauge d*
  obtains *q* **where** *q tagged_division_of* (*cbox a b*)
    and *d fine q*
    and ∀ (*x*,*k*) ∈ *p*. *k* ⊆ *d*(*x*) ⟶ (*x*,*k*) ∈ *q*
proof −
  have *p*: *finite p p tagged_partial_division_of* (*cbox a b*)
    using *ptag tagged_division_of_def* **by** *blast*+
  have (∃ *q*. *q tagged_division_of* (⋃{*k*. ∃ *x*. (*x*,*k*) ∈ *p*}) ∧ *d fine q* ∧ (∀ (*x*,*k*) ∈ *p*. *k* ⊆ *d*(*x*) ⟶ (*x*,*k*) ∈ *q*))
    if *finite p p tagged_partial_division_of* (*cbox a b*) *gauge d* **for** *p*
    using *that*
  proof (*induct p*)
    case *empty*
    show *?case*
      by (*force simp add*: *fine_def*)
  next
    case (*insert xk p*)
    obtain *x k* **where** *xk*: *xk* = (*x*, *k*)
      using *surj_pair*[*of xk*] **by** *metis*
    obtain *q1* **where** *q1*: *q1 tagged_division_of* ⋃{*k*. ∃ *x*. (*x*, *k*) ∈ *p*}
              and *d fine q1*
              and *q1I*: ⋀*x k*. ⟦(*x*, *k*)∈*p*; *k* ⊆ *d x*⟧ ⟹ (*x*, *k*) ∈ *q1*
      using *case_prodD tagged_partial_division_subset*[*OF insert*(*4*) *subset_insertI*]
      by (*metis* (*mono_tags*, *lifting*) *insert.hyps*(*3*) *insert.prems*(*2*))
    have ∗: ⋃{*l*. ∃ *y*. (*y*,*l*) ∈ *insert xk p*} = *k* ∪ ⋃{*l*. ∃ *y*. (*y*,*l*) ∈ *p*}
      unfolding *xk* **by** *auto*
    note *p* = *tagged_partial_division_ofD*[*OF insert*(*4*)]
    obtain *u v* **where** *uv*: *k* = *cbox u v*
      using *p*(*4*) *xk* **by** *blast*
    have *finite* {*k*. ∃ *x*. (*x*, *k*) ∈ *p*}
      apply (*rule finite_subset*[*of _ snd ' p*])
      using *image_iff* **apply** *fastforce*
      using *insert.hyps*(*1*) **by** *blast*
    then have *int*: *interior* (*cbox u v*) ∩ *interior* (⋃{*k*. ∃ *x*. (*x*, *k*) ∈ *p*}) = {}
    proof (*rule Int_interior_Union_intervals*)
      show *open* (*interior* (*cbox u v*))
        by *auto*

```
      show ⋀T. T ∈ {k. ∃x. (x, k) ∈ p} ⟹ ∃a b. T = cbox a b
        using p(4) by auto
      show ⋀T. T ∈ {k. ∃x. (x, k) ∈ p} ⟹ interior (cbox u v) ∩ interior T =
{}
        by clarify (metis insert.hyps(2) insert_iff interior_cbox p(5) uv xk)
    qed
    show ?case
    proof (cases cbox u v ⊆ d x)
      case True
      have {(x, cbox u v)} tagged_division_of cbox u v
        by (simp add: p(2) uv xk tagged_division_of_self)
      then have {(x, cbox u v)} ∪ q1 tagged_division_of ⋃{k. ∃x. (x, k) ∈ insert
xk p}
        unfolding ∗ uv by (metis (no_types, lifting) int q1 tagged_division_Un)
      with True show ?thesis
        apply (rule_tac x={(x,cbox u v)} ∪ q1 in exI)
        using ‹d fine q1› fine_def q1I uv xk apply fastforce
        done
    next
      case False
      obtain q2 where q2: q2 tagged_division_of cbox u v d fine q2
        using fine_division_exists[OF assms(2)] by blast
      show ?thesis
        apply (rule_tac x=q2 ∪ q1 in exI)
        apply (intro conjI)
        unfolding ∗ uv
        apply (rule tagged_division_Un q2 q1 int fine_Un)+
          apply (auto intro: q1 q2 fine_Un ‹d fine q1› simp add: False q1I uv xk)
        done
    qed
  qed
  with p obtain q where q: q tagged_division_of ⋃{k. ∃x. (x, k) ∈ p} d fine q
∀(x, k)∈p. k ⊆ d x ⟶ (x, k) ∈ q
    by (meson ‹gauge d›)
  with ptag that show ?thesis by auto
qed
```

Covering lemma

Some technical lemmas used in the approximation results that follow. Proof of the covering lemma is an obvious multidimensional generalization of Lemma 3, p65 of Swartz's "Introduction to Gauge Integrals".

```
proposition covering_lemma:
  assumes S ⊆ cbox a b box a b ≠ {} gauge g
  obtains 𝒟 where
    countable 𝒟 ⋃𝒟 ⊆ cbox a b
    ⋀K. K ∈ 𝒟 ⟹ interior K ≠ {} ∧ (∃c d. K = cbox c d)
    pairwise (λA B. interior A ∩ interior B = {}) 𝒟
    ⋀K. K ∈ 𝒟 ⟹ ∃x ∈ S ∩ K. K ⊆ g x
```

$\bigwedge u\ v.\ cbox\ u\ v \in \mathcal{D} \Longrightarrow \exists n.\ \forall i \in Basis.\ v \cdot i - u \cdot i = (b \cdot i - a \cdot i) / 2\hat{}n$
$S \subseteq \bigcup \mathcal{D}$
proof –
have *aibi*: $\bigwedge i.\ i \in Basis \Longrightarrow a \cdot i \leq b \cdot i$ **and** *normab*: $0 < norm(b - a)$
using ‹$box\ a\ b \neq \{\}$› *box_eq_empty box_sing* **by** *fastforce+*
let $?K0 = \lambda(n, f::'a \Rightarrow nat).$
$cbox\ (\sum i \in Basis.\ (a \cdot i + (f\ i / 2\hat{}n) * (b \cdot i - a \cdot i)) *_R i)$
$(\sum i \in Basis.\ (a \cdot i + ((f\ i + 1) / 2\hat{}n) * (b \cdot i - a \cdot i)) *_R i)$
let $?D0 = ?K0$ ‘ $(SIGMA\ n:UNIV.\ Pi_E\ Basis\ (\lambda i::'a.\ lessThan\ (2\hat{}n)))$
obtain $\mathcal{D}0$ **where** *count*: *countable* $\mathcal{D}0$
and *sub*: $\bigcup \mathcal{D}0 \subseteq cbox\ a\ b$
and *int*: $\bigwedge K.\ K \in \mathcal{D}0 \Longrightarrow (interior\ K \neq \{\}) \wedge (\exists c\ d.\ K = cbox\ c\ d)$
and *intdj*: $\bigwedge A\ B.\ [\![A \in \mathcal{D}0;\ B \in \mathcal{D}0]\!] \Longrightarrow A \subseteq B \vee B \subseteq A \vee interior\ A \cap interior\ B = \{\}$
and *SK*: $\bigwedge x.\ x \in S \Longrightarrow \exists K \in \mathcal{D}0.\ x \in K \wedge K \subseteq g\ x$
and *cbox*: $\bigwedge u\ v.\ cbox\ u\ v \in \mathcal{D}0 \Longrightarrow \exists n.\ \forall i \in Basis.\ v \cdot i - u \cdot i = (b \cdot i - a \cdot i) / 2\hat{}n$
and *fin*: $\bigwedge K.\ K \in \mathcal{D}0 \Longrightarrow finite\ \{L \in \mathcal{D}0.\ K \subseteq L\}$
proof
show *countable ?D0*
by (*simp add*: *countable_PiE*)
next
show $\bigcup ?D0 \subseteq cbox\ a\ b$
apply (*simp add*: *UN_subset_iff*)
apply (*intro conjI allI ballI subset_box_imp*)
apply (*simp add*: *field_simps*)
apply (*auto intro*: *mult_right_mono aibi*)
apply (*force simp*: *aibi scaling_mono nat_less_real_le dest*: *PiE_mem intro*: *mult_right_mono*)
done
next
show $\bigwedge K.\ K \in ?D0 \Longrightarrow interior\ K \neq \{\} \wedge (\exists c\ d.\ K = cbox\ c\ d)$
using ‹$box\ a\ b \neq \{\}$›
by (*clarsimp simp*: *box_eq_empty*) (*fastforce simp add*: *field_split_simps dest*: *PiE_mem*)
next
have *realff*: $(real\ w) * 2\hat{}m < (real\ v) * 2\hat{}n \longleftrightarrow w * 2\hat{}m < v * 2\hat{}n$ **for** *m n v w*
using *of_nat_less_iff less_imp_of_nat_less* **by** *fastforce*
have $*$: $\forall v\ w.\ ?K0(m,v) \subseteq ?K0(n,w) \vee ?K0(n,w) \subseteq ?K0(m,v) \vee interior(?K0(m,v)) \cap interior(?K0(n,w)) = \{\}$
for *m n* — The symmetry argument requires a single HOL formula
proof (*rule linorder_wlog* [**where** *a=m* **and** *b=n*], *intro allI impI*)
fix *v w m* **and** *n::nat*
assume $m \leq n$ — WLOG we can assume $m \leq n$, when the first disjunct becomes impossible
have $?K0(n,w) \subseteq ?K0(m,v) \vee interior(?K0(m,v)) \cap interior(?K0(n,w)) = \{\}$
apply (*simp add*: *subset_box disjoint_interval*)

```
        apply (rule ccontr)
        apply (clarsimp simp add: aibi mult_le_cancel_right divide_le_cancel not_less
not_le)
        apply (drule_tac x=i in bspec, assumption)
        using ‹m≤n› realff [of _ _ 1+_] realff [of 1+_ _ 1+_]
        apply (auto simp: divide_simps add.commute not_le nat_le_iff_add realff)
         apply (simp_all add: power_add)
         apply (metis (no_types, hide_lams) mult_Suc mult_less_cancel2 not_less_eq
mult.assoc)
         apply (metis (no_types, hide_lams) mult_Suc mult_less_cancel2 not_less_eq
mult.assoc)
        done
         then show ?K0(m,v) ⊆ ?K0(n,w) ∨ ?K0(n,w) ⊆ ?K0(m,v) ∨ inte-
rior(?K0(m,v)) ∩ interior(?K0(n,w)) = {}
        by meson
    qed auto
   show ⋀A B. ⟦A ∈ ?D0; B ∈ ?D0⟧ ⟹ A ⊆ B ∨ B ⊆ A ∨ interior A ∩ interior
B = {}
      apply (erule imageE SigmaE)+
      using * by simp
  next
    show ∃K ∈ ?D0. x ∈ K ∧ K ⊆ g x if x ∈ S for x
    proof (simp only: bex_simps split_paired_Bex_Sigma)
      show ∃n. ∃f ∈ Basis →E {..<2 ^ n}. x ∈ ?K0(n,f) ∧ ?K0(n,f) ⊆ g x
      proof -
        obtain e where 0 < e
                   and e: ⋀y. (⋀i. i ∈ Basis ⟹ |x · i − y · i| ≤ e) ⟹ y ∈ g x
        proof -
          have x ∈ g x open (g x)
            using ‹gauge g› by (auto simp: gauge_def)
          then obtain ε where 0 < ε and ε: ball x ε ⊆ g x
            using openE by blast
          have norm (x − y) < ε
               if (⋀i. i ∈ Basis ⟹ |x · i − y · i| ≤ ε / (2 ∗ real DIM('a))) for y
          proof -
            have norm (x − y) ≤ (∑i∈Basis. |x · i − y · i|)
              by (metis (no_types, lifting) inner_diff_left norm_le_l1 sum.cong)
            also have ... ≤ DIM('a) ∗ (ε / (2 ∗ real DIM('a)))
              by (meson sum_bounded_above that)
            also have ... = ε / 2
              by (simp add: field_split_simps)
            also have ... < ε
              by (simp add: ‹0 < ε›)
            finally show ?thesis .
          qed
          then show ?thesis
               by (rule_tac e = ε / 2 / DIM('a) in that) (simp_all add: ‹0 < ε›
dist_norm subsetD [OF ε])
        qed
```

have *xab*: $x \in cbox\ a\ b$
using ‹$x \in S$› ‹$S \subseteq cbox\ a\ b$› **by** *blast*
obtain *n* **where** *n*: $norm\ (b - a)\ /\ 2\hat{}n < e$
using *real_arch_pow_inv* [*of e / norm*$(b - a)$ *1/2*] *normab* ‹$0 < e$›
by (*auto simp*: *field_split_simps*)
then have $norm\ (b - a) < e * 2\hat{}n$
by (*auto simp*: *field_split_simps*)
then have *bai*: $b \cdot i - a \cdot i < e * 2\ \hat{}\ n$ **if** $i \in Basis$ **for** *i*
proof −
have $b \cdot i - a \cdot i \le norm\ (b - a)$
by (*metis abs_of_nonneg dual_order.trans inner_diff_left linear norm_ge_zero Basis_le_norm that*)
also have ... $< e * 2\ \hat{}\ n$
using ‹$norm\ (b - a) < e * 2\ \hat{}\ n$› **by** *blast*
finally show *?thesis* **.**
qed
have *D*: $(a + n \le x \wedge x \le a + m) \Longrightarrow (a + n \le y \wedge y \le a + m) \Longrightarrow abs(x - y) \le m - n$
for *a m n x* **and** *y*::*real*
by *auto*
have $\forall i{\in}Basis.\ \exists k{<}2\ \hat{}\ n.\ (a \cdot i + real\ k * (b \cdot i - a \cdot i)\ /\ 2\ \hat{}\ n \le x \cdot i \wedge$
$x \cdot i \le a \cdot i + (real\ k + 1) * (b \cdot i - a \cdot i)\ /\ 2\ \hat{}\ n)$
proof
fix *i*::*'a* **assume** $i \in Basis$
consider $x \cdot i = b \cdot i \mid x \cdot i < b \cdot i$
using ‹$i \in Basis$› *mem_box*(*2*) *xab* **by** *force*
then show $\exists k{<}2\ \hat{}\ n.\ (a \cdot i + real\ k * (b \cdot i - a \cdot i)\ /\ 2\ \hat{}\ n \le x \cdot i \wedge$
$x \cdot i \le a \cdot i + (real\ k + 1) * (b \cdot i - a \cdot i)\ /\ 2\ \hat{}\ n)$
proof *cases*
case *1* **then show** *?thesis*
by (*rule_tac x = 2ˆn − 1* **in** *exI*) (*auto simp*: *algebra_simps field_split_simps of_nat_diff* ‹$i \in Basis$› *aibi*)
next
case *2*
then have *abi_less*: $a \cdot i < b \cdot i$
using ‹$i \in Basis$› *xab* **by** (*auto simp*: *mem_box*)
let $?k = nat\ \lfloor 2\ \hat{}\ n * (x \cdot i - a \cdot i)\ /\ (b \cdot i - a \cdot i) \rfloor$
show *?thesis*
proof (*intro exI conjI*)
show $?k < 2\ \hat{}\ n$
using *aibi xab* ‹$i \in Basis$›
by (*force simp*: *nat_less_iff floor_less_iff field_split_simps 2 mem_box*)
next
have $a \cdot i + real\ ?k * (b \cdot i - a \cdot i)\ /\ 2\ \hat{}\ n \le$
$a \cdot i + (2\ \hat{}\ n * (x \cdot i - a \cdot i)\ /\ (b \cdot i - a \cdot i)) * (b \cdot i - a \cdot i)\ /\ 2\ \hat{}\ n$
apply (*intro add_left_mono mult_right_mono divide_right_mono of_nat_floor*)

```
            using aibi [OF ‹i ∈ Basis›] xab 2
              apply (simp_all add: ‹i ∈ Basis› mem_box field_split_simps)
            done
          also have ... = x · i
            using abi_less by (simp add: field_split_simps)
          finally show a · i + real ?k * (b · i − a · i) / 2 ^ n ≤ x · i .
        next
          have x · i ≤ a · i + (2 ^ n * (x · i − a · i) / (b · i − a · i)) * (b · i
− a · i) / 2 ^ n
            using abi_less by (simp add: field_split_simps)
          also have ... ≤ a · i + (real ?k + 1) * (b · i − a · i) / 2 ^ n
                apply (intro add_left_mono mult_right_mono divide_right_mono
of_nat_floor)
            using aibi [OF ‹i ∈ Basis›] xab
              apply (auto simp: ‹i ∈ Basis› mem_box divide_simps)
            done
          finally show x · i ≤ a · i + (real ?k + 1) * (b · i − a · i) / 2 ^ n .
        qed
      qed
    qed
    then have ∃f∈Basis →E {..<2 ^ n}. x ∈ ?K0(n,f)
      apply (simp add: mem_box Bex_def)
      apply (clarify dest!: bchoice)
      apply (rule_tac x=restrict f Basis in exI, simp)
      done
    moreover have ⋀f. x ∈ ?K0(n,f) ⟹ ?K0(n,f) ⊆ g x
      apply (clarsimp simp add: mem_box)
      apply (rule e)
      apply (drule bspec D, assumption)+
      apply (erule order_trans)
      apply (simp add: divide_simps)
      using bai apply (force simp add: algebra_simps)
      done
    ultimately show ?thesis by auto
  qed
 qed
next
 show ⋀u v. cbox u v ∈ ?D0 ⟹ ∃n. ∀i ∈ Basis. v · i − u · i = (b · i − a ·
i) / 2^n
   by (force simp: eq_cbox box_eq_empty field_simps dest!: aibi)
next
 obtain j::'a where j ∈ Basis
   using nonempty_Basis by blast
 have finite {L ∈ ?D0. ?K0(n,f) ⊆ L} if f ∈ Basis →E {..<2 ^ n} for n f
 proof (rule finite_subset)
   let ?B = (λ(n, f::'a⇒nat). cbox (∑i∈Basis. (a · i + (f i) / 2^n * (b · i −
a · i)) *R i)
                              (∑i∈Basis. (a · i + ((f i) + 1) / 2^n * (b · i −
a · i)) *R i))
```

' $(SIGMA\ m{:}atMost\ n.\ Pi_E\ Basis\ (\lambda i{::}'a.\ lessThan\ (2\hat{}m)))$
have $?K0(m,g) \in ?B$ **if** $g \in Basis \rightarrow_E \{..<2\ \hat{}\ m\}\ ?K0(n,f) \subseteq ?K0(m,g)$
for $m\ g$
proof −
have dd: $w\ /\ m \le v\ /\ n \wedge (v+1)\ /\ n \le (w+1)\ /\ m$
$\Longrightarrow inverse\ n \le inverse\ m$ **for** $w\ m\ v\ n{::}real$
by (*auto simp*: *field_split_simps*)
have $bjaj$: $b \cdot j - a \cdot j > 0$
using ‹$j \in Basis$› ‹$box\ a\ b \neq \{\}$› *box_eq_empty*(*1*) **by** *fastforce*
have $((g\ j)\ /\ 2\ \hat{}\ m) * (b \cdot j - a \cdot j) \le ((f\ j)\ /\ 2\ \hat{}\ n) * (b \cdot j - a \cdot j) \wedge$
$(((f\ j) + 1)\ /\ 2\ \hat{}\ n) * (b \cdot j - a \cdot j) \le (((g\ j) + 1)\ /\ 2\ \hat{}\ m) * (b \cdot j$
$- a \cdot j)$
using *that* ‹$j \in Basis$› **by** (*simp add*: *subset_box field_split_simps aibi*)
then have $((g\ j)\ /\ 2\ \hat{}\ m) \le ((f\ j)\ /\ 2\ \hat{}\ n) \wedge$
$((real(f\ j) + 1)\ /\ 2\ \hat{}\ n) \le ((real(g\ j) + 1)\ /\ 2\ \hat{}\ m)$
by (*metis bjaj mult.commute of_nat_1 of_nat_add mult_le_cancel_iff2*)
then have $inverse\ (2\hat{}n) \le (inverse\ (2\hat{}m) :: real)$
by (*rule dd*)
then have $m \le n$
by *auto*
show *?thesis*
by (*rule imageI*) (*simp add*: ‹$m \le n$› *that*)
qed
then show $\{L \in ?D0.\ ?K0(n,f) \subseteq L\} \subseteq ?B$
by *auto*
show *finite ?B*
by (*intro finite_imageI finite_SigmaI finite_atMost finite_lessThan finite_PiE finite_Basis*)
qed
then show *finite* $\{L \in ?D0.\ K \subseteq L\}$ **if** $K \in ?D0$ **for** K
using *that* **by** *auto*
qed
let $?D1 = \{K \in \mathcal{D}0.\ \exists x \in S \cap K.\ K \subseteq g\ x\}$
obtain $\mathcal{D}$ **where** *count*: *countable* $\mathcal{D}$
and *sub*: $\bigcup\mathcal{D} \subseteq cbox\ a\ b\ \ S \subseteq \bigcup\mathcal{D}$
and *int*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow (interior\ K \neq \{\}) \wedge (\exists c\ d.\ K = cbox\ c\ d)$
and *intdj*: $\bigwedge A\ B.\ [\![A \in \mathcal{D}; B \in \mathcal{D}]\!] \Longrightarrow A \subseteq B \vee B \subseteq A \vee interior\ A$
$\cap\ interior\ B = \{\}$
and *SK*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow \exists x.\ x \in S \cap K \wedge K \subseteq g\ x$
and *cbox*: $\bigwedge u\ v.\ cbox\ u\ v \in \mathcal{D} \Longrightarrow \exists n.\ \forall i \in Basis.\ v \cdot i - u \cdot i = (b$
$\cdot\ i - a \cdot i)\ /\ 2\hat{}n$
and *fin*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow finite\ \{L.\ L \in \mathcal{D} \wedge K \subseteq L\}$
proof
show *countable ?D1* **using** *count countable_subset*
by (*simp add*: *count countable_subset*)
show $\bigcup\ ?D1 \subseteq cbox\ a\ b$
using *sub* **by** *blast*
show $S \subseteq \bigcup\ ?D1$
using *SK* **by** (*force simp*:)

show $\bigwedge K.\ K \in ?D1 \Longrightarrow (interior\ K \neq \{\}) \wedge (\exists c\ d.\ K = cbox\ c\ d)$
using *int* **by** *blast*
show $\bigwedge A\ B.\ [\![A \in ?D1;\ B \in ?D1]\!] \Longrightarrow A \subseteq B \vee B \subseteq A \vee interior\ A \cap interior\ B = \{\}$
using *intdj* **by** *blast*
show $\bigwedge K.\ K \in ?D1 \Longrightarrow \exists x.\ x \in S \cap K \wedge K \subseteq g\ x$
by *auto*
show $\bigwedge u\ v.\ cbox\ u\ v \in ?D1 \Longrightarrow \exists n.\ \forall i \in Basis.\ v \cdot i - u \cdot i = (b \cdot i - a \cdot i) / 2\hat{}n$
using *cbox* **by** *blast*
show $\bigwedge K.\ K \in ?D1 \Longrightarrow finite\ \{L.\ L \in ?D1 \wedge K \subseteq L\}$
using *fin* **by** *simp* (*metis* (*mono_tags*, *lifting*) *Collect_mono rev_finite_subset*)
qed
let $?\mathcal{D} = \{K \in \mathcal{D}.\ \forall K'.\ K' \in \mathcal{D} \wedge K \neq K' \longrightarrow \neg(K \subseteq K')\}$
show *?thesis*
proof (*rule that*)
show *countable* $?\mathcal{D}$
by (*blast intro*: *countable_subset* [*OF* _ *count*])
show $\bigcup ?\mathcal{D} \subseteq cbox\ a\ b$
using *sub* **by** *blast*
show $S \subseteq \bigcup ?\mathcal{D}$
proof *clarsimp*
fix x
assume $x \in S$
then obtain X **where** $x \in X\ X \in \mathcal{D}$ **using** ‹$S \subseteq \bigcup \mathcal{D}$› **by** *blast*
let $?R = \{(K,L).\ K \in \mathcal{D} \wedge L \in \mathcal{D} \wedge L \subset K\}$
have *irrR*: *irrefl ?R* **by** (*force simp*: *irrefl_def*)
have *traR*: *trans ?R* **by** (*force simp*: *trans_def*)
have *finR*: $\bigwedge x.\ finite\ \{y.\ (y,\ x) \in ?R\}$
by *simp* (*metis* (*mono_tags*, *lifting*) *fin* ‹$X \in \mathcal{D}$› *finite_subset mem_Collect_eq psubset_imp_subset subsetI*)
have $\{X \in \mathcal{D}.\ x \in X\} \neq \{\}$
using ‹$X \in \mathcal{D}$› ‹$x \in X$› **by** *blast*
then obtain Y **where** $Y \in \{X \in \mathcal{D}.\ x \in X\}\ \bigwedge Y'.\ (Y',\ Y) \in ?R \Longrightarrow Y' \notin \{X \in \mathcal{D}.\ x \in X\}$
by (*rule wfE_min'* [*OF wf_finite_segments* [*OF irrR traR finR*]]) *blast*
then show $\exists Y.\ Y \in \mathcal{D} \wedge (\forall K'.\ K' \in \mathcal{D} \wedge Y \neq K' \longrightarrow \neg\ Y \subseteq K') \wedge x \in Y$
by *blast*
qed
show $\bigwedge K.\ K \in ?\mathcal{D} \Longrightarrow interior\ K \neq \{\} \wedge (\exists c\ d.\ K = cbox\ c\ d)$
by (*blast intro*: *dest*: *int*)
show *pairwise* $(\lambda A\ B.\ interior\ A \cap interior\ B = \{\})\ ?\mathcal{D}$
using *intdj* **by** (*simp add*: *pairwise_def*) *metis*
show $\bigwedge K.\ K \in ?\mathcal{D} \Longrightarrow \exists x \in S \cap K.\ K \subseteq g\ x$
using *SK* **by** *force*
show $\bigwedge u\ v.\ cbox\ u\ v \in ?\mathcal{D} \Longrightarrow \exists n.\ \forall i{\in}Basis.\ v \cdot i - u \cdot i = (b \cdot i - a \cdot i) / 2\hat{}n$
using *cbox* **by** *force*

qed
qed

6.14.16 Division filter

Divisions over all gauges towards finer divisions.

definition *division_filter* :: $'a$::*euclidean_space set* ⇒ ($'a$ × $'a$ *set*) *set filter*
where *division_filter s* = (*INF g*∈{*g. gauge g*}. *principal* {*p. p tagged_division_of s* ∧ *g fine p*})

proposition *eventually_division_filter*:
(∀$_F$ *p in division_filter s. P p*) ⟷
(∃ *g. gauge g* ∧ (∀ *p. p tagged_division_of s* ∧ *g fine p* ⟶ *P p*))
unfolding *division_filter_def*
proof (*subst eventually_INF_base*; *clarsimp*)
fix *g1 g2* :: $'a$ ⇒ $'a$ *set* **show** *gauge g1* ⟹ *gauge g2* ⟹ ∃ *x. gauge x* ∧
{*p. p tagged_division_of s* ∧ *x fine p*} ⊆ {*p. p tagged_division_of s* ∧ *g1 fine p*}
∧
{*p. p tagged_division_of s* ∧ *x fine p*} ⊆ {*p. p tagged_division_of s* ∧ *g2 fine p*}
by (*intro exI*[*of _ λx. g1 x* ∩ *g2 x*]) (*auto simp*: *fine_Int*)
qed (*auto simp*: *eventually_principal*)

lemma *division_filter_not_empty*: *division_filter* (*cbox a b*) ≠ *bot*
unfolding *trivial_limit_def eventually_division_filter*
by (*auto elim*: *fine_division_exists*)

lemma *eventually_division_filter_tagged_division*:
eventually (λ*p. p tagged_division_of s*) (*division_filter s*)
unfolding *eventually_division_filter* **by** (*intro exI*[*of _ λx. ball x 1*]) *auto*

end

6.15 Henstock-Kurzweil Gauge Integration in Many Dimensions

theory *Henstock_Kurzweil_Integration*
imports
Lebesgue_Measure Tagged_Division
begin

lemma *norm_diff2*: ⟦*y* = *y1* + *y2*; *x* = *x1* + *x2*; *e* = *e1* + *e2*; *norm*(*y1* − *x1*) ≤ *e1*; *norm*(*y2* − *x2*) ≤ *e2*⟧
⟹ *norm*(*y*−*x*) ≤ *e*
using *norm_triangle_mono* [*of y1* − *x1 e1 y2* − *x2 e2*]
by (*simp add*: *add_diff_add*)

lemma *setcomp_dot1*: {*z. P* (*z* · (*i*,*0*))} = {(*x*,*y*). *P*(*x* · *i*)}
by *auto*

lemma *setcomp_dot2*: $\{z.\ P\ (z \cdot (0,i))\} = \{(x,y).\ P(y \cdot i)\}$
by *auto*

lemma *Sigma_Int_Paircomp1*: $(Sigma\ A\ B) \cap \{(x,\ y).\ P\ x\} = Sigma\ (A \cap \{x.\ P\ x\})\ B$
by *blast*

lemma *Sigma_Int_Paircomp2*: $(Sigma\ A\ B) \cap \{(x,\ y).\ P\ y\} = Sigma\ A\ (\lambda z.\ B\ z \cap \{y.\ P\ y\})$
by *blast*

6.15.1 Content (length, area, volume...) of an interval

abbreviation *content* :: *'a::euclidean_space set* ⇒ *real*
where *content s* ≡ *measure lborel s*

lemma *content_cbox_cases*:
content (*cbox a b*) = (*if* ∀ *i*∈*Basis*. *a*•*i* ≤ *b*•*i then prod* (λ*i*. *b*•*i* − *a*•*i*) *Basis else 0*)
by (*simp add*: *measure_lborel_cbox_eq inner_diff*)

lemma *content_cbox*: ∀ *i*∈*Basis*. *a*•*i* ≤ *b*•*i* ⟹ *content* (*cbox a b*) = (∏ *i*∈*Basis*. *b*•*i* − *a*•*i*)
unfolding *content_cbox_cases* **by** *simp*

lemma *content_cbox'*: *cbox a b* ≠ {} ⟹ *content* (*cbox a b*) = (∏ *i*∈*Basis*. *b*•*i* − *a*•*i*)
by (*simp add*: *box_ne_empty inner_diff*)

lemma *content_cbox_if*: *content* (*cbox a b*) = (*if cbox a b* = {} *then 0 else* ∏ *i*∈*Basis*. *b*•*i* − *a*•*i*)
by (*simp add*: *content_cbox'*)

lemma *content_cbox_cart*:
cbox a b ≠ {} ⟹ *content*(*cbox a b*) = *prod* (λ*i*. *b*$*i* − *a*$*i*) *UNIV*
by (*simp add*: *content_cbox_if Basis_vec_def cart_eq_inner_axis axis_eq_axis prod.UNION_disjoint*)

lemma *content_cbox_if_cart*:
content(*cbox a b*) = (*if cbox a b* = {} *then 0 else prod* (λ*i*. *b*$*i* − *a*$*i*) *UNIV*)
by (*simp add*: *content_cbox_cart*)

lemma *content_division_of*:
assumes *K* ∈ 𝒟 𝒟 *division_of S*
shows *content K* = (∏ *i* ∈ *Basis*. *interval_upperbound K* · *i* − *interval_lowerbound K* · *i*)
proof −
obtain *a b* **where** *K* = *cbox a b*
using *cbox_division_memE assms* **by** *metis*

then show *?thesis*
using *assms* **by** (*force simp*: *division_of_def content_cbox'*)
qed

lemma *content_real*: $a \le b \Longrightarrow$ *content* $\{a..b\} = b - a$
by *simp*

lemma *abs_eq_content*: $|y - x|$ = (*if* $x \le y$ *then content* $\{x..y\}$ *else content* $\{y..x\}$)
by (*auto simp*: *content_real*)

lemma *content_singleton*: *content* $\{a\} = 0$
by *simp*

lemma *content_unit*[*iff*]: *content* (*cbox 0* (*One*::'*a*::*euclidean_space*)) = *1*
by *simp*

lemma *content_pos_le* [*iff*]: $0 \le$ *content X*
by *simp*

corollary *content_nonneg* [*simp*]: ¬ *content* (*cbox a b*) < *0*
using *not_le* **by** *blast*

lemma *content_pos_lt*: ∀ *i*∈*Basis*. *a*•*i* < *b*•*i* ⟹ *0* < *content* (*cbox a b*)
by (*auto simp*: *less_imp_le inner_diff box_eq_empty intro*!: *prod_pos*)

lemma *content_eq_0*: *content* (*cbox a b*) = *0* ⟷ (∃ *i*∈*Basis*. *b*•*i* ≤ *a*•*i*)
by (*auto simp*: *content_cbox_cases not_le intro*: *less_imp_le antisym eq_refl*)

lemma *content_eq_0_interior*: *content* (*cbox a b*) = *0* ⟷ *interior*(*cbox a b*) = {}
unfolding *content_eq_0 interior_cbox box_eq_empty* **by** *auto*

lemma *content_pos_lt_eq*: *0* < *content* (*cbox a* (*b*::'*a*::*euclidean_space*)) ⟷ (∀ *i*∈*Basis*. *a*•*i* < *b*•*i*)
by (*auto simp add*: *content_cbox_cases less_le prod_nonneg*)

lemma *content_empty* [*simp*]: *content* {} = *0*
by *simp*

lemma *content_real_if* [*simp*]: *content* $\{a..b\}$ = (*if* $a \le b$ *then* $b - a$ *else 0*)
by (*simp add*: *content_real*)

lemma *content_subset*: *cbox a b* ⊆ *cbox c d* ⟹ *content* (*cbox a b*) ≤ *content* (*cbox c d*)
unfolding *measure_def*
by (*intro enn2real_mono emeasure_mono*) (*auto simp*: *emeasure_lborel_cbox_eq*)

lemma *content_lt_nz*: *0* < *content* (*cbox a b*) ⟷ *content* (*cbox a b*) ≠ *0*
unfolding *content_pos_lt_eq content_eq_0* **unfolding** *not_ex not_le* **by** *fastforce*

lemma *content_Pair*: *content* (*cbox* (*a*,*c*) (*b*,*d*)) = *content* (*cbox a b*) ∗ *content* (*cbox c d*)
unfolding *measure_lborel_cbox_eq Basis_prod_def*
apply (*subst prod.union_disjoint*)
apply (*auto simp*: *bex_Un ball_Un*)
apply (*subst* (*1 2*) *prod.reindex_nontrivial*)
apply *auto*
done

lemma *content_cbox_pair_eq0_D*:
content (*cbox* (*a*,*c*) (*b*,*d*)) = *0* ⟹ *content* (*cbox a b*) = *0* ∨ *content* (*cbox c d*) = *0*
by (*simp add*: *content_Pair*)

lemma *content_cbox_plus*:
fixes *x* :: *'a*::*euclidean_space*
shows *content*(*cbox x* (*x* + *h* $*_R$ *One*)) = (*if h* ≥ *0 then h* ^ *DIM*(*'a*) *else 0*)
by (*simp add*: *algebra_simps content_cbox_if box_eq_empty*)

lemma *content_0_subset*: *content*(*cbox a b*) = *0* ⟹ *s* ⊆ *cbox a b* ⟹ *content s* = *0*
using *emeasure_mono*[*of s cbox a b lborel*]
by (*auto simp*: *measure_def enn2real_eq_0_iff emeasure_lborel_cbox_eq*)

lemma *content_ball_pos*:
assumes *r* > *0*
shows *content* (*ball c r*) > *0*
proof −
from *rational_boxes*[*OF assms*, *of c*] **obtain** *a b* **where** *ab*: *c* ∈ *box a b box a b* ⊆ *ball c r*
by *auto*
from *ab* **have** *0* < *content* (*box a b*)
by (*subst measure_lborel_box_eq*) (*auto intro*!: *prod_pos simp*: *algebra_simps box_def*)
have *emeasure lborel* (*box a b*) ≤ *emeasure lborel* (*ball c r*)
using *ab* **by** (*intro emeasure_mono*) *auto*
also have *emeasure lborel* (*box a b*) = *ennreal* (*content* (*box a b*))
using *emeasure_lborel_box_finite*[*of a b*] **by** (*intro emeasure_eq_ennreal_measure*) *auto*
also have *emeasure lborel* (*ball c r*) = *ennreal* (*content* (*ball c r*))
using *emeasure_lborel_ball_finite*[*of c r*] **by** (*intro emeasure_eq_ennreal_measure*) *auto*
finally show *?thesis*
using ‹*content* (*box a b*) > *0*› **by** *simp*
qed

lemma *content_cball_pos*:
assumes *r* > *0*
shows *content* (*cball c r*) > *0*

proof −
from *rational_boxes*[*OF assms*, *of c*] **obtain** *a b* **where** *ab*: $c \in$ *box a b box a b* $\subseteq$ *ball c r*
by *auto*
from *ab* **have** *0* < *content* (*box a b*)
by (*subst measure_lborel_box_eq*) (*auto intro*!: *prod_pos simp*: *algebra_simps box_def*)
have *emeasure lborel* (*box a b*) ≤ *emeasure lborel* (*ball c r*)
using *ab* **by** (*intro emeasure_mono*) *auto*
also have . . . ≤ *emeasure lborel* (*cball c r*)
by (*intro emeasure_mono*) *auto*
also have *emeasure lborel* (*box a b*) = *ennreal* (*content* (*box a b*))
using *emeasure_lborel_box_finite*[*of a b*] **by** (*intro emeasure_eq_ennreal_measure*) *auto*
also have *emeasure lborel* (*cball c r*) = *ennreal* (*content* (*cball c r*))
using *emeasure_lborel_cball_finite*[*of c r*] **by** (*intro emeasure_eq_ennreal_measure*) *auto*
finally show *?thesis*
using ‹*content* (*box a b*) > *0*› **by** *simp*
qed

lemma *content_split*:
fixes *a* :: ′*a*::*euclidean_space*
assumes $k \in$ *Basis*
shows *content* (*cbox a b*) = *content*(*cbox a b* ∩ {*x*. *x*•*k* ≤ *c*}) + *content*(*cbox a b* ∩ {*x*. *x*•*k* ≥ *c*})
— Prove using measure theory
proof (*cases* ∀ *i*∈*Basis*. *a* • *i* ≤ *b* • *i*)
case *True*
have *1*: ⋀*X Y Z*. (∏ *i*∈*Basis*. *Z i* (*if i* = *k then X else Y i*)) = *Z k X* ∗ (∏ *i*∈*Basis*−{*k*}. *Z i* (*Y i*))
by (*simp add*: *if_distrib prod.delta_remove assms*)
note *simps* = *interval_split*[*OF assms*] *content_cbox_cases*
have *2*: (∏ *i*∈*Basis*. *b*•*i* − *a*•*i*) = (∏ *i*∈*Basis*−{*k*}. *b*•*i* − *a*•*i*) ∗ (*b*•*k* − *a*•*k*)
by (*metis* (*no_types*, *lifting*) *assms finite_Basis mult.commute prod.remove*)
have ⋀*x*. *min* (*b* • *k*) *c* = *max* (*a* • *k*) *c* ⟹
x ∗ (*b*•*k* − *a*•*k*) = *x* ∗ (*max* (*a* • *k*) *c* − *a* • *k*) + *x* ∗ (*b* • *k* − *max* (*a* • *k*) *c*)
by (*auto simp add*: *field_simps*)
moreover
have ∗∗: (∏ *i*∈*Basis*. ((∑ *i*∈*Basis*. (*if i* = *k then min* (*b* • *k*) *c else b* • *i*) $*_R$ *i*) • *i* − *a* • *i*)) =
(∏ *i*∈*Basis*. (*if i* = *k then min* (*b* • *k*) *c else b* • *i*) − *a* • *i*)
(∏ *i*∈*Basis*. *b* • *i* − ((∑ *i*∈*Basis*. (*if i* = *k then max* (*a* • *k*) *c else a* • *i*) $*_R$ *i*) • *i*)) =
(∏ *i*∈*Basis*. *b* • *i* − (*if i* = *k then max* (*a* • *k*) *c else a* • *i*))
by (*auto intro*!: *prod.cong*)
have ¬ *a* • *k* ≤ *c* ⟹ ¬ *c* ≤ *b* • *k* ⟹ *False*
unfolding *not_le* **using** *True assms* **by** *auto*
ultimately show *?thesis*

```
    using assms unfolding simps ** 1[of λi x. b•i − x] 1[of λi x. x − a•i] 2
    by auto
next
  case False
  then have cbox a b = {}
    unfolding box_eq_empty by (auto simp: not_le)
  then show ?thesis
    by (auto simp: not_le)
qed

lemma division_of_content_0:
  assumes content (cbox a b) = 0 d division_of (cbox a b) K ∈ d
  shows content K = 0
  unfolding forall_in_division[OF assms(2)]
  by (meson assms content_0_subset division_of_def)

lemma sum_content_null:
  assumes content (cbox a b) = 0
    and p tagged_division_of (cbox a b)
  shows (∑(x,K)∈p. content K *R f x) = (0::'a::real_normed_vector)
proof (rule sum.neutral, rule)
  fix y
  assume y: y ∈ p
  obtain x K where xk: y = (x, K)
    using surj_pair[of y] by blast
  then obtain c d where k: K = cbox c d K ⊆ cbox a b
    by (metis assms(2) tagged_division_ofD(3) tagged_division_ofD(4) y)
  have (λ(x',K'). content K' *R f x') y = content K *R f x
    unfolding xk by auto
  also have ... = 0
    using assms(1) content_0_subset k(2) by auto
  finally show (λ(x, k). content k *R f x) y = 0 .
qed

global_interpretation sum_content: operative plus 0 content
  rewrites comm_monoid_set.F plus 0 = sum
proof −
  interpret operative plus 0 content
    by standard (auto simp add: content_split [symmetric] content_eq_0_interior)
  show operative plus 0 content
    by standard
  show comm_monoid_set.F plus 0 = sum
    by (simp add: sum_def)
qed

lemma additive_content_division: d division_of (cbox a b) =⇒ sum content d =
content (cbox a b)
  by (fact sum_content.division)
```

lemma *additive_content_tagged_division*:
d tagged_division_of (*cbox a b*) $\Longrightarrow$ *sum* (λ(*x*,*l*). *content l*) *d* = *content* (*cbox a b*)
by (*fact sum_content.tagged_division*)

lemma *subadditive_content_division*:
assumes $\mathcal{D}$ *division_of S S* $\subseteq$ *cbox a b*
shows *sum content* $\mathcal{D}$ $\leq$ *content*(*cbox a b*)
proof −
have $\mathcal{D}$ *division_of* $\bigcup\mathcal{D}$ $\bigcup\mathcal{D}$ $\subseteq$ *cbox a b*
using *assms* **by** *auto*
then obtain $\mathcal{D}'$ **where** $\mathcal{D} \subseteq \mathcal{D}'$ $\mathcal{D}'$ *division_of cbox a b*
using *partial_division_extend_interval* **by** *metis*
then have *sum content* $\mathcal{D}$ $\leq$ *sum content* $\mathcal{D}'$
using *sum_mono2* **by** *blast*
also have ... $\leq$ *content*(*cbox a b*)
by (*simp add*: ‹$\mathcal{D}'$ *division_of cbox a b*› *additive_content_division less_eq_real_def*)
finally show *?thesis* **.**
qed

lemma *content_real_eq_0*: *content* {*a*..*b*::*real*} = *0* $\longleftrightarrow$ *a* $\geq$ *b*
by (*metis atLeastatMost_empty_iff2 content_empty content_real diff_self eq_iff le_cases le_iff_diff_le_0*)

lemma *property_empty_interval*: $\forall$ *a b*. *content* (*cbox a b*) = *0* $\longrightarrow$ *P* (*cbox a b*) $\Longrightarrow$ *P* {}
using *content_empty* **unfolding** *empty_as_interval* **by** *auto*

lemma *interval_bounds_nz_content* [*simp*]:
assumes *content* (*cbox a b*) $\neq$ *0*
shows *interval_upperbound* (*cbox a b*) = *b*
and *interval_lowerbound* (*cbox a b*) = *a*
by (*metis assms content_empty interval_bounds'*)+

6.15.2 Gauge integral

Case distinction to define it first on compact intervals first, then use a limit. This is only much later unified. In Fremlin: Measure Theory, Volume 4I this is generalized using residual sets.

definition *has_integral* :: (*'n*::*euclidean_space* $\Rightarrow$ *'b*::*real_normed_vector*) $\Rightarrow$ *'b* $\Rightarrow$ *'n set* $\Rightarrow$ *bool*
(**infixr** *has'_integral 46*)
where (*f has_integral I*) *s* $\longleftrightarrow$
(*if* $\exists$ *a b*. *s* = *cbox a b*
then ((λ*p*. $\sum$(*x*,*k*)$\in$*p*. *content k* $*_R$ *f x*) $\longlongrightarrow$ *I*) (*division_filter s*)
else ($\forall$ *e*>*0*. $\exists$ *B*>*0*. $\forall$ *a b*. *ball 0 B* $\subseteq$ *cbox a b* $\longrightarrow$
($\exists$ *z*. ((λ*p*. $\sum$(*x*,*k*)$\in$*p*. *content k* $*_R$ (*if x* $\in$ *s then f x else 0*)) $\longrightarrow$ *z*) (*division_filter* (*cbox a b*)) $\wedge$
norm (*z* − *I*) < *e*)))

lemma *has_integral_cbox*:
  (*f has_integral I*) (*cbox a b*) ⟷ ((λ*p*. ∑(*x*,*k*)∈*p*. *content k* $*_R$ *f x*) ⟶ *I*) (*division_filter* (*cbox a b*))
  by (*auto simp add*: *has_integral_def*)

lemma *has_integral*:
  (*f has_integral y*) (*cbox a b*) ⟷
    (∀ *e*>*0*. ∃ γ. *gauge* γ ∧
      (∀ 𝒟. 𝒟 *tagged_division_of* (*cbox a b*) ∧ γ *fine* 𝒟 ⟶
        norm (*sum* (λ(*x*,*k*). *content*(*k*) $*_R$ *f x*) 𝒟 − *y*) < *e*))
  by (*auto simp*: *dist_norm eventually_division_filter has_integral_def tendsto_iff*)

lemma *has_integral_real*:
  (*f has_integral y*) {*a*..*b*::*real*} ⟷
    (∀ *e*>*0*. ∃ γ. *gauge* γ ∧
      (∀ 𝒟. 𝒟 *tagged_division_of* {*a*..*b*} ∧ γ *fine* 𝒟 ⟶
        norm (*sum* (λ(*x*,*k*). *content*(*k*) $*_R$ *f x*) 𝒟 − *y*) < *e*))
  unfolding *box_real*[*symmetric*] **by** (*rule has_integral*)

lemma *has_integralD*[*dest*]:
  assumes (*f has_integral y*) (*cbox a b*)
    and *e* > *0*
  obtains γ
    where *gauge* γ
      and ⋀𝒟. 𝒟 *tagged_division_of* (*cbox a b*) ⟹ γ *fine* 𝒟 ⟹
        norm ((∑(*x*,*k*)∈𝒟. *content k* $*_R$ *f x*) − *y*) < *e*
  using *assms* **unfolding** *has_integral* **by** *auto*

lemma *has_integral_alt*:
  (*f has_integral y*) *i* ⟷
    (*if* ∃ *a b*. *i* = *cbox a b*
      then (*f has_integral y*) *i*
      else (∀ *e*>*0*. ∃ *B*>*0*. ∀ *a b*. *ball 0 B* ⊆ *cbox a b* ⟶
        (∃ *z*. ((λ*x*. *if x* ∈ *i then f x else 0*) *has_integral z*) (*cbox a b*) ∧ *norm* (*z* − *y*) < *e*)))
  by (*subst has_integral_def*) (*auto simp add*: *has_integral_cbox*)

lemma *has_integral_altD*:
  assumes (*f has_integral y*) *i*
    and ¬ (∃ *a b*. *i* = *cbox a b*)
    and *e*>*0*
  obtains *B* **where** *B* > *0*
    and ∀ *a b*. *ball 0 B* ⊆ *cbox a b* ⟶
      (∃ *z*. ((λ*x*. *if x* ∈ *i then f*(*x*) *else 0*) *has_integral z*) (*cbox a b*) ∧ *norm*(*z* − *y*) < *e*)
  using *assms has_integral_alt*[*of f y i*] **by** *auto*

definition *integrable_on* (**infixr** *integrable′_on 46*)

where *f integrable_on i* ⟷ (∃ *y*. (*f has_integral y*) *i*)

definition *integral i f* = (*SOME y*. (*f has_integral y*) *i* ∨ ¬ *f integrable_on i* ∧ *y*=*0*)

lemma *integrable_integral*[*intro*]: *f integrable_on i* ⟹ (*f has_integral* (*integral i f*)) *i*
unfolding *integrable_on_def integral_def* **by** (*metis* (*mono_tags*, *lifting*) *someI_ex*)

lemma *not_integrable_integral*: ¬ *f integrable_on i* ⟹ *integral i f* = *0*
unfolding *integrable_on_def integral_def* **by** *blast*

lemma *has_integral_integrable*[*dest*]: (*f has_integral i*) *s* ⟹ *f integrable_on s*
unfolding *integrable_on_def* **by** *auto*

lemma *has_integral_integral*: *f integrable_on s* ⟷ (*f has_integral* (*integral s f*)) *s*
by *auto*

6.15.3 Basic theorems about integrals

lemma *has_integral_eq_rhs*: (*f has_integral j*) *S* ⟹ *i* = *j* ⟹ (*f has_integral i*) *S*
by (*rule forw_subst*)

lemma *has_integral_unique_cbox*:
fixes *f* :: ′*n*::*euclidean_space* ⇒ ′*a*::*real_normed_vector*
shows (*f has_integral k1*) (*cbox a b*) ⟹ (*f has_integral k2*) (*cbox a b*) ⟹ *k1* = *k2*
by (*auto simp*: *has_integral_cbox intro*: *tendsto_unique*[*OF division_filter_not_empty*])

lemma *has_integral_unique*:
fixes *f* :: ′*n*::*euclidean_space* ⇒ ′*a*::*real_normed_vector*
assumes (*f has_integral k1*) *i* (*f has_integral k2*) *i*
shows *k1* = *k2*
proof (*rule ccontr*)
let *?e* = *norm* (*k1* − *k2*)/*2*
let *?F* = (λ*x*. *if x* ∈ *i then f x else 0*)
assume *k1* ≠ *k2*
then have *e*: *?e* > *0*
by *auto*
have *nonbox*: ¬ (∃ *a b*. *i* = *cbox a b*)
using ‹*k1* ≠ *k2*› *assms has_integral_unique_cbox* **by** *blast*
obtain *B1* **where** *B1*:
0 < *B1*
⋀*a b*. *ball 0 B1* ⊆ *cbox a b* ⟹
∃ *z*. (*?F has_integral z*) (*cbox a b*) ∧ *norm* (*z* − *k1*) < *norm* (*k1* − *k2*)/*2*
by (*rule has_integral_altD*[*OF assms*(*1*) *nonbox*,*OF e*]) *blast*
obtain *B2* **where** *B2*:
0 < *B2*

⋀a b. ball 0 B2 ⊆ cbox a b ⟹
  ∃ z. (?F has_integral z) (cbox a b) ∧ norm (z − k2) < norm (k1 − k2)/2
 by (rule has_integral_altD[OF assms(2) nonbox,OF e]) blast
obtain a b :: 'n **where** ab: ball 0 B1 ⊆ cbox a b ball 0 B2 ⊆ cbox a b
 by (metis Un_subset_iff bounded_Un bounded_ball bounded_subset_cbox_symmetric)
obtain w **where** w: (?F has_integral w) (cbox a b) norm (w − k1) < norm (k1 − k2)/2
 using B1(2)[OF ab(1)] **by** blast
obtain z **where** z: (?F has_integral z) (cbox a b) norm (z − k2) < norm (k1 − k2)/2
 using B2(2)[OF ab(2)] **by** blast
have z = w
 using has_integral_unique_cbox[OF w(1) z(1)] **by** auto
then have norm (k1 − k2) ≤ norm (z − k2) + norm (w − k1)
 using norm_triangle_ineq4 [of k1 − w k2 − z]
 by (auto simp add: norm_minus_commute)
also have ... < norm (k1 − k2)/2 + norm (k1 − k2)/2
 by (metis add_strict_mono z(2) w(2))
finally show False **by** auto
qed

lemma integral_unique [intro]: (f has_integral y) k ⟹ integral k f = y
 unfolding integral_def
 by (rule some_equality) (auto intro: has_integral_unique)

lemma has_integral_iff: (f has_integral i) S ⟷ (f integrable_on S ∧ integral S f = i)
 by blast

lemma eq_integralD: integral k f = y ⟹ (f has_integral y) k ∨ ¬ f integrable_on k ∧ y=0
 unfolding integral_def integrable_on_def
 apply (erule subst)
 apply (rule someI_ex)
 by blast

lemma has_integral_const [intro]:
 fixes a b :: 'a::euclidean_space
 shows ((λx. c) has_integral (content (cbox a b) *R c)) (cbox a b)
 using eventually_division_filter_tagged_division[of cbox a b]
  additive_content_tagged_division[of _ a b]
 by (auto simp: has_integral_cbox split_beta' scaleR_sum_left[symmetric]
   elim!: eventually_mono intro!: tendsto_cong[THEN iffD1, OF _ tendsto_const])

lemma has_integral_const_real [intro]:
 fixes a b :: real
 shows ((λx. c) has_integral (content {a..b} *R c)) {a..b}
 by (metis box_real(2) has_integral_const)

```
lemma has_integral_integrable_integral: (f has_integral i) s ⟷ f integrable_on s ∧
integral s f = i
  by blast

lemma integral_const [simp]:
  fixes a b :: 'a::euclidean_space
  shows integral (cbox a b) (λx. c) = content (cbox a b) *R c
  by (rule integral_unique) (rule has_integral_const)

lemma integral_const_real [simp]:
  fixes a b :: real
  shows integral {a..b} (λx. c) = content {a..b} *R c
  by (metis box_real(2) integral_const)

lemma has_integral_is_0_cbox:
  fixes f :: 'n::euclidean_space ⇒ 'a::real_normed_vector
  assumes ⋀x. x ∈ cbox a b ⟹ f x = 0
  shows (f has_integral 0) (cbox a b)
    unfolding has_integral_cbox
    using eventually_division_filter_tagged_division[of cbox a b] assms
    by (subst tendsto_cong[where g=λ_. 0])
      (auto elim!: eventually_mono intro!: sum.neutral simp: tag_in_interval)

lemma has_integral_is_0:
  fixes f :: 'n::euclidean_space ⇒ 'a::real_normed_vector
  assumes ⋀x. x ∈ S ⟹ f x = 0
  shows (f has_integral 0) S
proof (cases (∃ a b. S = cbox a b))
  case True with assms has_integral_is_0_cbox show ?thesis
    by blast
next
  case False
  have *: (λx. if x ∈ S then f x else 0) = (λx. 0)
    by (auto simp add: assms)
  show ?thesis
    using has_integral_is_0_cbox False
    by (subst has_integral_alt) (force simp add: *)
qed

lemma has_integral_0[simp]: ((λx::'n::euclidean_space. 0) has_integral 0) S
  by (rule has_integral_is_0) auto

lemma has_integral_0_eq[simp]: ((λx. 0) has_integral i) S ⟷ i = 0
  using has_integral_unique[OF has_integral_0] by auto

lemma has_integral_linear_cbox:
  fixes f :: 'n::euclidean_space ⇒ 'a::real_normed_vector
  assumes f: (f has_integral y) (cbox a b)
```

```
    and h: bounded_linear h
  shows ((h ∘ f) has_integral (h y)) (cbox a b)
proof -
  interpret bounded_linear h using h .
  show ?thesis
    unfolding has_integral_cbox using tendsto [OF f [unfolded has_integral_cbox]]
    by (simp add: sum scaleR split_beta')
qed

lemma has_integral_linear:
  fixes f :: 'n::euclidean_space ⇒ 'a::real_normed_vector
  assumes f: (f has_integral y) S
    and h: bounded_linear h
  shows ((h ∘ f) has_integral (h y)) S
proof (cases (∃ a b. S = cbox a b))
  case True with f h has_integral_linear_cbox show ?thesis
    by blast
next
  case False
  interpret bounded_linear h using h .
  from pos_bounded obtain B where B: 0 < B ⋀x. norm (h x) ≤ norm x * B
    by blast
  let ?S = λf x. if x ∈ S then f x else 0
  show ?thesis
  proof (subst has_integral_alt, clarsimp simp: False)
    fix e :: real
    assume e: e > 0
    have *: 0 < e/B using e B(1) by simp
    obtain M where M:
      M > 0
      ⋀a b. ball 0 M ⊆ cbox a b ⟹
        ∃ z. (?S f has_integral z) (cbox a b) ∧ norm (z − y) < e/B
      using has_integral_altD[OF f False *] by blast
    show ∃ B>0. ∀ a b. ball 0 B ⊆ cbox a b ⟶
      (∃ z. (?S(h ∘ f) has_integral z) (cbox a b) ∧ norm (z − h y) < e)
    proof (rule exI, intro allI conjI impI)
      show M > 0 using M by metis
    next
      fix a b::'n
      assume sb: ball 0 M ⊆ cbox a b
      obtain z where z: (?S f has_integral z) (cbox a b) norm (z − y) < e/B
        using M(2)[OF sb] by blast
      have *: ?S(h ∘ f) = h ∘ ?S f
        using zero by auto
      show ∃ z. (?S(h ∘ f) has_integral z) (cbox a b) ∧ norm (z − h y) < e
      proof (intro exI conjI)
        show (?S(h ∘ f) has_integral h z) (cbox a b)
          by (simp add: * has_integral_linear_cbox[OF z(1) h])
        show norm (h z − h y) < e
```

by (*metis B diff le_less_trans pos_less_divide_eq z*(*2*))
qed
qed
qed
qed

lemma *has_integral_scaleR_left*:
(*f has_integral y*) *S* $\Longrightarrow$ ((λx. *f x* $*_R$ *c*) *has_integral* (*y* $*_R$ *c*)) *S*
using *has_integral_linear*[*OF _ bounded_linear_scaleR_left*] **by** (*simp add*: *comp_def*)

lemma *integrable_on_scaleR_left*:
assumes *f integrable_on A*
shows (λx. *f x* $*_R$ *y*) *integrable_on A*
using *assms has_integral_scaleR_left* **unfolding** *integrable_on_def* **by** *blast*

lemma *has_integral_mult_left*:
fixes *c* :: _ :: *real_normed_algebra*
shows (*f has_integral y*) *S* $\Longrightarrow$ ((λx. *f x* $*$ *c*) *has_integral* (*y* $*$ *c*)) *S*
using *has_integral_linear*[*OF _ bounded_linear_mult_left*] **by** (*simp add*: *comp_def*)

lemma *has_integral_divide*:
fixes *c* :: _ :: *real_normed_div_algebra*
shows (*f has_integral y*) *S* $\Longrightarrow$ ((λx. *f x* / *c*) *has_integral* (*y* / *c*)) *S*
unfolding *divide_inverse* **by** (*simp add*: *has_integral_mult_left*)

The case analysis eliminates the condition *f integrable_on S* at the cost of the type class constraint *division_ring*

corollary *integral_mult_left* [*simp*]:
fixes *c*:: *'a*::{*real_normed_algebra*,*division_ring*}
shows *integral S* (λx. *f x* $*$ *c*) = *integral S f* $*$ *c*
proof (*cases f integrable_on S* $\vee$ *c* = *0*)
case *True* **then show** *?thesis*
by (*force intro*: *has_integral_mult_left*)
next
case *False* **then have** $\neg$ (λx. *f x* $*$ *c*) *integrable_on S*
using *has_integral_mult_left* [*of* (λx. *f x* $*$ *c*) _ *S inverse c*]
by (*auto simp add*: *mult.assoc*)
with *False* **show** *?thesis* **by** (*simp add*: *not_integrable_integral*)
qed

corollary *integral_mult_right* [*simp*]:
fixes *c*:: *'a*::{*real_normed_field*}
shows *integral S* (λx. *c* $*$ *f x*) = *c* $*$ *integral S f*
by (*simp add*: *mult.commute* [*of c*])

corollary *integral_divide* [*simp*]:
fixes *z* :: *'a*::*real_normed_field*
shows *integral S* (λx. *f x* / *z*) = *integral S* (λx. *f x*) / *z*
using *integral_mult_left* [*of S f inverse z*]

by (*simp add*: *divide_inverse_commute*)

lemma *has_integral_mult_right*:
fixes *c* :: *'a* :: *real_normed_algebra*
shows (*f has_integral y*) *i* ⟹ ((λ*x*. *c* ∗ *f x*) *has_integral* (*c* ∗ *y*)) *i*
using *has_integral_linear*[*OF* _ *bounded_linear_mult_right*] **by** (*simp add*: *comp_def*)

lemma *has_integral_cmul*: (*f has_integral k*) *S* ⟹ ((λ*x*. *c* ∗$_R$ *f x*) *has_integral* (*c* ∗$_R$ *k*)) *S*
unfolding *o_def*[*symmetric*]
by (*metis has_integral_linear bounded_linear_scaleR_right*)

lemma *has_integral_cmult_real*:
fixes *c* :: *real*
assumes *c* ≠ *0* ⟹ (*f has_integral x*) *A*
shows ((λ*x*. *c* ∗ *f x*) *has_integral c* ∗ *x*) *A*
proof (*cases c* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
from *has_integral_cmul*[*OF assms*[*OF this*], *of c*] **show** *?thesis*
unfolding *real_scaleR_def* .
qed

lemma *has_integral_neg*: (*f has_integral k*) *S* ⟹ ((λ*x*. −(*f x*)) *has_integral* −*k*) *S*
by (*drule_tac c*=−*1* **in** *has_integral_cmul*) *auto*

lemma *has_integral_neg_iff*: ((λ*x*. − *f x*) *has_integral k*) *S* ⟷ (*f has_integral* − *k*) *S*
using *has_integral_neg*[*of f* − *k*] *has_integral_neg*[*of* λ*x*. − *f x k*] **by** *auto*

lemma *has_integral_add_cbox*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*real_normed_vector*
assumes (*f has_integral k*) (*cbox a b*) (*g has_integral l*) (*cbox a b*)
shows ((λ*x*. *f x* + *g x*) *has_integral* (*k* + *l*)) (*cbox a b*)
using *assms*
unfolding *has_integral_cbox*
by (*simp add*: *split_beta' scaleR_add_right sum.distrib*[*abs_def*] *tendsto_add*)

lemma *has_integral_add*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*real_normed_vector*
assumes *f*: (*f has_integral k*) *S* **and** *g*: (*g has_integral l*) *S*
shows ((λ*x*. *f x* + *g x*) *has_integral* (*k* + *l*)) *S*
proof (*cases* ∃ *a b*. *S* = *cbox a b*)
case *True* **with** *has_integral_add_cbox assms* **show** *?thesis*
by *blast*
next
let *?S* = λ*f x*. *if x* ∈ *S then f x else 0*

```
  case False
  then show ?thesis
  proof (subst has_integral_alt, clarsimp, goal_cases)
    case (1 e)
    then have e2: e/2 > 0
      by auto
    obtain Bf where 0 < Bf
      and Bf: ⋀a b. ball 0 Bf ⊆ cbox a b ⟹
                  ∃z. (?S f has_integral z) (cbox a b) ∧ norm (z − k) < e/2
      using has_integral_altD[OF f False e2] by blast
    obtain Bg where 0 < Bg
      and Bg: ⋀a b. ball 0 Bg ⊆ (cbox a b) ⟹
                  ∃z. (?S g has_integral z) (cbox a b) ∧ norm (z − l) < e/2
      using has_integral_altD[OF g False e2] by blast
    show ?case
    proof (rule_tac x=max Bf Bg in exI, clarsimp simp add: max.strict_coboundedI1
‹0 < Bf›)
      fix a b
      assume ball 0 (max Bf Bg) ⊆ cbox a (b::'n)
      then have fs: ball 0 Bf ⊆ cbox a (b::'n) and gs: ball 0 Bg ⊆ cbox a (b::'n)
        by auto
      obtain w where w: (?S f has_integral w) (cbox a b) norm (w − k) < e/2
        using Bf[OF fs] by blast
      obtain z where z: (?S g has_integral z) (cbox a b) norm (z − l) < e/2
        using Bg[OF gs] by blast
      have *: ⋀x. (if x ∈ S then f x + g x else 0) = (?S f x) + (?S g x)
        by auto
      show ∃z. (?S(λx. f x + g x) has_integral z) (cbox a b) ∧ norm (z − (k +
l)) < e
      proof (intro exI conjI)
        show (?S(λx. f x + g x) has_integral (w + z)) (cbox a b)
        by (simp add: has_integral_add_cbox[OF w(1) z(1), unfolded *[symmetric]])
        show norm (w + z − (k + l)) < e
          by (metis dist_norm dist_triangle_add_half w(2) z(2))
      qed
    qed
  qed
qed

lemma has_integral_diff:
  (f has_integral k) S ⟹ (g has_integral l) S ⟹
    ((λx. f x − g x) has_integral (k − l)) S
  using has_integral_add[OF _ has_integral_neg, of f k S g l]
  by (auto simp: algebra_simps)

lemma integral_0 [simp]:
  integral S (λx::'n::euclidean_space. 0::'m::real_normed_vector) = 0
  by (rule integral_unique has_integral_0)+
```

lemma *integral_add*: *f integrable_on S* ⟹ *g integrable_on S* ⟹
integral S (λ*x*. *f x* + *g x*) = *integral S f* + *integral S g*
by (*rule integral_unique*) (*metis integrable_integral has_integral_add*)

lemma *integral_cmul* [*simp*]: *integral S* (λ*x*. *c* $*_R$ *f x*) = *c* $*_R$ *integral S f*
proof (*cases f integrable_on S* ∨ *c* = *0*)
case *True* **with** *has_integral_cmul integrable_integral* **show** *?thesis*
by *fastforce*
next
case *False* **then have** ¬ (λ*x*. *c* $*_R$ *f x*) *integrable_on S*
using *has_integral_cmul* [*of* (λ*x*. *c* $*_R$ *f x*) _ *S inverse c*] **by** *auto*
with *False* **show** *?thesis* **by** (*simp add*: *not_integrable_integral*)
qed

lemma *integral_mult*:
fixes *K*::*real*
shows *f integrable_on X* ⟹ *K* ∗ *integral X f* = *integral X* (λ*x*. *K* ∗ *f x*)
unfolding *real_scaleR_def*[*symmetric*] *integral_cmul* **..**

lemma *integral_neg* [*simp*]: *integral S* (λ*x*. − *f x*) = − *integral S f*
proof (*cases f integrable_on S*)
case *True* **then show** *?thesis*
by (*simp add*: *has_integral_neg integrable_integral integral_unique*)
next
case *False* **then have** ¬ (λ*x*. − *f x*) *integrable_on S*
using *has_integral_neg* [*of* (λ*x*. − *f x*) _ *S*] **by** *auto*
with *False* **show** *?thesis* **by** (*simp add*: *not_integrable_integral*)
qed

lemma *integral_diff*: *f integrable_on S* ⟹ *g integrable_on S* ⟹
integral S (λ*x*. *f x* − *g x*) = *integral S f* − *integral S g*
by (*rule integral_unique*) (*metis integrable_integral has_integral_diff*)

lemma *integrable_0*: (λ*x*. *0*) *integrable_on S*
unfolding *integrable_on_def* **using** *has_integral_0* **by** *auto*

lemma *integrable_add*: *f integrable_on S* ⟹ *g integrable_on S* ⟹ (λ*x*. *f x* + *g x*) *integrable_on S*
unfolding *integrable_on_def* **by**(*auto intro*: *has_integral_add*)

lemma *integrable_cmul*: *f integrable_on S* ⟹ (λ*x*. *c* $*_R$ *f*(*x*)) *integrable_on S*
unfolding *integrable_on_def* **by**(*auto intro*: *has_integral_cmul*)

lemma *integrable_on_scaleR_iff* [*simp*]:
fixes *c* :: *real*
assumes *c* ≠ *0*
shows (λ*x*. *c* $*_R$ *f x*) *integrable_on S* ⟷ *f integrable_on S*
using *integrable_cmul*[*of* λ*x*. *c* $*_R$ *f x S 1* / *c*] *integrable_cmul*[*of f S c*] ‹*c* ≠ *0*›
by *auto*

lemma *integrable_on_cmult_iff* [*simp*]:
  fixes *c* :: *real*
  assumes $c \neq 0$
  shows ($\lambda x.\ c * f\ x$) *integrable_on S* $\longleftrightarrow$ *f integrable_on S*
  using *integrable_on_scaleR_iff* [*of c f*] *assms* **by** *simp*

lemma *integrable_on_cmult_left*:
  assumes *f integrable_on S*
  shows ($\lambda x.$ *of_real c* $* f\ x$) *integrable_on S*
    using *integrable_cmul*[*of f S of_real c*] *assms*
    by (*simp add*: *scaleR_conv_of_real*)

lemma *integrable_neg*: *f integrable_on S* $\Longrightarrow$ ($\lambda x.\ -f(x)$) *integrable_on S*
  unfolding *integrable_on_def* **by**(*auto intro*: *has_integral_neg*)

lemma *integrable_neg_iff*: ($\lambda x.\ -f(x)$) *integrable_on S* $\longleftrightarrow$ *f integrable_on S*
  using *integrable_neg* **by** *fastforce*

lemma *integrable_diff*:
  f integrable_on S $\Longrightarrow$ *g integrable_on S* $\Longrightarrow$ ($\lambda x.\ f\ x - g\ x$) *integrable_on S*
  unfolding *integrable_on_def* **by**(*auto intro*: *has_integral_diff*)

lemma *integrable_linear*:
  f integrable_on S $\Longrightarrow$ *bounded_linear h* $\Longrightarrow$ ($h \circ f$) *integrable_on S*
  unfolding *integrable_on_def* **by**(*auto intro*: *has_integral_linear*)

lemma *integral_linear*:
  f integrable_on S $\Longrightarrow$ *bounded_linear h* $\Longrightarrow$ *integral S* ($h \circ f$) = *h* (*integral S f*)
  by (*meson has_integral_iff has_integral_linear*)

lemma *integrable_on_cnj_iff*:
  ($\lambda x.$ *cnj* (*f x*)) *integrable_on A* $\longleftrightarrow$ *f integrable_on A*
  using *integrable_linear*[*OF _ bounded_linear_cnj*, *of f A*]
        integrable_linear[*OF _ bounded_linear_cnj*, *of cnj* $\circ$ *f A*]
  by (*auto simp*: *o_def*)

lemma *integral_cnj*: *cnj* (*integral A f*) = *integral A* ($\lambda x.$ *cnj* (*f x*))
  by (*cases f integrable_on A*)
     (*simp_all add*: *integral_linear*[*OF _ bounded_linear_cnj*, *symmetric*]
                o_def integrable_on_cnj_iff not_integrable_integral)

lemma *integral_component_eq*[*simp*]:
  fixes *f* :: '*n*::*euclidean_space* $\Rightarrow$ '*m*::*euclidean_space*
  assumes *f integrable_on S*
  shows *integral S* ($\lambda x.\ f\ x \cdot k$) = *integral S f* $\cdot$ *k*
  unfolding *integral_linear*[*OF assms*(*1*) *bounded_linear_inner_left*,*unfolded o_def*]
..

lemma *has_integral_sum*:
  assumes *finite T*
    and $\bigwedge a.\ a \in T \Longrightarrow ((f\ a)\ has\_integral\ (i\ a))\ S$
  shows $((\lambda x.\ sum\ (\lambda a.\ f\ a\ x)\ T)\ has\_integral\ (sum\ i\ T))\ S$
  using *assms*(*1*) *subset_refl*[*of T*]
proof (*induct rule*: *finite_subset_induct*)
  case *empty*
  then show *?case* **by** *auto*
next
  case (*insert x F*)
  with *assms* **show** *?case*
    by (*simp add*: *has_integral_add*)
qed

lemma *integral_sum*:
  $[\![finite\ I;\ \bigwedge a.\ a \in I \Longrightarrow f\ a\ integrable\_on\ S]\!] \Longrightarrow$
   $integral\ S\ (\lambda x.\ \sum a{\in}I.\ f\ a\ x) = (\sum a{\in}I.\ integral\ S\ (f\ a))$
  by (*simp add*: *has_integral_sum integrable_integral integral_unique*)

lemma *integrable_sum*:
  $[\![finite\ I;\ \bigwedge a.\ a \in I \Longrightarrow f\ a\ integrable\_on\ S]\!] \Longrightarrow (\lambda x.\ \sum a{\in}I.\ f\ a\ x)\ integrable\_on$ S
  unfolding *integrable_on_def* **using** *has_integral_sum*[*of I*] **by** *metis*

lemma *has_integral_eq*:
  assumes $\bigwedge x.\ x \in s \Longrightarrow f\ x = g\ x$
    and (*f has_integral k*) *s*
  shows (*g has_integral k*) *s*
  using *has_integral_diff*[*OF assms*(*2*), *of* $\lambda x.\ f\ x - g\ x\ 0$]
  using *has_integral_is_0*[*of s* $\lambda x.\ f\ x - g\ x$]
  using *assms*(*1*)
  by *auto*

lemma *integrable_eq*: $[\![f\ integrable\_on\ s;\ \bigwedge x.\ x \in s \Longrightarrow f\ x = g\ x]\!] \Longrightarrow g\ integrable\_on\ s$
  unfolding *integrable_on_def*
  using *has_integral_eq*[*of s f g*] *has_integral_eq* **by** *blast*

lemma *has_integral_cong*:
  assumes $\bigwedge x.\ x \in s \Longrightarrow f\ x = g\ x$
  shows (*f has_integral i*) *s* = (*g has_integral i*) *s*
  using *has_integral_eq*[*of s f g*] *has_integral_eq*[*of s g f*] *assms*
  by *auto*

lemma *integral_cong*:
  assumes $\bigwedge x.\ x \in s \Longrightarrow f\ x = g\ x$
  shows *integral s f* = *integral s g*
  unfolding *integral_def*
by (*metis* (*full_types*, *hide_lams*) *assms has_integral_cong integrable_eq*)

lemma *integrable_on_cmult_left_iff* [*simp*]:
  assumes *c ≠ 0*
  shows (*λx. of_real c ∗ f x*) *integrable_on s* ⟷ *f integrable_on s*
    (**is** *?lhs = ?rhs*)
proof
  assume *?lhs*
  then have (*λx. of_real* (*1 / c*) ∗ (*of_real c ∗ f x*)) *integrable_on s*
    using *integrable_cmul*[*of λx. of_real c ∗ f x s 1 / of_real c*]
    by (*simp add*: *scaleR_conv_of_real*)
  then have (*λx.* (*of_real* (*1 / c*) ∗ *of_real c ∗ f x*)) *integrable_on s*
    by (*simp add*: *algebra_simps*)
  with ‹*c ≠ 0*› **show** *?rhs*
    by (*metis* (*no_types*, *lifting*) *integrable_eq mult.left_neutral nonzero_divide_eq_eq of_real_1 of_real_mult*)
qed (*blast intro*: *integrable_on_cmult_left*)

lemma *integrable_on_cmult_right*:
  fixes *f* :: _ ⇒ *'b* :: {*comm_ring*,*real_algebra_1*,*real_normed_vector*}
  assumes *f integrable_on s*
  shows (*λx. f x ∗ of_real c*) *integrable_on s*
using *integrable_on_cmult_left* [*OF assms*] **by** (*simp add*: *mult.commute*)

lemma *integrable_on_cmult_right_iff* [*simp*]:
  fixes *f* :: _ ⇒ *'b* :: {*comm_ring*,*real_algebra_1*,*real_normed_vector*}
  assumes *c ≠ 0*
  shows (*λx. f x ∗ of_real c*) *integrable_on s* ⟷ *f integrable_on s*
using *integrable_on_cmult_left_iff* [*OF assms*] **by** (*simp add*: *mult.commute*)

lemma *integrable_on_cdivide*:
  fixes *f* :: _ ⇒ *'b* :: *real_normed_field*
  assumes *f integrable_on s*
  shows (*λx. f x / of_real c*) *integrable_on s*
by (*simp add*: *integrable_on_cmult_right divide_inverse assms flip*: *of_real_inverse*)

lemma *integrable_on_cdivide_iff* [*simp*]:
  fixes *f* :: _ ⇒ *'b* :: *real_normed_field*
  assumes *c ≠ 0*
  shows (*λx. f x / of_real c*) *integrable_on s* ⟷ *f integrable_on s*
by (*simp add*: *divide_inverse assms flip*: *of_real_inverse*)

lemma *has_integral_null* [*intro*]: *content*(*cbox a b*) = *0* ⟹ (*f has_integral 0*) (*cbox a b*)
  unfolding *has_integral_cbox*
  using *eventually_division_filter_tagged_division*[*of cbox a b*]
  by (*subst tendsto_cong*[**where** *g*=*λ_. 0*]) (*auto elim*: *eventually_mono intro*: *sum_content_null*)

lemma *has_integral_null_real* [*intro*]: *content* {*a..b*::*real*} = *0* ⟹ (*f has_integral 0*) {*a..b*}

by (*metis box_real*(*2*) *has_integral_null*)

lemma *has_integral_null_eq*[*simp*]: *content* (*cbox a b*) = *0* ⟹ (*f has_integral i*) (*cbox a b*) ⟷ *i* = *0*
by (*auto simp add*: *has_integral_null dest*!: *integral_unique*)

lemma *integral_null* [*simp*]: *content* (*cbox a b*) = *0* ⟹ *integral* (*cbox a b*) *f* = *0*
by (*metis has_integral_null integral_unique*)

lemma *integrable_on_null* [*intro*]: *content* (*cbox a b*) = *0* ⟹ *f integrable_on* (*cbox a b*)
by (*simp add*: *has_integral_integrable*)

lemma *has_integral_empty*[*intro*]: (*f has_integral 0*) {}
by (*meson ex_in_conv has_integral_is_0*)

lemma *has_integral_empty_eq*[*simp*]: (*f has_integral i*) {} ⟷ *i* = *0*
by (*auto simp add*: *has_integral_empty has_integral_unique*)

lemma *integrable_on_empty*[*intro*]: *f integrable_on* {}
unfolding *integrable_on_def* **by** *auto*

lemma *integral_empty*[*simp*]: *integral* {} *f* = *0*
by (*rule integral_unique*) (*rule has_integral_empty*)

lemma *has_integral_refl*[*intro*]:
fixes *a* :: *'a*::*euclidean_space*
shows (*f has_integral 0*) (*cbox a a*)
and (*f has_integral 0*) {*a*}
proof −
show (*f has_integral 0*) (*cbox a a*)
by (*rule has_integral_null*) *simp*
then show (*f has_integral 0*) {*a*}
by *simp*
qed

lemma *integrable_on_refl*[*intro*]: *f integrable_on cbox a a*
unfolding *integrable_on_def* **by** *auto*

lemma *integral_refl* [*simp*]: *integral* (*cbox a a*) *f* = *0*
by (*rule integral_unique*) *auto*

lemma *integral_singleton* [*simp*]: *integral* {*a*} *f* = *0*
by *auto*

lemma *integral_blinfun_apply*:
assumes *f integrable_on s*
shows *integral s* (λ*x*. *blinfun_apply h* (*f x*)) = *blinfun_apply h* (*integral s f*)
by (*subst integral_linear*[*symmetric*, *OF assms blinfun*.*bounded_linear_right*]) (*simp*

add: *o_def*)

lemma *blinfun_apply_integral*:
assumes *f integrable_on s*
shows *blinfun_apply* (*integral s f*) *x* = *integral s* (λ*y*. *blinfun_apply* (*f y*) *x*)
by (*metis* (*no_types*, *lifting*) *assms blinfun.prod_left.rep_eq integral_blinfun_apply integral_cong*)

lemma *has_integral_componentwise_iff*:
fixes *f* :: ′*a* :: *euclidean_space* ⇒ ′*b* :: *euclidean_space*
shows (*f has_integral y*) *A* ⟷ (∀ *b*∈*Basis*. ((λ*x*. *f x* · *b*) *has_integral* (*y* · *b*)) *A*)
proof *safe*
fix *b* :: ′*b* **assume** (*f has_integral y*) *A*
from *has_integral_linear*[*OF this*(*1*) *bounded_linear_inner_left*, *of b*]
show ((λ*x*. *f x* · *b*) *has_integral* (*y* · *b*)) *A* **by** (*simp add*: *o_def*)
next
assume (∀ *b*∈*Basis*. ((λ*x*. *f x* · *b*) *has_integral* (*y* · *b*)) *A*)
hence ∀ *b*∈*Basis*. (((λ*x*. *x* $*_R$ *b*) ∘ (λ*x*. *f x* · *b*)) *has_integral* ((*y* · *b*) $*_R$ *b*)) *A*
by (*intro ballI has_integral_linear*) (*simp_all add*: *bounded_linear_scaleR_left*)
hence ((λ*x*. ∑ *b*∈*Basis*. (*f x* · *b*) $*_R$ *b*) *has_integral* (∑ *b*∈*Basis*. (*y* · *b*) $*_R$ *b*)) *A*
by (*intro has_integral_sum*) (*simp_all add*: *o_def*)
thus (*f has_integral y*) *A* **by** (*simp add*: *euclidean_representation*)
qed

lemma *has_integral_componentwise*:
fixes *f* :: ′*a* :: *euclidean_space* ⇒ ′*b* :: *euclidean_space*
shows (⋀*b*. *b* ∈ *Basis* ⟹ ((λ*x*. *f x* · *b*) *has_integral* (*y* · *b*)) *A*) ⟹ (*f has_integral y*) *A*
by (*subst has_integral_componentwise_iff*) *blast*

lemma *integrable_componentwise_iff*:
fixes *f* :: ′*a* :: *euclidean_space* ⇒ ′*b* :: *euclidean_space*
shows *f integrable_on A* ⟷ (∀ *b*∈*Basis*. (λ*x*. *f x* · *b*) *integrable_on A*)
proof
assume *f integrable_on A*
then obtain *y* **where** (*f has_integral y*) *A* **by** (*auto simp*: *integrable_on_def*)
hence (∀ *b*∈*Basis*. ((λ*x*. *f x* · *b*) *has_integral* (*y* · *b*)) *A*)
by (*subst* (*asm*) *has_integral_componentwise_iff*)
thus (∀ *b*∈*Basis*. (λ*x*. *f x* · *b*) *integrable_on A*) **by** (*auto simp*: *integrable_on_def*)
next
assume (∀ *b*∈*Basis*. (λ*x*. *f x* · *b*) *integrable_on A*)
then obtain *y* **where** ∀ *b*∈*Basis*. ((λ*x*. *f x* · *b*) *has_integral y b*) *A*
unfolding *integrable_on_def* **by** (*subst* (*asm*) *bchoice_iff*) *blast*
hence ∀ *b*∈*Basis*. (((λ*x*. *x* $*_R$ *b*) ∘ (λ*x*. *f x* · *b*)) *has_integral* (*y b* $*_R$ *b*)) *A*
by (*intro ballI has_integral_linear*) (*simp_all add*: *bounded_linear_scaleR_left*)
hence ((λ*x*. ∑ *b*∈*Basis*. (*f x* · *b*) $*_R$ *b*) *has_integral* (∑ *b*∈*Basis*. *y b* $*_R$ *b*)) *A*
by (*intro has_integral_sum*) (*simp_all add*: *o_def*)

```
 thus f integrable_on A by (auto simp: integrable_on_def o_def euclidean_representation)
qed

lemma integrable_componentwise:
  fixes f :: 'a :: euclidean_space ⇒ 'b :: euclidean_space
  shows (⋀b. b ∈ Basis ⟹ (λx. f x · b) integrable_on A) ⟹ f integrable_on A
  by (subst integrable_componentwise_iff) blast

lemma integral_componentwise:
  fixes f :: 'a :: euclidean_space ⇒ 'b :: euclidean_space
  assumes f integrable_on A
  shows integral A f = (∑b∈Basis. integral A (λx. (f x · b) *R b))
proof -
 from assms have integrable: ∀b∈Basis. (λx. x *R b) ∘ (λx. (f x · b)) integrable_on
A
    by (subst (asm) integrable_componentwise_iff, intro integrable_linear ballI)
      (simp_all add: bounded_linear_scaleR_left)
  have integral A f = integral A (λx. ∑b∈Basis. (f x · b) *R b)
    by (simp add: euclidean_representation)
  also from integrable have ... = (∑a∈Basis. integral A (λx. (f x · a) *R a))
    by (subst integral_sum) (simp_all add: o_def)
  finally show ?thesis .
qed

lemma integrable_component:
  f integrable_on A ⟹ (λx. f x · (y :: 'b :: euclidean_space)) integrable_on A
 by (drule integrable_linear[OF _ bounded_linear_inner_left[of y]]) (simp add: o_def)
```

6.15.4 Cauchy-type criterion for integrability

```
proposition integrable_Cauchy:
  fixes f :: 'n::euclidean_space ⇒ 'a::{real_normed_vector,complete_space}
  shows f integrable_on cbox a b ⟷
        (∀e>0. ∃γ. gauge γ ∧
          (∀𝒟1 𝒟2. 𝒟1 tagged_division_of (cbox a b) ∧ γ fine 𝒟1 ∧
            𝒟2 tagged_division_of (cbox a b) ∧ γ fine 𝒟2 ⟶
           norm ((∑(x,K)∈𝒟1. content K *R f x) − (∑(x,K)∈𝒟2. content K *R
f x)) < e))
  (is ?l = (∀e>0. ∃γ. ?P e γ))
proof (intro iffI allI impI)
  assume ?l
  then obtain y
    where y: ⋀e. e > 0 ⟹
        ∃γ. gauge γ ∧
            (∀𝒟. 𝒟 tagged_division_of cbox a b ∧ γ fine 𝒟 ⟶
                norm ((∑(x,K) ∈ 𝒟. content K *R f x) − y) < e)
    by (auto simp: integrable_on_def has_integral)
  show ∃γ. ?P e γ if e > 0 for e
  proof -
```

have $e/2 > 0$ **using** *that* **by** *auto*
with y **obtain** γ **where** *gauge* γ
and γ: $\bigwedge\mathcal{D}.\ \mathcal{D}$ *tagged_division_of cbox a b* $\wedge\ \gamma$ *fine* $\mathcal{D} \Longrightarrow$
norm $((\sum(x,K)\in\mathcal{D}.\ content\ K *_R f\ x) - y) < e/2$
by *meson*
show *?thesis*
apply (*rule_tac* $x=\gamma$ **in** *exI*, *clarsimp simp*: ‹*gauge* γ›)
by (*blast intro*!: γ *dist_triangle_half_l*[**where** $y=y$,*unfolded dist_norm*])
qed
next
assume $\forall e>0.\ \exists\gamma.\ ?P\ e\ \gamma$
then have $\forall n::nat.\ \exists\gamma.\ ?P\ (1\ /\ (n + 1))\ \gamma$
by *auto*
then obtain γ :: *nat* $\Rightarrow$ *'n* $\Rightarrow$ *'n set* **where** γ:
$\bigwedge m.\ gauge\ (\gamma\ m)$
$\bigwedge m\ \mathcal{D}1\ \mathcal{D}2.\ [\![\mathcal{D}1$ *tagged_division_of cbox a b*;
$\gamma\ m$ *fine* $\mathcal{D}1$; $\mathcal{D}2$ *tagged_division_of cbox a b*; $\gamma\ m$ *fine* $\mathcal{D}2]\!]$
$\Longrightarrow norm\ ((\sum(x,K) \in \mathcal{D}1.\ content\ K *_R f\ x) - (\sum(x,K) \in \mathcal{D}2.\ content\ K *_R f\ x))$
$< 1\ /\ (m + 1)$
by *metis*
have *gauge* $(\lambda x.\ \bigcap\{\gamma\ i\ x\ |i.\ i \in \{0..n\}\})$ **for** n
using γ **by** (*intro gauge_Inter*) *auto*
then have $\forall n.\ \exists p.\ p$ *tagged_division_of* (*cbox a b*) $\wedge$ $(\lambda x.\ \bigcap\{\gamma\ i\ x\ |i.\ i \in \{0..n\}\})$ *fine* p
by (*meson fine_division_exists*)
then obtain p **where** p: $\bigwedge z.\ p\ z$ *tagged_division_of cbox a b*
$\bigwedge z.\ (\lambda x.\ \bigcap\{\gamma\ i\ x\ |i.\ i \in \{0..z\}\})$ *fine* $p\ z$
by *meson*
have *dp*: $\bigwedge i\ n.\ i\le n \Longrightarrow \gamma\ i$ *fine* $p\ n$
using p **unfolding** *fine_Inter*
using *atLeastAtMost_iff* **by** *blast*
have *Cauchy* $(\lambda n.\ sum\ (\lambda(x,K).\ content\ K *_R (f\ x))\ (p\ n))$
proof (*rule CauchyI*)
fix *e*::*real*
assume $0 < e$
then obtain N **where** $N \neq 0$ **and** N: *inverse* (*real* N) $< e$
using *real_arch_inverse*[*of e*] **by** *blast*
show $\exists M.\ \forall m\ge M.\ \forall n\ge M.\ norm\ ((\sum(x,K) \in p\ m.\ content\ K *_R f\ x) - (\sum(x,K) \in p\ n.\ content\ K *_R f\ x)) < e$
proof (*intro exI allI impI*)
fix $m\ n$
assume *mn*: $N \le m\ N \le n$
have *norm* $((\sum(x,K) \in p\ m.\ content\ K *_R f\ x) - (\sum(x,K) \in p\ n.\ content\ K *_R f\ x)) < 1\ /\ (real\ N + 1)$
by (*simp add*: $p(1)$ *dp mn* γ)
also have ... $< e$
using N ‹$N \neq 0$› ‹$0 < e$› **by** (*auto simp*: *field_simps*)
finally show *norm* $((\sum(x,K) \in p\ m.\ content\ K *_R f\ x) - (\sum(x,K) \in p\ n.$

```
content K *R f x)) < e .
    qed
  qed
  then obtain y where y: ∃ no. ∀ n≥no. norm ((∑ (x,K) ∈ p n. content K *R f
x) − y) < r if r > 0 for r
    by (auto simp: convergent_eq_Cauchy[symmetric] dest: LIMSEQ_D)
  show ?l
    unfolding integrable_on_def has_integral
  proof (rule_tac x=y in exI, clarify)
    fix e :: real
    assume e>0
    then have e2: e/2 > 0 by auto
    then obtain N1::nat where N1: N1 ≠ 0 inverse (real N1) < e/2
      using real_arch_inverse by blast
    obtain N2::nat where N2: ⋀n. n ≥ N2 ⟹ norm ((∑ (x,K) ∈ p n. content
K *R f x) − y) < e/2
      using y[OF e2] by metis
    show ∃ γ. gauge γ ∧
              (∀ 𝒟. 𝒟 tagged_division_of (cbox a b) ∧ γ fine 𝒟 ⟶
                norm ((∑ (x,K) ∈ 𝒟. content K *R f x) − y) < e)
    proof (intro exI conjI allI impI)
      show gauge (γ (N1+N2))
        using γ by auto
      show norm ((∑ (x,K) ∈ q. content K *R f x) − y) < e
           if q tagged_division_of cbox a b ∧ γ (N1+N2) fine q for q
      proof (rule norm_triangle_half_r)
        have norm ((∑ (x,K) ∈ p (N1+N2). content K *R f x) − (∑ (x,K) ∈ q.
content K *R f x))
             < 1 / (real (N1+N2) + 1)
          by (rule γ; simp add: dp p that)
        also have ... < e/2
          using N1 ‹0 < e› by (auto simp: field_simps intro: less_le_trans)
       finally show norm ((∑ (x,K) ∈ p (N1+N2). content K *R f x) − (∑ (x,K)
∈ q. content K *R f x)) < e/2 .
        show norm ((∑ (x,K) ∈ p (N1+N2). content K *R f x) − y) < e/2
          using N2 le_add_same_cancel2 by blast
      qed
    qed
  qed
qed
```

6.15.5 Additivity of integral on abutting intervals

```
lemma tagged_division_split_left_inj_content:
  assumes 𝒟: 𝒟 tagged_division_of S
    and (x1, K1) ∈ 𝒟 (x2, K2) ∈ 𝒟 K1 ≠ K2 K1 ∩ {x. x·k ≤ c} = K2 ∩ {x.
x·k ≤ c} k ∈ Basis
  shows content (K1 ∩ {x. x·k ≤ c}) = 0
proof −
```

from *tagged_division_ofD(4)[OF D ‹(x1, K1) ∈ D›]* **obtain** *a b* **where** *K1*: *K1 = cbox a b*
by *auto*
then have *interior (K1 ∩ {x. x · k ≤ c}) = {}*
by (*metis tagged_division_split_left_inj assms*)
then show *?thesis*
unfolding *K1 interval_split[OF ‹k ∈ Basis›]* **by** (*auto simp*: *content_eq_0_interior*)
qed

lemma *tagged_division_split_right_inj_content*:
assumes *D*: *D tagged_division_of S*
and *(x1, K1) ∈ D (x2, K2) ∈ D K1 ≠ K2 K1 ∩ {x. x·k ≥ c} = K2 ∩ {x. x·k ≥ c} k ∈ Basis*
shows *content (K1 ∩ {x. x·k ≥ c}) = 0*
proof −
from *tagged_division_ofD(4)[OF D ‹(x1, K1) ∈ D›]* **obtain** *a b* **where** *K1*: *K1 = cbox a b*
by *auto*
then have *interior (K1 ∩ {x. c ≤ x · k}) = {}*
by (*metis tagged_division_split_right_inj assms*)
then show *?thesis*
unfolding *K1 interval_split[OF ‹k ∈ Basis›]*
by (*auto simp*: *content_eq_0_interior*)
qed

proposition *has_integral_split*:
fixes *f* :: *'a::euclidean_space ⇒ 'b::real_normed_vector*
assumes *fi*: (*f has_integral i*) (*cbox a b ∩ {x. x·k ≤ c}*)
and *fj*: (*f has_integral j*) (*cbox a b ∩ {x. x·k ≥ c}*)
and *k*: *k ∈ Basis*
shows (*f has_integral (i + j)*) (*cbox a b*)
unfolding *has_integral*
proof *clarify*
fix *e::real*
assume *0 < e*
then have *e*: *e/2 > 0*
by *auto*
obtain *γ1* **where** *γ1*: *gauge γ1*
and *γ1norm*:
$\bigwedge\mathcal{D}.\ [\![\mathcal{D}\ tagged\_division\_of\ cbox\ a\ b \cap \{x.\ x \cdot k \le c\};\ \gamma 1\ fine\ \mathcal{D}]\!]$
$\Longrightarrow norm\ ((\sum (x,K) \in \mathcal{D}.\ content\ K *_R f\ x) - i) < e/2$
apply (*rule has_integralD[OF fi[unfolded interval_split[OF k]] e]*)
apply (*simp add*: *interval_split[symmetric] k*)
done
obtain *γ2* **where** *γ2*: *gauge γ2*
and *γ2norm*:
$\bigwedge\mathcal{D}.\ [\![\mathcal{D}\ tagged\_division\_of\ cbox\ a\ b \cap \{x.\ c \le x \cdot k\};\ \gamma 2\ fine\ \mathcal{D}]\!]$
$\Longrightarrow norm\ ((\sum (x,\ k) \in \mathcal{D}.\ content\ k *_R f\ x) - j) < e/2$

```
      apply (rule has_integralD[OF fj[unfolded interval_split[OF k]] e])
      apply (simp add: interval_split[symmetric] k)
      done
  let ?γ = λx. if x·k = c then (γ1 x ∩ γ2 x) else ball x |x·k − c| ∩ γ1 x ∩ γ2 x
  have gauge ?γ
    using γ1 γ2 unfolding gauge_def by auto
  then show ∃γ. gauge γ ∧
              (∀D. D tagged_division_of cbox a b ∧ γ fine D ⟶
                  norm ((∑(x, k)∈D. content k *R f x) − (i + j)) < e)
  proof (rule_tac x=?γ in exI, safe)
    fix p
    assume p: p tagged_division_of (cbox a b) and ?γ fine p
    have ab_eqp: cbox a b = ⋃{K. ∃x. (x, K) ∈ p}
      using p by blast
    have xk_le_c: x·k ≤ c if as: (x,K) ∈ p and K: K ∩ {x. x·k ≤ c} ≠ {} for x K
    proof (rule ccontr)
      assume **: ¬ x · k ≤ c
      then have K ⊆ ball x |x · k − c|
        using ‹?γ fine p› as by (fastforce simp: not_le algebra_simps)
      with K obtain y where y: y ∈ ball x |x · k − c| y·k ≤ c
        by blast
      then have |x · k − y · k| < |x · k − c|
        using Basis_le_norm[OF k, of x − y]
        by (auto simp add: dist_norm inner_diff_left intro: le_less_trans)
      with y show False
        using ** by (auto simp add: field_simps)
    qed
    have xk_ge_c: x·k ≥ c if as: (x,K) ∈ p and K: K ∩ {x. x·k ≥ c} ≠ {} for x
K
    proof (rule ccontr)
      assume **: ¬ x · k ≥ c
      then have K ⊆ ball x |x · k − c|
        using ‹?γ fine p› as by (fastforce simp: not_le algebra_simps)
      with K obtain y where y: y ∈ ball x |x · k − c| y·k ≥ c
        by blast
      then have |x · k − y · k| < |x · k − c|
        using Basis_le_norm[OF k, of x − y]
        by (auto simp add: dist_norm inner_diff_left intro: le_less_trans)
      with y show False
        using ** by (auto simp add: field_simps)
    qed
    have fin_finite: finite {(x,f K) | x K. (x,K) ∈ s ∧ P x K}
      if finite s for s and f :: 'a set ⇒ 'a set and P :: 'a ⇒ 'a set ⇒ bool
    proof −
      from that have finite ((λ(x,K). (x, f K)) ` s)
        by auto
      then show ?thesis
        by (rule rev_finite_subset) auto
    qed
```

{ fix $\mathcal{G}$ :: *'a set* ⇒ *'a set*
fix *i* :: *'a* × *'a set*
assume $i \in (\lambda(x, k).\ (x, \mathcal{G}\ k))$ ‘ $p - \{(x, \mathcal{G}\ k)\ |x\ k.\ (x, k) \in p \wedge \mathcal{G}\ k \neq \{\}\}$
then obtain *x K* **where** *xk*: $i = (x, \mathcal{G}\ K)$ $(x,K) \in p$
$(x, \mathcal{G}\ K) \notin \{(x, \mathcal{G}\ K)\ |x\ K.\ (x,K) \in p \wedge \mathcal{G}\ K \neq \{\}\}$
by *auto*
have *content* $(\mathcal{G}\ K) = 0$
using *xk* **using** *content_empty* **by** *auto*
then have $(\lambda(x,K).$ *content K* $*_R$ *f x*$)$ $i = 0$
unfolding *xk split_conv* **by** *auto*
} **note** [*simp*] = *this*
have *finite p*
using *p* **by** *blast*
let *?M1* = $\{(x, K \cap \{x.\ x \cdot k \leq c\})\ |x\ K.\ (x,K) \in p \wedge K \cap \{x.\ x \cdot k \leq c\} \neq \{\}\}$
have *γ1_fine*: *γ1 fine ?M1*
using ‹*?γ fine p*› **by** (*fastforce simp*: *fine_def split*: *if_split_asm*)
have *norm* $((\sum(x, k) \in ?M1.$ *content k* $*_R$ *f x*$) - i) < e/2$
proof (*rule γ1norm* [*OF tagged_division_ofI γ1_fine*])
show *finite ?M1*
by (*rule fin_finite*) (*use p* **in** *blast*)
show $\bigcup\{k.\ \exists x.\ (x, k) \in ?M1\} = cbox\ a\ b \cap \{x.\ x \cdot k \leq c\}$
by (*auto simp*: *ab_eqp*)

fix *x L*
assume *xL*: $(x, L) \in ?M1$
then obtain *x' L'* **where** *xL'*: $x = x'$ $L = L' \cap \{x.\ x \cdot k \leq c\}$
$(x', L') \in p$ $L' \cap \{x.\ x \cdot k \leq c\} \neq \{\}$
by *blast*
then obtain *a' b'* **where** *ab'*: $L' = cbox\ a'\ b'$
using *p* **by** *blast*
show $x \in L$ $L \subseteq cbox\ a\ b \cap \{x.\ x \cdot k \leq c\}$
using *p xk_le_c xL'* **by** *auto*
show $\exists a\ b.\ L = cbox\ a\ b$
using *p xL' ab'* **by** (*auto simp add*: *interval_split*[*OF k*,**where** *c*=*c*])

fix *y R*
assume *yR*: $(y, R) \in ?M1$
then obtain *y' R'* **where** *yR'*: $y = y'$ $R = R' \cap \{x.\ x \cdot k \leq c\}$
$(y', R') \in p$ $R' \cap \{x.\ x \cdot k \leq c\} \neq \{\}$
by *blast*
assume *as*: $(x, L) \neq (y, R)$
show *interior L* ∩ *interior R* = {}
proof (*cases* $L' = R' \longrightarrow x' = y'$)
case *False*
have *interior R'* = {}
by (*metis* (*no_types*) *False Pair_inject inf.idem tagged_division_ofD*(*5*) [*OF p*] *xL'*(*3*) *yR'*(*3*))
then show *?thesis*

```
          using yR' by simp
      next
        case True
        then have L' ≠ R'
          using as unfolding xL' yR' by auto
        have interior L' ∩ interior R' = {}
         by (metis (no_types) Pair_inject ‹L' ≠ R'› p tagged_division_ofD(5) xL'(3)
yR'(3))
        then show ?thesis
          using xL'(2) yR'(2) by auto
      qed
    qed
    moreover
    let ?M2 = {(x,K ∩ {x. x·k ≥ c}) |x K. (x,K) ∈ p ∧ K ∩ {x. x·k ≥ c} ≠ {}}
    have γ2_fine: γ2 fine ?M2
      using ‹?γ fine p› by (fastforce simp: fine_def split: if_split_asm)
    have norm ((∑(x, k)∈?M2. content k *R f x) − j) < e/2
    proof (rule γ2norm [OF tagged_division_ofI γ2_fine])
      show finite ?M2
        by (rule fin_finite) (use p in blast)
      show ⋃{k. ∃x. (x, k) ∈ ?M2} = cbox a b ∩ {x. x·k ≥ c}
        by (auto simp: ab_eqp)

      fix x L
      assume xL: (x, L) ∈ ?M2
      then obtain x' L' where xL': x = x' L = L' ∩ {x. x · k ≥ c}
                              (x', L') ∈ p L' ∩ {x. x · k ≥ c} ≠ {}
        by blast
      then obtain a' b' where ab': L' = cbox a' b'
        using p by blast
      show x ∈ L L ⊆ cbox a b ∩ {x. x · k ≥ c}
        using p xk_ge_c xL' by auto
      show ∃a b. L = cbox a b
        using p xL' ab' by (auto simp add: interval_split[OF k,where c=c])

      fix y R
      assume yR: (y, R) ∈ ?M2
      then obtain y' R' where yR': y = y' R = R' ∩ {x. x · k ≥ c}
                              (y', R') ∈ p R' ∩ {x. x · k ≥ c} ≠ {}
        by blast
      assume as: (x, L) ≠ (y, R)
      show interior L ∩ interior R = {}
      proof (cases L' = R' ⟶ x' = y')
        case False
        have interior R' = {}
         by (metis (no_types) False Pair_inject inf.idem tagged_division_ofD(5) [OF
p] xL'(3) yR'(3))
        then show ?thesis
          using yR' by simp
```

next
case *True*
then have $L' \neq R'$
using *as* **unfolding** xL' yR' **by** *auto*
have *interior* $L' \cap$ *interior* $R' = \{\}$
by (*metis* (*no_types*) *Pair_inject* ‹$L' \neq R'$› *p tagged_division_ofD*(*5*) xL'(*3*) yR'(*3*))
then show *?thesis*
using xL'(*2*) yR'(*2*) **by** *auto*
qed
qed
ultimately
have *norm* $(((\sum (x,K) \in ?M1.\ content\ K *_R f\ x) - i) + ((\sum (x,K) \in ?M2.\ content\ K *_R f\ x) - j)) < e/2 + e/2$
using *norm_add_less* **by** *blast*
moreover have $((\sum (x,K) \in ?M1.\ content\ K *_R f\ x) - i) +$
$((\sum (x,K) \in ?M2.\ content\ K *_R f\ x) - j) =$
$(\sum (x,\ ka) \in p.\ content\ ka *_R f\ x) - (i + j)$
proof −
have *eq0*: $\bigwedge x\ y.\ x = (0::real) \Longrightarrow x *_R (y::'b) = 0$
by *auto*
have *cont_eq*: $\bigwedge g.\ (\lambda(x,l).\ content\ l *_R f\ x) \circ (\lambda(x,l).\ (x,g\ l)) = (\lambda(x,l).\ content\ (g\ l) *_R f\ x)$
by *auto*
have ∗: $\bigwedge \mathcal{G} :: 'a\ set \Rightarrow 'a\ set.$
$(\sum (x,K) \in \{(x, \mathcal{G}\ K) \mid x\ K.\ (x,K) \in p \wedge \mathcal{G}\ K \neq \{\}\}.\ content\ K *_R f\ x) =$
$(\sum (x,K) \in (\lambda(x,K).\ (x, \mathcal{G}\ K))\ `\ p.\ content\ K *_R f\ x)$
by (*rule sum.mono_neutral_left*) (*auto simp*: ‹*finite p*›)
have $((\sum (x,\ k) \in ?M1.\ content\ k *_R f\ x) - i) + ((\sum (x,\ k) \in ?M2.\ content\ k *_R f\ x) - j) =$
$(\sum (x,\ k) \in ?M1.\ content\ k *_R f\ x) + (\sum (x,\ k) \in ?M2.\ content\ k *_R f\ x) - (i + j)$
by *auto*
moreover have $\ldots = (\sum (x,K) \in p.\ content\ (K \cap \{x.\ x \cdot k \leq c\}) *_R f\ x) +$
$(\sum (x,K) \in p.\ content\ (K \cap \{x.\ c \leq x \cdot k\}) *_R f\ x) - (i + j)$
unfolding ∗
apply (*subst* (*1 2*) *sum.reindex_nontrivial*)
apply (*auto intro*!: *k p eq0 tagged_division_split_left_inj_content tagged_division_split_right_inj_content*
simp: *cont_eq* ‹*finite p*›)
done
moreover have $\bigwedge x.\ x \in p \Longrightarrow (\lambda(a,B).\ content\ (B \cap \{a.\ a \cdot k \leq c\}) *_R f\ a)\ x +$
$(\lambda(a,B).\ content\ (B \cap \{a.\ c \leq a \cdot k\}) *_R f\ a)\ x =$
$(\lambda(a,B).\ content\ B *_R f\ a)\ x$
proof *clarify*
fix *a B*
assume $(a,\ B) \in p$

```
        with p obtain u v where uv: B = cbox u v by blast
        then show content (B ∩ {x. x · k ≤ c}) *R f a + content (B ∩ {x. c ≤ x
· k}) *R f a = content B *R f a
          by (auto simp: scaleR_left_distrib uv content_split[OF k,of u v c])
      qed
      ultimately show ?thesis
        by (auto simp: sum.distrib[symmetric])
      qed
    ultimately show norm ((∑(x, k)∈p. content k *R f x) − (i + j)) < e
      by auto
  qed
qed
```

6.15.6 A sort of converse, integrability on subintervals

```
lemma has_integral_separate_sides:
  fixes f :: 'a::euclidean_space ⇒ 'b::real_normed_vector
  assumes f: (f has_integral i) (cbox a b)
    and e > 0
    and k: k ∈ Basis
  obtains d where gauge d
    ∀ p1 p2. p1 tagged_division_of (cbox a b ∩ {x. x·k ≤ c}) ∧ d fine p1 ∧
        p2 tagged_division_of (cbox a b ∩ {x. x·k ≥ c}) ∧ d fine p2 ⟶
        norm ((sum (λ(x,k). content k *R f x) p1 + sum (λ(x,k). content k *R f
x) p2) − i) < e
proof −
  obtain γ where d: gauge γ
      ⋀p. ⟦p tagged_division_of cbox a b; γ fine p⟧
          ⟹ norm ((∑(x, k)∈p. content k *R f x) − i) < e
    using has_integralD[OF f ‹e > 0›] by metis
  { fix p1 p2
    assume tdiv1: p1 tagged_division_of (cbox a b) ∩ {x. x · k ≤ c} and γ fine p1
    note p1=tagged_division_ofD[OF this(1)]
    assume tdiv2: p2 tagged_division_of (cbox a b) ∩ {x. c ≤ x · k} and γ fine p2
    note p2=tagged_division_ofD[OF this(1)]
    note tagged_division_Un_interval[OF tdiv1 tdiv2]
    note p12 = tagged_division_ofD[OF this] this
    { fix a b
      assume ab: (a, b) ∈ p1 ∩ p2
      have (a, b) ∈ p1
        using ab by auto
      obtain u v where uv: b = cbox u v
        using ‹(a, b) ∈ p1› p1(4) by moura
      have b ⊆ {x. x·k = c}
        using ab p1(3)[of a b] p2(3)[of a b] by fastforce
      moreover
      have interior {x::'a. x · k = c} = {}
      proof (rule ccontr)
        assume ¬ ?thesis
```

then obtain *x* **where** *x*: $x \in$ *interior* $\{x::'a.\ x \cdot k = c\}$
by *auto*
then obtain ε **where** $0 < \varepsilon$ **and** ε: *ball* $x\ \varepsilon \subseteq \{x.\ x \cdot k = c\}$
using *mem_interior* **by** *metis*
have *x*: $x \cdot k = c$
using *x interior_subset* **by** *fastforce*
have $*$: $\bigwedge i.\ i \in Basis \Longrightarrow |(x - (x + (\varepsilon/2) *_R k)) \cdot i| = (if\ i = k\ then\ \varepsilon/2\ else\ 0)$
using ‹$0 < \varepsilon$› *k* **by** (*auto simp*: *inner_simps inner_not_same_Basis*)
have $(\sum i{\in}Basis.\ |(x - (x + (\varepsilon/2\ ) *_R k)) \cdot i|) =$
$(\sum i{\in}Basis.\ (if\ i = k\ then\ \varepsilon/2\ else\ 0))$
using $*$ **by** (*blast intro*: *sum.cong*)
also have $\ldots < \varepsilon$
by (*subst sum.delta*) (*use* ‹$0 < \varepsilon$› **in** *auto*)
finally have $x + (\varepsilon/2) *_R k \in$ *ball* $x\ \varepsilon$
unfolding *mem_ball dist_norm* **by**(*rule le_less_trans*[*OF norm_le_l1*])
then have $x + (\varepsilon/2) *_R k \in \{x.\ x \cdot k = c\}$
using ε **by** *auto*
then show *False*
using ‹$0 < \varepsilon$› *x k* **by** (*auto simp*: *inner_simps*)
qed
ultimately have *content* $b = 0$
unfolding *uv content_eq_0_interior*
using *interior_mono* **by** *blast*
then have *content* $b *_R f\ a = 0$
by *auto*
}
then have *norm* $((\sum (x, k){\in}p1.\ content\ k *_R f\ x) + (\sum (x, k){\in}p2.\ content\ k *_R f\ x) - i) =$
norm $((\sum (x, k){\in}p1 \cup p2.\ content\ k *_R f\ x) - i)$
by (*subst sum.union_inter_neutral*) (*auto simp*: *p1 p2*)
also have $\ldots < e$
using *d*(*2*) *p12* **by** (*simp add*: *fine_Un k* ‹γ *fine p1*› ‹γ *fine p2*›)
finally have *norm* $((\sum (x, k){\in}p1.\ content\ k *_R f\ x) + (\sum (x, k){\in}p2.\ content\ k *_R f\ x) - i) < e$ **.**
}
then show *?thesis*
using *d*(*1*) *that* **by** *auto*
qed

lemma *integrable_split* [*intro*]:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::{real_normed_vector,complete_space}*
assumes *f*: *f integrable_on cbox a b*
and *k*: $k \in Basis$
shows *f integrable_on* (*cbox a b* $\cap \{x.\ x \cdot k \le c\}$) (**is** *?thesis1*)
and *f integrable_on* (*cbox a b* $\cap \{x.\ x \cdot k \ge c\}$) (**is** *?thesis2*)
proof −
obtain *y* **where** *y*: (*f has_integral y*) (*cbox a b*)
using *f* **by** *blast*

```
  define a' where a' = (∑ i∈Basis. (if i = k then max (a·k) c else a·i)*R i)
  define b' where b' = (∑ i∈Basis. (if i = k then min (b·k) c else b·i)*R i)
  have ∃ d. gauge d ∧
          (∀ p1 p2. p1 tagged_division_of cbox a b ∩ {x. x · k ≤ c} ∧ d fine p1 ∧
                 p2 tagged_division_of cbox a b ∩ {x. x · k ≤ c} ∧ d fine p2 ⟶
                   norm ((∑ (x,K) ∈ p1. content K *R f x) − (∑ (x,K) ∈ p2.
content K *R f x)) < e)
    if e > 0 for e
  proof −
    have e/2 > 0 using that by auto
  with has_integral_separate_sides[OF y this k, of c]
  obtain d
    where gauge d
      and d: ⋀p1 p2. ⟦p1 tagged_division_of cbox a b ∩ {x. x · k ≤ c}; d fine p1;
                     p2 tagged_division_of cbox a b ∩ {x. c ≤ x · k}; d fine p2⟧
              ⟹ norm ((∑ (x,K)∈p1. content K *R f x) + (∑ (x,K)∈p2. content
K *R f x) − y) < e/2
    by metis
  show ?thesis
    proof (rule_tac x=d in exI, clarsimp simp add: ‹gauge d›)
      fix p1 p2
      assume as: p1 tagged_division_of (cbox a b) ∩ {x. x · k ≤ c} d fine p1
                 p2 tagged_division_of (cbox a b) ∩ {x. x · k ≤ c} d fine p2
      show norm ((∑ (x, k)∈p1. content k *R f x) − (∑ (x, k)∈p2. content k *R
f x)) < e
      proof (rule fine_division_exists[OF ‹gauge d›, of a' b])
        fix p
        assume p tagged_division_of cbox a' b d fine p
        then show ?thesis
          using as norm_triangle_half_l[OF d[of p1 p] d[of p2 p]]
          unfolding interval_split[OF k] b'_def[symmetric] a'_def[symmetric]
          by (auto simp add: algebra_simps)
      qed
    qed
  qed
  with f show ?thesis1
    by (simp add: interval_split[OF k] integrable_Cauchy)
  have ∃ d. gauge d ∧
          (∀ p1 p2. p1 tagged_division_of cbox a b ∩ {x. x · k ≥ c} ∧ d fine p1 ∧
                 p2 tagged_division_of cbox a b ∩ {x. x · k ≥ c} ∧ d fine p2 ⟶
                   norm ((∑ (x,K) ∈ p1. content K *R f x) − (∑ (x,K) ∈ p2.
content K *R f x)) < e)
    if e > 0 for e
  proof −
    have e/2 > 0 using that by auto
  with has_integral_separate_sides[OF y this k, of c]
  obtain d
    where gauge d
      and d: ⋀p1 p2. ⟦p1 tagged_division_of cbox a b ∩ {x. x · k ≤ c}; d fine p1;
```

p2 tagged_division_of cbox a b ∩ {*x*. *c* ≤ *x* · *k*}; *d fine p2*⟧
⟹ *norm* ((∑(*x*,*K*)∈*p1*. *content K* $*_R$ *f x*) + (∑(*x*,*K*)∈*p2*. *content K* $*_R$ *f x*) − *y*) < *e*/*2*
by *metis*
show *?thesis*
proof (*rule_tac x*=*d* **in** *exI*, *clarsimp simp add*: ‹*gauge d*›)
fix *p1 p2*
assume *as*: *p1 tagged_division_of* (*cbox a b*) ∩ {*x*. *x* · *k* ≥ *c*} *d fine p1*
p2 tagged_division_of (*cbox a b*) ∩ {*x*. *x* · *k* ≥ *c*} *d fine p2*
show *norm* ((∑(*x*, *k*)∈*p1*. *content k* $*_R$ *f x*) − (∑(*x*, *k*)∈*p2*. *content k* $*_R$ *f x*)) < *e*
proof (*rule fine_division_exists*[*OF* ‹*gauge d*›, *of a b'*])
fix *p*
assume *p tagged_division_of cbox a b' d fine p*
then show *?thesis*
using *as norm_triangle_half_l*[*OF d*[*of p p1*] *d*[*of p p2*]]
unfolding *interval_split*[*OF k*] *b'_def*[*symmetric*] *a'_def*[*symmetric*]
by (*auto simp add*: *algebra_simps*)
qed
qed
qed
with *f* **show** *?thesis2*
by (*simp add*: *interval_split*[*OF k*] *integrable_Cauchy*)
qed

lemma *operative_integralI*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*banach*
shows *operative* (*lift_option* (+)) (*Some 0*)
(λ*i*. *if f integrable_on i then Some* (*integral i f*) *else None*)
proof −
interpret *comm_monoid lift_option plus Some* (*0*::*'b*)
by (*rule comm_monoid_lift_option*)
(*rule add.comm_monoid_axioms*)
show *?thesis*
proof
fix *a b c*
fix *k* :: *'a*
assume *k*: *k* ∈ *Basis*
show (*if f integrable_on cbox a b then Some* (*integral* (*cbox a b*) *f*) *else None*) =
lift_option (+) (*if f integrable_on cbox a b* ∩ {*x*. *x* · *k* ≤ *c*} *then Some* (*integral* (*cbox a b* ∩ {*x*. *x* · *k* ≤ *c*}) *f*) *else None*)
(*if f integrable_on cbox a b* ∩ {*x*. *c* ≤ *x* · *k*} *then Some* (*integral* (*cbox a b* ∩ {*x*. *c* ≤ *x* · *k*}) *f*) *else None*)
proof (*cases f integrable_on cbox a b*)
case *True*
with *k* **show** *?thesis*
by (*auto simp*: *integrable_split intro*: *integral_unique* [*OF has_integral_split*[*OF _ _ k*]])

```
    next
    case False
      have ¬ (f integrable_on cbox a b ∩ {x. x · k ≤ c}) ∨ ¬ ( f integrable_on cbox
a b ∩ {x. c ≤ x · k})
      proof (rule ccontr)
        assume ¬ ?thesis
        then have f integrable_on cbox a b
          unfolding integrable_on_def
          apply (rule_tac x=integral (cbox a b ∩ {x. x · k ≤ c}) f + integral (cbox
a b ∩ {x. x · k ≥ c}) f in exI)
          apply (auto intro: has_integral_split[OF _ _ k])
          done
        then show False
          using False by auto
      qed
      then show ?thesis
        using False by auto
    qed
  next
    fix a b :: 'a
    assume box a b = {}
    then show (if f integrable_on cbox a b then Some (integral (cbox a b) f) else
None) = Some 0
      using has_integral_null_eq
      by (auto simp: integrable_on_null content_eq_0_interior)
  qed
qed
```

6.15.7 Bounds on the norm of Riemann sums and the integral itself

```
lemma dsum_bound:
  assumes p: p division_of (cbox a b)
    and norm c ≤ e
  shows norm (sum (λl. content l *R c) p) ≤ e * content(cbox a b)
proof -
  have sumeq: (∑ i∈p. |content i|) = sum content p
    by simp
  have e: 0 ≤ e
    using assms(2) norm_ge_zero order_trans by blast
  have norm (sum (λl. content l *R c) p) ≤ (∑ i∈p. norm (content i *R c))
    using norm_sum by blast
  also have ... ≤ e * (∑ i∈p. |content i|)
   by (simp add: sum_distrib_left[symmetric] mult.commute assms(2) mult_right_mono
sum_nonneg)
  also have ... ≤ e * content (cbox a b)
    by (metis additive_content_division p eq_iff sumeq)
  finally show ?thesis .
qed
```

lemma *rsum_bound*:
  assumes *p*: *p tagged_division_of* (*cbox a b*)
    and ∀ *x*∈*cbox a b*. *norm* (*f x*) ≤ *e*
   shows *norm* (*sum* (λ(*x*,*k*). *content k* $*_R$ *f x*) *p*) ≤ *e* ∗ *content* (*cbox a b*)
proof (*cases cbox a b* = {})
  case *True* **show** *?thesis*
   using *p* **unfolding** *True tagged_division_of_trivial* **by** *auto*
next
  case *False*
  then have *e*: *e* ≥ *0*
   by (*meson ex_in_conv assms*(*2*) *norm_ge_zero order_trans*)
  have *sum_le*: *sum* (*content* ∘ *snd*) *p* ≤ *content* (*cbox a b*)
   unfolding *additive_content_tagged_division*[*OF p*, *symmetric*] *split_def*
   by (*auto intro*: *eq_refl*)
  have *con*: ⋀*xk*. *xk* ∈ *p* ⟹ *0* ≤ *content* (*snd xk*)
   using *tagged_division_ofD*(*4*) [*OF p*] *content_pos_le*
   by *force*
  have *norm* (*sum* (λ(*x*,*k*). *content k* $*_R$ *f x*) *p*) ≤ (∑ *i*∈*p*. *norm* (*case i of* (*x*, *k*) ⇒ *content k* $*_R$ *f x*))
   by (*rule norm_sum*)
  also have ... ≤ *e* ∗ *content* (*cbox a b*)
  proof −
   have ⋀*xk*. *xk* ∈ *p* ⟹ *norm* (*f* (*fst xk*)) ≤ *e*
    using *assms*(*2*) *p tag_in_interval* **by** *force*
   moreover have (∑ *i*∈*p*. |*content* (*snd i*)| ∗ *e*) ≤ *e* ∗ *content* (*cbox a b*)
    unfolding *sum_distrib_right*[*symmetric*]
    using *con sum_le* **by** (*auto simp*: *mult.commute intro*: *mult_left_mono* [*OF _ e*])
   ultimately show *?thesis*
    unfolding *split_def norm_scaleR*
    by (*metis* (*no_types*, *lifting*) *mult_left_mono*[*OF _ abs_ge_zero*] *order_trans*[*OF sum_mono*])
  qed
  finally show *?thesis* **.**
qed

lemma *rsum_diff_bound*:
  assumes *p tagged_division_of* (*cbox a b*)
   and ∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e*
  shows *norm* (*sum* (λ(*x*,*k*). *content k* $*_R$ *f x*) *p* − *sum* (λ(*x*,*k*). *content k* $*_R$ *g x*) *p*) ≤
     e ∗ *content* (*cbox a b*)
  using *order_trans*[*OF _ rsum_bound*[*OF assms*]]
  by (*simp add*: *split_def scaleR_diff_right sum_subtractf eq_refl*)

lemma *has_integral_bound*:
  fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*real_normed_vector*
  assumes *0* ≤ *B*

```
    and f: (f has_integral i) (cbox a b)
    and ⋀x. x∈cbox a b ⟹ norm (f x) ≤ B
  shows norm i ≤ B * content (cbox a b)
proof (rule ccontr)
  assume ¬ ?thesis
  then have norm i − B * content (cbox a b) > 0
    by auto
  with f[unfolded has_integral]
  obtain γ where gauge γ and γ:
    ⋀p. ⟦p tagged_division_of cbox a b; γ fine p⟧
        ⟹ norm ((∑(x, K)∈p. content K *R f x) − i) < norm i − B * content
(cbox a b)
    by metis
  then obtain p where p: p tagged_division_of cbox a b and γ fine p
    using fine_division_exists by blast
  have ⋀s B. norm s ≤ B ⟹ ¬ norm (s − i) < norm i − B
    unfolding not_less
   by (metis diff_left_mono dist_commute dist_norm norm_triangle_ineq2 order_trans)
  then show False
    using γ [OF p ‹γ fine p›] rsum_bound[OF p] assms by metis
qed

corollary integrable_bound:
  fixes f :: 'a::euclidean_space ⇒ 'b::real_normed_vector
  assumes 0 ≤ B
    and f integrable_on (cbox a b)
    and ⋀x. x∈cbox a b ⟹ norm (f x) ≤ B
  shows norm (integral (cbox a b) f) ≤ B * content (cbox a b)
by (metis integrable_integral has_integral_bound assms)
```

6.15.8 Similar theorems about relationship among components

```
lemma rsum_component_le:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes p: p tagged_division_of (cbox a b)
    and ⋀x. x ∈ cbox a b ⟹ (f x)·i ≤ (g x)·i
  shows (∑(x, K)∈p. content K *R f x) · i ≤ (∑(x, K)∈p. content K *R g x)
· i
unfolding inner_sum_left
proof (rule sum_mono, clarify)
  fix x K
  assume ab: (x, K) ∈ p
  with p obtain u v where K: K = cbox u v
    by blast
  then show (content K *R f x) · i ≤ (content K *R g x) · i
   by (metis ab assms inner_scaleR_left measure_nonneg mult_left_mono tag_in_interval)
qed
```

lemma *has_integral_component_le*:
fixes *f g* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *k*: *k* ∈ *Basis*
assumes (*f has_integral i*) *S* (*g has_integral j*) *S*
and *f_le_g*: ⋀*x*. *x* ∈ *S* ⟹ (*f x*)•*k* ≤ (*g x*)•*k*
shows *i*•*k* ≤ *j*•*k*
proof −
have *ik_le_jk*: *i*•*k* ≤ *j*•*k*
if *f_i*: (*f has_integral i*) (*cbox a b*)
and *g_j*: (*g has_integral j*) (*cbox a b*)
and *le*: ∀ *x*∈*cbox a b*. (*f x*)•*k* ≤ (*g x*)•*k*
for *a b i* **and** *j* :: *'b* **and** *f g* :: *'a* ⇒ *'b*
proof (*rule ccontr*)
assume ¬ *?thesis*
then have ∗: *0* < (*i*•*k* − *j*•*k*) / *3*
by *auto*
obtain *γ1* **where** *gauge γ1*
and *γ1*: ⋀*p*. ⟦*p tagged_division_of cbox a b*; *γ1 fine p*⟧
⟹ *norm* ((∑(*x*, *k*)∈*p*. *content k* ∗*R* *f x*) − *i*) < (*i* • *k* − *j* • *k*) / *3*
using *f_i*[*unfolded has_integral*,*rule_format*,*OF* ∗] **by** *fastforce*
obtain *γ2* **where** *gauge γ2*
and *γ2*: ⋀*p*. ⟦*p tagged_division_of cbox a b*; *γ2 fine p*⟧
⟹ *norm* ((∑(*x*, *k*)∈*p*. *content k* ∗*R* *g x*) − *j*) < (*i* • *k* − *j* • *k*) / *3*
using *g_j*[*unfolded has_integral*,*rule_format*,*OF* ∗] **by** *fastforce*
obtain *p* **where** *p*: *p tagged_division_of cbox a b* **and** *γ1 fine p γ2 fine p*
using *fine_division_exists*[*OF gauge_Int*[*OF* ‹*gauge γ1*› ‹*gauge γ2*›], *of a b*]
unfolding *fine_Int*
by *metis*
then have |((∑(*x*, *k*)∈*p*. *content k* ∗*R* *f x*) − *i*) • *k*| < (*i* • *k* − *j* • *k*) / *3*
|((∑(*x*, *k*)∈*p*. *content k* ∗*R* *g x*) − *j*) • *k*| < (*i* • *k* − *j* • *k*) / *3*
using *le_less_trans*[*OF Basis_le_norm*[*OF k*]] *k γ1 γ2* **by** *metis*+
then show *False*
unfolding *inner_simps*
using *rsum_component_le*[*OF p*] *le*
by (*fastforce simp add*: *abs_real_def split*: *if_split_asm*)
qed
show *?thesis*
proof (*cases* ∃ *a b*. *S* = *cbox a b*)
case *True*
with *ik_le_jk assms* **show** *?thesis*
by *auto*
next
case *False*
show *?thesis*
proof (*rule ccontr*)
assume ¬ *i*•*k* ≤ *j*•*k*
then have *ij*: (*i*•*k* − *j*•*k*) / *3* > *0*
by *auto*
obtain *B1* **where** *0* < *B1*

```
          and B1: ⋀a b. ball 0 B1 ⊆ cbox a b ⟹
                  ∃z. ((λx. if x ∈ S then f x else 0) has_integral z) (cbox a b) ∧
                     norm (z − i) < (i · k − j · k) / 3
        using has_integral_altD[OF _ False ij] assms by blast
      obtain B2 where 0 < B2
          and B2: ⋀a b. ball 0 B2 ⊆ cbox a b ⟹
                  ∃z. ((λx. if x ∈ S then g x else 0) has_integral z) (cbox a b) ∧
                     norm (z − j) < (i · k − j · k) / 3
        using has_integral_altD[OF _ False ij] assms by blast
      have bounded (ball 0 B1 ∪ ball (0::'a) B2)
        unfolding bounded_Un by(rule conjI bounded_ball)+
      from bounded_subset_cbox_symmetric[OF this]
      obtain a b::'a where ab: ball 0 B1 ⊆ cbox a b ball 0 B2 ⊆ cbox a b
        by (meson Un_subset_iff)
      then obtain w1 w2 where int_w1: ((λx. if x ∈ S then f x else 0) has_integral
w1) (cbox a b)
                        and norm_w1:  norm (w1 − i) < (i · k − j · k) / 3
                          and int_w2: ((λx. if x ∈ S then g x else 0) has_integral w2)
(cbox a b)
                        and norm_w2: norm (w2 − j) < (i · k − j · k) / 3
        using B1 B2 by blast
      have ∗: ⋀w1 w2 j i::real .|w1 − i| < (i − j) / 3 ⟹ |w2 − j| < (i − j) / 3
⟹ w1 ≤ w2 ⟹ False
        by (simp add: abs_real_def split: if_split_asm)
      have |(w1 − i) · k| < (i · k − j · k) / 3
          |(w2 − j) · k| < (i · k − j · k) / 3
        using Basis_le_norm k le_less_trans norm_w1 norm_w2 by blast+
      moreover
      have w1·k ≤ w2·k
        using ik_le_jk int_w1 int_w2 f_le_g by auto
      ultimately show False
        unfolding inner_simps by(rule ∗)
    qed
  qed
qed

lemma integral_component_le:
  fixes g f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes k ∈ Basis
    and f integrable_on S g integrable_on S
    and ⋀x. x ∈ S ⟹ (f x)·k ≤ (g x)·k
  shows (integral S f)·k ≤ (integral S g)·k
  using has_integral_component_le assms by blast

lemma has_integral_component_nonneg:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes k ∈ Basis
    and (f has_integral i) S
    and ⋀x. x ∈ S ⟹ 0 ≤ (f x)·k
```

shows *0* ≤ *i*·*k*
using *has_integral_component_le*[*OF assms*(*1*) *has_integral_0 assms*(*2*)]
using *assms*(*3*−)
by *auto*

lemma *integral_component_nonneg*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
assumes *k* ∈ *Basis*
and ⋀*x*. *x* ∈ *S* ⟹ *0* ≤ (*f x*)·*k*
shows *0* ≤ (*integral S f*)·*k*
proof (*cases f integrable_on S*)
case *True* **show** *?thesis*
using *True assms has_integral_component_nonneg* **by** *blast*
next
case *False* **then show** *?thesis* **by** (*simp add*: *not_integrable_integral*)
qed

lemma *has_integral_component_neg*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
assumes *k* ∈ *Basis*
and (*f has_integral i*) *S*
and ⋀*x*. *x* ∈ *S* ⟹ (*f x*)·*k* ≤ *0*
shows *i*·*k* ≤ *0*
using *has_integral_component_le*[*OF assms*(*1*,*2*) *has_integral_0*] *assms*(*2*−)
by *auto*

lemma *has_integral_component_lbound*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
assumes (*f has_integral i*) (*cbox a b*)
and ∀ *x*∈*cbox a b*. *B* ≤ *f*(*x*)·*k*
and *k* ∈ *Basis*
shows *B* ∗ *content* (*cbox a b*) ≤ *i*·*k*
using *has_integral_component_le*[*OF assms*(*3*) *has_integral_const assms*(*1*),*of* (∑ *i*∈*Basis*. *B* $*_R$ *i*)::′*b*] *assms*(*2*−)
by (*auto simp add*: *field_simps*)

lemma *has_integral_component_ubound*:
fixes *f*::′*a*::*euclidean_space* => ′*b*::*euclidean_space*
assumes (*f has_integral i*) (*cbox a b*)
and ∀ *x*∈*cbox a b*. *f x*·*k* ≤ *B*
and *k* ∈ *Basis*
shows *i*·*k* ≤ *B* ∗ *content* (*cbox a b*)
using *has_integral_component_le*[*OF assms*(*3*,*1*) *has_integral_const*, *of* ∑ *i*∈*Basis*. *B* $*_R$ *i*] *assms*(*2*−)
by (*auto simp add*: *field_simps*)

lemma *integral_component_lbound*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
assumes *f integrable_on cbox a b*

and $\forall x \in cbox\ a\ b.\ B \leq f(x) \cdot k$
and $k \in Basis$
shows $B * content\ (cbox\ a\ b) \leq (integral(cbox\ a\ b)\ f) \cdot k$
using *assms has_integral_component_lbound* **by** *blast*

lemma *integral_component_lbound_real*:
assumes *f integrable_on* $\{a\ ::real..b\}$
and $\forall x \in \{a..b\}.\ B \leq f(x) \cdot k$
and $k \in Basis$
shows $B * content\ \{a..b\} \leq (integral\ \{a..b\}\ f) \cdot k$
using *assms*
by (*metis box_real*(*2*) *integral_component_lbound*)

lemma *integral_component_ubound*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *f integrable_on cbox a b*
and $\forall x \in cbox\ a\ b.\ f\ x \cdot k \leq B$
and $k \in Basis$
shows $(integral\ (cbox\ a\ b)\ f) \cdot k \leq B * content\ (cbox\ a\ b)$
using *assms has_integral_component_ubound* **by** *blast*

lemma *integral_component_ubound_real*:
fixes f :: *real* ⇒ *'a::euclidean_space*
assumes *f integrable_on* $\{a..b\}$
and $\forall x \in \{a..b\}.\ f\ x \cdot k \leq B$
and $k \in Basis$
shows $(integral\ \{a..b\}\ f) \cdot k \leq B * content\ \{a..b\}$
using *assms*
by (*metis box_real*(*2*) *integral_component_ubound*)

6.15.9 Uniform limit of integrable functions is integrable

lemma *real_arch_invD*:
$0 < (e::real) \Longrightarrow (\exists n::nat.\ n \neq 0 \wedge 0 < inverse\ (real\ n) \wedge inverse\ (real\ n) < e)$
by (*subst*(*asm*) *real_arch_inverse*)

lemma *integrable_uniform_limit*:
fixes f :: *'a::euclidean_space* ⇒ *'b::banach*
assumes $\bigwedge e.\ e > 0 \Longrightarrow \exists g.\ (\forall x \in cbox\ a\ b.\ norm\ (f\ x - g\ x) \leq e) \wedge g$ *integrable_on cbox a b*
shows *f integrable_on cbox a b*
proof (*cases content* (*cbox a b*) > *0*)
case *False* **then show** *?thesis*
using *has_integral_null* **by** (*simp add*: *content_lt_nz integrable_on_def*)
next
case *True*
have $1\ /\ (real\ n + 1) > 0$ **for** n

by *auto*
then have $\exists g.$ ($\forall x \in cbox\ a\ b.\ norm\ (f\ x\ -\ g\ x) \leq 1\ /\ (real\ n\ +\ 1)$) $\wedge$ *g integrable_on cbox a b* **for** *n*
using *assms* **by** *blast*
then obtain *g* **where** *g_near_f*: $\bigwedge n\ x.\ x \in cbox\ a\ b \Longrightarrow norm\ (f\ x\ -\ g\ n\ x) \leq 1\ /\ (real\ n\ +\ 1)$
and *int_g*: $\bigwedge n.$ *g n integrable_on cbox a b*
by *metis*
then obtain *h* **where** *h*: $\bigwedge n.$ (*g n has_integral h n*) (*cbox a b*)
unfolding *integrable_on_def* **by** *metis*
have *Cauchy h*
unfolding *Cauchy_def*
proof *clarify*
fix *e* :: *real*
assume $e>0$
then have $e/4\ /\ content\ (cbox\ a\ b) > 0$
using *True* **by** (*auto simp*: *field_simps*)
then obtain *M* **where** $M \neq 0$ **and** *M*: $1\ /\ (real\ M) < e/4\ /\ content\ (cbox\ a\ b)$
by (*metis inverse_eq_divide real_arch_inverse*)
show $\exists M.\ \forall m \geq M.\ \forall n \geq M.\ dist\ (h\ m)\ (h\ n) < e$
proof (*rule exI* [**where** $x=M$], *clarify*)
fix *m n*
assume *m*: $M \leq m$ **and** *n*: $M \leq n$
have $e/4>0$ **using** ‹$e>0$› **by** *auto*
then obtain *gm gn* **where** *gauge gm gauge gn*
and *gm*: $\bigwedge \mathcal{D}.\ \mathcal{D}$ *tagged_division_of cbox a b* $\wedge$ *gm fine* $\mathcal{D}$
$\Longrightarrow norm\ ((\sum (x,K) \in \mathcal{D}.\ content\ K *_R g\ m\ x) - h\ m) < e/4$
and *gn*: $\bigwedge \mathcal{D}.\ \mathcal{D}$ *tagged_division_of cbox a b* $\wedge$ *gn fine* $\mathcal{D} \Longrightarrow$
$norm\ ((\sum (x,K) \in \mathcal{D}.\ content\ K *_R g\ n\ x) - h\ n) < e/4$
using *h*[*unfolded has_integral*] **by** *meson*
then obtain $\mathcal{D}$ **where** $\mathcal{D}$: $\mathcal{D}$ *tagged_division_of cbox a b* ($\lambda x.\ gm\ x \cap gn\ x$) *fine* $\mathcal{D}$
by (*metis* (*full_types*) *fine_division_exists gauge_Int*)
have *triangle3*: $norm\ (i1\ -\ i2) < e$
if *no*: $norm(s2\ -\ s1) \leq e/2$ $norm\ (s1\ -\ i1) < e/4$ $norm\ (s2\ -\ i2) < e/4$ **for** *s1 s2 i1* **and** *i2*::'*b*
proof –
have $norm\ (i1\ -\ i2) \leq norm\ (i1\ -\ s1) + norm\ (s1\ -\ s2) + norm\ (s2\ -\ i2)$
using *norm_triangle_ineq*[*of i1 − s1 s1 − i2*]
using *norm_triangle_ineq*[*of s1 − s2 s2 − i2*]
by (*auto simp*: *algebra_simps*)
also have $\ldots < e$
using *no* **by** (*auto simp*: *algebra_simps norm_minus_commute*)
finally show *?thesis* **.**
qed
have *finep*: *gm fine* $\mathcal{D}$ *gn fine* $\mathcal{D}$

```
        using fine_Int 𝒟  by auto
      have norm_le: norm (g n x − g m x) ≤ 2 / real M if x: x ∈ cbox a b for x
      proof −
        have norm (f x − g n x) + norm (f x − g m x) ≤ 1 / (real n + 1) + 1 /
(real m + 1)
          using g_near_f[OF x, of n] g_near_f[OF x, of m] by simp
        also have ... ≤ 1 / (real M) + 1 / (real M)
          using ‹M ≠ 0› m n by (intro add_mono; force simp: field_split_simps)
        also have ... = 2 / real M
          by auto
        finally show norm (g n x − g m x) ≤ 2 / real M
          using norm_triangle_le[of g n x − f x f x − g m x 2 / real M]
          by (auto simp: algebra_simps simp add: norm_minus_commute)
      qed
      have norm ((∑(x,K) ∈ 𝒟. content K *R g n x) − (∑(x,K) ∈ 𝒟. content
K *R g m x)) ≤ 2 / real M * content (cbox a b)
        by (blast intro: norm_le rsum_diff_bound[OF 𝒟(1), where e=2 / real M])
      also have ... ≤ e/2
        using M True
        by (auto simp: field_simps)
      finally have le_e2: norm ((∑(x,K) ∈ 𝒟. content K *R g n x) − (∑(x,K)
∈ 𝒟. content K *R g m x)) ≤ e/2 .
      then show dist (h m) (h n) < e
        unfolding dist_norm using gm gn 𝒟 finep by (auto intro!: triangle3)
    qed
  qed
  then obtain s where s: h ⟶ s
    using convergent_eq_Cauchy[symmetric] by blast
  show ?thesis
    unfolding integrable_on_def has_integral
  proof (rule_tac x=s in exI, clarify)
    fix e::real
    assume e: 0 < e
    then have e/3 > 0 by auto
    then obtain N1 where N1: ∀n≥N1. norm (h n − s) < e/3
      using LIMSEQ_D [OF s] by metis
    from e True have e/3 / content (cbox a b) > 0
      by (auto simp: field_simps)
    then obtain N2 :: nat
        where N2 ≠ 0 and N2: 1 / (real N2) < e/3 / content (cbox a b)
      by (metis inverse_eq_divide real_arch_inverse)
    obtain g' where gauge g'
            and g': ⋀𝒟. 𝒟 tagged_division_of cbox a b ∧ g' fine 𝒟 ⟹
                norm ((∑(x,K) ∈ 𝒟. content K *R g (N1 + N2) x) − h (N1 +
N2)) < e/3
      by (metis h has_integral ‹e/3 > 0›)
    have *: norm (sf − s) < e
        if no: norm (sf − sg) ≤ e/3 norm(h − s) < e/3 norm (sg − h) < e/3 for
sf sg h
```

proof −
have *norm* (*sf* − *s*) ≤ *norm* (*sf* − *sg*) + *norm* (*sg* − *h*) + *norm* (*h* − *s*)
using *norm_triangle_ineq*[*of sf* − *sg sg* − *s*]
using *norm_triangle_ineq*[*of sg* − *h h* − *s*]
by (*auto simp*: *algebra_simps*)
also have ... < *e*
using *no* **by** (*auto simp*: *algebra_simps norm_minus_commute*)
finally show *?thesis* **.**
qed
{ fix $\mathcal{D}$
assume *ptag*: $\mathcal{D}$ *tagged_division_of* (*cbox a b*) **and** *g′ fine* $\mathcal{D}$
then have *norm_less*: *norm* ((∑(*x*,*K*) ∈ $\mathcal{D}$. *content K* $*_R$ *g* (*N1* + *N2*) *x*) − *h* (*N1* + *N2*)) < *e*/*3*
using *g′* **by** *blast*
have *content* (*cbox a b*) < *e*/*3* ∗ (*of_nat N2*)
using ‹*N2* ≠ *0*› *N2* **using** *True* **by** (*auto simp*: *field_split_simps*)
moreover have *e*/*3* ∗ *of_nat N2* ≤ *e*/*3* ∗ (*of_nat* (*N1* + *N2*) + *1*)
using ‹*e*>*0*› **by** *auto*
ultimately have *content* (*cbox a b*) < *e*/*3* ∗ (*of_nat* (*N1* + *N2*) + *1*)
by *linarith*
then have *le_e3*: *1* / (*real* (*N1* + *N2*) + *1*) ∗ *content* (*cbox a b*) ≤ *e*/*3*
unfolding *inverse_eq_divide*
by (*auto simp*: *field_simps*)
have *ne3*: *norm* (*h* (*N1* + *N2*) − *s*) < *e*/*3*
using *N1* **by** *auto*
have *norm* ((∑(*x*,*K*) ∈ $\mathcal{D}$. *content K* $*_R$ *f x*) − (∑(*x*,*K*) ∈ $\mathcal{D}$. *content K* $*_R$ *g* (*N1* + *N2*) *x*))
≤ *1* / (*real* (*N1* + *N2*) + *1*) ∗ *content* (*cbox a b*)
by (*blast intro*: *g_near_f rsum_diff_bound*[*OF ptag*])
then have *norm* ((∑(*x*,*K*) ∈ $\mathcal{D}$. *content K* $*_R$ *f x*) − *s*) < *e*
by (*rule* ∗[*OF order_trans* [*OF _ le_e3*] *ne3 norm_less*])
}
then show ∃ *d*. *gauge d* ∧
(∀ $\mathcal{D}$. $\mathcal{D}$ *tagged_division_of cbox a b* ∧ *d fine* $\mathcal{D}$ ⟶ *norm* ((∑(*x*,*K*) ∈ $\mathcal{D}$. *content K* $*_R$ *f x*) − *s*) < *e*)
by (*blast intro*: *g′* ‹*gauge g′*›)
qed
qed

lemmas *integrable_uniform_limit_real* = *integrable_uniform_limit* [**where** *′a*=*real*, *simplified*]

6.15.10 Negligible sets

definition *negligible* (*s*:: *′a*::*euclidean_space set*) ⟷
(∀ *a b*. ((*indicator s* :: *′a*⇒*real*) *has_integral 0*) (*cbox a b*))

Negligibility of hyperplane

lemma *content_doublesplit*:

fixes *a* :: *'a::euclidean_space*
assumes *0* < *e*
and *k*: *k* ∈ *Basis*
obtains *d* **where** *0* < *d* **and** *content* (*cbox a b* ∩ {*x*. |*x*·*k* − *c*| ≤ *d*}) < *e*
proof *cases*
assume ∗: *a* · *k* ≤ *c* ∧ *c* ≤ *b* · *k* ∧ (∀ *j*∈*Basis*. *a* · *j* ≤ *b* · *j*)
define *a'* **where** *a' d* = (∑ *j*∈*Basis*. (*if j* = *k then max* (*a*·*j*) (*c* − *d*) *else a*·*j*) ∗$_R$ *j*) **for** *d*
define *b'* **where** *b' d* = (∑ *j*∈*Basis*. (*if j* = *k then min* (*b*·*j*) (*c* + *d*) *else b*·*j*) ∗$_R$ *j*) **for** *d*

have ((λ*d*. ∏ *j*∈*Basis*. (*b' d* − *a' d*) · *j*) ⟶ (∏ *j*∈*Basis*. (*b' 0* − *a' 0*) · *j*)) (*at_right 0*)
by (*auto simp*: *b'_def a'_def intro*!: *tendsto_min tendsto_max tendsto_eq_intros*)
also have (∏ *j*∈*Basis*. (*b' 0* − *a' 0*) · *j*) = *0*
using *k* ∗
by (*intro prod_zero bexI*[*OF _ k*])
(*auto simp*: *b'_def a'_def inner_diff inner_sum_left inner_not_same_Basis intro*!: *sum.cong*)
also have ((λ*d*. ∏ *j*∈*Basis*. (*b' d* − *a' d*) · *j*) ⟶ *0*) (*at_right 0*) =
((λ*d*. *content* (*cbox a b* ∩ {*x*. |*x*·*k* − *c*| ≤ *d*})) ⟶ *0*) (*at_right 0*)
proof (*intro tendsto_cong eventually_at_rightI*)
fix *d* :: *real* **assume** *d*: *d* ∈ {*0*<..<*1*}
have *cbox a b* ∩ {*x*. |*x*·*k* − *c*| ≤ *d*} = *cbox* (*a' d*) (*b' d*) **for** *d*
using ∗ *d k* **by** (*auto simp add*: *cbox_def set_eq_iff Int_def ball_conj_distrib abs_diff_le_iff a'_def b'_def*)
moreover have *j* ∈ *Basis* ⟹ *a' d* · *j* ≤ *b' d* · *j* **for** *j*
using ∗ *d k* **by** (*auto simp*: *a'_def b'_def*)
ultimately show (∏ *j*∈*Basis*. (*b' d* − *a' d*) · *j*) = *content* (*cbox a b* ∩ {*x*. |*x*·*k* − *c*| ≤ *d*})
by *simp*
qed *simp*
finally have ((λ*d*. *content* (*cbox a b* ∩ {*x*. |*x* · *k* − *c*| ≤ *d*})) ⟶ *0*) (*at_right 0*) **.**
from *order_tendstoD*(*2*)[*OF this* ‹*0*<*e*›]
obtain *d'* **where** *0* < *d'* **and** *d'*: ⋀*y*. *y* > *0* ⟹ *y* < *d'* ⟹ *content* (*cbox a b* ∩ {*x*. |*x* · *k* − *c*| ≤ *y*}) < *e*
by (*subst* (*asm*) *eventually_at_right*[*of _ 1*]) *auto*
show *?thesis*
by (*rule that*[*of d'/2*], *insert* ‹*0*<*d'*› *d'*[*of d'/2*], *auto*)
next
assume ∗: ¬ (*a* · *k* ≤ *c* ∧ *c* ≤ *b* · *k* ∧ (∀ *j*∈*Basis*. *a* · *j* ≤ *b* · *j*))
then have (∃ *j*∈*Basis*. *b* · *j* < *a* · *j*) ∨ (*c* < *a* · *k* ∨ *b* · *k* < *c*)
by (*auto simp*: *not_le*)
show *thesis*
proof *cases*
assume ∃ *j*∈*Basis*. *b* · *j* < *a* · *j*
then have [*simp*]: *cbox a b* = {}
using *box_ne_empty*(*1*)[*of a b*] **by** *auto*

```
    show ?thesis
      by (rule that[of 1]) (simp_all add: ‹0<e›)
  next
    assume ¬ (∃j∈Basis. b · j < a · j)
    with ∗ have c < a · k ∨ b · k < c
      by auto
    then show thesis
    proof
      assume c: c < a · k
      moreover have x ∈ cbox a b =⇒ c ≤ x · k for x
        using k c by (auto simp: cbox_def)
      ultimately have cbox a b ∩ {x. |x · k − c| ≤ (a · k − c)/2} = {}
        using k by (auto simp: cbox_def)
      with ‹0<e› c that[of (a · k − c)/2] show ?thesis
        by auto
    next
      assume c: b · k < c
      moreover have x ∈ cbox a b =⇒ x · k ≤ c for x
        using k c by (auto simp: cbox_def)
      ultimately have cbox a b ∩ {x. |x · k − c| ≤ (c − b · k)/2} = {}
        using k by (auto simp: cbox_def)
      with ‹0<e› c that[of (c − b · k)/2] show ?thesis
        by auto
    qed
  qed
qed


proposition negligible_standard_hyperplane[intro]:
  fixes k :: 'a::euclidean_space
  assumes k: k ∈ Basis
  shows negligible {x. x·k = c}
  unfolding negligible_def has_integral
proof clarsimp
  fix a b and e::real assume e > 0
  with k obtain d where 0 < d and d: content (cbox a b ∩ {x. |x · k − c| ≤
d}) < e
    by (metis content_doublesplit)
  let ?i = indicator {x::'a. x·k = c} :: 'a⇒real
  show ∃γ. gauge γ ∧
           (∀D. D tagged_division_of cbox a b ∧ γ fine D −→
                |∑(x,K) ∈ D. content K ∗ ?i x| < e)
  proof (intro exI, safe)
    show gauge (λx. ball x d)
      using ‹0 < d› by blast
  next
    fix D
    assume p: D tagged_division_of (cbox a b) (λx. ball x d) fine D
    have content L = content (L ∩ {x. |x · k − c| ≤ d})
```

```
      if (x, L) ∈ 𝒟 ?i x ≠ 0 for x L
    proof –
      have xk: x·k = c
        using that by (simp add: indicator_def split: if_split_asm)
      have L ⊆ {x. |x · k − c| ≤ d}
      proof
        fix y
        assume y: y ∈ L
        have L ⊆ ball x d
          using p(2) that(1) by auto
        then have norm (x − y) < d
          by (simp add: dist_norm subset_iff y)
        then have |(x − y) · k| < d
          using k norm_bound_Basis_lt by blast
        then show y ∈ {x. |x · k − c| ≤ d}
          unfolding inner_simps xk by auto
      qed
      then show content L = content (L ∩ {x. |x · k − c| ≤ d})
        by (metis inf.orderE)
    qed
    then have *: (∑(x,K)∈𝒟. content K * ?i x) = (∑(x,K)∈𝒟. content (K ∩
{x. |x·k − c| ≤ d}) *R ?i x)
      by (force simp add: split_paired_all intro!: sum.cong [OF refl])
    note p'= tagged_division_ofD[OF p(1)] and p''=division_of_tagged_division[OF
p(1)]
    have (∑(x,K)∈𝒟. content (K ∩ {x. |x · k − c| ≤ d}) * indicator {x. x · k
= c} x) < e
    proof –
     have (∑(x,K)∈𝒟. content (K ∩ {x. |x · k − c| ≤ d}) * ?i x) ≤ (∑(x,K)∈𝒟.
content (K ∩ {x. |x · k − c| ≤ d}))
        by (force simp add: indicator_def intro!: sum_mono)
      also have ... < e
      proof (subst sum.over_tagged_division_lemma[OF p(1)])
        fix u v::'a
        assume box u v = {}
        then have *: content (cbox u v) = 0
          unfolding content_eq_0_interior by simp
        have cbox u v ∩ {x. |x · k − c| ≤ d} ⊆ cbox u v
          by auto
        then have content (cbox u v ∩ {x. |x · k − c| ≤ d}) ≤ content (cbox u v)
          unfolding interval_doublesplit[OF k] by (rule content_subset)
        then show content (cbox u v ∩ {x. |x · k − c| ≤ d}) = 0
          unfolding * interval_doublesplit[OF k]
          by (blast intro: antisym)
      next
        have (∑l∈snd ` 𝒟. content (l ∩ {x. |x · k − c| ≤ d})) =
          sum content ((λl. l ∩ {x. |x · k − c| ≤ d})`{l∈snd ` 𝒟. l ∩ {x. |x · k −
c| ≤ d} ≠ {}})
        proof (subst (2) sum.reindex_nontrivial)
```

fix x y **assume** $x \in \{l \in snd$ ‘ $\mathcal{D}.\ l \cap \{x.\ |x \cdot k - c| \leq d\} \neq \{\}\}$ $y \in \{l \in snd$ ‘ $\mathcal{D}.\ l \cap \{x.\ |x \cdot k - c| \leq d\} \neq \{\}\}$
$x \neq y$ **and** *eq*: $x \cap \{x.\ |x \cdot k - c| \leq d\} = y \cap \{x.\ |x \cdot k - c| \leq d\}$
then obtain x' y' **where** $(x', x) \in \mathcal{D}$ $x \cap \{x.\ |x \cdot k - c| \leq d\} \neq \{\}$ $(y', y) \in \mathcal{D}$ $y \cap \{x.\ |x \cdot k - c| \leq d\} \neq \{\}$
by (*auto*)
from $p'(5)$[*OF* ‹$(x', x) \in \mathcal{D}$› ‹$(y', y) \in \mathcal{D}$›] ‹$x \neq y$› **have** *interior* $(x \cap y) = \{\}$
by *auto*
moreover have *interior* $((x \cap \{x.\ |x \cdot k - c| \leq d\}) \cap (y \cap \{x.\ |x \cdot k - c| \leq d\})) \subseteq$ *interior* $(x \cap y)$
by (*auto intro*: *interior_mono*)
ultimately have *interior* $(x \cap \{x.\ |x \cdot k - c| \leq d\}) = \{\}$
by (*auto simp*: *eq*)
then show *content* $(x \cap \{x.\ |x \cdot k - c| \leq d\}) = 0$
using $p'(4)$[*OF* ‹$(x', x) \in \mathcal{D}$›] **by** (*auto simp*: *interval_doublesplit*[*OF k*] *content_eq_0_interior simp del*: *interior_Int*)
qed (*insert* $p'(1)$, *auto intro*!: *sum.mono_neutral_right*)
also have ... $\leq$ *norm* ($\sum l \in (\lambda l.\ l \cap \{x.\ |x \cdot k - c| \leq d\})$‘$\{l \in snd$ ‘ $\mathcal{D}.\ l \cap \{x.\ |x \cdot k - c| \leq d\} \neq \{\}\}$. *content* $l *_R 1::real$)
by *simp*
also have ... $\leq 1 *$ *content* (*cbox a b* $\cap \{x.\ |x \cdot k - c| \leq d\}$)
using *division_doublesplit*[*OF* p'' *k*, *unfolded interval_doublesplit*[*OF k*]]
unfolding *interval_doublesplit*[*OF k*] **by** (*intro dsum_bound*) *auto*
also have ... $< e$
using *d* **by** *simp*
finally show ($\sum K \in snd$ ‘ $\mathcal{D}$. *content* $(K \cap \{x.\ |x \cdot k - c| \leq d\})) < e$ **.**
qed
finally show ($\sum (x, K) \in \mathcal{D}$. *content* $(K \cap \{x.\ |x \cdot k - c| \leq d\}) * ?i\ x) < e$ **.**
qed
then show $|\sum (x, K) \in \mathcal{D}$. *content* $K * ?i\ x| < e$
unfolding $*$ **by** (*simp add*: *sum_nonneg split*: *prod.split*)
qed
qed

corollary *negligible_standard_hyperplane_cart*:
fixes k :: $'a$::*finite*
shows *negligible* $\{x.\ x\$k = (0::real)\}$
by (*simp add*: *cart_eq_inner_axis negligible_standard_hyperplane*)

Hence the main theorem about negligible sets

lemma *has_integral_negligible_cbox*:
fixes f :: $'b$::*euclidean_space* $\Rightarrow$ $'a$::*real_normed_vector*
assumes *negs*: *negligible S*
and *0*: $\bigwedge x.\ x \notin S \Longrightarrow f\ x = 0$
shows (*f has_integral 0*) (*cbox a b*)
unfolding *has_integral*
proof *clarify*

```
  fix e::real
  assume e > 0
  then have nn_gt0: e/2 / ((real n+1) * (2 ^ n)) > 0 for n
    by simp
  then have ∃γ. gauge γ ∧
               (∀D. D tagged_division_of cbox a b ∧ γ fine D ⟶
                    |∑(x,K) ∈ D. content K *R indicator S x|
                    < e/2 / ((real n + 1) * 2 ^ n)) for n
    using negs [unfolded negligible_def has_integral] by auto
  then obtain γ where
    gd: ⋀n. gauge (γ n)
    and γ: ⋀n D. ⟦D tagged_division_of cbox a b; γ n fine D⟧
             ⟹ |∑(x,K) ∈ D. content K *R indicator S x| < e/2 / ((real n +
1) * 2 ^ n)
    by metis
  show ∃γ. gauge γ ∧
            (∀D. D tagged_division_of cbox a b ∧ γ fine D ⟶
                 norm ((∑(x,K) ∈ D. content K *R f x) − 0) < e)
  proof (intro exI, safe)
    show gauge (λx. γ (nat ⌊norm (f x)⌋) x)
      using gd by (auto simp: gauge_def)

    show norm ((∑(x,K) ∈ D. content K *R f x) − 0) < e
      if D tagged_division_of (cbox a b) (λx. γ (nat ⌊norm (f x)⌋) x) fine D for D
    proof (cases D = {})
      case True with ‹0 < e› show ?thesis by simp
    next
      case False
      obtain N where Max ((λ(x, K). norm (f x)) ` D) ≤ real N
        using real_arch_simple by blast
      then have N: ⋀x. x ∈ (λ(x, K). norm (f x)) ` D ⟹ x ≤ real N
      by (meson Max_ge that(1) dual_order.trans finite_imageI tagged_division_of_finite)
       have ∀i. ∃q. q tagged_division_of (cbox a b) ∧ (γ i) fine q ∧ (∀(x,K) ∈ D.
K ⊆ (γ i) x ⟶ (x, K) ∈ q)
        by (auto intro: tagged_division_finer[OF that(1) gd])
      from choice[OF this]
      obtain q where q: ⋀n. q n tagged_division_of cbox a b
                        ⋀n. γ n fine q n
                        ⋀n x K. ⟦(x, K) ∈ D; K ⊆ γ n x⟧ ⟹ (x, K) ∈ q n
        by fastforce
      have finite D
        using that(1) by blast
      then have sum_le_inc: ⟦finite T; ⋀x y. (x,y) ∈ T ⟹ (0::real) ≤ g(x,y);
                        ⋀y. y∈D ⟹ ∃x. (x,y) ∈ T ∧ f(y) ≤ g(x,y)⟧ ⟹ sum f D ≤
sum g T for f g T
        by (rule sum_le_included[of D T g snd f]; force)
     have norm (∑(x,K) ∈ D. content K *R f x) ≤ (∑(x,K) ∈ D. norm (content
K *R f x))
        unfolding split_def by (rule norm_sum)
```

also have ... ≤ (∑(*i*, *j*) ∈ *Sigma* {..*N* + *1*} *q*.
(*real i* + *1*) ∗ (*case j of* (*x*, *K*) ⇒ *content K* $*_R$ *indicator S x*))
proof (*rule sum_le_inc*, *safe*)
show *finite* (*Sigma* {..*N*+*1*} *q*)
by (*meson finite_SigmaI finite_atMost tagged_division_of_finite q*(*1*))
next
fix *x K*
assume *xk*: (*x*, *K*) ∈ 𝒟
define *n* **where** *n* = *nat* ⌊*norm* (*f x*)⌋
have ∗: *norm* (*f x*) ∈ (λ(*x*, *K*). *norm* (*f x*)) ' 𝒟
using *xk* **by** *auto*
have *nfx*: *real n* ≤ *norm* (*f x*) *norm* (*f x*) ≤ *real n* + *1*
unfolding *n_def* **by** *auto*
then have *n* ∈ {*0*..*N* + *1*}
using *N*[*OF* ∗] **by** *auto*
moreover have *K* ⊆ γ (*nat* ⌊*norm* (*f x*)⌋) *x*
using *that*(*2*) *xk* **by** *auto*
moreover then have (*x*, *K*) ∈ *q* (*nat* ⌊*norm* (*f x*)⌋)
by (*simp add*: *q*(*3*) *xk*)
moreover then have (*x*, *K*) ∈ *q n*
using *n_def* **by** *blast*
moreover
have *norm* (*content K* $*_R$ *f x*) ≤ (*real n* + *1*) ∗ (*content K* ∗ *indicator S x*)
proof (*cases x* ∈ *S*)
case *False*
then show *?thesis* **by** (*simp add*: *0*)
next
case *True*
have ∗: *content K* ≥ *0*
using *tagged_division_ofD*(*4*)[*OF that*(*1*) *xk*] **by** *auto*
moreover have *content K* ∗ *norm* (*f x*) ≤ *content K* ∗ (*real n* + *1*)
by (*simp add*: *mult_left_mono nfx*(*2*))
ultimately show *?thesis*
using *nfx True* **by** (*auto simp*: *field_simps*)
qed
ultimately show ∃ *y*. (*y*, *x*, *K*) ∈ (*Sigma* {..*N* + *1*} *q*) ∧ *norm* (*content K* $*_R$ *f x*) ≤
(*real y* + *1*) ∗ (*content K* $*_R$ *indicator S x*)
by *force*
qed *auto*
also have ... = (∑*i*≤*N* + *1*. ∑*j*∈*q i*. (*real i* + *1*) ∗ (*case j of* (*x*, *K*) ⇒ *content K* $*_R$ *indicator S x*))
using *q*(*1*) **by** (*intro sum_Sigma_product* [*symmetric*]) *auto*
also have ... ≤ (∑*i*≤*N* + *1*. (*real i* + *1*) ∗ |∑(*x*,*K*) ∈ *q i*. *content K* $*_R$ *indicator S x*|)
by (*rule sum_mono*) (*simp add*: *sum_distrib_left* [*symmetric*])
also have ... ≤ (∑*i*≤*N* + *1*. *e*/*2*/*2* ˆ *i*)
proof (*rule sum_mono*)

```
      show (real i + 1) ∗ |∑(x,K) ∈ q i. content K ∗R indicator S x| ≤ e/2/2
^ i
        if i ∈ {..N + 1} for i
        using γ[of q i i] q by (simp add: divide_simps mult.left_commute)
    qed
    also have ... = e/2 ∗ (∑i≤N + 1. (1/2) ^ i)
    unfolding sum_distrib_left by (metis divide_inverse inverse_eq_divide power_one_over)
    also have ... < e/2 ∗ 2
    proof (rule mult_strict_left_mono)
      have sum (power (1/2)) {..N + 1} = sum (power (1/2::real)) {..<N +
2}
        using lessThan_Suc_atMost by auto
      also have ... < 2
        by (auto simp: geometric_sum)
      finally show sum (power (1/2::real)) {..N + 1} < 2 .
    qed (use ‹0 < e› in auto)
    finally show ?thesis by auto
  qed
 qed
qed


proposition has_integral_negligible:
  fixes f :: 'b::euclidean_space ⇒ 'a::real_normed_vector
  assumes negs: negligible S
    and ⋀x. x ∈ (T − S) ⟹ f x = 0
  shows (f has_integral 0) T
proof (cases ∃ a b. T = cbox a b)
  case True
  then have ((λx. if x ∈ T then f x else 0) has_integral 0) T
    using assms by (auto intro!: has_integral_negligible_cbox)
  then show ?thesis
    by (rule has_integral_eq [rotated]) auto
next
  case False
  let ?f = (λx. if x ∈ T then f x else 0)
  have ((λx. if x ∈ T then f x else 0) has_integral 0) T
    apply (auto simp: False has_integral_alt [of ?f])
    apply (rule_tac x=1 in exI, auto)
    apply (rule_tac x=0 in exI, simp add: has_integral_negligible_cbox [OF negs]
assms)
    done
  then show ?thesis
    by (rule_tac f=?f in has_integral_eq) auto
qed

lemma
  assumes negligible S
  shows integrable_negligible: f integrable_on S and integral_negligible: integral S f
```

= *0*
  using *has_integral_negligible* [*OF assms*]
  by (*auto simp*: *has_integral_iff*)

lemma *has_integral_spike*:
  fixes *f* :: *'b*::*euclidean_space* ⇒ *'a*::*real_normed_vector*
  assumes *negligible S*
    and *gf*: ⋀*x*. *x* ∈ *T* − *S* ⟹ *g x* = *f x*
    and *fint*: (*f has_integral y*) *T*
  shows (*g has_integral y*) *T*
proof −
  have ∗: (*g has_integral y*) (*cbox a b*)
      if (*f has_integral y*) (*cbox a b*) ∀ *x* ∈ *cbox a b* − *S*. *g x* = *f x* **for** *a b f* **and** *g*:: *'b* ⇒ *'a* **and** *y*
  proof −
    have ((λ*x*. *f x* + (*g x* − *f x*)) *has_integral* (*y* + *0*)) (*cbox a b*)
      using *that* **by** (*intro has_integral_add has_integral_negligible*) (*auto intro*!: ‹*negligible S*›)
    then show *?thesis*
      by *auto*
  qed
  have §: ⋀*a b z*. ⟦⋀*x*. *x* ∈ *T* ∧ *x* ∉ *S* ⟹ *g x* = *f x*;
                  ((λ*x*. *if x* ∈ *T then f x else 0*) *has_integral z*) (*cbox a b*)⟧
                 ⟹ ((λ*x*. *if x* ∈ *T then g x else 0*) *has_integral z*) (*cbox a b*)
      by (*auto dest*!: ∗[**where** *f*=λ*x*. *if x*∈*T then f x else 0* **and** *g*=λ*x*. *if x* ∈ *T then g x else 0*])
  show *?thesis*
    using *fint gf*
    apply (*subst has_integral_alt*)
    apply (*subst* (*asm*) *has_integral_alt*)
    apply (*auto split*: *if_split_asm*)
     apply (*blast dest*: ∗)
    using § **by** *meson*
qed

lemma *has_integral_spike_eq*:
  assumes *negligible S*
    and *gf*: ⋀*x*. *x* ∈ *T* − *S* ⟹ *g x* = *f x*
  shows (*f has_integral y*) *T* ⟷ (*g has_integral y*) *T*
    using *has_integral_spike* [*OF* ‹*negligible S*›] *gf*
    by *metis*

lemma *integrable_spike*:
  assumes *f integrable_on T negligible S* ⋀*x*. *x* ∈ *T* − *S* ⟹ *g x* = *f x*
    shows *g integrable_on T*
  using *assms* **unfolding** *integrable_on_def* **by** (*blast intro*: *has_integral_spike*)

lemma *integral_spike*:
  assumes *negligible S*

```
    and ⋀x. x ∈ T − S ⟹ g x = f x
  shows integral T f = integral T g
  using has_integral_spike_eq[OF assms]
    by (auto simp: integral_def integrable_on_def)
```

6.15.11 Some other trivialities about negligible sets

```
lemma negligible_subset:
  assumes negligible s t ⊆ s
  shows negligible t
  unfolding negligible_def
    by (metis (no_types) Diff_iff assms contra_subsetD has_integral_negligible indi-
cator_simps(2))

lemma negligible_diff[intro?]:
  assumes negligible s
  shows negligible (s − t)
  using assms by (meson Diff_subset negligible_subset)

lemma negligible_Int:
  assumes negligible s ∨ negligible t
  shows negligible (s ∩ t)
  using assms negligible_subset by force

lemma negligible_Un:
  assumes negligible S and T: negligible T
  shows negligible (S ∪ T)
proof −
  have (indicat_real (S ∪ T) has_integral 0) (cbox a b)
    if S0: (indicat_real S has_integral 0) (cbox a b)
      and (indicat_real T has_integral 0) (cbox a b) for a b
  proof (subst has_integral_spike_eq[OF T])
    show indicat_real S x = indicat_real (S ∪ T) x if x ∈ cbox a b − T for x
      by (metis Diff_iff Un_iff indicator_def that)
    show (indicat_real S has_integral 0) (cbox a b)
      by (simp add: S0)
  qed
  with assms show ?thesis
    unfolding negligible_def by blast
qed

lemma negligible_Un_eq[simp]: negligible (s ∪ t) ⟷ negligible s ∧ negligible t
  using negligible_Un negligible_subset by blast

lemma negligible_sing[intro]: negligible {a::'a::euclidean_space}
  using negligible_standard_hyperplane[OF SOME_Basis, of a · (SOME i. i ∈ Ba-
sis)] negligible_subset by blast

lemma negligible_insert[simp]: negligible (insert a s) ⟷ negligible s
```

by (*metis insert_is_Un negligible_Un_eq negligible_sing*)

lemma *negligible_empty*[*iff*]: *negligible* {}
using *negligible_insert* **by** *blast*

Useful in this form for backchaining

lemma *empty_imp_negligible*: $S = \{\} \Longrightarrow$ *negligible S*
by *simp*

lemma *negligible_finite*[*intro*]:
assumes *finite s*
shows *negligible s*
using *assms* **by** (*induct s*) *auto*

lemma *negligible_Union*[*intro*]:
assumes *finite* $\mathcal{T}$
and $\bigwedge t.\ t \in \mathcal{T} \Longrightarrow$ *negligible t*
shows *negligible*($\bigcup \mathcal{T}$)
using *assms* **by** *induct auto*

lemma *negligible*: *negligible S* $\longleftrightarrow$ ($\forall$ *T*. (*indicat_real S has_integral 0*) *T*)
proof (*intro iffI allI*)
fix *T*
assume *negligible S*
then show (*indicator S has_integral 0*) *T*
by (*meson Diff_iff has_integral_negligible indicator_simps*(*2*))
qed (*simp add*: *negligible_def*)

6.15.12 Finite case of the spike theorem is quite commonly needed

lemma *has_integral_spike_finite*:
assumes *finite S*
and $\bigwedge x.\ x \in T - S \Longrightarrow g\ x = f\ x$
and (*f has_integral y*) *T*
shows (*g has_integral y*) *T*
using *assms has_integral_spike negligible_finite* **by** *blast*

lemma *has_integral_spike_finite_eq*:
assumes *finite S*
and $\bigwedge x.\ x \in T - S \Longrightarrow g\ x = f\ x$
shows ((*f has_integral y*) *T* $\longleftrightarrow$ (*g has_integral y*) *T*)
by (*metis assms has_integral_spike_finite*)

lemma *integrable_spike_finite*:
assumes *finite S*
and $\bigwedge x.\ x \in T - S \Longrightarrow g\ x = f\ x$
and *f integrable_on T*
shows *g integrable_on T*

```
  using assms has_integral_spike_finite by blast

lemma has_integral_bound_spike_finite:
  fixes f :: 'a::euclidean_space ⇒ 'b::real_normed_vector
  assumes 0 ≤ B finite S
      and f: (f has_integral i) (cbox a b)
      and leB: ⋀x. x ∈ cbox a b − S ⟹ norm (f x) ≤ B
    shows norm i ≤ B ∗ content (cbox a b)
proof −
  define g where g ≡ (λx. if x ∈ S then 0 else f x)
  then have ⋀x. x ∈ cbox a b − S ⟹ norm (g x) ≤ B
    using leB by simp
  moreover have (g has_integral i) (cbox a b)
    using has_integral_spike_finite [OF ‹finite S› _ f]
    by (simp add: g_def)
  ultimately show ?thesis
    by (simp add: ‹0 ≤ B› g_def has_integral_bound)
qed

corollary has_integral_bound_real:
  fixes f :: real ⇒ 'b::real_normed_vector
  assumes 0 ≤ B finite S
      and (f has_integral i) {a..b}
      and ⋀x. x ∈ {a..b} − S ⟹ norm (f x) ≤ B
    shows norm i ≤ B ∗ content {a..b}
  by (metis assms box_real(2) has_integral_bound_spike_finite)
```

6.15.13 In particular, the boundary of an interval is negligible

```
lemma negligible_frontier_interval: negligible(cbox (a::'a::euclidean_space) b − box
a b)
proof −
  let ?A = ⋃((λk. {x. x•k = a•k} ∪ {x::'a. x•k = b•k}) ` Basis)
  have negligible ?A
    by (force simp add: negligible_Union[OF finite_imageI])
  moreover have cbox a b − box a b ⊆ ?A
    by (force simp add: mem_box)
  ultimately show ?thesis
    by (rule negligible_subset)
qed

lemma has_integral_spike_interior:
  assumes f: (f has_integral y) (cbox a b) and gf: ⋀x. x ∈ box a b ⟹ g x = f x
  shows (g has_integral y) (cbox a b)
  by (meson Diff_iff gf has_integral_spike[OF negligible_frontier_interval _ f])

lemma has_integral_spike_interior_eq:
  assumes ⋀x. x ∈ box a b ⟹ g x = f x
```

shows (*f has_integral y*) (*cbox a b*) ⟷ (*g has_integral y*) (*cbox a b*)
by (*metis assms has_integral_spike_interior*)

lemma *integrable_spike_interior*:
assumes ⋀*x*. *x* ∈ *box a b* ⟹ *g x* = *f x*
and *f integrable_on cbox a b*
shows *g integrable_on cbox a b*
using *assms has_integral_spike_interior_eq* **by** *blast*

6.15.14 Integrability of continuous functions

lemma *operative_approximableI*:
fixes *f* :: *'b::euclidean_space* ⇒ *'a::banach*
assumes *0* ≤ *e*
shows *operative conj True* (λ*i*. ∃ *g*. (∀ *x*∈*i*. *norm* (*f x* − *g* (*x*::*'b*)) ≤ *e*) ∧ *g integrable_on i*)
proof −
interpret *comm_monoid conj True*
by *standard auto*
show *?thesis*
proof (*standard*, *safe*)
fix *a b* :: *'b*
show ∃ *g*. (∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable_on cbox a b*
if *box a b* = {} **for** *a b*
using *assms that*
by (*metis content_eq_0_interior integrable_on_null interior_cbox norm_zero right_minus_eq*)
{
fix *c g* **and** *k* :: *'b*
assume *fg*: ∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e* **and** *g*: *g integrable_on cbox a b*
assume *k*: *k* ∈ *Basis*
show ∃ *g*. (∀ *x*∈*cbox a b* ∩ {*x*. *x* · *k* ≤ *c*}. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable_on cbox a b* ∩ {*x*. *x* · *k* ≤ *c*}
∃ *g*. (∀ *x*∈*cbox a b* ∩ {*x*. *c* ≤ *x* · *k*}. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable_on cbox a b* ∩ {*x*. *c* ≤ *x* · *k*}
using *fg g k* **by** *auto*
}
show ∃ *g*. (∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable_on cbox a b*
if *fg1*: ∀ *x*∈*cbox a b* ∩ {*x*. *x* · *k* ≤ *c*}. *norm* (*f x* − *g1 x*) ≤ *e*
and *g1*: *g1 integrable_on cbox a b* ∩ {*x*. *x* · *k* ≤ *c*}
and *fg2*: ∀ *x*∈*cbox a b* ∩ {*x*. *c* ≤ *x* · *k*}. *norm* (*f x* − *g2 x*) ≤ *e*
and *g2*: *g2 integrable_on cbox a b* ∩ {*x*. *c* ≤ *x* · *k*}
and *k*: *k* ∈ *Basis*
for *c k g1 g2*
proof −
let *?g* = λ*x*. *if x·k* = *c then f x else if x·k* ≤ *c then g1 x else g2 x*
show ∃ *g*. (∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable_on cbox a b*
proof (*intro exI conjI ballI*)

show *norm* (*f x* − *?g x*) ≤ *e* **if** *x* ∈ *cbox a b* **for** *x*
by (*auto simp*: *that assms fg1 fg2*)
show *?g integrable_on cbox a b*
proof −
have *?g integrable_on cbox a b* ∩ {*x*. *x* · *k* ≤ *c*} *?g integrable_on cbox a b* ∩ {*x*. *x* · *k* ≥ *c*}
by(*rule integrable_spike*[*OF* _ *negligible_standard_hyperplane*[*of k c*]], *use k g1 g2* **in** *auto*)+
with *has_integral_split*[*OF* _ _ *k*] **show** *?thesis*
unfolding *integrable_on_def* **by** *blast*
qed
qed
qed
qed
qed

lemma *comm_monoid_set_F_and*: *comm_monoid_set.F* (∧) *True f s* ⟷ (*finite s* ⟶ (∀ *x*∈*s*. *f x*))
proof −
interpret *bool*: *comm_monoid_set* ‹(∧)› *True* **..**
show *?thesis*
by (*induction s rule*: *infinite_finite_induct*) *auto*
qed

lemma *approximable_on_division*:
fixes *f* :: *'b*::*euclidean_space* ⇒ *'a*::*banach*
assumes *0* ≤ *e*
and *d*: *d division_of* (*cbox a b*)
and *f*: ∀ *i*∈*d*. ∃ *g*. (∀ *x*∈*i*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable_on i*
obtains *g* **where** ∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e g integrable_on cbox a b*
proof −
interpret *operative conj True* λ*i*. ∃ *g*. (∀ *x*∈*i*. *norm* (*f x* − *g* (*x*::*'b*)) ≤ *e*) ∧ *g integrable_on i*
using ‹*0* ≤ *e*› **by** (*rule operative_approximableI*)
from *f local.division* [*OF d*] *that* **show** *thesis*
by *auto*
qed

lemma *integrable_continuous*:
fixes *f* :: *'b*::*euclidean_space* ⇒ *'a*::*banach*
assumes *continuous_on* (*cbox a b*) *f*
shows *f integrable_on cbox a b*
proof (*rule integrable_uniform_limit*)
fix *e* :: *real*
assume *e*: *e* > *0*
then obtain *d* **where** *0* < *d* **and** *d*: ⋀*x x'*. ⟦*x* ∈ *cbox a b*; *x'* ∈ *cbox a b*; *dist x' x* < *d*⟧ ⟹ *dist* (*f x'*) (*f x*) < *e*
using *compact_uniformly_continuous*[*OF assms compact_cbox*] **unfolding** *uniformly_continuous_on_def* **by** *metis*

obtain *p* **where** *ptag*: *p tagged_division_of cbox a b* **and** *finep*: (*λx. ball x d*) *fine p*
  using *fine_division_exists*[*OF gauge_ball*[*OF* ‹*0* < *d*›], *of a b*] **.**
have ∗: ∀ *i*∈*snd* ‘ *p*. ∃ *g*. (∀ *x*∈*i*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable_on i*
proof (*safe*, *unfold snd_conv*)
  fix *x l*
  assume *as*: (*x*, *l*) ∈ *p*
  obtain *a b* **where** *l*: *l* = *cbox a b*
    using *as ptag* **by** *blast*
  then have *x*: *x* ∈ *cbox a b*
    using *as ptag* **by** *auto*
  show ∃ *g*. (∀ *x*∈*l*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable_on l*
  proof (*intro exI conjI strip*)
    show (*λy*. *f x*) *integrable_on l*
      unfolding *integrable_on_def l* **by** *blast*
  next
    fix *y*
    assume *y*: *y* ∈ *l*
    then have *y* ∈ *ball x d*
      using *as finep* **by** *fastforce*
    then show *norm* (*f y* − *f x*) ≤ *e*
      using *d x y as l*
        by (*metis dist_commute dist_norm less_imp_le mem_ball ptag subsetCE tagged_division_ofD*(*3*))
  qed
qed
from *e* **have** *e* ≥ *0*
  by *auto*
from *approximable_on_division*[*OF this division_of_tagged_division*[*OF ptag*] ∗]
show ∃ *g*. (∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable_on cbox a b*
  by *metis*
qed

lemma *integrable_continuous_interval*:
  fixes *f* :: ′*b*::*ordered_euclidean_space* ⇒ ′*a*::*banach*
  assumes *continuous_on* {*a*..*b*} *f*
  shows *f integrable_on* {*a*..*b*}
  by (*metis assms integrable_continuous interval_cbox*)

lemmas *integrable_continuous_real* = *integrable_continuous_interval*[**where** ′*b*=*real*]

lemma *integrable_continuous_closed_segment*:
  fixes *f* :: *real* ⇒ ′*a*::*banach*
  assumes *continuous_on* (*closed_segment a b*) *f*
  shows *f integrable_on* (*closed_segment a b*)
  using *assms*
  by (*auto intro*!: *integrable_continuous_interval simp*: *closed_segment_eq_real_ivl*)

6.15.15 Specialization of additivity to one dimension

6.15.16 A useful lemma allowing us to factor out the content size

```
lemma has_integral_factor_content:
  (f has_integral i) (cbox a b) ⟷
    (∀ e>0. ∃ d. gauge d ∧ (∀ p. p tagged_division_of (cbox a b) ∧ d fine p ⟶
      norm (sum (λ(x,k). content k *R f x) p − i) ≤ e ∗ content (cbox a b)))
proof (cases content (cbox a b) = 0)
  case True
  have ⋀e p. p tagged_division_of cbox a b ⟹ norm ((∑ (x, k)∈p. content k *R
f x)) ≤ e ∗ content (cbox a b)
    unfolding sum_content_null[OF True] True by force
  moreover have i = 0
    if ⋀e. e > 0 ⟹ ∃ d. gauge d ∧
              (∀ p. p tagged_division_of cbox a b ∧
                   d fine p ⟶
                   norm ((∑ (x, k)∈p. content k *R f x) − i) ≤ e ∗ content (cbox a
b))
    using that [of 1]
   by (force simp add: True sum_content_null[OF True] intro: fine_division_exists[of
_ a b])
  ultimately show ?thesis
    unfolding has_integral_null_eq[OF True]
    by (force simp add: )
next
  case False
  then have F: 0 < content (cbox a b)
    using zero_less_measure_iff by blast
  let ?P = λe opp. ∃ d. gauge d ∧
    (∀ p. p tagged_division_of (cbox a b) ∧ d fine p ⟶ opp (norm ((∑ (x, k)∈p.
content k *R f x) − i)) e)
  show ?thesis
  proof (subst has_integral, safe)
    fix e :: real
    assume e: e > 0
    show ?P (e ∗ content (cbox a b)) (≤) if §[rule_format]: ∀ ε>0. ?P ε (<)
      using § [of e ∗ content (cbox a b)]
      by (meson F e less_imp_le mult_pos_pos)
    show ?P e (<) if §[rule_format]:  ∀ ε>0. ?P (ε ∗ content (cbox a b)) (≤)
      using § [of e/2 / content (cbox a b)]
        using F e by (force simp add: algebra_simps)
  qed
qed

lemma has_integral_factor_content_real:
  (f has_integral i) {a..b::real} ⟷
    (∀ e>0. ∃ d. gauge d ∧ (∀ p. p tagged_division_of {a..b}  ∧ d fine p ⟶
      norm (sum (λ(x,k). content k *R f x) p − i) ≤ e ∗ content {a..b} ))
```

unfolding *box_real*[*symmetric*]
by (*rule has_integral_factor_content*)

6.15.17 Fundamental theorem of calculus

lemma *interval_bounds_real*:
fixes *q b* :: *real*
assumes $a \leq b$
shows *Sup* {*a..b*} = *b*
and *Inf* {*a..b*} = *a*
using *assms* **by** *auto*

theorem *fundamental_theorem_of_calculus*:
fixes *f* :: *real* ⇒ *'a*::*banach*
assumes $a \leq b$
and *vecd*: ⋀*x*. *x* ∈ {*a..b*} ⟹ (*f has_vector_derivative f' x*) (*at x within* {*a..b*})
shows (*f' has_integral* (*f b* − *f a*)) {*a..b*}
unfolding *has_integral_factor_content box_real*[*symmetric*]
proof *safe*
fix *e* :: *real*
assume *e* > *0*
then have ∀ *x*. ∃ *d*>*0*. *x* ∈ {*a..b*} ⟶
(∀ *y*∈{*a..b*}. *norm* (*y*−*x*) < *d* ⟶ *norm* (*f y* − *f x* − (*y*−*x*) $*_R$ *f' x*) ≤ *e* ∗ *norm* (*y*−*x*))
using *vecd*[*unfolded has_vector_derivative_def has_derivative_within_alt*] **by** *blast*
then obtain *d* **where** *d*: ⋀*x*. *0* < *d x*
⋀*x y*. ⟦*x* ∈ {*a..b*}; *y* ∈ {*a..b*}; *norm* (*y*−*x*) < *d x*⟧
⟹ *norm* (*f y* − *f x* − (*y*−*x*) $*_R$ *f' x*) ≤ *e* ∗ *norm* (*y*−*x*)
by *metis*
show ∃ *d*. *gauge d* ∧ (∀ *p*. *p tagged_division_of* (*cbox a b*) ∧ *d fine p* ⟶
norm ((∑ (*x*, *k*)∈*p*. *content k* $*_R$ *f' x*) − (*f b* − *f a*)) ≤ *e* ∗ *content* (*cbox a b*))
proof (*rule exI*, *safe*)
show *gauge* (λ*x*. *ball x* (*d x*))
using *d*(*1*) *gauge_ball_dependent* **by** *blast*
next
fix *p*
assume *ptag*: *p tagged_division_of cbox a b* **and** *finep*: (λ*x*. *ball x* (*d x*)) *fine p*
have *ba*: *b* − *a* = (∑ (*x*,*K*)∈*p*. *Sup K* − *Inf K*) *f b* − *f a* = (∑ (*x*,*K*)∈*p*. *f*(*Sup K*) − *f*(*Inf K*))
using *additive_tagged_division_1*[**where** *f*= λ*x*. *x*] *additive_tagged_division_1*[**where** *f*= *f*]
‹*a* ≤ *b*› *ptag* **by** *auto*
have *norm* (∑ (*x*, *K*) ∈ *p*. (*content K* $*_R$ *f' x*) − (*f* (*Sup K*) − *f* (*Inf K*)))
≤ (∑ *n*∈*p*. *e* ∗ (*case n of* (*x*, *k*) ⇒ *Sup k* − *Inf k*))
proof (*rule sum_norm_le*,*safe*)
fix *x K*
assume (*x*, *K*) ∈ *p*
then have *x* ∈ *K* **and** *kab*: *K* ⊆ *cbox a b*

```
        using ptag by blast+
      then obtain u v where k: K = cbox u v and x ∈ K and kab: K ⊆ cbox a b
        using ptag ‹(x, K) ∈ p› by auto
      have u ≤ v
        using ‹x ∈ K› unfolding k by auto
      have ball: ∀y∈K. y ∈ ball x (d x)
        using finep ‹(x, K) ∈ p› by blast
      have u ∈ K v ∈ K
        by (simp_all add: ‹u ≤ v› k)
      have norm ((v − u) *R f' x − (f v − f u)) = norm (f u − f x − (u − x)
*R f' x − (f v − f x − (v − x) *R f' x))
        by (auto simp add: algebra_simps)
      also have ... ≤ norm (f u − f x − (u − x) *R f' x) + norm (f v − f x − (v
− x) *R f' x)
        by (rule norm_triangle_ineq4)
      finally have norm ((v − u) *R f' x − (f v − f u)) ≤
        norm (f u − f x − (u − x) *R f' x) + norm (f v − f x − (v − x) *R f' x) .
      also have . . . ≤ e * norm (u − x) + e * norm (v − x)
      proof (rule add_mono)
        show norm (f u − f x − (u − x) *R f' x) ≤ e * norm (u − x)
        proof (rule d)
          show norm (u − x) < d x
            using ‹u ∈ K› ball by (auto simp add: dist_real_def)
        qed (use ‹x ∈ K› ‹u ∈ K› kab in auto)
        show norm (f v − f x − (v − x) *R f' x) ≤ e * norm (v − x)
        proof (rule d)
          show norm (v − x) < d x
            using ‹v ∈ K› ball by (auto simp add: dist_real_def)
        qed (use ‹x ∈ K› ‹v ∈ K› kab in auto)
      qed
      also have . . . ≤ e * (Sup K − Inf K)
        using ‹x ∈ K› by (auto simp: k interval_bounds_real[OF ‹u ≤ v›] field_simps)
      finally show norm (content K *R f' x − (f (Sup K) − f (Inf K))) ≤ e *
(Sup K − Inf K)
        using ‹u ≤ v› by (simp add: k)
    qed
    with ‹a ≤ b› show norm ((∑(x, K)∈p. content K *R f' x) − (f b − f a)) ≤
e * content (cbox a b)
      by (auto simp: ba split_def sum_subtractf [symmetric] sum_distrib_left)
  qed
qed

lemma ident_has_integral:
  fixes a::real
  assumes a ≤ b
  shows ((λx. x) has_integral (b^2 − a^2)/2) {a..b}
proof −
  have ((λx. x) has_integral inverse 2 * b^2 − inverse 2 * a^2) {a..b}
    unfolding power2_eq_square
```

by (*rule fundamental_theorem_of_calculus* [*OF assms*] *derivative_eq_intros* | *simp*)+
then show *?thesis*
by (*simp add*: *field_simps*)
qed

lemma *integral_ident* [*simp*]:
fixes *a*::*real*
assumes $a \leq b$
shows *integral* {*a*..*b*} ($\lambda x.$ *x*) = (*if* $a \leq b$ *then* $(b^2 - a^2)/2$ *else 0*)
by (*metis assms ident_has_integral integral_unique*)

lemma *ident_integrable_on*:
fixes *a*::*real*
shows ($\lambda x.$ *x*) *integrable_on* {*a*..*b*}
by (*metis atLeastatMost_empty_iff integrable_on_def has_integral_empty ident_has_integral*)

lemma *integral_sin* [*simp*]:
fixes *a*::*real*
assumes $a \leq b$ **shows** *integral* {*a*..*b*} *sin* = *cos a* − *cos b*
proof −
have (*sin has_integral* (− *cos b* − − *cos a*)) {*a*..*b*}
proof (*rule fundamental_theorem_of_calculus*)
show (($\lambda a.$ − *cos a*) *has_vector_derivative sin x*) (*at x within* {*a*..*b*}) **for** *x*
unfolding *has_field_derivative_iff_has_vector_derivative* [*symmetric*]
by (*rule derivative_eq_intros* | *force*)+
qed (*use assms* **in** *auto*)
then show *?thesis*
by (*simp add*: *integral_unique*)
qed

lemma *integral_cos* [*simp*]:
fixes *a*::*real*
assumes $a \leq b$ **shows** *integral* {*a*..*b*} *cos* = *sin b* − *sin a*
proof −
have (*cos has_integral* (*sin b* − *sin a*)) {*a*..*b*}
proof (*rule fundamental_theorem_of_calculus*)
show (*sin has_vector_derivative cos x*) (*at x within* {*a*..*b*}) **for** *x*
unfolding *has_field_derivative_iff_has_vector_derivative* [*symmetric*]
by (*rule derivative_eq_intros* | *force*)+
qed (*use assms* **in** *auto*)
then show *?thesis*
by (*simp add*: *integral_unique*)
qed

lemma *has_integral_sin_nx*: (($\lambda x.$ *sin*(*real_of_int n* ∗ *x*)) *has_integral 0*) {−*pi*..*pi*}
proof (*cases n* = *0*)
case *False*
have (($\lambda x.$ *sin* (*n* ∗ *x*)) *has_integral* (− *cos* (*n* ∗ *pi*)/*n* − − *cos* (*n* ∗ − *pi*)/*n*))

{−*pi*..*pi*}
  proof (*rule fundamental_theorem_of_calculus*)
    show ((λ*x*. − *cos* (*n* ∗ *x*) / *n*) *has_vector_derivative sin* (*n* ∗ *a*)) (*at a within* {−*pi*..*pi*})
      if *a* ∈ {−*pi*..*pi*} **for** *a* :: *real*
      using *that False*
      unfolding *has_vector_derivative_def*
      by (*intro derivative_eq_intros* | *force*)+
  qed *auto*
  then show *?thesis*
    by *simp*
qed *auto*

lemma *integral_sin_nx*:
  integral {−*pi*..*pi*} (λ*x*. *sin*(*x* ∗ *real_of_int n*)) = *0*
  using *has_integral_sin_nx* [*of n*] **by** (*force simp*: *mult.commute*)

lemma *has_integral_cos_nx*:
  ((λ*x*. *cos*(*real_of_int n* ∗ *x*)) *has_integral* (*if n* = *0 then 2* ∗ *pi else 0*)) {−*pi*..*pi*}
proof (*cases n* = *0*)
  case *True*
  then show *?thesis*
    using *has_integral_const_real* [*of 1*::*real* −*pi pi*] **by** *auto*
next
  case *False*
  have ((λ*x*. *cos* (*n* ∗ *x*)) *has_integral* (*sin* (*n* ∗ *pi*)/*n* − *sin* (*n* ∗ − *pi*)/*n*)) {−*pi*..*pi*}
  proof (*rule fundamental_theorem_of_calculus*)
    show ((λ*x*. *sin* (*n* ∗ *x*) / *n*) *has_vector_derivative cos* (*n* ∗ *x*)) (*at x within* {−*pi*..*pi*})
      if *x* ∈ {−*pi*..*pi*}
      for *x* :: *real*
      using *that False*
      unfolding *has_vector_derivative_def*
      by (*intro derivative_eq_intros* | *force*)+
  qed *auto*
  with *False* **show** *?thesis*
    by (*simp add*: *mult.commute*)
qed

lemma *integral_cos_nx*:
  integral {−*pi*..*pi*} (λ*x*. *cos*(*x* ∗ *real_of_int n*)) = (*if n* = *0 then 2* ∗ *pi else 0*)
  using *has_integral_cos_nx* [*of n*] **by** (*force simp*: *mult.commute*)

6.15.18 Taylor series expansion

lemma *mvt_integral*:
  fixes *f*::′*a*::*real_normed_vector*⇒′*b*::*banach*
  assumes *f* ′[*derivative_intros*]:

$\bigwedge x.\ x \in S \Longrightarrow (f\ has\_derivative\ f'\ x)\ (at\ x\ within\ S)$
assumes *line_in*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow x + t *_R y \in S$
shows $f\ (x + y) - f\ x = integral\ \{0..1\}\ (\lambda t.\ f'\ (x + t *_R y)\ y)$ (**is** *?th1*)
proof −
from *assms* **have** *subset*: $(\lambda xa.\ x + xa *_R y)$ ‘ $\{0..1\} \subseteq S$ **by** *auto*
note [*derivative_intros*] =
has_derivative_subset[*OF _ subset*]
has_derivative_in_compose[**where** $f=(\lambda xa.\ x + xa *_R y)$ **and** $g = f$]
note [*continuous_intros*] =
continuous_on_compose2[**where** $f=(\lambda xa.\ x + xa *_R y)$]
continuous_on_subset[*OF _ subset*]
have $\bigwedge t.\ t \in \{0..1\} \Longrightarrow$
$((\lambda t.\ f\ (x + t *_R y))\ has\_vector\_derivative\ f'\ (x + t *_R y)\ y)$
$(at\ t\ within\ \{0..1\})$
using *assms*
by (*auto simp*: *has_vector_derivative_def*
linear_cmul[*OF has_derivative_linear*[*OF f'*], *symmetric*]
intro!: *derivative_eq_intros*)
from *fundamental_theorem_of_calculus*[*rule_format*, *OF _ this*]
show *?th1*
by (*auto intro*!: *integral_unique*[*symmetric*])
qed

lemma (**in** *bounded_bilinear*) *sum_prod_derivatives_has_vector_derivative*:
assumes $p>0$
and *f0*: $Df\ 0 = f$
and *Df*: $\bigwedge m\ t.\ m < p \Longrightarrow a \le t \Longrightarrow t \le b \Longrightarrow$
$(Df\ m\ has\_vector\_derivative\ Df\ (Suc\ m)\ t)\ (at\ t\ within\ \{a..b\})$
and *g0*: $Dg\ 0 = g$
and *Dg*: $\bigwedge m\ t.\ m < p \Longrightarrow a \le t \Longrightarrow t \le b \Longrightarrow$
$(Dg\ m\ has\_vector\_derivative\ Dg\ (Suc\ m)\ t)\ (at\ t\ within\ \{a..b\})$
and *ivl*: $a \le t\ t \le b$
shows $((\lambda t.\ \sum i<p.\ (-1)\hat{\ }i *_R prod\ (Df\ i\ t)\ (Dg\ (p - Suc\ i)\ t))$
has_vector_derivative
$prod\ (f\ t)\ (Dg\ p\ t) - (-1)\hat{\ }p *_R prod\ (Df\ p\ t)\ (g\ t))$
$(at\ t\ within\ \{a..b\})$
using *assms*
proof *cases*
assume *p*: $p \neq 1$
define p' **where** $p' = p - 2$
from *assms p* **have** p': $\{..<p\} = \{..Suc\ p'\}\ p = Suc\ (Suc\ p')$
by (*auto simp*: *p'_def*)
have $*$: $\bigwedge i.\ i \le p' \Longrightarrow Suc\ (Suc\ p' - i) = (Suc\ (Suc\ p') - i)$
by *auto*
let $?f = \lambda i.\ (-1)\ \hat{\ }\ i *_R (prod\ (Df\ i\ t)\ (Dg\ ((p - i))\ t))$
have $(\sum i<p.\ (-1)\ \hat{\ }\ i *_R (prod\ (Df\ i\ t)\ (Dg\ (Suc\ (p - Suc\ i))\ t) +$
$prod\ (Df\ (Suc\ i)\ t)\ (Dg\ (p - Suc\ i)\ t))) =$
$(\sum i \le (Suc\ p').\ ?f\ i - ?f\ (Suc\ i))$
by (*auto simp*: *algebra_simps* $p'(2)$ *numeral_2_eq_2* $*$ *lessThan_Suc_atMost*)

```
  also note sum_telescope
  finally
  have (∑ i<p. (−1) ^ i *R (prod (Df i t) (Dg (Suc (p − Suc i)) t) +
    prod (Df (Suc i) t) (Dg (p − Suc i) t)))
    = prod (f t) (Dg p t) − (− 1) ^ p *R prod (Df p t) (g t)
    unfolding p'[symmetric]
    by (simp add: assms)
  thus ?thesis
    using assms
    by (auto intro!: derivative_eq_intros has_vector_derivative)
qed (auto intro!: derivative_eq_intros has_vector_derivative)

lemma
  fixes f::real⇒'a::banach
  assumes p>0
  and f0: Df 0 = f
  and Df: ⋀m t. m < p ⟹ a ≤ t ⟹ t ≤ b ⟹
    (Df m has_vector_derivative Df (Suc m) t) (at t within {a..b})
  and ivl: a ≤ b
  defines i ≡ λx. ((b − x) ^ (p − 1) / fact (p − 1)) *R Df p x
  shows Taylor_has_integral:
    (i has_integral f b − (∑ i<p. ((b−a) ^ i / fact i) *R Df i a)) {a..b}
  and Taylor_integral:
    f b = (∑ i<p. ((b−a) ^ i / fact i) *R Df i a) + integral {a..b} i
  and Taylor_integrable:
    i integrable_on {a..b}
proof goal_cases
  case 1
  interpret bounded_bilinear scaleR::real⇒'a⇒'a
    by (rule bounded_bilinear_scaleR)
  define g where g s = (b − s)^(p − 1)/fact (p − 1) for s
  define Dg where [abs_def]:
    Dg n s = (if n < p then (−1)^n * (b − s)^(p − 1 − n) / fact (p − 1 − n)
else 0) for n s
  have g0: Dg 0 = g
    using ‹p > 0›
    by (auto simp add: Dg_def field_split_simps g_def split: if_split_asm)
  {
    fix m
    assume p > Suc m
    hence p − Suc m = Suc (p − Suc (Suc m))
      by auto
    hence real (p − Suc m) * fact (p − Suc (Suc m)) = fact (p − Suc m)
      by auto
  } note fact_eq = this
  have Dg: ⋀m t. m < p ⟹ a ≤ t ⟹ t ≤ b ⟹
    (Dg m has_vector_derivative Dg (Suc m) t) (at t within {a..b})
    unfolding Dg_def
   by (auto intro!: derivative_eq_intros simp: has_vector_derivative_def fact_eq field_split_simps)
```

let *?sum* = λ*t*. ∑ *i*<*p*. (− *1*) ˆ *i* $*_R$ *Dg i t* $*_R$ *Df* (*p* − *Suc i*) *t*
from *sum_prod_derivatives_has_vector_derivative*[*of* _ *Dg* _ _ _ *Df*,
OF ‹*p* > *0*› *g0 Dg f0 Df*]
have *deriv*: ⋀*t*. *a* ≤ *t* ⟹ *t* ≤ *b* ⟹
(*?sum has_vector_derivative*
g t $*_R$ *Df p t* − (− *1*) ˆ *p* $*_R$ *Dg p t* $*_R$ *f t*) (*at t within* {*a*..*b*})
by *auto*
from *fundamental_theorem_of_calculus*[*rule_format*, *OF* ‹*a* ≤ *b*› *deriv*]
have (*i has_integral ?sum b* − *?sum a*) {*a*..*b*}
using *atLeastatMost_empty*′[*simp del*]
by (*simp add*: *i_def g_def Dg_def*)
also
have *one*: (− *1*) ˆ *p*′ ∗ (− *1*) ˆ *p*′ = (*1*::*real*)
and {..<*p*} ∩ {*i*. *p* = *Suc i*} = {*p* − *1*}
for *p*′
using ‹*p* > *0*›
by (*auto simp*: *power_mult_distrib*[*symmetric*])
then have *?sum b* = *f b*
using *Suc_pred*′[*OF* ‹*p* > *0*›]
by (*simp add*: *diff_eq_eq Dg_def power_0_left le_Suc_eq if_distrib*
if_distribR sum.If_cases f0)
also
have {..<*p*} = (λ*x*. *p* − *x* − *1*) ‘ {..<*p*}
proof *safe*
fix *x*
assume *x* < *p*
thus *x* ∈ (λ*x*. *p* − *x* − *1*) ‘ {..<*p*}
by (*auto intro*!: *image_eqI*[**where** *x* = *p* − *x* − *1*])
qed *simp*
from _ *this*
have *?sum a* = (∑ *i*<*p*. ((*b*−*a*) ˆ *i* / *fact i*) $*_R$ *Df i a*)
by (*rule sum.reindex_cong*) (*auto simp add*: *inj_on_def Dg_def one*)
finally show *c*: *?case* **.**
case *2* **show** *?case* **using** *c integral_unique*
by (*metis* (*lifting*) *add.commute diff_eq_eq integral_unique*)
case *3* **show** *?case* **using** *c* **by** *force*
qed

6.15.19 Only need trivial subintervals if the interval itself is trivial

proposition *division_of_nontrivial*:
fixes 𝒟 :: ′*a*::*euclidean_space set set*
assumes *sdiv*: 𝒟 *division_of* (*cbox a b*)
and *cont0*: *content* (*cbox a b*) ≠ *0*
shows {*k*. *k* ∈ 𝒟 ∧ *content k* ≠ *0*} *division_of* (*cbox a b*)
using *sdiv*
proof (*induction card* 𝒟 *arbitrary*: 𝒟 *rule*: *less_induct*)
case *less*

```
  note 𝒟 = division_ofD[OF less.prems]
  {
    presume *: {k ∈ 𝒟. content k ≠ 0} ≠ 𝒟 ⟹ ?case
    then show ?case
      using less.prems by fastforce
  }
  assume noteq: {k ∈ 𝒟. content k ≠ 0} ≠ 𝒟
  then obtain K c d where K ∈ 𝒟 and contk: content K = 0 and keq: K =
cbox c d
    using 𝒟(4) by blast
  then have card 𝒟 > 0
    unfolding card_gt_0_iff using less by auto
  then have card: card (𝒟 − {K}) < card 𝒟
    using less ‹K ∈ 𝒟› by (simp add: 𝒟(1))
  have closed: closed (⋃(𝒟 − {K}))
    using less.prems by auto
  have x islimpt ⋃(𝒟 − {K}) if x ∈ K for x
    unfolding islimpt_approachable
  proof (intro allI impI)
    fix e::real
    assume e > 0
    obtain i where i: c·i = d·i i∈Basis
      using contk 𝒟(3) [OF ‹K ∈ 𝒟›] unfolding box_ne_empty keq
      by (meson content_eq_0 dual_order.antisym)
    then have xi: x·i = d·i
      using ‹x ∈ K› unfolding keq mem_box by (metis antisym)
   define y where y = (∑j∈Basis. (if j = i then if c·i ≤ (a·i + b·i)/2 then c·i
+
      min e (b·i − c·i)/2 else c·i − min e (c·i − a·i)/2 else x·j) *R j)
    show ∃x'∈⋃(𝒟 − {K}). x' ≠ x ∧ dist x' x < e
    proof (intro bexI conjI)
      have d ∈ cbox c d
        using 𝒟(3)[OF ‹K ∈ 𝒟›] by (simp add: box_ne_empty(1) keq mem_box(2))
      then have d ∈ cbox a b
        using 𝒟(2)[OF ‹K ∈ 𝒟›] by (auto simp: keq)
      then have di: a · i ≤ d · i ∧ d · i ≤ b · i
        using ‹i ∈ Basis› mem_box(2) by blast
      then have xyi: y·i ≠ x·i
        unfolding y_def i xi using ‹e > 0› cont0 ‹i ∈ Basis›
        by (auto simp: content_eq_0 elim!: ballE[of _ _ i])
      then show y ≠ x
        unfolding euclidean_eq_iff [where 'a='a] using i by auto
      have norm (y−x) ≤ (∑b∈Basis. |(y − x) · b|)
        by (rule norm_le_l1)
      also have ... = |(y − x) · i| + (∑b ∈ Basis − {i}. |(y − x) · b|)
        by (meson finite_Basis i(2) sum.remove)
      also have ... <  e + sum (λi. 0) Basis
      proof (rule add_less_le_mono)
        show |(y−x) · i| < e
```

using *di* ‹*e* > *0*› *y_def i xi* **by** (*auto simp*: *inner_simps*)
show ($\sum$ *i*∈*Basis* − {*i*}. |(*y*−*x*) · *i*|) ≤ ($\sum$ *i*∈*Basis*. *0*)
unfolding *y_def* **by** (*auto simp*: *inner_simps*)
qed
finally have *norm* (*y*−*x*) < *e* + *sum* (λ*i*. *0*) *Basis* **.**
then show *dist y x* < *e*
unfolding *dist_norm* **by** *auto*
have *y* ∉ *K*
unfolding *keq mem_box* **using** *i*(*1*) *i*(*2*) *xi xyi* **by** *fastforce*
moreover have *y* ∈ ⋃𝒟
using *subsetD*[*OF* 𝒟(*2*)[*OF* ‹*K* ∈ 𝒟›] ‹*x* ∈ *K*›] ‹*e* > *0*› *di i*
by (*auto simp*: 𝒟 *mem_box y_def field_simps elim*!: *ballE*[*of* _ _ *i*])
ultimately show *y* ∈ ⋃(𝒟 − {*K*}) **by** *auto*
qed
qed
then have *K* ⊆ ⋃(𝒟 − {*K*})
using *closed closed_limpt* **by** *blast*
then have ⋃(𝒟 − {*K*}) = *cbox a b*
unfolding 𝒟(*6*)[*symmetric*] **by** *auto*
then have 𝒟 − {*K*} *division_of cbox a b*
by (*metis Diff_subset less.prems division_of_subset* 𝒟(*6*))
then have {*ka* ∈ 𝒟 − {*K*}. *content ka* ≠ *0*} *division_of* (*cbox a b*)
using *card less.hyps* **by** *blast*
moreover have {*ka* ∈ 𝒟 − {*K*}. *content ka* ≠ *0*} = {*K* ∈ 𝒟. *content K* ≠ *0*}
using *contk* **by** *auto*
ultimately show *?case* **by** *auto*
qed

6.15.20 Integrability on subintervals

lemma *operative_integrableI*:
fixes *f* :: *'b*::*euclidean_space* ⇒ *'a*::*banach*
assumes *0* ≤ *e*
shows *operative conj True* (λ*i*. *f integrable_on i*)
proof −
interpret *comm_monoid conj True*
proof qed
show *?thesis*
proof
show ⋀*a b*. *box a b* = {} ⟹ (*f integrable_on cbox a b*) = *True*
by (*simp add*: *content_eq_0_interior integrable_on_null*)
show ⋀*a b c k*.
k ∈ *Basis* ⟹
(*f integrable_on cbox a b*) ⟷
(*f integrable_on cbox a b* ∩ {*x*. *x* · *k* ≤ *c*} ∧ *f integrable_on cbox a b* ∩ {*x*. *c* ≤ *x* · *k*})
unfolding *integrable_on_def* **by** (*auto intro*!: *has_integral_split*)
qed
qed

```
lemma integrable_subinterval:
  fixes f :: 'b::euclidean_space ⇒ 'a::banach
  assumes f: f integrable_on cbox a b
    and cd: cbox c d ⊆ cbox a b
  shows f integrable_on cbox c d
proof -
  interpret operative conj True λi. f integrable_on i
    using order_refl by (rule operative_integrableI)
  show ?thesis
  proof (cases cbox c d = {})
    case True
    then show ?thesis
      using division [symmetric] f by (auto simp: comm_monoid_set_F_and)
  next
    case False
    then show ?thesis
     by (metis cd comm_monoid_set_F_and division division_of_finite f partial_division_extend_1)
  qed
qed

lemma integrable_subinterval_real:
  fixes f :: real ⇒ 'a::banach
  assumes f integrable_on {a..b}
    and {c..d} ⊆ {a..b}
  shows f integrable_on {c..d}
  by (metis assms box_real(2) integrable_subinterval)
```

6.15.21 Combining adjacent intervals in 1 dimension

```
lemma has_integral_combine:
  fixes a b c :: real and j :: 'a::banach
  assumes a ≤ c
      and c ≤ b
      and ac: (f has_integral i) {a..c}
      and cb: (f has_integral j) {c..b}
  shows (f has_integral (i + j)) {a..b}
proof -
  interpret operative_real lift_option plus Some 0
    λi. if f integrable_on i then Some (integral i f) else None
    using operative_integralI by (rule operative_realI)
  from ‹a ≤ c› ‹c ≤ b› ac cb coalesce_less_eq
  have *: lift_option (+)
             (if f integrable_on {a..c} then Some (integral {a..c} f) else None)
             (if f integrable_on {c..b} then Some (integral {c..b} f) else None) =
            (if f integrable_on {a..b} then Some (integral {a..b} f) else None)
    by (auto simp: split: if_split_asm)
  then have f integrable_on cbox a b
    using ac cb by (auto split: if_split_asm)
```

with ∗
show *?thesis*
using *ac cb* **by** (*auto simp add*: *integrable_on_def integral_unique split*: *if_split_asm*)
qed

lemma *integral_combine*:
fixes *f* :: *real* ⇒ *'a*::*banach*
assumes *a* ≤ *c*
and *c* ≤ *b*
and *ab*: *f integrable_on* {*a*..*b*}
shows *integral* {*a*..*c*} *f* + *integral* {*c*..*b*} *f* = *integral* {*a*..*b*} *f*
proof −
have (*f has_integral integral* {*a*..*c*} *f*) {*a*..*c*}
using *ab* ‹*c* ≤ *b*› *integrable_subinterval_real* **by** *fastforce*
moreover
have (*f has_integral integral* {*c*..*b*} *f*) {*c*..*b*}
using *ab* ‹*a* ≤ *c*› *integrable_subinterval_real* **by** *fastforce*
ultimately have (*f has_integral integral* {*a*..*c*} *f* + *integral* {*c*..*b*} *f*) {*a*..*b*}
using ‹*a* ≤ *c*› ‹*c* ≤ *b*› *has_integral_combine* **by** *blast*
then show *?thesis*
by (*simp add*: *has_integral_integrable_integral*)
qed

lemma *integrable_combine*:
fixes *f* :: *real* ⇒ *'a*::*banach*
assumes *a* ≤ *c*
and *c* ≤ *b*
and *f integrable_on* {*a*..*c*}
and *f integrable_on* {*c*..*b*}
shows *f integrable_on* {*a*..*b*}
using *assms*
unfolding *integrable_on_def*
by (*auto intro*!: *has_integral_combine*)

lemma *integral_minus_sets*:
fixes *f*::*real* ⇒ *'a*::*banach*
shows *c* ≤ *a* ⟹ *c* ≤ *b* ⟹ *f integrable_on* {*c* .. *max a b*} ⟹
integral {*c* .. *a*} *f* − *integral* {*c* .. *b*} *f* =
(*if a* ≤ *b then* − *integral* {*a* .. *b*} *f else integral* {*b* .. *a*} *f*)
using *integral_combine*[*of c a b f*] *integral_combine*[*of c b a f*]
by (*auto simp*: *algebra_simps max_def*)

lemma *integral_minus_sets'*:
fixes *f*::*real* ⇒ *'a*::*banach*
shows *c* ≥ *a* ⟹ *c* ≥ *b* ⟹ *f integrable_on* {*min a b* .. *c*} ⟹
integral {*a* .. *c*} *f* − *integral* {*b* .. *c*} *f* =
(*if a* ≤ *b then integral* {*a* .. *b*} *f else* − *integral* {*b* .. *a*} *f*)
using *integral_combine*[*of b a c f*] *integral_combine*[*of a b c f*]
by (*auto simp*: *algebra_simps min_def*)

6.15.22 Reduce integrability to "local" integrability

lemma *integrable_on_little_subintervals*:
fixes f :: $'b$::*euclidean_space* ⇒ $'a$::*banach*
assumes ∀ *x*∈*cbox a b*. ∃ *d*>*0*. ∀ *u v*. *x* ∈ *cbox u v* ∧ *cbox u v* ⊆ *ball x d* ∧ *cbox u v* ⊆ *cbox a b* ⟶
f integrable_on cbox u v
shows *f integrable_on cbox a b*
proof –
interpret *operative conj True* λ*i*. *f integrable_on i*
using *order_refl* **by** (*rule operative_integrableI*)
have ∀ *x*. ∃ *d*>*0*. *x*∈*cbox a b* ⟶ (∀ *u v*. *x* ∈ *cbox u v* ∧ *cbox u v* ⊆ *ball x d* ∧ *cbox u v* ⊆ *cbox a b* ⟶
f integrable_on cbox u v)
using *assms* **by** (*metis zero_less_one*)
then obtain *d* **where** *d*: ⋀*x*. *0* < *d x*
⋀*x u v*. ⟦*x* ∈ *cbox a b*; *x* ∈ *cbox u v*; *cbox u v* ⊆ *ball x* (*d x*); *cbox u v* ⊆ *cbox a b*⟧
⟹ *f integrable_on cbox u v*
by *metis*
obtain *p* **where** *p*: *p tagged_division_of cbox a b* (λ*x*. *ball x* (*d x*)) *fine p*
using *fine_division_exists*[*OF gauge_ball_dependent*,*of d a b*] *d*(*1*) **by** *blast*
then have *sndp*: *snd* ‘ *p division_of cbox a b*
by (*metis division_of_tagged_division*)
have *f integrable_on k* **if** (*x*, *k*) ∈ *p* **for** *x k*
using *tagged_division_ofD*(*2*–*4*)[*OF p*(*1*) *that*] *fineD*[*OF p*(*2*) *that*] *d*[*of x*] **by** *auto*
then show *?thesis*
unfolding *division* [*symmetric*, *OF sndp*] *comm_monoid_set_F_and*
by *auto*
qed

6.15.23 Second FTC or existence of antiderivative

lemma *integrable_const*[*intro*]: (λ*x*. *c*) *integrable_on cbox a b*
unfolding *integrable_on_def* **by** *blast*

lemma *integral_has_vector_derivative_continuous_at*:
fixes f :: *real* ⇒ $'a$::*banach*
assumes *f*: *f integrable_on* {*a*..*b*}
and *x*: *x* ∈ {*a*..*b*} – *S*
and *finite S*
and *fx*: *continuous* (*at x within* ({*a*..*b*} – *S*)) *f*
shows ((λ*u*. *integral* {*a*..*u*} *f*) *has_vector_derivative f x*) (*at x within* ({*a*..*b*} – *S*))
proof –
let *?I* = λ*a b*. *integral* {*a*..*b*} *f*
{ **fix** *e*::*real*
assume *e* > *0*
obtain *d* **where** *d*>*0* **and** *d*: ⋀*x'*. ⟦*x'* ∈ {*a*..*b*} – *S*; |*x'* – *x*| < *d*⟧ ⟹ *norm*(*f*

x′ − *f x*) ≤ *e*
using ‹*e*>*0*› *fx* **by** (*auto simp*: *continuous_within_eps_delta dist_norm less_imp_le*)
have *norm* (*integral* {*a*..*y*} *f* − *integral* {*a*..*x*} *f* − (*y*−*x*) ∗*R* *f x*) ≤ *e* ∗ |*y* − *x*| (**is** *?lhs* ≤ *?rhs*)
if *y*: *y* ∈ {*a*..*b*} − *S* **and** *yx*: |*y* − *x*| < *d* **for** *y*
proof (*cases y* < *x*)
case *False*
have *f integrable_on* {*a*..*y*}
using *f y* **by** (*simp add*: *integrable_subinterval_real*)
then have *Idiff*: *?I a y* − *?I a x* = *?I x y*
using *False x* **by** (*simp add*: *algebra_simps integral_combine*)
have *fux_int*: ((λ*u*. *f u* − *f x*) *has_integral integral* {*x*..*y*} *f* − (*y*−*x*) ∗*R* *f x*) {*x*..*y*}
proof (*rule has_integral_diff*)
show (*f has_integral integral* {*x*..*y*} *f*) {*x*..*y*}
using *x y* **by** (*auto intro*: *integrable_integral* [*OF integrable_subinterval_real* [*OF f*]])
show ((λ*u*. *f x*) *has_integral* (*y* − *x*) ∗*R* *f x*) {*x*..*y*}
using *has_integral_const_real* [*of f x x y*] *False* **by** *simp*
qed
have *?lhs* ≤ *e* ∗ *content* {*x*..*y*}
using *yx False d x y* ‹*e*>*0*› *assms*
by (*intro has_integral_bound_real*[**where** *f*=(λ*u*. *f u* − *f x*)]) (*auto simp*: *Idiff fux_int*)
also have ... ≤ *?rhs*
using *False* **by** *auto*
finally show *?thesis* **.**
next
case *True*
have *f integrable_on* {*a*..*x*}
using *f x* **by** (*simp add*: *integrable_subinterval_real*)
then have *Idiff*: *?I a x* − *?I a y* = *?I y x*
using *True x y* **by** (*simp add*: *algebra_simps integral_combine*)
have *fux_int*: ((λ*u*. *f u* − *f x*) *has_integral integral* {*y*..*x*} *f* − (*x* − *y*) ∗*R* *f x*) {*y*..*x*}
proof (*rule has_integral_diff*)
show (*f has_integral integral* {*y*..*x*} *f*) {*y*..*x*}
using *x y* **by** (*auto intro*: *integrable_integral* [*OF integrable_subinterval_real* [*OF f*]])
show ((λ*u*. *f x*) *has_integral* (*x* − *y*) ∗*R* *f x*) {*y*..*x*}
using *has_integral_const_real* [*of f x y x*] *True* **by** *simp*
qed
have *norm* (*integral* {*a*..*x*} *f* − *integral* {*a*..*y*} *f* − (*x* − *y*) ∗*R* *f x*) ≤ *e* ∗ *content* {*y*..*x*}
using *yx True d x y* ‹*e*>*0*› *assms*
by (*intro has_integral_bound_real*[**where** *f*=(λ*u*. *f u* − *f x*)]) (*auto simp*: *Idiff fux_int*)
also have ... ≤ *e* ∗ |*y* − *x*|
using *True* **by** *auto*

finally have *norm* (*integral* {*a..x*} *f* − *integral* {*a..y*} *f* − (*x* − *y*) $*_R$ *f x*) ≤ *e* ∗ |*y* − *x*| **.**
then show *?thesis*
by (*simp add*: *algebra_simps norm_minus_commute*)
qed
then have ∃ *d*>*0*. ∀ *y*∈{*a..b*} − *S*. |*y* − *x*| < *d* ⟶ *norm* (*integral* {*a..y*} *f* − *integral* {*a..x*} *f* − (*y*−*x*) $*_R$ *f x*) ≤ *e* ∗ |*y* − *x*|
using ‹*d*>*0*› **by** *blast*
}
then show *?thesis*
by (*simp add*: *has_vector_derivative_def has_derivative_within_alt bounded_linear_scaleR_left*)
qed

lemma *integral_has_vector_derivative*:
fixes *f* :: *real* ⇒ ′*a*::*banach*
assumes *continuous_on* {*a..b*} *f*
and *x* ∈ {*a..b*}
shows ((λ*u*. *integral* {*a..u*} *f*) *has_vector_derivative f*(*x*)) (*at x within* {*a..b*})
using *assms integral_has_vector_derivative_continuous_at* [*OF integrable_continuous_real*]
by (*fastforce simp*: *continuous_on_eq_continuous_within*)

lemma *integral_has_real_derivative*:
assumes *continuous_on* {*a..b*} *g*
assumes *t* ∈ {*a..b*}
shows ((λ*x*. *integral* {*a..x*} *g*) *has_real_derivative g t*) (*at t within* {*a..b*})
using *integral_has_vector_derivative*[*of a b g t*] *assms*
by (*auto simp*: *has_field_derivative_iff_has_vector_derivative*)

lemma *antiderivative_continuous*:
fixes *q b* :: *real*
assumes *continuous_on* {*a..b*} *f*
obtains *g* **where** ⋀*x*. *x* ∈ {*a..b*} ⟹ (*g has_vector_derivative* (*f x*::_::*banach*)) (*at x within* {*a..b*})
using *integral_has_vector_derivative*[*OF assms*] **by** *auto*

6.15.24 Combined fundamental theorem of calculus

lemma *antiderivative_integral_continuous*:
fixes *f* :: *real* ⇒ ′*a*::*banach*
assumes *continuous_on* {*a..b*} *f*
obtains *g* **where** ∀ *u*∈{*a..b*}. ∀ *v* ∈ {*a..b*}. *u* ≤ *v* ⟶ (*f has_integral* (*g v* − *g u*)) {*u..v*}
proof −
obtain *g*
where *g*: ⋀*x*. *x* ∈ {*a..b*} ⟹ (*g has_vector_derivative f x*) (*at x within* {*a..b*})
using *antiderivative_continuous*[*OF assms*] **by** *metis*
have (*f has_integral g v* − *g u*) {*u..v*} **if** *u* ∈ {*a..b*} *v* ∈ {*a..b*} *u* ≤ *v* **for** *u v*
proof −

```
    have ⋀x. x ∈ cbox u v ⟹ (g has_vector_derivative f x) (at x within cbox u v)
      by (metis atLeastAtMost_iff atLeastatMost_subset_iff box_real(2) g
          has_vector_derivative_within_subset subsetCE that(1,2))
    then show ?thesis
      by (metis box_real(2) that(3) fundamental_theorem_of_calculus)
  qed
  then show ?thesis
    using that by blast
qed
```

6.15.25 General "twiddling" for interval-to-interval function image

```
lemma has_integral_twiddle:
  assumes 0 < r
    and hg: ⋀x. h(g x) = x
    and gh: ⋀x. g(h x) = x
    and contg: ⋀x. continuous (at x) g
    and g: ⋀u v. ∃w z. g ` cbox u v = cbox w z
    and h: ⋀u v. ∃w z. h ` cbox u v = cbox w z
    and r: ⋀u v. content(g ` cbox u v) = r * content (cbox u v)
    and intfi: (f has_integral i) (cbox a b)
  shows ((λx. f(g x)) has_integral (1 / r) *R i) (h ` cbox a b)
proof (cases cbox a b = {})
  case True
  then show ?thesis
    using intfi by auto
next
  case False
  obtain w z where wz: h ` cbox a b = cbox w z
    using h by blast
  have inj: inj g inj h
    using hg gh injI by metis+
  from h obtain ha hb where h_eq: h ` cbox a b = cbox ha hb by blast
  have ∃d. gauge d ∧ (∀p. p tagged_division_of h ` cbox a b ∧ d fine p
          ⟶ norm ((∑(x, k)∈p. content k *R f (g x)) − (1 / r) *R i) < e)
    if e > 0 for e
  proof −
    have e * r > 0 using that ‹0 < r› by simp
    with intfi[unfolded has_integral]
    obtain d where gauge d
          and d: ⋀p. p tagged_division_of cbox a b ∧ d fine p
                ⟹ norm ((∑(x, k)∈p. content k *R f x) − i) < e * r
      by metis
    define d' where d' x = g −` d (g x) for x
    show ?thesis
    proof (rule_tac x=d' in exI, safe)
      show gauge d'
      using ‹gauge d› continuous_open_vimage[OF _ contg] by (auto simp: gauge_def
```

```
d'_def)
  next
    fix p
    assume ptag: p tagged_division_of h ` cbox a b and finep: d' fine p
    note p = tagged_division_ofD[OF ptag]
    have gab: g y ∈ cbox a b if y ∈ K (x, K) ∈ p for x y K
      by (metis hg inj(2) inj_image_mem_iff p(3) subsetCE that that)
    have gimp: (λ(x,K). (g x, g ` K)) ` p tagged_division_of (cbox a b) ∧
            d fine (λ(x, k). (g x, g ` k)) ` p
      unfolding tagged_division_of
    proof safe
      show finite ((λ(x, k). (g x, g ` k)) ` p)
        using ptag by auto
      show d fine (λ(x, k). (g x, g ` k)) ` p
        using finep unfolding fine_def d'_def by auto
    next
      fix x k
      assume xk: (x, k) ∈ p
      show g x ∈ g ` k
        using p(2)[OF xk] by auto
      show ∃ u v. g ` k = cbox u v
        using p(4)[OF xk] using assms(5−6) by auto
      fix x' K' u
      assume xk': (x', K') ∈ p and u: u ∈ interior (g ` k) u ∈ interior (g ` K')
      have interior k ∩ interior K' ≠ {}
      proof
        assume interior k ∩ interior K' = {}
        moreover have u ∈ g ` (interior k ∩ interior K')
          using interior_image_subset[OF ‹inj g› contg] u
          unfolding image_Int[OF inj(1)] by blast
        ultimately show False by blast
      qed
      then have same: (x, k) = (x', K')
        using ptag xk' xk by blast
      then show g x = g x'
        by auto
      show g u ∈ g ` K'if u ∈ k for u
        using that same by auto
      show g u ∈ g ` kif u ∈ K' for u
        using that same by auto
    next
      fix x
      assume x ∈ cbox a b
      then have h x ∈ ⋃{k. ∃ x. (x, k) ∈ p}
        using p(6) by auto
      then obtain X y where h x ∈ X (y, X) ∈ p by blast
      then show x ∈ ⋃{k. ∃ x. (x, k) ∈ (λ(x, k). (g x, g ` k)) ` p}
        by clarsimp (metis (no_types, lifting) gh image_eqI pair_imageI)
    qed (use gab in auto)
```

```
      have *: inj_on (λ(x, k). (g x, g ` k)) p
        using inj(1) unfolding inj_on_def by fastforce
      have (∑(x,K)∈(λ(y,L). (g y, g ` L)) ` p. content K *R f x)
          = (∑u∈p. case case u of (x,K) ⇒ (g x, g ` K) of (y,L) ⇒ content L *R
f y)
        by (metis (mono_tags, lifting) * sum.reindex_cong)
      also have ... = (∑(x,K)∈p. r *R content K *R f (g x))
        using r by (auto intro!: * sum.cong simp: bij_betw_def dest!: p(4))
      finally
      have (∑(x, K)∈(λ(x,K). (g x, g ` K)) ` p. content K *R f x) − i = r *R
(∑(x,K)∈p. content K *R f (g x)) − i
        by (simp add: scaleR_right.sum split_def)
      also have ... = r *R ((∑(x,K)∈p. content K *R f (g x)) − (1 / r) *R i)
        using ‹0 < r› by (auto simp: scaleR_diff_right)
      finally show norm ((∑(x,K)∈p. content K *R f (g x)) − (1 / r) *R i) < e
        using d[OF gimp] ‹0 < r› by auto
    qed
  qed
  then show ?thesis
    by (auto simp: h_eq has_integral)
qed
```

6.15.26 Special case of a basic affine transformation

```
lemma AE_lborel_inner_neq:
  assumes k: k ∈ Basis
  shows AE x in lborel. x · k ≠ c
proof −
  interpret finite_product_sigma_finite λ_. lborel Basis
    proof qed simp
    have emeasure lborel {x∈space lborel. x · k = c}
        = emeasure (ΠM j::'a∈Basis. lborel) (ΠE j∈Basis. if j = k then {c} else
UNIV)
    using k
   by (auto simp add: lborel_eq[where 'a='a] emeasure_distr intro!: arg_cong2[where
f=emeasure])
       (auto simp: space_PiM PiE_iff extensional_def split: if_split_asm)
  also have ... = (∏j∈Basis. emeasure lborel (if j = k then {c} else UNIV))
    by (intro measure_times) auto
  also have ... = 0
    by (intro prod_zero bexI[OF _ k]) auto
  finally show ?thesis
    by (subst AE_iff_measurable[OF _ refl]) auto
qed

lemma content_image_stretch_interval:
  fixes m :: 'a::euclidean_space ⇒ real
  defines s f x ≡ (∑k::'a∈Basis. (f k * (x·k)) *R k)
  shows content (s m ` cbox a b) = |∏k∈Basis. m k| * content (cbox a b)
```

```
proof cases
  have s[measurable]: s f ∈ borel →M borel for f
    by (auto simp: s_def[abs_def])
  assume m: ∀k∈Basis. m k ≠ 0
  then have s_comp_s: s (λk. 1 / m k) ∘ s m = id s m ∘ s (λk. 1 / m k) = id
    by (auto simp: s_def[abs_def] fun_eq_iff euclidean_representation)
  then have inv (s (λk. 1 / m k)) = s m bij (s (λk. 1 / m k))
    by (auto intro: inv_unique_comp o_bij)
  then have eq: s m ` cbox a b = s (λk. 1 / m k) -` cbox a b
    using bij_vimage_eq_inv_image[OF ‹bij (s (λk. 1 / m k))›, of cbox a b] by auto
  show ?thesis
    using m unfolding eq measure_def
    by (subst lborel_affine_euclidean[where c=m and t=0])
     (simp_all add: emeasure_density measurable_sets_borel[OF s] abs_prod nn_integral_cmult
                s_def[symmetric] emeasure_distr vimage_comp s_comp_s enn2real_mult
prod_nonneg)
next
  assume ¬ (∀k∈Basis. m k ≠ 0)
  then obtain k where k: k ∈ Basis m k = 0 by auto
  then have [simp]: (∏k∈Basis. m k) = 0
    by (intro prod_zero) auto
  have emeasure lborel {x∈space lborel. x ∈ s m ` cbox a b} = 0
  proof (rule emeasure_eq_0_AE)
    show AE x in lborel. x ∉ s m ` cbox a b
      using AE_lborel_inner_neq[OF ‹k∈Basis›]
    proof eventually_elim
      show x · k ≠ 0 ⟹ x ∉ s m ` cbox a b  for x
        using k by (auto simp: s_def[abs_def] cbox_def)
    qed
  qed
  then show ?thesis
    by (simp add: measure_def)
qed

lemma interval_image_affinity_interval:
  ∃u v. (λx. m *R (x::'a::euclidean_space) + c) ` cbox a b = cbox u v
  unfolding image_affinity_cbox
  by auto

lemma content_image_affinity_cbox:
  content((λx::'a::euclidean_space. m *R x + c) ` cbox a b) =
    |m| ^ DIM('a) * content (cbox a b) (is ?l = ?r)
proof (cases cbox a b = {})
  case True then show ?thesis by simp
next
  case False
  show ?thesis
  proof (cases m ≥ 0)
    case True
```

with ‹*cbox a b* ≠ {}› **have** *cbox* (*m* $*_R$ *a* + *c*) (*m* $*_R$ *b* + *c*) ≠ {}
by (*simp add*: *box_ne_empty inner_left_distrib mult_left_mono*)
moreover from *True* **have** ∗: ⋀*i*. (*m* $*_R$ *b* + *c*) · *i* − (*m* $*_R$ *a* + *c*) · *i* = *m* $*_R$ (*b*−*a*) · *i*
by (*simp add*: *inner_simps field_simps*)
ultimately show *?thesis*
by (*simp add*: *image_affinity_cbox True content_cbox′*
prod.distrib inner_diff_left)
next
case *False*
with ‹*cbox a b* ≠ {}› **have** *cbox* (*m* $*_R$ *b* + *c*) (*m* $*_R$ *a* + *c*) ≠ {}
by (*simp add*: *box_ne_empty inner_left_distrib mult_left_mono*)
moreover from *False* **have** ∗: ⋀*i*. (*m* $*_R$ *a* + *c*) · *i* − (*m* $*_R$ *b* + *c*) · *i* = (−*m*) $*_R$ (*b*−*a*) · *i*
by (*simp add*: *inner_simps field_simps*)
ultimately show *?thesis* **using** *False*
by (*simp add*: *image_affinity_cbox content_cbox′*
prod.distrib[*symmetric*] *inner_diff_left flip*: *prod_constant*)
qed
qed

lemma *has_integral_affinity*:
fixes *a* :: ′*a*::*euclidean_space*
assumes (*f has_integral i*) (*cbox a b*)
and *m* ≠ *0*
shows ((λ*x*. *f*(*m* $*_R$ *x* + *c*)) *has_integral* (*1* / (|*m*| ˆ *DIM*(′*a*))) $*_R$ *i*) ((λ*x*. (*1* / *m*) $*_R$ *x* + −((*1* / *m*) $*_R$ *c*)) ‘ *cbox a b*)
proof (*rule has_integral_twiddle*)
show ∃ *w z*. (λ*x*. (*1* / *m*) $*_R$ *x* + − ((*1* / *m*) $*_R$ *c*)) ‘ *cbox u v* = *cbox w z*
∃ *w z*. (λ*x*. *m* $*_R$ *x* + *c*) ‘ *cbox u v* = *cbox w z* **for** *u v*
using *interval_image_affinity_interval* **by** *blast*+
show *content* ((λ*x*. *m* $*_R$ *x* + *c*) ‘ *cbox u v*) = |*m*| ˆ *DIM*(′*a*) ∗ *content* (*cbox u v*) **for** *u v*
using *content_image_affinity_cbox* **by** *blast*
qed (*use assms zero_less_power* **in** ‹*auto simp*: *field_simps*›)

lemma *integrable_affinity*:
assumes *f integrable_on cbox a b*
and *m* ≠ *0*
shows (λ*x*. *f*(*m* $*_R$ *x* + *c*)) *integrable_on* ((λ*x*. (*1* / *m*) $*_R$ *x* + −((*1*/*m*) $*_R$ *c*)) ‘ *cbox a b*)
using *has_integral_affinity assms*
unfolding *integrable_on_def* **by** *blast*

lemmas *has_integral_affinity01* = *has_integral_affinity* [*of* _ _ *0 1*::*real*, *simplified*]

lemma *integrable_on_affinity*:
assumes *m* ≠ *0 f integrable_on* (*cbox a b*)
shows (λ*x*. *f* (*m* $*_R$ *x* + *c*)) *integrable_on* ((λ*x*. (*1* / *m*) $*_R$ *x* − ((*1* / *m*) $*_R$

c)) ‘ *cbox a b*)
proof –
  from *assms* **obtain** *I* **where** (*f has_integral I*) (*cbox a b*)
    by (*auto simp*: *integrable_on_def*)
  from *has_integral_affinity*[*OF this assms*(*1*), *of c*] **show** *?thesis*
    by (*auto simp*: *integrable_on_def*)
qed

lemma *has_integral_cmul_iff*:
  assumes $c \neq 0$
  shows $((\lambda x.\ c *_R f\ x)$ *has_integral* $(c *_R I))\ A \longleftrightarrow$ (*f has_integral I*) *A*
  using *assms has_integral_cmul*[*of f I A c*]
    has_integral_cmul[*of* $\lambda x.\ c *_R f\ x\ c *_R I\ A$ *inverse c*] **by** (*auto simp*: *field_simps*)

lemma *has_integral_affinity'*:
  fixes *a* :: *'a*::*euclidean_space*
  assumes (*f has_integral i*) (*cbox a b*) **and** $m > 0$
  shows $((\lambda x.\ f(m *_R x + c))$ *has_integral* $(i /_R m$ ^ *DIM*(*'a*)))
    (*cbox* $((a - c) /_R m)\ ((b - c) /_R m))$
proof (*cases cbox a b* = {})
  case *True*
  hence (*cbox* $((a - c) /_R m)\ ((b - c) /_R m)) = \{\}$
    using ‹$m > 0$› **unfolding** *box_eq_empty* **by** (*auto simp*: *algebra_simps*)
  with *True* **and** *assms* **show** *?thesis* **by** *simp*
next
  case *False*
  have $((\lambda x.\ f\ (m *_R x + c))$ *has_integral* $(1 / |m|$ ^ *DIM*(*'a*)) $*_R i)$
    $((\lambda x.\ (1 / m) *_R x + - ((1 / m) *_R c))$ ‘ *cbox a b*)
    using *assms* **by** (*intro has_integral_affinity*) *auto*
  also have $((\lambda x.\ (1 / m) *_R x + - ((1 / m) *_R c))$ ‘ *cbox a b*) =
      $((\lambda x.\ - ((1 / m) *_R c) + x)$ ‘ $(\lambda x.\ (1 / m) *_R x)$ ‘ *cbox a b*)
    by (*simp add*: *image_image algebra_simps*)
  also have $(\lambda x.\ (1 / m) *_R x)$ ‘ *cbox a b* = *cbox* $((1 / m) *_R a)\ ((1 / m) *_R b)$ **using** ‹$m > 0$› *False*
    by (*subst image_smult_cbox*) *simp_all*
  also have $(\lambda x.\ - ((1 / m) *_R c) + x)$ ‘ ... = *cbox* $((a - c) /_R m)\ ((b - c) /_R m)$
    by (*subst cbox_translation* [*symmetric*]) (*simp add*: *field_simps vector_add_divide_simps*)
  finally show *?thesis* **using** ‹$m > 0$› **by** (*simp add*: *field_simps*)
qed

lemma *has_integral_affinity_iff*:
  fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *real_normed_vector*
  assumes $m > 0$
  shows $((\lambda x.\ f\ (m *_R x + c))$ *has_integral* $(I /_R m$ ^ *DIM*(*'a*)))
      (*cbox* $((a - c) /_R m)\ ((b - c) /_R m)) \longleftrightarrow$
    (*f has_integral I*) (*cbox a b*) (**is** *?lhs* = *?rhs*)
proof

assume *?lhs*
from *has_integral_affinity'*[*OF this*, *of 1 / m −c /$_R$ m*] **and** ‹*m > 0*›
show *?rhs* **by** (*simp add*: *vector_add_divide_simps*) (*simp add*: *field_simps*)
next
assume *?rhs*
from *has_integral_affinity'*[*OF this*, *of m c*] **and** ‹*m > 0*›
show *?lhs* **by** *simp*
qed

6.15.27 Special case of stretching coordinate axes separately

lemma *has_integral_stretch*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*real_normed_vector*
assumes (*f has_integral i*) (*cbox a b*)
and ∀ *k*∈*Basis*. *m k* ≠ *0*
shows ((λ*x*. *f* (∑ *k*∈*Basis*. (*m k* ∗ (*x*·*k*))∗$_R$ *k*)) *has_integral*
((*1*/ |*prod m Basis*|) ∗$_R$ *i*)) ((λ*x*. (∑ *k*∈*Basis*. (*1 / m k* ∗ (*x*·*k*))∗$_R$ *k*)) ‘ *cbox a b*)
apply (*rule has_integral_twiddle*[**where** *f*=*f*])
unfolding *zero_less_abs_iff content_image_stretch_interval*
unfolding *image_stretch_interval empty_as_interval euclidean_eq_iff* [**where** *'a*=*'a*]
using *assms*
by *auto*

lemma *has_integral_stretch_real*:
fixes *f* :: *real* ⇒ *'b*::*real_normed_vector*
assumes (*f has_integral i*) {*a*..*b*} **and** *m* ≠ *0*
shows ((λ*x*. *f* (*m* ∗ *x*)) *has_integral* (*1* / |*m*|) ∗$_R$ *i*) ((λ*x*. *x* / *m*) ‘ {*a*..*b*})
using *has_integral_stretch* [*of f i a b* λ*b*. *m*] *assms* **by** *simp*

lemma *integrable_stretch*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*real_normed_vector*
assumes *f integrable_on cbox a b*
and ∀ *k*∈*Basis*. *m k* ≠ *0*
shows (λ*x*::*'a*. *f* (∑ *k*∈*Basis*. (*m k* ∗ (*x*·*k*))∗$_R$ *k*)) *integrable_on*
((λ*x*. ∑ *k*∈*Basis*. (*1 / m k* ∗ (*x*·*k*))∗$_R$ *k*) ‘ *cbox a b*)
using *assms* **unfolding** *integrable_on_def*
by (*force dest*: *has_integral_stretch*)

lemma *vec_lambda_eq_sum*:
(χ *k*. *f k* (*x* $ *k*)) = (∑ *k*∈*Basis*. (*f* (*axis_index k*) (*x* · *k*)) ∗$_R$ *k*) (**is** *?lhs* = *?rhs*)
proof −
have *?lhs* = (χ *k*. *f k* (*x* · *axis k 1*))
by (*simp add*: *cart_eq_inner_axis*)
also have ... = (∑ *u*∈*UNIV*. *f u* (*x* · *axis u 1*) ∗$_R$ *axis u 1*)
by (*simp add*: *vec_eq_iff axis_def if_distrib cong*: *if_cong*)
also have ... = *?rhs*
by (*simp add*: *Basis_vec_def UNION_singleton_eq_range sum.reindex axis_eq_axis*

inj_on_def)
  finally show *?thesis* **.**
qed

lemma *has_integral_stretch_cart*:
  fixes *m* :: *'n::finite* ⇒ *real*
  assumes *f*: (*f has_integral i*) (*cbox a b*) **and** *m*: ⋀*k*. *m k* ≠ *0*
  shows ((λ*x*. *f*(χ *k*. *m k* ∗ *x*$*k*)) *has_integral i* /$_R$ |*prod m UNIV*|)
      ((λ*x*. χ *k*. *x*$*k* / *m k*) ‘ (*cbox a b*))
proof −
  have ∗: ∀ *k*:: *real*^*'n* ∈ *Basis*. *m* (*axis_index k*) ≠ *0*
    using *axis_index* **by** (*simp add*: *m*)
  have *eqp*: (∏ *k*:: *real*^*'n* ∈ *Basis*. *m* (*axis_index k*)) = *prod m UNIV*
   by (*simp add*: *Basis_vec_def UNION_singleton_eq_range prod.reindex axis_eq_axis inj_on_def*)
  show *?thesis*
    using *has_integral_stretch* [*OF f* ∗] *vec_lambda_eq_sum* [**where** *f*=λ*i x*. *m i* ∗ *x*] *vec_lambda_eq_sum* [**where** *f*=λ*i x*. *x* / *m i*]
    by (*simp add*: *field_simps eqp*)
qed

lemma *image_stretch_interval_cart*:
  fixes *m* :: *'n::finite* ⇒ *real*
  shows (λ*x*. χ *k*. *m k* ∗ *x*$*k*) ‘ *cbox a b* =
      (*if cbox a b* = {} *then* {}
       else cbox (χ *k*. *min* (*m k* ∗ *a*$*k*) (*m k* ∗ *b*$*k*)) (χ *k*. *max* (*m k* ∗ *a*$*k*) (*m k* ∗ *b*$*k*)))
proof −
  have ∗: (∑ *k*∈*Basis*. *min* (*m* (*axis_index k*) ∗ (*a* · *k*)) (*m* (*axis_index k*) ∗ (*b* · *k*)) ∗$_R$ *k*)
      = (χ *k*. *min* (*m k* ∗ *a* $ *k*) (*m k* ∗ *b* $ *k*))
     (∑ *k*∈*Basis*. *max* (*m* (*axis_index k*) ∗ (*a* · *k*)) (*m* (*axis_index k*) ∗ (*b* · *k*)) ∗$_R$ *k*)
      = (χ *k*. *max* (*m k* ∗ *a* $ *k*) (*m k* ∗ *b* $ *k*))
   apply (*simp_all add*: *Basis_vec_def cart_eq_inner_axis UNION_singleton_eq_range sum.reindex axis_eq_axis inj_on_def*)
    apply (*simp_all add*: *vec_eq_iff axis_def if_distrib cong*: *if_cong*)
    done
  show *?thesis*
   by (*simp add*: *vec_lambda_eq_sum* [**where** *f*=λ*i x*. *m i* ∗ *x*] *image_stretch_interval eq_cbox* ∗)
qed

6.15.28 even more special cases

lemma *uminus_interval_vector*[*simp*]:
  fixes *a b* :: *'a::euclidean_space*
  shows *uminus* ‘ *cbox a b* = *cbox* (−*b*) (−*a*)
proof −

have *x* ∈ *uminus* ‘ *cbox a b* **if** *x* ∈ *cbox* (− *b*) (− *a*) **for** *x*
proof −
have −*x* ∈ *cbox a b*
using *that* **by** (*auto simp*: *mem_box*)
then show *?thesis*
by *force*
qed
then show *?thesis*
by (*auto simp*: *mem_box*)
qed

lemma *has_integral_reflect_lemma*[*intro*]:
assumes (*f has_integral i*) (*cbox a b*)
shows ((λ*x*. *f*(−*x*)) *has_integral i*) (*cbox* (−*b*) (−*a*))
using *has_integral_affinity*[*OF assms*, *of* −*1 0*]
by *auto*

lemma *has_integral_reflect_lemma_real*[*intro*]:
assumes (*f has_integral i*) {*a*..*b*::*real*}
shows ((λ*x*. *f*(−*x*)) *has_integral i*) {−*b* .. −*a*}
using *assms*
unfolding *box_real*[*symmetric*]
by (*rule has_integral_reflect_lemma*)

lemma *has_integral_reflect*[*simp*]:
((λ*x*. *f* (−*x*)) *has_integral i*) (*cbox* (−*b*) (−*a*)) ⟷ (*f has_integral i*) (*cbox a b*)
by (*auto dest*: *has_integral_reflect_lemma*)

lemma *has_integral_reflect_real*[*simp*]:
fixes *a b*::*real*
shows ((λ*x*. *f* (−*x*)) *has_integral i*) {−*b*..−*a*} ⟷ (*f has_integral i*) {*a*..*b*}
by (*metis has_integral_reflect interval_cbox*)

lemma *integrable_reflect*[*simp*]: (λ*x*. *f*(−*x*)) *integrable_on cbox* (−*b*) (−*a*) ⟷ *f integrable_on cbox a b*
unfolding *integrable_on_def* **by** *auto*

lemma *integrable_reflect_real*[*simp*]: (λ*x*. *f*(−*x*)) *integrable_on* {−*b* .. −*a*} ⟷ *f integrable_on* {*a*..*b*::*real*}
unfolding *box_real*[*symmetric*]
by (*rule integrable_reflect*)

lemma *integral_reflect*[*simp*]: *integral* (*cbox* (−*b*) (−*a*)) (λ*x*. *f* (−*x*)) = *integral* (*cbox a b*) *f*
unfolding *integral_def* **by** *auto*

lemma *integral_reflect_real*[*simp*]: *integral* {−*b* .. −*a*} (λ*x*. *f* (−*x*)) = *integral* {*a*..*b*::*real*} *f*
unfolding *box_real*[*symmetric*]

by (*rule integral_reflect*)

6.15.29 Stronger form of FCT; quite a tedious proof

lemma *split_minus*[*simp*]: (λ(*x*, *k*). *f x k*) *x* − (λ(*x*, *k*). *g x k*) *x* = (λ(*x*, *k*). *f x k* − *g x k*) *x*
by (*simp add*: *split_def*)

theorem *fundamental_theorem_of_calculus_interior*:
fixes *f* :: *real* ⇒ ′*a*::*real_normed_vector*
assumes *a* ≤ *b*
and *contf*: *continuous_on* {*a*..*b*} *f*
and *derf*: ⋀*x*. *x* ∈ {*a* <..< *b*} ⟹ (*f has_vector_derivative f* ′ *x*) (*at x*)
shows (*f* ′ *has_integral* (*f b* − *f a*)) {*a*..*b*}
proof (*cases a* = *b*)
case *True*
then have ∗: *cbox a b* = {*b*} *f b* − *f a* = *0*
by (*auto simp add*: *order_antisym*)
with *True* **show** *?thesis* **by** *auto*
next
case *False*
with ‹*a* ≤ *b*› **have** *ab*: *a* < *b* **by** *arith*
show *?thesis*
unfolding *has_integral_factor_content_real*
proof (*intro allI impI*)
fix *e* :: *real*
assume *e*: *e* > *0*
then have *eba8*: (*e* ∗ (*b*−*a*)) / *8* > *0*
using *ab* **by** (*auto simp add*: *field_simps*)
note *derf_exp* = *derf*[*unfolded has_vector_derivative_def has_derivative_at_alt*, *THEN conjunct2*, *rule_format*]
thm *derf_exp*
have *bounded*: ⋀*x*. *x* ∈ {*a*<..<*b*} ⟹ *bounded_linear* (λ*u*. *u* ∗$_R$ *f* ′ *x*)
by (*simp add*: *bounded_linear_scaleR_left*)
have ∀ *x* ∈ *box a b*. ∃ *d*>*0*. ∀ *y*. *norm* (*y*−*x*) < *d* ⟶ *norm* (*f y* − *f x* − (*y*−*x*) ∗$_R$ *f* ′ *x*) ≤ *e*/*2* ∗ *norm* (*y*−*x*)
(**is** ∀ *x* ∈ *box a b*. *?Q x*) — The explicit quantifier is required by the following step
proof
fix *x* **assume** *x* ∈ *box a b*
with *e* **show** *?Q x*
using *derf_exp* [*of x e*/*2*] **by** *auto*
qed
then obtain *d* **where** *d*: ⋀*x*. *0* < *d x*
⋀*x y*. ⟦*x* ∈ *box a b*; *norm* (*y*−*x*) < *d x*⟧ ⟹ *norm* (*f y* − *f x* − (*y*−*x*) ∗$_R$ *f* ′ *x*) ≤ *e*/*2* ∗ *norm* (*y*−*x*)
unfolding *bgauge_existence_lemma* **by** *metis*
have *bounded* (*f* ‘ *cbox a b*)
using *compact_cbox assms* **by** (*auto simp*: *compact_imp_bounded compact_continuous_image*)

then obtain B
where $0 < B$ **and** B: $\bigwedge x.\ x \in f$ ‘ *cbox a b* $\Longrightarrow$ *norm* $x \leq B$
unfolding *bounded_pos* **by** *metis*
obtain *da* **where** $0 < da$
and *da*: $\bigwedge c.\ [\![a \leq c; \{a..c\} \subseteq \{a..b\}; \{a..c\} \subseteq$ *ball a da*$]\!]$
$\Longrightarrow$ *norm* (*content* $\{a..c\} *_R f'\ a - (f\ c - f\ a)) \leq (e * (b-a))\ /\ 4$
proof –
have *continuous* (*at a within* $\{a..b\}$) f
using *contf continuous_on_eq_continuous_within* **by** *force*
with *eba8* **obtain** k **where** $0 < k$
and k: $\bigwedge x.\ [\![x \in \{a..b\}; 0 <$ *norm* $(x-a)$; *norm* $(x-a) < k]\!] \Longrightarrow$ *norm* $(f\ x - f\ a) < e * (b-a)\ /\ 8$
unfolding *continuous_within Lim_within dist_norm* **by** *metis*
obtain l **where** l: $0 < l$ *norm* $(l *_R f'\ a) \leq e * (b-a)\ /\ 8$
proof (*cases* $f'\ a = 0$)
case *True* **with** *ab e that* **show** *?thesis* **by** *auto*
next
case *False*
show *?thesis*
proof
show *norm* $((e * (b - a)\ /\ 8\ /$ *norm* $(f'\ a)) *_R f'\ a) \leq e * (b - a)\ /\ 8$
$0 < e * (b - a)\ /\ 8\ /$ *norm* $(f'\ a)$
using *False ab e* **by** (*auto simp add*: *field_simps*)
qed
qed
have *norm* (*content* $\{a..c\} *_R f'\ a - (f\ c - f\ a)) \leq e * (b-a)\ /\ 4$
if $a \leq c$ $\{a..c\} \subseteq \{a..b\}$ **and** *bmin*: $\{a..c\} \subseteq$ *ball a* (*min k l*) **for** c
proof –
have *minkl*: $|a - x| <$ *min k l* **if** $x \in \{a..c\}$ **for** x
using *bmin dist_real_def that* **by** *auto*
then have *lel*: $|c - a| \leq |l|$
using *that* **by** *force*
have *norm* $((c - a) *_R f'\ a - (f\ c - f\ a)) \leq$ *norm* $((c - a) *_R f'\ a) +$ *norm* $(f\ c - f\ a)$
by (*rule norm_triangle_ineq4*)
also have ... $\leq e * (b-a)\ /\ 8 + e * (b-a)\ /\ 8$
proof (*rule add_mono*)
have *norm* $((c - a) *_R f'\ a) \leq$ *norm* $(l *_R f'\ a)$
by (*auto intro*: *mult_right_mono* [*OF lel*])
also have ... $\leq e * (b-a)\ /\ 8$
by (*rule l*)
finally show *norm* $((c - a) *_R f'\ a) \leq e * (b-a)\ /\ 8$.
next
have *norm* $(f\ c - f\ a) < e * (b-a)\ /\ 8$
proof (*cases* $a = c$)
case *True* **then show** *?thesis*
using *eba8* **by** *auto*
next

```
          case False show ?thesis
            by (rule k) (use minkl ‹a ≤ c› that False in auto)
        qed
        then show norm (f c − f a) ≤ e ∗ (b−a) / 8 by simp
      qed
      finally show norm (content {a..c} ∗R f′ a − (f c − f a)) ≤ e ∗ (b−a) / 4
        unfolding content_real[OF ‹a ≤ c›] by auto
    qed
    then show ?thesis
      by (rule_tac da=min k l in that) (auto simp: l ‹0 < k›)
  qed
  obtain db where 0 < db
    and db: ⋀c. ⟦c ≤ b; {c..b} ⊆ {a..b}; {c..b} ⊆ ball b db⟧
                  ⟹ norm (content {c..b} ∗R f′ b − (f b − f c)) ≤ (e ∗ (b−a))
/ 4
  proof −
    have continuous (at b within {a..b}) f
      using contf continuous_on_eq_continuous_within by force
    with eba8 obtain k
      where 0 < k
        and k: ⋀x. ⟦x ∈ {a..b}; 0 < norm(x−b); norm(x−b) < k⟧
                ⟹ norm (f b − f x) < e ∗ (b−a) / 8
       unfolding continuous_within Lim_within dist_norm norm_minus_commute
by metis
    obtain l where l: 0 < l norm (l ∗R f′ b) ≤ (e ∗ (b−a)) / 8
    proof (cases f′ b = 0)
      case True thus ?thesis
        using ab e that by auto
    next
      case False show ?thesis
      proof
        show norm ((e ∗ (b − a) / 8 / norm (f′ b)) ∗R f′ b) ≤ e ∗ (b − a) / 8
             0 < e ∗ (b − a) / 8 / norm (f′ b)
          using False ab e by (auto simp add: field_simps)
      qed
    qed
    have norm (content {c..b} ∗R f′ b − (f b − f c)) ≤ e ∗ (b−a) / 4
      if c ≤ b {c..b} ⊆ {a..b} and bmin: {c..b} ⊆ ball b (min k l) for c
    proof −
      have minkl: |b − x| < min k l if x ∈ {c..b} for x
        using bmin dist_real_def that by auto
      then have lel: |b − c| ≤ |l|
        using that by force
       have norm ((b − c) ∗R f′ b − (f b − f c)) ≤ norm ((b − c) ∗R f′ b) +
norm (f b − f c)
        by (rule norm_triangle_ineq4)
      also have ... ≤ e ∗ (b−a) / 8 + e ∗ (b−a) / 8
      proof (rule add_mono)
        have norm ((b − c) ∗R f′ b) ≤ norm (l ∗R f′ b)
```

```
          by (auto intro: mult_right_mono [OF lel])
        also have ... ≤ e * (b−a) / 8
          by (rule l)
        finally show norm ((b − c) *R f′ b) ≤ e * (b−a) / 8 .
      next
        have norm (f b − f c) < e * (b−a) / 8
        proof (cases b = c)
          case True with eba8 show ?thesis
            by auto
        next
          case False show ?thesis
            by (rule k) (use minkl ‹c ≤ b› that False in auto)
        qed
        then show norm (f b − f c) ≤ e * (b−a) / 8 by simp
      qed
      finally show norm (content {c..b} *R f′ b − (f b − f c)) ≤ e * (b−a) / 4
        unfolding content_real[OF ‹c ≤ b›] by auto
    qed
    then show ?thesis
      by (rule_tac db=min k l in that) (auto simp: l ‹0 < k›)
  qed
  let ?d = (λx. ball x (if x=a then da else if x=b then db else d x))
  show ∃d. gauge d ∧ (∀p. p tagged_division_of {a..b} ∧ d fine p ⟶
           norm ((∑(x,K)∈p. content K *R f′ x) − (f b − f a)) ≤ e * content
{a..b})
  proof (rule exI, safe)
    show gauge ?d
      using ab ‹db > 0› ‹da > 0› d(1) by (auto intro: gauge_ball_dependent)
  next
    fix p
    assume ptag: p tagged_division_of {a..b} and fine: ?d fine p
    let ?A = {t. fst t ∈ {a, b}}
    note p = tagged_division_ofD[OF ptag]
    have pA: p = (p ∩ ?A) ∪ (p − ?A) finite (p ∩ ?A) finite (p − ?A) (p ∩ ?A)
∩ (p − ?A) = {}
      using ptag fine by auto
    have le_xz: ⋀w x y z::real. y ≤ z/2 ⟹ w − x ≤ z/2 ⟹ w + y ≤ x + z
      by arith
    have non: False if xk: (x,K) ∈ p and x ≠ a x ≠ b
      and less: e * (Sup K − Inf K)/2 < norm (content K *R f′ x − (f (Sup
K) − f (Inf K)))
    for x K
    proof −
      obtain u v where k: K = cbox u v
        using p(4) xk by blast
      then have u ≤ v and uv: {u, v} ⊆ cbox u v
        using p(2)[OF xk] by auto
      then have result: e * (v − u)/2 < norm ((v − u) *R f′ x − (f v − f u))
        using less[unfolded k box_real interval_bounds_real content_real] by auto
```

```
    then have x ∈ box a b
      using p(2) p(3) ‹x ≠ a› ‹x ≠ b› xk by fastforce
    with d have ∗: ⋀y. norm (y−x) < d x
            ⟹ norm (f y − f x − (y−x) ∗R f′ x) ≤ e/2 ∗ norm (y−x)
      by metis
    have xd: norm (u − x) < d x norm (v − x) < d x
      using fineD[OF fine xk] ‹x ≠ a› ‹x ≠ b› uv
      by (auto simp add: k subset_eq dist_commute dist_real_def)
    have norm ((v − u) ∗R f′ x − (f v − f u)) =
          norm ((f u − f x − (u − x) ∗R f′ x) − (f v − f x − (v − x) ∗R f′ x))
      by (rule arg_cong[where f=norm]) (auto simp: scaleR_left.diff)
    also have . . . ≤ e/2 ∗ norm (u − x) + e/2 ∗ norm (v − x)
      by (metis norm_triangle_le_diff add_mono ∗ xd)
    also have . . . ≤ e/2 ∗ norm (v − u)
      using p(2)[OF xk] by (auto simp add: field_simps k)
    also have . . . < norm ((v − u) ∗R f′ x − (f v − f u))
      using result by (simp add: ‹u ≤ v›)
    finally have e ∗ (v − u)/2 < e ∗ (v − u)/2
      using uv by auto
    then show False by auto
  qed
   have norm (∑(x, K)∈p − ?A. content K ∗R f′ x − (f (Sup K) − f (Inf
K)))
        ≤ (∑(x, K)∈p − ?A. norm (content K ∗R f′ x − (f (Sup K) − f (Inf
K))))
    by (auto intro: sum_norm_le)
  also have ... ≤ (∑n∈p − ?A. e ∗ (case n of (x, k) ⇒ Sup k − Inf k)/2)
    using non by (fastforce intro: sum_mono)
  finally have I: norm (∑(x, k)∈p − ?A.
              content k ∗R f′ x − (f (Sup k) − f (Inf k)))
          ≤ (∑n∈p − ?A. e ∗ (case n of (x, k) ⇒ Sup k − Inf k))/2
    by (simp add: sum_divide_distrib)
   have II: norm (∑(x, k)∈p ∩ ?A. content k ∗R f′ x − (f (Sup k) − f (Inf
k))) −
          (∑n∈p ∩ ?A. e ∗ (case n of (x, k) ⇒ Sup k − Inf k))
          ≤ (∑n∈p − ?A. e ∗ (case n of (x, k) ⇒ Sup k − Inf k))/2
  proof −
    have ge0: 0 ≤ e ∗ (Sup k − Inf k) if xkp: (x, k) ∈ p ∩ ?A for x k
    proof −
      obtain u v where uv: k = cbox u v
        by (meson Int_iff xkp p(4))
      with p(2) that uv have cbox u v ≠ {}
        by blast
      then show 0 ≤ e ∗ ((Sup k) − (Inf k))
        unfolding uv using e by (auto simp add: field_simps)
    qed
    let ?B = λx. {t ∈ p. fst t = x ∧ content (snd t) ≠ 0}
    let ?C = {t ∈ p. fst t ∈ {a, b} ∧ content (snd t) ≠ 0}
    have norm (∑(x, k)∈p ∩ {t. fst t ∈ {a, b}}. content k ∗R f′ x − (f (Sup
```

$k) - f\ (Inf\ k))) \le e * (b-a)/2$
proof −
have ∗: $\bigwedge S\ f\ e.\ sum\ f\ S = sum\ f\ (p \cap ?C) \Longrightarrow norm\ (sum\ f\ (p \cap ?C)) \le e \Longrightarrow norm\ (sum\ f\ S) \le e$
by *auto*
have *1*: $content\ K *_R (f'\ x) - (f\ ((Sup\ K)) - f\ ((Inf\ K))) = 0$
if $(x,K) \in p \cap \{t.\ fst\ t \in \{a,\ b\}\} - p \cap ?C$ **for** $x\ K$
proof −
have *xk*: $(x,K) \in p$ **and** *k0*: $content\ K = 0$
using *that* **by** *auto*
then obtain $u\ v$ **where** *uv*: $K = cbox\ u\ v$
using $p(4)$ **by** *blast*
then have $u = v$
using $xk\ k0\ p(2)$ **by** *force*
then show $content\ K *_R (f'\ x) - (f\ ((Sup\ K)) - f\ ((Inf\ K))) = 0$
using *xk* **unfolding** *uv* **by** *auto*
qed
have *2*: $norm(\sum (x,K) \in p \cap ?C.\ content\ K *_R f'\ x - (f\ (Sup\ K) - f\ (Inf\ K))) \le e * (b-a)/2$
proof −
have *norm_le*: $norm\ (sum\ f\ S) \le e$
if $\bigwedge x\ y.\ [\![x \in S;\ y \in S]\!] \Longrightarrow x = y\ \bigwedge x.\ x \in S \Longrightarrow norm\ (f\ x) \le e\ e > 0$
for $S\ f$ **and** $e :: real$
proof $(cases\ S = \{\})$
case *True*
with *that* **show** *?thesis* **by** *auto*
next
case *False* **then obtain** x **where** $x \in S$
by *auto*
then have $S = \{x\}$
using $that(1)$ **by** *auto*
then show *?thesis*
using $\langle x \in S \rangle\ that(2)$ **by** *auto*
qed
have ∗: $p \cap ?C = ?B\ a \cup ?B\ b$
by *blast*
then have $norm\ (\sum (x,K) \in p \cap ?C.\ content\ K *_R f'\ x - (f\ (Sup\ K) - f\ (Inf\ K))) =$
$norm\ (\sum (x,K) \in ?B\ a \cup ?B\ b.\ content\ K *_R f'\ x - (f\ (Sup\ K) - f\ (Inf\ K)))$
by *simp*
also have ... $= norm\ ((\sum (x,K) \in ?B\ a.\ content\ K *_R f'\ x - (f\ (Sup\ K) - f\ (Inf\ K))) +$
$(\sum (x,K) \in ?B\ b.\ content\ K *_R f'\ x - (f\ (Sup\ K) - f\ (Inf\ K))))$
using $p(1)\ ab\ e$ **by** (*subst sum.union_disjoint*) *auto*
also have ... $\le e * (b - a) / 4 + e * (b - a) / 4$
proof (*rule norm_triangle_le* [*OF add_mono*])

```
      have pa: ∃v. k = cbox a v ∧ a ≤ v if (a, k) ∈ p for k
        using p(2) p(3) p(4) that by fastforce
       show norm (∑(x,K) ∈ ?B a. content K *R f' x − (f (Sup K) − f
(Inf K))) ≤ e * (b − a) / 4
      proof (intro norm_le; clarsimp)
        fix K K'
        assume K: (a, K) ∈ p (a, K') ∈ p and ne0: content K ≠ 0 content
K' ≠ 0
          with pa obtain v v' where v: K = cbox a v a ≤ v and v': K' =
cbox a v' a ≤ v'
          by blast
        let ?v = min v v'
        have box a ?v ⊆ K ∩ K'
          unfolding v v' by (auto simp add: mem_box)
        then have interior (box a (min v v')) ⊆ interior K ∩ interior K'
          using interior_Int interior_mono by blast
        moreover have (a + ?v)/2 ∈ box a ?v
          using ne0 unfolding v v' content_eq_0 not_le
          by (auto simp add: mem_box)
        ultimately have (a + ?v)/2 ∈ interior K ∩ interior K'
          unfolding interior_open[OF open_box] by auto
        then show K = K'
          using p(5)[OF K] by auto
      next
        fix K
        assume K: (a, K) ∈ p and ne0: content K ≠ 0
        show norm (content c *R f' a − (f (Sup c) − f (Inf c))) * 4 ≤ e *
(b−a)
          if (a, c) ∈ p and ne0: content c ≠ 0 for c
        proof −
          obtain v where v: c = cbox a v and a ≤ v
            using pa[OF ‹(a, c) ∈ p›] by metis
          then have a ∈ {a..v} v ≤ b
            using p(3)[OF ‹(a, c) ∈ p›] by auto
          moreover have {a..v} ⊆ ball a da
              using fineD[OF ‹?d fine p› ‹(a, c) ∈ p›] by (simp add: v split:
if_split_asm)
          ultimately show ?thesis
            unfolding v interval_bounds_real[OF ‹a ≤ v›] box_real
            using da ‹a ≤ v› by auto
        qed
      qed (use ab e in auto)
    next
      have pb: ∃v. k = cbox v b ∧ b ≥ v if (b, k) ∈ p for k
        using p(2) p(3) p(4) that by fastforce
       show norm (∑(x,K) ∈ ?B b. content K *R f' x − (f (Sup K) − f
(Inf K))) ≤ e * (b − a) / 4
      proof (intro norm_le; clarsimp)
        fix K K'
```

assume *K*: (*b*, *K*) ∈ *p* (*b*, *K'*) ∈ *p* **and** *ne0*: *content K* ≠ *0 content K'* ≠ *0*
with *pb* **obtain** *v v'* **where** *v*: *K* = *cbox v b v* ≤ *b* **and** *v'*: *K'* = *cbox v' b v'* ≤ *b*
by *blast*
let *?v* = *max v v'*
have *box ?v b* ⊆ *K* ∩ *K'*
unfolding *v v'* **by** (*auto simp*: *mem_box*)
then have *interior* (*box* (*max v v'*) *b*) ⊆ *interior K* ∩ *interior K'*
using *interior_Int interior_mono* **by** *blast*
moreover have ((*b* + *?v*)/*2*) ∈ *box ?v b*
using *ne0* **unfolding** *v v' content_eq_0 not_le* **by** (*auto simp*: *mem_box*)
ultimately have ((*b* + *?v*)/*2*) ∈ *interior K* ∩ *interior K'*
unfolding *interior_open*[*OF open_box*] **by** *auto*
then show *K* = *K'*
using *p*(*5*)[*OF K*] **by** *auto*
next
fix *K*
assume *K*: (*b*, *K*) ∈ *p* **and** *ne0*: *content K* ≠ *0*
show *norm* (*content c* $*_R$ *f' b* − (*f* (*Sup c*) − *f* (*Inf c*))) ∗ *4* ≤ *e* ∗ (*b*−*a*)
if (*b*, *c*) ∈ *p* **and** *ne0*: *content c* ≠ *0* **for** *c*
proof −
obtain *v* **where** *v*: *c* = *cbox v b* **and** *v* ≤ *b*
using ‹(*b*, *c*) ∈ *p*› *pb* **by** *blast*
then have *v* ≥ *ab* ∈ {*v*.. *b*}
using *p*(*3*)[*OF* ‹(*b*, *c*) ∈ *p*›] **by** *auto*
moreover have {*v*..*b*} ⊆ *ball b db*
using *fineD*[*OF* ‹*?d fine p*› ‹(*b*, *c*) ∈ *p*›] *box_real*(*2*) *v False* **by** *force*
ultimately show *?thesis*
using *db v* **by** *auto*
qed
qed (*use ab e* **in** *auto*)
qed
also have ... = *e* ∗ (*b*−*a*)/*2*
by *simp*
finally show *norm* (∑(*x*,*k*)∈*p* ∩ *?C*.
content k $*_R$ *f' x* − (*f* (*Sup k*) − *f* (*Inf k*))) ≤ *e* ∗ (*b*−*a*)/*2* .
qed
show *norm* (∑(*x*, *k*)∈*p* ∩ *?A*. *content k* $*_R$ *f' x* − (*f* ((*Sup k*)) − *f* ((*Inf k*)))) ≤ *e* ∗ (*b*−*a*)/*2*
apply (*rule* ∗ [*OF sum.mono_neutral_right*[*OF pA*(*2*)]])
using *1 2* **by** (*auto simp*: *split_paired_all*)
qed
also have ... = (∑*n*∈*p*. *e* ∗ (*case n of* (*x*, *k*) ⇒ *Sup k* − *Inf k*))/*2*
unfolding *sum_distrib_left*[*symmetric*]
by (*subst additive_tagged_division_1*[*OF* ‹*a* ≤ *b*› *ptag*]) *auto*
finally have *norm_le*: *norm* (∑(*x*,*K*)∈*p* ∩ {*t*. *fst t* ∈ {*a*, *b*}}. *content K*

```
*R f' x - (f (Sup K) - f (Inf K)))
                ≤ (∑n∈p. e * (case n of (x, K) ⇒ Sup K - Inf K))/2 .
        have le2: ⋀x s1 s2::real. 0 ≤ s1 ⟹ x ≤ (s1 + s2)/2 ⟹ x - s1 ≤ s2/2
          by auto
        show ?thesis
          apply (rule le2 [OF sum_nonneg])
          using ge0 apply force
               by (metis (no_types, lifting) Diff_Diff_Int Diff_subset norm_le p(1)
sum.subset_diff)
      qed
      note * = additive_tagged_division_1[OF assms(1) ptag, symmetric]
      have norm (∑(x,K)∈p ∩ ?A ∪ (p - ?A). content K *R f' x - (f (Sup K)
- f (Inf K)))
               ≤ e * (∑(x,K)∈p ∩ ?A ∪ (p - ?A). Sup K - Inf K)
        unfolding sum_distrib_left
        unfolding sum.union_disjoint[OF pA(2-)]
        using le_xz norm_triangle_le I II by blast
      then
      show norm ((∑(x,K)∈p. content K *R f' x) - (f b - f a)) ≤ e * content
{a..b}
      by (simp only: content_real[OF ‹a ≤ b›] *[of λx. x] *[of f] sum_subtractf[symmetric]
split_minus pA(1) [symmetric])
    qed
  qed
qed
```

6.15.30 Stronger form with finite number of exceptional points

```
lemma fundamental_theorem_of_calculus_interior_strong:
 fixes f :: real ⇒ 'a::banach
 assumes finite S
   and a ≤ b ⋀x. x ∈ {a <..< b} - S ⟹ (f has_vector_derivative f'(x)) (at x)
   and continuous_on {a .. b} f
 shows (f' has_integral (f b - f a)) {a .. b}
  using assms
proof (induction arbitrary: a b)
case empty
  then show ?case
    using fundamental_theorem_of_calculus_interior by force
next
case (insert x S)
  show ?case
  proof (cases x ∈ {a<..<b})
    case False then show ?thesis
      using insert by blast
  next
    case True then have a < x x < b
      by auto
    have (f' has_integral f x - f a) {a..x} (f' has_integral f b - f x) {x..b}
```

using ‹*continuous_on* {*a*..*b*} *f*› ‹*a* < *x*› ‹*x* < *b*› *continuous_on_subset* **by** (*force simp*: *intro*!: *insert*)+
then have (*f'* *has_integral f x* − *f a* + (*f b* − *f x*)) {*a*..*b*}
using ‹*a* < *x*› ‹*x* < *b*› *has_integral_combine less_imp_le* **by** *blast*
then show *?thesis*
by *simp*
qed
qed

corollary *fundamental_theorem_of_calculus_strong*:
fixes *f* :: *real* ⇒ *'a*::*banach*
assumes *finite S*
and *a* ≤ *b*
and *vec*: ⋀*x*. *x* ∈ {*a*..*b*} − *S* ⟹ (*f has_vector_derivative f'*(*x*)) (*at x*)
and *continuous_on* {*a*..*b*} *f*
shows (*f'* *has_integral* (*f b* − *f a*)) {*a*..*b*}
by (*rule fundamental_theorem_of_calculus_interior_strong* [*OF* ‹*finite S*›]) (*force simp*: *assms*)+

proposition *indefinite_integral_continuous_left*:
fixes *f*:: *real* ⇒ *'a*::*banach*
assumes *intf*: *f integrable_on* {*a*..*b*} **and** *a* < *c* *c* ≤ *b* *e* > *0*
obtains *d* **where** *d* > *0*
and ∀ *t*. *c* − *d* < *t* ∧ *t* ≤ *c* ⟶ *norm* (*integral* {*a*..*c*} *f* − *integral* {*a*..*t*} *f*) < *e*
proof −
obtain *w* **where** *w* > *0* **and** *w*: ⋀*t*. ⟦*c* − *w* < *t*; *t* < *c*⟧ ⟹ *norm* (*f c*) ∗ *norm*(*c* − *t*) < *e*/*3*
proof (*cases f c* = *0*)
case *False*
hence *e3*: *0* < *e*/*3* / *norm* (*f c*) **using** ‹*e*>*0*› **by** *simp*
moreover have *norm* (*f c*) ∗ *norm* (*c* − *t*) < *e*/*3*
if *t* < *c* **and** *c* − *e*/*3* / *norm* (*f c*) < *t* **for** *t*
proof −
have *norm* (*c* − *t*) < *e*/*3* / *norm* (*f c*)
using *that* **by** *auto*
then show *norm* (*f c*) ∗ *norm* (*c* − *t*) < *e*/*3*
by (*metis e3 mult.commute norm_not_less_zero pos_less_divide_eq zero_less_divide_iff*)
qed
ultimately show *?thesis*
using *that* **by** *auto*
next
case *True* **then show** *?thesis*
using ‹*e* > *0*› *that* **by** *auto*
qed

let *?SUM* = λ*p*. (∑(*x*,*K*) ∈ *p*. *content K* ∗$_R$ *f x*)
have *e3*: *e*/*3* > *0*
using ‹*e* > *0*› **by** *auto*

```
  have f integrable_on {a..c}
    using ‹a < c› ‹c ≤ b› by (auto intro: integrable_subinterval_real[OF intf])
  then obtain d1 where gauge d1 and d1:
      ⋀p. ⟦p tagged_division_of {a..c}; d1 fine p⟧ ⟹ norm (?SUM p − integral
{a..c} f) < e/3
    using integrable_integral has_integral_real e3 by metis
  define d where [abs_def]: d x = ball x w ∩ d1 x for x
  have gauge d
    unfolding d_def using ‹w > 0› ‹gauge d1› by auto
  then obtain k where 0 < k and k: ball c k ⊆ d c
    by (meson gauge_def open_contains_ball)

  let ?d = min k (c − a)/2
  show thesis
  proof (intro that[of ?d] allI impI, safe)
    show ?d > 0
      using ‹0 < k› ‹a < c› by auto
  next
    fix t
    assume t: c − ?d < t t ≤ c
    show norm (integral ({a..c}) f − integral ({a..t}) f) < e
    proof (cases t < c)
      case False with ‹t ≤ c› show ?thesis
        by (simp add: ‹e > 0›)
    next
      case True
      have f integrable_on {a..t}
        using ‹t < c› ‹c ≤ b› by (auto intro: integrable_subinterval_real[OF intf])
      then obtain d2 where d2: gauge d2
        ⋀p. p tagged_division_of {a..t} ∧ d2 fine p ⟹ norm (?SUM p − integral
{a..t} f) < e/3
        using integrable_integral has_integral_real e3 by metis
      define d3 where d3 x = (if x ≤ t then d1 x ∩ d2 x else d1 x) for x
      have gauge d3
        using ‹gauge d1› ‹gauge d2› unfolding d3_def gauge_def by auto
      then obtain p where ptag: p tagged_division_of {a..t} and pfine: d3 fine p
        by (metis box_real(2) fine_division_exists)
      note p' = tagged_division_ofD[OF ptag]
      have pt: (x,K)∈p ⟹ x ≤ t for x K
        by (meson atLeastAtMost_iff p'(2) p'(3) subsetCE)
      with pfine have d2 fine p
        unfolding fine_def d3_def by fastforce
      then have d2_fin: norm (?SUM p − integral {a..t} f) < e/3
        using d2(2) ptag by auto
      have eqs: {a..c} ∩ {x. x ≤ t} = {a..t} {a..c} ∩ {x. x ≥ t} = {t..c}
        using t by (auto simp add: field_simps)
      have p ∪ {(c, {t..c})} tagged_division_of {a..c}
      proof (intro tagged_division_Un_interval_real)
        show {(c, {t..c})} tagged_division_of {a..c} ∩ {x. t ≤ x · 1}
```

```
        using ‹t ≤ c› by (auto simp: eqs tagged_division_of_self_real)
    qed (auto simp: eqs ptag)
    moreover
    have d1 fine p ∪ {(c, {t..c})}
      unfolding fine_def
    proof safe
      fix x K y
      assume (x,K) ∈ p and y ∈ K then show y ∈ d1 x
        by (metis Int_iff d3_def subsetD fineD pfine)
    next
      fix x assume x ∈ {t..c}
      then have dist c x < k
        using t(1) by (auto simp add: field_simps dist_real_def)
      with k show x ∈ d1 c
        unfolding d_def by auto
    qed
    ultimately have d1_fin: norm (?SUM(p ∪ {(c, {t..c})}) − integral {a..c}
f) < e/3
      using d1 by metis
    have SUMEQ: ?SUM(p ∪ {(c, {t..c})}) = (c − t) *R f c + ?SUM p
    proof −
      have ?SUM(p ∪ {(c, {t..c})}) = (content{t..c} *R f c) + ?SUM p
      proof (subst sum.union_disjoint)
        show p ∩ {(c, {t..c})} = {}
          using ‹t < c› pt by force
      qed (use p'(1) in auto)
      also have ... = (c − t) *R f c + ?SUM p
        using ‹t ≤ c› by auto
      finally show ?thesis .
    qed
    have c − k < t
      using ‹k>0› t(1) by (auto simp add: field_simps)
    moreover have k ≤ w
    proof (rule ccontr)
      assume ¬ k ≤ w
      then have c + (k + w) / 2 ∉ d c
        by (auto simp add: field_simps not_le not_less dist_real_def d_def)
      then have c + (k + w) / 2 ∉ ball c k
        using k by blast
      then show False
        using ‹0 < w› ‹¬ k ≤ w› dist_real_def by auto
    qed
    ultimately have cwt: c − w < t
      by (auto simp add: field_simps)
    have eq: integral {a..c} f − integral {a..t} f = −(((c − t) *R f c + ?SUM
p) −
          integral {a..c} f) + (?SUM p − integral {a..t} f) + (c − t) *R f c
      by auto
    have norm (integral {a..c} f − integral {a..t} f) < e/3 + e/3 + e/3
```

```
        unfolding eq
      proof (intro norm_triangle_lt add_strict_mono)
        show norm (− ((c − t) *R f c + ?SUM p − integral {a..c} f)) < e/3
          by (metis SUMEQ d1_fin norm_minus_cancel)
        show norm (?SUM p − integral {a..t} f) < e/3
          using d2_fin by blast
        show norm ((c − t) *R f c) < e/3
          using w cwt ‹t < c› by simp (simp add: field_simps)
      qed
      then show ?thesis by simp
    qed
  qed
qed

lemma indefinite_integral_continuous_right:
  fixes f :: real ⇒ 'a::banach
  assumes f integrable_on {a..b}
    and a ≤ c
    and c < b
    and e > 0
  obtains d where 0 < d
    and ∀ t. c ≤ t ∧ t < c + d ⟶ norm (integral {a..c} f − integral {a..t} f)
< e
proof −
  have intm: (λx. f (− x)) integrable_on {−b .. −a} − b < − c − c ≤ − a
    using assms by auto
  from indefinite_integral_continuous_left[OF intm ‹e>0›]
  obtain d where 0 < d
    and d: ⋀t. ⟦− c − d < t; t ≤ −c⟧
             ⟹ norm (integral {− b..− c} (λx. f (−x)) − integral {− b..t} (λx. f
(−x))) < e
    by metis
  let ?d = min d (b − c)
  show ?thesis
  proof (intro that[of ?d] allI impI)
    show 0 < ?d
      using ‹0 < d› ‹c < b› by auto
    fix t :: real
    assume t: c ≤ t ∧ t < c + ?d
    have *: integral {a..c} f = integral {a..b} f − integral {c..b} f
        integral {a..t} f = integral {a..b} f − integral {t..b} f
      using assms t by (auto simp: algebra_simps integral_combine)
    have (− c) − d < (− t) − t ≤ − c
      using t by auto
    from d[OF this] show norm (integral {a..c} f − integral {a..t} f) < e
      by (auto simp add: algebra_simps norm_minus_commute *)
  qed
qed
```

lemma *indefinite_integral_continuous_1*:
  fixes *f* :: *real* ⇒ *'a*::*banach*
  assumes *f integrable_on* {*a..b*}
  shows *continuous_on* {*a..b*} (λ*x*. *integral* {*a..x*} *f*)
proof −
  have ∃ *d*>*0*. ∀ *x'*∈{*a..b*}. *dist x' x* < *d* ⟶ *dist* (*integral* {*a..x'*} *f*) (*integral* {*a..x*} *f*) < *e*
      if *x*: *x* ∈ {*a..b*} **and** *e* > *0* **for** *x e* :: *real*
  proof (*cases a* = *b*)
    case *True*
    with *that* **show** *?thesis* **by** *force*
  next
    case *False*
    with *x* **have** *a* < *b* **by** *force*
    with *x* **consider** *x* = *a* | *x* = *b* | *a* < *x x* < *b*
      by *force*
    then show *?thesis*
    proof *cases*
      case *1* **then show** *?thesis*
      by (*force simp*: *dist_norm algebra_simps intro*: *indefinite_integral_continuous_right* [*OF assms* _ ‹*a* < *b*› ‹*e* > *0*›])
    next
      case *2* **then show** *?thesis*
        by (*force simp*: *dist_norm norm_minus_commute algebra_simps intro*: *indefinite_integral_continuous_left* [*OF assms* ‹*a* < *b*› _ ‹*e* > *0*›])
    next
      case *3*
      obtain *d1* **where** *0* < *d1*
        and *d1*: ⋀*t*. ⟦*x* − *d1* < *t*; *t* ≤ *x*⟧ ⟹ *norm* (*integral* {*a..x*} *f* − *integral* {*a..t*} *f*) < *e*
        using *3* **by** (*auto intro*: *indefinite_integral_continuous_left* [*OF assms* ‹*a* < *x*› _ ‹*e* > *0*›])
      obtain *d2* **where** *0* < *d2*
        and *d2*: ⋀*t*. ⟦*x* ≤ *t*; *t* < *x* + *d2*⟧ ⟹ *norm* (*integral* {*a..x*} *f* − *integral* {*a..t*} *f*) < *e*
        using *3* **by** (*auto intro*: *indefinite_integral_continuous_right* [*OF assms* _ ‹*x* < *b*› ‹*e* > *0*›])
      show *?thesis*
      proof (*intro exI ballI conjI impI*)
        show *0* < *min d1 d2*
          using ‹*0* < *d1*› ‹*0* < *d2*› **by** *simp*
        show *dist* (*integral* {*a..y*} *f*) (*integral* {*a..x*} *f*) < *e*
            if *y* ∈ {*a..b*} *dist y x* < *min d1 d2* **for** *y*
        proof (*cases y* < *x*)
          case *True*
          with *that d1* **show** *?thesis* **by** (*auto simp*: *dist_commute dist_norm*)
        next
          case *False*
          with *that d2* **show** *?thesis*

```
            by (auto simp: dist_commute dist_norm)
          qed
        qed
      qed
  qed
  then show ?thesis
    by (auto simp: continuous_on_iff)
qed

lemma indefinite_integral_continuous_1':
  fixes f::real ⇒ 'a::banach
  assumes f integrable_on {a..b}
  shows continuous_on {a..b} (λx. integral {x..b} f)
proof -
  have integral {a..b} f - integral {a..x} f = integral {x..b} f if x ∈ {a..b} for x
    using integral_combine[OF _ _ assms, of x] that
    by (auto simp: algebra_simps)
  with _ show ?thesis
   by (rule continuous_on_eq) (auto intro!: continuous_intros indefinite_integral_continuous_1
assms)
qed

theorem integral_has_vector_derivative':
  fixes f :: real ⇒ 'b::banach
  assumes continuous_on {a..b} f
    and x ∈ {a..b}
  shows ((λu. integral {u..b} f) has_vector_derivative - f x) (at x within {a..b})
proof -
  have *: integral {x..b} f = integral {a .. b} f - integral {a .. x} f if a ≤ x x ≤
b for x
     using integral_combine[of a x b for x, OF that integrable_continuous_real[OF
assms(1)]]
    by (simp add: algebra_simps)
  show ?thesis
    using ‹x ∈ _› *
    by (rule has_vector_derivative_transform)
      (auto intro!: derivative_eq_intros assms integral_has_vector_derivative)
qed

lemma integral_has_real_derivative':
  assumes continuous_on {a..b} g
  assumes t ∈ {a..b}
  shows ((λx. integral {x..b} g) has_real_derivative -g t) (at t within {a..b})
  using integral_has_vector_derivative'[OF assms]
  by (auto simp: has_field_derivative_iff_has_vector_derivative)
```

6.15.31 This doesn't directly involve integration, but that gives an easy proof

lemma *has_derivative_zero_unique_strong_interval*:
  fixes *f* :: *real* ⇒ *'a*::*banach*
  assumes *finite k*
    and *contf*: *continuous_on* {*a*..*b*} *f*
    and *f a* = *y*
    and *fder*: $\bigwedge$*x*. *x* ∈ {*a*..*b*} − *k* ⟹ (*f has_derivative* (λ*h*. *0*)) (*at x within* {*a*..*b*})
    and *x*: *x* ∈ {*a*..*b*}
  shows *f x* = *y*
proof −
  have *a* ≤ *b a* ≤ *x*
    using *assms* **by** *auto*
  have ((λ*x*. *0*::*'a*) *has_integral f x* − *f a*) {*a*..*x*}
  proof (*rule fundamental_theorem_of_calculus_interior_strong*[*OF* ‹*finite k*› ‹*a* ≤ *x*›]; *clarify?*)
    have {*a*..*x*} ⊆ {*a*..*b*}
      using *x* **by** *auto*
    then show *continuous_on* {*a*..*x*} *f*
      by (*rule continuous_on_subset*[*OF contf*])
    show (*f has_vector_derivative 0*) (*at z*) **if** *z*: *z* ∈ {*a*<..<*x*} **and** *notin*: *z* ∉ *k* **for** *z*
      unfolding *has_vector_derivative_def*
    proof (*simp add*: *at_within_open*[*OF z*, *symmetric*])
      show (*f has_derivative* (λ*x*. *0*)) (*at z within* {*a*<..<*x*})
        by (*rule has_derivative_subset* [*OF fder*]) (*use x z notin* **in** *auto*)
    qed
  qed
  from *has_integral_unique*[*OF has_integral_0 this*]
  show *?thesis*
    unfolding *assms* **by** *auto*
qed

6.15.32 Generalize a bit to any convex set

lemma *has_derivative_zero_unique_strong_convex*:
  fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*banach*
  assumes *convex S finite K*
    and *contf*: *continuous_on S f*
    and *c* ∈ *S f c* = *y*
    and *derf*: $\bigwedge$*x*. *x* ∈ *S* − *K* ⟹ (*f has_derivative* (λ*h*. *0*)) (*at x within S*)
    and *x* ∈ *S*
  shows *f x* = *y*
proof (*cases x* = *c*)
  case *True* **with** ‹*f c* = *y*› **show** *?thesis*
    by *blast*
next
  case *False*
  let *?φ* = λ*u*. (*1* − *u*) $*_R$ *c* + *u* $*_R$ *x*

```
  have contf': continuous_on {0 ..1} (f ∘ ?φ)
  proof (rule continuous_intros continuous_on_subset[OF contf])+
    show (λu. (1 − u) *R c + u *R x) ` {0..1} ⊆ S
     using ‹convex S› ‹x ∈ S› ‹c ∈ S› by (auto simp add: convex_alt algebra_simps)
  qed
  have t = u if ?φ t = ?φ u for t u
  proof −
    from that have (t − u) *R x = (t − u) *R c
      by (auto simp add: algebra_simps)
    then show ?thesis
      using ‹x ≠ c› by auto
  qed
  then have eq: (SOME t. ?φ t = ?φ u) = u for u
    by blast
  then have (?φ −` K) ⊆ (λz. SOME t. ?φ t = z) ` K
    by (clarsimp simp: image_iff) (metis (no_types) eq)
  then have fin: finite (?φ −` K)
    by (rule finite_surj[OF ‹finite K›])

  have derf': ((λu. f (?φ u)) has_derivative (λh. 0)) (at t within {0..1})
              if t ∈ {0..1} − {t. ?φ t ∈ K} for t
  proof −
    have df: (f has_derivative (λh. 0)) (at (?φ t) within ?φ ` {0..1})
      using ‹convex S› ‹x ∈ S› ‹c ∈ S› that
      by (auto simp add: convex_alt algebra_simps intro: has_derivative_subset [OF
derf])
   have (f ∘ ?φ has_derivative (λx. 0) ∘ (λz. (0 − z *R c) + z *R x)) (at t within
{0..1})
      by (rule derivative_eq_intros df | simp)+
    then show ?thesis
      unfolding o_def .
  qed
  have (f ∘ ?φ) 1 = y
    apply (rule has_derivative_zero_unique_strong_interval[OF fin contf'])
    unfolding o_def using ‹f c = y› derf' by auto
  then show ?thesis
    by auto
qed
```

Also to any open connected set with finite set of exceptions. Could generalize to locally convex set with limpt-free set of exceptions.

```
lemma has_derivative_zero_unique_strong_connected:
  fixes f :: 'a::euclidean_space ⇒ 'b::banach
  assumes connected S
    and open S
    and finite K
    and contf: continuous_on S f
    and c ∈ S
    and f c = y
```

```
    and derf: ⋀x. x ∈ S − K ⟹ (f has_derivative (λh. 0)) (at x within S)
    and x ∈ S
  shows f x = y
proof −
  have ∃ e>0. ball x e ⊆ (S ∩ f −‘ {f x}) if x ∈ S for x
  proof −
    obtain e where 0 < e and e: ball x e ⊆ S
      using ‹x ∈ S› ‹open S› open_contains_ball by blast
    have ball x e ⊆ {u ∈ S. f u ∈ {f x}}
    proof safe
      fix y
      assume y: y ∈ ball x e
      then show y ∈ S
        using e by auto
      show f y = f x
       proof (rule has_derivative_zero_unique_strong_convex[OF convex_ball ‹finite
K›])
        show continuous_on (ball x e) f
          using contf continuous_on_subset e by blast
        show (f has_derivative (λh. 0)) (at u within ball x e)
            if u ∈ ball x e − K for u
          by (metis Diff_iff contra_subsetD derf e has_derivative_subset that)
      qed (use y e ‹0 < e› in auto)
    qed
    then show ∃ e>0. ball x e ⊆ (S ∩ f −‘ {f x})
      using ‹0 < e› by blast
  qed
  then have openin (top_of_set S) (S ∩ f −‘ {y})
    by (auto intro!: open_openin_trans[OF ‹open S›] simp: open_contains_ball)
  moreover have closedin (top_of_set S) (S ∩ f −‘ {y})
    by (force intro!: continuous_closedin_preimage [OF contf])
  ultimately have (S ∩ f −‘ {y}) = {} ∨ (S ∩ f −‘ {y}) = S
    using ‹connected S› by (simp add: connected_clopen)
  then show ?thesis
    using ‹x ∈ S› ‹f c = y› ‹c ∈ S› by auto
qed

lemma has_derivative_zero_connected_constant:
  fixes f :: 'a::euclidean_space ⇒ 'b::banach
  assumes connected S
      and open S
      and finite k
      and continuous_on S f
      and ∀ x∈(S − k). (f has_derivative (λh. 0)) (at x within S)
    obtains c where ⋀x. x ∈ S ⟹ f(x) = c
proof (cases S = {})
  case True
  then show ?thesis
    by (metis empty_iff that)
```

next
  case *False*
  then obtain *c* **where** $c \in S$
    by (*metis equals0I*)
  then show *?thesis*
    by (*metis has_derivative_zero_unique_strong_connected assms that*)
qed

lemma *DERIV_zero_connected_constant*:
  fixes *f* :: *'a*::{*real_normed_field*,*euclidean_space*} ⇒ *'a*
  assumes *connected S*
    and *open S*
    and *finite K*
    and *continuous_on S f*
    and ∀ *x*∈(*S* − *K*). *DERIV f x* :> *0*
  obtains *c* **where** $\bigwedge x.\ x \in S \Longrightarrow f(x) = c$
  using *has_derivative_zero_connected_constant* [*OF assms*(*1*−*4*)] *assms*
  by (*metis DERIV_const has_derivative_const Diff_iff at_within_open frechet_derivative_at has_field_derivative_def*)

6.15.33 Integrating characteristic function of an interval

lemma *has_integral_restrict_open_subinterval*:
  fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*banach*
  assumes *intf*: (*f has_integral i*) (*cbox c d*)
    and *cb*: *cbox c d* ⊆ *cbox a b*
  shows ((λ*x*. *if x* ∈ *box c d then f x else 0*) *has_integral i*) (*cbox a b*)
proof (*cases cbox c d* = {})
  case *True*
  then have *box c d* = {}
    by (*metis bot.extremum_uniqueI box_subset_cbox*)
  then show *?thesis*
    using *True intf* **by** *auto*
next
  case *False*
  then obtain *p* **where** *pdiv*: *p division_of cbox a b* **and** *inp*: *cbox c d* ∈ *p*
    using *cb partial_division_extend_1* **by** *blast*
  define *g* **where** [*abs_def*]: *g x* = (*if x* ∈*box c d then f x else 0*) **for** *x*
  interpret *operative lift_option plus Some* (*0* :: *'b*)
    λ*i*. *if g integrable_on i then Some* (*integral i g*) *else None*
    by (*fact operative_integralI*)
  note *operat* = *division* [*OF pdiv*, *symmetric*]
  show *?thesis*
  proof (*cases* (*g has_integral i*) (*cbox a b*))
    case *True* **then show** *?thesis*
      by (*simp add*: *g_def*)
  next
    case *False*
    have *iterate*:*F* (λ*i*. *if g integrable_on i then Some* (*integral i g*) *else None*) (*p*

− {*cbox c d*}) = *Some 0*
  proof (*intro neutral ballI*)
   fix *x*
   assume *x*: *x* ∈ *p* − {*cbox c d*}
   then have *x* ∈ *p*
    by *auto*
   then obtain *u v* **where** *uv*: *x* = *cbox u v*
    using *pdiv* **by** *blast*
   have *interior x* ∩ *interior* (*cbox c d*) = {}
    using *pdiv inp x* **by** *blast*
   then have (*g has_integral 0*) *x*
    unfolding *uv* **using** *has_integral_spike_interior*[**where** *f*=λ*x*. *0*]
     by (*metis* (*no_types*, *lifting*) *disjoint_iff_not_equal g_def has_integral_0_eq interior_cbox*)
   then show (*if g integrable_on x then Some* (*integral x g*) *else None*) = *Some 0*
    by *auto*
  qed
  interpret *comm_monoid_set lift_option plus Some* (*0* :: ′*b*)
  by (*intro comm_monoid_set.intro comm_monoid_lift_option add.comm_monoid_axioms*)
  have *intg*: *g integrable_on cbox c d*
   using *integrable_spike_interior*[**where** *f*=*f*]
   by (*meson g_def has_integral_integrable intf*)
  moreover have *integral* (*cbox c d*) *g* = *i*
  proof (*rule has_integral_unique*[*OF has_integral_spike_interior intf*])
   show ⋀*x*. *x* ∈ *box c d* ⟹ *f x* = *g x*
    by (*auto simp*: *g_def*)
   show (*g has_integral integral* (*cbox c d*) *g*) (*cbox c d*)
    by (*rule integrable_integral*[*OF intg*])
  qed
  ultimately have *F* (λ*A*. *if g integrable_on A then Some* (*integral A g*) *else None*) *p* = *Some i*
  by (*metis* (*full_types*, *lifting*) *division_of_finite inp iterate pdiv remove right_neutral*)
  then
  have (*g has_integral i*) (*cbox a b*)
  by (*metis integrable_on_def integral_unique operat option.inject option.simps*(*3*))
  with *False* **show** *?thesis*
   by *blast*
 qed
qed

lemma *has_integral_restrict_closed_subinterval*:
 fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*banach*
 assumes (*f has_integral i*) (*cbox c d*)
  and *cbox c d* ⊆ *cbox a b*
 shows ((λ*x*. *if x* ∈ *cbox c d then f x else 0*) *has_integral i*) (*cbox a b*)
proof −
 note *has_integral_restrict_open_subinterval*[*OF assms*]

```
  note * = has_integral_spike[OF negligible_frontier_interval _ this]
  show ?thesis
    by (rule *[of c d]) (use box_subset_cbox[of c d] in auto)
qed

lemma has_integral_restrict_closed_subintervals_eq:
  fixes f :: 'a::euclidean_space ⇒ 'b::banach
  assumes cbox c d ⊆ cbox a b
  shows ((λx. if x ∈ cbox c d then f x else 0) has_integral i) (cbox a b) ⟷ (f
has_integral i) (cbox c d)
  (is ?l = ?r)
proof (cases cbox c d = {})
  case False
  let ?g = λx. if x ∈ cbox c d then f x else 0
  show ?thesis
  proof
    assume ?l
    then have ?g integrable_on cbox c d
      using assms has_integral_integrable integrable_subinterval by blast
    then have f integrable_on cbox c d
      by (rule integrable_eq) auto
    moreover then have i = integral (cbox c d) f
     by (meson ‹((λx. if x ∈ cbox c d then f x else 0) has_integral i) (cbox a b)› assms
has_integral_restrict_closed_subinterval has_integral_unique integrable_integral)
    ultimately show ?r by auto
  next
    assume ?r then show ?l
      by (rule has_integral_restrict_closed_subinterval[OF _ assms])
  qed
qed auto
```

Hence we can apply the limit process uniformly to all integrals.

```
lemma has_integral':
  fixes f :: 'n::euclidean_space ⇒ 'a::banach
  shows (f has_integral i) S ⟷
   (∀ e>0. ∃ B>0. ∀ a b. ball 0 B ⊆ cbox a b ⟶
     (∃ z. ((λx. if x ∈ S then f(x) else 0) has_integral z) (cbox a b) ∧ norm(z −
i) < e))
  (is ?l ⟷ (∀ e>0. ?r e))
proof (cases ∃ a b. S = cbox a b)
  case False then show ?thesis
    by (simp add: has_integral_alt)
next
  case True
  then obtain a b where S: S = cbox a b
    by blast
  obtain B where  0 < B and B: ⋀x. x ∈ cbox a b ⟹ norm x ≤ B
    using bounded_cbox[unfolded bounded_pos] by blast
  show ?thesis
```

```
proof safe
  fix e :: real
  assume ?l and e > 0
  have ((λx. if x ∈ S then f x else 0) has_integral i) (cbox c d)
    if ball 0 (B+1) ⊆ cbox c d for c d
      unfolding S using B that
      by (force intro: ‹?l›[unfolded S] has_integral_restrict_closed_subinterval)
  then show ?r e
   by (meson ‹0 < B› ‹0 < e› add_pos_pos le_less_trans zero_less_one norm_pths(2))
next
  assume as: ∀ e>0. ?r e
  then obtain C
    where C: ⋀a b. ball 0 C ⊆ cbox a b ⟹
             ∃ z. ((λx. if x ∈ S then f x else 0) has_integral z) (cbox a b)
    by (meson zero_less_one)
  define c :: 'n where c = (∑ i∈Basis. (− max B C) *R i)
  define d :: 'n where d = (∑ i∈Basis. max B C *R i)
  have c · i ≤ x · i ∧ x · i ≤ d · i if norm x ≤ B i ∈ Basis for x i
    using that and Basis_le_norm[OF ‹i∈Basis›, of x]
    by (auto simp add: field_simps sum_negf c_def d_def)
  then have c_d: cbox a b ⊆ cbox c d
    by (meson B mem_box(2) subsetI)
  have c · i ≤ x · i ∧ x · i ≤ d · i
    if x: norm (0 − x) < C and i: i ∈ Basis for x i
      using Basis_le_norm[OF i, of x] x i by (auto simp: sum_negf c_def d_def)
    then have ball 0 C ⊆ cbox c d
      by (auto simp: mem_box dist_norm)
  with C obtain y where y: (f has_integral y) (cbox a b)
    using c_d has_integral_restrict_closed_subintervals_eq S by blast
  have y = i
  proof (rule ccontr)
    assume y ≠ i
    then have 0 < norm (y − i)
      by auto
    from as[rule_format,OF this]
    obtain C where C: ⋀a b. ball 0 C ⊆ cbox a b ⟹
       ∃ z. ((λx. if x ∈ S then f x else 0) has_integral z) (cbox a b) ∧ norm (z−i)
< norm (y−i)
      by auto
    define c :: 'n where c = (∑ i∈Basis. (− max B C) *R i)
    define d :: 'n where d = (∑ i∈Basis. max B C *R i)
    have c · i ≤ x · i ∧ x · i ≤ d · i
      if norm x ≤ B and i ∈ Basis for x i
        using that Basis_le_norm[of i x] by (auto simp add: field_simps sum_negf
c_def d_def)
      then have c_d: cbox a b ⊆ cbox c d
      by (simp add: B mem_box(2) subset_eq)
   have c · i ≤ x · i ∧ x · i ≤ d · i if norm (0 − x) < C and i ∈ Basis for x i
      using Basis_le_norm[of i x] that by (auto simp: sum_negf c_def d_def)
```

```
      then have ball 0 C ⊆ cbox c d
        by (auto simp: mem_box dist_norm)
      with C obtain z where z: (f has_integral z) (cbox a b) norm (z−i) < norm
(y−i)
        using has_integral_restrict_closed_subintervals_eq[OF c_d] S by blast
      moreover then have z = y
        by (blast intro: has_integral_unique[OF _ y])
      ultimately show False
        by auto
    qed
    then show ?l
      using y by (auto simp: S)
  qed
qed

lemma has_integral_le:
  fixes f :: 'n::euclidean_space ⇒ real
  assumes fg: (f has_integral i) S (g has_integral j) S
    and le: ⋀x. x ∈ S ⟹ f x ≤ g x
  shows i ≤ j
  using has_integral_component_le[OF _ fg, of 1] le  by auto

lemma integral_le:
  fixes f :: 'n::euclidean_space ⇒ real
  assumes f integrable_on S
    and g integrable_on S
    and ⋀x. x ∈ S ⟹ f x ≤ g x
  shows integral S f ≤ integral S g
  by (rule has_integral_le[OF assms(1,2)[unfolded has_integral_integral] assms(3)])

lemma has_integral_nonneg:
  fixes f :: 'n::euclidean_space ⇒ real
  assumes (f has_integral i) S
    and ⋀x. x ∈ S ⟹ 0 ≤ f x
  shows 0 ≤ i
  using has_integral_component_nonneg[of 1 f i S]
  unfolding o_def
  using assms
  by auto

lemma integral_nonneg:
  fixes f :: 'n::euclidean_space ⇒ real
  assumes f: f integrable_on S and 0: ⋀x. x ∈ S ⟹ 0 ≤ f x
  shows 0 ≤ integral S f
  by (rule has_integral_nonneg[OF f[unfolded has_integral_integral] 0])
```

Hence a general restriction property.

```
lemma has_integral_restrict [simp]:
  fixes f :: 'a :: euclidean_space ⇒ 'b :: banach
```

assumes $S \subseteq T$
shows $((\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0)\ has\_integral\ i)\ T \longleftrightarrow (f\ has\_integral\ i)\ S$
proof −
have ∗: $\bigwedge x.\ (if\ x \in T\ then\ if\ x \in S\ then\ f\ x\ else\ 0\ else\ 0) = (if\ x{\in}S\ then\ f\ x\ else\ 0)$
using *assms* **by** *auto*
show *?thesis*
apply (*subst*(*2*) *has_integral′*)
apply (*subst has_integral′*)
apply (*simp add*: ∗)
done
qed

corollary *has_integral_restrict_UNIV*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
shows $((\lambda x.\ if\ x \in s\ then\ f\ x\ else\ 0)\ has\_integral\ i)\ UNIV \longleftrightarrow (f\ has\_integral\ i)\ s$
by *auto*

lemma *has_integral_restrict_Int*:
fixes $f :: 'a :: euclidean\_space \Rightarrow 'b :: banach$
shows $((\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0)\ has\_integral\ i)\ T \longleftrightarrow (f\ has\_integral\ i)\ (S \cap T)$
proof −
have $((\lambda x.\ if\ x \in T\ then\ if\ x \in S\ then\ f\ x\ else\ 0\ else\ 0)\ has\_integral\ i)\ UNIV =$
$((\lambda x.\ if\ x \in S \cap T\ then\ f\ x\ else\ 0)\ has\_integral\ i)\ UNIV$
by (*rule has_integral_cong*) *auto*
then show *?thesis*
using *has_integral_restrict_UNIV* **by** *fastforce*
qed

lemma *integral_restrict_Int*:
fixes $f :: 'a :: euclidean\_space \Rightarrow 'b :: banach$
shows $integral\ T\ (\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0) = integral\ (S \cap T)\ f$
by (*metis* (*no_types*, *lifting*) *has_integral_cong has_integral_restrict_Int integrable_integral integral_unique not_integrable_integral*)

lemma *integrable_restrict_Int*:
fixes $f :: 'a :: euclidean\_space \Rightarrow 'b :: banach$
shows $(\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0)\ integrable\_on\ T \longleftrightarrow f\ integrable\_on\ (S \cap T)$
using *has_integral_restrict_Int* **by** *fastforce*

lemma *has_integral_on_superset*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
assumes f: $(f\ has\_integral\ i)\ S$
and $\bigwedge x.\ x \notin S \Longrightarrow f\ x = 0$
and $S \subseteq T$
shows $(f\ has\_integral\ i)\ T$
proof −

have (λx. *if* x ∈ S *then* f x *else* 0) = (λx. *if* x ∈ T *then* f x *else* 0)
  using *assms* **by** *fastforce*
 with f **show** *?thesis*
  by (*simp only*: *has_integral_restrict_UNIV* [*symmetric*, *of* f])
qed

lemma *integrable_on_superset*:
 fixes f :: 'n::*euclidean_space* ⇒ 'a::*banach*
 assumes f *integrable_on* S
  and ⋀x. x ∉ S ⟹ f x = 0
  and S ⊆ t
 shows f *integrable_on* t
 using *assms*
 unfolding *integrable_on_def*
 by (*auto intro*:*has_integral_on_superset*)

lemma *integral_restrict_UNIV*:
 fixes f :: 'n::*euclidean_space* ⇒ 'a::*banach*
 shows *integral UNIV* (λx. *if* x ∈ S *then* f x *else* 0) = *integral* S f
 by (*simp add*: *integral_restrict_Int*)

lemma *integrable_restrict_UNIV*:
 fixes f :: 'n::*euclidean_space* ⇒ 'a::*banach*
 shows (λx. *if* x ∈ s *then* f x *else* 0) *integrable_on UNIV* ⟷ f *integrable_on* s
 unfolding *integrable_on_def*
 by *auto*

lemma *has_integral_subset_component_le*:
 fixes f :: 'n::*euclidean_space* ⇒ 'm::*euclidean_space*
 assumes k: k ∈ *Basis*
  and *as*: S ⊆ T (f *has_integral* i) S (f *has_integral* j) T ⋀x. x∈T ⟹ 0 ≤
f(x)•k
 shows i•k ≤ j•k
proof −
 have §: ((λx. *if* x ∈ S *then* f x *else* 0) *has_integral* i) *UNIV*
    ((λx. *if* x ∈ T *then* f x *else* 0) *has_integral* j) *UNIV*
  by (*simp_all add*: *assms*)
 show *?thesis*
  using *as* **by** (*force intro*!: *has_integral_component_le*[*OF* k §])
qed

6.15.34 Integrals on set differences

lemma *has_integral_setdiff*:
 fixes f :: 'a::*euclidean_space* ⇒ 'b::*banach*
 assumes S: (f *has_integral* i) S **and** T: (f *has_integral* j) T
  and *neg*: *negligible* (T − S)
 shows (f *has_integral* (i − j)) (S − T)
proof −

show *?thesis*
  unfolding *has_integral_restrict_UNIV* [*symmetric*, *of f*]
 proof (*rule has_integral_spike* [*OF neg*])
  have *eq*: (λ*x*. (*if x* ∈ *S then f x else 0*) − (*if x* ∈ *T then f x else 0*)) =
    (λ*x*. *if x* ∈ *T* − *S then* − *f x else if x* ∈ *S* − *T then f x else 0*)
   by (*force simp add*:)
  have ((λ*x*. *if x* ∈ *S then f x else 0*) *has_integral i*) *UNIV*
   ((λ*x*. *if x* ∈ *T then f x else 0*) *has_integral j*) *UNIV*
   using *S T has_integral_restrict_UNIV* **by** *auto*
  from *has_integral_diff* [*OF this*]
  show ((λ*x*. *if x* ∈ *T* − *S then* − *f x else if x* ∈ *S* − *T then f x else 0*)
     has_integral i−*j*) *UNIV*
   by (*simp add*: *eq*)
 qed *force*
qed

lemma *integral_setdiff*:
 fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*banach*
 assumes *f integrable_on S f integrable_on T negligible*(*T* − *S*)
 shows *integral* (*S* − *T*) *f* = *integral S f* − *integral T f*
 by (*rule integral_unique*) (*simp add*: *assms has_integral_setdiff integrable_integral*)

lemma *integrable_setdiff*:
 fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*banach*
 assumes (*f has_integral i*) *S* (*f has_integral j*) *T negligible* (*T* − *S*)
 shows *f integrable_on* (*S* − *T*)
 using *has_integral_setdiff* [*OF assms*]
 by (*simp add*: *has_integral_iff*)

lemma *negligible_setdiff* [*simp*]: *T* ⊆ *S* ⟹ *negligible* (*T* − *S*)
 by (*metis Diff_eq_empty_iff negligible_empty*)

lemma *negligible_on_intervals*: *negligible s* ⟷ (∀ *a b*. *negligible*(*s* ∩ *cbox a b*)) (**is** *?l* ⟷ *?r*)
proof
 assume *R*: *?r*
 show *?l*
  unfolding *negligible_def*
 proof *safe*
  fix *a b*
  have *negligible* (*s* ∩ *cbox a b*)
   by (*simp add*: *R*)
  then show (*indicator s has_integral 0*) (*cbox a b*)
   by (*meson Diff_iff Int_iff has_integral_negligible indicator_simps*(*2*))
 qed
qed (*simp add*: *negligible_Int*)

lemma *negligible_translation*:
 assumes *negligible S*

```
    shows negligible ((+) c ` S)
proof -
  have inj: inj ((+) c)
    by simp
  show ?thesis
  using assms
  proof (clarsimp simp: negligible_def)
    fix a b
    assume ∀x y. (indicator S has_integral 0) (cbox x y)
    then have *: (indicator S has_integral 0) (cbox (a-c) (b-c))
      by (meson Diff_iff assms has_integral_negligible indicator_simps(2))
    have eq: indicator ((+) c ` S) = (λx. indicator S (x - c))
      by (force simp add: indicator_def)
    show (indicator ((+) c ` S) has_integral 0) (cbox a b)
      using has_integral_affinity [OF *, of 1 -c]
            cbox_translation [of c -c+a -c+b]
      by (simp add: eq) (simp add: ac_simps)
  qed
qed

lemma negligible_translation_rev:
  assumes negligible ((+) c ` S)
    shows negligible S
by (metis negligible_translation [OF assms, of -c] translation_galois)

lemma has_integral_spike_set_eq:
  fixes f :: 'n::euclidean_space ⇒ 'a::banach
  assumes negligible {x ∈ S - T. f x ≠ 0} negligible {x ∈ T - S. f x ≠ 0}
  shows (f has_integral y) S ⟷ (f has_integral y) T
proof -
  have ((λx. if x ∈ S then f x else 0) has_integral y) UNIV =
        ((λx. if x ∈ T then f x else 0) has_integral y) UNIV
  proof (rule has_integral_spike_eq)
    show negligible ({x ∈ S - T. f x ≠ 0} ∪ {x ∈ T - S. f x ≠ 0})
      by (rule negligible_Un [OF assms])
  qed auto
  then show ?thesis
    by (simp add: has_integral_restrict_UNIV)
qed

corollary integral_spike_set:
  fixes f :: 'n::euclidean_space ⇒ 'a::banach
  assumes negligible {x ∈ S - T. f x ≠ 0} negligible {x ∈ T - S. f x ≠ 0}
  shows integral S f = integral T f
  using has_integral_spike_set_eq [OF assms]
  by (metis eq_integralD integral_unique)

lemma integrable_spike_set:
  fixes f :: 'n::euclidean_space ⇒ 'a::banach
```

assumes *f*: *f integrable_on S* **and** *neg*: *negligible* {*x* ∈ *S* − *T*. *f x* ≠ *0*} *negligible* {*x* ∈ *T* − *S*. *f x* ≠ *0*}
shows *f integrable_on T*
using *has_integral_spike_set_eq* [*OF neg*] *f* **by** *blast*

lemma *integrable_spike_set_eq*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*banach*
assumes *negligible* ((*S* − *T*) ∪ (*T* − *S*))
shows *f integrable_on S* ⟷ *f integrable_on T*
by (*blast intro*: *integrable_spike_set assms negligible_subset*)

lemma *integrable_on_insert_iff*: *f integrable_on* (*insert x X*) ⟷ *f integrable_on X*
for *f*::_ ⇒ *'a*::*banach*
by (*rule integrable_spike_set_eq*) (*auto simp*: *insert_Diff_if*)

lemma *has_integral_interior*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *banach*
shows *negligible*(*frontier S*) ⟹ (*f has_integral y*) (*interior S*) ⟷ (*f has_integral y*) *S*
by (*rule has_integral_spike_set_eq* [*OF empty_imp_negligible negligible_subset*])
(*use interior_subset* **in** ‹*auto simp*: *frontier_def closure_def*›)

lemma *has_integral_closure*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *banach*
shows *negligible*(*frontier S*) ⟹ (*f has_integral y*) (*closure S*) ⟷ (*f has_integral y*) *S*
by (*rule has_integral_spike_set_eq* [*OF negligible_subset empty_imp_negligible*]) (*auto simp*: *closure_Un_frontier*)

lemma *has_integral_open_interval*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *banach*
shows (*f has_integral y*) (*box a b*) ⟷ (*f has_integral y*) (*cbox a b*)
unfolding *interior_cbox* [*symmetric*]
by (*metis frontier_cbox has_integral_interior negligible_frontier_interval*)

lemma *integrable_on_open_interval*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *banach*
shows *f integrable_on box a b* ⟷ *f integrable_on cbox a b*
by (*simp add*: *has_integral_open_interval integrable_on_def*)

lemma *integral_open_interval*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *banach*
shows *integral*(*box a b*) *f* = *integral*(*cbox a b*) *f*
by (*metis has_integral_integrable_integral has_integral_open_interval not_integrable_integral*)

6.15.35 More lemmas that are useful later

lemma *has_integral_subset_le*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *real*

```
  assumes s ⊆ t
    and (f has_integral i) s
    and (f has_integral j) t
    and ∀x∈t. 0 ≤ f x
  shows i ≤ j
  using has_integral_subset_component_le[OF _ assms(1), of 1 f i j]
  using assms
  by auto

lemma integral_subset_component_le:
  fixes f :: 'n::euclidean_space ⇒ 'm::euclidean_space
  assumes k ∈ Basis
    and s ⊆ t
    and f integrable_on s
    and f integrable_on t
    and ∀x ∈ t. 0 ≤ f x • k
  shows (integral s f)•k ≤ (integral t f)•k
  by (meson assms has_integral_subset_component_le integrable_integral)

lemma integral_subset_le:
  fixes f :: 'n::euclidean_space ⇒ real
  assumes s ⊆ t
    and f integrable_on s
    and f integrable_on t
    and ∀x ∈ t. 0 ≤ f x
  shows integral s f ≤ integral t f
  using assms has_integral_subset_le by blast

lemma has_integral_alt':
  fixes f :: 'n::euclidean_space ⇒ 'a::banach
  shows (f has_integral i) s ⟷
        (∀a b. (λx. if x ∈ s then f x else 0) integrable_on cbox a b) ∧
        (∀e>0. ∃B>0. ∀a b. ball 0 B ⊆ cbox a b ⟶
          norm (integral (cbox a b) (λx. if x ∈ s then f x else 0) − i) < e)
  (is ?l = ?r)
proof
  assume rhs: ?r
  show ?l
  proof (subst has_integral', intro allI impI)
    fix e::real
    assume e > 0
    from rhs[THEN conjunct2,rule_format,OF this]
    show ∃B>0. ∀a b. ball 0 B ⊆ cbox a b ⟶
                (∃z. ((λx. if x ∈ s then f x else 0) has_integral z)
                     (cbox a b) ∧ norm (z − i) < e)
      by (simp add: has_integral_iff rhs)
  qed
next
  let ?Φ = λe a b. ∃z. ((λx. if x ∈ s then f x else 0) has_integral z) (cbox a b) ∧
```

norm (*z* − *i*) < *e*
  assume *?l*
  then have *lhs*: ∃ *B*>*0*. ∀ *a b*. *ball 0 B* ⊆ *cbox a b* ⟶ *?Φ e a b* **if** *e* > *0* **for** *e*
    using *that has_integral'*[*of f*] **by** *auto*
  let *?f* = λ*x*. *if x* ∈ *s then f x else 0*
  show *?r*
  proof (*intro conjI allI impI*)
    fix *a b* :: *'n*
    from *lhs*[*OF zero_less_one*]
    obtain *B* **where** *0* < *B* **and** *B*: ⋀*a b*. *ball 0 B* ⊆ *cbox a b* ⟹ *?Φ 1 a b*
      by *blast*
    let *?a* = ∑ *i*∈*Basis*. *min* (*a*•*i*) (−*B*) $*_R$ *i*::*'n*
    let *?b* = ∑ *i*∈*Basis*. *max* (*b*•*i*) *B* $*_R$ *i*::*'n*
    show *?f integrable_on cbox a b*
    proof (*rule integrable_subinterval*[*of _ ?a ?b*])
      have *?a* • *i* ≤ *x* • *i* ∧ *x* • *i* ≤ *?b* • *i* **if** *norm* (*0* − *x*) < *B i* ∈ *Basis* **for** *x i*
        using *Basis_le_norm*[*of i x*] *that* **by** (*auto simp add*:*field_simps*)
      then have *ball 0 B* ⊆ *cbox ?a ?b*
        by (*auto simp*: *mem_box dist_norm*)
      then show *?f integrable_on cbox ?a ?b*
        unfolding *integrable_on_def* **using** *B* **by** *blast*
      show *cbox a b* ⊆ *cbox ?a ?b*
        by (*force simp*: *mem_box*)
    qed

    fix *e* :: *real*
    assume *e* > *0*
    with *lhs* **show** ∃ *B*>*0*. ∀ *a b*. *ball 0 B* ⊆ *cbox a b* ⟶
      norm (*integral* (*cbox a b*) (λ*x*. *if x* ∈ *s then f x else 0*) − *i*) < *e*
      by (*metis* (*no_types*, *lifting*) *has_integral_integrable_integral*)
  qed
qed

6.15.36 Continuity of the integral (for a 1-dimensional interval)

lemma *integrable_alt*:
  fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*banach*
  shows *f integrable_on s* ⟷
    (∀ *a b*. (λ*x*. *if x* ∈ *s then f x else 0*) *integrable_on cbox a b*) ∧
    (∀ *e*>*0*. ∃ *B*>*0*. ∀ *a b c d*. *ball 0 B* ⊆ *cbox a b* ∧ *ball 0 B* ⊆ *cbox c d* ⟶
    norm (*integral* (*cbox a b*) (λ*x*. *if x* ∈ *s then f x else 0*) −
      integral (*cbox c d*) (λ*x*. *if x* ∈ *s then f x else 0*)) < *e*)
  (**is** *?l* = *?r*)
proof
  let *?F* = λ*x*. *if x* ∈ *s then f x else 0*
  assume *?l*
  then obtain *y* **where** *intF*: ⋀*a b*. *?F integrable_on cbox a b*
        and *y*: ⋀*e*. *0* < *e* ⟹

```
      ∃B>0. ∀a b. ball 0 B ⊆ cbox a b ⟶ norm (integral (cbox a b) ?F −
y) < e
    unfolding integrable_on_def has_integral_alt'[of f] by auto
  show ?r
  proof (intro conjI allI impI intF)
    fix e::real
    assume e > 0
    then have e/2 > 0
      by auto
    obtain B where 0 < B
      and B: ⋀a b. ball 0 B ⊆ cbox a b ⟹ norm (integral (cbox a b) ?F − y) <
e/2
      using ‹0 < e/2› y by blast
    show ∃B>0. ∀a b c d. ball 0 B ⊆ cbox a b ∧ ball 0 B ⊆ cbox c d ⟶
              norm (integral (cbox a b) ?F − integral (cbox c d) ?F) < e
    proof (intro conjI exI impI allI, rule ‹0 < B›)
      fix a b c d::'n
      assume sub: ball 0 B ⊆ cbox a b ∧ ball 0 B ⊆ cbox c d
      show norm (integral (cbox a b) ?F − integral (cbox c d) ?F) < e
        using sub by (auto intro: norm_triangle_half_l dest: B)
    qed
  qed
next
  let ?F = λx. if x ∈ s then f x else 0
  assume rhs: ?r
  let ?cube = λn. cbox (∑i∈Basis. − real n *R i::'n) (∑i∈Basis. real n *R i)
  have Cauchy (λn. integral (?cube n) ?F)
    unfolding Cauchy_def
  proof (intro allI impI)
    fix e::real
    assume e > 0
    with rhs obtain B where 0 < B
      and B: ⋀a b c d. ball 0 B ⊆ cbox a b ∧ ball 0 B ⊆ cbox c d
                 ⟹ norm (integral (cbox a b) ?F − integral (cbox c d) ?F) < e
      by blast
    obtain N where N: B ≤ real N
      using real_arch_simple by blast
    have ball 0 B ⊆ ?cube n if n: n ≥ N for n
    proof −
      have sum ((*R) (− real n)) Basis • i ≤ x • i ∧
            x • i ≤ sum ((*R) (real n)) Basis • i
        if norm x < B i ∈ Basis for x i::'n
            using Basis_le_norm[of i x] n N that by (auto simp add: field_simps
sum_negf)
      then show ?thesis
        by (auto simp: mem_box dist_norm)
    qed
   then show ∃M. ∀m≥M. ∀n≥M. dist (integral (?cube m) ?F) (integral (?cube
n) ?F) < e
```

by (*fastforce simp add*: *dist_norm intro*!: *B*)
qed
then obtain *i* **where** *i*: (λ*n*. *integral* (*?cube n*) *?F*) ⟶ *i*
using *convergent_eq_Cauchy* **by** *blast*
have ∃ *B*>*0*. ∀ *a b*. *ball 0 B* ⊆ *cbox a b* ⟶ *norm* (*integral* (*cbox a b*) *?F* − *i*) < *e*
if *e* > *0* **for** *e*
proof −
have ∗: *e*/*2* > *0* **using** *that* **by** *auto*
then obtain *N* **where** *N*: ⋀*n*. *N* ≤ *n* ⟹ *norm* (*i* − *integral* (*?cube n*) *?F*) < *e*/*2*
using *i*[*THEN LIMSEQ_D*, *simplified norm_minus_commute*] **by** *meson*
obtain *B* **where** *0* < *B*
and *B*: ⋀*a b c d*. ⟦*ball 0 B* ⊆ *cbox a b*; *ball 0 B* ⊆ *cbox c d*⟧ ⟹
norm (*integral* (*cbox a b*) *?F* − *integral* (*cbox c d*) *?F*) < *e*/*2*
using *rhs* ∗ **by** *meson*
let *?B* = *max* (*real N*) *B*
show *?thesis*
proof (*intro exI conjI allI impI*)
show *0* < *?B*
using ‹*B* > *0*› **by** *auto*
fix *a b* :: ′*n*
assume *ball 0 ?B* ⊆ *cbox a b*
moreover obtain *n* **where** *n*: *max* (*real N*) *B* ≤ *real n*
using *real_arch_simple* **by** *blast*
moreover have *ball 0 B* ⊆ *?cube n*
proof
fix *x* :: ′*n*
assume *x*: *x* ∈ *ball 0 B*
have ⟦*norm* (*0* − *x*) < *B*; *i* ∈ *Basis*⟧
⟹ *sum* ((∗$_R$) (−*n*)) *Basis* · *i*≤ *x* · *i* ∧ *x* · *i* ≤ *sum* ((∗$_R$) *n*) *Basis* · *i* **for** *i*
using *Basis_le_norm*[*of i x*] *n* **by** (*auto simp add*: *field_simps sum_negf*)
then show *x* ∈ *?cube n*
using *x* **by** (*auto simp*: *mem_box dist_norm*)
qed
ultimately show *norm* (*integral* (*cbox a b*) *?F* − *i*) < *e*
using *norm_triangle_half_l* [*OF B N*] **by** *force*
qed
qed
then show *?l* **unfolding** *integrable_on_def has_integral_alt*′[*of f*]
using *rhs* **by** *blast*
qed

lemma *integrable_altD*:
fixes *f* :: ′*n*::*euclidean_space* ⇒ ′*a*::*banach*
assumes *f integrable_on s*
shows ⋀*a b*. (λ*x*. *if x* ∈ *s then f x else 0*) *integrable_on cbox a b*
and ⋀*e*. *e* > *0* ⟹ ∃ *B*>*0*. ∀ *a b c d*. *ball 0 B* ⊆ *cbox a b* ∧ *ball 0 B* ⊆ *cbox c*

$d \longrightarrow$
norm (*integral* (*cbox a b*) (λx. *if* $x \in s$ *then f x else 0*) − *integral* (*cbox c d*) (λx. *if* $x \in s$ *then f x else 0*)) < *e*
using *assms*[*unfolded integrable_alt*[*of f*]] **by** *auto*

lemma *integrable_alt_subset*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::banach*
shows
f integrable_on S ⟷
(∀ *a b*. (λx. *if* $x \in S$ *then f x else 0*) *integrable_on cbox a b*) ∧
(∀ *e*>*0*. ∃ *B*>*0*. ∀ *a b c d*.
ball 0 B ⊆ *cbox a b* ∧ *cbox a b* ⊆ *cbox c d*
⟶ *norm*(*integral* (*cbox a b*) (λx. *if* $x \in S$ *then f x else 0*) −
integral (*cbox c d*) (λx. *if* $x \in S$ *then f x else 0*)) < *e*)
(**is** _ = *?rhs*)
proof −
let *?g* = λx. *if* $x \in S$ *then f x else 0*
have *f integrable_on S* ⟷
(∀ *a b*. *?g integrable_on cbox a b*) ∧
(∀ *e*>*0*. ∃ *B*>*0*. ∀ *a b c d*. *ball 0 B* ⊆ *cbox a b* ∧ *ball 0 B* ⊆ *cbox c d* ⟶
norm (*integral* (*cbox a b*) *?g* − *integral* (*cbox c d*) *?g*) < *e*)
by (*rule integrable_alt*)
also have ... = *?rhs*
proof −
{ **fix** *e* :: *real*
assume *e*: ⋀*e*. *e*>*0* ⟹ ∃ *B*>*0*. ∀ *a b c d*. *ball 0 B* ⊆ *cbox a b* ∧ *cbox a b* ⊆ *cbox c d* ⟶
norm (*integral* (*cbox a b*) *?g* − *integral* (*cbox c d*) *?g*) < *e*
and *e* > *0*
obtain *B* **where** *B* > *0*
and *B*: ⋀*a b c d*. ⟦*ball 0 B* ⊆ *cbox a b*; *cbox a b* ⊆ *cbox c d*⟧ ⟹
norm (*integral* (*cbox a b*) *?g* − *integral* (*cbox c d*) *?g*) < *e*/*2*
using ‹*e* > *0*› *e* [*of e*/*2*] **by** *force*
have ∃ *B*>*0*. ∀ *a b c d*.
ball 0 B ⊆ *cbox a b* ∧ *ball 0 B* ⊆ *cbox c d* ⟶
norm (*integral* (*cbox a b*) *?g* − *integral* (*cbox c d*) *?g*) < *e*
proof (*intro exI allI conjI impI*)
fix *a b c d* :: *'a*
let *?α* = ∑ *i*∈*Basis*. *max* (*a* · *i*) (*c* · *i*) $*_R$ *i*
let *?β* = ∑ *i*∈*Basis*. *min* (*b* · *i*) (*d* · *i*) $*_R$ *i*
show *norm* (*integral* (*cbox a b*) *?g* − *integral* (*cbox c d*) *?g*) < *e*
if *ball*: *ball 0 B* ⊆ *cbox a b* ∧ *ball 0 B* ⊆ *cbox c d*
proof −
have *B'*: *norm* (*integral* (*cbox a b* ∩ *cbox c d*) *?g* − *integral* (*cbox x y*) *?g*) < *e*/*2*
if *cbox a b* ∩ *cbox c d* ⊆ *cbox x y* **for** *x y*
using *B* [*of ?α ?β x y*] *ball that* **by** (*simp add*: *Int_interval* [*symmetric*])
show *?thesis*

using *B′ [of a b] B′ [of c d] norm_triangle_half_r* **by** *blast*
qed
qed (*use* ‹*B* > *0*› **in** *auto*)}
then show *?thesis*
by *force*
qed
finally show *?thesis* **.**
qed

lemma *integrable_on_subcbox*:
fixes *f* :: *′n::euclidean_space* ⇒ *′a::banach*
assumes *intf*: *f integrable_on S*
and *sub*: *cbox a b* ⊆ *S*
shows *f integrable_on cbox a b*
proof −
have (λ*x*. *if x* ∈ *S then f x else 0*) *integrable_on cbox a b*
by (*simp add*: *intf integrable_altD*(*1*))
then show *?thesis*
by (*metis* (*mono_tags*) *sub integrable_restrict_Int le_inf_iff order_refl subset_antisym*)
qed

6.15.37 A straddling criterion for integrability

lemma *integrable_straddle_interval*:
fixes *f* :: *′n::euclidean_space* ⇒ *real*
assumes ⋀*e*. *e*>*0* ⟹ ∃ *g h i j*. (*g has_integral i*) (*cbox a b*) ∧ (*h has_integral j*) (*cbox a b*) ∧
|*i* − *j*| < *e* ∧ (∀ *x*∈*cbox a b*. (*g x*) ≤ *f x* ∧ *f x* ≤ *h x*)
shows *f integrable_on cbox a b*
proof −
have ∃ *d*. *gauge d* ∧
(∀ *p1 p2*. *p1 tagged_division_of cbox a b* ∧ *d fine p1* ∧
p2 tagged_division_of cbox a b ∧ *d fine p2* ⟶
|(∑ (*x*,*K*)∈*p1*. *content K* $*_R$ *f x*) − (∑ (*x*,*K*)∈*p2*. *content K* $*_R$ *f x*)| < *e*)
if *e* > *0* **for** *e*
proof −
have *e*: *e*/*3* > *0*
using *that* **by** *auto*
then obtain *g h i j* **where** *ij*: |*i* − *j*| < *e*/*3*
and (*g has_integral i*) (*cbox a b*)
and (*h has_integral j*) (*cbox a b*)
and *fgh*: ⋀*x*. *x* ∈ *cbox a b* ⟹ *g x* ≤ *f x* ∧ *f x* ≤ *h x*
using *assms real_norm_def* **by** *metis*
then obtain *d1 d2* **where** *gauge d1 gauge d2*
and *d1*: ⋀*p*. ⟦*p tagged_division_of cbox a b*; *d1 fine p*⟧ ⟹
|(∑ (*x*,*K*)∈*p*. *content K* $*_R$ *g x*) − *i*| < *e*/*3*
and *d2*: ⋀*p*. ⟦*p tagged_division_of cbox a b*; *d2 fine p*⟧ ⟹
|(∑ (*x*,*K*) ∈ *p*. *content K* $*_R$ *h x*) − *j*| < *e*/*3*

```
      by (metis e has_integral real_norm_def)
    have |(∑(x,K) ∈ p1. content K *R f x) − (∑(x,K) ∈ p2. content K *R f x)|
< e
      if p1: p1 tagged_division_of cbox a b and 11: d1 fine p1 and 21: d2 fine p1
        and p2: p2 tagged_division_of cbox a b and 12: d1 fine p2 and 22: d2 fine
p2 for p1 p2
    proof −
      have *: ⋀g1 g2 h1 h2 f1 f2.
                ⟦|g2 − i| < e/3; |g1 − i| < e/3; |h2 − j| < e/3; |h1 − j| < e/3;
                 g1 − h2 ≤ f1 − f2; f1 − f2 ≤ h1 − g2⟧
                ⟹ |f1 − f2| < e
        using ‹e > 0› ij by arith
      have 0: (∑(x, k)∈p1. content k *R f x) − (∑(x, k)∈p1. content k *R g x)
≥ 0
              0 ≤ (∑(x, k)∈p2. content k *R h x) − (∑(x, k)∈p2. content k *R f
x)
             (∑(x, k)∈p2. content k *R f x) − (∑(x, k)∈p2. content k *R g x) ≥
0
              0 ≤ (∑(x, k)∈p1. content k *R h x) − (∑(x, k)∈p1. content k *R f
x)
        unfolding sum_subtractf[symmetric]
          apply (auto intro!: sum_nonneg)
            apply (meson fgh measure_nonneg mult_left_mono tag_in_interval that
sum_nonneg)+
        done
      show ?thesis
      proof (rule *)
        show |(∑(x,K) ∈ p2. content K *R g x) − i| < e/3
          by (rule d1[OF p2 12])
        show |(∑(x,K) ∈ p1. content K *R g x) − i| < e/3
          by (rule d1[OF p1 11])
        show |(∑(x,K) ∈ p2. content K *R h x) − j| < e/3
          by (rule d2[OF p2 22])
        show |(∑(x,K) ∈ p1. content K *R h x) − j| < e/3
          by (rule d2[OF p1 21])
      qed (use 0 in auto)
    qed
    then show ?thesis
      by (rule_tac x=λx. d1 x ∩ d2 x in exI)
        (auto simp: fine_Int intro: ‹gauge d1› ‹gauge d2› d1 d2)
  qed
  then show ?thesis
    by (simp add: integrable_Cauchy)
qed

lemma integrable_straddle:
  fixes f :: 'n::euclidean_space ⇒ real
  assumes ⋀e. e>0 ⟹ ∃g h i j. (g has_integral i) s ∧ (h has_integral j) s ∧
                   |i − j| < e ∧ (∀x∈s. g x ≤ f x ∧ f x ≤ h x)
```

shows *f integrable_on s*
proof −
let *?fs* = (λ*x*. *if x* ∈ *s then f x else 0*)
have *?fs integrable_on cbox a b* **for** *a b*
proof (*rule integrable_straddle_interval*)
fix *e*::*real*
assume *e* > *0*
then have ∗: *e*/*4* > *0*
by *auto*
with *assms* **obtain** *g h i j* **where** *g*: (*g has_integral i*) *s* **and** *h*: (*h has_integral j*) *s*
and *ij*: |*i* − *j*| < *e*/*4*
and *fgh*: ⋀*x*. *x* ∈ *s* ⟹ *g x* ≤ *f x* ∧ *f x* ≤ *h x*
by *metis*
let *?gs* = (λ*x*. *if x* ∈ *s then g x else 0*)
let *?hs* = (λ*x*. *if x* ∈ *s then h x else 0*)
obtain *Bg* **where** *Bg*: ⋀*a b*. *ball 0 Bg* ⊆ *cbox a b* ⟹ |*integral* (*cbox a b*) *?gs* − *i*| < *e*/*4*
and *int_g*: ⋀*a b*. *?gs integrable_on cbox a b*
using *g* ∗ **unfolding** *has_integral_alt′ real_norm_def* **by** *meson*
obtain *Bh* **where**
Bh: ⋀*a b*. *ball 0 Bh* ⊆ *cbox a b* ⟹ |*integral* (*cbox a b*) *?hs* − *j*| < *e*/*4*
and *int_h*: ⋀*a b*. *?hs integrable_on cbox a b*
using *h* ∗ **unfolding** *has_integral_alt′ real_norm_def* **by** *meson*
define *c* **where** *c* = (∑*i*∈*Basis*. *min* (*a*•*i*) (− (*max Bg Bh*)) ∗$_R$ *i*)
define *d* **where** *d* = (∑*i*∈*Basis*. *max* (*b*•*i*) (*max Bg Bh*) ∗$_R$ *i*)
have ⟦*norm* (*0* − *x*) < *Bg*; *i* ∈ *Basis*⟧ ⟹ *c* • *i* ≤ *x* • *i* ∧ *x* • *i* ≤ *d* • *i* **for** *x i*
using *Basis_le_norm*[*of i x*] **unfolding** *c_def d_def* **by** *auto*
then have *ballBg*: *ball 0 Bg* ⊆ *cbox c d*
by (*auto simp*: *mem_box dist_norm*)
have ⟦*norm* (*0* − *x*) < *Bh*; *i* ∈ *Basis*⟧ ⟹ *c* • *i* ≤ *x* • *i* ∧ *x* • *i* ≤ *d* • *i* **for** *x i*
using *Basis_le_norm*[*of i x*] **unfolding** *c_def d_def* **by** *auto*
then have *ballBh*: *ball 0 Bh* ⊆ *cbox c d*
by (*auto simp*: *mem_box dist_norm*)
have *ab_cd*: *cbox a b* ⊆ *cbox c d*
by (*auto simp*: *c_def d_def subset_box_imp*)
have ∗∗: ⋀*ch cg ag ah*::*real*. ⟦|*ah* − *ag*| ≤ |*ch* − *cg*|; |*cg* − *i*| < *e*/*4*; |*ch* − *j*| < *e*/*4*⟧
⟹ |*ag* − *ah*| < *e*
using *ij* **by** *arith*
show ∃ *g h i j*. (*g has_integral i*) (*cbox a b*) ∧ (*h has_integral j*) (*cbox a b*) ∧ |*i* − *j*| < *e* ∧
(∀ *x*∈*cbox a b*. *g x* ≤ (*if x* ∈ *s then f x else 0*) ∧
(*if x* ∈ *s then f x else 0*) ≤ *h x*)
proof (*intro exI ballI conjI*)
have *eq*: ⋀*x f g*. (*if x* ∈ *s then f x else 0*) − (*if x* ∈ *s then g x else 0*) =
(*if x* ∈ *s then f x* − *g x else* (*0*::*real*))
by *auto*
have *int_hg*: (λ*x*. *if x* ∈ *s then h x* − *g x else 0*) *integrable_on cbox a b*

```
                (λx. if x ∈ s then h x − g x else 0) integrable_on cbox c d
    by (metis (no_types) integrable_diff g h has_integral_integrable integrable_altD(1))+
    show (?gs has_integral integral (cbox a b) ?gs) (cbox a b)
         (?hs has_integral integral (cbox a b) ?hs) (cbox a b)
      by (intro integrable_integral int_g int_h)+
    then have integral (cbox a b) ?gs ≤ integral (cbox a b) ?hs
      using fgh by (force intro: has_integral_le)
    then have 0 ≤ integral (cbox a b) ?hs − integral (cbox a b) ?gs
      by simp
    then have |integral (cbox a b) ?hs − integral (cbox a b) ?gs|
            ≤ |integral (cbox c d) ?hs − integral (cbox c d) ?gs|
      apply (simp add: integral_diff [symmetric] int_g int_h)
     apply (subst abs_of_nonneg[OF integral_nonneg[OF integrable_diff, OF int_h
int_g]])
     using fgh apply (force simp: eq intro!: integral_subset_le [OF ab_cd int_hg])+
      done
    then show |integral (cbox a b) ?gs − integral (cbox a b) ?hs| < e
      using ** Bg ballBg Bh ballBh by blast
    show ⋀x. x ∈ cbox a b ⟹ ?gs x ≤ ?fs x ⋀x. x ∈ cbox a b ⟹ ?fs x ≤ ?hs
x
      using fgh by auto
  qed
qed
then have int_f: ?fs integrable_on cbox a b for a b
  by simp
have ∃B>0. ∀a b c d.
               ball 0 B ⊆ cbox a b ∧ ball 0 B ⊆ cbox c d ⟶
               abs (integral (cbox a b) ?fs − integral (cbox c d) ?fs) < e
    if 0 < e for e
proof −
  have *: e/3 > 0
    using that by auto
  with assms obtain g h i j where g: (g has_integral i) s and h: (h has_integral
j) s
               and ij: |i − j| < e/3
               and fgh: ⋀x. x ∈ s ⟹ g x ≤ f x ∧ f x ≤ h x
    by metis
  let ?gs = (λx. if x ∈ s then g x else 0)
  let ?hs = (λx. if x ∈ s then h x else 0)
  obtain Bg where Bg > 0
             and Bg: ⋀a b. ball 0 Bg ⊆ cbox a b ⟹ |integral (cbox a b) ?gs − i|
< e/3
            and int_g: ⋀a b. ?gs integrable_on cbox a b
    using g * unfolding has_integral_alt' real_norm_def by meson
  obtain Bh where Bh > 0
             and Bh: ⋀a b. ball 0 Bh ⊆ cbox a b ⟹ |integral (cbox a b) ?hs − j|
< e/3
            and int_h: ⋀a b. ?hs integrable_on cbox a b
    using h * unfolding has_integral_alt' real_norm_def by meson
```

{ fix *a b c d* :: *'n*
assume *as*: *ball 0* (*max Bg Bh*) ⊆ *cbox a b ball 0* (*max Bg Bh*) ⊆ *cbox c d*
have ∗∗: *ball 0 Bg* ⊆ *ball* (*0*::*'n*) (*max Bg Bh*) *ball 0 Bh* ⊆ *ball* (*0*::*'n*) (*max Bg Bh*)
by *auto*
have ∗: ⋀*ga gc ha hc fa fc*. ⟦|*ga* − *i*| < *e*/*3*; |*gc* − *i*| < *e*/*3*; |*ha* − *j*| < *e*/*3*; |*hc* − *j*| < *e*/*3*; *ga* ≤ *fa*; *fa* ≤ *ha*; *gc* ≤ *fc*; *fc* ≤ *hc*⟧ ⟹
|*fa* − *fc*| < *e*
using *ij* **by** *arith*
have *abs* (*integral* (*cbox a b*) (λ*x*. *if x* ∈ *s then f x else 0*) − *integral* (*cbox c d*)
(λ*x*. *if x* ∈ *s then f x else 0*)) < *e*
proof (*rule* ∗)
show |*integral* (*cbox a b*) *?gs* − *i*| < *e*/*3*
using ∗∗ *Bg as* **by** *blast*
show |*integral* (*cbox c d*) *?gs* − *i*| < *e*/*3*
using ∗∗ *Bg as* **by** *blast*
show |*integral* (*cbox a b*) *?hs* − *j*| < *e*/*3*
using ∗∗ *Bh as* **by** *blast*
show |*integral* (*cbox c d*) *?hs* − *j*| < *e*/*3*
using ∗∗ *Bh as* **by** *blast*
qed (*use int_f int_g int_h fgh* **in** ‹*simp_all add*: *integral_le*›)
}
then show *?thesis*
apply (*rule_tac x*=*max Bg Bh* **in** *exI*)
using ‹*Bg* > *0*› **by** *auto*
qed
then show *?thesis*
unfolding *integrable_alt*[*of f*] *real_norm_def* **by** (*blast intro*: *int_f*)
qed

6.15.38 Adding integrals over several sets

lemma *has_integral_Un*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*banach*
assumes *f*: (*f has_integral i*) *S* (*f has_integral j*) *T*
and *neg*: *negligible* (*S* ∩ *T*)
shows (*f has_integral* (*i* + *j*)) (*S* ∪ *T*)
unfolding *has_integral_restrict_UNIV*[*symmetric*, *of f*]
proof (*rule has_integral_spike*[*OF neg*])
let *?f* = λ*x*. (*if x* ∈ *S then f x else 0*) + (*if x* ∈ *T then f x else 0*)
show (*?f has_integral i* + *j*) *UNIV*
by (*simp add*: *f has_integral_add*)
qed *auto*

lemma *integral_Un* [*simp*]:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*banach*
assumes *f integrable_on S f integrable_on T negligible* (*S* ∩ *T*)
shows *integral* (*S* ∪ *T*) *f* = *integral S f* + *integral T f*

```
  by (simp add: has_integral_Un assms integrable_integral integral_unique)

lemma integrable_Un:
  fixes f :: 'a::euclidean_space ⇒ 'b :: banach
  assumes negligible (A ∩ B) f integrable_on A f integrable_on B
  shows   f integrable_on (A ∪ B)
proof –
  from assms obtain y z where (f has_integral y) A (f has_integral z) B
     by (auto simp: integrable_on_def)
   from has_integral_Un[OF this assms(1)] show ?thesis by (auto simp: inte-
grable_on_def)
qed

lemma integrable_Un':
  fixes f :: 'a::euclidean_space ⇒ 'b :: banach
  assumes f integrable_on A f integrable_on B negligible (A ∩ B) C = A ∪ B
  shows   f integrable_on C
  using integrable_Un[of A B f] assms by simp

lemma has_integral_Union:
  fixes f :: 'n::euclidean_space ⇒ 'a::banach
  assumes 𝒯: finite 𝒯
    and int: ⋀S. S ∈ 𝒯 ⟹ (f has_integral (i S)) S
    and neg: pairwise (λS S'. negligible (S ∩ S')) 𝒯
  shows (f has_integral (sum i 𝒯)) (⋃𝒯)
proof –
  let ?𝒰 = ((λ(a,b). a ∩ b) ` {(a,b). a ∈ 𝒯 ∧ b ∈ {y. y ∈ 𝒯 ∧ a ≠ y}})
  have ((λx. if x ∈ ⋃𝒯 then f x else 0) has_integral sum i 𝒯) UNIV
  proof (rule has_integral_spike)
    show negligible (⋃ ?𝒰)
    proof (rule negligible_Union)
      have finite (𝒯 × 𝒯)
        by (simp add: 𝒯)
      moreover have {(a, b). a ∈ 𝒯 ∧ b ∈ {y ∈ 𝒯. a ≠ y}} ⊆ 𝒯 × 𝒯
        by auto
      ultimately show finite ?𝒰
        by (blast intro: finite_subset[of _ 𝒯 × 𝒯])
      show ⋀t. t ∈ ?𝒰 ⟹ negligible t
        using neg unfolding pairwise_def by auto
    qed
  next
    show (if x ∈ ⋃𝒯 then f x else 0) = (∑A∈𝒯. if x ∈ A then f x else 0)
      if x ∈ UNIV – (⋃ ?𝒰) for x
    proof clarsimp
      fix S assume S ∈ 𝒯 x ∈ S
      moreover then have ∀b∈𝒯. x ∈ b ⟷ b = S
        using that by blast
      ultimately show f x = (∑A∈𝒯. if x ∈ A then f x else 0)
        by (simp add: sum.delta[OF 𝒯])
```

qed
next
show (($\lambda x.$ $\sum A{\in}\mathcal{T}.$ *if* $x \in A$ *then* $f\ x$ *else* 0) *has_integral* ($\sum A{\in}\mathcal{T}.$ $i\ A$)) *UNIV*
using *int* **by** (*simp add*: *has_integral_restrict_UNIV has_integral_sum* [*OF* $\mathcal{T}$])
qed
then show *?thesis*
using *has_integral_restrict_UNIV* **by** *blast*
qed

In particular adding integrals over a division, maybe not of an interval.

lemma *has_integral_combine_division*:
fixes f :: *'n::euclidean_space* $\Rightarrow$ *'a::banach*
assumes $\mathcal{D}$ *division_of* S
and $\bigwedge k.\ k \in \mathcal{D} \Longrightarrow$ (f *has_integral* ($i\ k$)) k
shows (f *has_integral* (*sum* $i\ \mathcal{D}$)) S
proof −
note $\mathcal{D}$ = *division_ofD*[*OF assms*(*1*)]
have *neg*: *negligible* ($S \cap s'$) **if** $S \in \mathcal{D}$ $s' \in \mathcal{D}$ $S \neq s'$ **for** $S\ s'$
proof −
obtain $a\ c\ b\ \mathcal{D}$ **where** *obt*: S = *cbox* $a\ b$ s' = *cbox* $c\ \mathcal{D}$
by (*meson* ‹$S \in \mathcal{D}$› ‹$s' \in \mathcal{D}$› $\mathcal{D}$(*4*))
from $\mathcal{D}$(*5*)[*OF that*] **show** *?thesis*
unfolding *obt interior_cbox*
by (*metis* (*no_types*, *lifting*) *Diff_empty Int_interval box_Int_box negligible_frontier_interval*)
qed
show *?thesis*
unfolding $\mathcal{D}$(*6*)[*symmetric*]
by (*auto intro*: $\mathcal{D}$ *neg assms has_integral_Union pairwiseI*)
qed

lemma *integral_combine_division_bottomup*:
fixes f :: *'n::euclidean_space* $\Rightarrow$ *'a::banach*
assumes $\mathcal{D}$ *division_of* S $\bigwedge k.\ k \in \mathcal{D} \Longrightarrow f$ *integrable_on* k
shows *integral* $S\ f$ = *sum* ($\lambda i.$ *integral* $i\ f$) $\mathcal{D}$
by (*meson assms integral_unique has_integral_combine_division has_integral_integrable_integral*)

lemma *has_integral_combine_division_topdown*:
fixes f :: *'n::euclidean_space* $\Rightarrow$ *'a::banach*
assumes f: f *integrable_on* S
and $\mathcal{D}$: $\mathcal{D}$ *division_of* K
and $K \subseteq S$
shows (f *has_integral* (*sum* ($\lambda i.$ *integral* $i\ f$) $\mathcal{D}$)) K
proof −
have f *integrable_on* L **if** $L \in \mathcal{D}$ **for** L
proof −
have $L \subseteq S$
using ‹$K \subseteq S$› $\mathcal{D}$ *that* **by** *blast*
then show f *integrable_on* L

```
      using that by (metis (no_types) f 𝒟 division_ofD(4) integrable_on_subcbox)
  qed
  then show ?thesis
    by (meson 𝒟 has_integral_combine_division has_integral_integrable_integral)
qed

lemma integral_combine_division_topdown:
  fixes f :: 'n::euclidean_space ⇒ 'a::banach
  assumes f integrable_on S
    and 𝒟 division_of S
  shows integral S f = sum (λi. integral i f) 𝒟
  using assms has_integral_combine_division_topdown by blast

lemma integrable_combine_division:
  fixes f :: 'n::euclidean_space ⇒ 'a::banach
  assumes 𝒟: 𝒟 division_of S
    and f: ⋀i. i ∈ 𝒟 ⟹ f integrable_on i
  shows f integrable_on S
  using f unfolding integrable_on_def by (metis has_integral_combine_division[OF
𝒟])

lemma integrable_on_subdivision:
  fixes f :: 'n::euclidean_space ⇒ 'a::banach
  assumes 𝒟: 𝒟 division_of i
    and f: f integrable_on S
    and i ⊆ S
  shows f integrable_on i
proof -
  have f integrable_on i if i ∈ 𝒟 for i
proof -
  have i ⊆ S
    using assms that by auto
  then show f integrable_on i
    using that by (metis (no_types) 𝒟 f division_ofD(4) integrable_on_subcbox)
qed
  then show ?thesis
    using 𝒟 integrable_combine_division by blast
qed
```

6.15.39 Also tagged divisions

```
lemma has_integral_combine_tagged_division:
  fixes f :: 'n::euclidean_space ⇒ 'a::banach
  assumes p tagged_division_of S
    and ⋀x k. (x,k) ∈ p ⟹ (f has_integral (i k)) k
  shows (f has_integral (∑(x,k)∈p. i k)) S
proof -
  have *: (f has_integral (∑k∈snd`p. integral k f)) S
  proof -
```

have *snd ‘ p division_of S*
  by (*simp add*: *assms*(*1*) *division_of_tagged_division*)
with *assms* **show** *?thesis*
  by (*metis* (*mono_tags*, *lifting*) *has_integral_combine_division has_integral_integrable_integral imageE prod.collapse*)
qed
also have ($\sum$ *k*∈*snd‘p*. *integral k f*) = ($\sum$ (*x*, *k*)∈*p*. *integral k f*)
  by (*intro sum.over_tagged_division_lemma*[*OF assms*(*1*), *symmetric*] *integral_null*)
    (*simp add*: *content_eq_0_interior*)
finally show *?thesis*
  using *assms* **by** (*auto simp add*: *has_integral_iff intro*!: *sum.cong*)
qed

lemma *integral_combine_tagged_division_bottomup*:
  fixes *f* :: *′n*::*euclidean_space* ⇒ *′a*::*banach*
  assumes *p*: *p tagged_division_of* (*cbox a b*)
    and *f*: ⋀*x k*. (*x*,*k*)∈*p* ⟹ *f integrable_on k*
  shows *integral* (*cbox a b*) *f* = *sum* (λ(*x*,*k*). *integral k f*) *p*
  by (*simp add*: *has_integral_combine_tagged_division*[*OF p*] *integral_unique f integrable_integral*)

lemma *has_integral_combine_tagged_division_topdown*:
  fixes *f* :: *′n*::*euclidean_space* ⇒ *′a*::*banach*
  assumes *f*: *f integrable_on cbox a b*
    and *p*: *p tagged_division_of* (*cbox a b*)
  shows (*f has_integral* (*sum* (λ(*x*,*K*). *integral K f*) *p*)) (*cbox a b*)
proof −
  have (*f has_integral integral K f*) *K* **if** (*x*,*K*) ∈ *p* **for** *x K*
    by (*metis assms integrable_integral integrable_on_subcbox tagged_division_ofD*(*3*,*4*) *that*)
  then show *?thesis*
    by (*simp add*: *has_integral_combine_tagged_division p*)
qed

lemma *integral_combine_tagged_division_topdown*:
  fixes *f* :: *′n*::*euclidean_space* ⇒ *′a*::*banach*
  assumes *f integrable_on cbox a b*
    and *p tagged_division_of* (*cbox a b*)
  shows *integral* (*cbox a b*) *f* = *sum* (λ(*x*,*k*). *integral k f*) *p*
  using *assms* **by** (*auto intro*: *integral_unique* [*OF has_integral_combine_tagged_division_topdown*])

6.15.40 Henstock’s lemma

lemma *Henstock_lemma_part1*:
  fixes *f* :: *′n*::*euclidean_space* ⇒ *′a*::*banach*
  assumes *intf*: *f integrable_on cbox a b*
    and *e* > *0*
    and *gauge d*
    and *less_e*: ⋀*p*. ⟦*p tagged_division_of* (*cbox a b*); *d fine p*⟧ ⟹

```
                    norm (sum (λ(x,K). content K *R f x) p − integral(cbox a b) f)
< e
  and p: p tagged_partial_division_of (cbox a b) d fine p
 shows norm (sum (λ(x,K). content K *R f x − integral K f) p) ≤ e (is ?lhs ≤
e)
proof (rule field_le_epsilon)
 fix k :: real
 assume k > 0
 let ?SUM = λp. (∑(x,K) ∈ p. content K *R f x)
 note p' = tagged_partial_division_ofD[OF p(1)]
 have ⋃(snd ` p) ⊆ cbox a b
   using p'(3) by fastforce
 then obtain q where q: snd ` p ⊆ q and qdiv: q division_of cbox a b
  by (meson p(1) partial_division_extend_interval partial_division_of_tagged_division)
 note q' = division_ofD[OF qdiv]
 define r where r = q − snd ` p
 have snd ` p ∩ r = {}
   unfolding r_def by auto
 have finite r
   using q' unfolding r_def by auto
 have ∃p. p tagged_division_of i ∧ d fine p ∧
      norm (?SUM p − integral i f) < k / (real (card r) + 1)
   if i∈r for i
 proof −
   have gt0: k / (real (card r) + 1) > 0 using ‹k > 0› by simp
   have i: i ∈ q
     using that unfolding r_def by auto
   then obtain u v where uv: i = cbox u v
     using q'(4) by blast
   then have cbox u v ⊆ cbox a b
     using i q'(2) by auto
   then have f integrable_on cbox u v
     by (rule integrable_subinterval[OF intf])
   with integrable_integral[OF this, unfolded has_integral[of f]]
   obtain dd where gauge dd and dd:
     ⋀𝒟. ⟦𝒟 tagged_division_of cbox u v; dd fine 𝒟⟧ ⟹
   norm (?SUM 𝒟 − integral (cbox u v) f) < k / (real (card r) + 1)
     using gt0 by auto
   with gauge_Int[OF ‹gauge d› ‹gauge dd›]
   obtain qq where qq: qq tagged_division_of cbox u v (λx. d x ∩ dd x) fine qq
     using fine_division_exists by blast
   with dd[of qq] show ?thesis
     by (auto simp: fine_Int uv)
 qed
 then obtain qq where qq: ⋀i. i ∈ r ⟹ qq i tagged_division_of i ∧
     d fine qq i ∧ norm (?SUM (qq i) − integral i f) < k / (real (card r) + 1)
   by metis

 let ?p = p ∪ ⋃(qq ` r)
```

```
have norm (?SUM ?p − integral (cbox a b) f) < e
proof (rule less_e)
  show d fine ?p
    by (metis (mono_tags, hide_lams) qq fine_Un fine_Union imageE p(2))
  note ptag = tagged_partial_division_of_Union_self [OF p(1)]
  have p ∪ ⋃(qq ‘ r) tagged_division_of ⋃(snd ‘ p) ∪ ⋃r
  proof (rule tagged_division_Un[OF ptag tagged_division_Union [OF ‹finite r›]])
    show ⋀i. i ∈ r ⟹ qq i tagged_division_of i
      using qq by auto
    show ⋀i1 i2. ⟦i1 ∈ r; i2 ∈ r; i1 ≠ i2⟧ ⟹ interior i1 ∩ interior i2 = {}
      by (simp add: q′(5) r_def)
    show interior (⋃(snd ‘ p)) ∩ interior (⋃r) = {}
    proof (rule Int_interior_Union_intervals [OF ‹finite r›])
      show open (interior (⋃(snd ‘ p)))
        by blast
      show ⋀T. T ∈ r ⟹ ∃ a b. T = cbox a b
        by (simp add: q′(4) r_def)
      have interior T ∩ interior (⋃(snd ‘ p)) = {} if T ∈ r for T
      proof (rule Int_interior_Union_intervals)
        show ⋀U. U ∈ snd ‘ p ⟹ ∃ a b. U = cbox a b
          using q q′(4) by blast
        show ⋀U. U ∈ snd ‘ p ⟹ interior T ∩ interior U = {}
          by (metis DiffE q q′(5) r_def subsetD that)
      qed (use p′ in auto)
      then show ⋀T. T ∈ r ⟹ interior (⋃(snd ‘ p)) ∩ interior T = {}
        by (metis Int_commute)
    qed
  qed
  moreover have ⋃(snd ‘ p) ∪ ⋃r = cbox a b and {qq i |i. i ∈ r} = qq ‘ r
    using qdiv q unfolding Union_Un_distrib[symmetric] r_def by auto
  ultimately show ?p tagged_division_of (cbox a b)
    by fastforce
qed
then have norm (?SUM p + (?SUM (⋃(qq ‘ r))) − integral (cbox a b) f) < e
proof (subst sum.union_inter_neutral[symmetric, OF ‹finite p›], safe)
  show content L *R f x = 0 if (x, L) ∈ p (x, L) ∈ qq K K ∈ r for x K L
  proof −
    obtain u v where uv: L = cbox u v
      using ‹(x,L) ∈ p› p′(4) by blast
    have L ⊆ K
      using qq[OF that(3)] tagged_division_ofD(3) ‹(x,L) ∈ qq K› by metis
    have L ∈ snd ‘ p
      using ‹(x,L) ∈ p› image_iff by fastforce
    then have L ∈ q K ∈ q L ≠ K
      using that(1,3) q(1) unfolding r_def by auto
    with q′(5) have interior L = {}
      using interior_mono[OF ‹L ⊆ K›] by blast
    then show content L *R f x = 0
      unfolding uv content_eq_0_interior[symmetric] by auto
```

```
    qed
    show finite (⋃(qq ` r))
      by (meson finite_UN qq ‹finite r› tagged_division_of_finite)
  qed
  moreover have content M *R f x = 0
      if x: (x,M) ∈ qq K (x,M) ∈ qq L and KL: qq K ≠ qq L and r: K ∈ r L ∈ r
    for x M K L
  proof -
    note kl = tagged_division_ofD(3,4)[OF qq[THEN conjunct1]]
    obtain u v where uv: M = cbox u v
      using ‹(x, M) ∈ qq L› ‹L ∈ r› kl(2) by blast
    have empty: interior (K ∩ L) = {}
      by (metis DiffD1 interior_Int q'(5) r_def KL r)
    have interior M = {}
      by (metis (no_types, lifting) Int_assoc empty inf.absorb_iff2 interior_Int kl(1)
subset_empty x r)
    then show content M *R f x = 0
      unfolding uv content_eq_0_interior[symmetric]
      by auto
  qed
  ultimately have norm (?SUM p + sum ?SUM (qq ` r) − integral (cbox a b) f)
< e
    apply (subst (asm) sum.Union_comp)
    using qq by (force simp: split_paired_all)+
  moreover have content M *R f x = 0
       if K ∈ r L ∈ r K ≠ L qq K = qq L (x, M) ∈ qq K for K L x M
    using tagged_division_ofD(6) qq that by (metis (no_types, lifting))
  ultimately have less_e: norm (?SUM p + sum (?SUM ∘ qq) r − integral (cbox
a b) f) < e
  proof (subst (asm) sum.reindex_nontrivial [OF ‹finite r›])
    qed (auto simp: split_paired_all sum.neutral)
  have norm_le: norm (cp − ip) ≤ e + k
                if norm ((cp + cr) − i) < e norm (cr − ir) < k ip + ir = i
                for ir ip i cr cp::'a
  proof -
    from that show ?thesis
      using norm_triangle_le[of cp + cr − i − (cr − ir)]
      unfolding that(3)[symmetric] norm_minus_cancel
      by (auto simp add: algebra_simps)
  qed

  have ?lhs =  norm (?SUM p − (∑(x, k)∈p. integral k f))
    unfolding split_def sum_subtractf ..
  also have ... ≤ e + k
  proof (rule norm_le[OF less_e])
    have lessk: k * real (card r) / (1 + real (card r)) < k
      using ‹k>0› by (auto simp add: field_simps)
   have norm (sum (?SUM ∘ qq) r − (∑k∈r. integral k f)) ≤ (∑x∈r. k / (real
(card r) + 1))
```

unfolding *sum_subtractf*[*symmetric*] **by** (*force dest*: *qq intro*!: *sum_norm_le*)
also have ... < *k*
by (*simp add*: *lessk add.commute mult.commute*)
finally show *norm* (*sum* (*?SUM* ∘ *qq*) *r* − (∑ *k*∈*r*. *integral k f*)) < *k* **.**
next
from *q*(*1*) **have** [*simp*]: *snd* ‘ *p* ∪ *q* = *q* **by** *auto*
have *integral l f* = *0*
if *inp*: (*x*, *l*) ∈ *p* (*y*, *m*) ∈ *p* **and** *ne*: (*x*, *l*) ≠ (*y*, *m*) **and** *l* = *m* **for** *x l y m*
proof −
obtain *u v* **where** *uv*: *l* = *cbox u v*
using *inp p*′(*4*) **by** *blast*
have *content* (*cbox u v*) = *0*
unfolding *content_eq_0_interior* **using** *that p*(*1*) *uv*
by (*auto dest*: *tagged_partial_division_ofD*)
then show *?thesis*
using *uv* **by** *blast*
qed
then have (∑ (*x*, *K*)∈*p*. *integral K f*) = (∑ *K*∈*snd* ‘ *p*. *integral K f*)
apply (*subst sum.reindex_nontrivial* [*OF* ‹*finite p*›])
unfolding *split_paired_all split_def* **by** *auto*
then show (∑ (*x*, *k*)∈*p*. *integral k f*) + (∑ *k*∈*r*. *integral k f*) = *integral* (*cbox a b*) *f*
unfolding *integral_combine_division_topdown*[*OF intf qdiv*] *r_def*
using *q*′(*1*) *p*′(*1*) *sum.union_disjoint* [*of snd* ‘ *p q* − *snd* ‘ *p*, *symmetric*]
by *simp*
qed
finally show *?lhs* ≤ *e* + *k* **.**
qed

lemma *Henstock_lemma_part2*:
fixes *f* :: ′*m*::*euclidean_space* ⇒ ′*n*::*euclidean_space*
assumes *fed*: *f integrable_on cbox a b e* > *0 gauge d*
and *less_e*: ⋀𝒟. ⟦𝒟 *tagged_division_of* (*cbox a b*); *d fine* 𝒟⟧ ⟹
norm (*sum* (λ(*x*,*k*). *content k* $*_R$ *f x*) 𝒟 − *integral* (*cbox a b*) *f*) < *e*
and *tag*: *p tagged_partial_division_of* (*cbox a b*)
and *d fine p*
shows *sum* (λ(*x*,*k*). *norm* (*content k* $*_R$ *f x* − *integral k f*)) *p* ≤ *2* ∗ *real* (*DIM*(′*n*)) ∗ *e*
proof −
have *finite p*
using *tag tagged_partial_division_ofD* **by** *blast*
then show *?thesis*
unfolding *split_def*
proof (*rule sum_norm_allsubsets_bound*)
fix *Q*
assume *Q*: *Q* ⊆ *p*
then have *fine*: *d fine Q*
by (*simp add*: ‹*d fine p*› *fine_subset*)

show *norm* (∑*x*∈*Q*. *content* (*snd x*) $*_R$ *f* (*fst x*) − *integral* (*snd x*) *f*) ≤ *e*
apply (*rule Henstock_lemma_part1*[*OF fed less_e*, *unfolded split_def*])
using *Q tag tagged_partial_division_subset* **by** (*force simp add*: *fine*)+
qed
qed

lemma *Henstock_lemma*:
fixes *f* :: *'m*::*euclidean_space* ⇒ *'n*::*euclidean_space*
assumes *intf*: *f integrable_on cbox a b*
and *e* > *0*
obtains γ **where** *gauge* γ
and ⋀*p*. ⟦*p tagged_partial_division_of* (*cbox a b*); γ *fine p*⟧ ⟹
sum (λ(*x*,*k*). *norm*(*content k* $*_R$ *f x* − *integral k f*)) *p* < *e*
proof −
have ∗: *e*/(*2* ∗ (*real DIM*(*'n*) + *1*)) > *0* **using** ‹*e* > *0*› **by** *simp*
with *integrable_integral*[*OF intf*, *unfolded has_integral*]
obtain γ **where** *gauge* γ
and γ: ⋀𝒟. ⟦𝒟 *tagged_division_of cbox a b*; γ *fine* 𝒟⟧ ⟹
norm ((∑(*x*,*K*)∈𝒟. *content K* $*_R$ *f x*) − *integral* (*cbox a b*) *f*)
< *e*/(*2* ∗ (*real DIM*(*'n*) + *1*))
by *metis*
show *thesis*
proof (*rule that* [*OF* ‹*gauge* γ›])
fix *p*
assume *p*: *p tagged_partial_division_of cbox a b* γ *fine p*
have (∑(*x*,*K*)∈*p*. *norm* (*content K* $*_R$ *f x* − *integral K f*))
≤ *2* ∗ *real DIM*(*'n*) ∗ (*e*/(*2* ∗ (*real DIM*(*'n*) + *1*)))
using *Henstock_lemma_part2*[*OF intf* ∗ ‹*gauge* γ› γ *p*] **by** *metis*
also have ... < *e*
using ‹*e* > *0*› **by** (*auto simp add*: *field_simps*)
finally
show (∑(*x*,*K*)∈*p*. *norm* (*content K* $*_R$ *f x* − *integral K f*)) < *e* **.**
qed
qed

6.15.41 Monotone convergence (bounded interval first)

lemma *bounded_increasing_convergent*:
fixes *f* :: *nat* ⇒ *real*
shows ⟦*bounded* (*range f*); ⋀*n*. *f n* ≤ *f* (*Suc n*)⟧ ⟹ ∃ *l*. *f* ⟶ *l*
using *Bseq_mono_convergent*[*of f*] *incseq_Suc_iff*[*of f*]
by (*auto simp*: *image_def Bseq_eq_bounded convergent_def incseq_def*)

lemma *monotone_convergence_interval*:
fixes *f* :: *nat* ⇒ *'n*::*euclidean_space* ⇒ *real*
assumes *intf*: ⋀*k*. (*f k*) *integrable_on cbox a b*
and *le*: ⋀*k x*. *x* ∈ *cbox a b* ⟹ (*f k x*) ≤ *f* (*Suc k*) *x*
and *fg*: ⋀*x*. *x* ∈ *cbox a b* ⟹ ((λ*k*. *f k x*) ⟶ *g x*) *sequentially*
and *bou*: *bounded* (*range* (λ*k*. *integral* (*cbox a b*) (*f k*)))

shows *g integrable_on cbox a b* ∧ ((λ*k*. *integral* (*cbox a b*) (*f k*)) ⟶ *integral* (*cbox a b*) *g*) *sequentially*
proof (*cases content* (*cbox a b*) = *0*)
  case *True* **then show** *?thesis*
    by *auto*
next
  case *False*
  have *fg1*: (*f k x*) ≤ (*g x*) **if** *x*: *x* ∈ *cbox a b* **for** *x k*
  proof −
    have ∀$_F$ *j in sequentially*. *f k x* ≤ *f j x*
    proof (*rule eventually_sequentiallyI* [*of k*])
      show ⋀*j*. *k* ≤ *j* ⟹ *f k x* ≤ *f j x*
        using *le x* **by** (*force intro*: *transitive_stepwise_le*)
    qed
    then show *f k x* ≤ *g x*
      using *tendsto_lowerbound* [*OF fg*] *x trivial_limit_sequentially* **by** *blast*
  qed
  have *int_inc*: ⋀*n*. *integral* (*cbox a b*) (*f n*) ≤ *integral* (*cbox a b*) (*f* (*Suc n*))
    by (*metis integral_le intf le*)
  then obtain *i* **where** *i*: (λ*k*. *integral* (*cbox a b*) (*f k*)) ⟶ *i*
    using *bounded_increasing_convergent bou* **by** *blast*
  have ⋀*k*. ∀$_F$ *x in sequentially*. *integral* (*cbox a b*) (*f k*) ≤ *integral* (*cbox a b*) (*f x*)
    unfolding *eventually_sequentially*
    by (*force intro*: *transitive_stepwise_le int_inc*)
  then have *i'*: ⋀*k*. (*integral*(*cbox a b*) (*f k*)) ≤ *i*
    using *tendsto_le* [*OF trivial_limit_sequentially i*] **by** *blast*
  have (*g has_integral i*) (*cbox a b*)
    unfolding *has_integral real_norm_def*
  proof *clarify*
    fix *e*::*real*
    assume *e*: *e* > *0*
    have ⋀*k*. (∃γ. *gauge* γ ∧ (∀𝒟. 𝒟 *tagged_division_of* (*cbox a b*) ∧ γ *fine* 𝒟 ⟶
      abs ((∑(*x*,*K*)∈𝒟. *content K* ∗$_R$ *f k x*) − *integral* (*cbox a b*) (*f k*)) < *e*/*2* ^ (*k* + *2*)))
      using *intf e* **by** (*auto simp*: *has_integral_integral has_integral*)
    then obtain *c* **where** *c*: ⋀*x*. *gauge* (*c x*)
        ⋀*x* 𝒟. ⟦𝒟 *tagged_division_of cbox a b*; *c x fine* 𝒟⟧ ⟹
          abs ((∑(*u*,*K*)∈𝒟. *content K* ∗$_R$ *f x u*) − *integral* (*cbox a b*) (*f x*))
          < *e*/*2* ^ (*x* + *2*)
      by *metis*

    have ∃*r*. ∀*k*≥*r*. *0* ≤ *i* − (*integral* (*cbox a b*) (*f k*)) ∧ *i* − (*integral* (*cbox a b*) (*f k*)) < *e*/*4*
    proof −
      have *e*/*4* > *0*
        using *e* **by** *auto*
      show *?thesis*
        using *LIMSEQ_D* [*OF i* ‹*e*/*4* > *0*›] *i'* **by** *auto*

qed
then obtain r **where** r: $\bigwedge k.\ r \leq k \Longrightarrow 0 \leq i - integral\ (cbox\ a\ b)\ (f\ k)$
$\bigwedge k.\ r \leq k \Longrightarrow i - integral\ (cbox\ a\ b)\ (f\ k) < e/4$
by *metis*
have $\exists n \geq r.\ \forall k \geq n.\ 0 \leq (g\ x) - (f\ k\ x) \wedge (g\ x) - (f\ k\ x) < e/(4 * content(cbox\ a\ b))$
if $x \in cbox\ a\ b$ **for** x
proof –
have $e/(4 * content\ (cbox\ a\ b)) > 0$
by (*simp add*: *False content_lt_nz e*)
with *fg that LIMSEQ_D*
obtain N **where** $\forall n \geq N.\ norm\ (f\ n\ x - g\ x) < e/(4 * content\ (cbox\ a\ b))$
by *metis*
then show $\exists n \geq r.\ \forall k \geq n.\ 0 \leq g\ x - f\ k\ x \wedge g\ x - f\ k\ x < e/(4 * content\ (cbox\ a\ b))$
apply (*rule_tac x=N + r* **in** *exI*)
using *fg1*[*OF that*] **by** (*auto simp add*: *field_simps*)
qed
then obtain m **where** *r_le_m*: $\bigwedge x.\ x \in cbox\ a\ b \Longrightarrow r \leq m\ x$
and m: $\bigwedge x\ k.\ [\![x \in cbox\ a\ b;\ m\ x \leq k]\!]$
$\Longrightarrow 0 \leq g\ x - f\ k\ x \wedge g\ x - f\ k\ x < e/(4 * content\ (cbox\ a\ b))$
by *metis*
define d **where** $d\ x = c\ (m\ x)\ x$ **for** x
show $\exists \gamma.\ gauge\ \gamma\ \wedge$
$(\forall \mathcal{D}.\ \mathcal{D}\ tagged\_division\_of\ cbox\ a\ b\ \wedge$
$\gamma\ fine\ \mathcal{D} \longrightarrow abs\ ((\sum (x,K) \in \mathcal{D}.\ content\ K *_R g\ x) - i) < e)$
proof (*rule exI*, *safe*)
show *gauge d*
using *c*(*1*) **unfolding** *gauge_def d_def* **by** *auto*
next
fix $\mathcal{D}$
assume *ptag*: $\mathcal{D}$ *tagged_division_of* (*cbox a b*) **and** *d fine* $\mathcal{D}$
note *p′=tagged_division_ofD*[*OF ptag*]
obtain s **where** s: $\bigwedge x.\ x \in \mathcal{D} \Longrightarrow m\ (fst\ x) \leq s$
by (*metis finite_imageI finite_nat_set_iff_bounded_le p′*(*1*) *rev_image_eqI*)
have $*$: $|a - d| < e$ **if** $|a - b| \leq e/4$ $|b - c| < e/2$ $|c - d| < e/4$ **for** $a\ b\ c\ d$
using *that norm_triangle_lt*[*of* $a - b\ b - c\ 3 * e/4$]
norm_triangle_lt[*of* $a - b + (b - c)\ c - d\ e$]
by (*auto simp add*: *algebra_simps*)
show $|(\sum (x, k) \in \mathcal{D}.\ content\ k *_R g\ x) - i| < e$
proof (*rule* $*$)
have $|(\sum (x,K) \in \mathcal{D}.\ content\ K *_R g\ x) - (\sum (x,K) \in \mathcal{D}.\ content\ K *_R f\ (m\ x)\ x)|$
$\leq (\sum i \in \mathcal{D}.\ |(case\ i\ of\ (x, K) \Rightarrow content\ K *_R g\ x) - (case\ i\ of\ (x, K) \Rightarrow content\ K *_R f\ (m\ x)\ x)|)$
by (*metis* (*mono_tags*) *sum_subtractf sum_abs*)
also have ... $\leq (\sum (x, k) \in \mathcal{D}.\ content\ k * (e/(4 * content\ (cbox\ a\ b))))$
proof (*rule sum_mono*, *simp add*: *split_paired_all*)

fix *x K*
assume *xk*: (*x*,*K*) ∈ 𝒟
with *ptag* **have** *x*: *x* ∈ *cbox a b*
by *blast*
then have *abs* (*content K* ∗ (*g x* − *f* (*m x*) *x*)) ≤ *content K* ∗ (*e*/(*4* ∗ *content* (*cbox a b*)))
by (*metis m*[*OF x*] *mult_nonneg_nonneg abs_of_nonneg less_eq_real_def measure_nonneg mult_left_mono order_refl*)
then show |*content K* ∗ *g x* − *content K* ∗ *f* (*m x*) *x*| ≤ *content K* ∗ *e*/(*4* ∗ *content* (*cbox a b*))
by (*simp add*: *algebra_simps*)
qed
also have ... = (*e*/(*4* ∗ *content* (*cbox a b*))) ∗ (∑(*x*, *k*)∈𝒟. *content k*)
by (*simp add*: *sum_distrib_left sum_divide_distrib split_def mult.commute*)
also have ... ≤ *e*/*4*
by (*metis False additive_content_tagged_division* [*OF ptag*] *nonzero_mult_divide_mult_cancel_right order_refl times_divide_eq_left*)
finally show |(∑(*x*,*K*)∈𝒟. *content K* ∗$_R$ *g x*) − (∑(*x*,*K*)∈𝒟. *content K* ∗$_R$ *f* (*m x*) *x*)| ≤ *e*/*4* **.**

next
have *norm* ((∑(*x*,*K*)∈𝒟. *content K* ∗$_R$ *f* (*m x*) *x*) − (∑(*x*,*K*)∈𝒟. *integral K* (*f* (*m x*))))
≤ *norm* (∑*j* = *0*..*s*. ∑(*x*,*K*)∈{*xk* ∈ 𝒟. *m* (*fst xk*) = *j*}. *content K* ∗$_R$ *f* (*m x*) *x* − *integral K* (*f* (*m x*)))
apply (*subst sum.group*)
using *s* **by** (*auto simp*: *sum_subtractf split_def p'*(*1*))
also have ... < *e*/*2*
proof −
have *norm* (∑*j* = *0*..*s*. ∑(*x*, *k*)∈{*xk* ∈ 𝒟. *m* (*fst xk*) = *j*}. *content k* ∗$_R$ *f* (*m x*) *x* − *integral k* (*f* (*m x*)))
≤ (∑*i* = *0*..*s*. *e*/*2* ^ (*i* + *2*))
proof (*rule sum_norm_le*)
fix *t*
assume *t* ∈ {*0*..*s*}
have *norm* (∑(*x*,*k*)∈{*xk* ∈ 𝒟. *m* (*fst xk*) = *t*}. *content k* ∗$_R$ *f* (*m x*) *x* − *integral k* (*f* (*m x*))) =
norm (∑(*x*,*k*)∈{*xk* ∈ 𝒟. *m* (*fst xk*) = *t*}. *content k* ∗$_R$ *f t x* − *integral k* (*f t*))
by (*force intro*!: *sum.cong arg_cong*[**where** *f*=*norm*])
also have ... ≤ *e*/*2* ^ (*t* + *2*)
proof (*rule Henstock_lemma_part1* [*OF intf*])
show {*xk* ∈ 𝒟. *m* (*fst xk*) = *t*} *tagged_partial_division_of cbox a b*
proof (*rule tagged_partial_division_subset*[*of* 𝒟])
show 𝒟 *tagged_partial_division_of cbox a b*
using *ptag tagged_division_of_def* **by** *blast*
qed *auto*
show *c t fine* {*xk* ∈ 𝒟. *m* (*fst xk*) = *t*}
using ‹*d fine* 𝒟› **by** (*auto simp*: *fine_def d_def*)

```
          qed (use c e in auto)
          finally show norm (∑(x,K)∈{xk ∈ 𝒟. m (fst xk) = t}. content K *R
f (m x) x −
                    integral K (f (m x))) ≤ e/2 ^ (t + 2) .
        qed
        also have ... = (e/2/2) * (∑i = 0..s. (1/2) ^ i)
          by (simp add: sum_distrib_left field_simps)
        also have ... < e/2
          by (simp add: sum_gp mult_strict_left_mono[OF _ e])
        finally show norm (∑j = 0..s. ∑(x, k)∈{xk ∈ 𝒟.
          m (fst xk) = j}. content k *R f (m x) x − integral k (f (m x))) < e/2 .
      qed
        finally show |(∑(x,K)∈𝒟. content K *R f (m x) x) − (∑(x,K)∈𝒟.
integral K (f (m x)))| < e/2
        by simp
    next
      have comb: integral (cbox a b) (f y) = (∑(x, k)∈𝒟. integral k (f y)) for y
        using integral_combine_tagged_division_topdown[OF intf ptag] by metis
      have f_le: ⋀y m n. ⟦y ∈ cbox a b; n≥m⟧ ⟹ f m y ≤ f n y
        using le by (auto intro: transitive_stepwise_le)
      have (∑(x, k)∈𝒟. integral k (f r)) ≤ (∑(x, K)∈𝒟. integral K (f (m x)))
      proof (rule sum_mono, simp add: split_paired_all)
        fix x K
        assume xK: (x, K) ∈ 𝒟
        show integral K (f r) ≤ integral K (f (m x))
        proof (rule integral_le)
          show f r integrable_on K
            by (metis integrable_on_subcbox intf p'(3) p'(4) xK)
          show f (m x) integrable_on K
           by (metis elementary_interval integrable_on_subdivision intf p'(3) p'(4)
xK)
          show f r y ≤ f (m x) y if y ∈ K for y
            using that r_le_m[of x] p'(2−3)[OF xK] f_le by auto
        qed
      qed
        moreover have (∑(x, K)∈𝒟. integral K (f (m x))) ≤ (∑(x, k)∈𝒟.
integral k (f s))
      proof (rule sum_mono, simp add: split_paired_all)
        fix x K
        assume xK: (x, K) ∈ 𝒟
        show integral K (f (m x)) ≤ integral K (f s)
        proof (rule integral_le)
          show f (m x) integrable_on K
           by (metis elementary_interval integrable_on_subdivision intf p'(3) p'(4)
xK)
          show f s integrable_on K
            by (metis integrable_on_subcbox intf p'(3) p'(4) xK)
          show f (m x) y ≤ f s y if y ∈ K for y
            using that s xK f_le p'(3) by fastforce
```

```
        qed
      qed
      moreover have 0 ≤ i − integral (cbox a b) (f r) i − integral (cbox a b) (f
r) < e/4
        using r by auto
      ultimately show |(∑(x,K)∈D. integral K (f (m x))) − i| < e/4
        using comb i'[of s] by auto
    qed
  qed
 qed
 with i integral_unique show ?thesis
   by blast
qed

lemma monotone_convergence_increasing:
 fixes f :: nat ⇒ 'n::euclidean_space ⇒ real
 assumes int_f: ⋀k. (f k) integrable_on S
   and ⋀k x. x ∈ S ⟹ (f k x) ≤ (f (Suc k) x)
   and fg: ⋀x. x ∈ S ⟹ ((λk. f k x) ⟶ g x) sequentially
   and bou: bounded (range (λk. integral S (f k)))
 shows g integrable_on S ∧ ((λk. integral S (f k)) ⟶ integral S g) sequentially
proof −
 have lem: g integrable_on S ∧ ((λk. integral S (f k)) ⟶ integral S g) sequentially
   if f0: ⋀k x. x ∈ S ⟹ 0 ≤ f k x
   and int_f: ⋀k. (f k) integrable_on S
   and le: ⋀k x. x ∈ S ⟹ f k x ≤ f (Suc k) x
   and lim: ⋀x. x ∈ S ⟹ ((λk. f k x) ⟶ g x) sequentially
   and bou: bounded (range(λk. integral S (f k)))
   for f :: nat ⇒ 'n::euclidean_space ⇒ real and g S
 proof −
   have fg: (f k x) ≤ (g x) if x ∈ S for x k
   proof −
     have ⋀xa. k ≤ xa ⟹ f k x ≤ f xa x
       using le by (force intro: transitive_stepwise_le that)
     then show ?thesis
       using tendsto_lowerbound [OF lim [OF that]] eventually_sequentiallyI by
force
   qed
   obtain i where i: (λk. integral S (f k)) ⟶ i
     using bounded_increasing_convergent [OF bou] le int_f integral_le by blast
   have i': (integral S (f k)) ≤ i for k
   proof −
     have ⋀k. ⋀x. x ∈ S ⟹ ∀n≥k. f k x ≤ f n x
       using le by (force intro: transitive_stepwise_le)
     then show ?thesis
     using tendsto_lowerbound [OF i eventually_sequentiallyI trivial_limit_sequentially]
       by (meson int_f integral_le)
   qed
   let ?f = (λk x. if x ∈ S then f k x else 0)
```

```
let ?g = (λx. if x ∈ S then g x else 0)
have int: ?f k integrable_on cbox a b for a b k
  by (simp add: int_f integrable_altD(1))
have int': ⋀k a b. f k integrable_on cbox a b ∩ S
  using int by (simp add: Int_commute integrable_restrict_Int)
have g: ?g integrable_on cbox a b ∧
        (λk. integral (cbox a b) (?f k)) ⟶ integral (cbox a b) ?g for a b
proof (rule monotone_convergence_interval)
  have norm (integral (cbox a b) (?f k)) ≤ norm (integral S (f k)) for k
  proof -
    have 0 ≤ integral (cbox a b) (?f k)
     by (metis (no_types) integral_nonneg Int_iff f0 inf_commute integral_restrict_Int
int')
    moreover have 0 ≤ integral S (f k)
      by (simp add: integral_nonneg f0 int_f)
    moreover have integral (S ∩ cbox a b) (f k) ≤ integral S (f k)
      by (metis f0 inf_commute int' int_f integral_subset_le le_inf_iff order_refl)
    ultimately show ?thesis
      by (simp add: integral_restrict_Int)
  qed
  moreover obtain B where ⋀x. x ∈ range (λk. integral S (f k)) ⟹ norm
x ≤ B
    using bou unfolding bounded_iff by blast
  ultimately show bounded (range (λk. integral (cbox a b) (?f k)))
    unfolding bounded_iff by (blast intro: order_trans)
qed (use int le lim in auto)
moreover have ∃B>0. ∀a b. ball 0 B ⊆ cbox a b ⟶ norm (integral (cbox a
b) ?g − i) < e
  if 0 < e for e
proof -
  have e/4>0
    using that by auto
 with LIMSEQ_D [OF i] obtain N where N: ⋀n. n ≥ N ⟹ norm (integral
S (f n) − i) < e/4
    by metis
  with int_f[of N, unfolded has_integral_integral has_integral_alt'[of f N]]
  obtain B where 0 < B and B:
    ⋀a b. ball 0 B ⊆ cbox a b ⟹ norm (integral (cbox a b) (?f N) − integral
S (f N)) < e/4
    by (meson ‹0 < e/4›)
  have norm (integral (cbox a b) ?g − i) < e if ab: ball 0 B ⊆ cbox a b for a
b
  proof -
     obtain M where M: ⋀n. n ≥ M ⟹ abs (integral (cbox a b) (?f n) −
integral (cbox a b) ?g) < e/2
       using ‹e > 0› g by (fastforce simp add: dest!: LIMSEQ_D [where r =
e/2])
     have *: ⋀α β g. ⟦|α − i| < e/2; |β − g| < e/2; α ≤ β; β ≤ i⟧ ⟹ |g −
i| < e
```

```
        unfolding real_inner_1_right by arith
      show norm (integral (cbox a b) ?g − i) < e
        unfolding real_norm_def
      proof (rule ∗)
        show |integral (cbox a b) (?f N) − i| < e/2
        proof (rule abs_triangle_half_l)
          show |integral (cbox a b) (?f N) − integral S (f N)| < e/2/2
            using B[OF ab] by simp
          show abs (i − integral S (f N)) < e/2/2
            using N by (simp add: abs_minus_commute)
        qed
        show |integral (cbox a b) (?f (M + N)) − integral (cbox a b) ?g| < e/2
          by (metis le_add1 M[of M + N])
        show integral (cbox a b) (?f N) ≤ integral (cbox a b) (?f (M + N))
        proof (intro ballI integral_le[OF int int])
          fix x assume x ∈ cbox a b
          have (f m x) ≤ (f n x) if x ∈ S n ≥ m for m n
          proof (rule transitive_stepwise_le [OF ‹n ≥ m› order_refl])
            show ⋀u y z. ⟦f u x ≤ f y x; f y x ≤ f z x⟧ ⟹ f u x ≤ f z x
              using dual_order.trans by blast
          qed (simp add: le ‹x ∈ S›)
          then show (?f N)x ≤ (?f (M+N))x
            by auto
        qed
        have integral (cbox a b ∩ S) (f (M + N)) ≤ integral S (f (M + N))
          by (metis Int_lower1 f0 inf_commute int' int_f integral_subset_le)
        then have integral (cbox a b) (?f (M + N)) ≤ integral S (f (M + N))
          by (metis (no_types) inf_commute integral_restrict_Int)
        also have ... ≤ i
          using i'[of M + N] by auto
        finally show integral (cbox a b) (?f (M + N)) ≤ i .
      qed
    qed
    then show ?thesis
      using ‹0 < B› by blast
  qed
  ultimately have (g has_integral i) S
    unfolding has_integral_alt' by auto
  then show ?thesis
    using has_integral_integrable_integral i integral_unique by metis
qed

have sub: ⋀k. integral S (λx. f k x − f 0 x) = integral S (f k) − integral S (f 0)
  by (simp add: integral_diff int_f)
have ∗: ⋀x m n. x ∈ S ⟹ n≥m ⟹ f m x ≤ f n x
  using assms(2) by (force intro: transitive_stepwise_le)
have gf: (λx. g x − f 0 x) integrable_on S ∧ ((λk. integral S (λx. f (Suc k) x −
f 0 x)) ⟶
  integral S (λx. g x − f 0 x)) sequentially
```

proof (*rule lem*)
  show $\bigwedge k$. (λx. *f* (*Suc k*) *x* − *f 0 x*) *integrable_on S*
    by (*simp add*: *integrable_diff int_f*)
  show (λk. *f* (*Suc k*) *x* − *f 0 x*) ⟶ *g x* − *f 0 x* **if** *x* ∈ *S* **for** *x*
  proof −
    have (λn. *f* (*Suc n*) *x*) ⟶ *g x*
      using *LIMSEQ_ignore_initial_segment*[*OF fg*[*OF* ‹*x* ∈ *S*›], *of 1*] **by** *simp*
    then show *?thesis*
      by (*simp add*: *tendsto_diff*)
  qed
  show *bounded* (*range* (λk. *integral S* (λx. *f* (*Suc k*) *x* − *f 0 x*)))
  proof −
    obtain *B* **where** *B*: $\bigwedge k$. *norm* (*integral S* (*f k*)) ≤ *B*
      using *bou* **by** (*auto simp*: *bounded_iff*)
    then have *norm* (*integral S* (λx. *f* (*Suc k*) *x* − *f 0 x*))
          ≤ *B* + *norm* (*integral S* (*f 0*)) **for** *k*
      unfolding *sub* **by** (*meson add_le_cancel_right norm_triangle_le_diff*)
    then show *?thesis*
      unfolding *bounded_iff* **by** *blast*
  qed
qed (*use* ∗ **in** *auto*)
then have (λx. *integral S* (λxa. *f* (*Suc x*) *xa* − *f 0 xa*) + *integral S* (*f 0*))
        ⟶ *integral S* (λx. *g x* − *f 0 x*) + *integral S* (*f 0*)
  by (*auto simp add*: *tendsto_add*)
moreover have (λx. *g x* − *f 0 x* + *f 0 x*) *integrable_on S*
  using *gf integrable_add int_f* [*of 0*] **by** *metis*
ultimately show *?thesis*
  by (*simp add*: *integral_diff int_f LIMSEQ_imp_Suc sub*)
qed

lemma *has_integral_monotone_convergence_increasing*:
  fixes *f* :: *nat* ⇒ *'a*::*euclidean_space* ⇒ *real*
  assumes *f*: $\bigwedge k$. (*f k has_integral x k*) *s*
  assumes $\bigwedge k\ x$. *x* ∈ *s* ⟹ *f k x* ≤ *f* (*Suc k*) *x*
  assumes $\bigwedge x$. *x* ∈ *s* ⟹ (λk. *f k x*) ⟶ *g x*
  assumes *x* ⟶ *x'*
  shows (*g has_integral x'*) *s*
proof −
  have *x_eq*: *x* = (λi. *integral s* (*f i*))
    by (*simp add*: *integral_unique*[*OF f*])
  then have *x*: *range*(λk. *integral s* (*f k*)) = *range x*
    by *auto*
  have ∗: *g integrable_on s* ∧ (λk. *integral s* (*f k*)) ⟶ *integral s g*
  proof (*intro monotone_convergence_increasing allI ballI assms*)
    show *bounded* (*range*(λk. *integral s* (*f k*)))
      using *x convergent_imp_bounded assms* **by** *metis*
  qed (*use f* **in** *auto*)
  then have *integral s g* = *x'*
    by (*intro LIMSEQ_unique*[*OF* _ ‹*x* ⟶ *x'*›]) (*simp add*: *x_eq*)

with ∗ **show** *?thesis*
by (*simp add*: *has_integral_integral*)
qed

lemma *monotone_convergence_decreasing*:
fixes f :: *nat* ⇒ *'n*::*euclidean_space* ⇒ *real*
assumes *intf*: $\bigwedge k.$ (f k) *integrable_on* S
and *le*: $\bigwedge k\ x.\ x \in S \Longrightarrow f\ (Suc\ k)\ x \leq f\ k\ x$
and *fg*: $\bigwedge x.\ x \in S \Longrightarrow ((\lambda k.\ f\ k\ x) \longrightarrow g\ x)$ *sequentially*
and *bou*: *bounded* (*range*($\lambda k.$ *integral* S (f k)))
shows g *integrable_on* S ∧ ($\lambda k.$ *integral* S (f k)) ⟶ *integral* S g
proof −
have ∗: *range*($\lambda k.$ *integral* S ($\lambda x.$ − f k x)) = ($*_R$) (− *1*) ‘ (*range*($\lambda k.$ *integral* S (f k)))
by *force*
have ($\lambda x.$ − g x) *integrable_on* S ∧ ($\lambda k.$ *integral* S ($\lambda x.$ − f k x)) ⟶ *integral* S ($\lambda x.$ − g x)
proof (*rule monotone_convergence_increasing*)
show $\bigwedge k.$ ($\lambda x.$ − f k x) *integrable_on* S
by (*blast intro*: *integrable_neg intf*)
show $\bigwedge k\ x.\ x \in S \Longrightarrow - f\ k\ x \leq - f\ (Suc\ k)\ x$
by (*simp add*: *le*)
show $\bigwedge x.\ x \in S \Longrightarrow (\lambda k.\ - f\ k\ x) \longrightarrow - g\ x$
by (*simp add*: *fg tendsto_minus*)
show *bounded* (*range*($\lambda k.$ *integral* S ($\lambda x.$ − f k x)))
using ∗ *bou bounded_scaling* **by** *auto*
qed
then show *?thesis*
by (*force dest*: *integrable_neg tendsto_minus*)
qed

lemma *integral_norm_bound_integral*:
fixes f :: *'n*::*euclidean_space* ⇒ *'a*::*banach*
assumes *int_f*: f *integrable_on* S
and *int_g*: g *integrable_on* S
and *le_g*: $\bigwedge x.\ x \in S \Longrightarrow$ *norm* (f x) ≤ g x
shows *norm* (*integral* S f) ≤ *integral* S g
proof −
have *norm*: *norm* $\eta \leq y + e$
if *norm* $\zeta \leq x$ **and** $|x - y| < e/2$ **and** *norm* $(\zeta - \eta) < e/2$
for e x y **and** ζ η :: *'a*
proof −
have *norm* $(\eta - \zeta) < e/2$
by (*metis norm_minus_commute that*(*3*))
moreover have $x \leq y + e/2$
using *that*(*2*) **by** *linarith*
ultimately show *?thesis*
using *that*(*1*) *le_less_trans*[*OF norm_triangle_sub*[*of* η ζ]] **by** (*auto simp*: *less_imp_le*)

```
  qed
  have lem: norm (integral(cbox a b) f) ≤ integral (cbox a b) g
    if f: f integrable_on cbox a b
    and g: g integrable_on cbox a b
    and nle: ⋀x. x ∈ cbox a b ⟹ norm (f x) ≤ g x
    for f :: 'n ⇒ 'a and g a b
  proof (rule field_le_epsilon)
    fix e :: real
    assume e > 0
    then have e: e/2 > 0
      by auto
    with integrable_integral[OF f,unfolded has_integral[of f]]
    obtain γ where γ: gauge γ
            ⋀𝒟. 𝒟 tagged_division_of cbox a b ∧ γ fine 𝒟
        ⟹ norm ((∑(x, k)∈𝒟. content k *R f x) − integral (cbox a b) f) < e/2
      by meson
    moreover
    from integrable_integral[OF g,unfolded has_integral[of g]] e
    obtain δ where δ: gauge δ
            ⋀𝒟. 𝒟 tagged_division_of cbox a b ∧ δ fine 𝒟
        ⟹ norm ((∑(x, k)∈𝒟. content k *R g x) − integral (cbox a b) g) < e/2
      by meson
    ultimately have gauge (λx. γ x ∩ δ x)
      using gauge_Int by blast
    with fine_division_exists obtain 𝒟
      where p: 𝒟 tagged_division_of cbox a b (λx. γ x ∩ δ x) fine 𝒟
      by metis
    have γ fine 𝒟 δ fine 𝒟
      using fine_Int p(2) by blast+
    show norm (integral (cbox a b) f) ≤ integral (cbox a b) g + e
    proof (rule norm)
      have norm (content K *R f x) ≤ content K *R g x if (x, K) ∈ 𝒟 for x K
      proof−
        have K: x ∈ K K ⊆ cbox a b
          using ‹(x, K) ∈ 𝒟› p(1) by blast+
        obtain u v where K = cbox u v
          using ‹(x, K) ∈ 𝒟› p(1) by blast
        moreover have content K * norm (f x) ≤ content K * g x
          by (meson K(1) K(2) content_pos_le mult_left_mono nle subsetD)
        then show ?thesis
          by simp
      qed
      then show norm (∑(x, k)∈𝒟. content k *R f x) ≤ (∑(x, k)∈𝒟. content k
*R g x)
        by (simp add: sum_norm_le split_def)
      show norm ((∑(x, k)∈𝒟. content k *R f x) − integral (cbox a b) f) < e/2
        using ‹γ fine 𝒟› γ p(1) by simp
      show |(∑(x, k)∈𝒟. content k *R g x) − integral (cbox a b) g| < e/2
        using ‹δ fine 𝒟› δ p(1) by simp
```

qed
qed
show *?thesis*
proof (*rule field_le_epsilon*)
fix *e* :: *real*
assume *e* > *0*
then have *e*: *e*/*2* > *0*
by *auto*
let *?f* = (λ*x*. *if x* ∈ *S then f x else 0*)
let *?g* = (λ*x*. *if x* ∈ *S then g x else 0*)
have *f*: *?f integrable_on cbox a b* **and** *g*: *?g integrable_on cbox a b* **for** *a b*
using *int_f int_g integrable_altD* **by** *auto*
obtain *Bf* **where** *0* < *Bf*
and *Bf*: ⋀*a b*. *ball 0 Bf* ⊆ *cbox a b* ⟹
∃ *z*. (*?f has_integral z*) (*cbox a b*) ∧ *norm* (*z* − *integral S f*) < *e*/*2*
using *integrable_integral* [*OF int_f*,*unfolded has_integral'*[*of f*]] *e that* **by** *blast*
obtain *Bg* **where** *0* < *Bg*
and *Bg*: ⋀*a b*. *ball 0 Bg* ⊆ *cbox a b* ⟹
∃ *z*. (*?g has_integral z*) (*cbox a b*) ∧ *norm* (*z* − *integral S g*) < *e*/*2*
using *integrable_integral* [*OF int_g*,*unfolded has_integral'*[*of g*]] *e that* **by** *blast*
obtain *a b*::*'n* **where** *ab*: *ball 0 Bf* ∪ *ball 0 Bg* ⊆ *cbox a b*
using *ball_max_Un* **by** (*metis bounded_ball bounded_subset_cbox_symmetric*)
have *ball 0 Bf* ⊆ *cbox a b*
using *ab* **by** *auto*
with *Bf* **obtain** *z* **where** *int_fz*: (*?f has_integral z*) (*cbox a b*) **and** *z*: *norm* (*z* − *integral S f*) < *e*/*2*
by *meson*
have *ball 0 Bg* ⊆ *cbox a b*
using *ab* **by** *auto*
with *Bg* **obtain** *w* **where** *int_gw*: (*?g has_integral w*) (*cbox a b*) **and** *w*: *norm* (*w* − *integral S g*) < *e*/*2*
by *meson*
show *norm* (*integral S f*) ≤ *integral S g* + *e*
proof (*rule norm*)
show *norm* (*integral* (*cbox a b*) *?f*) ≤ *integral* (*cbox a b*) *?g*
by (*simp add*: *le_g lem*[*OF f g*, *of a b*])
show |*integral* (*cbox a b*) *?g* − *integral S g*| < *e*/*2*
using *int_gw integral_unique w* **by** *auto*
show *norm* (*integral* (*cbox a b*) *?f* − *integral S f*) < *e*/*2*
using *int_fz integral_unique z* **by** *blast*
qed
qed
qed

lemma *continuous_on_imp_absolutely_integrable_on*:
fixes *f*::*real* ⇒ *'a*::*banach*
shows *continuous_on* {*a*..*b*} *f* ⟹
norm (*integral* {*a*..*b*} *f*) ≤ *integral* {*a*..*b*} (λ*x*. *norm* (*f x*))
by (*intro integral_norm_bound_integral integrable_continuous_real continuous_on_norm*)

auto

lemma *integral_bound*:
  fixes *f*::*real* ⇒ *'a*::*banach*
  assumes $a \leq b$
  assumes *continuous_on* {*a* .. *b*} *f*
  assumes ⋀*t*. *t* ∈ {*a* .. *b*} ⟹ *norm* (*f t*) ≤ *B*
  shows *norm* (*integral* {*a* .. *b*} *f*) ≤ *B* ∗ (*b* − *a*)
proof −
  note *continuous_on_imp_absolutely_integrable_on*[*OF assms*(*2*)]
  also have *integral* {*a*..*b*} (λ*x*. *norm* (*f x*)) ≤ *integral* {*a*..*b*} (λ_. *B*)
    by (*rule integral_le*)
      (*auto intro*!: *integrable_continuous_real continuous_intros assms*)
  also have ... = *B* ∗ (*b* − *a*) **using** *assms* **by** *simp*
  finally show *?thesis* **.**
qed

lemma *integral_norm_bound_integral_component*:
  fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*banach*
  fixes *g* :: *'n* ⇒ *'b*::*euclidean_space*
  assumes *f*: *f integrable_on S* **and** *g*: *g integrable_on S*
    and *fg*: ⋀*x*. *x* ∈ *S* ⟹ *norm*(*f x*) ≤ (*g x*)•*k*
  shows *norm* (*integral S f*) ≤ (*integral S g*)•*k*
proof −
  have *norm* (*integral S f*) ≤ *integral S* ((λ*x*. *x* • *k*) ∘ *g*)
    using *integral_norm_bound_integral*[*OF f integrable_linear*[*OF g*]]
    by (*simp add*: *bounded_linear_inner_left fg*)
  then show *?thesis*
    unfolding *o_def integral_component_eq*[*OF g*] **.**
qed

lemma *has_integral_norm_bound_integral_component*:
  fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*banach*
  fixes *g* :: *'n* ⇒ *'b*::*euclidean_space*
  assumes *f*: (*f has_integral i*) *S*
    and *g*: (*g has_integral j*) *S*
    and ⋀*x*. *x* ∈ *S* ⟹ *norm* (*f x*) ≤ (*g x*)•*k*
  shows *norm i* ≤ *j*•*k*
  using *integral_norm_bound_integral_component*[*of f S g k*]
  unfolding *integral_unique*[*OF f*] *integral_unique*[*OF g*]
  using *assms*
  by *auto*

lemma *uniformly_convergent_improper_integral*:
  fixes *f* :: *'b* ⇒ *real* ⇒ *'a* :: {*banach*}
  assumes *deriv*: ⋀*x*. *x* ≥ *a* ⟹ (*G has_field_derivative g x*) (*at x within* {*a*..})
  assumes *integrable*: ⋀*a' b x*. *x* ∈ *A* ⟹ *a'* ≥ *a* ⟹ *b* ≥ *a'* ⟹ *f x integrable_on* {*a'*..*b*}

assumes *G*: *convergent G*
assumes *le*: $\bigwedge y\ x.\ y \in A \Longrightarrow x \geq a \Longrightarrow norm\ (f\ y\ x) \leq g\ x$
shows *uniformly_convergent_on A* ($\lambda b\ x.$ *integral* {*a..b*} (*f x*))
proof (*intro Cauchy_uniformly_convergent uniformly_Cauchy_onI′*, *goal_cases*)
case (*1* ε)
from *G* **have** *Cauchy G*
by (*auto intro*!: *convergent_Cauchy*)
with *1* **obtain** *M* **where** *M*: *dist* (*G* (*real m*)) (*G* (*real n*)) $< \varepsilon$ **if** $m \geq M\ n \geq M$ **for** *m n*
by (*force simp*: *Cauchy_def*)
define *M′* **where** $M' = max\ (nat\ \lceil a \rceil)\ M$

show *?case*
proof (*rule exI*[*of _ M′*], *safe*, *goal_cases*)
case (*1 x m n*)
have *M′*: $M' \geq a\ M' \geq M$ **unfolding** *M′_def* **by** *linarith+*
have *int_g*: (*g has_integral* (*G* (*real n*) − *G* (*real m*))) {*real m..real n*}
using *1 M′* **by** (*intro fundamental_theorem_of_calculus*)
(*auto simp*: *has_field_derivative_iff_has_vector_derivative* [*symmetric*]

intro!: *DERIV_subset*[*OF deriv*])
have *int_f*: *f x integrable_on* {*a′..real n*} **if** $a' \geq a$ **for** *a′*
using *that 1* **by** (*cases* $a' \leq$ *real n*) (*auto intro*: *integrable*)

have *dist* (*integral* {*a..real m*} (*f x*)) (*integral* {*a..real n*} (*f x*)) =
norm (*integral* {*a..real n*} (*f x*) − *integral* {*a..real m*} (*f x*))
by (*simp add*: *dist_norm norm_minus_commute*)
also have *integral* {*a..real m*} (*f x*) + *integral* {*real m..real n*} (*f x*) =
integral {*a..real n*} (*f x*)
using *M′* **and** *1* **by** (*intro integral_combine int_f*) *auto*
hence *integral* {*a..real n*} (*f x*) − *integral* {*a..real m*} (*f x*) =
integral {*real m..real n*} (*f x*)
by (*simp add*: *algebra_simps*)
also have *norm* ... $\leq$ *integral* {*real m..real n*} *g*
using *le 1 M′ int_f int_g* **by** (*intro integral_norm_bound_integral*) *auto*
also from *int_g* **have** *integral* {*real m..real n*} *g* = *G* (*real n*) − *G* (*real m*)
by (*simp add*: *has_integral_iff*)
also have ... $\leq$ *dist* (*G m*) (*G n*)
by (*simp add*: *dist_norm*)
also from *1* **and** *M′* **have** ... $< \varepsilon$
by (*intro M*) *auto*
finally show *?case* **.**
qed
qed

lemma *uniformly_convergent_improper_integral′*:
fixes *f* :: *′b* ⇒ *real* ⇒ *′a* :: {*banach*, *real_normed_algebra*}
assumes *deriv*: $\bigwedge x.\ x \geq a \Longrightarrow$ (*G has_field_derivative g x*) (*at x within* {*a..*})
assumes *integrable*: $\bigwedge a'\ b\ x.\ x \in A \Longrightarrow a' \geq a \Longrightarrow b \geq a' \Longrightarrow$ *f x integrable_on*

{a′..b}
  assumes *G*: *convergent G*
  assumes *le*: *eventually* (λ*x*. ∀ *y*∈*A*. *norm* (*f y x*) ≤ *g x*) *at_top*
  shows *uniformly_convergent_on A* (λ*b x*. *integral* {*a*..*b*} (*f x*))
proof −
  from *le* **obtain** *a″* **where** *le*: ⋀*y x*. *y* ∈ *A* ⟹ *x* ≥ *a″* ⟹ *norm* (*f y x*) ≤ *g x*
    by (*auto simp*: *eventually_at_top_linorder*)
  define *a′* **where** *a′* = *max a a″*

  have *uniformly_convergent_on A* (λ*b x*. *integral* {*a′*..*real b*} (*f x*))
  proof (*rule uniformly_convergent_improper_integral*)
    fix *t* **assume** *t*: *t* ≥ *a′*
    hence (*G has_field_derivative g t*) (*at t within* {*a*..})
      by (*intro deriv*) (*auto simp*: *a′_def*)
    moreover have {*a′*..} ⊆ {*a*..} **unfolding** *a′_def* **by** *auto*
    ultimately show (*G has_field_derivative g t*) (*at t within* {*a′*..})
      by (*rule DERIV_subset*)
  qed (*insert le*, *auto simp*: *a′_def intro*: *integrable G*)
  hence *uniformly_convergent_on A* (λ*b x*. *integral* {*a*..*a′*} (*f x*) + *integral* {*a′*..*real b*} (*f x*))
    (**is** *?P*) **by** (*intro uniformly_convergent_add*) *auto*
  also have *eventually* (λ*x*. ∀ *y*∈*A*. *integral* {*a*..*a′*} (*f y*) + *integral* {*a′*..*x*} (*f y*) =
                    integral {*a*..*x*} (*f y*)) *sequentially*
     by (*intro eventually_mono* [*OF eventually_ge_at_top*[*of nat* ⌈*a′*⌉]] *ballI integral_combine*)
       (*auto simp*: *a′_def intro*: *integrable*)
  hence *?P* ⟷ *?thesis*
    by (*intro uniformly_convergent_cong*) *simp_all*
  finally show *?thesis* .
qed

6.15.42 differentiation under the integral sign

lemma *integral_continuous_on_param*:
  fixes *f*::′*a*::*topological_space* ⇒ ′*b*::*euclidean_space* ⇒ ′*c*::*banach*
  assumes *cont_fx*: *continuous_on* (*U* × *cbox a b*) (λ(*x*, *t*). *f x t*)
  shows *continuous_on U* (λ*x*. *integral* (*cbox a b*) (*f x*))
proof *cases*
  assume *content* (*cbox a b*) ≠ *0*
  then have *ne*: *cbox a b* ≠ {} **by** *auto*

  note [*continuous_intros*] =
    continuous_on_compose2[*OF cont_fx*, **where** *f*=λ*y*. *Pair x y* **for** *x*,
      unfolded split_beta fst_conv snd_conv]

  show *?thesis*
    unfolding *continuous_on_def*
  proof (*safe intro*!: *tendstoI*)

```
    fix e'::real and x
    assume e' > 0
    define e where e = e' / (content (cbox a b) + 1)
   have e > 0 using ‹e' > 0› by (auto simp: e_def intro!: divide_pos_pos add_nonneg_pos)
    assume x ∈ U
    from continuous_on_prod_compactE[OF cont_fx compact_cbox ‹x ∈ U› ‹0 < e›]
    obtain X0 where X0: x ∈ X0 open X0
      and fx_bound: ⋀y t. y ∈ X0 ∩ U ⟹ t ∈ cbox a b ⟹ norm (f y t − f x t)
≤ e
      unfolding split_beta fst_conv snd_conv dist_norm
      by metis
    have ∀F y in at x within U. y ∈ X0 ∩ U
      using X0(1) X0(2) eventually_at_topological by auto
     then show ∀F y in at x within U. dist (integral (cbox a b) (f y)) (integral
(cbox a b) (f x)) < e'
    proof eventually_elim
      case (elim y)
      have dist (integral (cbox a b) (f y)) (integral (cbox a b) (f x)) =
        norm (integral (cbox a b) (λt. f y t − f x t))
        using elim ‹x ∈ U›
        unfolding dist_norm
        by (subst integral_diff)
           (auto intro!: integrable_continuous continuous_intros)
      also have ... ≤ e ∗ content (cbox a b)
        using elim ‹x ∈ U›
        by (intro integrable_bound)
           (auto intro!: fx_bound ‹x ∈ U › less_imp_le[OF ‹0 < e›]
              integrable_continuous continuous_intros)
      also have ... < e'
        using ‹0 < e'› ‹e > 0›
        by (auto simp: e_def field_split_simps)
      finally show dist (integral (cbox a b) (f y)) (integral (cbox a b) (f x)) < e' .
    qed
  qed
qed (auto intro!: continuous_on_const)

lemma leibniz_rule:
  fixes f::'a::banach ⇒ 'b::euclidean_space ⇒ 'c::banach
  assumes fx: ⋀x t. x ∈ U ⟹ t ∈ cbox a b ⟹
    ((λx. f x t) has_derivative blinfun_apply (fx x t)) (at x within U)
  assumes integrable_f2: ⋀x. x ∈ U ⟹ f x integrable_on cbox a b
  assumes cont_fx: continuous_on (U × (cbox a b)) (λ(x, t). fx x t)
  assumes [intro]: x0 ∈ U
  assumes convex U
  shows
    ((λx. integral (cbox a b) (f x)) has_derivative integral (cbox a b) (fx x0)) (at x0
within U)
    (is (?F has_derivative ?dF) _)
proof cases
```

```
  assume content (cbox a b) ≠ 0
  then have ne: cbox a b ≠ {} by auto
  note [continuous_intros] =
    continuous_on_compose2[OF cont_fx, where f=λy. Pair x y for x,
      unfolded split_beta fst_conv snd_conv]
  show ?thesis
  proof (intro has_derivativeI bounded_linear_scaleR_left tendstoI, fold norm_conv_dist)
    have cont_f1: ⋀t. t ∈ cbox a b ⟹ continuous_on U (λx. f x t)
     by (auto simp: continuous_on_eq_continuous_within intro!: has_derivative_continuous
fx)
    note [continuous_intros] = continuous_on_compose2[OF cont_f1]
    fix e'::real
    assume e' > 0
    define e where e = e' / (content (cbox a b) + 1)
   have e > 0 using ‹e' > 0› by (auto simp: e_def intro!: divide_pos_pos add_nonneg_pos)
    from continuous_on_prod_compactE[OF cont_fx compact_cbox ‹x0 ∈ U› ‹e > 0›]
    obtain X0 where X0: x0 ∈ X0 open X0
      and fx_bound: ⋀x t. x ∈ X0 ∩ U ⟹ t ∈ cbox a b ⟹ norm (fx x t − fx x0
t) ≤ e
      unfolding split_beta fst_conv snd_conv
      by (metis dist_norm)

    note eventually_closed_segment[OF ‹open X0› ‹x0 ∈ X0›, of U]
    moreover
    have ∀F x in at x0 within U. x ∈ X0
      using ‹open X0› ‹x0 ∈ X0› eventually_at_topological by blast
    moreover have ∀F x in at x0 within U. x ≠ x0
      by (auto simp: eventually_at_filter)
    moreover have ∀F x in at x0 within U. x ∈ U
      by (auto simp: eventually_at_filter)
    ultimately
     show ∀F x in at x0 within U. norm ((?F x − ?F x0 − ?dF (x − x0)) /R
norm (x − x0)) < e'
    proof eventually_elim
      case (elim x)
      from elim have 0 < norm (x − x0) by simp
      have closed_segment x0 x ⊆ U
        by (rule ‹convex U›[unfolded convex_contains_segment, rule_format, OF ‹x0
∈ U› ‹x ∈ U›])
      from elim have [intro]: x ∈ U by auto
      have ?F x − ?F x0 − ?dF (x − x0) =
        integral (cbox a b) (λy. f x y − f x0 y − fx x0 y (x − x0))
        (is _ = ?id)
        using ‹x ≠ x0›
        by (subst blinfun_apply_integral integral_diff,
            auto intro!: integrable_diff integrable_f2 continuous_intros
              intro: integrable_continuous)+
      also
      {
```

```
      fix t assume t: t ∈ (cbox a b)
      have seg: ⋀t. t ∈ {0..1} ⟹ x0 + t *R (x − x0) ∈ X0 ∩ U
        using ‹closed_segment x0 x ⊆ U›
          ‹closed_segment x0 x ⊆ X0›
        by (force simp: closed_segment_def algebra_simps)
      from t have deriv:
        ((λx. f x t) has_derivative (fx y t)) (at y within X0 ∩ U)
        if y ∈ X0 ∩ U for y
        unfolding has_vector_derivative_def[symmetric]
        using that ‹x ∈ X0›
        by (intro has_derivative_subset[OF fx]) auto
      have ⋀x. x ∈ X0 ∩ U ⟹ onorm (blinfun_apply (fx x t) − (fx x0 t)) ≤ e
        using fx_bound t
     by (auto simp add: norm_blinfun_def fun_diff_def blinfun.bilinear_simps[symmetric])
      from differentiable_bound_linearization[OF seg deriv this] X0
      have norm (f x t − f x0 t − fx x0 t (x − x0)) ≤ e ∗ norm (x − x0)
        by (auto simp add: ac_simps)
    }
    then have norm ?id ≤ integral (cbox a b) (λ_. e ∗ norm (x − x0))
      by (intro integral_norm_bound_integral)
        (auto intro!: continuous_intros integrable_diff integrable_f2
          intro: integrable_continuous)
    also have ... = content (cbox a b) ∗ e ∗ norm (x − x0)
      by simp
    also have ... < e′ ∗ norm (x − x0)
    proof (intro mult_strict_right_mono[OF _ ‹0 < norm (x − x0)›])
      show content (cbox a b) ∗ e < e′
        using ‹e′ > 0› by (simp add: divide_simps e_def not_less)
    qed
    finally have norm (?F x − ?F x0 − ?dF (x − x0)) < e′ ∗ norm (x − x0) .
    then show ?case
      by (auto simp: divide_simps)
  qed
 qed (rule blinfun.bounded_linear_right)
qed (auto intro!: derivative_eq_intros simp: blinfun.bilinear_simps)

lemma has_vector_derivative_eq_has_derivative_blinfun:
  (f has_vector_derivative f′) (at x within U) ⟷
    (f has_derivative blinfun_scaleR_left f′) (at x within U)
  by (simp add: has_vector_derivative_def)

lemma leibniz_rule_vector_derivative:
  fixes f::real ⇒ ′b::euclidean_space ⇒ ′c::banach
  assumes fx: ⋀x t. x ∈ U ⟹ t ∈ cbox a b ⟹
      ((λx. f x t) has_vector_derivative (fx x t)) (at x within U)
  assumes integrable_f2: ⋀x. x ∈ U ⟹ (f x) integrable_on cbox a b
  assumes cont_fx: continuous_on (U × cbox a b) (λ(x, t). fx x t)
  assumes U: x0 ∈ U convex U
  shows ((λx. integral (cbox a b) (f x)) has_vector_derivative integral (cbox a b) (fx
```

```
x0))
    (at x0 within U)
proof -
  note [continuous_intros] =
    continuous_on_compose2[OF cont_fx, where f=λy. Pair x y for x,
      unfolded split_beta fst_conv snd_conv]
  show ?thesis
    unfolding has_vector_derivative_eq_has_derivative_blinfun
  proof (rule has_derivative_eq_rhs [OF leibniz_rule[OF _ integrable_f2 _ U]])
    show continuous_on (U × cbox a b) (λ(x, t). blinfun_scaleR_left (fx x t))
      using cont_fx by (auto simp: split_beta intro!: continuous_intros)
    show blinfun_apply (integral (cbox a b) (λt. blinfun_scaleR_left (fx x0 t))) =
        blinfun_apply (blinfun_scaleR_left (integral (cbox a b) (fx x0)))
    by (subst integral_linear[symmetric])
      (auto simp: has_vector_derivative_def o_def
        intro!: integrable_continuous U continuous_intros bounded_linear_intros)
  qed (use fx in ‹auto simp: has_vector_derivative_def›)
qed

lemma has_field_derivative_eq_has_derivative_blinfun:
  (f has_field_derivative f') (at x within U) ⟷ (f has_derivative blinfun_mult_right
f') (at x within U)
  by (simp add: has_field_derivative_def)

lemma leibniz_rule_field_derivative:
  fixes f::'a::{real_normed_field, banach} ⇒ 'b::euclidean_space ⇒ 'a
  assumes fx: ⋀x t. x ∈ U ⟹ t ∈ cbox a b ⟹ ((λx. f x t) has_field_derivative
fx x t) (at x within U)
  assumes integrable_f2: ⋀x. x ∈ U ⟹ (f x) integrable_on cbox a b
  assumes cont_fx: continuous_on (U × (cbox a b)) (λ(x, t). fx x t)
  assumes U: x0 ∈ U convex U
  shows ((λx. integral (cbox a b) (f x)) has_field_derivative integral (cbox a b) (fx
x0)) (at x0 within U)
proof -
  note [continuous_intros] =
    continuous_on_compose2[OF cont_fx, where f=λy. Pair x y for x,
      unfolded split_beta fst_conv snd_conv]
  have *: blinfun_mult_right (integral (cbox a b) (fx x0)) =
    integral (cbox a b) (λt. blinfun_mult_right (fx x0 t))
    by (subst integral_linear[symmetric])
     (auto simp: has_vector_derivative_def o_def
       intro!: integrable_continuous U continuous_intros bounded_linear_intros)
  show ?thesis
    unfolding has_field_derivative_eq_has_derivative_blinfun
  proof (rule has_derivative_eq_rhs [OF leibniz_rule[OF _ integrable_f2 _ U, where
fx=λx t. blinfun_mult_right (fx x t)]])
    show continuous_on (U × cbox a b) (λ(x, t). blinfun_mult_right (fx x t))
      using cont_fx by (auto simp: split_beta intro!: continuous_intros)
    show blinfun_apply (integral (cbox a b) (λt. blinfun_mult_right (fx x0 t))) =
```

blinfun_apply (*blinfun_mult_right* (*integral* (*cbox a b*) (*fx x0*)))
by (*subst integral_linear*[*symmetric*])
(*auto simp*: *has_vector_derivative_def o_def*
intro!: *integrable_continuous U continuous_intros bounded_linear_intros*)
qed (*use fx* **in** ⟨*auto simp*: *has_field_derivative_def*⟩)
qed

lemma *leibniz_rule_field_differentiable*:
fixes *f*::*'a*::{*real_normed_field*, *banach*} ⇒ *'b*::*euclidean_space* ⇒ *'a*
assumes ⋀*x t*. *x* ∈ *U* ⟹ *t* ∈ *cbox a b* ⟹ ((λ*x*. *f x t*) *has_field_derivative fx x t*) (*at x within U*)
assumes ⋀*x*. *x* ∈ *U* ⟹ (*f x*) *integrable_on cbox a b*
assumes *continuous_on* (*U* × (*cbox a b*)) (λ(*x*, *t*). *fx x t*)
assumes *x0* ∈ *U convex U*
shows (λ*x*. *integral* (*cbox a b*) (*f x*)) *field_differentiable at x0 within U*
using *leibniz_rule_field_derivative*[*OF assms*] **by** (*auto simp*: *field_differentiable_def*)

6.15.43 Exchange uniform limit and integral

lemma *uniform_limit_integral_cbox*:
fixes *f*::*'a* ⇒ *'b*::*euclidean_space* ⇒ *'c*::*banach*
assumes *u*: *uniform_limit* (*cbox a b*) *f g F*
assumes *c*: ⋀*n*. *continuous_on* (*cbox a b*) (*f n*)
assumes [*simp*]: *F* ≠ *bot*
obtains *I J* **where**
⋀*n*. (*f n has_integral I n*) (*cbox a b*)
(*g has_integral J*) (*cbox a b*)
(*I* ⟶ *J*) *F*
proof −
have *fi*[*simp*]: *f n integrable_on* (*cbox a b*) **for** *n*
by (*auto intro*!: *integrable_continuous assms*)
then obtain *I* **where** *I*: ⋀*n*. (*f n has_integral I n*) (*cbox a b*)
by *atomize_elim* (*auto simp*: *integrable_on_def intro*!: *choice*)

moreover
have *gi*[*simp*]: *g integrable_on* (*cbox a b*)
by (*auto intro*!: *integrable_continuous uniform_limit_theorem*[*OF _ u*] *eventuallyI c*)
then obtain *J* **where** *J*: (*g has_integral J*) (*cbox a b*)
by *blast*

moreover
have (*I* ⟶ *J*) *F*
proof *cases*
assume *content* (*cbox a b*) = *0*
hence *I* = (λ_. *0*) *J* = *0*
by (*auto intro*!: *has_integral_unique I J*)
thus *?thesis* **by** *simp*
next

```
    assume content_nonzero: content (cbox a b) ≠ 0
    show ?thesis
    proof (rule tendstoI)
      fix e::real
      assume e > 0
      define e' where e' = e/2
      with ‹e > 0› have e' > 0 by simp
     then have ∀F n in F. ∀x∈cbox a b. norm (f n x − g x) < e' / content (cbox
a b)
      using u content_nonzero by (auto simp: uniform_limit_iff dist_norm zero_less_measure_iff)
      then show ∀F n in F. dist (I n) J < e
      proof eventually_elim
        case (elim n)
        have I n = integral (cbox a b) (f n)
            J = integral (cbox a b) g
          using I[of n] J by (simp_all add: integral_unique)
        then have dist (I n) J = norm (integral (cbox a b) (λx. f n x − g x))
          by (simp add: integral_diff dist_norm)
        also have ... ≤ integral (cbox a b) (λx. (e' / content (cbox a b)))
          using elim
          by (intro integral_norm_bound_integral) (auto intro!: integrable_diff)
        also have ... < e
          using ‹0 < e›
          by (simp add: e'_def)
        finally show ?case .
      qed
    qed
  qed
  ultimately show ?thesis ..
qed

lemma uniform_limit_integral:
  fixes f::'a ⇒ 'b::ordered_euclidean_space ⇒ 'c::banach
  assumes u: uniform_limit {a..b} f g F
  assumes c: ⋀n. continuous_on {a..b} (f n)
  assumes [simp]: F ≠ bot
  obtains I J where
    ⋀n. (f n has_integral I n) {a..b}
    (g has_integral J) {a..b}
    (I ⟶ J) F
  by (metis interval_cbox assms uniform_limit_integral_cbox)
```

6.15.44 Integration by parts

```
lemma integration_by_parts_interior_strong:
  fixes prod :: _ ⇒ _ ⇒ 'b :: banach
  assumes bilinear: bounded_bilinear (prod)
  assumes s: finite s and le: a ≤ b
  assumes cont [continuous_intros]: continuous_on {a..b} f continuous_on {a..b}
```

g
assumes *deriv*: $\bigwedge x.\ x \in \{a<..<b\} - s \Longrightarrow (f\ has\_vector\_derivative\ f'\ x)\ (at\ x)$
$\bigwedge x.\ x \in \{a<..<b\} - s \Longrightarrow (g\ has\_vector\_derivative\ g'\ x)\ (at\ x)$
assumes *int*: $((\lambda x.\ prod\ (f\ x)\ (g'\ x))\ has\_integral$
$(prod\ (f\ b)\ (g\ b) - prod\ (f\ a)\ (g\ a) - y))\ \{a..b\}$
shows $((\lambda x.\ prod\ (f'\ x)\ (g\ x))\ has\_integral\ y)\ \{a..b\}$
proof −
interpret *bounded_bilinear prod* **by** *fact*
have $((\lambda x.\ prod\ (f\ x)\ (g'\ x) + prod\ (f'\ x)\ (g\ x))\ has\_integral$
$(prod\ (f\ b)\ (g\ b) - prod\ (f\ a)\ (g\ a)))\ \{a..b\}$
using *deriv* **by** (*intro fundamental_theorem_of_calculus_interior_strong*[*OF s le*])
(*auto intro*!: *continuous_intros continuous_on has_vector_derivative*)
from *has_integral_diff* [*OF this int*] **show** *?thesis* **by** (*simp add*: *algebra_simps*)
qed

lemma *integration_by_parts_interior*:
fixes *prod* :: _ ⇒ _ ⇒ *'b* :: *banach*
assumes *bounded_bilinear* (*prod*) $a \le b$
continuous_on {*a..b*} *f continuous_on* {*a..b*} *g*
assumes $\bigwedge x.\ x \in \{a<..<b\} \Longrightarrow (f\ has\_vector\_derivative\ f'\ x)\ (at\ x)$
$\bigwedge x.\ x \in \{a<..<b\} \Longrightarrow (g\ has\_vector\_derivative\ g'\ x)\ (at\ x)$
assumes $((\lambda x.\ prod\ (f\ x)\ (g'\ x))\ has\_integral\ (prod\ (f\ b)\ (g\ b) - prod\ (f\ a)\ (g\ a) - y))\ \{a..b\}$
shows $((\lambda x.\ prod\ (f'\ x)\ (g\ x))\ has\_integral\ y)\ \{a..b\}$
by (*rule integration_by_parts_interior_strong*[*of* _ {} _ _ *f g f' g'*]) (*insert assms*, *simp_all*)

lemma *integration_by_parts*:
fixes *prod* :: _ ⇒ _ ⇒ *'b* :: *banach*
assumes *bounded_bilinear* (*prod*) $a \le b$
continuous_on {*a..b*} *f continuous_on* {*a..b*} *g*
assumes $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (f\ has\_vector\_derivative\ f'\ x)\ (at\ x)$
$\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g\ has\_vector\_derivative\ g'\ x)\ (at\ x)$
assumes $((\lambda x.\ prod\ (f\ x)\ (g'\ x))\ has\_integral\ (prod\ (f\ b)\ (g\ b) - prod\ (f\ a)\ (g\ a) - y))\ \{a..b\}$
shows $((\lambda x.\ prod\ (f'\ x)\ (g\ x))\ has\_integral\ y)\ \{a..b\}$
by (*rule integration_by_parts_interior*[*of* _ _ _ *f g f' g'*]) (*insert assms*, *simp_all*)

lemma *integrable_by_parts_interior_strong*:
fixes *prod* :: _ ⇒ _ ⇒ *'b* :: *banach*
assumes *bilinear*: *bounded_bilinear* (*prod*)
assumes *s*: *finite s* **and** *le*: $a \le b$
assumes *cont* [*continuous_intros*]: *continuous_on* {*a..b*} *f continuous_on* {*a..b*} *g*
assumes *deriv*: $\bigwedge x.\ x \in \{a<..<b\} - s \Longrightarrow (f\ has\_vector\_derivative\ f'\ x)\ (at\ x)$
$\bigwedge x.\ x \in \{a<..<b\} - s \Longrightarrow (g\ has\_vector\_derivative\ g'\ x)\ (at\ x)$
assumes *int*: $(\lambda x.\ prod\ (f\ x)\ (g'\ x))\ integrable\_on\ \{a..b\}$
shows $(\lambda x.\ prod\ (f'\ x)\ (g\ x))\ integrable\_on\ \{a..b\}$
proof −

from *int* **obtain** *I* **where** $((\lambda x.\ prod\ (f\ x)\ (g'\ x))\ has\_integral\ I)\ \{a..b\}$
unfolding *integrable_on_def* **by** *blast*
hence $((\lambda x.\ prod\ (f\ x)\ (g'\ x))\ has\_integral\ (prod\ (f\ b)\ (g\ b) - prod\ (f\ a)\ (g\ a) - (prod\ (f\ b)\ (g\ b) - prod\ (f\ a)\ (g\ a) - I)))\ \{a..b\}$ **by** *simp*
from *integration_by_parts_interior_strong*[*OF assms*(*1*−*7*) *this*]
show *?thesis* **unfolding** *integrable_on_def* **by** *blast*
qed

lemma *integrable_by_parts_interior*:
fixes $prod :: \_ \Rightarrow \_ \Rightarrow 'b :: banach$
assumes $bounded\_bilinear\ (prod)\ a \leq b$
$continuous\_on\ \{a..b\}\ f\ continuous\_on\ \{a..b\}\ g$
assumes $\bigwedge x.\ x \in \{a<..<b\} \Longrightarrow (f\ has\_vector\_derivative\ f'\ x)\ (at\ x)$
$\bigwedge x.\ x \in \{a<..<b\} \Longrightarrow (g\ has\_vector\_derivative\ g'\ x)\ (at\ x)$
assumes $(\lambda x.\ prod\ (f\ x)\ (g'\ x))\ integrable\_on\ \{a..b\}$
shows $(\lambda x.\ prod\ (f'\ x)\ (g\ x))\ integrable\_on\ \{a..b\}$
by (*rule integrable_by_parts_interior_strong*[*of* _ {} _ _ *f g f' g'*]) (*insert assms*, *simp_all*)

lemma *integrable_by_parts*:
fixes $prod :: \_ \Rightarrow \_ \Rightarrow 'b :: banach$
assumes $bounded\_bilinear\ (prod)\ a \leq b$
$continuous\_on\ \{a..b\}\ f\ continuous\_on\ \{a..b\}\ g$
assumes $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (f\ has\_vector\_derivative\ f'\ x)\ (at\ x)$
$\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g\ has\_vector\_derivative\ g'\ x)\ (at\ x)$
assumes $(\lambda x.\ prod\ (f\ x)\ (g'\ x))\ integrable\_on\ \{a..b\}$
shows $(\lambda x.\ prod\ (f'\ x)\ (g\ x))\ integrable\_on\ \{a..b\}$
by (*rule integrable_by_parts_interior_strong*[*of* _ {} _ _ *f g f' g'*]) (*insert assms*, *simp_all*)

6.15.45 Integration by substitution

lemma *has_integral_substitution_general*:
fixes $f :: real \Rightarrow 'a::euclidean\_space$ **and** $g :: real \Rightarrow real$
assumes *s*: *finite s* **and** *le*: $a \leq b$
and *subset*: $g\ `\ \{a..b\} \subseteq \{c..d\}$
and *f* [*continuous_intros*]: $continuous\_on\ \{c..d\}\ f$
and *g* [*continuous_intros*]: $continuous\_on\ \{a..b\}\ g$
and *deriv* [*derivative_intros*]:
$\bigwedge x.\ x \in \{a..b\} - s \Longrightarrow (g\ has\_field\_derivative\ g'\ x)\ (at\ x\ within\ \{a..b\})$
shows $((\lambda x.\ g'\ x *_R f\ (g\ x))\ has\_integral\ (integral\ \{g\ a..g\ b\}\ f - integral\ \{g\ b..g\ a\}\ f))\ \{a..b\}$
proof −
let $?F = \lambda x.\ integral\ \{c..g\ x\}\ f$
have *cont_int*: $continuous\_on\ \{a..b\}\ ?F$
by (*rule continuous_on_compose2*[*OF* _ *g subset*] *indefinite_integral_continuous_1 f integrable_continuous_real*)+
have *deriv*: $(((\lambda x.\ integral\ \{c..x\}\ f) \circ g)\ has\_vector\_derivative\ g'\ x *_R f\ (g\ x))\ (at\ x\ within\ \{a..b\})$ **if** $x \in \{a..b\} - s$ **for** *x*

proof (*rule has_vector_derivative_eq_rhs* [*OF vector_diff_chain_within refl*])
show (*g has_vector_derivative g′ x*) (*at x within* {*a..b*})
using *deriv has_field_derivative_iff_has_vector_derivative that* **by** *blast*
show ((λ*x*. *integral* {*c..x*} *f*) *has_vector_derivative f* (*g x*))
(*at* (*g x*) *within g* ‘ {*a..b*})
using *that le subset*
by (*blast intro*: *has_vector_derivative_within_subset integral_has_vector_derivative f*)
qed
have *deriv*: (*?F has_vector_derivative g′ x* $*_R$ *f* (*g x*))
(*at x*) **if** *x* ∈ {*a..b*} − (*s* ∪ {*a*,*b*}) **for** *x*
using *deriv*[*of x*] *that* **by** (*simp add*: *at_within_Icc_at o_def*)
have ((λ*x*. *g′ x* $*_R$ *f* (*g x*)) *has_integral* (*?F b* − *?F a*)) {*a..b*}
using *le cont_int s deriv cont_int*
by (*intro fundamental_theorem_of_calculus_interior_strong*[*of s* ∪ {*a*,*b*}]) *simp_all*
also
from *subset* **have** *g x* ∈ {*c..d*} **if** *x* ∈ {*a..b*} **for** *x* **using** *that* **by** *blast*
from *this*[*of a*] *this*[*of b*] *le* **have** *cd*: *c* ≤ *g a g b* ≤ *d c* ≤ *g b g a* ≤ *d* **by** *auto*
have *integral* {*c..g b*} *f* − *integral* {*c..g a*} *f* = *integral* {*g a..g b*} *f* − *integral* {*g b..g a*} *f*
proof *cases*
assume *g a* ≤ *g b*
note *le* = *le this*
from *cd* **have** *integral* {*c..g a*} *f* + *integral* {*g a..g b*} *f* = *integral* {*c..g b*} *f*
by (*intro integral_combine integrable_continuous_real continuous_on_subset*[*OF f*] *le*) *simp_all*
with *le* **show** *?thesis*
by (*cases g a* = *g b*) (*simp_all add*: *algebra_simps*)
next
assume *less*: ¬*g a* ≤ *g b*
then have *g a* ≥ *g b* **by** *simp*
note *le* = *le this*
from *cd* **have** *integral* {*c..g b*} *f* + *integral* {*g b..g a*} *f* = *integral* {*c..g a*} *f*
by (*intro integral_combine integrable_continuous_real continuous_on_subset*[*OF f*] *le*) *simp_all*
with *less* **show** *?thesis*
by (*simp_all add*: *algebra_simps*)
qed
finally show *?thesis* **.**
qed

lemma *has_integral_substitution_strong*:
fixes *f* :: *real* ⇒ ′*a*::*euclidean_space* **and** *g* :: *real* ⇒ *real*
assumes *s*: *finite s* **and** *le*: *a* ≤ *b g a* ≤ *g b*
and *subset*: *g* ‘ {*a..b*} ⊆ {*c..d*}
and *f* [*continuous_intros*]: *continuous_on* {*c..d*} *f*
and *g* [*continuous_intros*]: *continuous_on* {*a..b*} *g*
and *deriv* [*derivative_intros*]:
⋀*x*. *x* ∈ {*a..b*} − *s* ⟹ (*g has_field_derivative g′ x*) (*at x within* {*a..b*})

shows ((λx. g′ x *R f (g x)) *has_integral* (*integral* {g a..g b} f)) {a..b}
using *has_integral_substitution_general*[*OF s le*(*1*) *subset f g deriv*] *le*(*2*)
by (*cases g a = g b*) *auto*

lemma *has_integral_substitution*:
fixes f :: *real* ⇒ ′a::*euclidean_space* **and** g :: *real* ⇒ *real*
assumes a ≤ b g a ≤ g b g ‘ {a..b} ⊆ {c..d}
and *continuous_on* {c..d} f
and ⋀x. x ∈ {a..b} ⟹ (g *has_field_derivative* g′ x) (*at* x *within* {a..b})
shows ((λx. g′ x *R f (g x)) *has_integral* (*integral* {g a..g b} f)) {a..b}
by (*intro has_integral_substitution_strong*[*of* {} a b g c d] *assms*)
(*auto intro*: *DERIV_continuous_on assms*)

lemma *integral_shift*:
fixes f :: *real* ⇒ ′a::*euclidean_space*
assumes *cont*: *continuous_on* {a + c..b + c} f
shows *integral* {a..b} (f ∘ (λx. x + c)) = *integral* {a + c..b + c} f
proof (*cases* a ≤ b)
case *True*
have ((λx. *1* *R f (x + c)) *has_integral integral* {a+c..b+c} f) {a..b}
using *True cont*
by (*intro has_integral_substitution*[**where** c = a + c **and** d = b + c])
(*auto intro*!: *derivative_eq_intros*)
thus *?thesis* **by** (*simp add*: *has_integral_iff o_def*)
qed *auto*

6.15.46 Compute a double integral using iterated integrals and switching the order of integration

lemma *continuous_on_imp_integrable_on_Pair1*:
fixes f :: _ ⇒ ′b::*banach*
assumes *con*: *continuous_on* (*cbox* (a,c) (b,d)) f **and** x: x ∈ *cbox* a b
shows (λy. f (x, y)) *integrable_on* (*cbox* c d)
proof −
have f ∘ (λy. (x, y)) *integrable_on* (*cbox* c d)
proof (*intro integrable_continuous continuous_on_compose* [*OF* _ *continuous_on_subset* [*OF con*]])
show *continuous_on* (*cbox* c d) (*Pair* x)
by (*simp add*: *continuous_on_Pair*)
show *Pair* x ‘ *cbox* c d ⊆ *cbox* (a,c) (b,d)
using x **by** *blast*
qed
then show *?thesis*
by (*simp add*: *o_def*)
qed

lemma *integral_integrable_2dim*:
fixes f :: (′a::*euclidean_space* * ′b::*euclidean_space*) ⇒ ′c::*banach*
assumes *continuous_on* (*cbox* (a,c) (b,d)) f

shows (*λx. integral* (*cbox c d*) (*λy. f* (*x*,*y*))) *integrable_on cbox a b*
proof (*cases content*(*cbox c d*) = *0*)
case *True*
then show *?thesis*
by (*simp add*: *True integrable_const*)
next
case *False*
have *uc*: *uniformly_continuous_on* (*cbox* (*a*,*c*) (*b*,*d*)) *f*
by (*simp add*: *assms compact_cbox compact_uniformly_continuous*)
{ fix *x*::*'a* **and** *e*::*real*
assume *x*: *x* ∈ *cbox a b* **and** *e*: *0* < *e*
then have *e2_gt*: *0* < *e*/*2* / *content* (*cbox c d*) **and** *e2_less*: *e*/*2* / *content* (*cbox c d*) ∗ *content* (*cbox c d*) < *e*
by (*auto simp*: *False content_lt_nz e*)
then obtain *dd*
where *dd*: ⋀*x x'*. ⟦*x*∈*cbox* (*a*, *c*) (*b*, *d*); *x'*∈*cbox* (*a*, *c*) (*b*, *d*); *norm* (*x'* − *x*) < *dd*⟧
⟹ *norm* (*f x'* − *f x*) ≤ *e*/(*2* ∗ *content* (*cbox c d*)) *dd*>*0*
using *uc* [*unfolded uniformly_continuous_on_def*, *THEN spec*, *of e*/(*2* ∗ *content* (*cbox c d*))]
by (*auto simp*: *dist_norm intro*: *less_imp_le*)
have ∃ *delta*>*0*. ∀ *x'*∈*cbox a b*. *norm* (*x'* − *x*) < *delta* ⟶ *norm* (*integral* (*cbox c d*) (*λu. f* (*x'*, *u*) − *f* (*x*, *u*))) < *e*
using *dd e2_gt assms x*
apply (*rule_tac x*=*dd* **in** *exI*)
apply *clarify*
apply (*rule le_less_trans* [*OF integrable_bound e2_less*])
apply (*auto intro*: *integrable_diff continuous_on_imp_integrable_on_Pair1*)
done
} note ∗ = *this*
show *?thesis*
proof (*rule integrable_continuous*)
show *continuous_on* (*cbox a b*) (*λx. integral* (*cbox c d*) (*λy. f* (*x*, *y*)))
by (*simp add*: ∗ *continuous_on_iff dist_norm integral_diff* [*symmetric*] *continuous_on_imp_integrable_on_Pair1* [*OF assms*])
qed
qed

lemma *integral_split*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::{*real_normed_vector*,*complete_space*}
assumes *f*: *f integrable_on* (*cbox a b*)
and *k*: *k* ∈ *Basis*
shows *integral* (*cbox a b*) *f* =
integral (*cbox a b* ∩ {*x. x*•*k* ≤ *c*}) *f* +
integral (*cbox a b* ∩ {*x. x*•*k* ≥ *c*}) *f*
using *k f*
by (*auto simp*: *has_integral_integral intro*: *integral_unique* [*OF has_integral_split*])

lemma *integral_swap_operativeI*:

```
  fixes f :: ('a::euclidean_space * 'b::euclidean_space) ⇒ 'c::banach
  assumes f: continuous_on s f and e: 0 < e
    shows operative conj True
           (λk. ∀ a b c d.
                cbox (a,c) (b,d) ⊆ k ∧ cbox (a,c) (b,d) ⊆ s
                ⟶ norm(integral (cbox (a,c) (b,d)) f −
                         integral (cbox a b) (λx. integral (cbox c d) (λy. f((x,y)))))
                     ≤ e * content (cbox (a,c) (b,d)))
proof (standard, auto)
  fix a::'a and c::'b and b::'a and d::'b and u::'a and v::'a and w::'b and z::'b
  assume *: box (a, c) (b, d) = {}
     and cb1: cbox (u, w) (v, z) ⊆ cbox (a, c) (b, d)
     and cb2: cbox (u, w) (v, z) ⊆ s
  then have c0: content (cbox (a, c) (b, d)) = 0
    using * unfolding content_eq_0_interior by simp
  have c0': content (cbox (u, w) (v, z)) = 0
    by (fact content_0_subset [OF c0 cb1])
   show norm (integral (cbox (u,w) (v,z)) f − integral (cbox u v) (λx. integral
(cbox w z) (λy. f (x, y))))
          ≤ e * content (cbox (u,w) (v,z))
    using content_cbox_pair_eq0_D [OF c0']
    by (force simp add: c0')
next
  fix a::'a and c::'b and b::'a and d::'b
  and M::real and i::'a and j::'b
  and u::'a and v::'a and w::'b and z::'b
  assume ij: (i,j) ∈ Basis
     and n1: ∀ a' b' c' d'.
                cbox (a',c') (b',d') ⊆ cbox (a,c) (b,d) ∧
                 cbox (a',c') (b',d') ⊆ {x. x · (i,j) ≤ M} ∧ cbox (a',c') (b',d') ⊆ s
⟶
                  norm (integral (cbox (a',c') (b',d')) f − integral (cbox a' b') (λx.
integral (cbox c' d') (λy. f (x,y))))
                ≤ e * content (cbox (a',c') (b',d'))
     and n2: ∀ a' b' c' d'.
                cbox (a',c') (b',d') ⊆ cbox (a,c) (b,d) ∧
                 cbox (a',c') (b',d') ⊆ {x. M ≤ x · (i,j)} ∧ cbox (a',c') (b',d') ⊆ s
⟶
                  norm (integral (cbox (a',c') (b',d')) f − integral (cbox a' b') (λx.
integral (cbox c' d') (λy. f (x,y))))
                ≤ e * content (cbox (a',c') (b',d'))
     and subs: cbox (u,w) (v,z) ⊆ cbox (a,c) (b,d)  cbox (u,w) (v,z) ⊆ s
  have fcont: continuous_on (cbox (u, w) (v, z)) f
    using assms(1) continuous_on_subset  subs(2) by blast
  then have fint: f integrable_on cbox (u, w) (v, z)
    by (metis integrable_continuous)
  consider i ∈ Basis j=0 | j ∈ Basis i=0  using ij
    by (auto simp: Euclidean_Space.Basis_prod_def)
 then show norm (integral (cbox (u,w) (v,z)) f − integral (cbox u v) (λx. integral
```

(*cbox w z*) (λ*y*. *f* (*x*,*y*))))
  ≤ *e* ∗ *content* (*cbox* (*u*,*w*) (*v*,*z*)) (**is** *?normle*)
proof *cases*
case *1*
then have *e*: *e* ∗ *content* (*cbox* (*u*, *w*) (*v*, *z*)) =
  e ∗ (*content* (*cbox u v* ∩ {*x*. *x* · *i* ≤ *M*}) ∗ *content* (*cbox w z*)) +
  e ∗ (*content* (*cbox u v* ∩ {*x*. *M* ≤ *x* · *i*}) ∗ *content* (*cbox w z*))
by (*simp add*: *content_split* [**where** *c*=*M*] *content_Pair algebra_simps*)
have ∗: *integral* (*cbox u v*) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*, *y*))) =
  integral (*cbox u v* ∩ {*x*. *x* · *i* ≤ *M*}) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*, *y*))) +
  integral (*cbox u v* ∩ {*x*. *M* ≤ *x* · *i*}) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*, *y*)))
using *1 f subs integral_integrable_2dim continuous_on_subset*
by (*blast intro*: *integral_split*)
show *?normle*
apply (*rule norm_diff2* [*OF integral_split* [**where** *c*=*M*, *OF fint ij*] ∗ *e*])
using *1 subs*
apply (*simp_all add*: *cbox_Pair_eq setcomp_dot1* [*of* λ*u*. *M*≤*u*] *setcomp_dot1* [*of* λ*u*. *u*≤*M*] *Sigma_Int_Paircomp1*)
apply (*simp_all add*: *interval_split ij flip*: *cbox_Pair_eq content_Pair*)
apply (*force simp flip*: *interval_split intro*!: *n1* [*rule_format*])
apply (*force simp flip*: *interval_split intro*!: *n2* [*rule_format*])
done
next
case *2*
then have *e*: *e* ∗ *content* (*cbox* (*u*, *w*) (*v*, *z*)) =
  e ∗ (*content* (*cbox u v*) ∗ *content* (*cbox w z* ∩ {*x*. *x* · *j* ≤ *M*})) +
  e ∗ (*content* (*cbox u v*) ∗ *content* (*cbox w z* ∩ {*x*. *M* ≤ *x* · *j*}))
by (*simp add*: *content_split* [**where** *c*=*M*] *content_Pair algebra_simps*)
have (λ*x*. *integral* (*cbox w z* ∩ {*x*. *x* · *j* ≤ *M*}) (λ*y*. *f* (*x*, *y*))) *integrable_on cbox u v*
  (λ*x*. *integral* (*cbox w z* ∩ {*x*. *M* ≤ *x* · *j*}) (λ*y*. *f* (*x*, *y*))) *integrable_on cbox u v*
using *2 subs*
apply (*simp_all add*: *interval_split*)
apply (*rule integral_integrable_2dim* [*OF continuous_on_subset* [*OF f*]]; *auto simp*: *cbox_Pair_eq interval_split* [*symmetric*])+
done
with *2* **have** ∗: *integral* (*cbox u v*) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*, *y*))) =
  integral (*cbox u v*) (λ*x*. *integral* (*cbox w z* ∩ {*x*. *x* · *j* ≤ *M*}) (λ*y*. *f* (*x*, *y*))) +
  integral (*cbox u v*) (λ*x*. *integral* (*cbox w z* ∩ {*x*. *M* ≤ *x* · *j*}) (λ*y*. *f* (*x*, *y*)))
by (*simp add*: *integral_add* [*symmetric*] *integral_split* [*symmetric*]
  continuous_on_imp_integrable_on_Pair1 [*OF fcont*] *cong*: *integral_cong*)
show *?normle*
apply (*rule norm_diff2* [*OF integral_split* [**where** *c*=*M*, *OF fint ij*] ∗ *e*])
using *2 subs*

```
    apply (simp_all add: cbox_Pair_eq setcomp_dot2 [of λu. M≤u] setcomp_dot2
[of λu. u≤M] Sigma_Int_Paircomp2)
    apply (simp_all add: interval_split ij flip: cbox_Pair_eq content_Pair)
    apply (force simp flip: interval_split intro!: n1 [rule_format])
    apply (force simp flip: interval_split intro!: n2 [rule_format])
    done
  qed
qed

lemma integral_Pair_const:
    integral (cbox (a,c) (b,d)) (λx. k) =
     integral (cbox a b) (λx. integral (cbox c d) (λy. k))
  by (simp add: content_Pair)

lemma integral_prod_continuous:
  fixes f :: ('a::euclidean_space * 'b::euclidean_space) ⇒ 'c::banach
  assumes continuous_on (cbox (a, c) (b, d)) f (is continuous_on ?CBOX f)
    shows integral (cbox (a, c) (b, d)) f = integral (cbox a b) (λx. integral (cbox
c d) (λy. f (x, y)))
proof (cases content ?CBOX = 0)
  case True
  then show ?thesis
    by (auto simp: content_Pair)
next
  case False
  then have cbp: content ?CBOX > 0
    using content_lt_nz by blast
  have norm (integral ?CBOX f − integral (cbox a b) (λx. integral (cbox c d) (λy.
f (x,y)))) = 0
  proof (rule dense_eq0_I, simp)
    fix e :: real
    assume 0 < e
    with ‹content ?CBOX > 0› have 0 < e/content ?CBOX
      by simp
    have f_int_acbd: f integrable_on ?CBOX
      by (rule integrable_continuous [OF assms])
    { fix p
      assume p: p division_of ?CBOX
      then have finite p
        by blast
      define e' where e' = e/content ?CBOX
      with ‹0 < e› ‹0 < e/content ?CBOX›
      have 0 < e'
        by simp
      interpret operative conj True
           λk. ∀ a' b' c' d'.
                cbox (a', c') (b', d') ⊆ k ∧ cbox (a', c') (b', d') ⊆ ?CBOX
                ⟶ norm (integral (cbox (a', c') (b', d')) f −
                       integral (cbox a' b') (λx. integral (cbox c' d') (λy. f ((x, y)))))
```

$\leq e' * content\ (cbox\ (a', c')\ (b', d'))$
using *assms* ‹0 < e'› **by** (*rule integral_swap_operativeI*)
have *norm* (*integral ?CBOX f* − *integral* (*cbox a b*) (λ*x*. *integral* (*cbox c d*) (λ*y*. *f* (*x*, *y*))))
≤ *e'* ∗ *content ?CBOX*
if ⋀*t u v w z*. *t* ∈ *p* ⟹ *cbox* (*u*, *w*) (*v*, *z*) ⊆ *t* ⟹ *cbox* (*u*, *w*) (*v*, *z*) ⊆ *?CBOX*
⟹ *norm* (*integral* (*cbox* (*u*, *w*) (*v*, *z*)) *f* −
integral (*cbox u v*) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*, *y*))))
≤ *e'* ∗ *content* (*cbox* (*u*, *w*) (*v*, *z*))
using *that division* [*of p* (*a*, *c*) (*b*, *d*)] *p* ‹*finite p*› **by** (*auto simp add*: *comm_monoid_set_F_and*)
with *False* **have** *norm* (*integral ?CBOX f* − *integral* (*cbox a b*) (λ*x*. *integral* (*cbox c d*) (λ*y*. *f* (*x*, *y*))))
≤ *e*
if ⋀*t u v w z*. *t* ∈ *p* ⟹ *cbox* (*u*, *w*) (*v*, *z*) ⊆ *t* ⟹ *cbox* (*u*, *w*) (*v*, *z*) ⊆ *?CBOX*
⟹ *norm* (*integral* (*cbox* (*u*, *w*) (*v*, *z*)) *f* −
integral (*cbox u v*) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*, *y*))))
≤ *e* ∗ *content* (*cbox* (*u*, *w*) (*v*, *z*)) / *content ?CBOX*
using *that* **by** (*simp add*: *e'_def*)
} **note** *op_acbd* = *this*
{ **fix** *k*::*real* **and** 𝒟 **and** *u*::*'a* **and** *v w* **and** *z*::*'b* **and** *t1 t2 l*
assume *k*: *0* < *k*
and *nf*: ⋀*x y u v*.
⟦*x* ∈ *cbox a b*; *y* ∈ *cbox c d*; *u* ∈ *cbox a b*; *v*∈*cbox c d*; *norm* (*u*−*x*, *v*−*y*) < *k*⟧
⟹ *norm* (*f*(*u*,*v*) − *f*(*x*,*y*)) < *e*/(*2* ∗ (*content ?CBOX*))
and *p_acbd*: 𝒟 *tagged_division_of cbox* (*a*,*c*) (*b*,*d*)
and *fine*: (λ*x*. *ball x k*) *fine* 𝒟 ((*t1*,*t2*), *l*) ∈ 𝒟
and *uwvz_sub*: *cbox* (*u*,*w*) (*v*,*z*) ⊆ *l cbox* (*u*,*w*) (*v*,*z*) ⊆ *cbox* (*a*,*c*) (*b*,*d*)
have *t*: *t1* ∈ *cbox a b t2* ∈ *cbox c d*
by (*meson fine p_acbd cbox_Pair_iff tag_in_interval*)+
have *ls*: *l* ⊆ *ball* (*t1*,*t2*) *k*
using *fine* **by** (*simp add*: *fine_def Ball_def*)
{ **fix** *x1 x2*
assume *xuvwz*: *x1* ∈ *cbox u v x2* ∈ *cbox w z*
then have *x*: *x1* ∈ *cbox a b x2* ∈ *cbox c d*
using *uwvz_sub* **by** *auto*
have *norm* (*x1* − *t1*, *x2* − *t2*) = *norm* (*t1* − *x1*, *t2* − *x2*)
by (*simp add*: *norm_Pair norm_minus_commute*)
also have *norm* (*t1* − *x1*, *t2* − *x2*) < *k*
using *xuvwz ls uwvz_sub* **unfolding** *ball_def*
by (*force simp add*: *cbox_Pair_eq dist_norm*)
finally have *norm* (*f* (*x1*,*x2*) − *f* (*t1*,*t2*)) ≤ *e*/*content ?CBOX*/*2*
using *nf* [*OF t x*] **by** *simp*
} **note** *nf'* = *this*
have *f_int_uwvz*: *f integrable_on cbox* (*u*,*w*) (*v*,*z*)
using *f_int_acbd uwvz_sub integrable_on_subcbox* **by** *blast*

have *f_int_uv*: $\bigwedge x.\ x \in cbox\ u\ v \Longrightarrow (\lambda y.\ f\ (x,y))$ *integrable_on cbox w z*
using *assms continuous_on_subset uwvz_sub*
by (*blast intro*: *continuous_on_imp_integrable_on_Pair1*)
have *1*: *norm* (*integral* (*cbox* (*u*,*w*) (*v*,*z*)) *f* − *integral* (*cbox* (*u*,*w*) (*v*,*z*)) ($\lambda x.\ f\ (t1,t2)$))
$\le e * content\ (cbox\ (u,w)\ (v,z)) / content\ ?CBOX/2$
using *cbp* ‹$0 < e/content\ ?CBOX$› *nf'*
apply (*simp only*: *integral_diff* [*symmetric*] *f_int_uwvz integrable_const*)
apply (*auto simp*: *integrable_diff f_int_uwvz integrable_const intro*: *order_trans* [*OF integrable_bound* [*of e*/*content ?CBOX*/*2*]])
done
have *int_integrable*: ($\lambda x.\ integral\ (cbox\ w\ z)\ (\lambda y.\ f\ (x,\ y))$) *integrable_on cbox u v*
using *assms integral_integrable_2dim continuous_on_subset uwvz_sub*(*2*) **by** *blast*
have *normint_wz*:
$\bigwedge x.\ x \in cbox\ u\ v \Longrightarrow$
norm (*integral* (*cbox w z*) ($\lambda y.\ f\ (x,\ y)$) − *integral* (*cbox w z*) ($\lambda y.\ f\ (t1,\ t2)$))
$\le e * content\ (cbox\ w\ z) / content\ (cbox\ (a,\ c)\ (b,\ d))/2$
using *cbp* ‹$0 < e/content\ ?CBOX$› *nf'*
apply (*simp only*: *integral_diff* [*symmetric*] *f_int_uv integrable_const*)
apply (*auto simp*: *integrable_diff f_int_uv integrable_const intro*: *order_trans* [*OF integrable_bound* [*of e*/*content ?CBOX*/*2*]])
done
have *norm* (*integral* (*cbox u v*)
($\lambda x.\ integral\ (cbox\ w\ z)\ (\lambda y.\ f\ (x,y))$ − *integral* (*cbox w z*) ($\lambda y.\ f\ (t1,t2)$)))
$\le e * content\ (cbox\ w\ z) / content\ ?CBOX/2 * content\ (cbox\ u\ v)$
using *cbp* ‹$0 < e/content\ ?CBOX$›
apply (*intro integrable_bound* [*OF* _ _ *normint_wz*])
apply (*auto simp*: *field_split_simps integrable_diff int_integrable integrable_const*)
done
also have ... $\le e * content\ (cbox\ (u,w)\ (v,z)) / content\ ?CBOX/2$
by (*simp add*: *content_Pair field_split_simps*)
finally have *2*: *norm* (*integral* (*cbox u v*) ($\lambda x.\ integral\ (cbox\ w\ z)\ (\lambda y.\ f\ (x,y))$) −
integral (*cbox u v*) ($\lambda x.\ integral\ (cbox\ w\ z)\ (\lambda y.\ f\ (t1,t2))$))
$\le e * content\ (cbox\ (u,w)\ (v,z)) / content\ ?CBOX/2$
by (*simp only*: *integral_diff* [*symmetric*] *int_integrable integrable_const*)
have *norm_xx*: $[\![x' = y';\ norm(x - x') \le e/2;\ norm(y - y') \le e/2]\!] \Longrightarrow norm(x - y) \le e$ **for** $x{::}'c$ **and** $y\ x'\ y'\ e$
using *norm_triangle_mono* [*of* $x{-}y'$ $e/2$ $y'{-}y$ $e/2$] *field_sum_of_halves*
by (*simp add*: *norm_minus_commute*)
have *norm* (*integral* (*cbox* (*u*,*w*) (*v*,*z*)) *f* − *integral* (*cbox u v*) ($\lambda x.\ integral\ (cbox\ w\ z)\ (\lambda y.\ f\ (x,y))$))
$\le e * content\ (cbox\ (u,w)\ (v,z)) / content\ ?CBOX$
by (*rule norm_xx* [*OF integral_Pair_const 1 2*])
} **note** $* = this$

have *norm* (*integral ?CBOX f* − *integral* (*cbox a b*) (λ*x*. *integral* (*cbox c d*) (λ*y*. *f* (*x*,*y*)))) ≤ *e*
if ∀ *x*∈*?CBOX*. ∀ *x*′∈*?CBOX*. *norm* (*x*′ − *x*) < *k* ⟶ *norm* (*f x*′ − *f x*) < *e* /(*2* ∗ *content* (*?CBOX*)) *0* < *k* **for** *k*
proof −
obtain *p* **where** *ptag*: *p tagged_division_of cbox* (*a*, *c*) (*b*, *d*)
and *fine*: (λ*x*. *ball x k*) *fine p*
using *fine_division_exists* ‹*0* < *k*› **by** *blast*
show *?thesis*
using *that fine ptag* ‹*0* < *k*›
by (*auto simp*: ∗ *intro*: *op_acbd* [*OF division_of_tagged_division* [*OF ptag*]])
qed
then show *norm* (*integral ?CBOX f* − *integral* (*cbox a b*) (λ*x*. *integral* (*cbox c d*) (λ*y*. *f* (*x*,*y*)))) ≤ *e*
using *compact_uniformly_continuous* [*OF assms compact_cbox*]
apply (*simp add*: *uniformly_continuous_on_def dist_norm*)
apply (*drule_tac x*=*e*/*2* / *content?CBOX* **in** *spec*)
using *cbp* ‹*0* < *e*› **by** (*auto simp*: *zero_less_mult_iff*)
qed
then show *?thesis*
by *simp*
qed

lemma *integral_swap_2dim*:
fixes *f* :: [′*a*::*euclidean_space*, ′*b*::*euclidean_space*] ⇒ ′*c*::*banach*
assumes *continuous_on* (*cbox* (*a*,*c*) (*b*,*d*)) (λ(*x*,*y*). *f x y*)
shows *integral* (*cbox* (*a*, *c*) (*b*, *d*)) (λ(*x*, *y*). *f x y*) = *integral* (*cbox* (*c*, *a*) (*d*, *b*)) (λ(*x*, *y*). *f y x*)
proof −
have ((λ(*x*, *y*). *f x y*) *has_integral integral* (*cbox* (*c*, *a*) (*d*, *b*)) (λ(*x*, *y*). *f y x*)) (*prod.swap* ‘ (*cbox* (*c*, *a*) (*d*, *b*)))
proof (*rule has_integral_twiddle* [*of 1 prod.swap prod.swap* λ(*x*,*y*). *f y x integral* (*cbox* (*c*, *a*) (*d*, *b*)) (λ(*x*, *y*). *f y x*), *simplified*])
show ⋀*u v*. *content* (*prod.swap* ‘ *cbox u v*) = *content* (*cbox u v*)
by (*metis content_Pair mult.commute old.prod.exhaust swap_cbox_Pair*)
show ((λ(*x*, *y*). *f y x*) *has_integral integral* (*cbox* (*c*, *a*) (*d*, *b*)) (λ(*x*, *y*). *f y x*)) (*cbox* (*c*, *a*) (*d*, *b*))
by (*simp add*: *assms integrable_continuous integrable_integral swap_continuous*)
qed (*use isCont_swap* **in** ‹*fastforce*+›)
then show *?thesis*
by *force*
qed

theorem *integral_swap_continuous*:
fixes *f* :: [′*a*::*euclidean_space*, ′*b*::*euclidean_space*] ⇒ ′*c*::*banach*
assumes *continuous_on* (*cbox* (*a*,*c*) (*b*,*d*)) (λ(*x*,*y*). *f x y*)
shows *integral* (*cbox a b*) (λ*x*. *integral* (*cbox c d*) (*f x*)) =
integral (*cbox c d*) (λ*y*. *integral* (*cbox a b*) (λ*x*. *f x y*))
proof −

have *integral* (*cbox a b*) (λ*x*. *integral* (*cbox c d*) (*f x*)) = *integral* (*cbox* (*a*, *c*) (*b*, *d*)) (λ(*x*, *y*). *f x y*)
  using *integral_prod_continuous* [*OF assms*] **by** *auto*
 also have ... = *integral* (*cbox* (*c*, *a*) (*d*, *b*)) (λ(*x*, *y*). *f y x*)
  by (*rule integral_swap_2dim* [*OF assms*])
 also have ... = *integral* (*cbox c d*) (λ*y*. *integral* (*cbox a b*) (λ*x*. *f x y*))
  using *integral_prod_continuous* [*OF swap_continuous*] *assms*
  by *auto*
 finally show *?thesis* .
qed

6.15.47 Definite integrals for exponential and power function

lemma *has_integral_exp_minus_to_infinity*:
 assumes *a*: *a* > *0*
 shows ((λ*x*::*real*. *exp* (−*a*∗*x*)) *has_integral exp* (−*a*∗*c*)/*a*) {*c*..}
proof −
 define *f* **where** *f* = (λ*k x*. *if x* ∈ {*c*..*real k*} *then exp* (−*a*∗*x*) *else 0*)
 {
  fix *k* :: *nat* **assume** *k*: *of_nat k* ≥ *c*
  from *k a*
  have ((λ*x*. *exp* (−*a*∗*x*)) *has_integral* (−*exp* (−*a*∗*real k*)/*a* − (−*exp* (−*a*∗*c*)/*a*))) {*c*..*real k*}
   by (*intro fundamental_theorem_of_calculus*)
    (*auto intro*!: *derivative_eq_intros*
      simp: *has_field_derivative_iff_has_vector_derivative* [*symmetric*])
  hence (*f k has_integral* (*exp* (−*a*∗*c*)/*a* − *exp* (−*a*∗*real k*)/*a*)) {*c*..} **unfolding** *f_def*
   by (*subst has_integral_restrict*) *simp_all*
 } **note** *has_integral_f* = *this*

 have [*simp*]: *f k* = (λ_. *0*) **if** *of_nat k* < *c* **for** *k* **using** *that* **by** (*auto simp*: *fun_eq_iff f_def*)
 have *integral_f*: *integral* {*c*..} (*f k*) =
      (*if real k* ≥ *c then exp* (−*a*∗*c*)/*a* − *exp* (−*a*∗*real k*)/*a else 0*)
  for *k* **using** *integral_unique*[*OF has_integral_f*[*of k*]] **by** *simp*

 have *A*: (λ*x*. *exp* (−*a*∗*x*)) *integrable_on* {*c*..} ∧
    (λ*k*. *integral* {*c*..} (*f k*)) ⟶ *integral* {*c*..} (λ*x*. *exp* (−*a*∗*x*))
 proof (*intro monotone_convergence_increasing allI ballI*)
  fix *k* ::*nat*
  have (λ*x*. *exp* (−*a*∗*x*)) *integrable_on* {*c*..*of_real k*} (**is** *?P*)
   unfolding *f_def* **by** (*auto intro*!: *continuous_intros integrable_continuous_real*)
  hence (*f k*) *integrable_on* {*c*..*of_real k*}
   by (*rule integrable_eq*) (*simp add*: *f_def*)
  then show *f k integrable_on* {*c*..}
   by (*rule integrable_on_superset*) (*auto simp*: *f_def*)
 next
  fix *x* **assume** *x*: *x* ∈ {*c*..}

have *sequentially* ≤ *principal* {*nat* ⌈*x*⌉..} **unfolding** *at_top_def* **by** (*simp add*: *Inf_lower*)
also have {*nat* ⌈*x*⌉..} ⊆ {*k*. *x* ≤ *real k*} **by** *auto*
also have *inf* (*principal* . . .) (*principal* {*k*. ¬*x* ≤ *real k*}) =
principal ({*k*. *x* ≤ *real k*} ∩ {*k*. ¬*x* ≤ *real k*}) **by** *simp*
also have {*k*. *x* ≤ *real k*} ∩ {*k*. ¬*x* ≤ *real k*} = {} **by** *blast*
finally have *inf sequentially* (*principal* {*k*. ¬*x* ≤ *real k*}) = *bot*
by (*simp add*: *inf*.*coboundedI1 bot_unique*)
with *x* **show** (λ*k*. *f k x*) ⟶ *exp* (−*a*∗*x*) **unfolding** *f_def*
by (*intro filterlim_If*) *auto*
next
have |*integral* {*c*..} (*f k*)| ≤ *exp* (−*a*∗*c*)/*a* **for** *k*
proof (*cases c* > *of_nat k*)
case *False*
hence *abs* (*integral* {*c*..} (*f k*)) = *abs* (*exp* (− (*a* ∗ *c*)) / *a* − *exp* (− (*a* ∗ *real k*)) / *a*)
by (*simp add*: *integral_f*)
also have *abs* (*exp* (− (*a* ∗ *c*)) / *a* − *exp* (− (*a* ∗ *real k*)) / *a*) =
exp (− (*a* ∗ *c*)) / *a* − *exp* (− (*a* ∗ *real k*)) / *a*
using *False a* **by** (*intro abs_of_nonneg*) (*simp_all add*: *field_simps*)
also have . . . ≤ *exp* (− *a* ∗ *c*) / *a* **using** *a* **by** *simp*
finally show *?thesis* **.**
qed (*insert a*, *simp_all add*: *integral_f*)
thus *bounded* (*range*(λ*k*. *integral* {*c*..} (*f k*)))
by (*intro boundedI*[*of* _ *exp* (−*a*∗*c*)/*a*]) *auto*
qed (*auto simp*: *f_def*)
have (λ*k*. *exp* (−*a*∗*c*)/*a* − *exp* (−*a* ∗ *of_nat k*)/*a*) ⟶ *exp* (−*a*∗*c*)/*a* − *0*/*a*
by (*intro tendsto_intros filterlim_compose*[*OF exp_at_bot*]
filterlim_tendsto_neg_mult_at_bot[*OF tendsto_const*] *filterlim_real_sequentially*)+
(*insert a*, *simp_all*)
moreover
from *eventually_gt_at_top*[*of nat* ⌈*c*⌉] **have** *eventually* (λ*k*. *of_nat k* > *c*) *sequentially*
by *eventually_elim linarith*
hence *eventually* (λ*k*. *exp* (−*a*∗*c*)/*a* − *exp* (−*a* ∗ *of_nat k*)/*a* = *integral* {*c*..} (*f k*)) *sequentially*
by *eventually_elim* (*simp add*: *integral_f*)
ultimately have (λ*k*. *integral* {*c*..} (*f k*)) ⟶ *exp* (−*a*∗*c*)/*a* − *0*/*a*
by (*rule Lim_transform_eventually*)
from *LIMSEQ_unique*[*OF conjunct2*[*OF A*] *this*]
have *integral* {*c*..} (λ*x*. *exp* (−*a*∗*x*)) = *exp* (−*a*∗*c*)/*a* **by** *simp*
with *conjunct1*[*OF A*] **show** *?thesis*
by (*simp add*: *has_integral_integral*)
qed

lemma *integrable_on_exp_minus_to_infinity*: *a* > *0* ⟹ (λ*x*. *exp* (−*a*∗*x*) :: *real*) *integrable_on* {*c*..}
using *has_integral_exp_minus_to_infinity*[*of a c*] **unfolding** *integrable_on_def* **by** *blast*

```
lemma has_integral_powr_from_0:
  assumes a: a > (−1::real) and c: c ≥ 0
  shows  ((λx. x powr a) has_integral (c powr (a+1) / (a+1))) {0..c}
proof (cases c = 0)
  case False
  define f where f = (λk x. if x ∈ {inverse (of_nat (Suc k))..c} then x powr a
else 0)
  define F where F = (λk. if inverse (of_nat (Suc k)) ≤ c then
                               c powr (a+1)/(a+1) − inverse (real (Suc k)) powr
(a+1)/(a+1) else 0)
  {
    fix k :: nat
    have (f k has_integral F k) {0..c}
    proof (cases inverse (of_nat (Suc k)) ≤ c)
      case True
      {
        fix x assume x: x ≥ inverse (1 + real k)
        have 0 < inverse (1 + real k) by simp
        also note x
        finally have x > 0 .
      } note x = this
      hence ((λx. x powr a) has_integral c powr (a + 1) / (a + 1) −
           inverse (real (Suc k)) powr (a + 1) / (a + 1)) {inverse (real (Suc
k))..c}
        using True a by (intro fundamental_theorem_of_calculus)
          (auto intro!: derivative_eq_intros continuous_on_powr' continuous_on_const
            simp: has_field_derivative_iff_has_vector_derivative [symmetric])
     with True show ?thesis unfolding f_def F_def by (subst has_integral_restrict)
simp_all
    next
      case False
      thus ?thesis unfolding f_def F_def by (subst has_integral_restrict) auto
    qed
  } note has_integral_f = this
  have integral_f: integral {0..c} (f k) = F k for k
    using has_integral_f[of k] by (rule integral_unique)

  have A: (λx. x powr a) integrable_on {0..c} ∧
          (λk. integral {0..c} (f k)) ⟶ integral {0..c} (λx. x powr a)
  proof (intro monotone_convergence_increasing ballI allI)
    fix k from has_integral_f[of k] show f k integrable_on {0..c}
      by (auto simp: integrable_on_def)
  next
    fix k :: nat and x :: real
    {
      assume x: inverse (real (Suc k)) ≤ x
       have inverse (real (Suc (Suc k))) ≤ inverse (real (Suc k)) by (simp add:
field_simps)
      also note x
```

finally have *inverse* (*real* (*Suc* (*Suc k*))) ≤ *x* **.**
}
thus *f k x* ≤ *f* (*Suc k*) *x* **by** (*auto simp*: *f_def simp del*: *of_nat_Suc*)
next
fix *x* **assume** *x*: *x* ∈ {*0*..*c*}
show (λ*k*. *f k x*) ⟶ *x powr a*
proof (*cases x* = *0*)
case *False*
with *x* **have** *x* > *0* **by** *simp*
from *order_tendstoD*(*2*)[*OF LIMSEQ_inverse_real_of_nat this*]
have *eventually* (λ*k*. *x powr a* = *f k x*) *sequentially*
by *eventually_elim* (*insert x*, *simp add*: *f_def*)
moreover have (λ_. *x powr a*) ⟶ *x powr a* **by** *simp*
ultimately show *?thesis* **by** (*blast intro*: *Lim_transform_eventually*)
qed (*simp_all add*: *f_def*)
next
{
fix *k*
from *a* **have** *F k* ≤ *c powr* (*a* + *1*) / (*a* + *1*)
by (*auto simp add*: *F_def divide_simps*)
also from *a* **have** *F k* ≥ *0*
by (*auto simp*: *F_def divide_simps simp del*: *of_nat_Suc intro*!: *powr_mono2*)
hence *F k* = *abs* (*F k*) **by** *simp*
finally have *abs* (*F k*) ≤ *c powr* (*a* + *1*) / (*a* + *1*) **.**
}
thus *bounded* (*range*(λ*k*. *integral* {*0*..*c*} (*f k*)))
by (*intro boundedI*[*of* _ *c powr* (*a*+*1*) / (*a*+*1*)]) (*auto simp*: *integral_f*)
qed

from *False c* **have** *c* > *0* **by** *simp*
from *order_tendstoD*(*2*)[*OF LIMSEQ_inverse_real_of_nat this*]
have *eventually* (λ*k*. *c powr* (*a* + *1*) / (*a* + *1*) − *inverse* (*real* (*Suc k*)) *powr* (*a*+*1*) / (*a*+*1*) =
integral {*0*..*c*} (*f k*)) *sequentially*
by *eventually_elim* (*simp add*: *integral_f F_def*)
moreover have (λ*k*. *c powr* (*a* + *1*) / (*a* + *1*) − *inverse* (*real* (*Suc k*)) *powr* (*a* + *1*) / (*a* + *1*))
⟶ *c powr* (*a* + *1*) / (*a* + *1*) − *0 powr* (*a* + *1*) / (*a* + *1*)
using *a* **by** (*intro tendsto_intros LIMSEQ_inverse_real_of_nat*) *auto*
hence (λ*k*. *c powr* (*a* + *1*) / (*a* + *1*) − *inverse* (*real* (*Suc k*)) *powr* (*a* + *1*) / (*a* + *1*))
⟶ *c powr* (*a* + *1*) / (*a* + *1*) **by** *simp*
ultimately have (λ*k*. *integral* {*0*..*c*} (*f k*)) ⟶ *c powr* (*a*+*1*) / (*a*+*1*)
by (*blast intro*: *Lim_transform_eventually*)
with *A* **have** *integral* {*0*..*c*} (λ*x*. *x powr a*) = *c powr* (*a*+*1*) / (*a*+*1*)
by (*blast intro*: *LIMSEQ_unique*)
with *A* **show** *?thesis* **by** (*simp add*: *has_integral_integral*)
qed (*simp_all add*: *has_integral_refl*)

```
lemma integrable_on_powr_from_0:
  assumes a: a > (−1::real) and c: c ≥ 0
  shows (λx. x powr a) integrable_on {0..c}
  using has_integral_powr_from_0[OF assms] unfolding integrable_on_def by blast

lemma has_integral_powr_to_inf:
  fixes a e :: real
  assumes e < −1 a > 0
  shows ((λx. x powr e) has_integral −(a powr (e + 1)) / (e + 1)) {a..}
proof −
  define f :: nat ⇒ real ⇒ real where f = (λn x. if x ∈ {a..n} then x powr e
else 0)
  define F where F = (λx. x powr (e + 1) / (e + 1))

  have has_integral_f: (f n has_integral (F n − F a)) {a..}
    if n: n ≥ a for n :: nat
  proof −
    from n assms have ((λx. x powr e) has_integral (F n − F a)) {a..n}
      by (intro fundamental_theorem_of_calculus) (auto intro!: derivative_eq_intros
          simp: has_field_derivative_iff_has_vector_derivative [symmetric] F_def)
    hence (f n has_integral (F n − F a)) {a..n}
      by (rule has_integral_eq [rotated]) (simp add: f_def)
    thus (f n has_integral (F n − F a)) {a..}
      by (rule has_integral_on_superset) (auto simp: f_def)
  qed
  have integral_f: integral {a..} (f n) = (if n ≥ a then F n − F a else 0) for n ::
nat
  proof (cases n ≥ a)
    case True
    with has_integral_f[OF this] show ?thesis by (simp add: integral_unique)
  next
    case False
    have (f n has_integral 0) {a} by (rule has_integral_refl)
    hence (f n has_integral 0) {a..}
      by (rule has_integral_on_superset) (insert False, simp_all add: f_def)
    with False show ?thesis by (simp add: integral_unique)
  qed

  have ∗: (λx. x powr e) integrable_on {a..} ∧
        (λn. integral {a..} (f n)) ⇢ integral {a..} (λx. x powr e)
  proof (intro monotone_convergence_increasing allI ballI)
    fix n :: nat
    from assms have (λx. x powr e) integrable_on {a..n}
      by (auto intro!: integrable_continuous_real continuous_intros)
    hence f n integrable_on {a..n}
      by (rule integrable_eq) (auto simp: f_def)
    thus f n integrable_on {a..}
      by (rule integrable_on_superset) (auto simp: f_def)
  next
```

fix *n* :: *nat* **and** *x* :: *real*
show *f n x* ≤ *f* (*Suc n*) *x* **by** (*auto simp*: *f_def*)
next
fix *x* :: *real* **assume** *x*: *x* ∈ {*a*..}
from *filterlim_real_sequentially*
have *eventually* (λ*n*. *real n* ≥ *x*) *at_top*
by (*simp add*: *filterlim_at_top*)
with *x* **have** *eventually* (λ*n*. *f n x* = *x powr e*) *at_top*
by (*auto elim*!: *eventually_mono simp*: *f_def*)
thus (λ*n*. *f n x*) ⟶ *x powr e* **by** (*simp add*: *tendsto_eventually*)
next
have *norm* (*integral* {*a*..} (*f n*)) ≤ −*F a* **for** *n* :: *nat*
proof (*cases n* ≥ *a*)
case *True*
with *assms* **have** *a powr* (*e* + *1*) ≥ *n powr* (*e* + *1*)
by (*intro powr_mono2'*) *simp_all*
with *assms* **show** *?thesis* **by** (*auto simp*: *divide_simps F_def integral_f*)
qed (*insert assms*, *simp add*: *integral_f F_def field_split_simps*)
thus *bounded* (*range*(λ*k*. *integral* {*a*..} (*f k*)))
unfolding *bounded_iff* **by** (*intro exI*[*of* _ −*F a*]) *auto*
qed

from *filterlim_real_sequentially*
have *eventually* (λ*n*. *real n* ≥ *a*) *at_top*
by (*simp add*: *filterlim_at_top*)
hence *eventually* (λ*n*. *F n* − *F a* = *integral* {*a*..} (*f n*)) *at_top*
by *eventually_elim* (*simp add*: *integral_f*)
moreover have (λ*n*. *F n* − *F a*) ⟶ *0* / (*e* + *1*) − *F a* **unfolding** *F_def*
by (*insert assms*, (*rule tendsto_intros filterlim_compose*[*OF tendsto_neg_powr*]
filterlim_ident filterlim_real_sequentially | *simp*)+)
hence (λ*n*. *F n* − *F a*) ⟶ −*F a* **by** *simp*
ultimately have (λ*n*. *integral* {*a*..} (*f n*)) ⟶ −*F a* **by** (*blast intro*: *Lim_transform_eventually*)
from *conjunct2*[*OF* ∗] **and** *this*
have *integral* {*a*..} (λ*x*. *x powr e*) = −*F a* **by** (*rule LIMSEQ_unique*)
with *conjunct1*[*OF* ∗] **show** *?thesis*
by (*simp add*: *has_integral_integral F_def*)
qed

lemma *has_integral_inverse_power_to_inf*:
fixes *a* :: *real* **and** *n* :: *nat*
assumes *n* > *1 a* > *0*
shows ((λ*x*. *1* / *x* ^ *n*) *has_integral 1* / (*real* (*n* − *1*) ∗ *a* ^ (*n* − *1*))) {*a*..}
proof −
from *assms* **have** *real_of_int* (−*int n*) < −*1* **by** *simp*
note *has_integral_powr_to_inf*[*OF this* ‹*a* > *0*›]
also have − (*a powr* (*real_of_int* (− *int n*) + *1*)) / (*real_of_int* (− *int n*) + *1*)
=
1 / (*real* (*n* − *1*) ∗ *a powr* (*real* (*n* − *1*))) **using** *assms*
by (*simp add*: *field_split_simps powr_add* [*symmetric*] *of_nat_diff*)

also from *assms* **have** *a powr* (*real* (*n* − *1*)) = *a* ^ (*n* − *1*)
by (*intro powr_realpow*)
finally show *?thesis*
by (*rule has_integral_eq* [*rotated*])
(*insert assms*, *simp_all add*: *powr_minus powr_realpow field_split_simps*)
qed

Adaption to ordered Euclidean spaces and the Cartesian Euclidean space

lemma *integral_component_eq_cart*[*simp*]:
fixes *f* :: *'n::euclidean_space* ⇒ *real^'m*
assumes *f integrable_on s*
shows *integral s* (λ*x*. *f x* $ *k*) = *integral s f* $ *k*
using *integral_linear*[*OF assms*(*1*) *bounded_linear_vec_nth*,*unfolded o_def*] **.**

lemma *content_closed_interval*:
fixes *a* :: *'a::ordered_euclidean_space*
assumes *a* ≤ *b*
shows *content* {*a*..*b*} = (∏ *i*∈*Basis*. *b*•*i* − *a*•*i*)
using *content_cbox*[*of a b*] *assms* **by** (*simp add*: *cbox_interval eucl_le*[**where** *'a*=*'a*])

lemma *integrable_const_ivl*[*intro*]:
fixes *a*::*'a::ordered_euclidean_space*
shows (λ*x*. *c*) *integrable_on* {*a*..*b*}
unfolding *cbox_interval*[*symmetric*] **by** (*rule integrable_const*)

lemma *integrable_on_subinterval*:
fixes *f* :: *'n::ordered_euclidean_space* ⇒ *'a::banach*
assumes *f integrable_on S* {*a*..*b*} ⊆ *S*
shows *f integrable_on* {*a*..*b*}
using *integrable_on_subcbox*[*of f S a b*] *assms* **by** (*simp add*: *cbox_interval*)

end

6.16 Radon-Nikodým Derivative

theory *Radon_Nikodym*
imports *Bochner_Integration*
begin

definition *diff_measure* :: *'a measure* ⇒ *'a measure* ⇒ *'a measure*
where
diff_measure M N = *measure_of* (*space M*) (*sets M*) (λ*A*. *emeasure M A* − *emeasure N A*)

lemma
shows *space_diff_measure*[*simp*]: *space* (*diff_measure M N*) = *space M*

```
    and sets_diff_measure[simp]: sets (diff_measure M N) = sets M
  by (auto simp: diff_measure_def)

lemma emeasure_diff_measure:
  assumes fin: finite_measure M finite_measure N and sets_eq: sets M = sets N
  assumes pos: ⋀A. A ∈ sets M ⟹ emeasure N A ≤ emeasure M A and A: A
∈ sets M
  shows emeasure (diff_measure M N) A = emeasure M A − emeasure N A (is _
= ?μ A)
  unfolding diff_measure_def
proof (rule emeasure_measure_of_sigma)
  show sigma_algebra (space M) (sets M) ..
  show positive (sets M) ?μ
    using pos by (simp add: positive_def)
  show countably_additive (sets M) ?μ
  proof (rule countably_additiveI)
    fix A :: nat ⇒ _ assume A: range A ⊆ sets M and disjoint_family A
    then have suminf:
      (∑i. emeasure M (A i)) = emeasure M (⋃i. A i)
      (∑i. emeasure N (A i)) = emeasure N (⋃i. A i)
      by (simp_all add: suminf_emeasure sets_eq)
    with A have (∑i. emeasure M (A i) − emeasure N (A i)) =
      (∑i. emeasure M (A i)) − (∑i. emeasure N (A i))
      using fin pos[of A _]
      by (intro ennreal_suminf_minus)
        (auto simp: sets_eq finite_measure.emeasure_eq_measure suminf_emeasure)
    then show (∑i. emeasure M (A i) − emeasure N (A i)) =
      emeasure M (⋃i. A i) − emeasure N (⋃i. A i)
      by (simp add: suminf)
  qed
qed fact
```

An equivalent characterization of sigma-finite spaces is the existence of integrable positive functions (or, still equivalently, the existence of a probability measure which is in the same measure class as the original measure).

```
proposition (in sigma_finite_measure) obtain_positive_integrable_function:
  obtains f::'a ⇒ real where
    f ∈ borel_measurable M
    ⋀x. f x > 0
    ⋀x. f x ≤ 1
    integrable M f
proof −
  obtain A :: nat ⇒ 'a set where range A ⊆ sets M (⋃i. A i) = space M ⋀i.
emeasure M (A i) ≠ ∞
    using sigma_finite by auto
  then have [measurable]:A n ∈ sets M for n by auto
   define g where g = (λx. (∑n. (1/2)^(Suc n) ∗ indicator (A n) x / (1+
measure M (A n))))
  have [measurable]: g ∈ borel_measurable M unfolding g_def by auto
```

have ∗: *summable* (λ*n*. (*1*/*2*) ^(*Suc n*) ∗ *indicator* (*A n*) *x* / (*1*+ *measure M* (*A n*))) **for** *x*
apply (*rule summable_comparison_test'*[*of* λ*n*. (*1*/*2*) ^(*Suc n*) *0*])
using *power_half_series summable_def* **by** (*auto simp add*: *indicator_def divide_simps*)
have *g x* ≤ (∑*n*. (*1*/*2*) ^(*Suc n*)) **for** *x* **unfolding** *g_def*
apply (*rule suminf_le*) **using** ∗ *power_half_series summable_def* **by** (*auto simp add*: *indicator_def divide_simps*)
then have *g_le_1*: *g x* ≤ *1* **for** *x* **using** *power_half_series sums_unique* **by** *fastforce*

have *g_pos*: *g x* > *0* **if** *x* ∈ *space M* **for** *x*
unfolding *g_def* **proof** (*subst suminf_pos_iff*[*OF* ∗[*of x*]], *auto*)
obtain *i* **where** *x* ∈ *A i* **using** ‹(⋃ *i*. *A i*) = *space M*› ‹*x* ∈ *space M*› **by** *auto*
then have *0* < (*1* / *2*) ^ *Suc i* ∗ *indicator* (*A i*) *x* / (*1* + *Sigma_Algebra.measure M* (*A i*))
unfolding *indicator_def* **apply** (*auto simp add*: *divide_simps*) **using** *measure_nonneg*[*of M A i*]
by (*auto*, *meson add_nonneg_nonneg linorder_not_le mult_nonneg_nonneg zero_le_numeral zero_le_one zero_le_power*)
then show ∃ *i*. *0* < (*1* / *2*) ^ *i* ∗ *indicator* (*A i*) *x* / (*2* + *2* ∗ *Sigma_Algebra.measure M* (*A i*))
by *auto*
qed

have *integrable M g*
unfolding *g_def* **proof** (*rule integrable_suminf*)
fix *n*
show *integrable M* (λ*x*. (*1* / *2*) ^ *Suc n* ∗ *indicator* (*A n*) *x* / (*1* + *Sigma_Algebra.measure M* (*A n*)))
using ‹*emeasure M* (*A n*) ≠ ∞›
by (*auto intro*!: *integrable_mult_right integrable_divide_zero integrable_real_indicator simp add*: *top.not_eq_extremum*)
next
show *AE x in M*. *summable* (λ*n*. *norm* ((*1* / *2*) ^ *Suc n* ∗ *indicator* (*A n*) *x* / (*1* + *Sigma_Algebra.measure M* (*A n*))))
using ∗ **by** *auto*
show *summable* (λ*n*. (∫ *x*. *norm* ((*1* / *2*) ^ *Suc n* ∗ *indicator* (*A n*) *x* / (*1* + *Sigma_Algebra.measure M* (*A n*))) ∂*M*))
apply (*rule summable_comparison_test'*[*of* λ*n*. (*1*/*2*) ^(*Suc n*) *0*], *auto*)
using *power_half_series summable_def* **apply** *auto*[*1*]
apply (*auto simp add*: *field_split_simps*) **using** *measure_nonneg*[*of M*] *not_less* **by** *fastforce*
qed

define *f* **where** *f* = (λ*x*. *if x* ∈ *space M then g x else 1*)
have *f x* > *0* **for** *x* **unfolding** *f_def* **using** *g_pos* **by** *auto*
moreover have *f x* ≤ *1* **for** *x* **unfolding** *f_def* **using** *g_le_1* **by** *auto*
moreover have [*measurable*]: *f* ∈ *borel_measurable M* **unfolding** *f_def* **by** *auto*
moreover have *integrable M f*

```
    apply (subst integrable_cong[of _ _ _ g]) unfolding f_def using ‹integrable M
g› by auto
  ultimately show (⋀f. f ∈ borel_measurable M ⟹ (⋀x. 0 < f x) ⟹ (⋀x. f x
≤ 1) ⟹ integrable M f ⟹ thesis) ⟹ thesis
    by (meson that)
qed

lemma (in sigma_finite_measure) Ex_finite_integrable_function:
  ∃ h∈borel_measurable M. integral^N M h ≠ ∞ ∧ (∀ x∈space M. 0 < h x ∧ h x <
∞)
proof −
  obtain A :: nat ⇒ 'a set where
    range[measurable]: range A ⊆ sets M and
    space: (⋃ i. A i) = space M and
    measure: ⋀i. emeasure M (A i) ≠ ∞ and
    disjoint: disjoint_family A
    using sigma_finite_disjoint by blast
  let ?B = λi. 2^Suc i ∗ emeasure M (A i)
  have [measurable]: ⋀i. A i ∈ sets M
    using range by fastforce+
  have ∀ i. ∃ x. 0 < x ∧ x < inverse (?B i)
  proof
    fix i show ∃ x. 0 < x ∧ x < inverse (?B i)
      using measure[of i]
       by (auto intro!: dense simp: ennreal_inverse_positive ennreal_mult_eq_top_iff
power_eq_top_ennreal)
  qed
  from choice[OF this] obtain n where n: ⋀i. 0 < n i
    ⋀i. n i < inverse (2^Suc i ∗ emeasure M (A i)) by auto
  { fix i have 0 ≤ n i using n(1)[of i] by auto } note pos = this
  let ?h = λx. ∑ i. n i ∗ indicator (A i) x
  show ?thesis
  proof (safe intro!: bexI[of _ ?h] del: notI)
    have integral^N M ?h = (∑ i. n i ∗ emeasure M (A i)) using pos
      by (simp add: nn_integral_suminf nn_integral_cmult_indicator)
    also have ... ≤ (∑ i. ennreal ((1/2)^Suc i))
    proof (intro suminf_le allI)
      fix N
      have n N ∗ emeasure M (A N) ≤ inverse (2^Suc N ∗ emeasure M (A N))
∗ emeasure M (A N)
        using n[of N] by (intro mult_right_mono) auto
      also have ... = (1/2)^Suc N ∗ (inverse (emeasure M (A N)) ∗ emeasure
M (A N))
        using measure[of N]
        by (simp add: ennreal_inverse_power divide_ennreal_def ennreal_inverse_mult
                  power_eq_top_ennreal less_top[symmetric] mult_ac
              del: power_Suc)
      also have ... ≤ inverse (ennreal 2) ^ Suc N
        using measure[of N]
```

by (*cases emeasure M* (*A N*); *cases emeasure M* (*A N*) = *0*)
(*auto simp*: *inverse_ennreal ennreal_mult*[*symmetric*] *divide_ennreal_def simp del*: *power_Suc*)
also have . . . = *ennreal* (*inverse 2* ^ *Suc N*)
by (*subst ennreal_power*[*symmetric*], *simp*) (*simp add*: *inverse_ennreal*)
finally show *n N* ∗ *emeasure M* (*A N*) ≤ *ennreal* ((*1*/*2*)^*Suc N*)
by *simp*
qed *auto*
also have . . . < *top*
unfolding *less_top*[*symmetric*]
by (*rule ennreal_suminf_neq_top*)
(*auto simp*: *summable_geometric summable_Suc_iff simp del*: *power_Suc*)
finally show $integral^N$ *M ?h* ≠ ∞
by (*auto simp*: *top_unique*)
next
{ **fix** *x* **assume** *x* ∈ *space M*
then obtain *i* **where** *x* ∈ *A i* **using** *space*[*symmetric*] **by** *auto*
with *disjoint n* **have** *?h x* = *n i*
by (*auto intro*!: *suminf_cmult_indicator intro*: *less_imp_le*)
then show *0* < *?h x* **and** *?h x* < ∞ **using** *n*[*of i*] **by** (*auto simp*: *less_top*[*symmetric*]) }
note *pos* = *this*
qed *measurable*
qed

6.16.1 Absolutely continuous

definition *absolutely_continuous* :: *'a measure* ⇒ *'a measure* ⇒ *bool* **where**
absolutely_continuous M N ⟷ *null_sets M* ⊆ *null_sets N*

lemma *absolutely_continuousI_count_space*: *absolutely_continuous* (*count_space A*) *M*
unfolding *absolutely_continuous_def* **by** (*auto simp*: *null_sets_count_space*)

lemma *absolutely_continuousI_density*:
f ∈ *borel_measurable M* ⟹ *absolutely_continuous M* (*density M f*)
by (*force simp add*: *absolutely_continuous_def null_sets_density_iff dest*: *AE_not_in*)

lemma *absolutely_continuousI_point_measure_finite*:
(⋀*x*. ⟦ *x* ∈ *A* ; *f x* ≤ *0* ⟧ ⟹ *g x* ≤ *0*) ⟹ *absolutely_continuous* (*point_measure A f*) (*point_measure A g*)
unfolding *absolutely_continuous_def* **by** (*force simp*: *null_sets_point_measure_iff*)

lemma *absolutely_continuousD*:
absolutely_continuous M N ⟹ *A* ∈ *sets M* ⟹ *emeasure M A* = *0* ⟹ *emeasure N A* = *0*
by (*auto simp*: *absolutely_continuous_def null_sets_def*)

lemma *absolutely_continuous_AE*:

assumes *sets_eq*: *sets M' = sets M*
and *absolutely_continuous M M' AE x in M. P x*
shows *AE x in M'. P x*
proof −
from ‹*AE x in M. P x*› **obtain** *N* **where** *N*: *N ∈ null_sets M {x∈space M. ¬ P x} ⊆ N*
unfolding *eventually_ae_filter* **by** *auto*
show *AE x in M'. P x*
proof (*rule AE_I'*)
show *{x∈space M'. ¬ P x} ⊆ N* **using** *sets_eq_imp_space_eq[OF sets_eq] N(2)* **by** *simp*
from ‹*absolutely_continuous M M'*› **show** *N ∈ null_sets M'*
using *N* **unfolding** *absolutely_continuous_def sets_eq null_sets_def* **by** *auto*
qed
qed

6.16.2 Existence of the Radon-Nikodym derivative

proposition
(**in** *finite_measure*) *Radon_Nikodym_finite_measure*:
assumes *finite_measure N* **and** *sets_eq[simp]*: *sets N = sets M*
assumes *absolutely_continuous M N*
shows *∃f ∈ borel_measurable M. density M f = N*
proof −
interpret *N*: *finite_measure N* **by** *fact*
define *G* **where** *G = {g ∈ borel_measurable M. ∀A∈sets M. (∫⁺x. g x ∗ indicator A x ∂M) ≤ N A}*
have [*measurable_dest*]: *f ∈ G ⟹ f ∈ borel_measurable M*
and *G_D*: *⋀A. f ∈ G ⟹ A ∈ sets M ⟹ (∫⁺x. f x ∗ indicator A x ∂M) ≤ N A* **for** *f*
by (*auto simp*: *G_def*)
note *this[measurable_dest]*
have (*λx. 0*) *∈ G* **unfolding** *G_def* **by** *auto*
hence *G ≠ {}* **by** *auto*
{ fix *f g* **assume** *f[measurable]*: *f ∈ G* **and** *g[measurable]*: *g ∈ G*
have (*λx. max (g x) (f x)*) *∈ G* (**is** *?max ∈ G*) **unfolding** *G_def*
proof *safe*
let *?A = {x ∈ space M. f x ≤ g x}*
have *?A ∈ sets M* **using** *f g* **unfolding** *G_def* **by** *auto*
fix *A* **assume** [*measurable*]: *A ∈ sets M*
have *union*: *((?A ∩ A) ∪ ((space M − ?A) ∩ A)) = A*
using *sets.sets_into_space[OF ‹A ∈ sets M›]* **by** *auto*
have *⋀x. x ∈ space M ⟹ max (g x) (f x) ∗ indicator A x =*
g x ∗ indicator (?A ∩ A) x + f x ∗ indicator ((space M − ?A) ∩ A) x
by (*auto simp*: *indicator_def max_def*)
hence *(∫⁺x. max (g x) (f x) ∗ indicator A x ∂M) =*
(∫⁺x. g x ∗ indicator (?A ∩ A) x ∂M) +
(∫⁺x. f x ∗ indicator ((space M − ?A) ∩ A) x ∂M)
by (*auto cong*: *nn_integral_cong intro!*: *nn_integral_add*)

```
      also have ... ≤ N (?A ∩ A) + N ((space M − ?A) ∩ A)
        using f g unfolding G_def by (auto intro!: add_mono)
      also have ... = N A
        using union by (subst plus_emeasure) auto
      finally show (∫⁺x. max (g x) (f x) ∗ indicator A x ∂M) ≤ N A .
    qed auto }
  note max_in_G = this
  { fix f assume  incseq f and f: ⋀i. f i ∈ G
    then have [measurable]: ⋀i. f i ∈ borel_measurable M by (auto simp: G_def)
    have (λx. SUP i. f i x) ∈ G unfolding G_def
    proof safe
      show (λx. SUP i. f i x) ∈ borel_measurable M by measurable
    next
      fix A assume A ∈ sets M
      have (∫⁺x. (SUP i. f i x) ∗ indicator A x ∂M) =
        (∫⁺x. (SUP i. f i x ∗ indicator A x) ∂M)
        by (intro nn_integral_cong) (simp split: split_indicator)
      also have ... = (SUP i. (∫⁺x. f i x ∗ indicator A x ∂M))
        using ‹incseq f› f ‹A ∈ sets M›
        by (intro nn_integral_monotone_convergence_SUP)
          (auto simp: G_def incseq_Suc_iff le_fun_def split: split_indicator)
      finally show (∫⁺x. (SUP i. f i x) ∗ indicator A x ∂M) ≤ N A
        using f ‹A ∈ sets M› by (auto intro!: SUP_least simp: G_D)
    qed }
  note SUP_in_G = this
  let ?y = SUP g ∈ G. integral^N M g
  have y_le: ?y ≤ N (space M) unfolding G_def
  proof (safe intro!: SUP_least)
    fix g assume ∀A∈sets M. (∫⁺x. g x ∗ indicator A x ∂M) ≤ N A
    from this[THEN bspec, OF sets.top] show integral^N M g ≤ N (space M)
      by (simp cong: nn_integral_cong)
  qed
   from ennreal_SUP_countable_SUP [OF ‹G ≠ {}›, of integral^N M] guess ys ..
note ys = this
  then have ∀n. ∃g. g∈G ∧ integral^N M g = ys n
  proof safe
    fix n assume range ys ⊆ integral^N M ‘ G
    hence ys n ∈ integral^N M ‘ G by auto
    thus ∃g. g∈G ∧ integral^N M g = ys n by auto
  qed
   from choice[OF this] obtain gs where ⋀i. gs i ∈ G ⋀n. integral^N M (gs n)
= ys n by auto
  hence y_eq: ?y = (SUP i. integral^N M (gs i)) using ys by auto
  let ?g = λi x. Max ((λn. gs n x) ‘ {..i})
  define f where [abs_def]: f x = (SUP i. ?g i x) for x
  let ?F = λA x. f x ∗ indicator A x
  have gs_not_empty: ⋀i x. (λn. gs n x) ‘ {..i} ≠ {} by auto
  { fix i have ?g i ∈ G
    proof (induct i)
```

```
      case 0 thus ?case by simp fact
    next
      case (Suc i)
      with Suc gs_not_empty ‹gs (Suc i) ∈ G› show ?case
        by (auto simp add: atMost_Suc intro!: max_in_G)
    qed }
  note g_in_G = this
  have incseq ?g using gs_not_empty
    by (auto intro!: incseq_SucI le_funI simp add: atMost_Suc)

  from SUP_in_G[OF this g_in_G] have [measurable]: f ∈ G unfolding f_def .
  then have [measurable]: f ∈ borel_measurable M unfolding G_def by auto

  have integral^N M f = (SUP i. integral^N M (?g i)) unfolding f_def
   using g_in_G ‹incseq ?g› by (auto intro!: nn_integral_monotone_convergence_SUP
simp: G_def)
  also have ... = ?y
  proof (rule antisym)
    show (SUP i. integral^N M (?g i)) ≤ ?y
      using g_in_G by (auto intro: SUP_mono)
    show ?y ≤ (SUP i. integral^N M (?g i)) unfolding y_eq
      by (auto intro!: SUP_mono nn_integral_mono Max_ge)
  qed
  finally have int_f_eq_y: integral^N M f = ?y .

  have upper_bound: ∀ A∈sets M. N A ≤ density M f A
  proof (rule ccontr)
    assume ¬ ?thesis
     then obtain A where A[measurable]: A ∈ sets M and f_less_N: density M f
A < N A
      by (auto simp: not_le)
    then have pos_A: 0 < M A
      using ‹absolutely_continuous M N›[THEN absolutely_continuousD, OF A]
      by (auto simp: zero_less_iff_neq_zero)

    define b where b = (N A − density M f A) / M A / 2
    with f_less_N pos_A have 0 < b b ≠ top
     by (auto intro!: diff_gr0_ennreal simp: zero_less_iff_neq_zero diff_eq_0_iff_ennreal
ennreal_divide_eq_top_iff)

    let ?f = λx. f x + b
    have nn_integral M f ≠ top
       using ‹f ∈ G›[THEN G_D, of space M] by (auto simp: top_unique cong:
nn_integral_cong)
    with ‹b ≠ top› interpret Mf: finite_measure density M ?f
      by (intro finite_measureI)
         (auto simp: field_simps mult_indicator_subset ennreal_mult_eq_top_iff
                emeasure_density nn_integral_cmult_indicator nn_integral_add
             cong: nn_integral_cong)
```

```
from unsigned_Hahn_decomposition[of density M ?f N A]
obtain Y where [measurable]: Y ∈ sets M and [simp]: Y ⊆ A
  and Y1: ⋀C. C ∈ sets M ⟹ C ⊆ Y ⟹ density M ?f C ≤ N C
  and Y2: ⋀C. C ∈ sets M ⟹ C ⊆ A ⟹ C ∩ Y = {} ⟹ N C ≤ density
M ?f C
  by auto

let ?f' = λx. f x + b * indicator Y x
have M Y ≠ 0
proof
  assume M Y = 0
  then have N Y = 0
   using ‹absolutely_continuous M N›[THEN absolutely_continuousD, of Y] by
auto
  then have N A = N (A − Y)
    by (subst emeasure_Diff) auto
  also have ... ≤ density M ?f (A − Y)
    by (rule Y2) auto
  also have ... ≤ density M ?f A − density M ?f Y
    by (subst emeasure_Diff) auto
  also have ... ≤ density M ?f A − 0
    by (intro ennreal_minus_mono) auto
  also have density M ?f A = b * M A + density M f A
  by (simp add: emeasure_density field_simps mult_indicator_subset nn_integral_add
nn_integral_cmult_indicator)
  also have ... < N A
    using f_less_N pos_A
    by (cases density M f A; cases M A; cases N A)
        (auto simp: b_def ennreal_less_iff ennreal_minus divide_ennreal en-
nreal_numeral[symmetric]
              ennreal_plus[symmetric] ennreal_mult[symmetric] field_simps
            simp del: ennreal_numeral ennreal_plus)
  finally show False
    by simp
qed
then have nn_integral M f < nn_integral M ?f'
  using ‹0 < b› ‹nn_integral M f ≠ top›
 by (simp add: nn_integral_add nn_integral_cmult_indicator ennreal_zero_less_mult_iff
zero_less_iff_neq_zero)
moreover
have ?f' ∈ G
  unfolding G_def
proof safe
  fix X assume [measurable]: X ∈ sets M
  have (∫+ x. ?f' x * indicator X x ∂M) = density M f (X − Y) + density
M ?f (X ∩ Y)
      by (auto simp add: emeasure_density nn_integral_add[symmetric] split:
split_indicator intro!: nn_integral_cong)
```

also have . . . $\leq$ *N* (*X* − *Y*) + *N* (*X* ∩ *Y*)
using *G_D*[*OF* ‹*f* ∈ *G*›] **by** (*intro add_mono Y1*) (*auto simp*: *emeasure_density*)
also have . . . = *N X*
by (*subst plus_emeasure*) (*auto intro*!: *arg_cong2*[**where** *f*=*emeasure*])
finally show ($\int^+$ *x*. *?f'* *x* ∗ *indicator X x* ∂*M*) $\leq$ *N X* **.**
qed *simp*
then have *nn_integral M ?f'* $\leq$ *?y*
by (*rule SUP_upper*)
ultimately show *False*
by (*simp add*: *int_f_eq_y*)
qed
show *?thesis*
proof (*intro bexI*[*of _ f*] *measure_eqI conjI antisym*)
fix *A* **assume** *A* ∈ *sets* (*density M f*) **then show** *emeasure* (*density M f*) *A* $\leq$ *emeasure N A*
by (*auto simp*: *emeasure_density intro*!: *G_D*[*OF* ‹*f* ∈ *G*›])
next
fix *A* **assume** *A*: *A* ∈ *sets* (*density M f*) **then show** *emeasure N A* $\leq$ *emeasure* (*density M f*) *A*
using *upper_bound* **by** *auto*
qed *auto*
qed

lemma (**in** *finite_measure*) *split_space_into_finite_sets_and_rest*:
assumes *ac*: *absolutely_continuous M N* **and** *sets_eq*[*simp*]: *sets N* = *sets M*
shows ∃ *B*::*nat*⇒*'a set*. *disjoint_family B* ∧ *range B* ⊆ *sets M* ∧ (∀ *i*. *N* (*B i*) ≠ ∞) ∧
(∀ *A*∈*sets M*. *A* ∩ (⋃ *i*. *B i*) = {} ⟶ (*emeasure M A* = *0* ∧ *N A* = *0*) ∨ (*emeasure M A* > *0* ∧ *N A* = ∞))
proof −
let *?Q* = {*Q*∈*sets M*. *N Q* ≠ ∞}
let *?a* = *SUP Q*∈*?Q*. *emeasure M Q*
have {} ∈ *?Q* **by** *auto*
then have *Q_not_empty*: *?Q* ≠ {} **by** *blast*
have *?a* $\leq$ *emeasure M* (*space M*) **using** *sets.sets_into_space*
by (*auto intro*!: *SUP_least emeasure_mono*)
then have *?a* ≠ ∞
using *finite_emeasure_space*
by (*auto simp*: *less_top*[*symmetric*] *top_unique simp del*: *SUP_eq_top_iff Sup_eq_top_iff*)
from *ennreal_SUP_countable_SUP* [*OF Q_not_empty*, *of emeasure M*]
obtain *Q''* **where** *range Q''* ⊆ *emeasure M* ‘ *?Q* **and** *a*: *?a* = (*SUP i*::*nat*. *Q'' i*)
by *auto*
then have ∀ *i*. ∃ *Q'*. *Q'' i* = *emeasure M Q'* ∧ *Q'* ∈ *?Q* **by** *auto*
from *choice*[*OF this*] **obtain** *Q'* **where** *Q'*: ⋀*i*. *Q'' i* = *emeasure M* (*Q' i*) ⋀*i*. *Q' i* ∈ *?Q*
by *auto*
then have *a_Lim*: *?a* = (*SUP i*. *emeasure M* (*Q' i*)) **using** *a* **by** *simp*

```
let ?O = λn. ⋃i≤n. Q' i
have Union: (SUP i. emeasure M (?O i)) = emeasure M (⋃i. ?O i)
proof (rule SUP_emeasure_incseq[of ?O])
  show range ?O ⊆ sets M using Q' by auto
  show incseq ?O by (fastforce intro!: incseq_SucI)
qed
have Q'_sets[measurable]: ⋀i. Q' i ∈ sets M using Q' by auto
have O_sets: ⋀i. ?O i ∈ sets M using Q' by auto
then have O_in_G: ⋀i. ?O i ∈ ?Q
proof (safe del: notI)
  fix i have Q' ` {..i} ⊆ sets M using Q' by auto
  then have N (?O i) ≤ (∑i≤i. N (Q' i))
    by (simp add: emeasure_subadditive_finite)
  also have ... < ∞ using Q' by (simp add: less_top)
  finally show N (?O i) ≠ ∞ by simp
qed auto
have O_mono: ⋀n. ?O n ⊆ ?O (Suc n) by fastforce
have a_eq: ?a = emeasure M (⋃i. ?O i) unfolding Union[symmetric]
proof (rule antisym)
  show ?a ≤ (SUP i. emeasure M (?O i)) unfolding a_Lim
    using Q' by (auto intro!: SUP_mono emeasure_mono)
  show (SUP i. emeasure M (?O i)) ≤ ?a
  proof (safe intro!: Sup_mono, unfold bex_simps)
    fix i
    have *: (⋃(Q' ` {..i})) = ?O i by auto
    then show ∃x. (x ∈ sets M ∧ N x ≠ ∞) ∧
      emeasure M (⋃(Q' ` {..i})) ≤ emeasure M x
      using O_in_G[of i] by (auto intro!: exI[of _ ?O i])
  qed
qed
let ?O_0 = (⋃i. ?O i)
have ?O_0 ∈ sets M using Q' by auto
have disjointed Q' i ∈ sets M for i
   using sets.range_disjointed_sets[of Q' M] using Q'_sets by (auto simp: sub-
set_eq)
note Q_sets = this
show ?thesis
proof (intro bexI exI conjI ballI impI allI)
  show disjoint_family (disjointed Q')
    by (rule disjoint_family_disjointed)
  show range (disjointed Q') ⊆ sets M
    using Q'_sets by (intro sets.range_disjointed_sets) auto
  { fix A assume A: A ∈ sets M A ∩ (⋃i. disjointed Q' i) = {}
    then have A1: A ∩ (⋃i. Q' i) = {}
      unfolding UN_disjointed_eq by auto
    show emeasure M A = 0 ∧ N A = 0 ∨ 0 < emeasure M A ∧ N A = ∞
    proof (rule disjCI, simp)
      assume *: emeasure M A = 0 ∨ N A ≠ top
      show emeasure M A = 0 ∧ N A = 0
```

```
      proof (cases emeasure M A = 0)
        case True
        with ac A have N A = 0
          unfolding absolutely_continuous_def by auto
        with True show ?thesis by simp
      next
        case False
        with ∗ have N A ≠ ∞ by auto
        with A have emeasure M ?O_0 + emeasure M A = emeasure M (?O_0 ∪
A)
          using Q' A1 by (auto intro!: plus_emeasure simp: set_eq_iff)
        also have ... = (SUP i. emeasure M (?O i ∪ A))
        proof (rule SUP_emeasure_incseq[of λi. ?O i ∪ A, symmetric, simplified])
          show range (λi. ?O i ∪ A) ⊆ sets M
            using ‹N A ≠ ∞› O_sets A by auto
        qed (fastforce intro!: incseq_SucI)
        also have ... ≤ ?a
        proof (safe intro!: SUP_least)
          fix i have ?O i ∪ A ∈ ?Q
          proof (safe del: notI)
            show ?O i ∪ A ∈ sets M using O_sets A by auto
            from O_in_G[of i] have N (?O i ∪ A) ≤ N (?O i) + N A
              using emeasure_subadditive[of ?O i N A] A O_sets by auto
            with O_in_G[of i] show N (?O i ∪ A) ≠ ∞
              using ‹N A ≠ ∞› by (auto simp: top_unique)
          qed
          then show emeasure M (?O i ∪ A) ≤ ?a by (rule SUP_upper)
        qed
        finally have emeasure M A = 0
            unfolding a_eq using measure_nonneg[of M A] by (simp add: emea-
sure_eq_measure)
        with ‹emeasure M A ≠ 0› show ?thesis by auto
      qed
    qed }
  { fix i
    have N (disjointed Q' i) ≤ N (Q' i)
      by (auto intro!: emeasure_mono simp: disjointed_def)
    then show N (disjointed Q' i) ≠ ∞
      using Q'(2)[of i] by (auto simp: top_unique) }
  qed
qed

proposition (in finite_measure) Radon_Nikodym_finite_measure_infinite:
  assumes absolutely_continuous M N and sets_eq: sets N = sets M
  shows ∃f∈borel_measurable M. density M f = N
proof -
  from split_space_into_finite_sets_and_rest[OF assms]
  obtain Q :: nat ⇒ 'a set
    where Q: disjoint_family Q range Q ⊆ sets M
```

```
    and in_Q0: ⋀A. A ∈ sets M ⟹ A ∩ (⋃i. Q i) = {} ⟹ emeasure M A = 0
∧ N A = 0 ∨ 0 < emeasure M A ∧ N A = ∞
    and Q_fin: ⋀i. N (Q i) ≠ ∞ by force
  from Q have Q_sets: ⋀i. Q i ∈ sets M by auto
  let ?N = λi. density N (indicator (Q i)) and ?M = λi. density M (indicator
(Q i))
  have ∀i. ∃f∈borel_measurable (?M i). density (?M i) f = ?N i
  proof (intro allI finite_measure.Radon_Nikodym_finite_measure)
    fix i
    from Q show finite_measure (?M i)
      by (auto intro!: finite_measureI cong: nn_integral_cong
              simp add: emeasure_density subset_eq sets_eq)
    from Q have emeasure (?N i) (space N) = emeasure N (Q i)
     by (simp add: sets_eq[symmetric] emeasure_density subset_eq cong: nn_integral_cong)
    with Q_fin show finite_measure (?N i)
      by (auto intro!: finite_measureI)
    show sets (?N i) = sets (?M i) by (simp add: sets_eq)
    have [measurable]: ⋀A. A ∈ sets M ⟹ A ∈ sets N by (simp add: sets_eq)
    show absolutely_continuous (?M i) (?N i)
      using ‹absolutely_continuous M N› ‹Q i ∈ sets M›
      by (auto simp: absolutely_continuous_def null_sets_density_iff sets_eq
              intro!: absolutely_continuous_AE[OF sets_eq])
  qed
  from choice[OF this[unfolded Bex_def]]
  obtain f where borel: ⋀i. f i ∈ borel_measurable M ⋀i x. 0 ≤ f i x
    and f_density: ⋀i. density (?M i) (f i) = ?N i
    by force
  { fix A i assume A: A ∈ sets M
     with Q borel have (∫⁺x. f i x ∗ indicator (Q i ∩ A) x ∂M) = emeasure
(density (?M i) (f i)) A
      by (auto simp add: emeasure_density nn_integral_density subset_eq
              intro!: nn_integral_cong split: split_indicator)
    also have ... = emeasure N (Q i ∩ A)
      using A Q by (simp add: f_density emeasure_restricted subset_eq sets_eq)
     finally have emeasure N (Q i ∩ A) = (∫⁺x. f i x ∗ indicator (Q i ∩ A) x
∂M) .. }
  note integral_eq = this
  let ?f = λx. (∑i. f i x ∗ indicator (Q i) x) + ∞ ∗ indicator (space M − (⋃i.
Q i)) x
  show ?thesis
  proof (safe intro!: bexI[of _ ?f])
    show ?f ∈ borel_measurable M using borel Q_sets
      by (auto intro!: measurable_If)
    show density M ?f = N
    proof (rule measure_eqI)
      fix A assume A ∈ sets (density M ?f)
      then have A ∈ sets M by simp
      have Qi: ⋀i. Q i ∈ sets M using Q by auto
      have [intro,simp]: ⋀i. (λx. f i x ∗ indicator (Q i ∩ A) x) ∈ borel_measurable
```

M

$\bigwedge i.$ *AE x in M*. $0 \le f\ i\ x * indicator\ (Q\ i \cap A)\ x$
using *borel Qi* ‹$A \in sets\ M$› **by** *auto*
have $(\int^+ x.\ ?f\ x * indicator\ A\ x\ \partial M) = (\int^+ x.\ (\sum i.\ f\ i\ x * indicator\ (Q\ i \cap A)\ x) + \infty * indicator\ ((space\ M - (\bigcup i.\ Q\ i)) \cap A)\ x\ \partial M)$
using *borel* **by** (*intro nn_integral_cong*) (*auto simp*: *indicator_def*)
also have ... $= (\int^+ x.\ (\sum i.\ f\ i\ x * indicator\ (Q\ i \cap A)\ x)\ \partial M) + \infty * emeasure\ M\ ((space\ M - (\bigcup i.\ Q\ i)) \cap A)$
using *borel Qi* ‹$A \in sets\ M$›
by (*subst nn_integral_add*)
(*auto simp add*: *nn_integral_cmult_indicator sets.Int intro*!: *suminf_0_le*)
also have ... $= (\sum i.\ N\ (Q\ i \cap A)) + \infty * emeasure\ M\ ((space\ M - (\bigcup i.\ Q\ i)) \cap A)$
by (*subst integral_eq*[*OF* ‹$A \in sets\ M$›], *subst nn_integral_suminf*) *auto*
finally have $(\int^+ x.\ ?f\ x * indicator\ A\ x\ \partial M) = (\sum i.\ N\ (Q\ i \cap A)) + \infty * emeasure\ M\ ((space\ M - (\bigcup i.\ Q\ i)) \cap A)$ **.**
moreover have $(\sum i.\ N\ (Q\ i \cap A)) = N\ ((\bigcup i.\ Q\ i) \cap A)$
using *Q Q_sets* ‹$A \in sets\ M$›
by (*subst suminf_emeasure*) (*auto simp*: *disjoint_family_on_def sets_eq*)
moreover
have $(space\ M - (\bigcup x.\ Q\ x)) \cap A \cap (\bigcup x.\ Q\ x) = \{\}$
by *auto*
then have $\infty * emeasure\ M\ ((space\ M - (\bigcup i.\ Q\ i)) \cap A) = N\ ((space\ M - (\bigcup i.\ Q\ i)) \cap A)$
using *in_Q0*[*of* $(space\ M - (\bigcup i.\ Q\ i)) \cap A$] ‹$A \in sets\ M$› *Q* **by** (*auto simp*: *ennreal_top_mult*)
moreover have $(space\ M - (\bigcup i.\ Q\ i)) \cap A \in sets\ M$ $((\bigcup i.\ Q\ i) \cap A) \in sets\ M$
using *Q_sets* ‹$A \in sets\ M$› **by** *auto*
moreover have $((\bigcup i.\ Q\ i) \cap A) \cup ((space\ M - (\bigcup i.\ Q\ i)) \cap A) = A$ $((\bigcup i.\ Q\ i) \cap A) \cap ((space\ M - (\bigcup i.\ Q\ i)) \cap A) = \{\}$
using ‹$A \in sets\ M$› *sets.sets_into_space* **by** *auto*
ultimately have $N\ A = (\int^+ x.\ ?f\ x * indicator\ A\ x\ \partial M)$
using *plus_emeasure*[*of* $(\bigcup i.\ Q\ i) \cap A\ N\ (space\ M - (\bigcup i.\ Q\ i)) \cap A$] **by** (*simp add*: *sets_eq*)
with ‹$A \in sets\ M$› *borel Q* **show** *emeasure* (*density M ?f*) $A = N\ A$
by (*auto simp*: *subset_eq emeasure_density*)
qed (*simp add*: *sets_eq*)
qed
qed

theorem (**in** *sigma_finite_measure*) *Radon_Nikodym*:
assumes *ac*: *absolutely_continuous M N* **assumes** *sets_eq*: *sets N = sets M*
shows $\exists f \in borel\_measurable\ M.\ density\ M\ f = N$
proof −
from *Ex_finite_integrable_function*
obtain *h* **where** *finite*: $integral^N\ M\ h \neq \infty$ **and**
borel: $h \in borel\_measurable\ M$ **and**
nn: $\bigwedge x.\ 0 \le h\ x$ **and**

```
    pos: ⋀x. x ∈ space M ⟹ 0 < h x and
    ⋀x. x ∈ space M ⟹ h x < ∞ by auto
  let ?T = λA. (∫+x. h x * indicator A x ∂M)
  let ?MT = density M h
  from borel finite nn interpret T: finite_measure ?MT
    by (auto intro!: finite_measureI cong: nn_integral_cong simp: emeasure_density)
  have absolutely_continuous ?MT N sets N = sets ?MT
  proof (unfold absolutely_continuous_def, safe)
    fix A assume A ∈ null_sets ?MT
    with borel have A ∈ sets M AE x in M. x ∈ A ⟶ h x ≤ 0
      by (auto simp add: null_sets_density_iff)
    with pos sets.sets_into_space have AE x in M. x ∉ A
      by (elim eventually_mono) (auto simp: not_le[symmetric])
    then have A ∈ null_sets M
      using ‹A ∈ sets M› by (simp add: AE_iff_null_sets)
    with ac show A ∈ null_sets N
      by (auto simp: absolutely_continuous_def)
  qed (auto simp add: sets_eq)
  from T.Radon_Nikodym_finite_measure_infinite[OF this]
  obtain f where f_borel: f ∈ borel_measurable M density ?MT f = N by auto
  with nn borel show ?thesis
    by (auto intro!: bexI[of _ λx. h x * f x] simp: density_density_eq)
qed
```

6.16.3 Uniqueness of densities

```
lemma finite_density_unique:
  assumes borel: f ∈ borel_measurable M g ∈ borel_measurable M
  assumes pos: AE x in M. 0 ≤ f x AE x in M. 0 ≤ g x
  and fin: integral^N M f ≠ ∞
  shows density M f = density M g ⟷ (AE x in M. f x = g x)
proof (intro iffI ballI)
  fix A assume eq: AE x in M. f x = g x
  with borel show density M f = density M g
    by (auto intro: density_cong)
next
  let ?P = λf A. ∫+ x. f x * indicator A x ∂M
  assume density M f = density M g
  with borel have eq: ∀A∈sets M. ?P f A = ?P g A
    by (simp add: emeasure_density[symmetric])
  from this[THEN bspec, OF sets.top] fin
  have g_fin: integral^N M g ≠ ∞ by (simp cong: nn_integral_cong)
  { fix f g assume borel: f ∈ borel_measurable M g ∈ borel_measurable M
      and pos: AE x in M. 0 ≤ f x AE x in M. 0 ≤ g x
      and g_fin: integral^N M g ≠ ∞ and eq: ∀A∈sets M. ?P f A = ?P g A
    let ?N = {x∈space M. g x < f x}
    have N: ?N ∈ sets M using borel by simp
    have ?P g ?N ≤ integral^N M g using pos
      by (intro nn_integral_mono_AE) (auto split: split_indicator)
```

then have *Pg_fin*: *?P g ?N* $\neq \infty$ **using** *g_fin* **by** (*auto simp*: *top_unique*)
have *?P* (λx. (*f x* $-$ *g x*)) *?N* = ($\int^+ x$. *f x* $*$ *indicator ?N x* $-$ *g x* $*$ *indicator ?N x* ∂M)
by (*auto intro*!: *nn_integral_cong simp*: *indicator_def*)
also have ... = *?P f ?N* $-$ *?P g ?N*
proof (*rule nn_integral_diff*)
show (λx. *f x* $*$ *indicator ?N x*) $\in$ *borel_measurable M* (λx. *g x* $*$ *indicator ?N x*) $\in$ *borel_measurable M*
using *borel N* **by** *auto*
show *AE x in M*. *g x* $*$ *indicator ?N x* $\leq$ *f x* $*$ *indicator ?N x*
using *pos* **by** (*auto split*: *split_indicator*)
qed *fact*
also have ... = *0*
unfolding *eq*[*THEN bspec*, *OF N*] **using** *Pg_fin* **by** *auto*
finally have *AE x in M*. *f x* $\leq$ *g x*
using *pos borel nn_integral_PInf_AE*[*OF borel*(*2*) *g_fin*]
by (*subst* (*asm*) *nn_integral_0_iff_AE*)
(*auto split*: *split_indicator simp*: *not_less ennreal_minus_eq_0*) **}**
from *this*[*OF borel pos g_fin eq*] *this*[*OF borel*(*2*,*1*) *pos*(*2*,*1*) *fin*] *eq*
show *AE x in M*. *f x* = *g x* **by** *auto*
qed

lemma (**in** *finite_measure*) *density_unique_finite_measure*:
assumes *borel*: *f* $\in$ *borel_measurable M f'* $\in$ *borel_measurable M*
assumes *pos*: *AE x in M*. *0* $\leq$ *f x AE x in M*. *0* $\leq$ *f' x*
assumes *f*: $\bigwedge A$. *A* $\in$ *sets M* $\Longrightarrow$ ($\int^+ x$. *f x* $*$ *indicator A x* ∂M) = ($\int^+ x$. *f' x* $*$ *indicator A x* ∂M)
(**is** $\bigwedge A$. *A* $\in$ *sets M* $\Longrightarrow$ *?P f A* = *?P f' A*)
shows *AE x in M*. *f x* = *f' x*
proof $-$
let *?D* = λf. *density M f*
let *?N* = λA. *?P f A* **and** *?N'* = λA. *?P f' A*
let *?f* = λA *x*. *f x* $*$ *indicator A x* **and** *?f'* = λA *x*. *f' x* $*$ *indicator A x*

have *ac*: *absolutely_continuous M* (*density M f*) *sets* (*density M f*) = *sets M*
using *borel* **by** (*auto intro*!: *absolutely_continuousI_density*)
from *split_space_into_finite_sets_and_rest*[*OF this*]
obtain *Q* :: *nat* $\Rightarrow$ *'a set*
where *Q*: *disjoint_family Q range Q* $\subseteq$ *sets M*
and *in_Q0*: $\bigwedge A$. *A* $\in$ *sets M* $\Longrightarrow$ *A* $\cap$ ($\bigcup i$. *Q i*) = {} $\Longrightarrow$ *emeasure M A* = *0* $\wedge$ *?D f A* = *0* $\vee$ *0* < *emeasure M A* $\wedge$ *?D f A* = ∞
and *Q_fin*: $\bigwedge i$. *?D f* (*Q i*) $\neq \infty$ **by** *force*
with *borel pos* **have** *in_Q0*: $\bigwedge A$. *A* $\in$ *sets M* $\Longrightarrow$ *A* $\cap$ ($\bigcup i$. *Q i*) = {} $\Longrightarrow$ *emeasure M A* = *0* $\wedge$ *?N A* = *0* $\vee$ *0* < *emeasure M A* $\wedge$ *?N A* = ∞
and *Q_fin*: $\bigwedge i$. *?N* (*Q i*) $\neq \infty$ **by** (*auto simp*: *emeasure_density subset_eq*)

from *Q* **have** *Q_sets*[*measurable*]: $\bigwedge i$. *Q i* $\in$ *sets M* **by** *auto*
let *?D* = {*x*$\in$*space M*. *f x* $\neq$ *f' x*}
have *?D* $\in$ *sets M* **using** *borel* **by** *auto*

have ∗: $\bigwedge i\, x\, A.\ \bigwedge y$::*ennreal*. *y* ∗ *indicator* (*Q i*) *x* ∗ *indicator A x* = *y* ∗ *indicator* (*Q i* ∩ *A*) *x*
unfolding *indicator_def* **by** *auto*
have ∀ *i*. *AE x in M*. *?f* (*Q i*) *x* = *?f′* (*Q i*) *x* **using** *borel Q_fin Q pos*
by (*intro finite_density_unique*[*THEN iffD1*] *allI*)
(*auto intro*!: *f measure_eqI simp*: *emeasure_density* ∗ *subset_eq*)
moreover have *AE x in M*. *?f* (*space M* − (⋃*i*. *Q i*)) *x* = *?f′* (*space M* − (⋃*i*. *Q i*)) *x*
proof (*rule AE_I′*)
{ **fix** *f* :: *′a* ⇒ *ennreal* **assume** *borel*: *f* ∈ *borel_measurable M*
and *eq*: $\bigwedge$*A*. *A* ∈ *sets M* ⟹ *?N A* = (∫⁺*x*. *f x* ∗ *indicator A x* ∂*M*)
let *?A* = λ*i*. (*space M* − (⋃*i*. *Q i*)) ∩ {*x* ∈ *space M*. *f x* < (*i*::*nat*)}
have (⋃*i*. *?A i*) ∈ *null_sets M*
proof (*rule null_sets_UN*)
fix *i* ::*nat* **have** *?A i* ∈ *sets M*
using *borel* **by** *auto*
have *?N* (*?A i*) ≤ (∫⁺*x*. (*i*::*ennreal*) ∗ *indicator* (*?A i*) *x* ∂*M*)
unfolding *eq*[*OF* ‹*?A i* ∈ *sets M*›]
by (*auto intro*!: *nn_integral_mono simp*: *indicator_def*)
also have . . . = *i* ∗ *emeasure M* (*?A i*)
using ‹*?A i* ∈ *sets M*› **by** (*auto intro*!: *nn_integral_cmult_indicator*)
also have . . . < ∞ **using** *emeasure_real*[*of ?A i*] **by** (*auto simp*: *ennreal_mult_less_top of_nat_less_top*)
finally have *?N* (*?A i*) ≠ ∞ **by** *simp*
then show *?A i* ∈ *null_sets M* **using** *in_Q0*[*OF* ‹*?A i* ∈ *sets M*›] ‹*?A i* ∈ *sets M*› **by** *auto*
qed
also have (⋃*i*. *?A i*) = (*space M* − (⋃*i*. *Q i*)) ∩ {*x*∈*space M*. *f x* ≠ ∞}
by (*auto simp*: *ennreal_Ex_less_of_nat less_top*[*symmetric*])
finally have (*space M* − (⋃*i*. *Q i*)) ∩ {*x*∈*space M*. *f x* ≠ ∞} ∈ *null_sets M* **by** *simp* }
from *this*[*OF borel*(*1*) *refl*] *this*[*OF borel*(*2*) *f*]
have (*space M* − (⋃*i*. *Q i*)) ∩ {*x*∈*space M*. *f x* ≠ ∞} ∈ *null_sets M* (*space M* − (⋃*i*. *Q i*)) ∩ {*x*∈*space M*. *f′ x* ≠ ∞} ∈ *null_sets M* **by** *simp_all*
then show ((*space M* − (⋃*i*. *Q i*)) ∩ {*x*∈*space M*. *f x* ≠ ∞}) ∪ ((*space M* − (⋃*i*. *Q i*)) ∩ {*x*∈*space M*. *f′ x* ≠ ∞}) ∈ *null_sets M* **by** (*rule null_sets.Un*)
show {*x* ∈ *space M*. *?f* (*space M* − (⋃*i*. *Q i*)) *x* ≠ *?f′* (*space M* − (⋃*i*. *Q i*)) *x*} ⊆
((*space M* − (⋃*i*. *Q i*)) ∩ {*x*∈*space M*. *f x* ≠ ∞}) ∪ ((*space M* − (⋃*i*. *Q i*)) ∩ {*x*∈*space M*. *f′ x* ≠ ∞}) **by** (*auto simp*: *indicator_def*)
qed
moreover have *AE x in M*. (*?f* (*space M* − (⋃*i*. *Q i*)) *x* = *?f′* (*space M* − (⋃*i*. *Q i*)) *x*) ⟶ (∀ *i*. *?f* (*Q i*) *x* = *?f′* (*Q i*) *x*) ⟶
?f (*space M*) *x* = *?f′* (*space M*) *x*
by (*auto simp*: *indicator_def*)
ultimately have *AE x in M*. *?f* (*space M*) *x* = *?f′* (*space M*) *x*
unfolding *AE_all_countable*[*symmetric*]
by *eventually_elim* (*auto split*: *if_split_asm simp*: *indicator_def*)
then show *AE x in M*. *f x* = *f′ x* **by** *auto*

qed

proposition (**in** *sigma_finite_measure*) *density_unique*:
assumes *f*: $f \in$ *borel_measurable M*
assumes *f'*: $f' \in$ *borel_measurable M*
assumes *density_eq*: *density M f* = *density M f'*
shows *AE x in M*. $f\ x = f'\ x$
proof −
obtain *h* **where** *h_borel*: $h \in$ *borel_measurable M*
and *fin*: $integral^N\ M\ h \neq \infty$ **and** *pos*: $\bigwedge x.\ x \in space\ M \Longrightarrow 0 < h\ x \wedge h\ x < \infty$ $\bigwedge x.\ 0 \leq h\ x$
using *Ex_finite_integrable_function* **by** *auto*
then have *h_nn*: *AE x in M*. $0 \leq h\ x$ **by** *auto*
let *?H* = *density M h*
interpret *h*: *finite_measure ?H*
using *fin h_borel pos*
by (*intro finite_measureI*) (*simp cong*: *nn_integral_cong emeasure_density add*: *fin*)
let *?fM* = *density M f*
let *?f'M* = *density M f'*
{ **fix** *A* **assume** $A \in$ *sets M*
then have $\{x \in space\ M.\ h\ x * indicator\ A\ x \neq 0\} = A$
using *pos*(*1*) *sets.sets_into_space* **by** (*force simp*: *indicator_def*)
then have $(\int^+ x.\ h\ x * indicator\ A\ x\ \partial M) = 0 \longleftrightarrow A \in null\_sets\ M$
using *h_borel* ‹$A \in sets\ M$› *h_nn* **by** (*subst nn_integral_0_iff*) *auto* }
note *h_null_sets* = *this*
{ **fix** *A* **assume** $A \in$ *sets M*
have $(\int^+ x.\ f\ x * (h\ x * indicator\ A\ x)\ \partial M) = (\int^+ x.\ h\ x * indicator\ A\ x\ \partial ?fM)$
using ‹$A \in sets\ M$› *h_borel h_nn f f'*
by (*intro nn_integral_density*[*symmetric*]) *auto*
also have ... $= (\int^+ x.\ h\ x * indicator\ A\ x\ \partial ?f'M)$
by (*simp_all add*: *density_eq*)
also have ... $= (\int^+ x.\ f'\ x * (h\ x * indicator\ A\ x)\ \partial M)$
using ‹$A \in sets\ M$› *h_borel h_nn f f'*
by (*intro nn_integral_density*) *auto*
finally have $(\int^+ x.\ h\ x * (f\ x * indicator\ A\ x)\ \partial M) = (\int^+ x.\ h\ x * (f'\ x * indicator\ A\ x)\ \partial M)$
by (*simp add*: *ac_simps*)
then have $(\int^+ x.\ (f\ x * indicator\ A\ x)\ \partial ?H) = (\int^+ x.\ (f'\ x * indicator\ A\ x)\ \partial ?H)$
using ‹$A \in sets\ M$› *h_borel h_nn f f'*
by (*subst* (*asm*) (*1 2*) *nn_integral_density*[*symmetric*]) *auto* }
then have *AE x in ?H*. $f\ x = f'\ x$ **using** *h_borel h_nn f f'*
by (*intro h.density_unique_finite_measure absolutely_continuous_AE*[*of M*]) *auto*
with *AE_space*[*of M*] *pos* **show** *AE x in M*. $f\ x = f'\ x$
unfolding *AE_density*[*OF h_borel*] **by** *auto*
qed

```
lemma (in sigma_finite_measure) density_unique_iff:
  assumes f: f ∈ borel_measurable M and f': f' ∈ borel_measurable M
  shows density M f = density M f' ⟷ (AE x in M. f x = f' x)
  using density_unique[OF assms] density_cong[OF f f'] by auto

lemma sigma_finite_density_unique:
  assumes borel: f ∈ borel_measurable M g ∈ borel_measurable M
  and fin: sigma_finite_measure (density M f)
  shows density M f = density M g ⟷ (AE x in M. f x = g x)
proof
  assume AE x in M. f x = g x with borel show density M f = density M g
    by (auto intro: density_cong)
next
  assume eq: density M f = density M g
  interpret f: sigma_finite_measure density M f by fact
  from f.sigma_finite_incseq guess A . note cover = this

  have AE x in M. ∀ i. x ∈ A i ⟶ f x = g x
    unfolding AE_all_countable
  proof
    fix i
   have density (density M f) (indicator (A i)) = density (density M g) (indicator
(A i))
      unfolding eq ..
    moreover have (∫+x. f x ∗ indicator (A i) x ∂M) ≠ ∞
     using cover(1) cover(3)[of i] borel by (auto simp: emeasure_density subset_eq)
    ultimately have AE x in M. f x ∗ indicator (A i) x = g x ∗ indicator (A i) x
      using borel cover(1)
       by (intro finite_density_unique[THEN iffD1]) (auto simp: density_density_eq
subset_eq)
    then show AE x in M. x ∈ A i ⟶ f x = g x
      by auto
  qed
  with AE_space show AE x in M. f x = g x
    apply eventually_elim
    using cover(2)[symmetric]
    apply auto
    done
qed

lemma (in sigma_finite_measure) sigma_finite_iff_density_finite':
  assumes f: f ∈ borel_measurable M
  shows sigma_finite_measure (density M f) ⟷ (AE x in M. f x ≠ ∞)
    (is sigma_finite_measure ?N ⟷ _)
proof
  assume sigma_finite_measure ?N
  then interpret N: sigma_finite_measure ?N .
  from N.Ex_finite_integrable_function obtain h where
    h: h ∈ borel_measurable M integral^N ?N h ≠ ∞ and
```

fin: ∀ *x*∈*space M*. *0* < *h x* ∧ *h x* < ∞
by *auto*
have *AE x in M*. *f x* ∗ *h x* ≠ ∞
proof (*rule AE_I'*)
have *integral*^{N} *?N h* = (∫$^+$*x*. *f x* ∗ *h x* ∂*M*)
using *f h* **by** (*auto intro*!: *nn_integral_density*)
then have (∫$^+$*x*. *f x* ∗ *h x* ∂*M*) ≠ ∞
using *h*(*2*) **by** *simp*
then show (λ*x*. *f x* ∗ *h x*) −' {∞} ∩ *space M* ∈ *null_sets M*
using *f h*(*1*) **by** (*auto intro*!: *nn_integral_PInf*[*unfolded infinity_ennreal_def*] *borel_measurable_vimage*)
qed *auto*
then show *AE x in M*. *f x* ≠ ∞
using *fin* **by** (*auto elim*!: *AE_Ball_mp simp*: *less_top ennreal_mult_less_top*)
next
assume *AE*: *AE x in M*. *f x* ≠ ∞
from *sigma_finite* **guess** *Q* **. note** *Q* = *this*
define *A* **where** *A i* =
f −' (*case i of 0* ⇒ {∞} | *Suc n* ⇒ {.. *ennreal*(*of_nat* (*Suc n*))}) ∩ *space M*
for *i*
{ fix *i j* **have** *A i* ∩ *Q j* ∈ *sets M*
unfolding *A_def* **using** *f Q*
apply (*rule_tac sets.Int*)
by (*cases i*) (*auto intro*: *measurable_sets*[*OF f*(*1*)]) **}**
note *A_in_sets* = *this*

show *sigma_finite_measure ?N*
proof (*standard*, *intro exI conjI ballI*)
show *countable* (*range* (λ(*i*, *j*). *A i* ∩ *Q j*))
by *auto*
show *range* (λ(*i*, *j*). *A i* ∩ *Q j*) ⊆ *sets* (*density M f*)
using *A_in_sets* **by** *auto*
next
have ⋃(*range* (λ(*i*, *j*). *A i* ∩ *Q j*)) = (⋃*i j*. *A i* ∩ *Q j*)
by *auto*
also have ... = (⋃*i*. *A i*) ∩ *space M* **using** *Q* **by** *auto*
also have (⋃*i*. *A i*) = *space M*
proof *safe*
fix *x* **assume** *x*: *x* ∈ *space M*
show *x* ∈ (⋃*i*. *A i*)
proof (*cases f x rule*: *ennreal_cases*)
case *top* **with** *x* **show** *?thesis* **unfolding** *A_def* **by** (*auto intro*: *exI*[*of _ 0*])
next
case (*real r*)
with *ennreal_Ex_less_of_nat*[*of f x*] **obtain** *n* :: *nat* **where** *f x* < *n*
by *auto*
also have *n* < (*Suc n* :: *ennreal*)
by *simp*
finally show *?thesis*

```
        using x real by (auto simp: A_def ennreal_of_nat_eq_real_of_nat intro!:
exI[of _ Suc n])
      qed
    qed (auto simp: A_def)
    finally show ⋃(range (λ(i, j). A i ∩ Q j)) = space ?N by simp
  next
    fix X assume X ∈ range (λ(i, j). A i ∩ Q j)
    then obtain i j where [simp]:X = A i ∩ Q j by auto
    have (∫+x. f x * indicator (A i ∩ Q j) x ∂M) ≠ ∞
    proof (cases i)
      case 0
      have AE x in M. f x * indicator (A i ∩ Q j) x = 0
        using AE by (auto simp: A_def ‹i = 0›)
      from nn_integral_cong_AE[OF this] show ?thesis by simp
    next
      case (Suc n)
      then have (∫+x. f x * indicator (A i ∩ Q j) x ∂M) ≤
        (∫+x. (Suc n :: ennreal) * indicator (Q j) x ∂M)
      by (auto intro!: nn_integral_mono simp: indicator_def A_def ennreal_of_nat_eq_real_of_nat)
      also have ... = Suc n * emeasure M (Q j)
        using Q by (auto intro!: nn_integral_cmult_indicator)
      also have ... < ∞
        using Q by (auto simp: ennreal_mult_less_top less_top of_nat_less_top)
      finally show ?thesis by simp
    qed
    then show emeasure ?N X ≠ ∞
      using A_in_sets Q f by (auto simp: emeasure_density)
  qed
qed

lemma (in sigma_finite_measure) sigma_finite_iff_density_finite:
  f ∈ borel_measurable M ⟹ sigma_finite_measure (density M f) ⟷ (AE x in
M. f x ≠ ∞)
  by (subst sigma_finite_iff_density_finite')
    (auto simp: max_def intro!: measurable_If)
```

6.16.4 Radon-Nikodym derivative

```
definition RN_deriv :: 'a measure ⇒ 'a measure ⇒ 'a ⇒ ennreal where
  RN_deriv M N =
    (if ∃f. f ∈ borel_measurable M ∧ density M f = N
      then SOME f. f ∈ borel_measurable M ∧ density M f = N
      else (λ_. 0))

lemma RN_derivI:
  assumes f ∈ borel_measurable M density M f = N
  shows density M (RN_deriv M N) = N
proof -
  have *: ∃f. f ∈ borel_measurable M ∧ density M f = N
```

using *assms* **by** *auto*
then have *density M* (*SOME f. f* ∈ *borel_measurable M* ∧ *density M f* = *N*) = *N*
by (*rule someI2_ex*) *auto*
with ∗ **show** *?thesis*
by (*auto simp*: *RN_deriv_def*)
qed

lemma *borel_measurable_RN_deriv*[*measurable*]: *RN_deriv M N* ∈ *borel_measurable M*
proof −
{ assume *ex*: ∃*f. f* ∈ *borel_measurable M* ∧ *density M f* = *N*
have *1*: (*SOME f. f* ∈ *borel_measurable M* ∧ *density M f* = *N*) ∈ *borel_measurable M*
using *ex* **by** (*rule someI2_ex*) *auto* **}**
from *this* **show** *?thesis*
by (*auto simp*: *RN_deriv_def*)
qed

lemma *density_RN_deriv_density*:
assumes *f*: *f* ∈ *borel_measurable M*
shows *density M* (*RN_deriv M* (*density M f*)) = *density M f*
by (*rule RN_derivI*[*OF f*]) *simp*

lemma (**in** *sigma_finite_measure*) *density_RN_deriv*:
absolutely_continuous M N ⟹ *sets N* = *sets M* ⟹ *density M* (*RN_deriv M N*) = *N*
by (*metis RN_derivI Radon_Nikodym*)

lemma (**in** *sigma_finite_measure*) *RN_deriv_nn_integral*:
assumes *N*: *absolutely_continuous M N sets N* = *sets M*
and *f*: *f* ∈ *borel_measurable M*
shows $integral^N$ *N f* = ($\int^+ x$. *RN_deriv M N x* ∗ *f x* ∂*M*)
proof −
have $integral^N$ *N f* = $integral^N$ (*density M* (*RN_deriv M N*)) *f*
using *N* **by** (*simp add*: *density_RN_deriv*)
also have . . . = ($\int^+ x$. *RN_deriv M N x* ∗ *f x* ∂*M*)
using *f* **by** (*simp add*: *nn_integral_density*)
finally show *?thesis* **by** *simp*
qed

lemma (**in** *sigma_finite_measure*) *RN_deriv_unique*:
assumes *f*: *f* ∈ *borel_measurable M*
and *eq*: *density M f* = *N*
shows *AE x in M. f x* = *RN_deriv M N x*
unfolding *eq*[*symmetric*]
by (*intro density_unique_iff* [*THEN iffD1*] *f borel_measurable_RN_deriv*
density_RN_deriv_density[*symmetric*])

lemma *RN_deriv_unique_sigma_finite*:
assumes *f*: *f* ∈ *borel_measurable M*
and *eq*: *density M f* = *N* **and** *fin*: *sigma_finite_measure N*
shows *AE x in M*. *f x* = *RN_deriv M N x*
using *fin* **unfolding** *eq*[*symmetric*]
by (*intro sigma_finite_density_unique*[*THEN iffD1*] *f borel_measurable_RN_deriv density_RN_deriv_density*[*symmetric*])

lemma (**in** *sigma_finite_measure*) *RN_deriv_distr*:
fixes *T* :: *'a* ⇒ *'b*
assumes *T*: *T* ∈ *measurable M M'* **and** *T'*: *T'* ∈ *measurable M' M*
and *inv*: ∀ *x*∈*space M*. *T'* (*T x*) = *x*
and *ac*[*simp*]: *absolutely_continuous* (*distr M M' T*) (*distr N M' T*)
and *N*: *sets N* = *sets M*
shows *AE x in M*. *RN_deriv* (*distr M M' T*) (*distr N M' T*) (*T x*) = *RN_deriv M N x*
proof (*rule RN_deriv_unique*)
have [*simp*]: *sets N* = *sets M* **by** *fact*
note *sets_eq_imp_space_eq*[*OF N*, *simp*]
have *measurable_N*[*simp*]: ⋀*M'*. *measurable N M'* = *measurable M M'* **by** (*auto simp*: *measurable_def*)
{ **fix** *A* **assume** *A* ∈ *sets M*
with *inv T T' sets.sets_into_space*[*OF this*]
have *T* −' *T'* −' *A* ∩ *T* −' *space M'* ∩ *space M* = *A*
by (*auto simp*: *measurable_def*) }
note *eq* = *this*[*simp*]
{ **fix** *A* **assume** *A* ∈ *sets M*
with *inv T T' sets.sets_into_space*[*OF this*]
have (*T'* ∘ *T*) −' *A* ∩ *space M* = *A*
by (*auto simp*: *measurable_def*) }
note *eq2* = *this*[*simp*]
let *?M'* = *distr M M' T* **and** *?N'* = *distr N M' T*
interpret *M'*: *sigma_finite_measure ?M'*
proof
from *sigma_finite_countable* **guess** *F* .. **note** *F* = *this*
show ∃ *A*. *countable A* ∧ *A* ⊆ *sets* (*distr M M' T*) ∧ ⋃ *A* = *space* (*distr M M' T*) ∧ (∀ *a*∈*A*. *emeasure* (*distr M M' T*) *a* ≠ ∞)
proof (*intro exI conjI ballI*)
show ∗: (λ*A*. *T'* −' *A* ∩ *space ?M'*) ' *F* ⊆ *sets ?M'*
using *F T'* **by** (*auto simp*: *measurable_def*)
show ⋃((λ*A*. *T'* −' *A* ∩ *space ?M'*)'*F*) = *space ?M'*
using *F T'*[*THEN measurable_space*] **by** (*auto simp*: *set_eq_iff*)
next
fix *X* **assume** *X* ∈ (λ*A*. *T'* −' *A* ∩ *space ?M'*)'*F*
then obtain *A* **where** [*simp*]: *X* = *T'* −' *A* ∩ *space ?M'* **and** *A* ∈ *F* **by** *auto*
have *X* ∈ *sets M'* **using** *F T'* ‹*A*∈*F*› **by** *auto*
moreover
have *Fi*: *A* ∈ *sets M* **using** *F* ‹*A*∈*F*› **by** *auto*

ultimately show *emeasure ?M′ X* ≠ ∞
using *F T T′* ‹*A*∈*F*› **by** (*simp add*: *emeasure_distr*)
qed (*insert F*, *auto*)
qed
have (*RN_deriv ?M′ ?N′*) ∘ *T* ∈ *borel_measurable M*
using *T ac* **by** *measurable*
then show (λ*x*. *RN_deriv ?M′ ?N′* (*T x*)) ∈ *borel_measurable M*
by (*simp add*: *comp_def*)

have *N* = *distr N M* (*T′* ∘ *T*)
by (*subst measure_of_of_measure*[*of N*, *symmetric*])
(*auto simp add*: *distr_def sets.sigma_sets_eq intro*!: *measure_of_eq sets.space_closed*)
also have ... = *distr* (*distr N M′ T*) *M T′*
using *T T′* **by** (*simp add*: *distr_distr*)
also have ... = *distr* (*density* (*distr M M′ T*) (*RN_deriv* (*distr M M′ T*) (*distr N M′ T*))) *M T′*
using *ac* **by** (*simp add*: *M′.density_RN_deriv*)
also have ... = *density M* (*RN_deriv* (*distr M M′ T*) (*distr N M′ T*) ∘ *T*)
by (*simp add*: *distr_density_distr*[*OF T T′*, *OF inv*])
finally show *density M* (λ*x*. *RN_deriv* (*distr M M′ T*) (*distr N M′ T*) (*T x*)) = *N*
by (*simp add*: *comp_def*)
qed

lemma (**in** *sigma_finite_measure*) *RN_deriv_finite*:
assumes *N*: *sigma_finite_measure N* **and** *ac*: *absolutely_continuous M N sets N* = *sets M*
shows *AE x in M*. *RN_deriv M N x* ≠ ∞
proof −
interpret *N*: *sigma_finite_measure N* **by** *fact*
from *N* **show** *?thesis*
using *sigma_finite_iff_density_finite*[*OF borel_measurable_RN_deriv*, *of N*] *density_RN_deriv*[*OF ac*]
by *simp*
qed

lemma (**in** *sigma_finite_measure*)
assumes *N*: *sigma_finite_measure N* **and** *ac*: *absolutely_continuous M N sets N* = *sets M*
and *f*: *f* ∈ *borel_measurable M*
shows *RN_deriv_integrable*: *integrable N f* ⟷
integrable M (λ*x*. *enn2real* (*RN_deriv M N x*) ∗ *f x*) (**is** *?integrable*)
and *RN_deriv_integral*: $integral^L$ *N f* = (∫ *x*. *enn2real* (*RN_deriv M N x*) ∗ *f x* ∂*M*) (**is** *?integral*)
proof −
note *ac*(*2*)[*simp*] **and** *sets_eq_imp_space_eq*[*OF ac*(*2*), *simp*]
interpret *N*: *sigma_finite_measure N* **by** *fact*

have *eq*: *density M* (*RN_deriv M N*) = *density M* (λ*x*. *enn2real* (*RN_deriv M N*

```
x))
  proof (rule density_cong)
    from RN_deriv_finite[OF assms(1,2,3)]
    show AE x in M. RN_deriv M N x = ennreal (enn2real (RN_deriv M N x))
      by eventually_elim (auto simp: less_top)
  qed (insert ac, auto)

  show ?integrable
    apply (subst density_RN_deriv[OF ac, symmetric])
    unfolding eq
    apply (intro integrable_real_density f AE_I2 enn2real_nonneg)
    apply (insert ac, auto)
    done

  show ?integral
    apply (subst density_RN_deriv[OF ac, symmetric])
    unfolding eq
    apply (intro integral_real_density f AE_I2 enn2real_nonneg)
    apply (insert ac, auto)
    done
qed

proposition (in sigma_finite_measure) real_RN_deriv:
  assumes finite_measure N
  assumes ac: absolutely_continuous M N sets N = sets M
  obtains D where D ∈ borel_measurable M
    and AE x in M. RN_deriv M N x = ennreal (D x)
    and AE x in N. 0 < D x
    and ⋀x. 0 ≤ D x
proof
  interpret N: finite_measure N by fact

  note RN = borel_measurable_RN_deriv density_RN_deriv[OF ac]

  let ?RN = λt. {x ∈ space M. RN_deriv M N x = t}

  show (λx. enn2real (RN_deriv M N x)) ∈ borel_measurable M
    using RN by auto

  have N (?RN ∞) = (∫+ x. RN_deriv M N x * indicator (?RN ∞) x ∂M)
    using RN(1) by (subst RN(2)[symmetric]) (auto simp: emeasure_density)
  also have ... = (∫+ x. ∞ * indicator (?RN ∞) x ∂M)
    by (intro nn_integral_cong) (auto simp: indicator_def)
  also have ... = ∞ * emeasure M (?RN ∞)
    using RN by (intro nn_integral_cmult_indicator) auto
  finally have eq: N (?RN ∞) = ∞ * emeasure M (?RN ∞) .
  moreover
  have emeasure M (?RN ∞) = 0
  proof (rule ccontr)
```

assume *emeasure M* $\{x \in$ *space M*. *RN_deriv M N x* $= \infty\} \neq 0$
then have *0* $<$ *emeasure M* $\{x \in$ *space M*. *RN_deriv M N x* $= \infty\}$
by (*auto simp*: *zero_less_iff_neq_zero*)
with *eq* **have** *N* (*?RN* ∞) $= \infty$ **by** (*simp add*: *ennreal_mult_eq_top_iff*)
with *N.emeasure_finite*[*of ?RN* ∞] *RN* **show** *False* **by** *auto*
qed
ultimately have *AE x in M*. *RN_deriv M N x* $< \infty$
using *RN* **by** (*intro AE_iff_measurable*[*THEN iffD2*]) (*auto simp*: *less_top*[*symmetric*])
then show *AE x in M*. *RN_deriv M N x* = *ennreal* (*enn2real* (*RN_deriv M N x*))
by *auto*
then have *eq*: *AE x in N*. *RN_deriv M N x* = *ennreal* (*enn2real* (*RN_deriv M N x*))
using *ac absolutely_continuous_AE* **by** *auto*

have *N* (*?RN 0*) = ($\int^+$ *x*. *RN_deriv M N x* $*$ *indicator* (*?RN 0*) *x* ∂M)
by (*subst RN*(*2*)[*symmetric*]) (*auto simp*: *emeasure_density*)
also have ... = ($\int^+$ *x*. *0* ∂M)
by (*intro nn_integral_cong*) (*auto simp*: *indicator_def*)
finally have *AE x in N*. *RN_deriv M N x* $\neq$ *0*
using *RN* **by** (*subst AE_iff_measurable*[*OF _ refl*]) (*auto simp*: *ac cong*: *sets_eq_imp_space_eq*)
with *eq* **show** *AE x in N*. *0* $<$ *enn2real* (*RN_deriv M N x*)
by (*auto simp*: *enn2real_positive_iff less_top*[*symmetric*] *zero_less_iff_neq_zero*)
qed (*rule enn2real_nonneg*)

lemma (**in** *sigma_finite_measure*) *RN_deriv_singleton*:
assumes *ac*: *absolutely_continuous M N sets N* = *sets M*
and *x*: $\{x\} \in$ *sets M*
shows *N* $\{x\}$ = *RN_deriv M N x* $*$ *emeasure M* $\{x\}$
proof −
from ‹$\{x\} \in$ *sets M*›
have *density M* (*RN_deriv M N*) $\{x\}$ = ($\int^+$*w*. *RN_deriv M N x* $*$ *indicator* $\{x\}$ *w* ∂M)
by (*auto simp*: *indicator_def emeasure_density intro*!: *nn_integral_cong*)
with *x density_RN_deriv*[*OF ac*] **show** *?thesis*
by (*auto simp*: *max_def*)
qed

end

theory *Set_Integral*
imports *Radon_Nikodym*
begin

definition *set_borel_measurable M A f* $\equiv$ (λx. *indicator A x* $*_R$ *f x*) $\in$ *borel_measurable*

M

definition *set_integrable M A f* ≡ *integrable M* (*λx. indicator A x* $*_R$ *f x*)

definition *set_lebesgue_integral M A f* ≡ *lebesgue_integral M* (*λx. indicator A x* $*_R$ *f x*)

syntax
_ascii_set_lebesgue_integral :: *pttrn* ⇒ *'a set* ⇒ *'a measure* ⇒ *real* ⇒ *real*
((*4LINT* (_):(_)/|(_)./ _) [*0*,*60*,*110*,*61*] *60*)

translations
LINT x:*A*|*M*. *f* == *CONST set_lebesgue_integral M A* (*λx. f*)

syntax
_lebesgue_borel_integral :: *pttrn* ⇒ *real* ⇒ *real*
((*2LBINT* _./ _) [*0*,*60*] *60*)

syntax
_set_lebesgue_borel_integral :: *pttrn* ⇒ *real set* ⇒ *real* ⇒ *real*
((*3LBINT* _:_./ _) [*0*,*60*,*61*] *60*)

lemma *set_integrable_cong*:
assumes *M* = *M'* *A* = *A'* ⋀*x*. *x* ∈ *A* ⟹ *f x* = *f' x*
shows *set_integrable M A f* = *set_integrable M' A' f'*
proof −
have (*λx. indicator A x* $*_R$ *f x*) = (*λx. indicator A' x* $*_R$ *f' x*)
using *assms* **by** (*auto simp*: *indicator_def*)
thus *?thesis* **by** (*simp add*: *set_integrable_def assms*)
qed

lemma *set_borel_measurable_sets*:
fixes *f* :: _ ⇒ _::*real_normed_vector*
assumes *set_borel_measurable M X f B* ∈ *sets borel X* ∈ *sets M*
shows *f* −' *B* ∩ *X* ∈ *sets M*
proof −
have *f* ∈ *borel_measurable* (*restrict_space M X*)
using *assms* **unfolding** *set_borel_measurable_def* **by** (*subst borel_measurable_restrict_space_iff*) *auto*
then have *f* −' *B* ∩ *space* (*restrict_space M X*) ∈ *sets* (*restrict_space M X*)
by (*rule measurable_sets*) *fact*
with ‹*X* ∈ *sets M*› **show** *?thesis*
by (*subst* (*asm*) *sets_restrict_space_iff*) (*auto simp*: *space_restrict_space*)

qed

lemma *set_lebesgue_integral_zero* [*simp*]: *set_lebesgue_integral M A* ($\lambda x.$ *0*) = *0*
  by (*auto simp*: *set_lebesgue_integral_def*)

lemma *set_lebesgue_integral_cong*:
  assumes $A \in$ *sets M* **and** $\forall x.\ x \in A \longrightarrow f\ x = g\ x$
  shows (*LINT x*:*A*|*M*. *f x*) = (*LINT x*:*A*|*M*. *g x*)
  unfolding *set_lebesgue_integral_def*
  using *assms*
  by (*metis indicator_simps*(*2*) *real_vector.scale_zero_left*)

lemma *set_lebesgue_integral_cong_AE*:
  assumes [*measurable*]: $A \in$ *sets M f* $\in$ *borel_measurable M g* $\in$ *borel_measurable M*
  assumes *AE x* $\in$ *A in M*. *f x* = *g x*
  shows *LINT x*:*A*|*M*. *f x* = *LINT x*:*A*|*M*. *g x*
proof−
  have *AE x in M*. *indicator A x* $*_R$ *f x* = *indicator A x* $*_R$ *g x*
    using *assms* **by** *auto*
  thus *?thesis*
  unfolding *set_lebesgue_integral_def* **by** (*intro integral_cong_AE*) *auto*
qed

lemma *set_integrable_cong_AE*:
    f $\in$ *borel_measurable M* $\Longrightarrow$ *g* $\in$ *borel_measurable M* $\Longrightarrow$
    AE x $\in$ *A in M*. *f x* = *g x* $\Longrightarrow$ *A* $\in$ *sets M* $\Longrightarrow$
    set_integrable M A f = *set_integrable M A g*
  unfolding *set_integrable_def*
  by (*rule integrable_cong_AE*) *auto*

lemma *set_integrable_subset*:
  fixes *M A B* **and** *f* :: _ ⇒ _ :: {*banach*, *second_countable_topology*}
  assumes *set_integrable M A f B* $\in$ *sets M B* $\subseteq$ *A*
  shows *set_integrable M B f*
proof −
  have *set_integrable M B* ($\lambda x.$ *indicator A x* $*_R$ *f x*)
    using *assms integrable_mult_indicator set_integrable_def* **by** *blast*
  with ‹*B* $\subseteq$ *A*› **show** *?thesis*
    unfolding *set_integrable_def*
    by (*simp add*: *indicator_inter_arith*[*symmetric*] *Int_absorb2*)
qed

lemma *set_integrable_restrict_space*:
  fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
  assumes *f*: *set_integrable M S f* **and** *T*: *T* $\in$ *sets* (*restrict_space M S*)
  shows *set_integrable M T f*
proof −
  obtain *T'* **where** *T_eq*: *T* = *S* $\cap$ *T'* **and** *T'* $\in$ *sets M*

```
    using T by (auto simp: sets_restrict_space)
  have ‹integrable M (λx. indicator T' x *R (indicator S x *R f x))›
    using ‹T' ∈ sets M› f integrable_mult_indicator set_integrable_def by blast
  then show ?thesis
    unfolding set_integrable_def
    unfolding T_eq indicator_inter_arith by (simp add: ac_simps)
qed

lemma set_integral_scaleR_right [simp]: LINT t:A|M. a *R f t = a *R (LINT
t:A|M. f t)
  unfolding set_lebesgue_integral_def
  by (subst integral_scaleR_right[symmetric]) (auto intro!: Bochner_Integration.integral_cong)

lemma set_integral_mult_right [simp]:
  fixes a :: 'a::{real_normed_field, second_countable_topology}
  shows LINT t:A|M. a * f t = a * (LINT t:A|M. f t)
  unfolding set_lebesgue_integral_def
  by (subst integral_mult_right_zero[symmetric]) auto

lemma set_integral_mult_left [simp]:
  fixes a :: 'a::{real_normed_field, second_countable_topology}
  shows LINT t:A|M. f t * a = (LINT t:A|M. f t) * a
  unfolding set_lebesgue_integral_def
  by (subst integral_mult_left_zero[symmetric]) auto

lemma set_integral_divide_zero [simp]:
  fixes a :: 'a::{real_normed_field, field, second_countable_topology}
  shows LINT t:A|M. f t / a = (LINT t:A|M. f t) / a
  unfolding set_lebesgue_integral_def
  by (subst integral_divide_zero[symmetric], intro Bochner_Integration.integral_cong)
     (auto split: split_indicator)

lemma set_integrable_scaleR_right [simp, intro]:
  shows (a ≠ 0 ⟹ set_integrable M A f) ⟹ set_integrable M A (λt. a *R f t)
  unfolding set_integrable_def
  unfolding scaleR_left_commute by (rule integrable_scaleR_right)

lemma set_integrable_scaleR_left [simp, intro]:
  fixes a :: _ :: {banach, second_countable_topology}
  shows (a ≠ 0 ⟹ set_integrable M A f) ⟹ set_integrable M A (λt. f t *R a)
  unfolding set_integrable_def
  using integrable_scaleR_left[of a M λx. indicator A x *R f x] by simp

lemma set_integrable_mult_right [simp, intro]:
  fixes a :: 'a::{real_normed_field, second_countable_topology}
  shows (a ≠ 0 ⟹ set_integrable M A f) ⟹ set_integrable M A (λt. a * f t)
```

unfolding *set_integrable_def*
using *integrable_mult_right*[*of a M λx. indicator A x* $*_R$ *f x*] **by** *simp*

lemma *set_integrable_mult_right_iff* [*simp*]:
fixes *a* :: *'a*::{*real_normed_field*, *second_countable_topology*}
assumes *a* ≠ *0*
shows *set_integrable M A* (*λt. a* ∗ *f t*) ⟷ *set_integrable M A f*
proof
assume *set_integrable M A* (*λt. a* ∗ *f t*)
then have *set_integrable M A* (*λt. 1/a* ∗ (*a* ∗ *f t*))
using *set_integrable_mult_right* **by** *blast*
then show *set_integrable M A f*
using *assms* **by** *auto*
qed *auto*

lemma *set_integrable_mult_left* [*simp*, *intro*]:
fixes *a* :: *'a*::{*real_normed_field*, *second_countable_topology*}
shows (*a* ≠ *0* ⟹ *set_integrable M A f*) ⟹ *set_integrable M A* (*λt. f t* ∗ *a*)
unfolding *set_integrable_def*
using *integrable_mult_left*[*of a M λx. indicator A x* $*_R$ *f x*] **by** *simp*

lemma *set_integrable_mult_left_iff* [*simp*]:
fixes *a* :: *'a*::{*real_normed_field*, *second_countable_topology*}
assumes *a* ≠ *0*
shows *set_integrable M A* (*λt. f t* ∗ *a*) ⟷ *set_integrable M A f*
using *assms* **by** (*subst set_integrable_mult_right_iff* [*symmetric*]) (*auto simp*: *mult.commute*)

lemma *set_integrable_divide* [*simp*, *intro*]:
fixes *a* :: *'a*::{*real_normed_field*, *field*, *second_countable_topology*}
assumes *a* ≠ *0* ⟹ *set_integrable M A f*
shows *set_integrable M A* (*λt. f t* / *a*)
proof −
have *integrable M* (*λx. indicator A x* $*_R$ *f x* / *a*)
using *assms* **unfolding** *set_integrable_def* **by** (*rule integrable_divide_zero*)
also have (*λx. indicator A x* $*_R$ *f x* / *a*) = (*λx. indicator A x* $*_R$ (*f x* / *a*))
by (*auto split*: *split_indicator*)
finally show *?thesis*
unfolding *set_integrable_def* **.**
qed

lemma *set_integrable_mult_divide_iff* [*simp*]:
fixes *a* :: *'a*::{*real_normed_field*, *second_countable_topology*}
assumes *a* ≠ *0*
shows *set_integrable M A* (*λt. f t* / *a*) ⟷ *set_integrable M A f*
by (*simp add*: *divide_inverse assms*)

lemma *set_integral_add* [*simp*, *intro*]:
fixes *f g* :: _ ⇒ _ :: {*banach*, *second_countable_topology*}

assumes *set_integrable M A f set_integrable M A g*
shows *set_integrable M A* ($\lambda x.\ f\ x + g\ x$)
and *LINT x:A|M. f x + g x* = (*LINT x:A|M. f x*) + (*LINT x:A|M. g x*)
using *assms* **unfolding** *set_integrable_def set_lebesgue_integral_def* **by** (*simp_all add*: *scaleR_add_right*)

lemma *set_integral_diff* [*simp*, *intro*]:
assumes *set_integrable M A f set_integrable M A g*
shows *set_integrable M A* ($\lambda x.\ f\ x - g\ x$) **and** *LINT x:A|M. f x − g x* =
(*LINT x:A|M. f x*) − (*LINT x:A|M. g x*)
using *assms* **unfolding** *set_integrable_def set_lebesgue_integral_def* **by** (*simp_all add*: *scaleR_diff_right*)

lemma *set_integral_uminus*: *set_integrable M A f* $\Longrightarrow$ *LINT x:A|M.* − *f x* = − (*LINT x:A|M. f x*)
unfolding *set_integrable_def set_lebesgue_integral_def*
by (*subst integral_minus*[*symmetric*]) *simp_all*

lemma *set_integral_complex_of_real*:
LINT x:A|M. complex_of_real (*f x*) = *of_real* (*LINT x:A|M. f x*)
unfolding *set_lebesgue_integral_def*
by (*subst integral_complex_of_real*[*symmetric*])
(*auto intro*!: *Bochner_Integration.integral_cong split*: *split_indicator*)

lemma *set_integral_mono*:
fixes *f g* :: _ ⇒ *real*
assumes *set_integrable M A f set_integrable M A g*
$\bigwedge x.\ x \in A \Longrightarrow f\ x \le g\ x$
shows (*LINT x:A|M. f x*) ≤ (*LINT x:A|M. g x*)
using *assms* **unfolding** *set_integrable_def set_lebesgue_integral_def*
by (*auto intro*: *integral_mono split*: *split_indicator*)

lemma *set_integral_mono_AE*:
fixes *f g* :: _ ⇒ *real*
assumes *set_integrable M A f set_integrable M A g*
AE x ∈ *A in M. f x* ≤ *g x*
shows (*LINT x:A|M. f x*) ≤ (*LINT x:A|M. g x*)
using *assms* **unfolding** *set_integrable_def set_lebesgue_integral_def*
by (*auto intro*: *integral_mono_AE split*: *split_indicator*)

lemma *set_integrable_abs*: *set_integrable M A f* $\Longrightarrow$ *set_integrable M A* ($\lambda x.\ |f\ x|$ :: *real*)
using *integrable_abs*[*of M* $\lambda x.\ f\ x * $ *indicator A x*]**unfolding** *set_integrable_def*
by (*simp add*: *abs_mult ac_simps*)

lemma *set_integrable_abs_iff*:
fixes *f* :: _ ⇒ *real*
shows *set_borel_measurable M A f* $\Longrightarrow$ *set_integrable M A* ($\lambda x.\ |f\ x|$) = *set_integrable M A f*

unfolding *set_integrable_def set_borel_measurable_def*
by (*subst* (*2*) *integrable_abs_iff*[*symmetric*]) (*simp_all add*: *abs_mult ac_simps*)

lemma *set_integrable_abs_iff'*:
fixes *f* :: _ ⇒ *real*
shows $f \in$ *borel_measurable M* $\Longrightarrow A \in$ *sets M* $\Longrightarrow$
set_integrable M A ($\lambda x.$ $|f\, x|$) = *set_integrable M A f*
by (*simp add*: *set_borel_measurable_def set_integrable_abs_iff*)

lemma *set_integrable_discrete_difference*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes *countable X*
assumes *diff*: $(A - B) \cup (B - A) \subseteq X$
assumes $\bigwedge x.\ x \in X \Longrightarrow$ *emeasure M* $\{x\} = 0$ $\bigwedge x.\ x \in X \Longrightarrow \{x\} \in$ *sets M*
shows *set_integrable M A f* ⟷ *set_integrable M B f*
unfolding *set_integrable_def*
proof (*rule integrable_discrete_difference*[**where** *X*=*X*])
show $\bigwedge x.\ x \in$ *space M* $\Longrightarrow x \notin X \Longrightarrow$ *indicator A x* $*_R$ *f x* = *indicator B x* $*_R$ *f x*
using *diff* **by** (*auto split*: *split_indicator*)
qed *fact+*

lemma *set_integral_discrete_difference*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes *countable X*
assumes *diff*: $(A - B) \cup (B - A) \subseteq X$
assumes $\bigwedge x.\ x \in X \Longrightarrow$ *emeasure M* $\{x\} = 0$ $\bigwedge x.\ x \in X \Longrightarrow \{x\} \in$ *sets M*
shows *set_lebesgue_integral M A f* = *set_lebesgue_integral M B f*
unfolding *set_lebesgue_integral_def*
proof (*rule integral_discrete_difference*[**where** *X*=*X*])
show $\bigwedge x.\ x \in$ *space M* $\Longrightarrow x \notin X \Longrightarrow$ *indicator A x* $*_R$ *f x* = *indicator B x* $*_R$ *f x*
using *diff* **by** (*auto split*: *split_indicator*)
qed *fact+*

lemma *set_integrable_Un*:
fixes *f g* :: _ ⇒ _ :: {*banach*, *second_countable_topology*}
assumes *f_A*: *set_integrable M A f* **and** *f_B*: *set_integrable M B f*
and [*measurable*]: $A \in$ *sets M* $B \in$ *sets M*
shows *set_integrable M* $(A \cup B)$ *f*
proof −
have *set_integrable M* $(A - B)$ *f*
using *f_A* **by** (*rule set_integrable_subset*) *auto*
with *f_B* **have** *integrable M* ($\lambda x.$ *indicator* $(A - B)$ *x* $*_R$ *f x* + *indicator B x* $*_R$ *f x*)
unfolding *set_integrable_def* **using** *integrable_add* **by** *blast*
then show *?thesis*
unfolding *set_integrable_def*
by (*rule integrable_cong*[*THEN iffD1*, *rotated 2*]) (*auto split*: *split_indicator*)

qed

lemma *set_integrable_empty* [*simp*]: *set_integrable M* {} *f*
  by (*auto simp*: *set_integrable_def*)

lemma *set_integrable_UN*:
  fixes *f* :: _ ⇒ _ :: {*banach*, *second_countable_topology*}
  assumes *finite I* ⋀*i*. *i*∈*I* ⟹ *set_integrable M* (*A i*) *f*
    ⋀*i*. *i*∈*I* ⟹ *A i* ∈ *sets M*
  shows *set_integrable M* (⋃*i*∈*I*. *A i*) *f*
  using *assms*
  by (*induct I*) (*auto simp*: *set_integrable_Un sets.finite_UN*)

lemma *set_integral_Un*:
  fixes *f* :: _ ⇒ _ :: {*banach*, *second_countable_topology*}
  assumes *A* ∩ *B* = {}
  and *set_integrable M A f*
  and *set_integrable M B f*
shows *LINT x*:*A*∪*B*|*M*. *f x* = (*LINT x*:*A*|*M*. *f x*) + (*LINT x*:*B*|*M*. *f x*)
  using *assms*
  unfolding *set_integrable_def set_lebesgue_integral_def*
  by (*auto simp add*: *indicator_union_arith indicator_inter_arith*[*symmetric*] *scaleR_add_left*)

lemma *set_integral_cong_set*:
  fixes *f* :: _ ⇒ _ :: {*banach*, *second_countable_topology*}
  assumes *set_borel_measurable M A f set_borel_measurable M B f*
    and *ae*: *AE x in M*. *x* ∈ *A* ⟷ *x* ∈ *B*
  shows *LINT x*:*B*|*M*. *f x* = *LINT x*:*A*|*M*. *f x*
  unfolding *set_lebesgue_integral_def*
proof (*rule integral_cong_AE*)
  show *AE x in M*. *indicator B x* $*_R$ *f x* = *indicator A x* $*_R$ *f x*
    using *ae* **by** (*auto simp*: *subset_eq split*: *split_indicator*)
qed (*use assms* **in** ‹*auto simp*: *set_borel_measurable_def*›)

proposition *set_borel_measurable_subset*:
  fixes *f* :: _ ⇒ _ :: {*banach*, *second_countable_topology*}
  assumes [*measurable*]: *set_borel_measurable M A f B* ∈ *sets M* **and** *B* ⊆ *A*
  shows *set_borel_measurable M B f*
proof−
  have *set_borel_measurable M B* (λ*x*. *indicator A x* $*_R$ *f x*)
    using *assms* **unfolding** *set_borel_measurable_def*
    using *borel_measurable_indicator borel_measurable_scaleR* **by** *blast*
  moreover have (λ*x*. *indicator B x* $*_R$ *indicator A x* $*_R$ *f x*) = (λ*x*. *indicator B x* $*_R$ *f x*)
    using ‹*B* ⊆ *A*› **by** (*auto simp*: *fun_eq_iff split*: *split_indicator*)
  ultimately show *?thesis*
    unfolding *set_borel_measurable_def* **by** *simp*
qed

lemma *set_integral_Un_AE*:
  fixes *f* :: _ ⇒ _ :: {*banach*, *second_countable_topology*}
  assumes *ae*: *AE x in M*. ¬ (*x* ∈ *A* ∧ *x* ∈ *B*) **and** [*measurable*]: *A* ∈ *sets M B* ∈ *sets M*
  and *set_integrable M A f*
  and *set_integrable M B f*
  shows *LINT x*:*A*∪*B*|*M*. *f x* = (*LINT x*:*A*|*M*. *f x*) + (*LINT x*:*B*|*M*. *f x*)
proof −
  have *f*: *set_integrable M* (*A* ∪ *B*) *f*
    by (*intro set_integrable_Un assms*)
  then have *f'*: *set_borel_measurable M* (*A* ∪ *B*) *f*
    using *integrable_iff_bounded set_borel_measurable_def set_integrable_def* **by** *blast*
  have *LINT x*:*A*∪*B*|*M*. *f x* = *LINT x*:(*A* − *A* ∩ *B*) ∪ (*B* − *A* ∩ *B*)|*M*. *f x*
  proof (*rule set_integral_cong_set*)
    show *AE x in M*. (*x* ∈ *A* − *A* ∩ *B* ∪ (*B* − *A* ∩ *B*)) = (*x* ∈ *A* ∪ *B*)
      using *ae* **by** *auto*
    show *set_borel_measurable M* (*A* − *A* ∩ *B* ∪ (*B* − *A* ∩ *B*)) *f*
      using *f'* **by** (*rule set_borel_measurable_subset*) *auto*
  qed *fact*
  also have ... = (*LINT x*:(*A* − *A* ∩ *B*)|*M*. *f x*) + (*LINT x*:(*B* − *A* ∩ *B*)|*M*. *f x*)
    by (*auto intro*!: *set_integral_Un set_integrable_subset*[*OF f*])
  also have ... = (*LINT x*:*A*|*M*. *f x*) + (*LINT x*:*B*|*M*. *f x*)
    using *ae*
    by (*intro arg_cong2*[**where** *f*=(+)] *set_integral_cong_set*)
      (*auto intro*!: *set_borel_measurable_subset*[*OF f'*])
  finally show *?thesis* **.**
qed

lemma *set_integral_finite_Union*:
  fixes *f* :: _ ⇒ _ :: {*banach*, *second_countable_topology*}
  assumes *finite I disjoint_family_on A I*
    and ⋀*i*. *i* ∈ *I* ⟹ *set_integrable M* (*A i*) *f* ⋀*i*. *i* ∈ *I* ⟹ *A i* ∈ *sets M*
  shows (*LINT x*:(⋃*i*∈*I*. *A i*)|*M*. *f x*) = (∑*i*∈*I*. *LINT x*:*A i*|*M*. *f x*)
  using *assms*
proof *induction*
  case (*insert x F*)
  then have *A x* ∩ ⋃(*A* ‘ *F*) = {}
    by (*meson disjoint_family_on_insert*)
  with *insert* **show** *?case*
   by (*simp add*: *set_integral_Un set_integrable_Un set_integrable_UN disjoint_family_on_insert*)
qed (*simp add*: *set_lebesgue_integral_def*)

lemma *pos_integrable_to_top*:
  fixes *l*::*real*
  assumes ⋀*i*. *A i* ∈ *sets M mono A*
  assumes *nneg*: ⋀*x i*. *x* ∈ *A i* ⟹ *0* ≤ *f x*
  and *intgbl*: ⋀*i*::*nat*. *set_integrable M* (*A i*) *f*

```
  and lim: (λi::nat. LINT x:A i|M. f x) ⟶ l
shows set_integrable M (⋃ i. A i) f
    unfolding set_integrable_def
  apply (rule integrable_monotone_convergence[where f = λi::nat. λx. indicator
(A i) x *R f x and x = l])
  apply (rule intgbl [unfolded set_integrable_def])
  prefer 3 apply (rule lim [unfolded set_lebesgue_integral_def])
  apply (rule AE_I2)
  using ‹mono A› apply (auto simp: mono_def nneg split: split_indicator) []
proof (rule AE_I2)
  { fix x assume x ∈ space M
    show (λi. indicator (A i) x *R f x) ⟶ indicator (⋃ i. A i) x *R f x
    proof cases
      assume ∃ i. x ∈ A i
      then guess i ..
      then have *: eventually (λi. x ∈ A i) sequentially
        using ‹x ∈ A i› ‹mono A› by (auto simp: eventually_sequentially mono_def)
      show ?thesis
        apply (intro tendsto_eventually)
        using *
        apply eventually_elim
        apply (auto split: split_indicator)
        done
    qed auto }
  then show (λx. indicator (⋃ i. A i) x *R f x) ∈ borel_measurable M
    apply (rule borel_measurable_LIMSEQ_real)
    apply assumption
    using intgbl set_integrable_def by blast
qed


lemma lebesgue_integral_countable_add:
  fixes f :: _ ⇒ 'a :: {banach, second_countable_topology}
  assumes meas[intro]: ⋀i::nat. A i ∈ sets M
    and disj: ⋀i j. i ≠ j ⟹ A i ∩ A j = {}
    and intgbl: set_integrable M (⋃ i. A i) f
  shows LINT x:(⋃ i. A i)|M. f x = (∑ i. (LINT x:(A i)|M. f x))
    unfolding set_lebesgue_integral_def
proof (subst integral_suminf[symmetric])
  show int_A: integrable M (λx. indicat_real (A i) x *R f x) for i
    using intgbl unfolding set_integrable_def [symmetric]
    by (rule set_integrable_subset) auto
  { fix x assume x ∈ space M
    have (λi. indicator (A i) x *R f x) sums (indicator (⋃ i. A i) x *R f x)
      by (intro sums_scaleR_left indicator_sums) fact }
  note sums = this

  have norm_f: ⋀i. set_integrable M (A i) (λx. norm (f x))
    using int_A[THEN integrable_norm] unfolding set_integrable_def by auto
```

show *AE x in M. summable* (λ*i. norm* (*indicator* (*A i*) *x* $*_R$ *f x*))
using *disj* **by** (*intro AE_I2*) (*auto intro*!: *summable_mult2 sums_summable*[*OF indicator_sums*])

show *summable* (λ*i. LINT x*|*M. norm* (*indicator* (*A i*) *x* $*_R$ *f x*))
proof (*rule summableI_nonneg_bounded*)
fix *n*
show *0* ≤ *LINT x*|*M. norm* (*indicator* (*A n*) *x* $*_R$ *f x*)
using *norm_f* **by** (*auto intro*!: *integral_nonneg_AE*)

have (∑*i*<*n. LINT x*|*M. norm* (*indicator* (*A i*) *x* $*_R$ *f x*)) = (∑*i*<*n. LINT x*:*A i*|*M. norm* (*f x*))
by (*simp add*: *abs_mult set_lebesgue_integral_def*)
also have ... = *set_lebesgue_integral M* (⋃*i*<*n. A i*) (λ*x. norm* (*f x*))
using *norm_f*
by (*subst set_integral_finite_Union*) (*auto simp*: *disjoint_family_on_def disj*)
also have ... ≤ *set_lebesgue_integral M* (⋃*i. A i*) (λ*x. norm* (*f x*))
using *intgbl*[*unfolded set_integrable_def*, *THEN integrable_norm*] *norm_f*
unfolding *set_lebesgue_integral_def set_integrable_def*
apply (*intro integral_mono set_integrable_UN*[*of* {..<*n*}, *unfolded set_integrable_def*])
apply (*auto split*: *split_indicator*)
done
finally show (∑*i*<*n. LINT x*|*M. norm* (*indicator* (*A i*) *x* $*_R$ *f x*)) ≤
set_lebesgue_integral M (⋃*i. A i*) (λ*x. norm* (*f x*))
by *simp*
qed
show *LINT x*|*M. indicator* (⋃(*A* ‘ *UNIV*)) *x* $*_R$ *f x* = *LINT x*|*M.* (∑*i. indicator* (*A i*) *x* $*_R$ *f x*)
apply (*rule Bochner_Integration.integral_cong*[*OF refl*])
apply (*subst suminf_scaleR_left*[*OF sums_summable*[*OF indicator_sums*, *OF disj*], *symmetric*])
using *sums_unique*[*OF indicator_sums*[*OF disj*]]
apply *auto*
done
qed

lemma *set_integral_cont_up*:
fixes *f* :: _ ⇒ ′*a* :: {*banach*, *second_countable_topology*}
assumes [*measurable*]: ⋀*i. A i* ∈ *sets M* **and** *A*: *incseq A*
and *intgbl*: *set_integrable M* (⋃*i. A i*) *f*
shows (λ*i. LINT x*:(*A i*)|*M. f x*) ⟶ *LINT x*:(⋃*i. A i*)|*M. f x*
unfolding *set_lebesgue_integral_def*
proof (*intro integral_dominated_convergence*[**where** *w*=λ*x. indicator* (⋃*i. A i*) *x* $*_R$ *norm* (*f x*)])
have *int_A*: ⋀*i. set_integrable M* (*A i*) *f*
using *intgbl* **by** (*rule set_integrable_subset*) *auto*
show ⋀*i.* (λ*x. indicator* (*A i*) *x* $*_R$ *f x*) ∈ *borel_measurable M*
using *int_A integrable_iff_bounded set_integrable_def* **by** *blast*

show (λx. *indicator* (⋃(A ‘ UNIV)) x $*_R$ f x) ∈ *borel_measurable M*
using *integrable_iff_bounded intgbl set_integrable_def* **by** *blast*
show *integrable M* (λx. *indicator* (⋃i. A i) x $*_R$ *norm* (f x))
using *int_A intgbl integrable_norm* **unfolding** *set_integrable_def*
by *fastforce*
{ **fix** x i **assume** x ∈ A i
with A **have** (λxa. *indicator* (A xa) x::*real*) ⟶ *1* ⟷ (λxa. *1*::*real*) ⟶ *1*
by (*intro filterlim_cong refl*)
(*fastforce simp*: *eventually_sequentially incseq_def subset_eq intro*!: *exI*[*of* _ i]) }
then show *AE x in M*. (λi. *indicator* (A i) x $*_R$ f x) ⟶ *indicator* (⋃i. A i) x $*_R$ f x
by (*intro AE_I2 tendsto_intros*) (*auto split*: *split_indicator*)
qed (*auto split*: *split_indicator*)

lemma *set_integral_cont_down*:
fixes f :: _ ⇒ 'a :: {*banach*, *second_countable_topology*}
assumes [*measurable*]: ⋀i. A i ∈ *sets M* **and** A: *decseq A*
and *int0*: *set_integrable M* (A *0*) f
shows (λi::*nat*. *LINT* x:(A i)|M. f x) ⟶ *LINT* x:(⋂i. A i)|M. f x
unfolding *set_lebesgue_integral_def*
proof (*rule integral_dominated_convergence*)
have *int_A*: ⋀i. *set_integrable M* (A i) f
using *int0* **by** (*rule set_integrable_subset*) (*insert A*, *auto simp*: *decseq_def*)
have *integrable M* (λc. *norm* (*indicat_real* (A *0*) c $*_R$ f c))
by (*metis* (*no_types*) *int0 integrable_norm set_integrable_def*)
then show *integrable M* (λx. *indicator* (A *0*) x $*_R$ *norm* (f x))
by *force*
have *set_integrable M* (⋂i. A i) f
using *int0* **by** (*rule set_integrable_subset*) (*insert A*, *auto simp*: *decseq_def*)
with *int_A* **show** (λx. *indicat_real* (⋂(A ‘ UNIV)) x $*_R$ f x) ∈ *borel_measurable M*
⋀i. (λx. *indicat_real* (A i) x $*_R$ f x) ∈ *borel_measurable M*
by (*auto simp*: *set_integrable_def*)
show ⋀i. *AE x in M*. *norm* (*indicator* (A i) x $*_R$ f x) ≤ *indicator* (A *0*) x $*_R$ *norm* (f x)
using A **by** (*auto split*: *split_indicator simp*: *decseq_def*)
{ **fix** x i **assume** x ∈ *space M* x ∉ A i
with A **have** (λi. *indicator* (A i) x::*real*) ⟶ *0* ⟷ (λi. *0*::*real*) ⟶ *0*
by (*intro filterlim_cong refl*)
(*auto split*: *split_indicator simp*: *eventually_sequentially decseq_def intro*!: *exI*[*of* _ i]) }
then show *AE x in M*. (λi. *indicator* (A i) x $*_R$ f x) ⟶ *indicator* (⋂i. A i) x $*_R$ f x
by (*intro AE_I2 tendsto_intros*) (*auto split*: *split_indicator*)
qed

lemma *set_integral_at_point*:
  fixes *a* :: *real*
  assumes *set_integrable M* {*a*} *f*
  and [*simp*]: {*a*} ∈ *sets M* **and** (*emeasure M*) {*a*} ≠ ∞
  shows (*LINT x*:{*a*} | *M*. *f x*) = *f a* ∗ *measure M* {*a*}
proof−
  have *set_lebesgue_integral M* {*a*} *f* = *set_lebesgue_integral M* {*a*} (%*x*. *f a*)
    by (*intro set_lebesgue_integral_cong*) *simp_all*
  then show *?thesis* **using** *assms*
    unfolding *set_lebesgue_integral_def* **by** *simp*
qed

abbreviation *complex_integrable* :: ′*a measure* ⇒ (′*a* ⇒ *complex*) ⇒ *bool* **where**
  complex_integrable M f ≡ *integrable M f*

abbreviation *complex_lebesgue_integral* :: ′*a measure* ⇒ (′*a* ⇒ *complex*) ⇒ *complex* (*integral*^{C}) **where**
  integral^C *M f* == *integral*^{L} *M f*

syntax
  _complex_lebesgue_integral :: *pttrn* ⇒ *complex* ⇒ ′*a measure* ⇒ *complex*
  (∫^{C} _. _ ∂_ [*60*,*61*] *110*)

translations
  ∫^{C}*x*. *f* ∂*M* == *CONST complex_lebesgue_integral M* (λ*x*. *f*)

syntax
  _ascii_complex_lebesgue_integral :: *pttrn* ⇒ ′*a measure* ⇒ *real* ⇒ *real*
  ((*3CLINT* _|_. _) [*0*,*110*,*60*] *60*)

translations
  CLINT x|*M*. *f* == *CONST complex_lebesgue_integral M* (λ*x*. *f*)

lemma *complex_integrable_cnj* [*simp*]:
  complex_integrable M (λ*x*. *cnj* (*f x*)) ⟷ *complex_integrable M f*
proof
  assume *complex_integrable M* (λ*x*. *cnj* (*f x*))
  then have *complex_integrable M* (λ*x*. *cnj* (*cnj* (*f x*)))
    by (*rule integrable_cnj*)
  then show *complex_integrable M f*
    by *simp*
qed *simp*

lemma *complex_of_real_integrable_eq*:
  complex_integrable M (λ*x*. *complex_of_real* (*f x*)) ⟷ *integrable M f*
proof
  assume *complex_integrable M* (λ*x*. *complex_of_real* (*f x*))
  then have *integrable M* (λ*x*. *Re* (*complex_of_real* (*f x*)))

  by (*rule integrable_Re*)
 then show *integrable M f*
  by *simp*
qed *simp*

abbreviation *complex_set_integrable* :: *'a measure* ⇒ *'a set* ⇒ (*'a* ⇒ *complex*) ⇒ *bool* **where**
 complex_set_integrable M A f ≡ *set_integrable M A f*

abbreviation *complex_set_lebesgue_integral* :: *'a measure* ⇒ *'a set* ⇒ (*'a* ⇒ *complex*) ⇒ *complex* **where**
 complex_set_lebesgue_integral M A f ≡ *set_lebesgue_integral M A f*

syntax
_ascii_complex_set_lebesgue_integral :: *pttrn* ⇒ *'a set* ⇒ *'a measure* ⇒ *real* ⇒ *real*
((*4CLINT* _:_|_. _) [*0*,*60*,*110*,*61*] *60*)

translations
CLINT x:*A*|*M*. *f* == *CONST complex_set_lebesgue_integral M A* (λ*x*. *f*)

lemma *set_measurable_continuous_on_ivl*:
 assumes *continuous_on* {*a*..*b*} (*f* :: *real* ⇒ *real*)
 shows *set_borel_measurable borel* {*a*..*b*} *f*
 unfolding *set_borel_measurable_def*
 by (*rule borel_measurable_continuous_on_indicator*[*OF* _ *assms*]) *simp*

This notation is from Sbastien Gouzel: His use is not directly in line with the notations in this file, they are more in line with sum, and more readable he thinks.

abbreviation *set_nn_integral M A f* ≡ *nn_integral M* (λ*x*. *f x* ∗ *indicator A x*)

syntax
_set_nn_integral :: *pttrn* => *'a set* => *'a measure* => *ereal* => *ereal*
((∫ ⁺((_)∈(_)./ _)/∂_) [*0*,*60*,*110*,*61*] *60*)

_set_lebesgue_integral :: *pttrn* => *'a set* => *'a measure* => *ereal* => *ereal*
((∫ ((_)∈(_)./ _)/∂_) [*0*,*60*,*110*,*61*] *60*)

translations
∫ ⁺*x* ∈ *A*. *f* ∂*M* == *CONST set_nn_integral M A* (λ*x*. *f*)
∫ *x* ∈ *A*. *f* ∂*M* == *CONST set_lebesgue_integral M A* (λ*x*. *f*)

lemma *nn_integral_disjoint_pair*:
 assumes [*measurable*]: *f* ∈ *borel_measurable M*
   B ∈ *sets M C* ∈ *sets M*
   B ∩ *C* = {}
 shows (∫ ⁺*x* ∈ *B* ∪ *C*. *f x* ∂*M*) = (∫ ⁺*x* ∈ *B*. *f x* ∂*M*) + (∫ ⁺*x* ∈ *C*. *f x* ∂*M*)
proof −

have *mes*: $\bigwedge D.\ D \in sets\ M \Longrightarrow (\lambda x.\ f\ x * indicator\ D\ x) \in borel\_measurable\ M$
by *simp*
have *pos*: $\bigwedge D.\ AE\ x\ in\ M.\ f\ x * indicator\ D\ x \geq 0$ **using** *assms*(*2*) **by** *auto*
have $\bigwedge x.\ f\ x * indicator\ (B \cup C)\ x = f\ x * indicator\ B\ x + f\ x * indicator\ C\ x$ **using** *assms*(*4*)
by (*auto split*: *split_indicator*)
then have $(\int^+ x.\ f\ x * indicator\ (B \cup C)\ x\ \partial M) = (\int^+ x.\ f\ x * indicator\ B\ x + f\ x * indicator\ C\ x\ \partial M)$
by *simp*
also have ... $= (\int^+ x.\ f\ x * indicator\ B\ x\ \partial M) + (\int^+ x.\ f\ x * indicator\ C\ x\ \partial M)$
by (*rule nn_integral_add*) (*auto simp add*: *assms mes pos*)
finally show *?thesis* **by** *simp*
qed

lemma *nn_integral_disjoint_pair_countspace*:
assumes $B \cap C = \{\}$
shows $(\int^+ x \in B \cup C.\ f\ x\ \partial count\_space\ UNIV) = (\int^+ x \in B.\ f\ x\ \partial count\_space\ UNIV) + (\int^+ x \in C.\ f\ x\ \partial count\_space\ UNIV)$
by (*rule nn_integral_disjoint_pair*) (*simp_all add*: *assms*)

lemma *nn_integral_null_delta*:
assumes $A \in sets\ M\ B \in sets\ M$
$(A - B) \cup (B - A) \in null\_sets\ M$
shows $(\int^+ x \in A.\ f\ x\ \partial M) = (\int^+ x \in B.\ f\ x\ \partial M)$
proof (*rule nn_integral_cong_AE*, *auto simp add*: *indicator_def*)
have $*$: $AE\ a\ in\ M.\ a \notin (A - B) \cup (B - A)$
using *assms*(*3*) *AE_not_in* **by** *blast*
then show $AE\ a\ in\ M.\ a \notin A \longrightarrow a \in B \longrightarrow f\ a = 0$
by *auto*
show $AE\ x{\in}A\ in\ M.\ x \notin B \longrightarrow f\ x = 0$
using $*$ **by** *auto*
qed

proposition *nn_integral_disjoint_family*:
assumes [*measurable*]: $f \in borel\_measurable\ M\ \bigwedge(n{::}nat).\ B\ n \in sets\ M$
and *disjoint_family B*
shows $(\int^+ x \in (\bigcup n.\ B\ n).\ f\ x\ \partial M) = (\sum n.\ (\int^+ x \in B\ n.\ f\ x\ \partial M))$
proof −
have $(\int^+ x.\ (\sum n.\ f\ x * indicator\ (B\ n)\ x)\ \partial M) = (\sum n.\ (\int^+ x.\ f\ x * indicator\ (B\ n)\ x\ \partial M))$
by (*rule nn_integral_suminf*) *simp*
moreover have $(\sum n.\ f\ x * indicator\ (B\ n)\ x) = f\ x * indicator\ (\bigcup n.\ B\ n)\ x$ **for** x
proof (*cases*)
assume $x \in (\bigcup n.\ B\ n)$
then obtain n **where** $x \in B\ n$ **by** *blast*
have *a*: *finite* $\{n\}$ **by** *simp*
have $\bigwedge i.\ i \neq n \Longrightarrow x \notin B\ i$ **using** ‹$x \in B\ n$› *assms*(*3*) *disjoint_family_on_def*

by (*metis IntI UNIV_I empty_iff*)
then have $\bigwedge i.\ i \notin \{n\} \Longrightarrow$ *indicator* $(B\ i)\ x = (0{::}ennreal)$ **using** *indicator_def* **by** *simp*
then have b: $\bigwedge i.\ i \notin \{n\} \Longrightarrow f\,x *$ *indicator* $(B\ i)\ x = (0{::}ennreal)$ **by** *simp*

define h **where** $h = (\lambda i.\ f\,x *$ *indicator* $(B\ i)\ x)$
then have $\bigwedge i.\ i \notin \{n\} \Longrightarrow h\ i = 0$ **using** b **by** *simp*
then have $(\sum i.\ h\ i) = (\sum i{\in}\{n\}.\ h\ i)$
by (*metis sums_unique*[*OF sums_finite*[*OF a*]])
then have $(\sum i.\ h\ i) = h\ n$ **by** *simp*
then have $(\sum n.\ f\ x *$ *indicator* $(B\ n)\ x) = f\ x *$ *indicator* $(B\ n)\ x$ **using** *h_def* **by** *simp*
then have $(\sum n.\ f\,x *$ *indicator* $(B\ n)\ x) = f\,x$ **using** ‹$x \in B\ n$› *indicator_def* **by** *simp*
then show *?thesis* **using** ‹$x \in (\bigcup n.\ B\ n)$› **by** *auto*
next
assume $x \notin (\bigcup n.\ B\ n)$
then have $\bigwedge n.\ f\,x *$ *indicator* $(B\ n)\ x = 0$ **by** *simp*
have $(\sum n.\ f\,x *$ *indicator* $(B\ n)\ x) = 0$
by (*simp add*: ‹$\bigwedge n.\ f\,x *$ *indicator* $(B\ n)\ x = 0$›)
then show *?thesis* **using** ‹$x \notin (\bigcup n.\ B\ n)$› **by** *auto*
qed
ultimately show *?thesis* **by** *simp*
qed

lemma *nn_set_integral_add*:
assumes [*measurable*]: $f \in$ *borel_measurable* M $g \in$ *borel_measurable* M
$A \in$ *sets* M
shows $(\int^+ x \in A.\ (f\ x + g\ x)\ \partial M) = (\int^+ x \in A.\ f\ x\ \partial M) + (\int^+ x \in A.\ g\ x\ \partial M)$
proof −
have $(\int^+ x \in A.\ (f\,x + g\ x)\ \partial M) = (\int^+ x.\ (f\,x *$ *indicator* $A\ x + g\ x *$ *indicator* $A\ x)\ \partial M)$
by (*auto simp add*: *indicator_def intro*!: *nn_integral_cong*)
also have ... $= (\int^+ x.\ f\ x *$ *indicator* $A\ x\ \partial M) + (\int^+ x.\ g\ x *$ *indicator* $A\ x\ \partial M)$
apply (*rule nn_integral_add*) **using** *assms*(*1*) *assms*(*2*) **by** *auto*
finally show *?thesis* **by** *simp*
qed

lemma *nn_set_integral_cong*:
assumes *AE* x *in* M. $f\,x = g\ x$
shows $(\int^+ x \in A.\ f\,x\ \partial M) = (\int^+ x \in A.\ g\ x\ \partial M)$
apply (*rule nn_integral_cong_AE*) **using** *assms*(*1*) **by** *auto*

lemma *nn_set_integral_set_mono*:
$A \subseteq B \Longrightarrow (\int^+ x \in A.\ f\,x\ \partial M) \le (\int^+ x \in B.\ f\,x\ \partial M)$
by (*auto intro*!: *nn_integral_mono split*: *split_indicator*)

lemma *nn_set_integral_mono*:
assumes [*measurable*]: $f \in$ *borel_measurable M* $g \in$ *borel_measurable M*
$A \in$ *sets M*
and *AE* $x \in A$ *in M*. $f\ x \leq g\ x$
shows $(\int^+ x \in A.\ f\ x\ \partial M) \leq (\int^+ x \in A.\ g\ x\ \partial M)$
by (*auto intro*!: *nn_integral_mono_AE split*: *split_indicator simp*: *assms*)

lemma *nn_set_integral_space* [*simp*]:
shows $(\int^+ x \in space\ M.\ f\ x\ \partial M) = (\int^+ x.\ f\ x\ \partial M)$
by (*metis* (*mono_tags*, *lifting*) *indicator_simps*(*1*) *mult.right_neutral nn_integral_cong*)

lemma *nn_integral_count_compose_inj*:
assumes *inj_on g A*
shows $(\int^+ x \in A.\ f\ (g\ x)\ \partial count\_space\ UNIV) = (\int^+ y \in g`A.\ f\ y\ \partial count\_space\ UNIV)$
proof −
have $(\int^+ x \in A.\ f\ (g\ x)\ \partial count\_space\ UNIV) = (\int^+ x.\ f\ (g\ x)\ \partial count\_space\ A)$
by (*auto simp add*: *nn_integral_count_space_indicator*[*symmetric*])
also have ... $= (\int^+ y.\ f\ y\ \partial count\_space\ (g`A))$
by (*simp add*: *assms nn_integral_bij_count_space inj_on_imp_bij_betw*)
also have ... $= (\int^+ y \in g`A.\ f\ y\ \partial count\_space\ UNIV)$
by (*auto simp add*: *nn_integral_count_space_indicator*[*symmetric*])
finally show *?thesis* **by** *simp*
qed

lemma *nn_integral_count_compose_bij*:
assumes *bij_betw g A B*
shows $(\int^+ x \in A.\ f\ (g\ x)\ \partial count\_space\ UNIV) = (\int^+ y \in B.\ f\ y\ \partial count\_space\ UNIV)$
proof −
have *inj_on g A* **using** *assms bij_betw_def* **by** *auto*
then have $(\int^+ x \in A.\ f\ (g\ x)\ \partial count\_space\ UNIV) = (\int^+ y \in g`A.\ f\ y\ \partial count\_space\ UNIV)$
by (*rule nn_integral_count_compose_inj*)
then show *?thesis* **using** *assms* **by** (*simp add*: *bij_betw_def*)
qed

lemma *set_integral_null_delta*:
fixes *f*::_ ⇒ _ :: {*banach*, *second_countable_topology*}
assumes [*measurable*]: *integrable M f* $A \in$ *sets M* $B \in$ *sets M*
and *null*: $(A - B) \cup (B - A) \in$ *null_sets M*
shows $(\int x \in A.\ f\ x\ \partial M) = (\int x \in B.\ f\ x\ \partial M)$
proof (*rule set_integral_cong_set*)
have ∗: *AE a in M*. $a \notin (A - B) \cup (B - A)$
using *null AE_not_in* **by** *blast*
then show *AE x in M*. $(x \in B) = (x \in A)$
by *auto*
qed (*simp_all add*: *set_borel_measurable_def*)

lemma *set_integral_space*:
  assumes *integrable M f*
  shows $(\int x \in space\ M.\ f\ x\ \partial M) = (\int x.\ f\ x\ \partial M)$
  by (*metis* (*no_types*, *lifting*) *indicator_simps*(*1*) *integral_cong scaleR_one set_lebesgue_integral_def*)

lemma *null_if_pos_func_has_zero_nn_int*:
  fixes f::$'a \Rightarrow$ *ennreal*
  assumes [*measurable*]: $f \in$ *borel_measurable M* $A \in$ *sets M*
    and *AE* $x{\in}A$ *in M*. $f\ x > 0$ $(\int^+ x{\in}A.\ f\ x\ \partial M) = 0$
  shows $A \in$ *null_sets M*
proof −
  have *AE x in M*. $f\ x * indicator\ A\ x = 0$
    by (*subst nn_integral_0_iff_AE*[*symmetric*], *auto simp add*: *assms*(*4*))
  then have *AE* $x{\in}A$ *in M*. *False* **using** *assms*(*3*) **by** *auto*
  then show $A \in$ *null_sets M* **using** *assms*(*2*) **by** (*simp add*: *AE_iff_null_sets*)
qed

lemma *null_if_pos_func_has_zero_int*:
  assumes [*measurable*]: *integrable M f* $A \in$ *sets M*
      and *AE* $x{\in}A$ *in M*. $f\ x > 0$ $(\int x{\in}A.\ f\ x\ \partial M) = (0::real)$
  shows $A \in$ *null_sets M*
proof −
  have *AE x in M*. $indicator\ A\ x * f\ x = 0$
    apply (*subst integral_nonneg_eq_0_iff_AE*[*symmetric*])
    using *assms integrable_mult_indicator*[*OF* ‹$A \in$ *sets M*› *assms*(*1*)]
    by (*auto simp*: *set_lebesgue_integral_def*)
  then have *AE* $x{\in}A$ *in M*. $f\ x = 0$ **by** *auto*
  then have *AE* $x{\in}A$ *in M*. *False* **using** *assms*(*3*) **by** *auto*
  then show $A \in$ *null_sets M* **using** *assms*(*2*) **by** (*simp add*: *AE_iff_null_sets*)
qed

The next lemma is a variant of *density_unique*. Note that it uses the notation for nonnegative set integrals introduced earlier.

lemma (**in** *sigma_finite_measure*) *density_unique2*:
  assumes [*measurable*]: $f \in$ *borel_measurable M* $f' \in$ *borel_measurable M*
  assumes *density_eq*: $\bigwedge A.\ A \in sets\ M \Longrightarrow (\int^+ x \in A.\ f\ x\ \partial M) = (\int^+ x \in A.\ f'\ x\ \partial M)$
  shows *AE x in M*. $f\ x = f'\ x$
proof (*rule density_unique*)
  show *density M f* = *density M f'*
    by (*intro measure_eqI*) (*auto simp*: *emeasure_density intro*!: *density_eq*)
qed (*auto simp add*: *assms*)

The next lemma implies the same statement for Banach-space valued functions using Hahn-Banach theorem and linear forms. Since they are not yet easily available, I only formulate it for real-valued functions.

lemma *density_unique_real*:
  fixes $f\ f'$::$\_ \Rightarrow$ *real*
  assumes *M*[*measurable*]: *integrable M f integrable M f'*

assumes *density_eq*: $\bigwedge A.\ A \in sets\ M \Longrightarrow (\int x \in A.\ f\ x\ \partial M) = (\int x \in A.\ f'\ x\ \partial M)$
shows $AE\ x\ in\ M.\ f\ x = f'\ x$
proof −
define A **where** $A = \{x \in space\ M.\ f\ x < f'\ x\}$
then have [*measurable*]: $A \in sets\ M$ **by** *simp*
have $(\int x \in A.\ (f'\ x - f\ x)\ \partial M) = (\int x \in A.\ f'\ x\ \partial M) - (\int x \in A.\ f\ x\ \partial M)$
using ‹$A \in sets\ M$› *M integrable_mult_indicator set_integrable_def* **by** *blast*
then have $(\int x \in A.\ (f'\ x - f\ x)\ \partial M) = 0$ **using** *assms*(*3*) **by** *simp*
then have $A \in null\_sets\ M$
using *A_def null_if_pos_func_has_zero_int*[**where** $?f = \lambda x.\ f'\ x - f\ x$ **and** $?A = A$] *assms* **by** *auto*
then have $AE\ x\ in\ M.\ x \notin A$ **by** (*simp add*: *AE_not_in*)
then have ∗: $AE\ x\ in\ M.\ f'\ x \le f\ x$ **unfolding** *A_def* **by** *auto*

define B **where** $B = \{x \in space\ M.\ f'\ x < f\ x\}$
then have [*measurable*]: $B \in sets\ M$ **by** *simp*
have $(\int x \in B.\ (f\ x - f'\ x)\ \partial M) = (\int x \in B.\ f\ x\ \partial M) - (\int x \in B.\ f'\ x\ \partial M)$
using ‹$B \in sets\ M$› *M integrable_mult_indicator set_integrable_def* **by** *blast*
then have $(\int x \in B.\ (f\ x - f'\ x)\ \partial M) = 0$ **using** *assms*(*3*) **by** *simp*
then have $B \in null\_sets\ M$
using *B_def null_if_pos_func_has_zero_int*[**where** $?f = \lambda x.\ f\ x - f'\ x$ **and** $?A = B$] *assms* **by** *auto*
then have $AE\ x\ in\ M.\ x \notin B$ **by** (*simp add*: *AE_not_in*)
then have $AE\ x\ in\ M.\ f'\ x \ge f\ x$ **unfolding** *B_def* **by** *auto*
then show *?thesis* **using** ∗ **by** *auto*
qed

The next lemma shows that L^1 convergence of a sequence of functions follows from almost everywhere convergence and the weaker condition of the convergence of the integrated norms (or even just the nontrivial inequality about them). Useful in a lot of contexts! This statement (or its variations) are known as Scheffe lemma.

The formalization is more painful as one should jump back and forth between reals and ereals and justify all the time positivity or integrability (thankfully, measurability is handled more or less automatically).

proposition *Scheffe_lemma1*:
assumes $\bigwedge n.\ integrable\ M\ (F\ n)$ $integrable\ M\ f$
$AE\ x\ in\ M.\ (\lambda n.\ F\ n\ x) \longrightarrow f\ x$
$limsup\ (\lambda n.\ \int^+ x.\ norm(F\ n\ x)\ \partial M) \le (\int^+ x.\ norm(f\ x)\ \partial M)$
shows $(\lambda n.\ \int^+ x.\ norm(F\ n\ x - f\ x)\ \partial M) \longrightarrow 0$
proof −
have [*measurable*]: $\bigwedge n.\ F\ n \in borel\_measurable\ M$ $f \in borel\_measurable\ M$
using *assms*(*1*) *assms*(*2*) **by** *simp_all*
define G **where** $G = (\lambda n\ x.\ norm(f\ x) + norm(F\ n\ x) - norm(F\ n\ x - f\ x))$
have [*measurable*]: $\bigwedge n.\ G\ n \in borel\_measurable\ M$ **unfolding** *G_def* **by** *simp*
have *G_pos*[*simp*]: $\bigwedge n\ x.\ G\ n\ x \ge 0$
unfolding *G_def* **by** (*metis ge_iff_diff_ge_0 norm_minus_commute norm_triangle_ineq4*)

have *finint*: $(\int^+ x.\ norm(f\ x)\ \partial M) \neq \infty$
using *has_bochner_integral_implies_finite_norm*[*OF has_bochner_integral_integrable*[*OF* ‹*integrable M f*›]]
by *simp*
then have *fin2*: $2 * (\int^+ x.\ norm(f\ x)\ \partial M) \neq \infty$
by (*auto simp*: *ennreal_mult_eq_top_iff*)

{
fix x **assume** $*$: $(\lambda n.\ F\ n\ x) \longrightarrow f\ x$
then have $(\lambda n.\ norm(F\ n\ x)) \longrightarrow norm(f\ x)$ **using** *tendsto_norm* **by** *blast*
moreover have $(\lambda n.\ norm(F\ n\ x - f\ x)) \longrightarrow 0$ **using** $*$ *Lim_null tendsto_norm_zero_iff* **by** *fastforce*
ultimately have *a*: $(\lambda n.\ norm(F\ n\ x) - norm(F\ n\ x - f\ x)) \longrightarrow norm(f\ x)$ **using** *tendsto_diff* **by** *fastforce*
have $(\lambda n.\ norm(f\ x) + (norm(F\ n\ x) - norm(F\ n\ x - f\ x))) \longrightarrow norm(f\ x) + norm(f\ x)$
by (*rule tendsto_add*) (*auto simp add*: *a*)
moreover have $\bigwedge n.\ G\ n\ x = norm(f\ x) + (norm(F\ n\ x) - norm(F\ n\ x - f\ x))$ **unfolding** *G_def* **by** *simp*
ultimately have $(\lambda n.\ G\ n\ x) \longrightarrow 2 * norm(f\ x)$ **by** *simp*
then have $(\lambda n.\ ennreal(G\ n\ x)) \longrightarrow ennreal(2 * norm(f\ x))$ **by** *simp*
then have *liminf* $(\lambda n.\ ennreal(G\ n\ x)) = ennreal(2 * norm(f\ x))$
using *sequentially_bot tendsto_iff_Liminf_eq_Limsup* **by** *blast*
}
then have *AE x in M*. *liminf* $(\lambda n.\ ennreal(G\ n\ x)) = ennreal(2 * norm(f\ x))$ **using** *assms*(*3*) **by** *auto*
then have $(\int^+ x.\ liminf\ (\lambda n.\ ennreal\ (G\ n\ x))\ \partial M) = (\int^+ x.\ 2 * ennreal(norm(f\ x))\ \partial M)$
by (*simp add*: *nn_integral_cong_AE ennreal_mult*)
also have ... $= 2 * (\int^+ x.\ norm(f\ x)\ \partial M)$ **by** (*rule nn_integral_cmult*) *auto*
finally have *int_liminf*: $(\int^+ x.\ liminf\ (\lambda n.\ ennreal\ (G\ n\ x))\ \partial M) = 2 * (\int^+ x.\ norm(f\ x)\ \partial M)$
by *simp*

have $(\int^+x.\ ennreal(norm(f\ x)) + ennreal(norm(F\ n\ x))\ \partial M) = (\int^+x.\ norm(f\ x)\ \partial M) + (\int^+x.\ norm(F\ n\ x)\ \partial M)$ **for** n
by (*rule nn_integral_add*) (*auto simp add*: *assms*)
then have *limsup* $(\lambda n.\ (\int^+x.\ ennreal(norm(f\ x)) + ennreal(norm(F\ n\ x))\ \partial M)) =$
limsup $(\lambda n.\ (\int^+x.\ norm(f\ x)\ \partial M) + (\int^+x.\ norm(F\ n\ x)\ \partial M))$
by *simp*
also have ... $= (\int^+x.\ norm(f\ x)\ \partial M) +$ *limsup* $(\lambda n.\ (\int^+x.\ norm(F\ n\ x)\ \partial M))$
by (*rule Limsup_const_add*, *auto simp add*: *finint*)
also have ... $\leq (\int^+x.\ norm(f\ x)\ \partial M) + (\int^+x.\ norm(f\ x)\ \partial M)$
using *assms*(*4*) **by** (*simp add*: *add_left_mono*)
also have ... $= 2 * (\int^+x.\ norm(f\ x)\ \partial M)$
unfolding *one_add_one*[*symmetric*] *distrib_right* **by** *simp*
ultimately have *a*: *limsup* $(\lambda n.\ (\int^+x.\ ennreal(norm(f\ x)) + ennreal(norm(F\ n\ x))\ \partial M)) \leq$

2 ∗ (∫ +x. norm(f x) ∂M) **by** simp

have le: ennreal (norm (F n x − f x)) ≤ ennreal (norm (f x)) + ennreal (norm (F n x)) **for** n x
by (simp add: norm_minus_commute norm_triangle_ineq4 ennreal_minus flip: ennreal_plus)
then have le2: (∫ + x. ennreal (norm (F n x − f x)) ∂M) ≤ (∫ + x. ennreal (norm (f x)) + ennreal (norm (F n x)) ∂M) **for** n
by (rule nn_integral_mono)

have 2 ∗ (∫ + x. norm(f x) ∂M) = (∫ + x. liminf (λn. ennreal (G n x)) ∂M)
by (simp add: int_liminf)
also have . . . ≤ liminf (λn. (∫ +x. G n x ∂M))
by (rule nn_integral_liminf) auto
also have liminf (λn. (∫ +x. G n x ∂M)) =
liminf (λn. (∫ +x. ennreal(norm(f x)) + ennreal(norm(F n x)) ∂M) − (∫ +x. norm(F n x − f x) ∂M))
proof (intro arg_cong[**where** f=liminf] ext)
fix n
have ⋀x. ennreal(G n x) = ennreal(norm(f x)) + ennreal(norm(F n x)) − ennreal(norm(F n x − f x))
unfolding G_def **by** (simp add: ennreal_minus flip: ennreal_plus)
moreover have (∫ +x. ennreal(norm(f x)) + ennreal(norm(F n x)) − ennreal(norm(F n x − f x)) ∂M)
= (∫ +x. ennreal(norm(f x)) + ennreal(norm(F n x)) ∂M) − (∫ +x. norm(F n x − f x) ∂M)
proof (rule nn_integral_diff)
from le **show** AE x in M. ennreal (norm (F n x − f x)) ≤ ennreal (norm (f x)) + ennreal (norm (F n x))
by simp
from le2 **have** (∫ +x. ennreal (norm (F n x − f x)) ∂M) < ∞ **using** assms(1) assms(2)
by (metis has_bochner_integral_implies_finite_norm integrable.simps Bochner_Integration.integrable_d
then show (∫ +x. ennreal (norm (F n x − f x)) ∂M) ≠ ∞ **by** simp
qed (auto simp add: assms)
ultimately show (∫ +x. G n x ∂M) = (∫ +x. ennreal(norm(f x)) + ennreal(norm(F n x)) ∂M) − (∫ +x. norm(F n x − f x) ∂M)
by simp
qed
finally have 2 ∗ (∫ + x. norm(f x) ∂M) + limsup (λn. (∫ +x. norm(F n x − f x) ∂M)) ≤
liminf (λn. (∫ +x. ennreal(norm(f x)) + ennreal(norm(F n x)) ∂M) − (∫ +x. norm(F n x − f x) ∂M)) +
limsup (λn. (∫ +x. norm(F n x − f x) ∂M))
by (intro add_mono) auto
also have . . . ≤ (limsup (λn. ∫ +x. ennreal(norm(f x)) + ennreal(norm(F n x)) ∂M) − limsup (λn. ∫ +x. norm (F n x − f x) ∂M)) +
limsup (λn. (∫ +x. norm(F n x − f x) ∂M))
by (intro add_mono liminf_minus_ennreal le2) auto

```
  also have ... = limsup (λn. (∫+x. ennreal(norm(f x)) + ennreal(norm(F n x))
∂M))
    by (intro diff_add_cancel_ennreal Limsup_mono always_eventually allI le2)
  also have ... ≤ 2 ∗ (∫+x. norm(f x) ∂M)
    by fact
  finally have limsup (λn. (∫+x. norm(F n x − f x) ∂M)) = 0
    using fin2 by simp
  then show ?thesis
    by (rule tendsto_0_if_Limsup_eq_0_ennreal)
qed

proposition Scheffe_lemma2:
  fixes F::nat ⇒ 'a ⇒ 'b::{banach, second_countable_topology}
  assumes ⋀ n::nat. F n ∈ borel_measurable M integrable M f
          AE x in M. (λn. F n x) ⟶ f x
          ⋀n. (∫+ x. norm(F n x) ∂M) ≤ (∫+ x. norm(f x) ∂M)
  shows (λn. ∫+ x. norm(F n x − f x) ∂M) ⟶ 0
proof (rule Scheffe_lemma1)
  fix n::nat
 have (∫+ x. norm(f x) ∂M) < ∞ using assms(2) by (metis has_bochner_integral_implies_finite_norm
integrable.cases)
  then have (∫+ x. norm(F n x) ∂M) < ∞ using assms(4)[of n] by auto
 then show integrable M (F n) by (subst integrable_iff_bounded, simp add: assms(1)[of
n])
qed (auto simp add: assms Limsup_bounded)

lemma tendsto_set_lebesgue_integral_at_right:
  fixes a b :: real and f :: real ⇒ 'a :: {banach,second_countable_topology}
  assumes a < b and sets: ⋀a'. a' ∈ {a<..b} ⟹ {a'..b} ∈ sets M
      and set_integrable M {a<..b} f
  shows  ((λa'. set_lebesgue_integral M {a'..b} f) ⟶
           set_lebesgue_integral M {a<..b} f) (at_right a)
proof (rule tendsto_at_right_sequentially[OF assms(1)], goal_cases)
  case (1 S)
  have eq: (⋃n. {S n..b}) = {a<..b}
  proof safe
    fix x n assume x ∈ {S n..b}
    with 1(1,2)[of n] show x ∈ {a<..b} by auto
  next
    fix x assume x ∈ {a<..b}
    with order_tendstoD[OF ‹S ⟶ a›, of x] show x ∈ (⋃n. {S n..b})
      by (force simp: eventually_at_top_linorder dest: less_imp_le)
  qed
   have (λn. set_lebesgue_integral M {S n..b} f) ⟶ set_lebesgue_integral M
(⋃n. {S n..b}) f
   by (rule set_integral_cont_up) (insert assms 1, auto simp: eq incseq_def decseq_def
less_imp_le)
  with eq show ?case by simp
qed
```

The next lemmas relate convergence of integrals over an interval to improper integrals.

lemma *tendsto_set_lebesgue_integral_at_left*:
fixes *a b* :: *real* **and** *f* :: *real* ⇒ *'a* :: {*banach*,*second_countable_topology*}
assumes $a < b$ **and** *sets*: ⋀*b'*. *b'* ∈ {*a*..<*b*} ⟹ {*a*..*b'*} ∈ *sets M*
and *set_integrable M* {*a*..<*b*} *f*
shows ((λ*b'*. *set_lebesgue_integral M* {*a*..*b'*} *f*) ⟶
set_lebesgue_integral M {*a*..<*b*} *f*) (*at_left b*)
proof (*rule tendsto_at_left_sequentially*[*OF assms*(*1*)], *goal_cases*)
case (*1 S*)
have *eq*: (⋃*n*. {*a*..*S n*}) = {*a*..<*b*}
proof *safe*
fix *x n* **assume** *x* ∈ {*a*..*S n*}
with *1*(*1*,*2*)[*of n*] **show** *x* ∈ {*a*..<*b*} **by** *auto*
next
fix *x* **assume** *x* ∈ {*a*..<*b*}
with *order_tendstoD*[*OF* ‹*S* ⟶ *b*›, *of x*] **show** *x* ∈ (⋃*n*. {*a*..*S n*})
by (*force simp*: *eventually_at_top_linorder dest*: *less_imp_le*)
qed
have (λ*n*. *set_lebesgue_integral M* {*a*..*S n*} *f*) ⟶ *set_lebesgue_integral M* (⋃*n*. {*a*..*S n*}) *f*
by (*rule set_integral_cont_up*) (*insert assms 1*, *auto simp*: *eq incseq_def decseq_def less_imp_le*)
with *eq* **show** *?case* **by** *simp*
qed

proposition *tendsto_set_lebesgue_integral_at_top*:
fixes *f* :: *real* ⇒ *'a*::{*banach*, *second_countable_topology*}
assumes *sets*: ⋀*b*. *b* ≥ *a* ⟹ {*a*..*b*} ∈ *sets M*
and *int*: *set_integrable M* {*a*..} *f*
shows ((λ*b*. *set_lebesgue_integral M* {*a*..*b*} *f*) ⟶ *set_lebesgue_integral M* {*a*..} *f*) *at_top*
proof (*rule tendsto_at_topI_sequentially*)
fix *X* :: *nat* ⇒ *real* **assume** *filterlim X at_top sequentially*
show (λ*n*. *set_lebesgue_integral M* {*a*..*X n*} *f*) ⟶ *set_lebesgue_integral M* {*a*..} *f*
unfolding *set_lebesgue_integral_def*
proof (*rule integral_dominated_convergence*)
show *integrable M* (λ*x*. *indicat_real* {*a*..} *x* $*_R$ *norm* (*f x*))
using *integrable_norm*[*OF int*[*unfolded set_integrable_def*]] **by** *simp*
show *AE x in M*. (λ*n*. *indicator* {*a*..*X n*} *x* $*_R$ *f x*) ⟶ *indicat_real* {*a*..} *x* $*_R$ *f x*
proof
fix *x*
from ‹*filterlim X at_top sequentially*›
have *eventually* (λ*n*. *x* ≤ *X n*) *sequentially*
unfolding *filterlim_at_top_ge*[**where** *c*=*x*] **by** *auto*
then show (λ*n*. *indicator* {*a*..*X n*} *x* $*_R$ *f x*) ⟶ *indicat_real* {*a*..} *x* $*_R$ *f x*

```
          by (intro tendsto_eventually) (auto split: split_indicator elim!: eventu-
ally_mono)
    qed
    fix n show AE x in M. norm (indicator {a..X n} x *R f x) ≤
                          indicator {a..} x *R norm (f x)
      by (auto split: split_indicator)
  next
    from int show (λx. indicat_real {a..} x *R f x) ∈ borel_measurable M
      by (simp add: set_integrable_def)
  next
    fix n :: nat
    from sets have {a..X n} ∈ sets M by (cases X n ≥ a) auto
    with int have set_integrable M {a..X n} f
      by (rule set_integrable_subset) auto
    thus (λx. indicat_real {a..X n} x *R f x) ∈ borel_measurable M
      by (simp add: set_integrable_def)
  qed
qed

proposition tendsto_set_lebesgue_integral_at_bot:
  fixes f :: real ⇒ 'a::{banach, second_countable_topology}
  assumes sets: ⋀a. a ≤ b ⟹ {a..b} ∈ sets M
      and int: set_integrable M {..b} f
     shows ((λa. set_lebesgue_integral M {a..b} f) ⟶ set_lebesgue_integral M
{..b} f) at_bot
proof (rule tendsto_at_botI_sequentially)
  fix X :: nat ⇒ real assume filterlim X at_bot sequentially
  show (λn. set_lebesgue_integral M {X n..b} f) ⟶ set_lebesgue_integral M
{..b} f
    unfolding set_lebesgue_integral_def
  proof (rule integral_dominated_convergence)
    show integrable M (λx. indicat_real {..b} x *R norm (f x))
      using integrable_norm[OF int[unfolded set_integrable_def]] by simp
    show AE x in M. (λn. indicator {X n..b} x *R f x) ⟶ indicat_real {..b}
x *R f x
    proof
      fix x
      from ‹filterlim X at_bot sequentially›
      have eventually (λn. x ≥ X n) sequentially
        unfolding filterlim_at_bot_le[where c=x] by auto
      then show (λn. indicator {X n..b} x *R f x) ⟶ indicat_real {..b} x *R
f x
          by (intro tendsto_eventually) (auto split: split_indicator elim!: eventu-
ally_mono)
    qed
    fix n show AE x in M. norm (indicator {X n..b} x *R f x) ≤
                          indicator {..b} x *R norm (f x)
      by (auto split: split_indicator)
  next
```

from *int* **show** $(\lambda x.\ indicat\_real\ \{..b\}\ x *_R f\ x) \in borel\_measurable\ M$
  by (*simp add*: *set_integrable_def*)
next
  fix n :: *nat*
  from *sets* **have** $\{X\ n..b\} \in sets\ M$ **by** (*cases* $X\ n \leq b$) *auto*
  with *int* **have** *set_integrable* $M\ \{X\ n..b\}\ f$
    by (*rule set_integrable_subset*) *auto*
  thus $(\lambda x.\ indicat\_real\ \{X\ n..b\}\ x *_R f\ x) \in borel\_measurable\ M$
    by (*simp add*: *set_integrable_def*)
qed
qed

end

6.17 Non-Denumerability of the Continuum

theory *Continuum_Not_Denumerable*
imports
  Complex_Main
  HOL−Library.Countable_Set
begin

6.17.1 Abstract

The following document presents a proof that the Continuum is uncountable. It is formalised in the Isabelle/Isar theorem proving system.

Theorem: The Continuum $\mathbb{R}$ is not denumerable. In other words, there does not exist a function $f: \mathbb{N} \Rightarrow \mathbb{R}$ such that f is surjective.

Outline: An elegant informal proof of this result uses Cantor's Diagonalisation argument. The proof presented here is not this one.

First we formalise some properties of closed intervals, then we prove the Nested Interval Property. This property relies on the completeness of the Real numbers and is the foundation for our argument. Informally it states that an intersection of countable closed intervals (where each successive interval is a subset of the last) is non-empty. We then assume a surjective function $f: \mathbb{N} \Rightarrow \mathbb{R}$ exists and find a real x such that x is not in the range of f by generating a sequence of closed intervals then using the Nested Interval Property.

theorem *real_non_denum*: $\nexists f :: nat \Rightarrow real.\ surj\ f$
proof
  assume $\exists f::nat \Rightarrow real.\ surj\ f$
  then obtain $f :: nat \Rightarrow real$ **where** *surj f* **..**

First we construct a sequence of nested intervals, ignoring *range f*.

  have $a < b \Longrightarrow \exists\, ka\ kb.\ ka < kb \wedge \{ka..kb\} \subseteq \{a..b\} \wedge c \notin \{ka..kb\}$ **for** $a\ b\ c$ :: *real*

by (*auto simp add*: *not_le cong*: *conj_cong*)
(*metis dense le_less_linear less_linear less_trans order_refl*)
then obtain *i j* **where** *ij*:
$a < b \Longrightarrow i\ a\ b\ c < j\ a\ b\ c$
$a < b \Longrightarrow \{i\ a\ b\ c\ ..\ j\ a\ b\ c\} \subseteq \{a\ ..\ b\}$
$a < b \Longrightarrow c \notin \{i\ a\ b\ c\ ..\ j\ a\ b\ c\}$
for *a b c* :: *real*
by *metis*

define *ivl* **where** *ivl* =
rec_nat (*f 0* + *1*, *f 0* + *2*) (λ*n x*. (*i* (*fst x*) (*snd x*) (*f n*), *j* (*fst x*) (*snd x*) (*f n*)))
define *I* **where** *I n* = {*fst* (*ivl n*) .. *snd* (*ivl n*)} **for** *n*

have *ivl* [*simp*]:
ivl 0 = (*f 0* + *1*, *f 0* + *2*)
⋀*n*. *ivl* (*Suc n*) = (*i* (*fst* (*ivl n*)) (*snd* (*ivl n*)) (*f n*), *j* (*fst* (*ivl n*)) (*snd* (*ivl n*)) (*f n*))
unfolding *ivl_def* **by** *simp_all*

This is a decreasing sequence of non-empty intervals.

have *less*: *fst* (*ivl n*) < *snd* (*ivl n*) **for** *n*
by (*induct n*) (*auto intro*!: *ij*)

have *decseq I*
unfolding *I_def decseq_Suc_iff ivl fst_conv snd_conv*
by (*intro ij allI less*)

Now we apply the finite intersection property of compact sets.

have *I 0* ∩ (⋂*i*. *I i*) ≠ {}
proof (*rule compact_imp_fip_image*)
fix *S* :: *nat set*
assume *fin*: *finite S*
have {} ⊂ *I* (*Max* (*insert 0 S*))
unfolding *I_def* **using** *less*[*of Max* (*insert 0 S*)] **by** *auto*
also have *I* (*Max* (*insert 0 S*)) ⊆ (⋂*i*∈*insert 0 S*. *I i*)
using *fin decseqD*[*OF* ‹*decseq I*›, *of* _ *Max* (*insert 0 S*)]
by (*auto simp*: *Max_ge_iff*)
also have (⋂*i*∈*insert 0 S*. *I i*) = *I 0* ∩ (⋂*i*∈*S*. *I i*)
by *auto*
finally show *I 0* ∩ (⋂*i*∈*S*. *I i*) ≠ {}
by *auto*
qed (*auto simp*: *I_def*)
then obtain *x* **where** *x* ∈ *I n* **for** *n*
by *blast*
moreover from ‹*surj f*› **obtain** *j* **where** *x* = *f j*
by *blast*
ultimately have *f j* ∈ *I* (*Suc j*)
by *blast*

with *ij*(*3*)[*OF less*] **show** *False*
  unfolding *I_def ivl fst_conv snd_conv* **by** *auto*
qed

lemma *uncountable_UNIV_real*: *uncountable* (*UNIV* :: *real set*)
 using *real_non_denum* **unfolding** *uncountable_def* **by** *auto*

lemma *bij_betw_open_intervals*:
 fixes *a b c d* :: *real*
 assumes *a* < *b c* < *d*
 shows ∃*f*. *bij_betw f* {*a*<..<*b*} {*c*<..<*d*}
proof −
 define *f* **where** *f a b c d x* = (*d* − *c*)/(*b* − *a*) ∗ (*x* − *a*) + *c* **for** *a b c d x* :: *real*
 {
  fix *a b c d x* :: *real*
  assume ∗: *a* < *b c* < *d a* < *x x* < *b*
  moreover from ∗ **have** (*d* − *c*) ∗ (*x* − *a*) < (*d* − *c*) ∗ (*b* − *a*)
   by (*intro mult_strict_left_mono*) *simp_all*
  moreover from ∗ **have** *0* < (*d* − *c*) ∗ (*x* − *a*) / (*b* − *a*)
   by *simp*
  ultimately have *f a b c d x* < *d c* < *f a b c d x*
   by (*simp_all add*: *f_def field_simps*)
 }
 with *assms* **have** *bij_betw* (*f a b c d*) {*a*<..<*b*} {*c*<..<*d*}
  by (*intro bij_betw_byWitness*[**where** *f'*=*f c d a b*]) (*auto simp*: *f_def*)
 then show *?thesis* **by** *auto*
qed

lemma *bij_betw_tan*: *bij_betw tan* {−*pi*/*2*<..<*pi*/*2*} *UNIV*
 using *arctan_ubound* **by** (*intro bij_betw_byWitness*[**where** *f'*=*arctan*]) (*auto simp*: *arctan arctan_tan*)

lemma *uncountable_open_interval*: *uncountable* {*a*<..<*b*} ⟷ *a* < *b* **for** *a b* :: *real*
proof
 show *a* < *b* **if** *uncountable* {*a*<..<*b*}
  using *uncountable_def that* **by** *force*
 show *uncountable* {*a*<..<*b*} **if** *a* < *b*
 proof −
  obtain *f* **where** *bij_betw f* {*a* <..< *b*} {−*pi*/*2*<..<*pi*/*2*}
   using *bij_betw_open_intervals*[*OF* ‹*a* < *b*›, *of* −*pi*/*2 pi*/*2*] **by** *auto*
  then show *?thesis*
   by (*metis bij_betw_tan uncountable_bij_betw uncountable_UNIV_real*)
 qed
qed

lemma *uncountable_half_open_interval_1*: *uncountable* {*a*..<*b*} ⟷ *a* < *b* **for** *a b* :: *real*

apply *auto*
using *atLeastLessThan_empty_iff*
apply *fastforce*
using *uncountable_open_interval* [*of a b*]
apply (*metis countable_Un_iff ivl_disj_un_singleton*(*3*))
done

lemma *uncountable_half_open_interval_2*: *uncountable* {*a*<..*b*} ⟷ *a* < *b* **for** *a b* :: *real*
apply *auto*
using *atLeastLessThan_empty_iff*
apply *fastforce*
using *uncountable_open_interval* [*of a b*]
apply (*metis countable_Un_iff ivl_disj_un_singleton*(*4*))
done

lemma *real_interval_avoid_countable_set*:
fixes *a b* :: *real* **and** *A* :: *real set*
assumes *a* < *b* **and** *countable A*
shows ∃ *x*∈{*a*<..<*b*}. *x* ∉ *A*
proof −
from ‹*countable A*› **have** ∗: *countable* (*A* ∩ {*a*<..<*b*})
by *auto*
with ‹*a* < *b*› **have** ¬ *countable* {*a*<..<*b*}
by (*simp add*: *uncountable_open_interval*)
with ∗ **have** *A* ∩ {*a*<..<*b*} ≠ {*a*<..<*b*}
by *auto*
then have *A* ∩ {*a*<..<*b*} ⊂ {*a*<..<*b*}
by (*intro psubsetI*) *auto*
then have ∃ *x*. *x* ∈ {*a*<..<*b*} − *A* ∩ {*a*<..<*b*}
by (*rule psubset_imp_ex_mem*)
then show *?thesis*
by *auto*
qed

lemma *uncountable_closed_interval*: *uncountable* {*a*..*b*} ⟷ *a* < *b* **for** *a b* :: *real*
apply (*rule iffI*)
apply (*metis atLeastAtMost_singleton atLeastatMost_empty countable_finite finite.emptyI finite_insert linorder_neqE_linordered_idom*)
using *real_interval_avoid_countable_set* **by** *fastforce*

lemma *open_minus_countable*:
fixes *S A* :: *real set*
assumes *countable A S* ≠ {} *open S*
shows ∃ *x*∈*S*. *x* ∉ *A*
proof −
obtain *x* **where** *x* ∈ *S*
using ‹*S* ≠ {}› **by** *auto*
then obtain *e* **where** *0* < *e* {*y*. *dist y x* < *e*} ⊆ *S*

using ‹*open S*› **by** (*auto simp*: *open_dist subset_eq*)
moreover have {*y*. *dist y x* < *e*} = {*x* − *e* <..< *x* + *e*}
by (*auto simp*: *dist_real_def*)
ultimately have *uncountable* (*S* − *A*)
using *uncountable_open_interval*[*of x* − *e x* + *e*] ‹*countable A*›
by (*intro uncountable_minus_countable*) (*auto dest*: *countable_subset*)
then show *?thesis*
unfolding *uncountable_def* **by** *auto*
qed

end

6.18 Homotopy of Maps

theory *Homotopy*
imports *Path_Connected Continuum_Not_Denumerable Product_Topology*
begin

definition *homotopic_with*
where
homotopic_with P X Y f g ≡
(∃ *h*. *continuous_map* (*prod_topology* (*top_of_set* {*0..1*::*real*}) *X*) *Y h* ∧
(∀ *x*. *h*(*0*, *x*) = *f x*) ∧
(∀ *x*. *h*(*1*, *x*) = *g x*) ∧
(∀ *t* ∈ {*0..1*}. *P*(λ*x*. *h*(*t*,*x*))))

p, *q* are functions *X* → *Y*, and the property *P* restricts all intermediate maps. We often just want to require that *P* fixes some subset, but to include the case of a loop homotopy, it is convenient to have a general property *P*.

abbreviation *homotopic_with_canon* ::
[(*'a*::*topological_space* ⇒ *'b*::*topological_space*) ⇒ *bool*, *'a set*, *'b set*, *'a* ⇒ *'b*, *'a* ⇒ *'b*] ⇒ *bool*
where
homotopic_with_canon P S T p q ≡ *homotopic_with P* (*top_of_set S*) (*top_of_set T*) *p q*

lemma *split_01*: {*0..1*::*real*} = {*0..1/2*} ∪ {*1/2..1*}
by *force*

lemma *split_01_prod*: {*0..1*::*real*} × *X* = ({*0..1/2*} × *X*) ∪ ({*1/2..1*} × *X*)
by *force*

lemma *image_Pair_const*: (λ*x*. (*x*, *c*)) ‘ *A* = *A* × {*c*}
by *auto*

lemma *fst_o_paired* [*simp*]: *fst* ∘ (λ(*x*,*y*). (*f x y*, *g x y*)) = (λ(*x*,*y*). *f x y*)
by *auto*

lemma *snd_o_paired* [*simp*]: *snd* ∘ (λ(*x*,*y*). (*f x y*, *g x y*)) = (λ(*x*,*y*). *g x y*)
  by *auto*

lemma *continuous_on_o_Pair*: ⟦*continuous_on* (*T* × *X*) *h*; *t* ∈ *T*⟧ ⟹ *continuous_on X* (*h* ∘ *Pair t*)
  by (*fast intro*: *continuous_intros elim*!: *continuous_on_subset*)

lemma *continuous_map_o_Pair*:
  assumes *h*: *continuous_map* (*prod_topology X Y*) *Z h* **and** *t*: *t* ∈ *topspace X*
  shows *continuous_map Y Z* (*h* ∘ *Pair t*)
  by (*intro continuous_map_compose* [*OF* _ *h*] *continuous_intros*; *simp add*: *t*)

6.18.1 Trivial properties

We often want to just localize the ending function equality or whatever.

proposition *homotopic_with*:
  assumes ⋀*h k*. (⋀*x*. *x* ∈ *topspace X* ⟹ *h x* = *k x*) ⟹ (*P h* ⟷ *P k*)
  shows *homotopic_with P X Y p q* ⟷
        (∃ *h*. *continuous_map* (*prod_topology* (*subtopology euclideanreal* {*0*..*1*}) *X*) *Y h* ∧
            (∀ *x* ∈ *topspace X*. *h*(*0*,*x*) = *p x*) ∧
            (∀ *x* ∈ *topspace X*. *h*(*1*,*x*) = *q x*) ∧
            (∀ *t* ∈ {*0*..*1*}. *P*(λ*x*. *h*(*t*, *x*))))
  unfolding *homotopic_with_def*
  apply (*rule iffI*, *blast*, *clarify*)
  apply (*rule_tac x*=λ(*u*,*v*). *if v* ∈ *topspace X then h*(*u*,*v*) *else if u* = *0 then p v else q v* **in** *exI*)
  apply *auto*
  using *continuous_map_eq* **apply** *fastforce*
  apply (*drule_tac x*=*t* **in** *bspec*, *force*)
  apply (*subst assms*; *simp*)
  done

lemma *homotopic_with_mono*:
  assumes *hom*: *homotopic_with P X Y f g*
    and *Q*: ⋀*h*. ⟦*continuous_map X Y h*; *P h*⟧ ⟹ *Q h*
  shows *homotopic_with Q X Y f g*
  using *hom* **unfolding** *homotopic_with_def*
  by (*force simp*: *o_def dest*: *continuous_map_o_Pair intro*: *Q*)

lemma *homotopic_with_imp_continuous_maps*:
    assumes *homotopic_with P X Y f g*
    shows *continuous_map X Y f* ∧ *continuous_map X Y g*
proof −
  obtain *h* :: *real* × *'a* ⇒ *'b*
    where *conth*: *continuous_map* (*prod_topology* (*top_of_set* {*0*..*1*}) *X*) *Y h*
      and *h*: ∀ *x*. *h* (*0*, *x*) = *f x* ∀ *x*. *h* (*1*, *x*) = *g x*
    using *assms* **by** (*auto simp*: *homotopic_with_def*)

have ∗: *t* ∈ {*0..1*} ⟹ *continuous_map X Y* (*h* ∘ (λ*x*. (*t*,*x*))) **for** *t*
by (*rule continuous_map_compose* [*OF _ conth*]) (*simp add*: *o_def continuous_map_pairwise*)
show *?thesis*
using *h* ∗[*of 0*] ∗[*of 1*] **by** (*simp add*: *continuous_map_eq*)
qed

lemma *homotopic_with_imp_continuous*:
assumes *homotopic_with_canon P X Y f g*
shows *continuous_on X f* ∧ *continuous_on X g*
by (*meson assms continuous_map_subtopology_eu homotopic_with_imp_continuous_maps*)

lemma *homotopic_with_imp_property*:
assumes *homotopic_with P X Y f g*
shows *P f* ∧ *P g*
proof
obtain *h* **where** *h*: ⋀*x*. *h*(*0*, *x*) = *f x* ⋀*x*. *h*(*1*, *x*) = *g x* **and** *P*: ⋀*t*. *t* ∈ {*0..1*::*real*} ⟹ *P*(λ*x*. *h*(*t*,*x*))
using *assms* **by** (*force simp*: *homotopic_with_def*)
show *P f P g*
using *P* [*of 0*] *P* [*of 1*] **by** (*force simp*: *h*)+
qed

lemma *homotopic_with_equal*:
assumes *P f P g* **and** *contf*: *continuous_map X Y f* **and** *fg*: ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *g x*
shows *homotopic_with P X Y f g*
unfolding *homotopic_with_def*
proof (*intro exI conjI allI ballI*)
let *?h* = λ(*t*::*real*,*x*). *if t* = *1 then g x else f x*
show *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) *X*) *Y ?h*
proof (*rule continuous_map_eq*)
show *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) *X*) *Y* (*f* ∘ *snd*)
by (*simp add*: *contf continuous_map_of_snd*)
qed (*auto simp*: *fg*)
show *P* (λ*x*. *?h* (*t*, *x*)) **if** *t* ∈ {*0..1*} **for** *t*
by (*cases t* = *1*) (*simp_all add*: *assms*)
qed *auto*

lemma *homotopic_with_imp_subset1*:
homotopic_with_canon P X Y f g ⟹ *f* ‘ *X* ⊆ *Y*
by (*simp add*: *homotopic_with_def image_subset_iff*) (*metis atLeastAtMost_iff order_refl zero_le_one*)

lemma *homotopic_with_imp_subset2*:
homotopic_with_canon P X Y f g ⟹ *g* ‘ *X* ⊆ *Y*
by (*simp add*: *homotopic_with_def image_subset_iff*) (*metis atLeastAtMost_iff order_refl zero_le_one*)

lemma *homotopic_with_subset_left*:
⟦*homotopic_with_canon P X Y f g*; $Z \subseteq X$⟧ ⟹ *homotopic_with_canon P Z Y f g*
unfolding *homotopic_with_def* **by** (*auto elim*!: *continuous_on_subset ex_forward*)

lemma *homotopic_with_subset_right*:
⟦*homotopic_with_canon P X Y f g*; $Y \subseteq Z$⟧ ⟹ *homotopic_with_canon P X Z f g*
unfolding *homotopic_with_def* **by** (*auto elim*!: *continuous_on_subset ex_forward*)

6.18.2 Homotopy with P is an equivalence relation

(on continuous functions mapping X into Y that satisfy P, though this only affects reflexivity)

lemma *homotopic_with_refl* [*simp*]: *homotopic_with P X Y f f* ⟷ *continuous_map X Y f* ∧ *P f*
by (*auto simp*: *homotopic_with_imp_continuous_maps intro*: *homotopic_with_equal dest*: *homotopic_with_imp_property*)

lemma *homotopic_with_symD*:
assumes *homotopic_with P X Y f g*
shows *homotopic_with P X Y g f*
proof −
let *?I01* = *subtopology euclideanreal* {*0*..*1*}
let *?j* = λ*y*. (*1* − *fst y*, *snd y*)
have *1*: *continuous_map* (*prod_topology ?I01 X*) (*prod_topology euclideanreal X*) *?j*
by (*intro continuous_intros*; *simp add*: *continuous_map_subtopology_fst prod_topology_subtopology*)
have ∗: *continuous_map* (*prod_topology ?I01 X*) (*prod_topology ?I01 X*) *?j*
proof −
have *continuous_map* (*prod_topology ?I01 X*) (*subtopology* (*prod_topology euclideanreal X*) ({*0*..*1*} × *topspace X*)) *?j*
by (*simp add*: *continuous_map_into_subtopology* [*OF 1*] *image_subset_iff*)
then show *?thesis*
by (*simp add*: *prod_topology_subtopology*(*1*))
qed
show *?thesis*
using *assms*
apply (*clarsimp simp add*: *homotopic_with_def*)
subgoal for *h*
by (*rule_tac x=h* ∘ (λ*y*. (*1* − *fst y*, *snd y*)) **in** *exI*) (*simp add*: *continuous_map_compose* [*OF* ∗])
done
qed

lemma *homotopic_with_sym*:
homotopic_with P X Y f g ⟷ *homotopic_with P X Y g f*
by (*metis homotopic_with_symD*)

```
proposition homotopic_with_trans:
  assumes homotopic_with P X Y f g  homotopic_with P X Y g h
  shows homotopic_with P X Y f h
proof −
 let ?X01 = prod_topology (subtopology euclideanreal {0..1}) X
 obtain k1 k2
  where contk1: continuous_map ?X01 Y k1 and contk2: continuous_map ?X01
Y k2
   and k12: ∀x. k1 (1, x) = g x ∀x. k2 (0, x) = g x
   ∀x. k1 (0, x) = f x ∀x. k2 (1, x) = h x
   and P:  ∀t∈{0..1}. P (λx. k1 (t, x)) ∀t∈{0..1}. P (λx. k2 (t, x))
  using assms by (auto simp: homotopic_with_def)
 define k where k ≡ λy. if fst y ≤ 1/2
                  then (k1 ∘ (λx. (2 *R fst x, snd x))) y
                  else (k2 ∘ (λx. (2 *R fst x −1, snd x))) y
 have keq: k1 (2 ∗ u, v) = k2 (2 ∗ u −1, v) if u = 1/2  for u v
  by (simp add: k12 that)
 show ?thesis
  unfolding homotopic_with_def
 proof (intro exI conjI)
  show continuous_map ?X01 Y k
   unfolding k_def
  proof (rule continuous_map_cases_le)
   show fst: continuous_map ?X01 euclideanreal fst
    using continuous_map_fst continuous_map_in_subtopology by blast
   show continuous_map ?X01 euclideanreal (λx. 1/2)
    by simp
   show continuous_map (subtopology ?X01 {y ∈ topspace ?X01. fst y ≤ 1/2})
Y
        (k1 ∘ (λx. (2 *R fst x, snd x)))
     apply (intro fst continuous_map_compose [OF _ contk1] continuous_intros
continuous_map_into_subtopology continuous_map_from_subtopology | simp)+
    by (force simp: prod_topology_subtopology)
   show continuous_map (subtopology ?X01 {y ∈ topspace ?X01. 1/2 ≤ fst y})
Y
        (k2 ∘ (λx. (2 *R fst x −1, snd x)))
     apply (intro fst continuous_map_compose [OF _ contk2] continuous_intros
continuous_map_into_subtopology continuous_map_from_subtopology | simp)+
    by (force simp: prod_topology_subtopology)
   show (k1 ∘ (λx. (2 *R fst x, snd x))) y = (k2 ∘ (λx. (2 *R fst x −1, snd
x))) y
    if y ∈ topspace ?X01 and fst y = 1/2 for y
    using that by (simp add: keq)
  qed
  show ∀x. k (0, x) = f x
   by (simp add: k12 k_def)
  show ∀x. k (1, x) = h x
   by (simp add: k12 k_def)
  show ∀t∈{0..1}. P (λx. k (t, x))
```

```
    proof
      fix t show t∈{0..1} ⟹ P (λx. k (t, x))
        by (cases t ≤ 1/2) (auto simp add: k_def P)
    qed
  qed
qed
```

```
lemma homotopic_with_id2:
  (⋀x. x ∈ topspace X ⟹ g (f x) = x) ⟹ homotopic_with (λx. True) X X (g ∘
f) id
   by (metis comp_apply continuous_map_id eq_id_iff homotopic_with_equal homo-
topic_with_symD)
```

6.18.3 Continuity lemmas

```
lemma homotopic_with_compose_continuous_map_left:
  ⟦homotopic_with p X1 X2 f g; continuous_map X2 X3 h; ⋀j. p j ⟹ q(h ∘ j)⟧
   ⟹ homotopic_with q X1 X3 (h ∘ f) (h ∘ g)
  unfolding homotopic_with_def
  apply clarify
  subgoal for k
    by (rule_tac x=h ∘ k in exI) (rule conjI continuous_map_compose | simp add:
o_def)+
  done
```

```
lemma homotopic_with_compose_continuous_map_right:
  assumes hom: homotopic_with p X2 X3 f g and conth: continuous_map X1 X2
h
    and q: ⋀j. p j ⟹ q(j ∘ h)
  shows homotopic_with q X1 X3 (f ∘ h) (g ∘ h)
proof -
  obtain k
   where contk: continuous_map (prod_topology (subtopology euclideanreal {0..1})
X2) X3 k
      and k: ∀x. k (0, x) = f x ∀x. k (1, x) = g x and p: ⋀t. t∈{0..1} ⟹ p
(λx. k (t, x))
    using hom unfolding homotopic_with_def by blast
   have hsnd: continuous_map (prod_topology (subtopology euclideanreal {0..1})
X1) X2 (h ∘ snd)
    by (rule continuous_map_compose [OF continuous_map_snd conth])
  let ?h = k ∘ (λ(t,x). (t,h x))
  show ?thesis
    unfolding homotopic_with_def
  proof (intro exI conjI allI ballI)
    have continuous_map (prod_topology (top_of_set {0..1}) X1)
     (prod_topology (top_of_set {0..1::real}) X2) (λ(t, x). (t, h x))
       by (metis (mono_tags, lifting) case_prod_beta' comp_def continuous_map_eq
continuous_map_fst continuous_map_pairedI hsnd)
     then show continuous_map (prod_topology (subtopology euclideanreal {0..1})
```

X1) *X3 ?h*
by (*intro conjI continuous_map_compose* [*OF _ contk*])
show *q* (λ*x*. *?h* (*t*, *x*)) **if** *t* ∈ {*0..1*} **for** *t*
using *q* [*OF p* [*OF that*]] **by** (*simp add*: *o_def*)
qed (*auto simp*: *k*)
qed

corollary *homotopic_compose*:
assumes *homotopic_with* (λ*x*. *True*) *X Y f f′ homotopic_with* (λ*x*. *True*) *Y Z g g′*
shows *homotopic_with* (λ*x*. *True*) *X Z* (*g* ∘ *f*) (*g′* ∘ *f′*)
proof (*rule homotopic_with_trans* [**where** *g* = *g* ∘ *f′*])
show *homotopic_with* (λ*x*. *True*) *X Z* (*g* ∘ *f*) (*g* ∘ *f′*)
using *assms* **by** (*simp add*: *homotopic_with_compose_continuous_map_left homotopic_with_imp_continuous_maps*)
show *homotopic_with* (λ*x*. *True*) *X Z* (*g* ∘ *f′*) (*g′* ∘ *f′*)
using *assms* **by** (*simp add*: *homotopic_with_compose_continuous_map_right homotopic_with_imp_continuous_maps*)
qed

proposition *homotopic_with_compose_continuous_right*:
⟦*homotopic_with_canon* (λ*f*. *p* (*f* ∘ *h*)) *X Y f g*; *continuous_on W h*; *h* ‘ *W* ⊆ *X*⟧
⟹ *homotopic_with_canon p W Y* (*f* ∘ *h*) (*g* ∘ *h*)
apply (*clarsimp simp add*: *homotopic_with_def*)
subgoal for *k*
apply (*rule_tac x*=*k* ∘ (λ*y*. (*fst y*, *h* (*snd y*))) **in** *exI*)
by (*intro conjI continuous_intros continuous_on_compose2* [**where** *f*=*snd* **and** *g*=*h*]; *fastforce simp*: *o_def elim*: *continuous_on_subset*)
done

proposition *homotopic_with_compose_continuous_left*:
⟦*homotopic_with_canon* (λ*f*. *p* (*h* ∘ *f*)) *X Y f g*; *continuous_on Y h*; *h* ‘ *Y* ⊆ *Z*⟧
⟹ *homotopic_with_canon p X Z* (*h* ∘ *f*) (*h* ∘ *g*)
apply (*clarsimp simp add*: *homotopic_with_def*)
subgoal for *k*
apply (*rule_tac x*=*h* ∘ *k* **in** *exI*)
by (*intro conjI continuous_intros continuous_on_compose* [**where** *f*=*snd* **and** *g*=*h*, *unfolded o_def*]; *fastforce simp*: *o_def elim*: *continuous_on_subset*)
done

lemma *homotopic_from_subtopology*:
homotopic_with P X X′ f g ⟹ *homotopic_with P* (*subtopology X s*) *X′ f g*
unfolding *homotopic_with_def*
by (*force simp add*: *continuous_map_from_subtopology prod_topology_subtopology*(*2*) *elim*!: *ex_forward*)

lemma *homotopic_on_emptyI*:

assumes *topspace X* = {} *P f P g*
shows *homotopic_with P X X′ f g*
unfolding *homotopic_with_def*
proof (*intro exI conjI ballI*)
show *P* (λ*x*. (λ(*t*,*x*). *if t* = *0 then f x else g x*) (*t*, *x*)) **if** *t* ∈ {*0*..*1*} **for** *t*::*real*
by (*cases t* = *0*, *auto simp*: *assms*)
qed (*auto simp*: *continuous_map_atin assms*)

lemma *homotopic_on_empty*:
topspace X = {} ⟹ (*homotopic_with P X X′ f g* ⟷ *P f* ∧ *P g*)
using *homotopic_on_emptyI homotopic_with_imp_property* **by** *metis*

lemma *homotopic_with_canon_on_empty* [*simp*]: *homotopic_with_canon* (λ*x*. *True*) {} *t f g*
by (*auto intro*: *homotopic_with_equal*)

lemma *homotopic_constant_maps*:
homotopic_with (λ*x*. *True*) *X X′* (λ*x*. *a*) (λ*x*. *b*) ⟷
topspace X = {} ∨ *path_component_of X′ a b* (**is** *?lhs* = *?rhs*)
proof (*cases topspace X* = {})
case *False*
then obtain *c* **where** *c*: *c* ∈ *topspace X*
by *blast*
have ∃ *g*. *continuous_map* (*top_of_set* {*0*..*1*::*real*}) *X′ g* ∧ *g 0* = *a* ∧ *g 1* = *b*
if *x* ∈ *topspace X* **and** *hom*: *homotopic_with* (λ*x*. *True*) *X X′* (λ*x*. *a*) (λ*x*. *b*) **for** *x*
proof −
obtain *h* :: *real* × *′a* ⇒ *′b*
where *conth*: *continuous_map* (*prod_topology* (*top_of_set* {*0*..*1*}) *X*) *X′ h*
and *h*: ⋀*x*. *h* (*0*, *x*) = *a* ⋀*x*. *h* (*1*, *x*) = *b*
using *hom* **by** (*auto simp*: *homotopic_with_def*)
have *cont*: *continuous_map* (*top_of_set* {*0*..*1*}) *X′* (*h* ∘ (λ*t*. (*t*, *c*)))
by (*rule continuous_map_compose* [*OF* _ *conth*] *continuous_intros c* | *simp*)+
then show *?thesis*
by (*force simp*: *h*)
qed
moreover have *homotopic_with* (λ*x*. *True*) *X X′* (λ*x*. *g 0*) (λ*x*. *g 1*)
if *x* ∈ *topspace X a* = *g 0 b* = *g 1 continuous_map* (*top_of_set* {*0*..*1*}) *X′ g*
for *x* **and** *g* :: *real* ⇒ *′b*
unfolding *homotopic_with_def*
by (*force intro*!: *continuous_map_compose continuous_intros c that*)
ultimately show *?thesis*
using *False* **by** (*auto simp*: *path_component_of_def pathin_def*)
qed (*simp add*: *homotopic_on_empty*)

proposition *homotopic_with_eq*:
assumes *h*: *homotopic_with P X Y f g*
and *f′*: ⋀*x*. *x* ∈ *topspace X* ⟹ *f′ x* = *f x*
and *g′*: ⋀*x*. *x* ∈ *topspace X* ⟹ *g′ x* = *g x*

```
    and P:  (⋀h k. (⋀x. x ∈ topspace X ⟹ h x = k x) ⟹ P h ⟷ P k)
  shows homotopic_with P X Y f' g'
  using h unfolding homotopic_with_def
  apply clarify
  subgoal for h
    apply (rule_tac x=λ(u,v). if v ∈ topspace X then h(u,v) else if u = 0 then f'
v else g' v in exI)
    apply (simp add: f' g', safe)
     apply (fastforce intro: continuous_map_eq)
    apply (subst P; fastforce)
    done
  done

lemma homotopic_with_prod_topology:
  assumes homotopic_with p X1 Y1 f f' and homotopic_with q X2 Y2 g g'
    and r: ⋀i j. ⟦p i; q j⟧ ⟹ r(λ(x,y). (i x, j y))
  shows homotopic_with r (prod_topology X1 X2) (prod_topology Y1 Y2)
                         (λz. (f(fst z),g(snd z))) (λz. (f'(fst z), g'(snd z)))
proof -
  obtain h
   where h: continuous_map (prod_topology (subtopology euclideanreal {0..1}) X1)
Y1 h
      and h0: ⋀x. h (0, x) = f x
      and h1: ⋀x. h (1, x) = f' x
      and p: ⋀t. ⟦0 ≤ t; t ≤ 1⟧ ⟹ p (λx. h (t,x))
    using assms unfolding homotopic_with_def by auto
  obtain k
   where k: continuous_map (prod_topology (subtopology euclideanreal {0..1}) X2)
Y2 k
      and k0: ⋀x. k (0, x) = g x
      and k1: ⋀x. k (1, x) = g' x
      and q: ⋀t. ⟦0 ≤ t; t ≤ 1⟧ ⟹ q (λx. k (t,x))
    using assms unfolding homotopic_with_def by auto
  let ?hk = λ(t,x,y). (h(t,x), k(t,y))
  show ?thesis
    unfolding homotopic_with_def
  proof (intro conjI allI exI)
   show continuous_map (prod_topology (subtopology euclideanreal {0..1}) (prod_topology
X1 X2))
                        (prod_topology Y1 Y2) ?hk
      unfolding continuous_map_pairwise case_prod_unfold
       by (rule conjI continuous_map_pairedI continuous_intros continuous_map_id
[unfolded id_def]
            continuous_map_fst_of [unfolded o_def] continuous_map_snd_of [unfolded
o_def]
          continuous_map_compose [OF _ h, unfolded o_def]
          continuous_map_compose [OF _ k, unfolded o_def])+
  next
    fix x
```

```
    show ?hk (0, x) = (f (fst x), g (snd x)) ?hk (1, x) = (f' (fst x), g' (snd x))
      by (auto simp: case_prod_beta h0 k0 h1 k1)
  qed (auto simp: p q r)
qed


lemma homotopic_with_product_topology:
  assumes ht: ⋀i. i ∈ I ⟹ homotopic_with (p i) (X i) (Y i) (f i) (g i)
    and pq: ⋀h. (⋀i. i ∈ I ⟹ p i (h i)) ⟹ q(λx. (λi∈I. h i (x i)))
  shows homotopic_with q (product_topology X I) (product_topology Y I)
                        (λz. (λi∈I. (f i) (z i))) (λz. (λi∈I. (g i) (z i)))
proof -
  obtain h
    where h: ⋀i. i ∈ I ⟹ continuous_map (prod_topology (subtopology euclidean-
real {0..1}) (X i)) (Y i) (h i)
      and h0: ⋀i x. i ∈ I ⟹ h i (0, x) = f i x
      and h1: ⋀i x. i ∈ I ⟹ h i (1, x) = g i x
      and p: ⋀i t. ⟦i ∈ I; t ∈ {0..1}⟧ ⟹ p i (λx. h i (t,x))
    using ht unfolding homotopic_with_def by metis
  show ?thesis
    unfolding homotopic_with_def
  proof (intro conjI allI exI)
    let ?h = λ(t,z). λi∈I. h i (t,z i)
   have continuous_map (prod_topology (subtopology euclideanreal {0..1}) (product_topology
X I))
                        (Y i) (λx. h i (fst x, snd x i)) if i ∈ I for i
    proof -
      have §: continuous_map (prod_topology (top_of_set {0..1}) (product_topology
X I)) (X i) (λx. snd x i)
        using continuous_map_componentwise continuous_map_snd that by fastforce
      show ?thesis
        unfolding continuous_map_pairwise case_prod_unfold
        by (intro conjI that § continuous_intros continuous_map_compose [OF _ h,
unfolded o_def])
    qed
    then show continuous_map (prod_topology (subtopology euclideanreal {0..1})
(product_topology X I))
          (product_topology Y I) ?h
      by (auto simp: continuous_map_componentwise case_prod_beta)
    show ?h (0, x) = (λi∈I. f i (x i)) ?h (1, x) = (λi∈I. g i (x i)) for x
      by (auto simp: case_prod_beta h0 h1)
    show ∀t∈{0..1}. q (λx. ?h (t, x))
      by (force intro: p pq)
  qed
qed
```

Homotopic triviality implicitly incorporates path-connectedness.

```
lemma homotopic_triviality:
  shows (∀f g. continuous_on S f ∧ f ` S ⊆ T ∧
```

continuous_on S g ∧ *g* ‘ *S* ⊆ *T*
⟶ *homotopic_with_canon* (λ*x*. *True*) *S T f g*) ⟷
(*S* = {} ∨ *path_connected T*) ∧
(∀ *f*. *continuous_on S f* ∧ *f* ‘ *S* ⊆ *T* ⟶ (∃ *c*. *homotopic_with_canon* (λ*x*. *True*) *S T f* (λ*x*. *c*)))
(**is** *?lhs* = *?rhs*)
proof (*cases S* = {} ∨ *T* = {})
case *True* **then show** *?thesis*
by (*auto simp*: *homotopic_on_emptyI*)
next
case *False* **show** *?thesis*
proof
assume *LHS* [*rule_format*]: *?lhs*
have *pab*: *path_component T a b* **if** *a* ∈ *T b* ∈ *T* **for** *a b*
proof −
have *homotopic_with_canon* (λ*x*. *True*) *S T* (λ*x*. *a*) (λ*x*. *b*)
by (*simp add*: *LHS image_subset_iff that*)
then show *?thesis*
using *False homotopic_constant_maps* [*of top_of_set S top_of_set T a b*] **by** *auto*
qed
moreover
have ∃ *c*. *homotopic_with_canon* (λ*x*. *True*) *S T f* (λ*x*. *c*) **if** *continuous_on S f f* ‘ *S* ⊆ *T* **for** *f*
using *False LHS continuous_on_const that* **by** *blast*
ultimately show *?rhs*
by (*simp add*: *path_connected_component*)
next
assume *RHS*: *?rhs*
with *False* **have** *T*: *path_connected T*
by *blast*
show *?lhs*
proof *clarify*
fix *f g*
assume *continuous_on S f f* ‘ *S* ⊆ *T continuous_on S g g* ‘ *S* ⊆ *T*
obtain *c d* **where** *c*: *homotopic_with_canon* (λ*x*. *True*) *S T f* (λ*x*. *c*) **and** *d*: *homotopic_with_canon* (λ*x*. *True*) *S T g* (λ*x*. *d*)
using *False* ‹*continuous_on S f*› ‹*f* ‘ *S* ⊆ *T*› *RHS* ‹*continuous_on S g*› ‹*g* ‘ *S* ⊆ *T*› **by** *blast*
then have *c* ∈ *T d* ∈ *T*
using *False homotopic_with_imp_continuous_maps* **by** *fastforce*+
with *T* **have** *path_component T c d*
using *path_connected_component* **by** *blast*
then have *homotopic_with_canon* (λ*x*. *True*) *S T* (λ*x*. *c*) (λ*x*. *d*)
by (*simp add*: *homotopic_constant_maps*)
with *c d* **show** *homotopic_with_canon* (λ*x*. *True*) *S T f g*
by (*meson homotopic_with_symD homotopic_with_trans*)
qed
qed

qed

6.18.4 Homotopy of paths, maintaining the same endpoints

definition *homotopic_paths* :: ['a set, real ⇒ 'a, real ⇒ 'a::topological_space] ⇒ bool
where
homotopic_paths s p q ≡
homotopic_with_canon (λr. *pathstart r* = *pathstart p* ∧ *pathfinish r* = *pathfinish p*) {0..1} *s p q*

lemma *homotopic_paths*:
homotopic_paths s p q ⟷
(∃ h. *continuous_on* ({0..1} × {0..1}) h ∧
h ‘ ({0..1} × {0..1}) ⊆ s ∧
(∀ x ∈ {0..1}. h(0,x) = p x) ∧
(∀ x ∈ {0..1}. h(1,x) = q x) ∧
(∀ t ∈ {0..1::real}. *pathstart*(h ∘ *Pair* t) = *pathstart* p ∧
pathfinish(h ∘ *Pair* t) = *pathfinish* p))
by (*auto simp*: *homotopic_paths_def homotopic_with pathstart_def pathfinish_def*)

proposition *homotopic_paths_imp_pathstart*:
homotopic_paths s p q ⟹ *pathstart p* = *pathstart q*
by (*metis* (*mono_tags*, *lifting*) *homotopic_paths_def homotopic_with_imp_property*)

proposition *homotopic_paths_imp_pathfinish*:
homotopic_paths s p q ⟹ *pathfinish p* = *pathfinish q*
by (*metis* (*mono_tags*, *lifting*) *homotopic_paths_def homotopic_with_imp_property*)

lemma *homotopic_paths_imp_path*:
homotopic_paths s p q ⟹ *path p* ∧ *path q*
using *homotopic_paths_def homotopic_with_imp_continuous_maps path_def continuous_map_subtopology_eu* **by** *blast*

lemma *homotopic_paths_imp_subset*:
homotopic_paths s p q ⟹ *path_image p* ⊆ s ∧ *path_image q* ⊆ s
by (*metis* (*mono_tags*) *continuous_map_subtopology_eu homotopic_paths_def homotopic_with_imp_continuous_maps path_image_def*)

proposition *homotopic_paths_refl* [*simp*]: *homotopic_paths s p p* ⟷ *path p* ∧ *path_image p* ⊆ s
by (*simp add*: *homotopic_paths_def path_def path_image_def*)

proposition *homotopic_paths_sym*: *homotopic_paths s p q* ⟹ *homotopic_paths s q p*
by (*metis* (*mono_tags*) *homotopic_paths_def homotopic_paths_imp_pathfinish homotopic_paths_imp_pathstart homotopic_with_symD*)

proposition *homotopic_paths_sym_eq*: *homotopic_paths s p q* ⟷ *homotopic_paths*

s q p
by (*metis homotopic_paths_sym*)

proposition *homotopic_paths_trans* [*trans*]:
assumes *homotopic_paths s p q homotopic_paths s q r*
shows *homotopic_paths s p r*
proof −
have *pathstart q* = *pathstart p pathfinish q* = *pathfinish p*
using *assms* **by** (*simp_all add*: *homotopic_paths_imp_pathstart homotopic_paths_imp_pathfinish*)
then have *homotopic_with_canon* (λ*f*. *pathstart f* = *pathstart p* ∧ *pathfinish f* = *pathfinish p*) {*0..1*} *s q r*
using ‹*homotopic_paths s q r*› *homotopic_paths_def* **by** *force*
then show *?thesis*
using *assms homotopic_paths_def homotopic_with_trans* **by** *blast*
qed

proposition *homotopic_paths_eq*:
⟦*path p*; *path_image p* ⊆ *s*; ⋀*t*. *t* ∈ {*0..1*} ⟹ *p t* = *q t*⟧ ⟹ *homotopic_paths s p q*
unfolding *homotopic_paths_def*
by (*rule homotopic_with_eq*)
(*auto simp*: *path_def pathstart_def pathfinish_def path_image_def elim*: *continuous_on_eq*)

proposition *homotopic_paths_reparametrize*:
assumes *path p*
and *pips*: *path_image p* ⊆ *s*
and *contf*: *continuous_on* {*0..1*} *f*
and *f01*:*f* ‘ {*0..1*} ⊆ {*0..1*}
and [*simp*]: *f*(*0*) = *0 f*(*1*) = *1*
and *q*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *q*(*t*) = *p*(*f t*)
shows *homotopic_paths s p q*
proof −
have *contp*: *continuous_on* {*0..1*} *p*
by (*metis* ‹*path p*› *path_def*)
then have *continuous_on* {*0..1*} (*p* ∘ *f*)
using *contf continuous_on_compose continuous_on_subset f01* **by** *blast*
then have *path q*
by (*simp add*: *path_def*) (*metis q continuous_on_cong*)
have *piqs*: *path_image q* ⊆ *s*
by (*metis* (*no_types*, *hide_lams*) *pips f01 image_subset_iff path_image_def q*)
have *fb0*: ⋀*a b*. ⟦*0* ≤ *a*; *a* ≤ *1*; *0* ≤ *b*; *b* ≤ *1*⟧ ⟹ *0* ≤ (*1* − *a*) ∗ *f b* + *a* ∗ *b*
using *f01* **by** *force*
have *fb1*: ⟦*0* ≤ *a*; *a* ≤ *1*; *0* ≤ *b*; *b* ≤ *1*⟧ ⟹ (*1* − *a*) ∗ *f b* + *a* ∗ *b* ≤ *1* **for** *a b*
using *f01* [*THEN subsetD*, *of f b*] **by** (*simp add*: *convex_bound_le*)
have *homotopic_paths s q p*
proof (*rule homotopic_paths_trans*)
show *homotopic_paths s q* (*p* ∘ *f*)
using *q* **by** (*force intro*: *homotopic_paths_eq* [*OF* ‹*path q*› *piqs*])

```
  next
    show homotopic_paths s (p ∘ f) p
      using pips [unfolded path_image_def]
      apply (simp add: homotopic_paths_def homotopic_with_def)
      apply (rule_tac x=p ∘ (λy. (1 − (fst y)) *R ((f ∘ snd) y) + (fst y) *R snd
y) in exI)
       apply (rule conjI contf continuous_intros continuous_on_subset [OF contp] |
simp)+
      by (auto simp: fb0 fb1 pathstart_def pathfinish_def)
  qed
  then show ?thesis
    by (simp add: homotopic_paths_sym)
qed

lemma homotopic_paths_subset: ⟦homotopic_paths s p q; s ⊆ t⟧ ⟹ homotopic_paths
t p q
  unfolding homotopic_paths by fast
```

A slightly ad-hoc but useful lemma in constructing homotopies.

```
lemma continuous_on_homotopic_join_lemma:
  fixes q :: [real,real] ⇒ 'a::topological_space
  assumes p: continuous_on ({0..1} × {0..1}) (λy. p (fst y) (snd y)) (is contin-
uous_on ?A ?p)
      and q: continuous_on ({0..1} × {0..1}) (λy. q (fst y) (snd y)) (is continu-
ous_on ?A ?q)
      and pf: ⋀t. t ∈ {0..1} ⟹ pathfinish(p t) = pathstart(q t)
    shows continuous_on ({0..1} × {0..1}) (λy. (p(fst y) +++ q(fst y)) (snd y))
proof −
  have §: (λt. p (fst t) (2 * snd t)) = ?p ∘ (λy. (fst y, 2 * snd y))
          (λt. q (fst t) (2 * snd t − 1)) = ?q ∘ (λy. (fst y, 2 * snd y − 1))
    by force+
  show ?thesis
    unfolding joinpaths_def
  proof (rule continuous_on_cases_le)
    show continuous_on {y ∈ ?A. snd y ≤ 1/2} (λt. p (fst t) (2 * snd t))
         continuous_on {y ∈ ?A. 1/2 ≤ snd y} (λt. q (fst t) (2 * snd t − 1))
         continuous_on ?A snd
      unfolding §
      by (rule continuous_intros continuous_on_subset [OF p] continuous_on_subset
[OF q] | force)+
  qed (use pf in ‹auto simp: mult.commute pathstart_def pathfinish_def›)
qed
```

Congruence properties of homotopy w.r.t. path-combining operations.

```
lemma homotopic_paths_reversepath_D:
      assumes homotopic_paths s p q
      shows   homotopic_paths s (reversepath p) (reversepath q)
  using assms
  apply (simp add: homotopic_paths_def homotopic_with_def, clarify)
```

apply (*rule_tac x=h* ∘ (*λx*. (*fst x*, *1* − *snd x*)) **in** *exI*)
apply (*rule conjI continuous_intros*)+
apply (*auto simp*: *reversepath_def pathstart_def pathfinish_def elim*!: *continuous_on_subset*)
done

proposition *homotopic_paths_reversepath*:
homotopic_paths s (*reversepath p*) (*reversepath q*) ⟷ *homotopic_paths s p q*
using *homotopic_paths_reversepath_D* **by** *force*

proposition *homotopic_paths_join*:
⟦*homotopic_paths s p p′*; *homotopic_paths s q q′*; *pathfinish p* = *pathstart q*⟧ ⟹ *homotopic_paths s* (*p* +++ *q*) (*p′* +++ *q′*)
apply (*clarsimp simp add*: *homotopic_paths_def homotopic_with_def*)
apply (*rename_tac k1 k2*)
apply (*rule_tac x*=(*λy*. ((*k1* ∘ *Pair* (*fst y*)) +++ (*k2* ∘ *Pair* (*fst y*))) (*snd y*)) **in** *exI*)
apply (*intro conjI continuous_intros continuous_on_homotopic_join_lemma*; *force simp*: *joinpaths_def pathstart_def pathfinish_def path_image_def*)
done

proposition *homotopic_paths_continuous_image*:
⟦*homotopic_paths s f g*; *continuous_on s h*; *h* ‘ *s* ⊆ *t*⟧ ⟹ *homotopic_paths t* (*h* ∘ *f*) (*h* ∘ *g*)
unfolding *homotopic_paths_def*
by (*simp add*: *homotopic_with_compose_continuous_map_left pathfinish_compose pathstart_compose*)

6.18.5 Group properties for homotopy of paths

So taking equivalence classes under homotopy would give the fundamental group

proposition *homotopic_paths_rid*:
assumes *path p path_image p* ⊆ *s*
shows *homotopic_paths s* (*p* +++ *linepath* (*pathfinish p*) (*pathfinish p*)) *p*
proof −
have §: *continuous_on* {*0..1*} (*λt::real*. *if t* ≤ *1/2 then 2* $*_R$ *t else 1*)
unfolding *split_01*
by (*rule continuous_on_cases continuous_intros* | *force simp*: *pathfinish_def joinpaths_def*)+
show *?thesis*
apply (*rule homotopic_paths_sym*)
using *assms* **unfolding** *pathfinish_def joinpaths_def*
by (*intro* § *continuous_on_cases continuous_intros homotopic_paths_reparametrize* [**where** *f* = *λt*. *if t* ≤ *1/2 then 2* $*_R$ *t else 1*]; *force*)
qed

proposition *homotopic_paths_lid*:

⟦path p; path_image p ⊆ s⟧ ⟹ homotopic_paths s (linepath (pathstart p) (pathstart p) +++ p) p
using homotopic_paths_rid [of reversepath p s]
by (metis homotopic_paths_reversepath path_image_reversepath path_reversepath pathfinish_linepath
pathfinish_reversepath reversepath_joinpaths reversepath_linepath)

proposition *homotopic_paths_assoc*:
⟦path p; path_image p ⊆ s; path q; path_image q ⊆ s; path r; path_image r ⊆ s; pathfinish p = pathstart q;
pathfinish q = pathstart r⟧
⟹ homotopic_paths s (p +++ (q +++ r)) ((p +++ q) +++ r)
apply (subst homotopic_paths_sym)
apply (rule homotopic_paths_reparametrize
[**where** f = λt. if t ≤ 1/2 then inverse 2 $*_R$ t
else if t ≤ 3 / 4 then t − (1 / 4)
else 2 $*_R$ t − 1])
apply (simp_all del: le_divide_eq_numeral1 add: subset_path_image_join)
apply (rule continuous_on_cases_1 continuous_intros | auto simp: joinpaths_def)+
done

proposition *homotopic_paths_rinv*:
assumes path p path_image p ⊆ s
shows homotopic_paths s (p +++ reversepath p) (linepath (pathstart p) (pathstart p))
proof −
have p: continuous_on {0..1} p
using assms **by** (auto simp: path_def)
let ?A = {0..1} × {0..1}
have continuous_on ?A (λx. (subpath 0 (fst x) p +++ reversepath (subpath 0 (fst x) p)) (snd x))
unfolding joinpaths_def subpath_def reversepath_def path_def add_0_right diff_0_right
proof (rule continuous_on_cases_le)
show continuous_on {x ∈ ?A. snd x ≤ 1/2} (λt. p (fst t * (2 * snd t)))
continuous_on {x ∈ ?A. 1/2 ≤ snd x} (λt. p (fst t * (1 − (2 * snd t − 1))))
continuous_on ?A snd
by (intro continuous_on_compose2 [OF p] continuous_intros; auto simp add: mult_le_one)+
qed (auto simp add: algebra_simps)
then show ?thesis
using assms
apply (subst homotopic_paths_sym_eq)
unfolding homotopic_paths_def homotopic_with_def
apply (rule_tac x=(λy. (subpath 0 (fst y) p +++ reversepath(subpath 0 (fst y) p)) (snd y)) **in** exI)
apply (force simp: mult_le_one path_defs joinpaths_def subpath_def reversepath_def)
done
qed

proposition *homotopic_paths_linv*:
assumes *path p path_image p* ⊆ *s*
shows *homotopic_paths s* (*reversepath p* +++ *p*) (*linepath* (*pathfinish p*) (*pathfinish p*))
using *homotopic_paths_rinv* [*of reversepath p s*] *assms* **by** *simp*

6.18.6 Homotopy of loops without requiring preservation of endpoints

definition *homotopic_loops* :: *'a*::*topological_space set* ⇒ (*real* ⇒ *'a*) ⇒ (*real* ⇒ *'a*) ⇒ *bool* **where**
homotopic_loops s p q ≡
homotopic_with_canon (λ*r*. *pathfinish r* = *pathstart r*) {*0*..*1*} *s p q*

lemma *homotopic_loops*:
homotopic_loops s p q ⟷
(∃ *h*. *continuous_on* ({*0*..*1*::*real*} × {*0*..*1*}) *h* ∧
image h ({*0*..*1*} × {*0*..*1*}) ⊆ *s* ∧
(∀ *x* ∈ {*0*..*1*}. *h*(*0*,*x*) = *p x*) ∧
(∀ *x* ∈ {*0*..*1*}. *h*(*1*,*x*) = *q x*) ∧
(∀ *t* ∈ {*0*..*1*}. *pathfinish*(*h* ∘ *Pair t*) = *pathstart*(*h* ∘ *Pair t*)))
by (*simp add*: *homotopic_loops_def pathstart_def pathfinish_def homotopic_with*)

proposition *homotopic_loops_imp_loop*:
homotopic_loops s p q ⟹ *pathfinish p* = *pathstart p* ∧ *pathfinish q* = *pathstart q*
using *homotopic_with_imp_property homotopic_loops_def* **by** *blast*

proposition *homotopic_loops_imp_path*:
homotopic_loops s p q ⟹ *path p* ∧ *path q*
unfolding *homotopic_loops_def path_def*
using *homotopic_with_imp_continuous_maps continuous_map_subtopology_eu* **by** *blast*

proposition *homotopic_loops_imp_subset*:
homotopic_loops s p q ⟹ *path_image p* ⊆ *s* ∧ *path_image q* ⊆ *s*
unfolding *homotopic_loops_def path_image_def*
by (*meson continuous_map_subtopology_eu homotopic_with_imp_continuous_maps*)

proposition *homotopic_loops_refl*:
homotopic_loops s p p ⟷
path p ∧ *path_image p* ⊆ *s* ∧ *pathfinish p* = *pathstart p*
by (*simp add*: *homotopic_loops_def path_image_def path_def*)

proposition *homotopic_loops_sym*: *homotopic_loops s p q* ⟹ *homotopic_loops s q p*
by (*simp add*: *homotopic_loops_def homotopic_with_sym*)

proposition *homotopic_loops_sym_eq*: *homotopic_loops s p q* ⟷ *homotopic_loops s q p*
by (*metis homotopic_loops_sym*)

proposition *homotopic_loops_trans*:
⟦*homotopic_loops s p q*; *homotopic_loops s q r*⟧ ⟹ *homotopic_loops s p r*
unfolding *homotopic_loops_def* **by** (*blast intro*: *homotopic_with_trans*)

proposition *homotopic_loops_subset*:
⟦*homotopic_loops s p q*; *s* ⊆ *t*⟧ ⟹ *homotopic_loops t p q*
by (*fastforce simp add*: *homotopic_loops*)

proposition *homotopic_loops_eq*:
⟦*path p*; *path_image p* ⊆ *s*; *pathfinish p* = *pathstart p*; ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*t*) = *q*(*t*)⟧
⟹ *homotopic_loops s p q*
unfolding *homotopic_loops_def path_image_def path_def pathstart_def pathfinish_def*
by (*auto intro*: *homotopic_with_eq* [*OF homotopic_with_refl* [**where** *f* = *p*, *THEN iffD2*]])

proposition *homotopic_loops_continuous_image*:
⟦*homotopic_loops s f g*; *continuous_on s h*; *h* ‘ *s* ⊆ *t*⟧ ⟹ *homotopic_loops t* (*h* ∘ *f*) (*h* ∘ *g*)
unfolding *homotopic_loops_def*
by (*simp add*: *homotopic_with_compose_continuous_map_left pathfinish_def pathstart_def*)

6.18.7 Relations between the two variants of homotopy

proposition *homotopic_paths_imp_homotopic_loops*:
⟦*homotopic_paths s p q*; *pathfinish p* = *pathstart p*; *pathfinish q* = *pathstart p*⟧ ⟹ *homotopic_loops s p q*
by (*auto simp*: *homotopic_with_def homotopic_paths_def homotopic_loops_def*)

proposition *homotopic_loops_imp_homotopic_paths_null*:
assumes *homotopic_loops s p* (*linepath a a*)
shows *homotopic_paths s p* (*linepath* (*pathstart p*) (*pathstart p*))
proof −
have *path p* **by** (*metis assms homotopic_loops_imp_path*)
have *ploop*: *pathfinish p* = *pathstart p* **by** (*metis assms homotopic_loops_imp_loop*)
have *pip*: *path_image p* ⊆ *s* **by** (*metis assms homotopic_loops_imp_subset*)
let *?A* = {*0..1*::*real*} × {*0..1*::*real*}
obtain *h* **where** *conth*: *continuous_on ?A h*
and *hs*: *h* ‘ *?A* ⊆ *s*
and [*simp*]: ⋀*x*. *x* ∈ {*0..1*} ⟹ *h*(*0*,*x*) = *p x*
and [*simp*]: ⋀*x*. *x* ∈ {*0..1*} ⟹ *h*(*1*,*x*) = *a*
and *ends*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *pathfinish* (*h* ∘ *Pair t*) = *pathstart* (*h* ∘ *Pair t*)
using *assms* **by** (*auto simp*: *homotopic_loops homotopic_with*)

have *conth0*: *path* (λ*u*. *h* (*u*, *0*))
unfolding *path_def*
proof (*rule continuous_on_compose* [*of* _ _ *h*, *unfolded o_def*])
show *continuous_on* ((λ*x*. (*x*, *0*)) ' {*0..1*}) *h*
by (*force intro*: *continuous_on_subset* [*OF conth*])
qed (*force intro*: *continuous_intros*)
have *pih0*: *path_image* (λ*u*. *h* (*u*, *0*)) ⊆ *s*
using *hs* **by** (*force simp*: *path_image_def*)
have *c1*: *continuous_on ?A* (λ*x*. *h* (*fst x* ∗ *snd x*, *0*))
proof (*rule continuous_on_compose* [*of* _ _ *h*, *unfolded o_def*])
show *continuous_on* ((λ*x*. (*fst x* ∗ *snd x*, *0*)) ' *?A*) *h*
by (*force simp*: *mult_le_one intro*: *continuous_on_subset* [*OF conth*])
qed (*force intro*: *continuous_intros*)+
have *c2*: *continuous_on ?A* (λ*x*. *h* (*fst x* − *fst x* ∗ *snd x*, *0*))
proof (*rule continuous_on_compose* [*of* _ _ *h*, *unfolded o_def*])
show *continuous_on* ((λ*x*. (*fst x* − *fst x* ∗ *snd x*, *0*)) ' *?A*) *h*
by (*auto simp*: *algebra_simps add_increasing2 mult_left_le intro*: *continuous_on_subset* [*OF conth*])
qed (*force intro*: *continuous_intros*)
have [*simp*]: ⋀*t*. ⟦*0* ≤ *t* ∧ *t* ≤ *1*⟧ ⟹ *h* (*t*, *1*) = *h* (*t*, *0*)
using *ends* **by** (*simp add*: *pathfinish_def pathstart_def*)
have *adhoc_le*: *c* ∗ *4* ≤ *1* + *c* ∗ (*d* ∗ *4*) **if** ¬ *d* ∗ *4* ≤ *3 0* ≤ *c c* ≤ *1* **for** *c d*::*real*
proof −
have *c* ∗ *3* ≤ *c* ∗ (*d* ∗ *4*) **using** *that less_eq_real_def* **by** *auto*
with ‹*c* ≤ *1*› **show** *?thesis* **by** *fastforce*
qed
have ∗: ⋀*p x*. ⟦*path p* ∧ *path*(*reversepath p*);
path_image p ⊆ *s* ∧ *path_image*(*reversepath p*) ⊆ *s*;
pathfinish p = *pathstart*(*linepath a a* +++ *reversepath p*) ∧
pathstart(*reversepath p*) = *a* ∧ *pathstart p* = *x*⟧
⟹ *homotopic_paths s* (*p* +++ *linepath a a* +++ *reversepath p*) (*linepath x x*)
by (*metis homotopic_paths_lid homotopic_paths_join*
homotopic_paths_trans homotopic_paths_sym homotopic_paths_rinv)
have *1*: *homotopic_paths s p* (*p* +++ *linepath* (*pathfinish p*) (*pathfinish p*))
using ‹*path p*› *homotopic_paths_rid homotopic_paths_sym pip* **by** *blast*
moreover have *homotopic_paths s* (*p* +++ *linepath* (*pathfinish p*) (*pathfinish p*))
(*linepath* (*pathstart p*) (*pathstart p*) +++ *p* +++ *linepath* (*pathfinish p*) (*pathfinish p*))
apply (*rule homotopic_paths_sym*)
using *homotopic_paths_lid* [*of p* +++ *linepath* (*pathfinish p*) (*pathfinish p*) *s*]
by (*metis 1 homotopic_paths_imp_path homotopic_paths_imp_pathstart homotopic_paths_imp_subset*)
moreover
have *homotopic_paths s* (*linepath* (*pathstart p*) (*pathstart p*) +++ *p* +++ *linepath* (*pathfinish p*) (*pathfinish p*))
((λ*u*. *h* (*u*, *0*)) +++ *linepath a a* +++ *reversepath* (λ*u*. *h* (*u*, *0*)))

```
    unfolding homotopic_paths_def homotopic_with_def
  proof (intro exI strip conjI)
    let ?h = λy. (subpath 0 (fst y) (λu. h (u, 0)) +++ (λu. h (Pair (fst y) u))
               +++ subpath (fst y) 0 (λu. h (u, 0))) (snd y)
    have continuous_on ?A ?h
        by (intro continuous_on_homotopic_join_lemma; simp add: path_defs join-
paths_def subpath_def conth c1 c2)
    moreover have ?h ‘ ?A ⊆ s
      unfolding joinpaths_def subpath_def
     by (force simp: algebra_simps mult_le_one mult_left_le intro: hs [THEN subsetD]
adhoc_le)
  ultimately show continuous_map (prod_topology (top_of_set {0..1}) (top_of_set
{0..1}))
                         (top_of_set s) ?h
    by (simp add: subpath_reversepath)
   qed (use ploop in ‹simp_all add: reversepath_def path_defs joinpaths_def o_def
subpath_def conth c1 c2›)
   moreover have homotopic_paths s ((λu. h (u, 0)) +++ linepath a a +++
reversepath (λu. h (u, 0)))
                                   (linepath (pathstart p) (pathstart p))
  proof (rule *; simp add: pih0 pathstart_def pathfinish_def conth0)
    show a = (linepath a a +++ reversepath (λu. h (u, 0))) 0 ∧ reversepath (λu.
h (u, 0)) 0 = a
      by (simp_all add: reversepath_def joinpaths_def)
  qed
  ultimately show ?thesis
    by (blast intro: homotopic_paths_trans)
qed

proposition homotopic_loops_conjugate:
  fixes s :: 'a::real_normed_vector set
  assumes path p path q and pip: path_image p ⊆ s and piq: path_image q ⊆ s
      and pq: pathfinish p = pathstart q and qloop: pathfinish q = pathstart q
    shows homotopic_loops s (p +++ q +++ reversepath p) q
proof −
  have contp: continuous_on {0..1} p  using ‹path p› [unfolded path_def] by blast
  have contq: continuous_on {0..1} q  using ‹path q› [unfolded path_def] by blast
  let ?A = {0..1::real} × {0..1::real}
  have c1: continuous_on ?A (λx. p ((1 − fst x) ∗ snd x + fst x))
  proof (rule continuous_on_compose [of _ _ p, unfolded o_def])
    show continuous_on ((λx. (1 − fst x) ∗ snd x + fst x) ‘ ?A) p
     by (auto intro: continuous_on_subset [OF contp] simp: algebra_simps add_increasing2
mult_right_le_one_le sum_le_prod1)
  qed (force intro: continuous_intros)
  have c2: continuous_on ?A (λx. p ((fst x − 1) ∗ snd x + 1))
  proof (rule continuous_on_compose [of _ _ p, unfolded o_def])
    show continuous_on ((λx. (fst x − 1) ∗ snd x + 1) ‘ ?A) p
     by (auto intro: continuous_on_subset [OF contp] simp: algebra_simps add_increasing2
mult_left_le_one_le)
```

qed (*force intro*: *continuous_intros*)

have *ps1*: $\bigwedge a\ b.\ [\![b * 2 \le 1;\ 0 \le b;\ 0 \le a;\ a \le 1]\!] \Longrightarrow p\ ((1 - a) * (2 * b) + a) \in s$
using *sum_le_prod1*
by (*force simp*: *algebra_simps add_increasing2 mult_left_le intro*: *pip* [*unfolded path_image_def*, *THEN subsetD*])
have *ps2*: $\bigwedge a\ b.\ [\![\neg\ 4 * b \le 3;\ b \le 1;\ 0 \le a;\ a \le 1]\!] \Longrightarrow p\ ((a - 1) * (4 * b - 3) + 1) \in s$
apply (*rule pip* [*unfolded path_image_def*, *THEN subsetD*])
apply (*rule image_eqI*, *blast*)
apply (*simp add*: *algebra_simps*)
by (*metis add_mono_thms_linordered_semiring*(*1*) *affine_ineq linear mult.commute mult.left_neutral mult_right_mono add.commute zero_le_numeral*)
have *qs*: $\bigwedge a\ b.\ [\![4 * b \le 3;\ \neg\ b * 2 \le 1]\!] \Longrightarrow q\ (4 * b - 2) \in s$
using *path_image_def piq* **by** *fastforce*
have *homotopic_loops s* (*p* +++ *q* +++ *reversepath p*) (*linepath* (*pathstart q*) (*pathstart q*) +++ *q* +++ *linepath* (*pathstart q*) (*pathstart q*))
unfolding *homotopic_loops_def homotopic_with_def*
proof (*intro exI strip conjI*)
let *?h* = (λ*y*. (*subpath* (*fst y*) *1 p* +++ *q* +++ *subpath 1* (*fst y*) *p*) (*snd y*))
have *continuous_on ?A* (λ*y*. *q* (*snd y*))
by (*force simp*: *contq intro*: *continuous_on_compose* [*of* _ _ *q*, *unfolded o_def*] *continuous_on_id continuous_on_snd*)
then have *continuous_on ?A ?h*
using *pq qloop*
by (*intro continuous_on_homotopic_join_lemma*) (*auto simp*: *path_defs joinpaths_def subpath_def c1 c2*)
then show *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) (*top_of_set* {*0..1*})) (*top_of_set s*) *?h*
by (*auto simp*: *joinpaths_def subpath_def ps1 ps2 qs*)
show *?h* (*1*,*x*) = (*linepath* (*pathstart q*) (*pathstart q*) +++ *q* +++ *linepath* (*pathstart q*) (*pathstart q*)) *x* **for** *x*
using *pq* **by** (*simp add*: *pathfinish_def subpath_refl*)
qed (*auto simp*: *subpath_reversepath*)
moreover have *homotopic_loops s* (*linepath* (*pathstart q*) (*pathstart q*) +++ *q* +++ *linepath* (*pathstart q*) (*pathstart q*)) *q*
proof −
have *homotopic_paths s* (*linepath* (*pathfinish q*) (*pathfinish q*) +++ *q*) *q*
using ‹*path q*› *homotopic_paths_lid qloop piq* **by** *auto*
hence *1*: $\bigwedge f$. *homotopic_paths s f q* ∨ ¬ *homotopic_paths s f* (*linepath* (*pathfinish q*) (*pathfinish q*) +++ *q*)
using *homotopic_paths_trans* **by** *blast*
hence *homotopic_paths s* (*linepath* (*pathfinish q*) (*pathfinish q*) +++ *q* +++ *linepath* (*pathfinish q*) (*pathfinish q*)) *q*
proof −
have *homotopic_paths s* (*q* +++ *linepath* (*pathfinish q*) (*pathfinish q*)) *q*

by (*simp add*: ‹*path q*› *homotopic_paths_rid piq*)
thus *?thesis*
by (*metis* (*no_types*) *1* ‹*path q*› *homotopic_paths_join homotopic_paths_rinv homotopic_paths_sym*
homotopic_paths_trans qloop pathfinish_linepath piq)
qed
thus *?thesis*
by (*metis* (*no_types*) *qloop homotopic_loops_sym homotopic_paths_imp_homotopic_loops homotopic_paths_imp_pathfinish homotopic_paths_sym*)
qed
ultimately show *?thesis*
by (*blast intro*: *homotopic_loops_trans*)
qed

lemma *homotopic_paths_loop_parts*:
assumes *loops*: *homotopic_loops S* (*p* +++ *reversepath q*) (*linepath a a*) **and** *path q*
shows *homotopic_paths S p q*
proof −
have *paths*: *homotopic_paths S* (*p* +++ *reversepath q*) (*linepath* (*pathstart p*) (*pathstart p*))
using *homotopic_loops_imp_homotopic_paths_null* [*OF loops*] **by** *simp*
then have *path p*
using ‹*path q*› *homotopic_loops_imp_path loops path_join path_join_path_ends path_reversepath* **by** *blast*
show *?thesis*
proof (*cases pathfinish p* = *pathfinish q*)
case *True*
have *pipq*: *path_image p* ⊆ *S path_image q* ⊆ *S*
by (*metis Un_subset_iff paths* ‹*path p*› ‹*path q*› *homotopic_loops_imp_subset homotopic_paths_imp_path loops*
path_image_join path_image_reversepath path_imp_reversepath path_join_eq)+
have *homotopic_paths S p* (*p* +++ (*linepath* (*pathfinish p*) (*pathfinish p*)))
using ‹*path p*› ‹*path_image p* ⊆ *S*› *homotopic_paths_rid homotopic_paths_sym* **by** *blast*
moreover have *homotopic_paths S* (*p* +++ (*linepath* (*pathfinish p*) (*pathfinish p*))) (*p* +++ (*reversepath q* +++ *q*))
by (*simp add*: *True* ‹*path p*› ‹*path q*› *pipq homotopic_paths_join homotopic_paths_linv homotopic_paths_sym*)
moreover have *homotopic_paths S* (*p* +++ (*reversepath q* +++ *q*)) ((*p* +++ *reversepath q*) +++ *q*)
by (*simp add*: *True* ‹*path p*› ‹*path q*› *homotopic_paths_assoc pipq*)
moreover have *homotopic_paths S* ((*p* +++ *reversepath q*) +++ *q*) (*linepath* (*pathstart p*) (*pathstart p*) +++ *q*)
by (*simp add*: ‹*path q*› *homotopic_paths_join paths pipq*)
moreover then have *homotopic_paths S* (*linepath* (*pathstart p*) (*pathstart p*) +++ *q*) *q*
by (*metis* ‹*path q*› *homotopic_paths_imp_path homotopic_paths_lid linepath_trivial path_join_path_ends pathfinish_def pipq*(*2*))

ultimately show *?thesis*
using *homotopic_paths_trans* **by** *metis*
next
case *False*
then show *?thesis*
using ‹*path q*› *homotopic_loops_imp_path loops path_join_path_ends* **by** *fastforce*
qed
qed

6.18.8 Homotopy of "nearby" function, paths and loops

lemma *homotopic_with_linear*:
fixes *f g* :: _ ⇒ *'b*::*real_normed_vector*
assumes *contf*: *continuous_on S f*
and *contg*:*continuous_on S g*
and *sub*: ⋀*x*. *x* ∈ *S* ⟹ *closed_segment* (*f x*) (*g x*) ⊆ *t*
shows *homotopic_with_canon* (λ*z*. *True*) *S t f g*
unfolding *homotopic_with_def*
apply (*rule_tac x*=λ*y*. ((*1* − (*fst y*)) $*_R$ *f*(*snd y*) + (*fst y*) $*_R$ *g*(*snd y*)) **in** *exI*)
using *sub closed_segment_def*
by (*fastforce intro*: *continuous_intros continuous_on_subset* [*OF contf*] *continuous_on_compose2* [**where** *g*=*f*]
continuous_on_subset [*OF contg*] *continuous_on_compose2* [**where** *g*=*g*])

lemma *homotopic_paths_linear*:
fixes *g h* :: *real* ⇒ *'a*::*real_normed_vector*
assumes *path g path h pathstart h* = *pathstart g pathfinish h* = *pathfinish g*
⋀*t*. *t* ∈ {*0..1*} ⟹ *closed_segment* (*g t*) (*h t*) ⊆ *S*
shows *homotopic_paths S g h*
using *assms*
unfolding *path_def*
apply (*simp add*: *closed_segment_def pathstart_def pathfinish_def homotopic_paths_def homotopic_with_def*)
apply (*rule_tac x*=λ*y*. ((*1* − (*fst y*)) $*_R$ (*g* ∘ *snd*) *y* + (*fst y*) $*_R$ (*h* ∘ *snd*) *y*) **in** *exI*)
apply (*intro conjI subsetI continuous_intros*; *force*)
done

lemma *homotopic_loops_linear*:
fixes *g h* :: *real* ⇒ *'a*::*real_normed_vector*
assumes *path g path h pathfinish g* = *pathstart g pathfinish h* = *pathstart h*
⋀*t x*. *t* ∈ {*0..1*} ⟹ *closed_segment* (*g t*) (*h t*) ⊆ *S*
shows *homotopic_loops S g h*
using *assms*
unfolding *path_defs homotopic_loops_def homotopic_with_def*
apply (*rule_tac x*=λ*y*. ((*1* − (*fst y*)) $*_R$ *g*(*snd y*) + (*fst y*) $*_R$ *h*(*snd y*)) **in** *exI*)
by (*force simp*: *closed_segment_def intro*!: *continuous_intros intro*: *continuous_on_compose2* [**where** *g*=*g*] *continuous_on_compose2* [**where** *g*=*h*])

lemma *homotopic_paths_nearby_explicit*:
assumes §: *path g path h pathstart h = pathstart g pathfinish h = pathfinish g*
and *no*: $\bigwedge t\ x.\ [\![t \in \{0..1\};\ x \notin S]\!] \Longrightarrow norm(h\ t - g\ t) < norm(g\ t - x)$
shows *homotopic_paths S g h*
proof (*rule homotopic_paths_linear* [*OF* §])
show $\bigwedge t.\ t \in \{0..1\} \Longrightarrow closed\_segment\ (g\ t)\ (h\ t) \subseteq S$
by (*metis no segment_bound*(*1*) *subsetI norm_minus_commute not_le*)
qed

lemma *homotopic_loops_nearby_explicit*:
assumes §: *path g path h pathfinish g = pathstart g pathfinish h = pathstart h*
and *no*: $\bigwedge t\ x.\ [\![t \in \{0..1\};\ x \notin S]\!] \Longrightarrow norm(h\ t - g\ t) < norm(g\ t - x)$
shows *homotopic_loops S g h*
proof (*rule homotopic_loops_linear* [*OF* §])
show $\bigwedge t.\ t \in \{0..1\} \Longrightarrow closed\_segment\ (g\ t)\ (h\ t) \subseteq S$
by (*metis no segment_bound*(*1*) *subsetI norm_minus_commute not_le*)
qed

lemma *homotopic_nearby_paths*:
fixes *g h* :: *real* ⇒ *'a*::*euclidean_space*
assumes *path g open S path_image g* ⊆ *S*
shows $\exists e.\ 0 < e\ \wedge$
$(\forall h.\ path\ h\ \wedge$
$pathstart\ h = pathstart\ g\ \wedge\ pathfinish\ h = pathfinish\ g\ \wedge$
$(\forall t \in \{0..1\}.\ norm(h\ t - g\ t) < e) \longrightarrow homotopic\_paths\ S\ g\ h)$
proof −
obtain *e* **where** *e* > *0* **and** *e*: $\bigwedge x\ y.\ x \in path\_image\ g \Longrightarrow y \in -\ S \Longrightarrow e \le dist\ x\ y$
using *separate_compact_closed* [*of path_image g* −*S*] *assms* **by** *force*
show *?thesis*
using *e* [*unfolded dist_norm*] ‹*e* > *0*›
by (*fastforce simp*: *path_image_def intro*!: *homotopic_paths_nearby_explicit assms exI*)
qed

lemma *homotopic_nearby_loops*:
fixes *g h* :: *real* ⇒ *'a*::*euclidean_space*
assumes *path g open S path_image g* ⊆ *S pathfinish g = pathstart g*
shows $\exists e.\ 0 < e\ \wedge$
$(\forall h.\ path\ h\ \wedge\ pathfinish\ h = pathstart\ h\ \wedge$
$(\forall t \in \{0..1\}.\ norm(h\ t - g\ t) < e) \longrightarrow homotopic\_loops\ S\ g\ h)$
proof −
obtain *e* **where** *e* > *0* **and** *e*: $\bigwedge x\ y.\ x \in path\_image\ g \Longrightarrow y \in -\ S \Longrightarrow e \le dist\ x\ y$
using *separate_compact_closed* [*of path_image g* −*S*] *assms* **by** *force*
show *?thesis*
using *e* [*unfolded dist_norm*] ‹*e* > *0*›
by (*fastforce simp*: *path_image_def intro*!: *homotopic_loops_nearby_explicit assms exI*)

qed

6.18.9 Homotopy and subpaths

lemma *homotopic_join_subpaths1*:
assumes *path g* **and** *pag*: *path_image g* ⊆ *s*
and *u*: *u* ∈ {*0..1*} **and** *v*: *v* ∈ {*0..1*} **and** *w*: *w* ∈ {*0..1*} *u* ≤ *v v* ≤ *w*
shows *homotopic_paths s* (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)
proof −
have *1*: *t* ∗ *2* ≤ *1* ⟹ *u* + *t* ∗ (*v* ∗ *2*) ≤ *v* + *t* ∗ (*u* ∗ *2*) **for** *t*
using *affine_ineq* ‹*u* ≤ *v*› **by** *fastforce*
have *2*: *t* ∗ *2* > *1* ⟹ *u* + (*2*∗*t* − *1*) ∗ *v* ≤ *v* + (*2*∗*t* − *1*) ∗ *w* **for** *t*
by (*metis add_mono_thms_linordered_semiring*(*1*) *diff_gt_0_iff_gt less_eq_real_def mult.commute mult_right_mono* ‹*u* ≤ *v*› ‹*v* ≤ *w*›)
have *t2*: ⋀*t*::*real*. *t*∗*2* = *1* ⟹ *t* = *1*/*2* **by** *auto*
have *homotopic_paths* (*path_image g*) (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)
proof (*cases w* = *u*)
case *True*
then show *?thesis*
by (*metis* ‹*path g*› *homotopic_paths_rinv path_image_subpath_subset path_subpath pathstart_subpath reversepath_subpath subpath_refl u v*)
next
case *False*
let *?f* = λ*t*. *if t* ≤ *1*/*2 then inverse*((*w* − *u*)) $*_R$ (*2* ∗ (*v* − *u*)) $*_R$ *t*
else inverse((*w* − *u*)) $*_R$ ((*v* − *u*) + (*w* − *v*) $*_R$ (*2* $*_R$ *t* − *1*))
show *?thesis*
proof (*rule homotopic_paths_sym* [*OF homotopic_paths_reparametrize* [**where** *f* = *?f*]])
show *path* (*subpath u w g*)
using *assms*(*1*) *path_subpath u w*(*1*) **by** *blast*
show *path_image* (*subpath u w g*) ⊆ *path_image g*
by (*meson path_image_subpath_subset u w*(*1*))
show *continuous_on* {*0..1*} *?f*
unfolding *split_01*
by (*rule continuous_on_cases continuous_intros* | *force simp*: *pathfinish_def joinpaths_def dest*!: *t2*)+
show *?f* ‘ {*0..1*} ⊆ {*0..1*}
using *False assms*
by (*force simp*: *field_simps not_le mult_left_mono affine_ineq dest*!: *1 2*)
show (*subpath u v g* +++ *subpath v w g*) *t* = *subpath u w g* (*?f t*) **if** *t* ∈ {*0..1*} **for** *t*
using *assms*
unfolding *joinpaths_def subpath_def* **by** (*auto simp add*: *divide_simps add.commute mult.commute mult.left_commute*)
qed (*use False* **in** *auto*)
qed
then show *?thesis*

by (*rule homotopic_paths_subset* [*OF* _ *pag*])
qed

lemma *homotopic_join_subpaths2*:
assumes *homotopic_paths s* (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)
shows *homotopic_paths s* (*subpath w v g* +++ *subpath v u g*) (*subpath w u g*)
by (*metis assms homotopic_paths_reversepath_D pathfinish_subpath pathstart_subpath reversepath_joinpaths reversepath_subpath*)

lemma *homotopic_join_subpaths3*:
assumes *hom*: *homotopic_paths s* (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)
and *path g* **and** *pag*: *path_image g* ⊆ *s*
and *u*: *u* ∈ {*0..1*} **and** *v*: *v* ∈ {*0..1*} **and** *w*: *w* ∈ {*0..1*}
shows *homotopic_paths s* (*subpath v w g* +++ *subpath w u g*) (*subpath v u g*)
proof −
have *homotopic_paths s* (*subpath u w g* +++ *subpath w v g*) ((*subpath u v g* +++ *subpath v w g*) +++ *subpath w v g*)
proof (*rule homotopic_paths_join*)
show *homotopic_paths s* (*subpath u w g*) (*subpath u v g* +++ *subpath v w g*)
using *hom homotopic_paths_sym_eq* **by** *blast*
show *homotopic_paths s* (*subpath w v g*) (*subpath w v g*)
by (*metis* ‹*path g*› *homotopic_paths_eq pag path_image_subpath_subset path_subpath subset_trans v w*)
qed *auto*
also have *homotopic_paths s* ((*subpath u v g* +++ *subpath v w g*) +++ *subpath w v g*) (*subpath u v g* +++ *subpath v w g* +++ *subpath w v g*)
by (*rule homotopic_paths_sym* [*OF homotopic_paths_assoc*])
(*use assms* **in** ‹*simp_all add*: *path_image_subpath_subset* [*THEN order_trans*]›)
also have *homotopic_paths s* (*subpath u v g* +++ *subpath v w g* +++ *subpath w v g*)
(*subpath u v g* +++ *linepath* (*pathfinish* (*subpath u v g*)) (*pathfinish* (*subpath u v g*)))
proof (*rule homotopic_paths_join*; *simp*)
show *path* (*subpath u v g*) ∧ *path_image* (*subpath u v g*) ⊆ *s*
by (*metis* ‹*path g*› *order.trans pag path_image_subpath_subset path_subpath u v*)
show *homotopic_paths s* (*subpath v w g* +++ *subpath w v g*) (*linepath* (*g v*) (*g v*))
by (*metis* (*no_types*, *lifting*) ‹*path g*› *homotopic_paths_linv order_trans pag path_image_subpath_subset path_subpath pathfinish_subpath reversepath_subpath v w*)
qed
also have *homotopic_paths s* (*subpath u v g* +++ *linepath* (*pathfinish* (*subpath u v g*)) (*pathfinish* (*subpath u v g*))) (*subpath u v g*)
proof (*rule homotopic_paths_rid*)
show *path* (*subpath u v g*)
using ‹*path g*› *path_subpath u v* **by** *blast*
show *path_image* (*subpath u v g*) ⊆ *s*
by (*meson* ‹*path g*› *order.trans pag path_image_subpath_subset u v*)

qed
finally have *homotopic_paths s* (*subpath u w g* +++ *subpath w v g*) (*subpath u v g*) **.**
then show *?thesis*
using *homotopic_join_subpaths2* **by** *blast*
qed

proposition *homotopic_join_subpaths*:
⟦*path g*; *path_image g* ⊆ *s*; *u* ∈ {*0..1*}; *v* ∈ {*0..1*}; *w* ∈ {*0..1*}⟧
⟹ *homotopic_paths s* (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)
using *le_cases3* [*of u v w*] *homotopic_join_subpaths1 homotopic_join_subpaths2 homotopic_join_subpaths3*
by *metis*

Relating homotopy of trivial loops to path-connectedness.

lemma *path_component_imp_homotopic_points*:
assumes *path_component S a b*
shows *homotopic_loops S* (*linepath a a*) (*linepath b b*)
proof −
obtain *g* :: *real* ⇒ ′*a* **where** *g*: *continuous_on* {*0..1*} *g g* ‘ {*0..1*} ⊆ *S g 0* = *a g 1* = *b*
using *assms* **by** (*auto simp*: *path_defs*)
then have *continuous_on* ({*0..1*} × {*0..1*}) (*g* ∘ *fst*)
by (*fastforce intro*!: *continuous_intros*)+
with *g* **show** *?thesis*
by (*auto simp add*: *homotopic_loops_def homotopic_with_def path_defs image_subset_iff*)
qed

lemma *homotopic_loops_imp_path_component_value*:
⟦*homotopic_loops S p q*; *0* ≤ *t*; *t* ≤ *1*⟧
⟹ *path_component S* (*p t*) (*q t*)
apply (*clarsimp simp add*: *homotopic_loops_def homotopic_with_def path_defs*)
apply (*rule_tac x*=*h* ∘ (λ*u*. (*u*, *t*)) **in** *exI*)
apply (*fastforce elim*!: *continuous_on_subset intro*!: *continuous_intros*)
done

lemma *homotopic_points_eq_path_component*:
homotopic_loops S (*linepath a a*) (*linepath b b*) ⟷ *path_component S a b*
by (*auto simp*: *path_component_imp_homotopic_points*
dest: *homotopic_loops_imp_path_component_value* [**where** *t*=*1*])

lemma *path_connected_eq_homotopic_points*:
path_connected S ⟷
(∀ *a b*. *a* ∈ *S* ∧ *b* ∈ *S* ⟶ *homotopic_loops S* (*linepath a a*) (*linepath b b*))
by (*auto simp*: *path_connected_def path_component_def homotopic_points_eq_path_component*)

6.18.10 Simply connected sets

defined as "all loops are homotopic (as loops)

```
definition simply_connected where
  simply_connected S ≡
      ∀ p q. path p ∧ pathfinish p = pathstart p ∧ path_image p ⊆ S ∧
            path q ∧ pathfinish q = pathstart q ∧ path_image q ⊆ S
            ⟶ homotopic_loops S p q

lemma simply_connected_empty [iff]: simply_connected {}
  by (simp add: simply_connected_def)

lemma simply_connected_imp_path_connected:
  fixes S :: _::real_normed_vector set
  shows simply_connected S ⟹ path_connected S
by (simp add: simply_connected_def path_connected_eq_homotopic_points)

lemma simply_connected_imp_connected:
  fixes S :: _::real_normed_vector set
  shows simply_connected S ⟹ connected S
by (simp add: path_connected_imp_connected simply_connected_imp_path_connected)

lemma simply_connected_eq_contractible_loop_any:
  fixes S :: _::real_normed_vector set
  shows simply_connected S ⟷
          (∀ p a. path p ∧ path_image p ⊆ S ∧ pathfinish p = pathstart p ∧ a ∈ S
                ⟶ homotopic_loops S p (linepath a a))
        (is ?lhs = ?rhs)
proof
  assume ?lhs then show ?rhs
    unfolding simply_connected_def by force
next
  assume ?rhs then show ?lhs
    unfolding simply_connected_def
   by (metis pathfinish_in_path_image subsetD  homotopic_loops_trans homotopic_loops_sym)
qed

lemma simply_connected_eq_contractible_loop_some:
  fixes S :: _::real_normed_vector set
  shows simply_connected S ⟷
              path_connected S ∧
              (∀ p. path p ∧ path_image p ⊆ S ∧ pathfinish p = pathstart p
                 ⟶ (∃ a. a ∈ S ∧ homotopic_loops S p (linepath a a)))
     (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
 using simply_connected_eq_contractible_loop_any by (blast intro: simply_connected_imp_path_connected)
next
  assume r: ?rhs
  note pa = r [THEN conjunct2, rule_format]
  show ?lhs
```

```
  proof (clarsimp simp add: simply_connected_eq_contractible_loop_any)
    fix p a
    assume path p and path_image p ⊆ S pathfinish p = pathstart p
      and a ∈ S
    with pa [of p] show homotopic_loops S p (linepath a a)
      using homotopic_loops_trans path_connected_eq_homotopic_points r by blast
  qed
qed

lemma simply_connected_eq_contractible_loop_all:
  fixes S :: _::real_normed_vector set
  shows simply_connected S ⟷
        S = {} ∨
        (∃ a ∈ S. ∀ p. path p ∧ path_image p ⊆ S ∧ pathfinish p = pathstart p
              ⟶ homotopic_loops S p (linepath a a))
       (is ?lhs = ?rhs)
proof (cases S = {})
  case True then show ?thesis by force
next
  case False
  then obtain a where a ∈ S by blast
  show ?thesis
  proof
    assume simply_connected S
    then show ?rhs
      using ‹a ∈ S› ‹simply_connected S› simply_connected_eq_contractible_loop_any
      by blast
  next
    assume ?rhs
    then show simply_connected S
      unfolding simply_connected_eq_contractible_loop_any
     by (meson False homotopic_loops_refl homotopic_loops_sym homotopic_loops_trans

          path_component_imp_homotopic_points path_component_refl)
  qed
qed

lemma simply_connected_eq_contractible_path:
  fixes S :: _::real_normed_vector set
  shows simply_connected S ⟷
           path_connected S ∧
           (∀ p. path p ∧ path_image p ⊆ S ∧ pathfinish p = pathstart p
            ⟶ homotopic_paths S p (linepath (pathstart p) (pathstart p)))
     (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    unfolding simply_connected_imp_path_connected
   by (metis simply_connected_eq_contractible_loop_some homotopic_loops_imp_homotopic_paths_null)
```

```
next
  assume ?rhs
  then show ?lhs
   using homotopic_paths_imp_homotopic_loops simply_connected_eq_contractible_loop_some
by fastforce
qed

lemma simply_connected_eq_homotopic_paths:
  fixes S :: _::real_normed_vector set
  shows simply_connected S ⟷
          path_connected S ∧
          (∀ p q. path p ∧ path_image p ⊆ S ∧
                path q ∧ path_image q ⊆ S ∧
                pathstart q = pathstart p ∧ pathfinish q = pathfinish p
                ⟶ homotopic_paths S p q)
         (is ?lhs = ?rhs)
proof
  assume ?lhs
  then have pc: path_connected S
       and ∗:  ⋀p. ⟦path p; path_image p ⊆ S;
                     pathfinish p = pathstart p⟧
                    ⟹ homotopic_paths S p (linepath (pathstart p) (pathstart p))
    by (auto simp: simply_connected_eq_contractible_path)
  have homotopic_paths S p q
       if path p path_image p ⊆ S path q
          path_image q ⊆ S pathstart q = pathstart p
          pathfinish q = pathfinish p for p q
  proof −
    have homotopic_paths S p (p +++ linepath (pathfinish p) (pathfinish p))
      by (simp add: homotopic_paths_rid homotopic_paths_sym that)
    also have homotopic_paths S (p +++ linepath (pathfinish p) (pathfinish p))
                                (p +++ reversepath q +++ q)
      using that
        by (metis homotopic_paths_join homotopic_paths_linv homotopic_paths_refl
homotopic_paths_sym_eq pathstart_linepath)
    also have homotopic_paths S (p +++ reversepath q +++ q)
                                ((p +++ reversepath q) +++ q)
      by (simp add: that homotopic_paths_assoc)
    also have homotopic_paths S ((p +++ reversepath q) +++ q)
                                (linepath (pathstart q) (pathstart q) +++ q)
      using ∗ [of p +++ reversepath q] that
      by (simp add: homotopic_paths_join path_image_join)
    also have homotopic_paths S (linepath (pathstart q) (pathstart q) +++ q) q
      using that homotopic_paths_lid by blast
    finally show ?thesis .
  qed
  then show ?rhs
    by (blast intro: pc ∗)
next
```

assume *?rhs*
then show *?lhs*
by (*force simp*: *simply_connected_eq_contractible_path*)
qed

proposition *simply_connected_Times*:
fixes *S* :: *'a*::*real_normed_vector set* **and** *T* :: *'b*::*real_normed_vector set*
assumes *S*: *simply_connected S* **and** *T*: *simply_connected T*
shows *simply_connected*(*S* × *T*)
proof −
have *homotopic_loops* (*S* × *T*) *p* (*linepath* (*a*, *b*) (*a*, *b*))
if *path p path_image p* ⊆ *S* × *T p 1* = *p 0 a* ∈ *S b* ∈ *T*
for *p a b*
proof −
have *path* (*fst* ∘ *p*)
by (*simp add*: *continuous_on_fst Path_Connected.path_continuous_image* [*OF* ‹*path p*›])
moreover have *path_image* (*fst* ∘ *p*) ⊆ *S*
using *that* **by** (*force simp add*: *path_image_def*)
ultimately have *p1*: *homotopic_loops S* (*fst* ∘ *p*) (*linepath a a*)
using *S that*
by (*simp add*: *simply_connected_eq_contractible_loop_any pathfinish_def pathstart_def*)
have *path* (*snd* ∘ *p*)
by (*simp add*: *continuous_on_snd Path_Connected.path_continuous_image* [*OF* ‹*path p*›])
moreover have *path_image* (*snd* ∘ *p*) ⊆ *T*
using *that* **by** (*force simp*: *path_image_def*)
ultimately have *p2*: *homotopic_loops T* (*snd* ∘ *p*) (*linepath b b*)
using *T that*
by (*simp add*: *simply_connected_eq_contractible_loop_any pathfinish_def pathstart_def*)
show *?thesis*
using *p1 p2* **unfolding** *homotopic_loops*
apply *clarify*
subgoal for *h k*
by (*rule_tac x*=λ*z*. (*h z*, *k z*) **in** *exI*) (*force intro*: *continuous_intros simp*: *path_defs*)
done
qed
with *assms* **show** *?thesis*
by (*simp add*: *simply_connected_eq_contractible_loop_any pathfinish_def pathstart_def*)
qed

6.18.11 Contractible sets

definition *contractible* **where**
contractible S ≡ ∃ *a*. *homotopic_with_canon* (λ*x*. *True*) *S S id* (λ*x*. *a*)

```
proposition contractible_imp_simply_connected:
  fixes S :: _::real_normed_vector set
  assumes contractible S shows simply_connected S
proof (cases S = {})
  case True then show ?thesis by force
next
  case False
  obtain a where a: homotopic_with_canon (λx. True) S S id (λx. a)
    using assms by (force simp: contractible_def)
  then have a ∈ S
   by (metis False homotopic_constant_maps homotopic_with_symD homotopic_with_trans
path_component_in_topspace topspace_euclidean_subtopology)
  have ∀ p. path p ∧
            path_image p ⊆ S ∧ pathfinish p = pathstart p ⟶
            homotopic_loops S p (linepath a a)
     using a apply (clarsimp simp add: homotopic_loops_def homotopic_with_def
path_defs)
    apply (rule_tac x=(h ∘ (λy. (fst y, (p ∘ snd) y))) in exI)
     apply (intro conjI continuous_on_compose continuous_intros; force elim: con-
tinuous_on_subset)
    done
  with ‹a ∈ S› show ?thesis
    by (auto simp add: simply_connected_eq_contractible_loop_all False)
qed

corollary contractible_imp_connected:
  fixes S :: _::real_normed_vector set
  shows contractible S ⟹ connected S
by (simp add: contractible_imp_simply_connected simply_connected_imp_connected)

lemma contractible_imp_path_connected:
  fixes S :: _::real_normed_vector set
  shows contractible S ⟹ path_connected S
by (simp add: contractible_imp_simply_connected simply_connected_imp_path_connected)

lemma nullhomotopic_through_contractible:
  fixes S :: _::topological_space set
  assumes f: continuous_on S f f ‘ S ⊆ T
      and g: continuous_on T g g ‘ T ⊆ U
      and T: contractible T
    obtains c where homotopic_with_canon (λh. True) S U (g ∘ f) (λx. c)
proof −
  obtain b where b: homotopic_with_canon (λx. True) T T id (λx. b)
    using assms by (force simp: contractible_def)
  have homotopic_with_canon (λf. True) T U (g ∘ id) (g ∘ (λx. b))
   by (metis Abstract_Topology.continuous_map_subtopology_eu b g homotopic_with_compose_continuous_map_left)
  then have homotopic_with_canon (λf. True) S U (g ∘ id ∘ f) (g ∘ (λx. b) ∘ f)
    by (simp add: f homotopic_with_compose_continuous_map_right)
```

then show *?thesis*
by (*simp add*: *comp_def that*)
qed

lemma *nullhomotopic_into_contractible*:
assumes *f*: *continuous_on S f f ‘ S* ⊆ *T*
and *T*: *contractible T*
obtains *c* **where** *homotopic_with_canon* (λ*h*. *True*) *S T f* (λ*x*. *c*)
by (*rule nullhomotopic_through_contractible* [*OF f*, *of id T*]) (*use assms* **in** *auto*)

lemma *nullhomotopic_from_contractible*:
assumes *f*: *continuous_on S f f ‘ S* ⊆ *T*
and *S*: *contractible S*
obtains *c* **where** *homotopic_with_canon* (λ*h*. *True*) *S T f* (λ*x*. *c*)
by (*auto simp*: *comp_def intro*: *nullhomotopic_through_contractible* [*OF continuous_on_id _ f S*])

lemma *homotopic_through_contractible*:
fixes *S* :: _::*real_normed_vector set*
assumes *continuous_on S f1 f1 ‘ S* ⊆ *T*
continuous_on T g1 g1 ‘ T ⊆ *U*
continuous_on S f2 f2 ‘ S ⊆ *T*
continuous_on T g2 g2 ‘ T ⊆ *U*
contractible T path_connected U
shows *homotopic_with_canon* (λ*h*. *True*) *S U* (*g1* ∘ *f1*) (*g2* ∘ *f2*)
proof −
obtain *c1* **where** *c1*: *homotopic_with_canon* (λ*h*. *True*) *S U* (*g1* ∘ *f1*) (λ*x*. *c1*)
by (*rule nullhomotopic_through_contractible* [*of S f1 T g1 U*]) (*use assms* **in** *auto*)
obtain *c2* **where** *c2*: *homotopic_with_canon* (λ*h*. *True*) *S U* (*g2* ∘ *f2*) (λ*x*. *c2*)
by (*rule nullhomotopic_through_contractible* [*of S f2 T g2 U*]) (*use assms* **in** *auto*)
have *S* = {} ∨ (∃ *t*. *path_connected t* ∧ *t* ⊆ *U* ∧ *c2* ∈ *t* ∧ *c1* ∈ *t*)
proof (*cases S* = {})
case *True* **then show** *?thesis* **by** *force*
next
case *False*
with *c1 c2* **have** *c1* ∈ *U c2* ∈ *U*
using *homotopic_with_imp_continuous_maps* **by** *fastforce+*
with ‹*path_connected U*› **show** *?thesis* **by** *blast*
qed
then have *homotopic_with_canon* (λ*h*. *True*) *S U* (λ*x*. *c2*) (λ*x*. *c1*)
by (*simp add*: *path_component homotopic_constant_maps*)
then show *?thesis*
using *c1 c2 homotopic_with_symD homotopic_with_trans* **by** *blast*
qed

lemma *homotopic_into_contractible*:
fixes *S* :: ′*a*::*real_normed_vector set* **and** *T*:: ′*b*::*real_normed_vector set*

assumes *f*: *continuous_on S f f ' S ⊆ T*
and *g*: *continuous_on S g g ' S ⊆ T*
and *T*: *contractible T*
shows *homotopic_with_canon* (λ*h*. *True*) *S T f g*
using *homotopic_through_contractible* [*of S f T id T g id*]
by (*simp add*: *assms contractible_imp_path_connected*)

lemma *homotopic_from_contractible*:
fixes *S* :: '*a*::*real_normed_vector set* **and** *T*:: '*b*::*real_normed_vector set*
assumes *f*: *continuous_on S f f ' S ⊆ T*
and *g*: *continuous_on S g g ' S ⊆ T*
and *contractible S path_connected T*
shows *homotopic_with_canon* (λ*h*. *True*) *S T f g*
using *homotopic_through_contractible* [*of S id S f T id g*]
by (*simp add*: *assms contractible_imp_path_connected*)

6.18.12 Starlike sets

definition *starlike S* ⟷ (∃ *a*∈*S*. ∀ *x*∈*S*. *closed_segment a x* ⊆ *S*)

lemma *starlike_UNIV* [*simp*]: *starlike UNIV*
by (*simp add*: *starlike_def*)

lemma *convex_imp_starlike*:
convex S ⟹ *S* ≠ {} ⟹ *starlike S*
unfolding *convex_contains_segment starlike_def* **by** *auto*

lemma *starlike_convex_tweak_boundary_points*:
fixes *S* :: '*a*::*euclidean_space set*
assumes *convex S S* ≠ {} **and** *ST*: *rel_interior S* ⊆ *T* **and** *TS*: *T* ⊆ *closure S*
shows *starlike T*
proof −
have *rel_interior S* ≠ {}
by (*simp add*: *assms rel_interior_eq_empty*)
then obtain *a* **where** *a*: *a* ∈ *rel_interior S* **by** *blast*
with *ST* **have** *a* ∈ *T* **by** *blast*
have ⋀*x*. *x* ∈ *T* ⟹ *open_segment a x* ⊆ *rel_interior S*
by (*rule rel_interior_closure_convex_segment* [*OF* ‹*convex S*› *a*]) (*use assms* **in** *auto*)
then have ∀ *x*∈*T*. *a* ∈ *T* ∧ *open_segment a x* ⊆ *T*
using *ST* **by** (*blast intro*: *a* ‹*a* ∈ *T*› *rel_interior_closure_convex_segment* [*OF* ‹*convex S*› *a*])
then show *?thesis*
unfolding *starlike_def* **using** *bexI* [*OF* _ ‹*a* ∈ *T*›]
by (*simp add*: *closed_segment_eq_open*)
qed

lemma *starlike_imp_contractible_gen*:
fixes *S* :: '*a*::*real_normed_vector set*

```
  assumes S: starlike S
    and P: ⋀a T. ⟦a ∈ S; 0 ≤ T; T ≤ 1⟧ ⟹ P(λx. (1 − T) *R x + T *R a)
   obtains a where homotopic_with_canon P S S (λx. x) (λx. a)
proof −
  obtain a where a ∈ S and a: ⋀x. x ∈ S ⟹ closed_segment a x ⊆ S
   using S by (auto simp: starlike_def)
  have ⋀t b. 0 ≤ t ∧ t ≤ 1 ⟹
           ∃u. (1 − t) *R b + t *R a = (1 − u) *R a + u *R b ∧ 0 ≤ u ∧ u ≤ 1
   by (metis add_diff_cancel_right' diff_ge_0_iff_ge le_add_diff_inverse pth_c(1))
  then have (λy. (1 − fst y) *R snd y + fst y *R a) ` ({0..1} × S) ⊆ S
   using a [unfolded closed_segment_def] by force
  then have homotopic_with_canon P S S (λx. x) (λx. a)
   using ‹a ∈ S›
   unfolding homotopic_with_def
   apply (rule_tac x=λy. (1 − (fst y)) *R snd y + (fst y) *R a in exI)
   apply (force simp add: P intro: continuous_intros)
   done
  then show ?thesis
   using that by blast
qed

lemma starlike_imp_contractible:
  fixes S :: 'a::real_normed_vector set
  shows starlike S ⟹ contractible S
using starlike_imp_contractible_gen contractible_def by (fastforce simp: id_def)

lemma contractible_UNIV [simp]: contractible (UNIV :: 'a::real_normed_vector
set)
  by (simp add: starlike_imp_contractible)

lemma starlike_imp_simply_connected:
  fixes S :: 'a::real_normed_vector set
  shows starlike S ⟹ simply_connected S
by (simp add: contractible_imp_simply_connected starlike_imp_contractible)

lemma convex_imp_simply_connected:
  fixes S :: 'a::real_normed_vector set
  shows convex S ⟹ simply_connected S
using convex_imp_starlike starlike_imp_simply_connected by blast

lemma starlike_imp_path_connected:
  fixes S :: 'a::real_normed_vector set
  shows starlike S ⟹ path_connected S
by (simp add: simply_connected_imp_path_connected starlike_imp_simply_connected)

lemma starlike_imp_connected:
  fixes S :: 'a::real_normed_vector set
  shows starlike S ⟹ connected S
by (simp add: path_connected_imp_connected starlike_imp_path_connected)
```

```
lemma is_interval_simply_connected_1:
  fixes S :: real set
  shows is_interval S ⟷ simply_connected S
using convex_imp_simply_connected is_interval_convex_1 is_interval_path_connected_1
simply_connected_imp_path_connected by auto

lemma contractible_empty [simp]: contractible {}
  by (simp add: contractible_def homotopic_on_emptyI)

lemma contractible_convex_tweak_boundary_points:
  fixes S :: 'a::euclidean_space set
  assumes convex S and TS: rel_interior S ⊆ T T ⊆ closure S
  shows contractible T
proof (cases S = {})
  case True
  with assms show ?thesis
    by (simp add: subsetCE)
next
  case False
  show ?thesis
   by (meson False assms starlike_convex_tweak_boundary_points starlike_imp_contractible)
qed

lemma convex_imp_contractible:
  fixes S :: 'a::real_normed_vector set
  shows convex S ⟹ contractible S
  using contractible_empty convex_imp_starlike starlike_imp_contractible by blast

lemma contractible_sing [simp]:
  fixes a :: 'a::real_normed_vector
  shows contractible {a}
by (rule convex_imp_contractible [OF convex_singleton])

lemma is_interval_contractible_1:
  fixes S :: real set
  shows  is_interval S ⟷ contractible S
using contractible_imp_simply_connected convex_imp_contractible is_interval_convex_1
    is_interval_simply_connected_1 by auto

lemma contractible_Times:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S: contractible S and T: contractible T
  shows contractible (S × T)
proof -
  obtain a h where conth: continuous_on ({0..1} × S) h
           and hsub: h ` ({0..1} × S) ⊆ S
           and [simp]: ⋀x. x ∈ S ⟹ h (0, x) = x
           and [simp]: ⋀x. x ∈ S ⟹  h (1::real, x) = a
```

using *S* **by** (*auto simp*: *contractible_def homotopic_with*)
obtain *b k* **where** *contk*: *continuous_on* ({*0..1*} × *T*) *k*
and *ksub*: *k* ` ({*0..1*} × *T*) ⊆ *T*
and [*simp*]: ⋀*x*. *x* ∈ *T* ⟹ *k* (*0*, *x*) = *x*
and [*simp*]: ⋀*x*. *x* ∈ *T* ⟹ *k* (*1*::*real*, *x*) = *b*
using *T* **by** (*auto simp*: *contractible_def homotopic_with*)
show *?thesis*
apply (*simp add*: *contractible_def homotopic_with*)
apply (*rule exI* [**where** *x*=*a*])
apply (*rule exI* [**where** *x*=*b*])
apply (*rule exI* [**where** *x* = λ*z*. (*h* (*fst z*, *fst*(*snd z*)), *k* (*fst z*, *snd*(*snd z*)))])
using *hsub ksub*
apply (*fastforce intro*!: *continuous_intros continuous_on_compose2* [*OF conth*] *continuous_on_compose2* [*OF contk*])
done
qed

6.18.13 Local versions of topological properties in general

definition *locally* :: (*'a*::*topological_space set* ⇒ *bool*) ⇒ *'a set* ⇒ *bool*
where
locally P S ≡
∀ *w x*. *openin* (*top_of_set S*) *w* ∧ *x* ∈ *w*
⟶ (∃ *u v*. *openin* (*top_of_set S*) *u* ∧ *P v* ∧ *x* ∈ *u* ∧ *u* ⊆ *v* ∧ *v* ⊆ *w*)

lemma *locallyI*:
assumes ⋀*w x*. ⟦*openin* (*top_of_set S*) *w*; *x* ∈ *w*⟧
⟹ ∃ *u v*. *openin* (*top_of_set S*) *u* ∧ *P v* ∧ *x* ∈ *u* ∧ *u* ⊆ *v* ∧ *v* ⊆ *w*
shows *locally P S*
using *assms* **by** (*force simp*: *locally_def*)

lemma *locallyE*:
assumes *locally P S openin* (*top_of_set S*) *w x* ∈ *w*
obtains *u v* **where** *openin* (*top_of_set S*) *u*
P v x ∈ *u u* ⊆ *v v* ⊆ *w*
using *assms* **unfolding** *locally_def* **by** *meson*

lemma *locally_mono*:
assumes *locally P S* ⋀*T*. *P T* ⟹ *Q T*
shows *locally Q S*
by (*metis assms locally_def*)

lemma *locally_open_subset*:
assumes *locally P S openin* (*top_of_set S*) *t*
shows *locally P t*
using *assms*
unfolding *locally_def*
by (*elim all_forward*) (*meson dual_order.trans openin_imp_subset openin_subset_trans openin_trans*)

lemma *locally_diff_closed*:
  [[*locally P S*; *closedin* (*top_of_set S*) *t*]] ⟹ *locally P* (*S* − *t*)
 using *locally_open_subset closedin_def* **by** *fastforce*

lemma *locally_empty* [*iff*]: *locally P* {}
 by (*simp add*: *locally_def openin_subtopology*)

lemma *locally_singleton* [*iff*]:
 fixes *a* :: ′*a*::*metric_space*
 shows *locally P* {*a*} ⟷ *P* {*a*}
proof −
 have ∀ *x*::*real*. ¬ *0* < *x* ⟹ *P* {*a*}
  using *zero_less_one* **by** *blast*
 then show *?thesis*
  unfolding *locally_def*
  by (*auto simp add*: *openin_euclidean_subtopology_iff subset_singleton_iff conj_disj_distribR*)
qed

lemma *locally_iff*:
  locally P S ⟷
  (∀ *T x*. *open T* ∧ *x* ∈ *S* ∩ *T* ⟶ (∃ *U*. *open U* ∧ (∃ *V*. *P V* ∧ *x* ∈ *S* ∩ *U* ∧ *S* ∩ *U* ⊆ *V* ∧ *V* ⊆ *S* ∩ *T*)))
 apply (*simp add*: *le_inf_iff locally_def openin_open*, *safe*)
  apply (*metis IntE IntI le_inf_iff*)
 apply (*metis IntI Int_subset_iff*)
 done

lemma *locally_Int*:
 assumes *S*: *locally P S* **and** *T*: *locally P T*
  and *P*: ⋀*S T*. *P S* ∧ *P T* ⟹ *P*(*S* ∩ *T*)
 shows *locally P* (*S* ∩ *T*)
 unfolding *locally_iff*
proof *clarify*
 fix *A x*
 assume *open A x* ∈ *A x* ∈ *S x* ∈ *T*
 then obtain *U1 V1 U2 V2*
  where *open U1 P V1 x* ∈ *S* ∩ *U1 S* ∩ *U1* ⊆ *V1* ∧ *V1* ⊆ *S* ∩ *A*
   open U2 P V2 x ∈ *T* ∩ *U2 T* ∩ *U2* ⊆ *V2* ∧ *V2* ⊆ *T* ∩ *A*
  using *S T* **unfolding** *locally_iff* **by** (*meson IntI*)
 then have *S* ∩ *T* ∩ (*U1* ∩ *U2*) ⊆ *V1* ∩ *V2 V1* ∩ *V2* ⊆ *S* ∩ *T* ∩ *A x* ∈ *S* ∩ *T* ∩ (*U1* ∩ *U2*)
  by *blast+*
 moreover have *P* (*V1* ∩ *V2*)
  by (*simp add*: *P* ‹*P V1*› ‹*P V2*›)
 ultimately show ∃ *U*. *open U* ∧ (∃ *V*. *P V* ∧ *x* ∈ *S* ∩ *T* ∩ *U* ∧ *S* ∩ *T* ∩ *U* ⊆ *V* ∧ *V* ⊆ *S* ∩ *T* ∩ *A*)
  using ‹*open U1*› ‹*open U2*› **by** *blast*
qed

lemma *locally_Times*:
  fixes S :: ('a::*metric_space*) *set* **and** T :: ('b::*metric_space*) *set*
  assumes *PS*: *locally P S* **and** *QT*: *locally Q T* **and** *R*: $\bigwedge S\ T.\ P\ S \wedge Q\ T \Longrightarrow R(S \times T)$
  shows *locally R* $(S \times T)$
    unfolding *locally_def*
proof (*clarify*)
  fix *W x y*
  assume *W*: *openin* (*top_of_set* $(S \times T)$) *W* **and** *xy*: $(x, y) \in W$
  then obtain *U V* **where** *openin* (*top_of_set S*) *U* $x \in U$
      openin (*top_of_set T*) *V* $y \in V$ $U \times V \subseteq W$
    using *Times_in_interior_subtopology* **by** *metis*
  then obtain *U1 U2 V1 V2*
      where *opeS*: *openin* (*top_of_set S*) *U1* $\wedge$ *P U2* $\wedge$ $x \in U1 \wedge U1 \subseteq U2 \wedge U2 \subseteq U$
      and *opeT*: *openin* (*top_of_set T*) *V1* $\wedge$ *Q V2* $\wedge$ $y \in V1 \wedge V1 \subseteq V2 \wedge V2 \subseteq V$
    by (*meson PS QT locallyE*)
  then have *openin* (*top_of_set* $(S \times T)$) $(U1 \times V1)$
    by (*simp add*: *openin_Times*)
  moreover have *R* $(U2 \times V2)$
    by (*simp add*: *R opeS opeT*)
  moreover have $U1 \times V1 \subseteq U2 \times V2 \wedge U2 \times V2 \subseteq W$
    using *opeS opeT* ‹$U \times V \subseteq W$› **by** *auto*
  ultimately show $\exists\ U\ V.$ *openin* (*top_of_set* $(S \times T)$) *U* $\wedge$ *R V* $\wedge$ $(x,y) \in U \wedge U \subseteq V \wedge V \subseteq W$
    using *opeS opeT* **by** *auto*
qed

proposition *homeomorphism_locally_imp*:
  fixes S :: 'a::*metric_space set* **and** T :: 'b::*t2_space set*
  assumes *S*: *locally P S* **and** *hom*: *homeomorphism S T f g*
    and *Q*: $\bigwedge S\ S'.\ [\![P\ S;\ homeomorphism\ S\ S'\ f\ g]\!] \Longrightarrow Q\ S'$
    shows *locally Q T*
proof (*clarsimp simp*: *locally_def*)
  fix *W y*
  assume $y \in W$ **and** *openin* (*top_of_set T*) *W*
  then obtain *A* **where** *T*: *open A* $W = T \cap A$
    by (*force simp*: *openin_open*)
  then have $W \subseteq T$ **by** *auto*
  have *f*: $\bigwedge x.\ x \in S \Longrightarrow g(f\ x) = x$ *f* ‘ $S = T$ *continuous_on S f*
   and *g*: $\bigwedge y.\ y \in T \Longrightarrow f(g\ y) = y$ *g* ‘ $T = S$ *continuous_on T g*
    using *hom* **by** (*auto simp*: *homeomorphism_def*)
  have *gw*: *g* ‘ $W = S \cap f$ −‘ *W*
    using ‹$W \subseteq T$› *g* **by** *force*
  have ∘: *openin* (*top_of_set S*) (*g* ‘ *W*)

```
  proof –
    have continuous_on S f
      using f(3) by blast
    then show openin (top_of_set S) (g ‘ W)
      by (simp add: gw Collect_conj_eq ‹openin (top_of_set T) W› continuous_on_open
f(2))
  qed
  then obtain U V
    where osu: openin (top_of_set S) U and uv: P V g y ∈ U U ⊆ V V ⊆ g ‘ W
    using S [unfolded locally_def, rule_format, of g ‘ W g y] ‹y ∈ W› by force
  have V ⊆ S using uv by (simp add: gw)
  have fv: f ‘ V = T ∩ {x. g x ∈ V}
    using ‹f ‘ S = T› f ‹V ⊆ S› by auto
  have f ‘ V ⊆ W
   using uv using Int_lower2 gw image_subsetI mem_Collect_eq subset_iff by auto
  have contvf: continuous_on V f
    using ‹V ⊆ S› continuous_on_subset f(3) by blast
  have contvg: continuous_on (f ‘ V) g
    using ‹f ‘ V ⊆ W› ‹W ⊆ T› continuous_on_subset [OF g(3)] by blast
  have V ⊆ g ‘ f ‘ V
     by (metis ‹V ⊆ S› hom homeomorphism_def homeomorphism_of_subsets or-
der_refl)
  then have homv: homeomorphism V (f ‘ V) f g
    using ‹V ⊆ S› f by (auto simp add: homeomorphism_def contvf contvg)
  have openin (top_of_set (g ‘ T)) U
    using ‹g ‘ T = S› by (simp add: osu)
  then have 1: openin (top_of_set T) (T ∩ g –‘ U)
    using ‹continuous_on T g› continuous_on_open [THEN iffD1] by blast
  have 2: ∃ V. Q V ∧ y ∈ (T ∩ g –‘ U) ∧ (T ∩ g –‘ U) ⊆ V ∧ V ⊆ W
  proof (intro exI conjI)
    show Q (f ‘ V)
      using Q homv ‹P V› by blast
    show y ∈ T ∩ g –‘ U
      using T(2) ‹y ∈ W› ‹g y ∈ U› by blast
    show T ∩ g –‘ U ⊆ f ‘ V
      using g(1) image_iff uv(3) by fastforce
    show f ‘ V ⊆ W
      using ‹f ‘ V ⊆ W› by blast
  qed
  show ∃ U. openin (top_of_set T) U ∧ (∃ v. Q v ∧ y ∈ U ∧ U ⊆ v ∧ v ⊆ W)
    by (meson 1 2)
qed

lemma homeomorphism_locally:
  fixes f:: 'a::metric_space ⇒ 'b::metric_space
  assumes hom: homeomorphism S T f g
      and eq: ⋀S T. homeomorphism S T f g ⟹ (P S ⟷ Q T)
    shows locally P S ⟷ locally Q T
     (is ?lhs = ?rhs)
```

proof
  assume *?lhs*
  then show *?rhs*
    using *eq hom homeomorphism_locally_imp* **by** *blast*
next
  assume *?rhs*
  then show *?lhs*
    using *eq homeomorphism_sym homeomorphism_symD* [*OF hom*]
    by (*blast intro*: *homeomorphism_locally_imp*)
qed

lemma *homeomorphic_locally*:
  fixes *S*:: *'a*::*metric_space set* **and** *T*:: *'b*::*metric_space set*
  assumes *hom*: *S homeomorphic T*
        and *iff*: $\bigwedge X\ Y.\ X$ *homeomorphic* $Y \Longrightarrow (P\ X \longleftrightarrow Q\ Y)$
    shows *locally P S* $\longleftrightarrow$ *locally Q T*
proof −
  obtain *f g* **where** *hom*: *homeomorphism S T f g*
    using *assms* **by** (*force simp*: *homeomorphic_def*)
  then show *?thesis*
    using *homeomorphic_def local.iff*
    by (*blast intro*!: *homeomorphism_locally*)
qed

lemma *homeomorphic_local_compactness*:
  fixes *S*:: *'a*::*metric_space set* **and** *T*:: *'b*::*metric_space set*
  shows *S homeomorphic T* $\Longrightarrow$ *locally compact S* $\longleftrightarrow$ *locally compact T*
by (*simp add*: *homeomorphic_compactness homeomorphic_locally*)

lemma *locally_translation*:
  fixes *P* :: *'a* :: *real_normed_vector set* ⇒ *bool*
  shows ($\bigwedge S.\ P\ ((+)\ a\ `\ S) = P\ S) \Longrightarrow$ *locally P* $((+)\ a\ `\ S) =$ *locally P S*
  using *homeomorphism_locally* [*OF homeomorphism_translation*]
  by (*metis* (*full_types*) *homeomorphism_image2*)

lemma *locally_injective_linear_image*:
  fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
  assumes *f*: *linear f inj f* **and** *iff*: $\bigwedge S.\ P\ (f\ `\ S) \longleftrightarrow Q\ S$
  shows *locally P* $(f\ `\ S) \longleftrightarrow$ *locally Q S*
  using *homeomorphism_locally* [*of f'S _ _ f*] *linear_homeomorphism_image* [*OF f*]
  by (*metis* (*no_types*, *lifting*) *homeomorphism_image2 iff*)

lemma *locally_open_map_image*:
  fixes *f* :: *'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector*
  assumes *P*: *locally P S*
      and *f*: *continuous_on S f*
      and *oo*: $\bigwedge T.$ *openin* (*top_of_set S*) $T \Longrightarrow$ *openin* (*top_of_set* $(f\ `\ S)$) $(f\ `\ T)$
      and *Q*: $\bigwedge T.\ [\![T \subseteq S;\ P\ T]\!] \Longrightarrow Q(f\ `\ T)$
    shows *locally Q* $(f\ `\ S)$

proof (*clarsimp simp add*: *locally_def*)
  fix *W y*
  assume *oiw*: *openin* (*top_of_set* (*f* ‘ *S*)) *W* **and** $y \in W$
  then have $W \subseteq f$ ‘ *S* **by** (*simp add*: *openin_euclidean_subtopology_iff*)
  have *oivf*: *openin* (*top_of_set S*) ($S \cap f$ −‘ *W*)
    by (*rule continuous_on_open* [*THEN iffD1*, *rule_format*, *OF f oiw*])
  then obtain *x* **where** $x \in S$ $f\ x = y$
    using ‹$W \subseteq f$ ‘ *S*› ‹$y \in W$› **by** *blast*
  then obtain *U V*
    where *openin* (*top_of_set S*) *U P V* $x \in U$ $U \subseteq V$ $V \subseteq S \cap f$ −‘ *W*
    using *P* [*unfolded locally_def*, *rule_format*, *of* ($S \cap f$ −‘ *W*) *x*] *oivf* ‹$y \in W$›
    by *auto*
  then have *openin* (*top_of_set* (*f* ‘ *S*)) (*f* ‘ *U*)
    by (*simp add*: *oo*)
  then show ∃ *X*. *openin* (*top_of_set* (*f* ‘ *S*)) *X* ∧ (∃ *Y*. *Q Y* ∧ $y \in X$ ∧ $X \subseteq Y$ ∧ $Y \subseteq W$)
    using *Q* ‹*P V*› ‹$U \subseteq V$› ‹$V \subseteq S \cap f$ −‘ *W*› ‹$f\ x = y$› ‹$x \in U$› **by** *blast*
qed

6.18.14 An induction principle for connected sets

proposition *connected_induction*:
  assumes *connected S*
    and *opD*: ⋀*T a*. ⟦*openin* (*top_of_set S*) *T*; $a \in T$⟧ ⟹ ∃ *z*. $z \in T$ ∧ *P z*
    and *opI*: ⋀*a*. $a \in S$
        ⟹ ∃ *T*. *openin* (*top_of_set S*) *T* ∧ $a \in T$ ∧
            (∀ $x \in T$. ∀ $y \in T$. *P x* ∧ *P y* ∧ *Q x* ⟶ *Q y*)
    and *etc*: $a \in S$ $b \in S$ *P a P b Q a*
   shows *Q b*
proof −
  let *?A* = {*b*. ∃ *T*. *openin* (*top_of_set S*) *T* ∧ $b \in T$ ∧ (∀ *x*∈*T*. *P x* ⟶ *Q x*)}
  let *?B* = {*b*. ∃ *T*. *openin* (*top_of_set S*) *T* ∧ $b \in T$ ∧ (∀ *x*∈*T*. *P x* ⟶ ¬ *Q x*)}
  have *1*: *openin* (*top_of_set S*) *?A*
    by (*subst openin_subopen*, *blast*)
  have *2*: *openin* (*top_of_set S*) *?B*
    by (*subst openin_subopen*, *blast*)
  have §: *?A* ∩ *?B* = {}
   by (*clarsimp simp*: *set_eq_iff*) (*metis* (*no_types*, *hide_lams*) *Int_iff opD openin_Int*)
  have ∗: *S* ⊆ *?A* ∪ *?B*
    by *clarsimp* (*meson opI*)
  have *?A* = {} ∨ *?B* = {}
    using ‹*connected S*› [*unfolded connected_openin*, *simplified*, *rule_format*, *OF 1* § ∗ *2*]
    by *blast*
  then show *?thesis*
    by *clarsimp* (*meson opI etc*)
qed

lemma *connected_equivalence_relation_gen*:

assumes *connected S*
and *etc*: $a \in S$ $b \in S$ $P\ a$ $P\ b$
and *trans*: $\bigwedge x\ y\ z.\ [\![R\ x\ y;\ R\ y\ z]\!] \Longrightarrow R\ x\ z$
and *opD*: $\bigwedge T\ a.\ [\![\mathit{openin}\ (\mathit{top\_of\_set}\ S)\ T;\ a \in T]\!] \Longrightarrow \exists z.\ z \in T \wedge P\ z$
and *opI*: $\bigwedge a.\ a \in S$
$\Longrightarrow \exists T.\ \mathit{openin}\ (\mathit{top\_of\_set}\ S)\ T \wedge a \in T \wedge$
$(\forall x \in T.\ \forall y \in T.\ P\ x \wedge P\ y \longrightarrow R\ x\ y)$
shows $R\ a\ b$
proof –
have $\bigwedge a\ b\ c.\ [\![a \in S;\ P\ a;\ b \in S;\ c \in S;\ P\ b;\ P\ c;\ R\ a\ b]\!] \Longrightarrow R\ a\ c$
apply (*rule connected_induction* [*OF* ‹*connected S*› *opD*], *simp_all*)
by (*meson trans opI*)
then show *?thesis* **by** (*metis etc opI*)
qed

lemma *connected_induction_simple*:
assumes *connected S*
and *etc*: $a \in S$ $b \in S$ $P\ a$
and *opI*: $\bigwedge a.\ a \in S$
$\Longrightarrow \exists T.\ \mathit{openin}\ (\mathit{top\_of\_set}\ S)\ T \wedge a \in T \wedge$
$(\forall x \in T.\ \forall y \in T.\ P\ x \longrightarrow P\ y)$
shows $P\ b$
by (*rule connected_induction* [*OF* ‹*connected S*› _, **where** $P = \lambda x.\ \mathit{True}$])
(*use opI etc* **in** *auto*)

lemma *connected_equivalence_relation*:
assumes *connected S*
and *etc*: $a \in S$ $b \in S$
and *sym*: $\bigwedge x\ y.\ [\![R\ x\ y;\ x \in S;\ y \in S]\!] \Longrightarrow R\ y\ x$
and *trans*: $\bigwedge x\ y\ z.\ [\![R\ x\ y;\ R\ y\ z;\ x \in S;\ y \in S;\ z \in S]\!] \Longrightarrow R\ x\ z$
and *opI*: $\bigwedge a.\ a \in S \Longrightarrow \exists T.\ \mathit{openin}\ (\mathit{top\_of\_set}\ S)\ T \wedge a \in T \wedge (\forall x \in T.$
$R\ a\ x)$
shows $R\ a\ b$
proof –
have $\bigwedge a\ b\ c.\ [\![a \in S;\ b \in S;\ c \in S;\ R\ a\ b]\!] \Longrightarrow R\ a\ c$
apply (*rule connected_induction_simple* [*OF* ‹*connected S*›], *simp_all*)
by (*meson local.sym local.trans opI openin_imp_subset subsetCE*)
then show *?thesis* **by** (*metis etc opI*)
qed

lemma *locally_constant_imp_constant*:
assumes *connected S*
and *opI*: $\bigwedge a.\ a \in S$
$\Longrightarrow \exists T.\ \mathit{openin}\ (\mathit{top\_of\_set}\ S)\ T \wedge a \in T \wedge (\forall x \in T.\ f\ x = f\ a)$
shows *f constant_on S*
proof –
have $\bigwedge x\ y.\ x \in S \Longrightarrow y \in S \Longrightarrow f\ x = f\ y$
apply (*rule connected_equivalence_relation* [*OF* ‹*connected S*›], *simp_all*)
by (*metis opI*)

```
  then show ?thesis
    by (metis constant_on_def)
qed

lemma locally_constant:
  assumes connected S
  shows locally (λU. f constant_on U) S ⟷ f constant_on S (is ?lhs = ?rhs)
proof
  assume ?lhs
  then have ⋀a. a ∈ S ⟹ ∃T. openin (top_of_set S) T ∧ a ∈ T ∧ (∀x∈T. f x
= f a)
    unfolding locally_def
   by (metis (mono_tags, hide_lams) constant_on_def constant_on_subset openin_subtopology_self)
  then show ?rhs
    using assms
    by (simp add: locally_constant_imp_constant)
next
  assume ?rhs then show ?lhs
   using assms by (metis constant_on_subset locallyI openin_imp_subset order_refl)
qed
```

6.18.15 Basic properties of local compactness

```
proposition locally_compact:
  fixes s :: 'a :: metric_space set
  shows
    locally compact s ⟷
     (∀x ∈ s. ∃u v. x ∈ u ∧ u ⊆ v ∧ v ⊆ s ∧
                    openin (top_of_set s) u ∧ compact v)
     (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    by (meson locallyE openin_subtopology_self)
next
  assume r [rule_format]: ?rhs
  have *: ∃u v.
              openin (top_of_set s) u ∧
              compact v ∧ x ∈ u ∧ u ⊆ v ∧ v ⊆ s ∩ T
        if open T x ∈ s x ∈ T for x T
  proof −
    obtain u v where uv: x ∈ u u ⊆ v v ⊆ s compact v openin (top_of_set s) u
      using r [OF ‹x ∈ s›] by auto
    obtain e where e>0 and e: cball x e ⊆ T
      using open_contains_cball ‹open T› ‹x ∈ T› by blast
    show ?thesis
      apply (rule_tac x=(s ∩ ball x e) ∩ u in exI)
      apply (rule_tac x=cball x e ∩ v in exI)
      using that ‹e > 0› e uv
```

```
    apply auto
    done
  qed
  show ?lhs
    by (rule locallyI) (metis * Int_iff openin_open)
qed

lemma locally_compactE:
  fixes S :: 'a :: metric_space set
  assumes locally compact S
  obtains u v where ⋀x. x ∈ S ⟹ x ∈ u x ∧ u x ⊆ v x ∧ v x ⊆ S ∧
                            openin (top_of_set S) (u x) ∧ compact (v x)
  using assms unfolding locally_compact by metis

lemma locally_compact_alt:
  fixes S :: 'a :: heine_borel set
  shows locally compact S ⟷
        (∀x ∈ S. ∃U. x ∈ U ∧
                  openin (top_of_set S) U ∧ compact(closure U) ∧ closure U ⊆ S)
       (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
   by (meson bounded_subset closure_minimal compact_closure compact_imp_bounded

              compact_imp_closed dual_order.trans locally_compactE)
next
  assume ?rhs then show ?lhs
    by (meson closure_subset locally_compact)
qed

lemma locally_compact_Int_cball:
  fixes S :: 'a :: heine_borel set
  shows locally compact S ⟷ (∀x ∈ S. ∃e. 0 < e ∧ closed(cball x e ∩ S))
       (is ?lhs = ?rhs)
proof
  assume L: ?lhs
  then have ⋀x U V e. ⟦U ⊆ V; V ⊆ S; compact V; 0 < e; cball x e ∩ S ⊆ U⟧
      ⟹ closed (cball x e ∩ S)
   by (metis compact_Int compact_cball compact_imp_closed inf.absorb_iff2 inf.assoc
inf.orderE)
  with L show ?rhs
    by (meson locally_compactE openin_contains_cball)
next
  assume R: ?rhs
  show ?lhs unfolding locally_compact
  proof
    fix x
    assume x ∈ S
```

then obtain *e* **where** *e>0* **and** *e*: *closed* (*cball x e* ∩ *S*)
using *R* **by** *blast*
then have *compact* (*cball x e* ∩ *S*)
by (*simp add*: *bounded_Int compact_eq_bounded_closed*)
moreover have ∀ *y*∈*ball x e* ∩ *S*. ∃ε>*0*. *cball y* ε ∩ *S* ⊆ *ball x e*
by (*meson Elementary_Metric_Spaces.open_ball IntD1 le_infI1 open_contains_cball_eq*)
moreover have *openin* (*top_of_set S*) (*ball x e* ∩ *S*)
by (*simp add*: *inf_commute openin_open_Int*)
ultimately show ∃ *U V*. *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *S* ∧ *openin* (*top_of_set S*) *U* ∧ *compact V*
by (*metis Int_iff* ‹*0* < *e*› ‹*x* ∈ *S*› *ball_subset_cball centre_in_ball inf_commute inf_le1 inf_mono order_refl*)
qed
qed

lemma *locally_compact_compact*:
fixes *S* :: *'a* :: *heine_borel set*
shows *locally compact S* ⟷
(∀ *K*. *K* ⊆ *S* ∧ *compact K*
⟶ (∃ *U V*. *K* ⊆ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *S* ∧
openin (*top_of_set S*) *U* ∧ *compact V*))
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then obtain *u v* **where**
uv: ⋀*x*. *x* ∈ *S* ⟹ *x* ∈ *u x* ∧ *u x* ⊆ *v x* ∧ *v x* ⊆ *S* ∧
openin (*top_of_set S*) (*u x*) ∧ *compact* (*v x*)
by (*metis locally_compactE*)
have ∗: ∃ *U V*. *K* ⊆ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *S* ∧ *openin* (*top_of_set S*) *U* ∧ *compact V*
if *K* ⊆ *S compact K* **for** *K*
proof −
have ⋀*C*. (∀ *c*∈*C*. *openin* (*top_of_set K*) *c*) ∧ *K* ⊆ ⋃ *C* ⟹
∃ *D*⊆*C*. *finite D* ∧ *K* ⊆ ⋃ *D*
using *that* **by** (*simp add*: *compact_eq_openin_cover*)
moreover have ∀ *c* ∈ (λ*x*. *K* ∩ *u x*) ‘ *K*. *openin* (*top_of_set K*) *c*
using *that* **by** *clarify* (*metis subsetD inf.absorb_iff2 openin_subset openin_subtopology_Int_subset topspace_euclidean_subtopology uv*)
moreover have *K* ⊆ ⋃((λ*x*. *K* ∩ *u x*) ‘ *K*)
using *that* **by** *clarsimp* (*meson subsetCE uv*)
ultimately obtain *D* **where** *D* ⊆ (λ*x*. *K* ∩ *u x*) ‘ *K finite D K* ⊆ ⋃ *D*
by *metis*
then obtain *T* **where** *T*: *T* ⊆ *K finite T K* ⊆ ⋃((λ*x*. *K* ∩ *u x*) ‘ *T*)
by (*metis finite_subset_image*)
have *Tuv*: ⋃(*u* ‘ *T*) ⊆ ⋃(*v* ‘ *T*)
using *T that* **by** (*force dest*!: *uv*)
moreover
have *openin* (*top_of_set S*) (⋃ (*u* ‘ *T*))
using *T that uv* **by** *fastforce*

```
    moreover
    have compact (⋃ (v ` T))
      by (meson T compact_UN subset_eq that(1) uv)
    moreover have ⋃ (v ` T) ⊆ S
      by (metis SUP_least T(1) subset_eq that(1) uv)
    ultimately show ?thesis
      using T by auto
  qed
  show ?rhs
    by (blast intro: *)
next
  assume ?rhs
  then show ?lhs
    apply (clarsimp simp add: locally_compact)
    apply (drule_tac x={x} in spec, simp)
    done
qed

lemma open_imp_locally_compact:
  fixes S :: 'a :: heine_borel set
  assumes open S
    shows locally compact S
proof -
  have *: ∃ U V. x ∈ U ∧ U ⊆ V ∧ V ⊆ S ∧ openin (top_of_set S) U ∧ compact
V
          if x ∈ S for x
  proof -
    obtain e where e>0 and e: cball x e ⊆ S
      using open_contains_cball assms ‹x ∈ S› by blast
    have ope: openin (top_of_set S) (ball x e)
      by (meson e open_ball ball_subset_cball dual_order.trans open_subset)
    show ?thesis
    proof (intro exI conjI)
      let ?U = ball x e
      let ?V = cball x e
      show x ∈ ?U ?U ⊆ ?V ?V ⊆ S compact ?V
        using ‹e > 0› e by auto
      show openin (top_of_set S) ?U
        using ope by blast
    qed
  qed
  show ?thesis
    unfolding locally_compact by (blast intro: *)
qed

lemma closed_imp_locally_compact:
  fixes S :: 'a :: heine_borel set
  assumes closed S
    shows locally compact S
```

proof −
  have ∗: ∃ *U V*. *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *S* ∧ *openin* (*top_of_set S*) *U* ∧ *compact V*
        if *x* ∈ *S* **for** *x*
    apply (*rule_tac x* = *S* ∩ *ball x 1* **in** *exI*, *rule_tac x* = *S* ∩ *cball x 1* **in** *exI*)
    using ‹*x* ∈ *S*› *assms* **by** *auto*
  show *?thesis*
    unfolding *locally_compact* **by** (*blast intro*: ∗)
qed

lemma *locally_compact_UNIV*: *locally compact* (*UNIV* :: ′*a* :: *heine_borel set*)
  by (*simp add*: *closed_imp_locally_compact*)

lemma *locally_compact_Int*:
  fixes *S* :: ′*a* :: *t2_space set*
  shows ⟦*locally compact S*; *locally compact t*⟧ ⟹ *locally compact* (*S* ∩ *t*)
by (*simp add*: *compact_Int locally_Int*)

lemma *locally_compact_closedin*:
  fixes *S* :: ′*a* :: *heine_borel set*
  shows ⟦*closedin* (*top_of_set S*) *t*; *locally compact S*⟧
      ⟹ *locally compact t*
  unfolding *closedin_closed*
  using *closed_imp_locally_compact locally_compact_Int* **by** *blast*

lemma *locally_compact_delete*:
    fixes *S* :: ′*a* :: *t1_space set*
    shows *locally compact S* ⟹ *locally compact* (*S* − {*a*})
  by (*auto simp*: *openin_delete locally_open_subset*)

lemma *locally_closed*:
  fixes *S* :: ′*a* :: *heine_borel set*
  shows *locally closed S* ⟷ *locally compact S*
      (**is** *?lhs* = *?rhs*)
proof
  assume *?lhs*
  then show *?rhs*
    unfolding *locally_def*
    apply (*elim all_forward imp_forward asm_rl exE*)
    apply (*rule_tac x* = *u* ∩ *ball x 1* **in** *exI*)
    apply (*rule_tac x* = *v* ∩ *cball x 1* **in** *exI*)
    apply (*force intro*: *openin_trans*)
    done
next
  assume *?rhs* **then show** *?lhs*
    using *compact_eq_bounded_closed locally_mono* **by** *blast*
qed

lemma *locally_compact_openin_Un*:

```
  fixes S :: 'a::euclidean_space set
  assumes LCS: locally compact S and LCT:locally compact T
      and opS: openin (top_of_set (S ∪ T)) S
      and opT: openin (top_of_set (S ∪ T)) T
    shows locally compact (S ∪ T)
proof -
  have ∃ e>0. closed (cball x e ∩ (S ∪ T)) if x ∈ S for x
  proof -
    obtain e1 where e1 > 0 and e1: closed (cball x e1 ∩ S)
      using LCS ‹x ∈ S› unfolding locally_compact_Int_cball by blast
    moreover obtain e2 where e2 > 0 and e2: cball x e2 ∩ (S ∪ T) ⊆ S
      by (meson ‹x ∈ S› opS openin_contains_cball)
    then have cball x e2 ∩ (S ∪ T) = cball x e2 ∩ S
      by force
    ultimately have closed (cball x (min e1 e2) ∩ (S ∪ T))
        by (metis (no_types, lifting) cball_min_Int closed_Int closed_cball inf_assoc
inf_commute)
    then show ?thesis
      by (metis ‹0 < e1› ‹0 < e2› min_def)
  qed
  moreover have ∃ e>0. closed (cball x e ∩ (S ∪ T)) if x ∈ T for x
  proof -
    obtain e1 where e1 > 0 and e1: closed (cball x e1 ∩ T)
      using LCT ‹x ∈ T› unfolding locally_compact_Int_cball by blast
    moreover obtain e2 where e2 > 0 and e2: cball x e2 ∩ (S ∪ T) ⊆ T
      by (meson ‹x ∈ T› opT openin_contains_cball)
    then have cball x e2 ∩ (S ∪ T) = cball x e2 ∩ T
      by force
    moreover have closed (cball x e1 ∩ (cball x e2 ∩ T))
      by (metis closed_Int closed_cball e1 inf_left_commute)
    ultimately show ?thesis
     by (rule_tac x=min e1 e2 in exI) (simp add: ‹0 < e2› cball_min_Int inf_assoc)
  qed
  ultimately show ?thesis
    by (force simp: locally_compact_Int_cball)
qed

lemma locally_compact_closedin_Un:
  fixes S :: 'a::euclidean_space set
  assumes LCS: locally compact S and LCT:locally compact T
      and clS: closedin (top_of_set (S ∪ T)) S
      and clT: closedin (top_of_set (S ∪ T)) T
    shows locally compact (S ∪ T)
proof -
  have ∃ e>0. closed (cball x e ∩ (S ∪ T)) if x ∈ S x ∈ T for x
  proof -
    obtain e1 where e1 > 0 and e1: closed (cball x e1 ∩ S)
      using LCS ‹x ∈ S› unfolding locally_compact_Int_cball by blast
    moreover
```

```
    obtain e2 where e2 > 0 and e2: closed (cball x e2 ∩ T)
      using LCT ‹x ∈ T› unfolding locally_compact_Int_cball by blast
    moreover have closed (cball x (min e1 e2) ∩ (S ∪ T))
    proof -
      have closed (cball x e1 ∩ (cball x e2 ∩ S))
        by (metis closed_Int closed_cball e1 inf_left_commute)
      then show ?thesis
       by (simp add: Int_Un_distrib cball_min_Int closed_Int closed_Un e2 inf_assoc)
    qed
    ultimately show ?thesis
      by (rule_tac x=min e1 e2 in exI) linarith
  qed
  moreover
  have ∃e>0. closed (cball x e ∩ (S ∪ T)) if x: x ∈ S x ∉ T for x
  proof -
    obtain e1 where e1 > 0 and e1: closed (cball x e1 ∩ S)
      using LCS ‹x ∈ S› unfolding locally_compact_Int_cball by blast
    moreover
    obtain e2 where e2>0 and cball x e2 ∩ (S ∪ T) ⊆ S − T
      using clT x by (fastforce simp: openin_contains_cball closedin_def)
    then have closed (cball x e2 ∩ T)
    proof -
      have {} = T − (T − cball x e2)
        using Diff_subset Int_Diff ‹cball x e2 ∩ (S ∪ T) ⊆ S − T› by auto
      then show ?thesis
        by (simp add: Diff_Diff_Int inf_commute)
    qed
    with e1 have closed ((cball x e1 ∩ cball x e2) ∩ (S ∪ T))
      apply (simp add: inf_commute inf_sup_distrib2)
      by (metis closed_Int closed_Un closed_cball inf_assoc inf_left_commute)
    then have closed (cball x (min e1 e2) ∩ (S ∪ T))
      by (simp add: cball_min_Int inf_commute)
    ultimately show ?thesis
      using ‹0 < e2› by (rule_tac x=min e1 e2 in exI) linarith
  qed
  moreover
  have ∃e>0. closed (cball x e ∩ (S ∪ T)) if x: x ∉ S x ∈ T for x
  proof -
    obtain e1 where e1 > 0 and e1: closed (cball x e1 ∩ T)
      using LCT ‹x ∈ T› unfolding locally_compact_Int_cball by blast
    moreover
    obtain e2 where e2>0 and cball x e2 ∩ (S ∪ T) ⊆ S ∪ T − S
      using clS x by (fastforce simp: openin_contains_cball closedin_def)
    then have closed (cball x e2 ∩ S)
     by (metis Diff_disjoint Int_empty_right closed_empty inf.left_commute inf.orderE
inf_sup_absorb)
    with e1 have closed ((cball x e1 ∩ cball x e2) ∩ (S ∪ T))
      apply (simp add: inf_commute inf_sup_distrib2)
      by (metis closed_Int closed_Un closed_cball inf_assoc inf_left_commute)
```

```
    then have closed (cball x (min e1 e2) ∩ (S ∪ T))
      by (auto simp: cball_min_Int)
    ultimately show ?thesis
      using ‹0 < e2› by (rule_tac x=min e1 e2 in exI) linarith
  qed
  ultimately show ?thesis
    by (auto simp: locally_compact_Int_cball)
qed

lemma locally_compact_Times:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  shows ⟦locally compact S; locally compact T⟧ ⟹ locally compact (S × T)
  by (auto simp: compact_Times locally_Times)

lemma locally_compact_compact_subopen:
  fixes S :: 'a :: heine_borel set
  shows
   locally compact S ⟷
    (∀ K T. K ⊆ S ∧ compact K ∧ open T ∧ K ⊆ T
       ⟶ (∃ U V. K ⊆ U ∧ U ⊆ V ∧ U ⊆ T ∧ V ⊆ S ∧
                  openin (top_of_set S) U ∧ compact V))
   (is ?lhs = ?rhs)
proof
  assume L: ?lhs
  show ?rhs
  proof clarify
    fix K :: 'a set and T :: 'a set
    assume K ⊆ S and compact K and open T and K ⊆ T
    obtain U V where K ⊆ U U ⊆ V V ⊆ S compact V
             and ope: openin (top_of_set S) U
      using L unfolding locally_compact_compact by (meson ‹K ⊆ S› ‹compact
K›)
    show ∃ U V. K ⊆ U ∧ U ⊆ V ∧ U ⊆ T ∧ V ⊆ S ∧
            openin (top_of_set S) U ∧ compact V
    proof (intro exI conjI)
      show K ⊆ U ∩ T
        by (simp add: ‹K ⊆ T› ‹K ⊆ U›)
      show U ∩ T ⊆ closure(U ∩ T)
        by (rule closure_subset)
      show closure (U ∩ T) ⊆ S
        by (metis ‹U ⊆ V› ‹V ⊆ S› ‹compact V› closure_closed closure_mono
compact_imp_closed inf.cobounded1 subset_trans)
      show openin (top_of_set S) (U ∩ T)
        by (simp add: ‹open T› ope openin_Int_open)
      show compact (closure (U ∩ T))
       by (meson Int_lower1 ‹U ⊆ V› ‹compact V› bounded_subset compact_closure
compact_eq_bounded_closed)
    qed auto
  qed
```

```
next
  assume ?rhs then show ?lhs
    unfolding locally_compact_compact
   by (metis open_openin openin_topspace subtopology_superset top.extremum topspace_euclidean_subtopology)
qed
```

6.18.16 Sura-Bura's results about compact components of sets

```
proposition Sura_Bura_compact:
  fixes S :: 'a::euclidean_space set
  assumes compact S and C: C ∈ components S
  shows C = ⋂{T. C ⊆ T ∧ openin (top_of_set S) T ∧
                           closedin (top_of_set S) T}
         (is C = ⋂ ?𝒯)
proof
  obtain x where x: C = connected_component_set S x and x ∈ S
    using C by (auto simp: components_def)
  have C ⊆ S
    by (simp add: C in_components_subset)
  have ⋂ ?𝒯 ⊆ connected_component_set S x
  proof (rule connected_component_maximal)
    have x ∈ C
      by (simp add: ‹x ∈ S› x)
    then show x ∈ ⋂ ?𝒯
      by blast
    have clo: closed (⋂ ?𝒯)
     by (simp add: ‹compact S› closed_Inter closedin_compact_eq compact_imp_closed)
    have False
      if K1: closedin (top_of_set (⋂ ?𝒯)) K1 and
         K2: closedin (top_of_set (⋂ ?𝒯)) K2 and
          K12_Int: K1 ∩ K2 = {} and K12_Un: K1 ∪ K2 = ⋂ ?𝒯 and K1 ≠ {}
K2 ≠ {}
        for K1 K2
    proof −
      have closed K1 closed K2
        using closedin_closed_trans clo K1 K2 by blast+
     then obtain V1 V2 where open V1 open V2 K1 ⊆ V1 K2 ⊆ V2 and V12:
V1 ∩ V2 = {}
        using separation_normal ‹K1 ∩ K2 = {}› by metis
      have SV12_ne: (S − (V1 ∪ V2)) ∩ (⋂ ?𝒯) ≠ {}
      proof (rule compact_imp_fip)
        show compact (S − (V1 ∪ V2))
          by (simp add: ‹open V1› ‹open V2› ‹compact S› compact_diff open_Un)
        show clo𝒯: closed T if T ∈ ?𝒯 for T
          using that ‹compact S›
          by (force intro: closedin_closed_trans simp add: compact_imp_closed)
        show (S − (V1 ∪ V2)) ∩ ⋂ℱ ≠ {} if finite ℱ and ℱ: ℱ ⊆ ?𝒯 for ℱ
        proof
```

assume *djo*: $(S - (V1 \cup V2)) \cap \bigcap \mathcal{F} = \{\}$
obtain *D* **where** *opeD*: *openin* (*top_of_set S*) *D*
and *cloD*: *closedin* (*top_of_set S*) *D*
and $C \subseteq D$ **and** *DV12*: $D \subseteq V1 \cup V2$
proof (*cases* $\mathcal{F} = \{\}$)
case *True*
with ‹$C \subseteq S$› *djo that* **show** *?thesis*
by *force*
next
case *False* **show** *?thesis*
proof
show *ope*: *openin* (*top_of_set S*) ($\bigcap \mathcal{F}$)
using *openin_Inter* ‹*finite* $\mathcal{F}$› *False* $\mathcal{F}$ **by** *blast*
then show *closedin* (*top_of_set S*) ($\bigcap \mathcal{F}$)
by (*meson clo*$\mathcal{T}$ $\mathcal{F}$ *closed_Inter closed_subset openin_imp_subset subset_eq*)
show $C \subseteq \bigcap \mathcal{F}$
using $\mathcal{F}$ **by** *auto*
show $\bigcap \mathcal{F} \subseteq V1 \cup V2$
using *ope djo openin_imp_subset* **by** *fastforce*
qed
qed
have *connected C*
by (*simp add*: *x*)
have *closed D*
using ‹*compact S*› *cloD closedin_closed_trans compact_imp_closed* **by** *blast*
have *cloV1*: *closedin* (*top_of_set D*) ($D \cap$ *closure V1*)
and *cloV2*: *closedin* (*top_of_set D*) ($D \cap$ *closure V2*)
by (*simp_all add*: *closedin_closed_Int*)
moreover have $D \cap$ *closure* $V1 = D \cap V1$ $D \cap$ *closure* $V2 = D \cap V2$
using ‹$D \subseteq V1 \cup V2$› ‹*open V1*› ‹*open V2*› *V12*
by (*auto simp add*: *closure_subset* [*THEN subsetD*] *closure_iff_nhds_not_empty*, *blast+*)
ultimately have *cloDV1*: *closedin* (*top_of_set D*) ($D \cap V1$)
and *cloDV2*: *closedin* (*top_of_set D*) ($D \cap V2$)
by *metis+*
then obtain *U1 U2* **where** *closed U1 closed U2*
and *D1*: $D \cap V1 = D \cap U1$ **and** *D2*: $D \cap V2 = D \cap U2$
by (*auto simp*: *closedin_closed*)
have $D \cap U1 \cap C \neq \{\}$
proof
assume $D \cap U1 \cap C = \{\}$
then have $*$: $C \subseteq D \cap V2$
using *D1 DV12* ‹$C \subseteq D$› **by** *auto*
have *1*: *openin* (*top_of_set S*) ($D \cap V2$)
by (*simp add*: ‹*open V2*› *opeD openin_Int_open*)
have *2*: *closedin* (*top_of_set S*) ($D \cap V2$)
using *cloD cloDV2 closedin_trans* **by** *blast*
have $\bigcap$ *?*$\mathcal{T} \subseteq D \cap V2$

```
          by (rule Inter_lower) (use * 1 2 in simp)
        then show False
          using K1 V12 ‹K1 ≠ {}› ‹K1 ⊆ V1› closedin_imp_subset by blast
      qed
      moreover have D ∩ U2 ∩ C ≠ {}
      proof
        assume D ∩ U2 ∩ C = {}
        then have *: C ⊆ D ∩ V1
          using D2 DV12 ‹C ⊆ D› by auto
        have 1: openin (top_of_set S) (D ∩ V1)
          by (simp add: ‹open V1› opeD openin_Int_open)
        have 2: closedin (top_of_set S) (D ∩ V1)
          using cloD cloDV1 closedin_trans by blast
        have ⋂ ?𝒯 ⊆ D ∩ V1
          by (rule Inter_lower) (use * 1 2 in simp)
        then show False
          using K2 V12 ‹K2 ≠ {}› ‹K2 ⊆ V2› closedin_imp_subset by blast
      qed
      ultimately show False
        using ‹connected C› [unfolded connected_closed, simplified, rule_format,
of concl: D ∩ U1 D ∩ U2]
        using ‹C ⊆ D› D1 D2 V12 DV12 ‹closed U1› ‹closed U2› ‹closed D›
        by blast
    qed
  qed
  show False
    by (metis (full_types) DiffE UnE Un_upper2 SV12_ne ‹K1 ⊆ V1› ‹K2 ⊆
V2› disjoint_iff_not_equal subsetCE sup_ge1 K12_Un)
 qed
 then show connected (⋂ ?𝒯)
   by (auto simp: connected_closedin_eq)
 show ⋂ ?𝒯 ⊆ S
   by (fastforce simp: C in_components_subset)
qed
with x show ⋂ ?𝒯 ⊆ C by simp
qed auto


corollary Sura_Bura_clopen_subset:
 fixes S :: 'a::euclidean_space set
 assumes S: locally compact S and C: C ∈ components S and compact C
   and U: open U C ⊆ U
 obtains K where openin (top_of_set S) K compact K C ⊆ K K ⊆ U
proof (rule ccontr)
 assume ¬ thesis
 with that have neg: ∄K. openin (top_of_set S) K ∧ compact K ∧ C ⊆ K ∧ K
⊆ U
   by metis
 obtain V K where C ⊆ V V ⊆ U V ⊆ K K ⊆ S compact K
```

and *opeSV*: *openin* (*top_of_set S*) *V*
using *S U* ‹*compact C*› **by** (*meson C in_components_subset locally_compact_compact_subopen*)
let *?𝒯* = {*T*. *C* ⊆ *T* ∧ *openin* (*top_of_set K*) *T* ∧ *compact T* ∧ *T* ⊆ *K*}
have *CK*: *C* ∈ *components K*
by (*meson C* ‹*C* ⊆ *V*› ‹*K* ⊆ *S*› ‹*V* ⊆ *K*› *components_intermediate_subset subset_trans*)
with ‹*compact K*›
have *C* = ⋂{*T*. *C* ⊆ *T* ∧ *openin* (*top_of_set K*) *T* ∧ *closedin* (*top_of_set K*) *T*}
by (*simp add*: *Sura_Bura_compact*)
then have *Ceq*: *C* = ⋂ *?𝒯*
by (*simp add*: *closedin_compact_eq* ‹*compact K*›)
obtain *W* **where** *open W* **and** *W*: *V* = *S* ∩ *W*
using *opeSV* **by** (*auto simp*: *openin_open*)
have −(*U* ∩ *W*) ∩ ⋂ *?𝒯* ≠ {}
proof (*rule closed_imp_fip_compact*)
show − (*U* ∩ *W*) ∩ ⋂*ℱ* ≠ {}
if *finite ℱ* **and** *ℱ*: *ℱ* ⊆ *?𝒯* **for** *ℱ*
proof (*cases ℱ* = {})
case *True*
have *False* **if** *U* = *UNIV W* = *UNIV*
proof −
have *V* = *S*
by (*simp add*: *W* ‹*W* = *UNIV*›)
with *neg* **show** *False*
using ‹*C* ⊆ *V*› ‹*K* ⊆ *S*› ‹*V* ⊆ *K*› ‹*V* ⊆ *U*› ‹*compact K*› **by** *auto*
qed
with *True* **show** *?thesis*
by *auto*
next
case *False*
show *?thesis*
proof
assume − (*U* ∩ *W*) ∩ ⋂*ℱ* = {}
then have *FUW*: ⋂*ℱ* ⊆ *U* ∩ *W*
by *blast*
have *C* ⊆ ⋂*ℱ*
using *ℱ* **by** *auto*
moreover have *compact* (⋂*ℱ*)
by (*metis* (*no_types*, *lifting*) *compact_Inter False mem_Collect_eq subsetCE ℱ*)
moreover have ⋂*ℱ* ⊆ *K*
using *False that*(*2*) **by** *fastforce*
moreover have *opeKF*: *openin* (*top_of_set K*) (⋂*ℱ*)
using *False ℱ* ‹*finite ℱ*› **by** *blast*
then have *opeVF*: *openin* (*top_of_set V*) (⋂*ℱ*)
using *W* ‹*K* ⊆ *S*› ‹*V* ⊆ *K*› *opeKF* ‹⋂*ℱ* ⊆ *K*› *FUW openin_subset_trans* **by** *fastforce*
then have *openin* (*top_of_set S*) (⋂*ℱ*)

```
        by (metis opeSV openin_trans)
      moreover have ⋂ℱ ⊆ U
        by (meson ‹V ⊆ U› opeVF dual_order.trans openin_imp_subset)
      ultimately show False
        using neg by blast
    qed
  qed
 qed (use ‹open W› ‹open U› in auto)
 with W Ceq ‹C ⊆ V› ‹C ⊆ U› show False
   by auto
qed


corollary Sura_Bura_clopen_subset_alt:
 fixes S :: 'a::euclidean_space set
 assumes S: locally compact S and C: C ∈ components S and compact C
    and opeSU: openin (top_of_set S) U and C ⊆ U
 obtains K where openin (top_of_set S) K compact K C ⊆ K K ⊆ U
proof –
 obtain V where open V U = S ∩ V
   using opeSU by (auto simp: openin_open)
 with ‹C ⊆ U› have C ⊆ V
   by auto
 then show ?thesis
   using Sura_Bura_clopen_subset [OF S C ‹compact C› ‹open V›]
   by (metis ‹U = S ∩ V› inf.bounded_iff openin_imp_subset that)
qed

corollary Sura_Bura:
 fixes S :: 'a::euclidean_space set
 assumes locally compact S C ∈ components S compact C
 shows C = ⋂ {K. C ⊆ K ∧ compact K ∧ openin (top_of_set S) K}
       (is C = ?rhs)
proof
 show ?rhs ⊆ C
 proof (clarsimp, rule ccontr)
   fix x
   assume *: ∀X. C ⊆ X ∧ compact X ∧ openin (top_of_set S) X ⟶ x ∈ X
     and x ∉ C
   obtain U V where open U open V {x} ⊆ U C ⊆ V U ∩ V = {}
     using separation_normal [of {x} C]
       by (metis Int_empty_left ‹x ∉ C› ‹compact C› closed_empty closed_insert
compact_imp_closed insert_disjoint(1))
   have x ∉ V
     using ‹U ∩ V = {}› ‹{x} ⊆ U› by blast
   then show False
     by (meson * Sura_Bura_clopen_subset ‹C ⊆ V› ‹open V› assms(1) assms(2)
assms(3) subsetCE)
 qed
```

qed *blast*

6.18.17 Special cases of local connectedness and path connectedness

lemma *locally_connected_1*:
assumes
$\bigwedge V\ x.\ [\![openin\ (top\_of\_set\ S)\ V;\ x \in V]\!] \Longrightarrow \exists\, U.\ openin\ (top\_of\_set\ S)\ U \land connected\ U \land x \in U \land U \subseteq V$
shows *locally connected S*
by (*metis assms locally_def*)

lemma *locally_connected_2*:
assumes *locally connected S*
openin (*top_of_set S*) *t*
$x \in t$
shows *openin* (*top_of_set S*) (*connected_component_set t x*)
proof −
{ **fix** *y* :: *'a*
let *?SS* = *top_of_set S*
assume *1*: *openin ?SS t*
$\forall w\ x.\ openin\ ?SS\ w \land x \in w \longrightarrow (\exists\, u.\ openin\ ?SS\ u \land (\exists\, v.\ connected\ v \land x \in u \land u \subseteq v \land v \subseteq w))$
and *connected_component t x y*
then have $y \in t$ **and** *y*: $y \in connected\_component\_set\ t\ x$
using *connected_component_subset* **by** *blast+*
obtain *F* **where**
$\forall x\ y.\ (\exists\, w.\ openin\ ?SS\ w \land (\exists\, u.\ connected\ u \land x \in w \land w \subseteq u \land u \subseteq y)) = (openin\ ?SS\ (F\ x\ y) \land (\exists\, u.\ connected\ u \land x \in F\ x\ y \land F\ x\ y \subseteq u \land u \subseteq y))$
by *moura*
then obtain *G* **where**
$\forall a\ A.\ (\exists\, U.\ openin\ ?SS\ U \land (\exists\, V.\ connected\ V \land a \in U \land U \subseteq V \land V \subseteq A)) = (openin\ ?SS\ (F\ a\ A) \land connected\ (G\ a\ A) \land a \in F\ a\ A \land F\ a\ A \subseteq G\ a\ A \land G\ a\ A \subseteq A)$
by *moura*
then have ∗: $openin\ ?SS\ (F\ y\ t) \land connected\ (G\ y\ t) \land y \in F\ y\ t \land F\ y\ t \subseteq G\ y\ t \land G\ y\ t \subseteq t$
using *1* ‹$y \in t$› **by** *presburger*
have $G\ y\ t \subseteq connected\_component\_set\ t\ y$
by (*metis* (*no_types*) ∗ *connected_component_eq_self connected_component_mono contra_subsetD*)
then have $\exists\, A.\ openin\ ?SS\ A \land y \in A \land A \subseteq connected\_component\_set\ t\ x$
by (*metis* (*no_types*) ∗ *connected_component_eq dual_order.trans y*)
}
then show *?thesis*
using *assms openin_subopen* **by** (*force simp*: *locally_def*)
qed

lemma *locally_connected_3*:

assumes $\bigwedge t\ x.\ [\![openin\ (top\_of\_set\ S)\ t;\ x \in t]\!]$
$\Longrightarrow openin\ (top\_of\_set\ S)$
$(connected\_component\_set\ t\ x)$
$openin\ (top\_of\_set\ S)\ v\ x \in v$
shows $\exists\, u.\ openin\ (top\_of\_set\ S)\ u \wedge connected\ u \wedge x \in u \wedge u \subseteq v$
using *assms connected_component_subset* **by** *fastforce*

lemma *locally_connected*:
$locally\ connected\ S \longleftrightarrow$
$(\forall\, v\ x.\ openin\ (top\_of\_set\ S)\ v \wedge x \in v$
$\longrightarrow (\exists\, u.\ openin\ (top\_of\_set\ S)\ u \wedge connected\ u \wedge x \in u \wedge u \subseteq v))$
by (*metis locally_connected_1 locally_connected_2 locally_connected_3*)

lemma *locally_connected_open_connected_component*:
$locally\ connected\ S \longleftrightarrow$
$(\forall\, t\ x.\ openin\ (top\_of\_set\ S)\ t \wedge x \in t$
$\longrightarrow openin\ (top\_of\_set\ S)\ (connected\_component\_set\ t\ x))$
by (*metis locally_connected_1 locally_connected_2 locally_connected_3*)

lemma *locally_path_connected_1*:
assumes
$\bigwedge v\ x.\ [\![openin\ (top\_of\_set\ S)\ v;\ x \in v]\!]$
$\Longrightarrow \exists\, u.\ openin\ (top\_of\_set\ S)\ u \wedge path\_connected\ u \wedge x \in u \wedge u \subseteq v$
shows *locally path_connected S*
by (*force simp add*: *locally_def dest*: *assms*)

lemma *locally_path_connected_2*:
assumes *locally path_connected S*
$openin\ (top\_of\_set\ S)\ t$
$x \in t$
shows $openin\ (top\_of\_set\ S)\ (path\_component\_set\ t\ x)$
proof –
{ **fix** $y :: 'a$
let $?SS = top\_of\_set\ S$
assume *1*: $openin\ ?SS\ t$
$\forall\, w\ x.\ openin\ ?SS\ w \wedge x \in w \longrightarrow (\exists\, u.\ openin\ ?SS\ u \wedge (\exists\, v.\ path\_connected\ v \wedge x \in u \wedge u \subseteq v \wedge v \subseteq w))$
and $path\_component\ t\ x\ y$
then have $y \in t$ **and** y: $y \in path\_component\_set\ t\ x$
using *path_component_mem*(*2*) **by** *blast+*
obtain F **where**
$\forall\, x\ y.\ (\exists\, w.\ openin\ ?SS\ w \wedge (\exists\, u.\ path\_connected\ u \wedge x \in w \wedge w \subseteq u \wedge u \subseteq y)) = (openin\ ?SS\ (F\ x\ y) \wedge (\exists\, u.\ path\_connected\ u \wedge x \in F\ x\ y \wedge F\ x\ y \subseteq u \wedge u \subseteq y))$
by *moura*
then obtain G **where**
$\forall\, a\ A.\ (\exists\, U.\ openin\ ?SS\ U \wedge (\exists\, V.\ path\_connected\ V \wedge a \in U \wedge U \subseteq V \wedge V \subseteq A)) = (openin\ ?SS\ (F\ a\ A) \wedge path\_connected\ (G\ a\ A) \wedge a \in F\ a\ A \wedge F\ a\ A \subseteq G\ a\ A \wedge G\ a\ A \subseteq A)$

by *moura*
then have ∗: *openin ?SS* (*F y t*) ∧ *path_connected* (*G y t*) ∧ *y* ∈ *F y t* ∧ *F y t* ⊆ *G y t* ∧ *G y t* ⊆ *t*
using *1* ‹*y* ∈ *t*› **by** *presburger*
have *G y t* ⊆ *path_component_set t y*
using ∗ *path_component_maximal rev_subsetD* **by** *blast*
then have ∃ *A*. *openin ?SS A* ∧ *y* ∈ *A* ∧ *A* ⊆ *path_component_set t x*
by (*metis* ∗ ‹*G y t* ⊆ *path_component_set t y*› *dual_order.trans path_component_eq y*)
}
then show *?thesis*
using *assms openin_subopen* **by** (*force simp*: *locally_def*)
qed

lemma *locally_path_connected_3*:
assumes ⋀*t x*. ⟦*openin* (*top_of_set S*) *t*; *x* ∈ *t*⟧
⟹ *openin* (*top_of_set S*) (*path_component_set t x*)
openin (*top_of_set S*) *v x* ∈ *v*
shows ∃ *u*. *openin* (*top_of_set S*) *u* ∧ *path_connected u* ∧ *x* ∈ *u* ∧ *u* ⊆ *v*
proof −
have *path_component v x x*
by (*meson assms*(*3*) *path_component_refl*)
then show *?thesis*
by (*metis assms mem_Collect_eq path_component_subset path_connected_path_component*)
qed

proposition *locally_path_connected*:
locally path_connected S ⟷
(∀ *V x*. *openin* (*top_of_set S*) *V* ∧ *x* ∈ *V*
⟶ (∃ *U*. *openin* (*top_of_set S*) *U* ∧ *path_connected U* ∧ *x* ∈ *U* ∧ *U* ⊆ *V*))
by (*metis locally_path_connected_1 locally_path_connected_2 locally_path_connected_3*)

proposition *locally_path_connected_open_path_component*:
locally path_connected S ⟷
(∀ *t x*. *openin* (*top_of_set S*) *t* ∧ *x* ∈ *t*
⟶ *openin* (*top_of_set S*) (*path_component_set t x*))
by (*metis locally_path_connected_1 locally_path_connected_2 locally_path_connected_3*)

lemma *locally_connected_open_component*:
locally connected S ⟷
(∀ *t c*. *openin* (*top_of_set S*) *t* ∧ *c* ∈ *components t*
⟶ *openin* (*top_of_set S*) *c*)
by (*metis components_iff locally_connected_open_connected_component*)

proposition *locally_connected_im_kleinen*:
locally connected S ⟷
(∀ *v x*. *openin* (*top_of_set S*) *v* ∧ *x* ∈ *v*
⟶ (∃ *u*. *openin* (*top_of_set S*) *u* ∧

```
              x ∈ u ∧ u ⊆ v ∧
              (∀ y. y ∈ u ⟶ (∃ c. connected c ∧ c ⊆ v ∧ x ∈ c ∧ y ∈ c))))
   (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    by (fastforce simp add: locally_connected)
next
  assume ?rhs
  have *: ∃ T. openin (top_of_set S) T ∧ x ∈ T ∧ T ⊆ c
       if openin (top_of_set S) t and c: c ∈ components t and x ∈ c for t c x
  proof −
    from that ‹?rhs› [rule_format, of t x]
    obtain u where u:
      openin (top_of_set S) u ∧ x ∈ u ∧ u ⊆ t ∧
       (∀ y. y ∈ u ⟶ (∃ c. connected c ∧ c ⊆ t ∧ x ∈ c ∧ y ∈ c))
      using in_components_subset by auto
    obtain F :: 'a set ⇒ 'a set ⇒ 'a where
      ∀ x y. (∃ z. z ∈ x ∧ y = connected_component_set x z) = (F x y ∈ x ∧ y =
connected_component_set x (F x y))
      by moura
    then have F: F t c ∈ t ∧ c = connected_component_set t (F t c)
      by (meson components_iff c)
    obtain G :: 'a set ⇒ 'a set ⇒ 'a where
        G: ∀ x y. (∃ z. z ∈ y ∧ z ∉ x) = (G x y ∈ y ∧ G x y ∉ x)
      by moura
     have G c u ∉ u ∨ G c u ∈ c
     using F by (metis (full_types) u connected_componentI connected_component_eq
mem_Collect_eq that(3))
    then show ?thesis
      using G u by auto
  qed
  show ?lhs
    unfolding locally_connected_open_component by (meson * openin_subopen)
qed

proposition locally_path_connected_im_kleinen:
  locally path_connected S ⟷
   (∀ v x. openin (top_of_set S) v ∧ x ∈ v
       ⟶ (∃ u. openin (top_of_set S) u ∧
              x ∈ u ∧ u ⊆ v ∧
              (∀ y. y ∈ u ⟶ (∃ p. path p ∧ path_image p ⊆ v ∧
                              pathstart p = x ∧ pathfinish p = y))))
   (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    apply (simp add: locally_path_connected path_connected_def)
    apply (erule all_forward ex_forward imp_forward conjE | simp)+
```

by (*meson dual_order.trans*)
next
assume *?rhs*
have ∗: ∃ *T*. *openin* (*top_of_set S*) *T* ∧
x ∈ *T* ∧ *T* ⊆ *path_component_set u z*
if *openin* (*top_of_set S*) *u* **and** *z* ∈ *u* **and** *c*: *path_component u z x* **for** *u z x*
proof −
have *x* ∈ *u*
by (*meson c path_component_mem*(*2*))
with *that* ‹*?rhs*› [*rule_format*, *of u x*]
obtain *U* **where** *U*:
openin (*top_of_set S*) *U* ∧ *x* ∈ *U* ∧ *U* ⊆ *u* ∧
(∀ *y*. *y* ∈ *U* ⟶ (∃ *p*. *path p* ∧ *path_image p* ⊆ *u* ∧ *pathstart p* = *x* ∧
pathfinish p = *y*))
by *blast*
show *?thesis*
by (*metis U c mem_Collect_eq path_component_def path_component_eq subsetI*)
qed
show *?lhs*
unfolding *locally_path_connected_open_path_component*
using ∗ *openin_subopen* **by** *fastforce*
qed

lemma *locally_path_connected_imp_locally_connected*:
locally path_connected S ⟹ *locally connected S*
using *locally_mono path_connected_imp_connected* **by** *blast*

lemma *locally_connected_components*:
⟦*locally connected S*; *c* ∈ *components S*⟧ ⟹ *locally connected c*
by (*meson locally_connected_open_component locally_open_subset openin_subtopology_self*)

lemma *locally_path_connected_components*:
⟦*locally path_connected S*; *c* ∈ *components S*⟧ ⟹ *locally path_connected c*
by (*meson locally_connected_open_component locally_open_subset locally_path_connected_imp_locally_conne*
openin_subtopology_self)

lemma *locally_path_connected_connected_component*:
locally path_connected S ⟹ *locally path_connected* (*connected_component_set S*
x)
by (*metis components_iff connected_component_eq_empty locally_empty locally_path_connected_components*

lemma *open_imp_locally_path_connected*:
fixes *S* :: ′*a* :: *real_normed_vector set*
assumes *open S*
shows *locally path_connected S*
proof (*rule locally_mono*)
show *locally convex S*
using *assms* **unfolding** *locally_def*
by (*meson open_ball centre_in_ball convex_ball openE open_subset openin_imp_subset*

openin_open_trans subset_trans)
  show $\bigwedge T$::'*a set. convex T* ⟹ *path_connected T*
    using *convex_imp_path_connected* **by** *blast*
qed

lemma *open_imp_locally_connected*:
  fixes *S* :: '*a* :: *real_normed_vector set*
  shows *open S* ⟹ *locally connected S*
by (*simp add*: *locally_path_connected_imp_locally_connected open_imp_locally_path_connected*)

lemma *locally_path_connected_UNIV*: *locally path_connected* (*UNIV*::'*a* :: *real_normed_vector set*)
  by (*simp add*: *open_imp_locally_path_connected*)

lemma *locally_connected_UNIV*: *locally connected* (*UNIV*::'*a* :: *real_normed_vector set*)
  by (*simp add*: *open_imp_locally_connected*)

lemma *openin_connected_component_locally_connected*:
    locally connected S
     ⟹ *openin* (*top_of_set S*) (*connected_component_set S x*)
  by (*metis connected_component_eq_empty locally_connected_2 openin_empty openin_subtopology_self*)

lemma *openin_components_locally_connected*:
    ⟦*locally connected S*; *c* ∈ *components S*⟧ ⟹ *openin* (*top_of_set S*) *c*
  using *locally_connected_open_component openin_subtopology_self* **by** *blast*

lemma *openin_path_component_locally_path_connected*:
  locally path_connected S
        ⟹ *openin* (*top_of_set S*) (*path_component_set S x*)
by (*metis* (*no_types*) *empty_iff locally_path_connected_2 openin_subopen openin_subtopology_self path_component_eq_empty*)

lemma *closedin_path_component_locally_path_connected*:
  assumes *locally path_connected S*
  shows *closedin* (*top_of_set S*) (*path_component_set S x*)
proof −
 have *openin* (*top_of_set S*) (⋃ ({*path_component_set S y* |*y*. *y* ∈ *S*} − {*path_component_set S x*}))
    using *locally_path_connected_2 assms* **by** *fastforce*
  then show *?thesis*
   by (*simp add*: *closedin_def path_component_subset complement_path_component_Union*)
qed

lemma *convex_imp_locally_path_connected*:
  fixes *S* :: '*a*:: *real_normed_vector set*
  assumes *convex S*
  shows *locally path_connected S*
proof (*clarsimp simp add*: *locally_path_connected*)

fix *V x*
assume *openin* (*top_of_set S*) *V* **and** $x \in V$
then obtain *T e* **where** $V = S \cap T$ $x \in S$ $0 < e$ *ball x e* $\subseteq T$
by (*metis Int_iff openE openin_open*)
then have *openin* (*top_of_set S*) ($S \cap$ *ball x e*) *path_connected* ($S \cap$ *ball x e*)
by (*simp_all add*: *assms convex_Int convex_imp_path_connected openin_open_Int*)
then show $\exists U$. *openin* (*top_of_set S*) *U* $\wedge$ *path_connected U* $\wedge$ $x \in U \wedge U \subseteq$
V
using ‹$0 < e$› ‹$V = S \cap T$› ‹*ball x e* $\subseteq T$› ‹$x \in S$› **by** *auto*
qed

lemma *convex_imp_locally_connected*:
fixes *S* :: *'a*:: *real_normed_vector set*
shows *convex S* $\Longrightarrow$ *locally connected S*
by (*simp add*: *locally_path_connected_imp_locally_connected convex_imp_locally_path_connected*)

6.18.18 Relations between components and path components

lemma *path_component_eq_connected_component*:
assumes *locally path_connected S*
shows (*path_component S x* = *connected_component S x*)
proof (*cases* $x \in S$)
case *True*
have *openin* (*top_of_set* (*connected_component_set S x*)) (*path_component_set S*
x)
proof (*rule openin_subset_trans*)
show *openin* (*top_of_set S*) (*path_component_set S x*)
by (*simp add*: *True assms locally_path_connected_2*)
show *connected_component_set S x* $\subseteq S$
by (*simp add*: *connected_component_subset*)
qed (*simp add*: *path_component_subset_connected_component*)
moreover have *closedin* (*top_of_set* (*connected_component_set S x*)) (*path_component_set*
S x)
proof (*rule closedin_subset_trans* [*of S*])
show *closedin* (*top_of_set S*) (*path_component_set S x*)
by (*simp add*: *assms closedin_path_component_locally_path_connected*)
show *connected_component_set S x* $\subseteq S$
by (*simp add*: *connected_component_subset*)
qed (*simp add*: *path_component_subset_connected_component*)
ultimately have ∗: *path_component_set S x* = *connected_component_set S x*
by (*metis connected_connected_component connected_clopen True path_component_eq_empty*)
then show *?thesis*
by *blast*
next
case *False* **then show** *?thesis*
by (*metis Collect_empty_eq_bot connected_component_eq_empty path_component_eq_empty*)
qed

lemma *path_component_eq_connected_component_set*:

locally path_connected S $\Longrightarrow$ (*path_component_set S x* = *connected_component_set S x*)
by (*simp add*: *path_component_eq_connected_component*)

lemma *locally_path_connected_path_component*:
locally path_connected S $\Longrightarrow$ *locally path_connected* (*path_component_set S x*)
using *locally_path_connected_connected_component path_component_eq_connected_component*
by *fastforce*

lemma *open_path_connected_component*:
fixes *S* :: *'a* :: *real_normed_vector set*
shows *open S* $\Longrightarrow$ *path_component S x* = *connected_component S x*
by (*simp add*: *path_component_eq_connected_component open_imp_locally_path_connected*)

lemma *open_path_connected_component_set*:
fixes *S* :: *'a* :: *real_normed_vector set*
shows *open S* $\Longrightarrow$ *path_component_set S x* = *connected_component_set S x*
by (*simp add*: *open_path_connected_component*)

proposition *locally_connected_quotient_image*:
assumes *lcS*: *locally connected S*
and *oo*: $\bigwedge$*T*. *T* ⊆ *f* ‘ *S*
$\Longrightarrow$ *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *T*) ⟷
openin (*top_of_set* (*f* ‘ *S*)) *T*
shows *locally connected* (*f* ‘ *S*)
proof (*clarsimp simp*: *locally_connected_open_component*)
fix *U C*
assume *opefSU*: *openin* (*top_of_set* (*f* ‘ *S*)) *U* **and** *C* ∈ *components U*
then have *C* ⊆ *U U* ⊆ *f* ‘ *S*
by (*meson in_components_subset openin_imp_subset*)+
then have *openin* (*top_of_set* (*f* ‘ *S*)) *C* ⟷
openin (*top_of_set S*) (*S* ∩ *f* −‘ *C*)
by (*auto simp*: *oo*)
moreover have *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *C*)
proof (*subst openin_subopen*, *clarify*)
fix *x*
assume *x* ∈ *S f x* ∈ *C*
show ∃ *T*. *openin* (*top_of_set S*) *T* ∧ *x* ∈ *T* ∧ *T* ⊆ (*S* ∩ *f* −‘ *C*)
proof (*intro conjI exI*)
show *openin* (*top_of_set S*) (*connected_component_set* (*S* ∩ *f* −‘ *U*) *x*)
proof (*rule ccontr*)
assume ∗∗: ¬ *openin* (*top_of_set S*) (*connected_component_set* (*S* ∩ *f* −‘ *U*) *x*)
then have *x* ∉ (*S* ∩ *f* −‘ *U*)
using ‹*U* ⊆ *f* ‘ *S*› *opefSU lcS locally_connected_2 oo* **by** *blast*
with ∗∗ **show** *False*
by (*metis* (*no_types*) *connected_component_eq_empty empty_iff openin_subopen*)
qed
next

show *x* ∈ *connected_component_set* (*S* ∩ *f* −‘ *U*) *x*
using ‹*C* ⊆ *U*› ‹*f x* ∈ *C*› ‹*x* ∈ *S*› **by** *auto*
next
have *contf*: *continuous_on S f*
by (*simp add*: *continuous_on_open oo openin_imp_subset*)
then have *continuous_on* (*connected_component_set* (*S* ∩ *f* −‘ *U*) *x*) *f*
by (*meson connected_component_subset continuous_on_subset inf.boundedE*)
then have *connected* (*f* ‘ *connected_component_set* (*S* ∩ *f* −‘ *U*) *x*)
by (*rule connected_continuous_image* [*OF* _ *connected_connected_component*])
moreover have *f* ‘ *connected_component_set* (*S* ∩ *f* −‘ *U*) *x* ⊆ *U*
using *connected_component_in* **by** *blast*
moreover have *C* ∩ *f* ‘ *connected_component_set* (*S* ∩ *f* −‘ *U*) *x* ≠ {}
using ‹*C* ⊆ *U*› ‹*f x* ∈ *C*› ‹*x* ∈ *S*› **by** *fastforce*
ultimately have *fC*: *f* ‘ (*connected_component_set* (*S* ∩ *f* −‘ *U*) *x*) ⊆ *C*
by (*rule components_maximal* [*OF* ‹*C* ∈ *components U*›])
have *cUC*: *connected_component_set* (*S* ∩ *f* −‘ *U*) *x* ⊆ (*S* ∩ *f* −‘ *C*)
using *connected_component_subset fC* **by** *blast*
have *connected_component_set* (*S* ∩ *f* −‘ *U*) *x* ⊆ *connected_component_set* (*S* ∩ *f* −‘ *C*) *x*
proof −
{ assume *x* ∈ *connected_component_set* (*S* ∩ *f* −‘ *U*) *x*
then have *?thesis*
using *cUC connected_component_idemp connected_component_mono* **by** *blast* **}**
then show *?thesis*
using *connected_component_eq_empty* **by** *auto*
qed
also have . . . ⊆ (*S* ∩ *f* −‘ *C*)
by (*rule connected_component_subset*)
finally show *connected_component_set* (*S* ∩ *f* −‘ *U*) *x* ⊆ (*S* ∩ *f* −‘ *C*) **.**
qed
qed
ultimately show *openin* (*top_of_set* (*f* ‘ *S*)) *C*
by *metis*
qed

The proof resembles that above but is not identical!

proposition *locally_path_connected_quotient_image*:
assumes *lcS*: *locally path_connected S*
and *oo*: ⋀*T*. *T* ⊆ *f* ‘ *S*
⟹ *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *T*) ⟷ *openin* (*top_of_set* (*f* ‘ *S*)) *T*
shows *locally path_connected* (*f* ‘ *S*)
proof (*clarsimp simp*: *locally_path_connected_open_path_component*)
fix *U y*
assume *opefSU*: *openin* (*top_of_set* (*f* ‘ *S*)) *U* **and** *y* ∈ *U*
then have *path_component_set U y* ⊆ *U U* ⊆ *f* ‘ *S*
by (*meson path_component_subset openin_imp_subset*)+
then have *openin* (*top_of_set* (*f* ‘ *S*)) (*path_component_set U y*) ⟷

```
            openin (top_of_set S) (S ∩ f −‘ path_component_set U y)
  proof −
    have path_component_set U y ⊆ f ‘ S
      using ‹U ⊆ f ‘ S› ‹path_component_set U y ⊆ U› by blast
    then show ?thesis
      using oo by blast
  qed
  moreover have openin (top_of_set S) (S ∩ f −‘ path_component_set U y)
  proof (subst openin_subopen, clarify)
    fix x
    assume x ∈ S and Uyfx: path_component U y (f x)
    then have f x ∈ U
      using path_component_mem by blast
   show ∃ T. openin (top_of_set S) T ∧ x ∈ T ∧ T ⊆ (S ∩ f −‘ path_component_set
U y)
    proof (intro conjI exI)
      show openin (top_of_set S) (path_component_set (S ∩ f −‘ U) x)
      proof (rule ccontr)
        assume **: ¬ openin (top_of_set S) (path_component_set (S ∩ f −‘ U) x)
        then have x ∉ (S ∩ f −‘ U)
       by (metis (no_types, lifting) ‹U ⊆ f ‘ S› opefSU lcS oo locally_path_connected_open_path_component)
        then show False
          using ** ‹path_component_set U y ⊆ U›  ‹x ∈ S› ‹path_component U y (f
x)› by blast
      qed
    next
      show x ∈ path_component_set (S ∩ f −‘ U) x
        by (simp add: ‹f x ∈ U› ‹x ∈ S› path_component_refl)
    next
      have contf: continuous_on S f
        by (simp add: continuous_on_open oo openin_imp_subset)
      then have continuous_on (path_component_set (S ∩ f −‘ U) x) f
        by (meson Int_lower1 continuous_on_subset path_component_subset)
      then have path_connected (f ‘ path_component_set (S ∩ f −‘ U) x)
        by (simp add: path_connected_continuous_image)
      moreover have f ‘ path_component_set (S ∩ f −‘ U) x ⊆ U
        using path_component_mem by fastforce
      moreover have f x ∈ f ‘ path_component_set (S ∩ f −‘ U) x
        by (force simp: ‹x ∈ S› ‹f x ∈ U› path_component_refl_eq)
    ultimately have f ‘ (path_component_set (S ∩ f −‘ U) x) ⊆ path_component_set
U (f x)
        by (meson path_component_maximal)
       also have  . . . ⊆ path_component_set U y
      by (simp add: Uyfx path_component_maximal path_component_subset path_component_sym)
    finally have fC: f ‘ (path_component_set (S ∩ f −‘ U) x) ⊆ path_component_set
U y .
    have cUC: path_component_set (S ∩ f −‘ U) x ⊆ (S ∩ f −‘ path_component_set
U y)
        using path_component_subset fC by blast
```

have *path_component_set* $(S \cap f -` U)$ $x \subseteq$ *path_component_set* $(S \cap f -`$ *path_component_set* U $y)$ x
proof −
have $\bigwedge a.$ *path_component_set* (*path_component_set* $(S \cap f -` U)$ $x)$ $a \subseteq$ *path_component_set* $(S \cap f -`$ *path_component_set* U $y)$ a
using *cUC path_component_mono* **by** *blast*
then show *?thesis*
using *path_component_path_component* **by** *blast*
qed
also have ... $\subseteq (S \cap f -`$ *path_component_set* U $y)$
by (*rule path_component_subset*)
finally show *path_component_set* $(S \cap f -` U)$ $x \subseteq (S \cap f -`$ *path_component_set* U $y)$.
qed
qed
ultimately show *openin* (*top_of_set* $(f$ ` $S))$ (*path_component_set* U y)
by *metis*
qed

6.18.19 Components, continuity, openin, closedin

lemma *continuous_on_components_gen*:
fixes f :: *'a::topological_space* ⇒ *'b::topological_space*
assumes $\bigwedge C.$ $C \in$ *components* $S \Longrightarrow$
openin (*top_of_set* S) C ∧ *continuous_on* C f
shows *continuous_on* S f
proof (*clarsimp simp*: *continuous_openin_preimage_eq*)
fix t :: *'b set*
assume *open* t
have ∗: $S \cap f -` t = (\bigcup c \in$ *components* $S.$ $c \cap f -` t)$
by *auto*
show *openin* (*top_of_set* S) $(S \cap f -` t)$
unfolding ∗ **using** ‹*open t*› *assms continuous_openin_preimage_gen openin_trans openin_Union* **by** *blast*
qed

lemma *continuous_on_components*:
fixes f :: *'a::topological_space* ⇒ *'b::topological_space*
assumes *locally connected* S $\bigwedge C.$ $C \in$ *components* $S \Longrightarrow$ *continuous_on* C f
shows *continuous_on* S f
proof (*rule continuous_on_components_gen*)
fix C
assume $C \in$ *components* S
then show *openin* (*top_of_set* S) C ∧ *continuous_on* C f
by (*simp add*: *assms openin_components_locally_connected*)
qed

lemma *continuous_on_components_eq*:
locally connected S

$\Longrightarrow$ (*continuous_on S f* $\longleftrightarrow$ ($\forall$ *c* $\in$ *components S*. *continuous_on c f*))
by (*meson continuous_on_components continuous_on_subset in_components_subset*)

lemma *continuous_on_components_open*:
fixes *S* :: *'a::real_normed_vector set*
assumes *open S*
$\bigwedge$*c*. *c* $\in$ *components S* $\Longrightarrow$ *continuous_on c f*
shows *continuous_on S f*
using *continuous_on_components open_imp_locally_connected assms* **by** *blast*

lemma *continuous_on_components_open_eq*:
fixes *S* :: *'a::real_normed_vector set*
shows *open S* $\Longrightarrow$ (*continuous_on S f* $\longleftrightarrow$ ($\forall$ *c* $\in$ *components S*. *continuous_on c f*))
using *continuous_on_subset in_components_subset*
by (*blast intro*: *continuous_on_components_open*)

lemma *closedin_union_complement_components*:
assumes *U*: *locally connected U*
and *S*: *closedin* (*top_of_set U*) *S*
and *cuS*: *c* $\subseteq$ *components*(*U* − *S*)
shows *closedin* (*top_of_set U*) (*S* $\cup$ $\bigcup$ *c*)
proof −
have *di*: ($\bigwedge$*S T*. *S* $\in$ *c* $\wedge$ *T* $\in$ *c'* $\Longrightarrow$ *disjnt S T*) $\Longrightarrow$ *disjnt* ($\bigcup$ *c*) ($\bigcup$ *c'*) **for** *c'*
by (*simp add*: *disjnt_def*) *blast*
have *S* $\subseteq$ *U*
using *S closedin_imp_subset* **by** *blast*
moreover have *U* − *S* = $\bigcup$ *c* $\cup$ $\bigcup$(*components* (*U* − *S*) − *c*)
by (*metis Diff_partition Union_components Union_Un_distrib assms*(*3*))
moreover have *disjnt* ($\bigcup$ *c*) ($\bigcup$(*components* (*U* − *S*) − *c*))
apply (*rule di*)
by (*metis di DiffD1 DiffD2 assms*(*3*) *components_nonoverlap disjnt_def subsetCE*)
ultimately have *eq*: *S* $\cup$ $\bigcup$ *c* = *U* − ($\bigcup$(*components*(*U* − *S*) − *c*))
by (*auto simp*: *disjnt_def*)
have $*$: *openin* (*top_of_set U*) ($\bigcup$(*components* (*U* − *S*) − *c*))
proof (*rule openin_Union* [*OF openin_trans* [*of U* − *S*]])
show *openin* (*top_of_set* (*U* − *S*)) *T* **if** *T* $\in$ *components* (*U* − *S*) − *c* **for** *T*
using *that* **by** (*simp add*: *U S locally_diff_closed openin_components_locally_connected*)
show *openin* (*top_of_set U*) (*U* − *S*) **if** *T* $\in$ *components* (*U* − *S*) − *c* **for** *T*
using *that* **by** (*simp add*: *openin_diff S*)
qed
have *closedin* (*top_of_set U*) (*U* − $\bigcup$ (*components* (*U* − *S*) − *c*))
by (*metis closedin_diff closedin_topspace topspace_euclidean_subtopology* $*$)
then have *openin* (*top_of_set U*) (*U* − (*U* − $\bigcup$(*components* (*U* − *S*) − *c*)))
by (*simp add*: *openin_diff*)
then show *?thesis*
by (*force simp*: *eq closedin_def*)
qed

lemma *closed_union_complement_components*:
fixes S :: $'a$::*real_normed_vector set*
assumes S: *closed* S **and** c: $c \subseteq$ *components*$(- S)$
shows *closed*$(S \cup \bigcup c)$
proof –
have *closedin* (*top_of_set UNIV*) $(S \cup \bigcup c)$
by (*metis Compl_eq_Diff_UNIV S c closed_closedin closedin_union_complement_components locally_connected_UNIV subtopology_UNIV*)
then show *?thesis* **by** *simp*
qed

lemma *closedin_Un_complement_component*:
fixes S :: $'a$::*real_normed_vector set*
assumes u: *locally connected* u
and S: *closedin* (*top_of_set* u) S
and c: $c \in$ *components*$(u - S)$
shows *closedin* (*top_of_set* u) $(S \cup c)$
proof –
have *closedin* (*top_of_set* u) $(S \cup \bigcup\{c\})$
using c **by** (*blast intro*: *closedin_union_complement_components* [*OF* u S])
then show *?thesis*
by *simp*
qed

lemma *closed_Un_complement_component*:
fixes S :: $'a$::*real_normed_vector set*
assumes S: *closed* S **and** c: $c \in$ *components*$(-S)$
shows *closed* $(S \cup c)$
by (*metis Compl_eq_Diff_UNIV S c closed_closedin closedin_Un_complement_component locally_connected_UNIV subtopology_UNIV*)

6.18.20 Existence of isometry between subspaces of same dimension

lemma *isometry_subset_subspace*:
fixes S :: $'a$::*euclidean_space set*
and T :: $'b$::*euclidean_space set*
assumes S: *subspace* S
and T: *subspace* T
and d: *dim* $S \leq$ *dim* T
obtains f **where** *linear* f f ' $S \subseteq T$ $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) = norm\ x$
proof –
obtain B **where** $B \subseteq S$ **and** *Borth*: *pairwise orthogonal* B
and *B1*: $\bigwedge x.\ x \in B \Longrightarrow norm\ x = 1$
and *independent* B *finite* B *card* B = *dim* S *span* $B = S$
by (*metis orthonormal_basis_subspace* [*OF* S] *independent_finite*)
obtain C **where** $C \subseteq T$ **and** *Corth*: *pairwise orthogonal* C
and *C1*: $\bigwedge x.\ x \in C \Longrightarrow norm\ x = 1$

and *independent C finite C card C = dim T span C = T*
by (*metis orthonormal_basis_subspace* [*OF T*] *independent_finite*)
obtain *fb* **where** *fb ‘ B ⊆ C inj_on fb B*
by (*metis* ‹*card B = dim S*› ‹*card C = dim T*› ‹*finite B*› ‹*finite C*› *card_le_inj d*)
then have *pairwise_orth_fb*: *pairwise* (*λv j. orthogonal* (*fb v*) (*fb j*)) *B*
using *Corth* **unfolding** *pairwise_def inj_on_def*
by (*blast intro*: *orthogonal_clauses*)
obtain *f* **where** *linear f* **and** *ffb*: ⋀*x. x ∈ B ⟹ f x = fb x*
using *linear_independent_extend* ‹*independent B*› **by** *fastforce*
have *span* (*f ‘ B*) ⊆ *span C*
by (*metis* ‹*fb ‘ B ⊆ C*› *ffb image_cong span_mono*)
then have *f ‘ S ⊆ T*
unfolding ‹*span B = S*› ‹*span C = T*› *span_linear_image*[*OF* ‹*linear f*›] **.**
have [*simp*]: ⋀*x. x ∈ B ⟹ norm* (*fb x*) = *norm x*
using *B1 C1* ‹*fb ‘ B ⊆ C*› **by** *auto*
have *norm* (*f x*) = *norm x* **if** *x ∈ S* **for** *x*
proof −
interpret *linear f* **by** *fact*
obtain *a* **where** *x*: *x* = (∑ *v ∈ B. a v* $*_R$ *v*)
using ‹*finite B*› ‹*span B = S*› ‹*x ∈ S*› *span_finite* **by** *fastforce*
have *norm* (*f x*)ˆ*2* = *norm* (∑ *v∈B. a v* $*_R$ *fb v*)ˆ*2* **by** (*simp add*: *sum scale ffb x*)
also have ... = (∑ *v∈B. norm* ((*a v* $*_R$ *fb v*))ˆ*2*)
proof (*rule norm_sum_Pythagorean* [*OF* ‹*finite B*›])
show *pairwise* (*λv j. orthogonal* (*a v* $*_R$ *fb v*) (*a j* $*_R$ *fb j*)) *B*
by (*rule pairwise_ortho_scaleR* [*OF pairwise_orth_fb*])
qed
also have ... = *norm x* ˆ*2*
by (*simp add*: *x pairwise_ortho_scaleR Borth norm_sum_Pythagorean* [*OF* ‹*finite B*›])
finally show *?thesis*
by (*simp add*: *norm_eq_sqrt_inner*)
qed
then show *?thesis*
by (*rule that* [*OF* ‹*linear f*› ‹*f ‘ S ⊆ T*›])
qed

proposition *isometries_subspaces*:
fixes *S* :: *'a::euclidean_space set*
and *T* :: *'b::euclidean_space set*
assumes *S*: *subspace S*
and *T*: *subspace T*
and *d*: *dim S = dim T*
obtains *f g* **where** *linear f linear g f ‘ S = T g ‘ T = S*
⋀*x. x ∈ S ⟹ norm*(*f x*) = *norm x*
⋀*x. x ∈ T ⟹ norm*(*g x*) = *norm x*
⋀*x. x ∈ S ⟹ g*(*f x*) = *x*
⋀*x. x ∈ T ⟹ f*(*g x*) = *x*

proof −
  obtain *B* **where** $B \subseteq S$ **and** *Borth*: *pairwise orthogonal B*
    and *B1*: $\bigwedge x.\ x \in B \Longrightarrow norm\ x = 1$
    and *independent B finite B card B = dim S span B = S*
   by (*metis orthonormal_basis_subspace* [*OF S*] *independent_finite*)
  obtain *C* **where** $C \subseteq T$ **and** *Corth*: *pairwise orthogonal C*
    and *C1*:$\bigwedge x.\ x \in C \Longrightarrow norm\ x = 1$
    and *independent C finite C card C = dim T span C = T*
   by (*metis orthonormal_basis_subspace* [*OF T*] *independent_finite*)
  obtain *fb* **where** *bij_betw fb B C*
   by (*metis* ‹*finite B*› ‹*finite C*› *bij_betw_iff_card* ‹*card B = dim S*› ‹*card C = dim T*› *d*)
  then have *pairwise_orth_fb*: *pairwise* ($\lambda v\ j.$ *orthogonal* (*fb v*) (*fb j*)) *B*
   using *Corth* **unfolding** *pairwise_def inj_on_def bij_betw_def*
   by (*blast intro*: *orthogonal_clauses*)
  obtain *f* **where** *linear f* **and** *ffb*: $\bigwedge x.\ x \in B \Longrightarrow f\ x = fb\ x$
   using *linear_independent_extend* ‹*independent B*› **by** *fastforce*
  interpret *f*: *linear f* **by** *fact*
  define *gb* **where** $gb \equiv inv\_into\ B\ fb$
  then have *pairwise_orth_gb*: *pairwise* ($\lambda v\ j.$ *orthogonal* (*gb v*) (*gb j*)) *C*
   using *Borth* ‹*bij_betw fb B C*› **unfolding** *pairwise_def bij_betw_def* **by** *force*
  obtain *g* **where** *linear g* **and** *ggb*: $\bigwedge x.\ x \in C \Longrightarrow g\ x = gb\ x$
   using *linear_independent_extend* ‹*independent C*› **by** *fastforce*
  interpret *g*: *linear g* **by** *fact*
  have *span* (*f* ‘ *B*) ⊆ *span C*
   by (*metis* ‹*bij_betw fb B C*› *bij_betw_imp_surj_on eq_iff ffb image_cong*)
  then have *f* ‘ *S* ⊆ *T*
   unfolding ‹*span B = S*› ‹*span C = T*› *span_linear_image*[*OF* ‹*linear f*›] **.**
  have [*simp*]: $\bigwedge x.\ x \in B \Longrightarrow norm\ (fb\ x) = norm\ x$
   using *B1 C1* ‹*bij_betw fb B C*› *bij_betw_imp_surj_on* **by** *fastforce*
  have *f* [*simp*]: *norm* (*f x*) = *norm x g* (*f x*) = *x* **if** $x \in S$ **for** *x*
  proof −
   obtain *a* **where** *x*: $x = (\sum v \in B.\ a\ v *_R v)$
    using ‹*finite B*› ‹*span B = S*› ‹$x \in S$› *span_finite* **by** *fastforce*
   have $f\ x = (\sum v \in B.\ f\ (a\ v *_R v))$
    using *linear_sum* [*OF* ‹*linear f*›] *x* **by** *auto*
   also have ... $= (\sum v \in B.\ a\ v *_R f\ v)$
    by (*simp add*: *f.sum f.scale*)
   also have ... $= (\sum v \in B.\ a\ v *_R fb\ v)$
    by (*simp add*: *ffb cong*: *sum.cong*)
   finally have ∗: $f\ x = (\sum v{\in}B.\ a\ v *_R fb\ v)$ **.**
   then have $(norm\ (f\ x))^2 = (norm\ (\sum v{\in}B.\ a\ v *_R fb\ v))^2$ **by** *simp*
   also have ... $= (\sum v{\in}B.\ norm\ ((a\ v *_R fb\ v))\hat{}2)$
   proof (*rule norm_sum_Pythagorean* [*OF* ‹*finite B*›])
    show *pairwise* ($\lambda v\ j.$ *orthogonal* ($a\ v *_R fb\ v$) ($a\ j *_R fb\ j$)) *B*
     by (*rule pairwise_ortho_scaleR* [*OF pairwise_orth_fb*])
   qed
   also have ... $= (norm\ x)^2$
     by (*simp add*: *x pairwise_ortho_scaleR Borth norm_sum_Pythagorean* [*OF*

‹finite B›])
    finally show *norm* (*f x*) = *norm x*
      by (*simp add*: *norm_eq_sqrt_inner*)
    have *g* (*f x*) = *g* (∑ *v*∈*B*. *a v* $*_R$ *fb v*) **by** (*simp add*: ∗)
    also have ... = (∑ *v*∈*B*. *g* (*a v* $*_R$ *fb v*))
      by (*simp add*: *g.sum g.scale*)
    also have ... = (∑ *v*∈*B*. *a v* $*_R$ *g* (*fb v*))
      by (*simp add*: *g.scale*)
    also have ... = (∑ *v*∈*B*. *a v* $*_R$ *v*)
    proof (*rule sum.cong* [*OF refl*])
      show *a x* $*_R$ *g* (*fb x*) = *a x* $*_R$ *x* **if** *x* ∈ *B* **for** *x*
        using *that* ‹*bij_betw fb B C*› *bij_betwE bij_betw_inv_into_left gb_def ggb* **by** *fastforce*
    qed
    also have ... = *x*
      using *x* **by** *blast*
    finally show *g* (*f x*) = *x* **.**
  qed
  have [*simp*]: ⋀*x*. *x* ∈ *C* ⟹ *norm* (*gb x*) = *norm x*
    by (*metis B1 C1* ‹*bij_betw fb B C*› *bij_betw_imp_surj_on gb_def inv_into_into*)
  have *g* [*simp*]: *f* (*g x*) = *x* **if** *x* ∈ *T* **for** *x*
  proof −
    obtain *a* **where** *x*: *x* = (∑ *v* ∈ *C*. *a v* $*_R$ *v*)
      using ‹*finite C*› ‹*span C* = *T*› ‹*x* ∈ *T*› *span_finite* **by** *fastforce*
    have *g x* = (∑ *v* ∈ *C*. *g* (*a v* $*_R$ *v*))
      by (*simp add*: *x g.sum*)
    also have ... = (∑ *v* ∈ *C*. *a v* $*_R$ *g v*)
      by (*simp add*: *g.scale*)
    also have ... = (∑ *v* ∈ *C*. *a v* $*_R$ *gb v*)
      by (*simp add*: *ggb cong*: *sum.cong*)
    finally have *f* (*g x*) = *f* (∑ *v*∈*C*. *a v* $*_R$ *gb v*) **by** *simp*
    also have ... = (∑ *v*∈*C*. *f* (*a v* $*_R$ *gb v*))
      by (*simp add*: *f.scale f.sum*)
    also have ... = (∑ *v*∈*C*. *a v* $*_R$ *f* (*gb v*))
      by (*simp add*: *f.scale f.sum*)
    also have ... = (∑ *v*∈*C*. *a v* $*_R$ *v*)
      using ‹*bij_betw fb B C*›
      by (*simp add*: *bij_betw_def gb_def bij_betw_inv_into_right ffb inv_into_into*)
    also have ... = *x*
      using *x* **by** *blast*
    finally show *f* (*g x*) = *x* **.**
  qed
  have *gim*: *g* ‘ *T* = *S*
    by (*metis* (*full_types*) *S T* ‹*f* ‘ *S* ⊆ *T*› *d dim_eq_span dim_image_le f*(*2*) *g.linear_axioms*
        image_iff linear_subspace_image span_eq_iff subset_iff)
  have *fim*: *f* ‘ *S* = *T*
    using ‹*g* ‘ *T* = *S*› *image_iff* **by** *fastforce*
  have [*simp*]: *norm* (*g x*) = *norm x* **if** *x* ∈ *T* **for** *x*

using *fim that* **by** *auto*
show *?thesis*
by (*rule that* [*OF* ‹*linear f*› ‹*linear g*›]) (*simp_all add*: *fim gim*)
qed

corollary *isometry_subspaces*:
fixes S :: $'a$::*euclidean_space set*
and T :: $'b$::*euclidean_space set*
assumes S: *subspace* S
and T: *subspace* T
and d: *dim* S = *dim* T
obtains f **where** *linear f f* ‘ $S = T$ $\bigwedge x.\ x \in S \Longrightarrow$ *norm*$(f\ x)$ = *norm* x
using *isometries_subspaces* [*OF assms*]
by *metis*

corollary *isomorphisms_UNIV_UNIV*:
assumes *DIM*$('M)$ = *DIM*$('N)$
obtains f::$'M$::*euclidean_space* ⇒$'N$::*euclidean_space* **and** g
where *linear f linear g*
$\bigwedge x.$ *norm*$(f\ x)$ = *norm* x $\bigwedge y.$ *norm*$(g\ y)$ = *norm* y
$\bigwedge x.\ g\ (f\ x) = x$ $\bigwedge y.\ f(g\ y) = y$
using *assms* **by** (*auto intro*: *isometries_subspaces* [*of UNIV*::$'M$ *set UNIV*::$'N$ *set*])

lemma *homeomorphic_subspaces*:
fixes S :: $'a$::*euclidean_space set*
and T :: $'b$::*euclidean_space set*
assumes S: *subspace* S
and T: *subspace* T
and d: *dim* S = *dim* T
shows S *homeomorphic* T
proof −
obtain $f\ g$ **where** *linear f linear g f* ‘ $S = T$ g ‘ $T = S$
$\bigwedge x.\ x \in S \Longrightarrow g(f\ x) = x$ $\bigwedge x.\ x \in T \Longrightarrow f(g\ x) = x$
by (*blast intro*: *isometries_subspaces* [*OF assms*])
then show *?thesis*
unfolding *homeomorphic_def homeomorphism_def*
apply (*rule_tac x=f* **in** *exI*, *rule_tac x=g* **in** *exI*)
apply (*auto simp*: *linear_continuous_on linear_conv_bounded_linear*)
done
qed

lemma *homeomorphic_affine_sets*:
assumes *affine S affine T aff_dim S* = *aff_dim T*
shows S *homeomorphic* T
proof (*cases* $S = \{\} \vee T = \{\}$)
case *True* **with** *assms aff_dim_empty homeomorphic_empty* **show** *?thesis*
by *metis*
next

case *False*
then obtain *a b* **where** *ab*: $a \in S$ $b \in T$ **by** *auto*
then have *ss*: *subspace* $((+)\ (-\ a)$ ‘ $S)$ *subspace* $((+)\ (-\ b)$ ‘ $T)$
using *affine_diffs_subspace assms* **by** *blast+*
have *dd*: *dim* $((+)\ (-\ a)$ ‘ $S)$ = *dim* $((+)\ (-\ b)$ ‘ $T)$
using *assms ab* **by** (*simp add*: *aff_dim_eq_dim* [*OF hull_inc*] *image_def*)
have *S homeomorphic* $((+)\ (-\ a)$ ‘ $S)$
by (*fact homeomorphic_translation*)
also have . . . *homeomorphic* $((+)\ (-\ b)$ ‘ $T)$
by (*rule homeomorphic_subspaces* [*OF ss dd*])
also have . . . *homeomorphic* *T*
using *homeomorphic_translation* [*of T* − *b*] **by** (*simp add*: *homeomorphic_sym* [*of T*])
finally show *?thesis* .
qed

6.18.21 Retracts, in a general sense, preserve (co)homotopic triviality)

locale *Retracts* =
fixes *s h t k*
assumes *conth*: *continuous_on s h*
and *imh*: *h* ‘ *s* = *t*
and *contk*: *continuous_on t k*
and *imk*: *k* ‘ $t \subseteq s$
and *idhk*: $\bigwedge y.\ y \in t \Longrightarrow h(k\ y) = y$

begin

lemma *homotopically_trivial_retraction_gen*:
assumes *P*: $\bigwedge f.$ ⟦*continuous_on U f*; *f* ‘ $U \subseteq t$; *Q f*⟧ $\Longrightarrow P(k \circ f)$
and *Q*: $\bigwedge f.$ ⟦*continuous_on U f*; *f* ‘ $U \subseteq s$; *P f*⟧ $\Longrightarrow Q(h \circ f)$
and *Qeq*: $\bigwedge h\ k.\ (\bigwedge x.\ x \in U \Longrightarrow h\ x = k\ x) \Longrightarrow Q\ h = Q\ k$
and *hom*: $\bigwedge f\ g.$ ⟦*continuous_on U f*; *f* ‘ $U \subseteq s$; *P f*;
continuous_on U g; *g* ‘ $U \subseteq s$; *P g*⟧
$\Longrightarrow$ *homotopic_with_canon P U s f g*
and *contf*: *continuous_on U f* **and** *imf*: *f* ‘ $U \subseteq t$ **and** *Qf*: *Q f*
and *contg*: *continuous_on U g* **and** *img*: *g* ‘ $U \subseteq t$ **and** *Qg*: *Q g*
shows *homotopic_with_canon Q U t f g*
proof −
have *continuous_on U* $(k \circ f)$
using *contf continuous_on_compose continuous_on_subset contk imf* **by** *blast*
moreover have $(k \circ f)$ ‘ $U \subseteq s$
using *imf imk* **by** *fastforce*
moreover have *P* $(k \circ f)$
by (*simp add*: *P Qf contf imf*)
moreover have *continuous_on U* $(k \circ g)$
using *contg continuous_on_compose continuous_on_subset contk img* **by** *blast*
moreover have $(k \circ g)$ ‘ $U \subseteq s$

using *img imk* **by** *fastforce*
moreover have $P\ (k \circ g)$
by (*simp add*: *P Qg contg img*)
ultimately have *homotopic_with_canon* $P\ U\ s\ (k \circ f)\ (k \circ g)$
by (*rule hom*)
then have *homotopic_with_canon* $Q\ U\ t\ (h \circ (k \circ f))\ (h \circ (k \circ g))$
apply (*rule homotopic_with_compose_continuous_left* [*OF homotopic_with_mono*])
using *Q* **by** (*auto simp*: *conth imh*)
then show *?thesis*
proof (*rule homotopic_with_eq*; *simp*)
show $\bigwedge h\ k.\ (\bigwedge x.\ x \in U \Longrightarrow h\ x = k\ x) \Longrightarrow Q\ h = Q\ k$
using *Qeq topspace_euclidean_subtopology* **by** *blast*
show $\bigwedge x.\ x \in U \Longrightarrow f\ x = h\ (k\ (f\ x))\ \bigwedge x.\ x \in U \Longrightarrow g\ x = h\ (k\ (g\ x))$
using *idhk imf img* **by** *auto*
qed
qed

lemma *homotopically_trivial_retraction_null_gen*:
assumes *P*: $\bigwedge f.\ [\![$*continuous_on* $U\ f;\ f\ `\ U \subseteq t;\ Q\ f]\!] \Longrightarrow P(k \circ f)$
and *Q*: $\bigwedge f.\ [\![$*continuous_on* $U\ f;\ f\ `\ U \subseteq s;\ P\ f]\!] \Longrightarrow Q(h \circ f)$
and *Qeq*: $\bigwedge h\ k.\ (\bigwedge x.\ x \in U \Longrightarrow h\ x = k\ x) \Longrightarrow Q\ h = Q\ k$
and *hom*: $\bigwedge f.\ [\![$*continuous_on* $U\ f;\ f\ `\ U \subseteq s;\ P\ f]\!]$
$\Longrightarrow \exists\, c.$ *homotopic_with_canon* $P\ U\ s\ f\ (\lambda x.\ c)$
and *contf*: *continuous_on* $U\ f$ **and** *imf*:$f\ `\ U \subseteq t$ **and** *Qf*: $Q\ f$
obtains *c* **where** *homotopic_with_canon* $Q\ U\ t\ f\ (\lambda x.\ c)$
proof −
have *feq*: $\bigwedge x.\ x \in U \Longrightarrow (h \circ (k \circ f))\ x = f\ x$ **using** *idhk imf* **by** *auto*
have *continuous_on* $U\ (k \circ f)$
using *contf continuous_on_compose continuous_on_subset contk imf* **by** *blast*
moreover have $(k \circ f)\ `\ U \subseteq s$
using *imf imk* **by** *fastforce*
moreover have $P\ (k \circ f)$
by (*simp add*: *P Qf contf imf*)
ultimately obtain *c* **where** *homotopic_with_canon* $P\ U\ s\ (k \circ f)\ (\lambda x.\ c)$
by (*metis hom*)
then have *homotopic_with_canon* $Q\ U\ t\ (h \circ (k \circ f))\ (h \circ (\lambda x.\ c))$
apply (*rule homotopic_with_compose_continuous_left* [*OF homotopic_with_mono*])
using *Q* **by** (*auto simp*: *conth imh*)
then have *homotopic_with_canon* $Q\ U\ t\ f\ (\lambda x.\ h\ c)$
proof (*rule homotopic_with_eq*)
show $\bigwedge x.\ x \in$ *topspace* (*top_of_set* U) $\Longrightarrow f\ x = (h \circ (k \circ f))\ x$
using *feq* **by** *auto*
show $\bigwedge h\ k.\ (\bigwedge x.\ x \in$ *topspace* (*top_of_set* U) $\Longrightarrow h\ x = k\ x) \Longrightarrow Q\ h = Q\ k$
using *Qeq topspace_euclidean_subtopology* **by** *blast*
qed *auto*
then show *?thesis*
using *that* **by** *blast*
qed

lemma *cohomotopically_trivial_retraction_gen*:
  assumes P: $\bigwedge f. [\![continuous\_on\ t\ f;\ f\ `\ t \subseteq U;\ Q\ f]\!] \Longrightarrow P(f \circ h)$
    and Q: $\bigwedge f. [\![continuous\_on\ s\ f;\ f\ `\ s \subseteq U;\ P\ f]\!] \Longrightarrow Q(f \circ k)$
    and Qeq: $\bigwedge h\ k.\ (\bigwedge x.\ x \in t \Longrightarrow h\ x = k\ x) \Longrightarrow Q\ h = Q\ k$
    and hom: $\bigwedge f\ g.\ [\![continuous\_on\ s\ f;\ f\ `\ s \subseteq U;\ P\ f;$
        $continuous\_on\ s\ g;\ g\ `\ s \subseteq U;\ P\ g]\!]$
        $\Longrightarrow homotopic\_with\_canon\ P\ s\ U\ f\ g$
    and $contf$: $continuous\_on\ t\ f$ **and** imf: $f\ `\ t \subseteq U$ **and** Qf: $Q\ f$
    and $contg$: $continuous\_on\ t\ g$ **and** img: $g\ `\ t \subseteq U$ **and** Qg: $Q\ g$
   shows $homotopic\_with\_canon\ Q\ t\ U\ f\ g$
proof −
  have feq: $\bigwedge x.\ x \in t \Longrightarrow (f \circ h \circ k)\ x = f\ x$ **using** *idhk imf* **by** *auto*
  have geq: $\bigwedge x.\ x \in t \Longrightarrow (g \circ h \circ k)\ x = g\ x$ **using** *idhk img* **by** *auto*
  have $continuous\_on\ s\ (f \circ h)$
   using *contf conth continuous_on_compose imh* **by** *blast*
  moreover have $(f \circ h)\ `\ s \subseteq U$
   using *imf imh* **by** *fastforce*
  moreover have $P\ (f \circ h)$
   by (*simp add*: *P Qf contf imf*)
  moreover have $continuous\_on\ s\ (g \circ h)$
   using *contg continuous_on_compose continuous_on_subset conth imh* **by** *blast*
  moreover have $(g \circ h)\ `\ s \subseteq U$
   using *img imh* **by** *fastforce*
  moreover have $P\ (g \circ h)$
   by (*simp add*: *P Qg contg img*)
  ultimately have $homotopic\_with\_canon\ P\ s\ U\ (f \circ h)\ (g \circ h)$
   by (*rule hom*)
  then have $homotopic\_with\_canon\ Q\ t\ U\ (f \circ h \circ k)\ (g \circ h \circ k)$
   apply (*rule homotopic_with_compose_continuous_right* [*OF homotopic_with_mono*])
   using Q **by** (*auto simp*: *contk imk*)
  then show *?thesis*
  proof (*rule homotopic_with_eq*)
   show $f\ x = (f \circ h \circ k)\ x\ g\ x = (g \circ h \circ k)\ x$
    if $x \in topspace\ (top\_of\_set\ t)$ **for** x
    using *feq geq that* **by** *force+*
  qed (*use Qeq topspace_euclidean_subtopology* **in** *blast*)
qed

lemma *cohomotopically_trivial_retraction_null_gen*:
  assumes P: $\bigwedge f. [\![continuous\_on\ t\ f;\ f\ `\ t \subseteq U;\ Q\ f]\!] \Longrightarrow P(f \circ h)$
    and Q: $\bigwedge f. [\![continuous\_on\ s\ f;\ f\ `\ s \subseteq U;\ P\ f]\!] \Longrightarrow Q(f \circ k)$
    and Qeq: $\bigwedge h\ k.\ (\bigwedge x.\ x \in t \Longrightarrow h\ x = k\ x) \Longrightarrow Q\ h = Q\ k$
    and hom: $\bigwedge f\ g.\ [\![continuous\_on\ s\ f;\ f\ `\ s \subseteq U;\ P\ f]\!]$
        $\Longrightarrow \exists\, c.\ homotopic\_with\_canon\ P\ s\ U\ f\ (\lambda x.\ c)$
    and $contf$: $continuous\_on\ t\ f$ **and** imf: $f\ `\ t \subseteq U$ **and** Qf: $Q\ f$
  obtains c **where** $homotopic\_with\_canon\ Q\ t\ U\ f\ (\lambda x.\ c)$
proof −
  have feq: $\bigwedge x.\ x \in t \Longrightarrow (f \circ h \circ k)\ x = f\ x$ **using** *idhk imf* **by** *auto*
  have $continuous\_on\ s\ (f \circ h)$

using *contf conth continuous_on_compose imh* **by** *blast*
moreover have $(f \circ h)$ ‘ $s \subseteq U$
using *imf imh* **by** *fastforce*
moreover have P $(f \circ h)$
by (*simp add*: P *Qf contf imf*)
ultimately obtain c **where** *homotopic_with_canon* P s U $(f \circ h)$ $(\lambda x.\ c)$
by (*metis hom*)
then have §: *homotopic_with_canon* Q t U $(f \circ h \circ k)$ $((\lambda x.\ c) \circ k)$
proof (*rule homotopic_with_compose_continuous_right* [*OF homotopic_with_mono*])
show $\bigwedge h.$ ⟦*continuous_map* (*top_of_set* s) (*top_of_set* U) h; P h⟧ $\Longrightarrow$ Q $(h \circ k)$
using Q **by** *auto*
qed (*auto simp*: *contk imk*)
moreover have *homotopic_with_canon* Q t U f $(\lambda x.\ c)$
using *homotopic_with_eq* [*OF* §] *feq Qeq* **by** *fastforce*
ultimately show *?thesis*
using *that* **by** *blast*
qed

end

lemma *simply_connected_retraction_gen*:
shows ⟦*simply_connected* S; *continuous_on* S h; h ‘ $S = T$;
continuous_on T k; k ‘ $T \subseteq S$; $\bigwedge y.\ y \in T \Longrightarrow h(k\ y) = y$⟧
$\Longrightarrow$ *simply_connected* T
apply (*simp add*: *simply_connected_def path_def path_image_def homotopic_loops_def*, *clarify*)
apply (*rule Retracts.homotopically_trivial_retraction_gen*
[*of* S h _ k _ $\lambda p.$ *pathfinish* p = *pathstart* p $\lambda p.$ *pathfinish* p = *pathstart* p])
apply (*simp_all add*: *Retracts_def pathfinish_def pathstart_def*)
done

lemma *homeomorphic_simply_connected*:
⟦S *homeomorphic* T; *simply_connected* S⟧ $\Longrightarrow$ *simply_connected* T
by (*auto simp*: *homeomorphic_def homeomorphism_def intro*: *simply_connected_retraction_gen*)

lemma *homeomorphic_simply_connected_eq*:
S *homeomorphic* T $\Longrightarrow$ (*simply_connected* S $\longleftrightarrow$ *simply_connected* T)
by (*metis homeomorphic_simply_connected homeomorphic_sym*)

6.18.22 Homotopy equivalence

6.18.23 Homotopy equivalence of topological spaces.

definition *homotopy_equivalent_space*
(**infix** *homotopy'_equivalent'_space 50*)
where X *homotopy_equivalent_space* Y ≡
(∃f g. *continuous_map* X Y f ∧
continuous_map Y X g ∧
homotopic_with $(\lambda x.\ True)$ X X $(g \circ f)$ *id* ∧
homotopic_with $(\lambda x.\ True)$ Y Y $(f \circ g)$ *id*)

lemma *homeomorphic_imp_homotopy_equivalent_space*:
X homeomorphic_space Y ⟹ *X homotopy_equivalent_space Y*
unfolding *homeomorphic_space_def homotopy_equivalent_space_def*
apply (*erule ex_forward*)+
by (*simp add*: *homotopic_with_equal homotopic_with_sym homeomorphic_maps_def*)

lemma *homotopy_equivalent_space_refl*:
X homotopy_equivalent_space X
by (*simp add*: *homeomorphic_imp_homotopy_equivalent_space homeomorphic_space_refl*)

lemma *homotopy_equivalent_space_sym*:
X homotopy_equivalent_space Y ⟷ *Y homotopy_equivalent_space X*
by (*meson homotopy_equivalent_space_def*)

lemma *homotopy_eqv_trans* [*trans*]:
assumes *1*: *X homotopy_equivalent_space Y* **and** *2*: *Y homotopy_equivalent_space U*
shows *X homotopy_equivalent_space U*
proof −
obtain *f1 g1* **where** *f1*: *continuous_map X Y f1*
and *g1*: *continuous_map Y X g1*
and *hom1*: *homotopic_with* (λ*x*. *True*) *X X* (*g1* ∘ *f1*) *id*
homotopic_with (λ*x*. *True*) *Y Y* (*f1* ∘ *g1*) *id*
using *1* **by** (*auto simp*: *homotopy_equivalent_space_def*)
obtain *f2 g2* **where** *f2*: *continuous_map Y U f2*
and *g2*: *continuous_map U Y g2*
and *hom2*: *homotopic_with* (λ*x*. *True*) *Y Y* (*g2* ∘ *f2*) *id*
homotopic_with (λ*x*. *True*) *U U* (*f2* ∘ *g2*) *id*
using *2* **by** (*auto simp*: *homotopy_equivalent_space_def*)
have *homotopic_with* (λ*f*. *True*) *X Y* (*g2* ∘ *f2* ∘ *f1*) (*id* ∘ *f1*)
using *f1 hom2*(*1*) *homotopic_with_compose_continuous_map_right* **by** *metis*
then have *homotopic_with* (λ*f*. *True*) *X Y* (*g2* ∘ (*f2* ∘ *f1*)) (*id* ∘ *f1*)
by (*simp add*: *o_assoc*)
then have *homotopic_with* (λ*x*. *True*) *X X*
(*g1* ∘ (*g2* ∘ (*f2* ∘ *f1*))) (*g1* ∘ (*id* ∘ *f1*))
by (*simp add*: *g1 homotopic_with_compose_continuous_map_left*)
moreover have *homotopic_with* (λ*x*. *True*) *X X* (*g1* ∘ *id* ∘ *f1*) *id*
using *hom1* **by** *simp*
ultimately have *SS*: *homotopic_with* (λ*x*. *True*) *X X* (*g1* ∘ *g2* ∘ (*f2* ∘ *f1*)) *id*
by (*metis comp_assoc homotopic_with_trans id_comp*)
have *homotopic_with* (λ*f*. *True*) *U Y* (*f1* ∘ *g1* ∘ *g2*) (*id* ∘ *g2*)
using *g2 hom1*(*2*) *homotopic_with_compose_continuous_map_right* **by** *fastforce*
then have *homotopic_with* (λ*f*. *True*) *U Y* (*f1* ∘ (*g1* ∘ *g2*)) (*id* ∘ *g2*)
by (*simp add*: *o_assoc*)
then have *homotopic_with* (λ*x*. *True*) *U U*
(*f2* ∘ (*f1* ∘ (*g1* ∘ *g2*))) (*f2* ∘ (*id* ∘ *g2*))
by (*simp add*: *f2 homotopic_with_compose_continuous_map_left*)
moreover have *homotopic_with* (λ*x*. *True*) *U U* (*f2* ∘ *id* ∘ *g2*) *id*

using *hom2* **by** *simp*
ultimately have *UU*: *homotopic_with* (λ*x*. *True*) *U U* (*f2* ∘ *f1* ∘ (*g1* ∘ *g2*)) *id*
by (*simp add*: *fun.map_comp hom2*(*2*) *homotopic_with_trans*)
show *?thesis*
unfolding *homotopy_equivalent_space_def*
by (*blast intro*: *f1 f2 g1 g2 continuous_map_compose SS UU*)
qed

lemma *deformation_retraction_imp_homotopy_equivalent_space*:
⟦*homotopic_with* (λ*x*. *True*) *X X* (*s* ∘ *r*) *id*; *retraction_maps X Y r s*⟧
⟹ *X homotopy_equivalent_space Y*
unfolding *homotopy_equivalent_space_def retraction_maps_def*
using *homotopic_with_id2* **by** *fastforce*

lemma *deformation_retract_imp_homotopy_equivalent_space*:
⟦*homotopic_with* (λ*x*. *True*) *X X r id*; *retraction_maps X Y r id*⟧
⟹ *X homotopy_equivalent_space Y*
using *deformation_retraction_imp_homotopy_equivalent_space* **by** *force*

lemma *deformation_retract_of_space*:
S ⊆ *topspace X* ∧
(∃ *r*. *homotopic_with* (λ*x*. *True*) *X X id r* ∧ *retraction_maps X* (*subtopology X S*) *r id*) ⟷
S retract_of_space X ∧ (∃ *f*. *homotopic_with* (λ*x*. *True*) *X X id f* ∧ *f* ‘ (*topspace X*) ⊆ *S*)
proof (*cases S* ⊆ *topspace X*)
case *True*
moreover have (∃ *r*. *homotopic_with* (λ*x*. *True*) *X X id r* ∧ *retraction_maps X* (*subtopology X S*) *r id*)
⟷ (*S retract_of_space X* ∧ (∃ *f*. *homotopic_with* (λ*x*. *True*) *X X id f* ∧ *f* ‘ *topspace X* ⊆ *S*))
unfolding *retract_of_space_def*
proof *safe*
fix *f r*
assume *f*: *homotopic_with* (λ*x*. *True*) *X X id f*
and *fS*: *f* ‘ *topspace X* ⊆ *S*
and *r*: *continuous_map X* (*subtopology X S*) *r*
and *req*: ∀ *x*∈*S*. *r x* = *x*
show ∃ *r*. *homotopic_with* (λ*x*. *True*) *X X id r* ∧ *retraction_maps X* (*subtopology X S*) *r id*
proof (*intro exI conjI*)
have *homotopic_with* (λ*x*. *True*) *X X f r*
proof (*rule homotopic_with_eq*)
show *homotopic_with* (λ*x*. *True*) *X X* (*r* ∘ *f*) (*r* ∘ *id*)
by (*metis continuous_map_into_fulltopology f homotopic_with_compose_continuous_map_left homotopic_with_symD r*)
show *f x* = (*r* ∘ *f*) *x* **if** *x* ∈ *topspace X* **for** *x*
using *that fS req* **by** *auto*
qed *auto*

then show *homotopic_with* (λ*x*. *True*) *X X id r*
by (*rule homotopic_with_trans* [*OF f*])
next
show *retraction_maps X* (*subtopology X S*) *r id*
by (*simp add*: *r req retraction_maps_def*)
qed
qed (*use True* **in** ‹*auto simp*: *retraction_maps_def topspace_subtopology_subset continuous_map_in_subtopology*›)
ultimately show *?thesis* **by** *simp*
qed (*auto simp*: *retract_of_space_def retraction_maps_def*)

6.18.24 Contractible spaces

The definition (which agrees with "contractible" on subsets of Euclidean space) is a little cryptic because we don't in fact assume that the constant "a" is in the space. This forces the convention that the empty space / set is contractible, avoiding some special cases.

definition *contractible_space* **where**
contractible_space X ≡ ∃ *a*. *homotopic_with* (λ*x*. *True*) *X X id* (λ*x*. *a*)

lemma *contractible_space_top_of_set* [*simp*]:*contractible_space* (*top_of_set S*) ⟷ *contractible S*
by (*auto simp*: *contractible_space_def contractible_def*)

lemma *contractible_space_empty*:
topspace X = {} ⟹ *contractible_space X*
unfolding *contractible_space_def homotopic_with_def*
apply (*rule_tac x*=*undefined* **in** *exI*)
apply (*rule_tac x*=λ(*t*,*x*). *if t* = *0 then x else undefined* **in** *exI*)
apply (*auto simp*: *continuous_map_on_empty*)
done

lemma *contractible_space_singleton*:
topspace X = {*a*} ⟹ *contractible_space X*
unfolding *contractible_space_def homotopic_with_def*
apply (*rule_tac x*=*a* **in** *exI*)
apply (*rule_tac x*=λ(*t*,*x*). *if t* = *0 then x else a* **in** *exI*)
apply (*auto intro*: *continuous_map_eq* [**where** *f* = λ*z*. *a*])
done

lemma *contractible_space_subset_singleton*:
topspace X ⊆ {*a*} ⟹ *contractible_space X*
by (*meson contractible_space_empty contractible_space_singleton subset_singletonD*)

lemma *contractible_space_subtopology_singleton*:
contractible_space(*subtopology X* {*a*})
by (*meson contractible_space_subset_singleton insert_subset path_connectedin_singleton path_connectedin_subtopology subsetI*)

lemma *contractible_space*:
contractible_space X ⟷
topspace X = {} ∨
(∃ *a* ∈ *topspace X*. *homotopic_with* (λ*x*. *True*) *X X id* (λ*x*. *a*))
proof (*cases topspace X* = {})
case *False*
then show *?thesis*
using *homotopic_with_imp_continuous_maps* **by** (*fastforce simp*: *contractible_space_def*)
qed (*simp add*: *contractible_space_empty*)

lemma *contractible_imp_path_connected_space*:
assumes *contractible_space X* **shows** *path_connected_space X*
proof (*cases topspace X* = {})
case *False*
have ∗: *path_connected_space X*
if *a* ∈ *topspace X* **and** *conth*: *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) *X*) *X h*
and *h*: ∀ *x*. *h* (*0*, *x*) = *x* ∀ *x*. *h* (*1*, *x*) = *a*
for *a* **and** *h* :: *real* × ′*a* ⇒ ′*a*
proof −
have *path_component_of X b a* **if** *b* ∈ *topspace X* **for** *b*
unfolding *path_component_of_def*
proof (*intro exI conjI*)
let *?g* = *h* ∘ (λ*x*. (*x*,*b*))
show *pathin X ?g*
unfolding *pathin_def*
proof (*rule continuous_map_compose* [*OF* _ *conth*])
show *continuous_map* (*top_of_set* {*0..1*}) (*prod_topology* (*top_of_set* {*0..1*}) *X*) (λ*x*. (*x*, *b*))
using *that* **by** (*auto intro*!: *continuous_intros*)
qed
qed (*use h* **in** *auto*)
then show *?thesis*
by (*metis path_component_of_equiv path_connected_space_iff_path_component*)
qed
show *?thesis*
using *assms False* **by** (*auto simp*: *contractible_space homotopic_with_def* ∗)
qed (*simp add*: *path_connected_space_topspace_empty*)

lemma *contractible_imp_connected_space*:
contractible_space X ⟹ *connected_space X*
by (*simp add*: *contractible_imp_path_connected_space path_connected_imp_connected_space*)

lemma *contractible_space_alt*:
contractible_space X ⟷ (∀ *a* ∈ *topspace X*. *homotopic_with* (λ*x*. *True*) *X X id* (λ*x*. *a*)) (**is** *?lhs* = *?rhs*)
proof
assume *X*: *?lhs*

then obtain *a* **where** *a*: *homotopic_with* (*λx. True*) *X X id* (*λx. a*)
by (*auto simp*: *contractible_space_def*)
show *?rhs*
proof
show *homotopic_with* (*λx. True*) *X X id* (*λx. b*) **if** *b* ∈ *topspace X* **for** *b*
proof (*rule homotopic_with_trans* [*OF a*])
show *homotopic_with* (*λx. True*) *X X* (*λx. a*) (*λx. b*)
using *homotopic_constant_maps path_connected_space_imp_path_component_of*
by (*metis* (*full_types*) *X a continuous_map_const contractible_imp_path_connected_space homotopic_with_imp_continuous_maps that*)
qed
qed
next
assume *R*: *?rhs*
then show *?lhs*
unfolding *contractible_space_def* **by** (*metis equals0I homotopic_on_emptyI*)
qed

lemma *compose_const* [*simp*]: *f* ∘ (*λx. a*) = (*λx. f a*) (*λx. a*) ∘ *g* = (*λx. a*)
by (*simp_all add*: *o_def*)

lemma *nullhomotopic_through_contractible_space*:
assumes *f*: *continuous_map X Y f* **and** *g*: *continuous_map Y Z g* **and** *Y*: *contractible_space Y*
obtains *c* **where** *homotopic_with* (*λh. True*) *X Z* (*g* ∘ *f*) (*λx. c*)
proof −
obtain *b* **where** *b*: *homotopic_with* (*λx. True*) *Y Y id* (*λx. b*)
using *Y* **by** (*auto simp*: *contractible_space_def*)
show *thesis*
using *homotopic_with_compose_continuous_map_right*
[*OF homotopic_with_compose_continuous_map_left* [*OF b g*] *f*]
by (*force simp add*: *that*)
qed

lemma *nullhomotopic_into_contractible_space*:
assumes *f*: *continuous_map X Y f* **and** *Y*: *contractible_space Y*
obtains *c* **where** *homotopic_with* (*λh. True*) *X Y f* (*λx. c*)
using *nullhomotopic_through_contractible_space* [*OF f _ Y*]
by (*metis continuous_map_id id_comp*)

lemma *nullhomotopic_from_contractible_space*:
assumes *f*: *continuous_map X Y f* **and** *X*: *contractible_space X*
obtains *c* **where** *homotopic_with* (*λh. True*) *X Y f* (*λx. c*)
using *nullhomotopic_through_contractible_space* [*OF _ f X*]
by (*metis comp_id continuous_map_id*)

lemma *homotopy_dominated_contractibility*:
assumes *f*: *continuous_map X Y f* **and** *g*: *continuous_map Y X g*

and *hom*: *homotopic_with* (λ*x*. *True*) *Y Y* (*f* ∘ *g*) *id* **and** *X*: *contractible_space X*
shows *contractible_space Y*
proof −
obtain *c* **where** *c*: *homotopic_with* (λ*h*. *True*) *X Y f* (λ*x*. *c*)
using *nullhomotopic_from_contractible_space* [*OF f X*] **.**
have *homotopic_with* (λ*x*. *True*) *Y Y* (*f* ∘ *g*) (λ*x*. *c*)
using *homotopic_with_compose_continuous_map_right* [*OF c g*] **by** *fastforce*
then have *homotopic_with* (λ*x*. *True*) *Y Y id* (λ*x*. *c*)
using *homotopic_with_trans* [*OF _ hom*] *homotopic_with_symD* **by** *blast*
then show *?thesis*
unfolding *contractible_space_def* **..**
qed

lemma *homotopy_equivalent_space_contractibility*:
X homotopy_equivalent_space Y ⟹ (*contractible_space X* ⟷ *contractible_space Y*)
unfolding *homotopy_equivalent_space_def*
by (*blast intro*: *homotopy_dominated_contractibility*)

lemma *homeomorphic_space_contractibility*:
X homeomorphic_space Y
⟹ (*contractible_space X* ⟷ *contractible_space Y*)
by (*simp add*: *homeomorphic_imp_homotopy_equivalent_space homotopy_equivalent_space_contractibility*)

lemma *contractible_eq_homotopy_equivalent_singleton_subtopology*:
contractible_space X ⟷
topspace X = {} ∨ (∃ *a* ∈ *topspace X*. *X homotopy_equivalent_space* (*subtopology X* {*a*}))(**is** *?lhs* = *?rhs*)
proof (*cases topspace X* = {})
case *False*
show *?thesis*
proof
assume *?lhs*
then obtain *a* **where** *a*: *homotopic_with* (λ*x*. *True*) *X X id* (λ*x*. *a*)
by (*auto simp*: *contractible_space_def*)
then have *a* ∈ *topspace X*
by (*metis False continuous_map_const homotopic_with_imp_continuous_maps*)
then have *homotopic_with* (λ*x*. *True*) (*subtopology X* {*a*}) (*subtopology X* {*a*}) *id* (λ*x*. *a*)
using *connectedin_absolute connectedin_sing contractible_space_alt contractible_space_subtopology_singl*
by *fastforce*
then have *X homotopy_equivalent_space subtopology X* {*a*}
unfolding *homotopy_equivalent_space_def* **using** ‹*a* ∈ *topspace X*›
by (*metis* (*full_types*) *a comp_id continuous_map_const continuous_map_id_subt empty_subsetI homotopic_with_symD*
id_comp insertI1 insert_subset topspace_subtopology_subset)
with ‹*a* ∈ *topspace X*› **show** *?rhs*
by *blast*

next
assume *?rhs*
then show *?lhs*
by (*meson False contractible_space_subtopology_singleton homotopy_equivalent_space_contractibility*)
qed
qed (*simp add*: *contractible_space_empty*)

lemma *contractible_space_retraction_map_image*:
assumes *retraction_map X Y f* **and** *X*: *contractible_space X*
shows *contractible_space Y*
proof −
obtain *g* **where** *f*: *continuous_map X Y f* **and** *g*: *continuous_map Y X g* **and** *fg*: ∀ *y* ∈ *topspace Y*. *f*(*g y*) = *y*
using *assms* **by** (*auto simp*: *retraction_map_def retraction_maps_def*)
obtain *a* **where** *a*: *homotopic_with* (λ*x*. *True*) *X X id* (λ*x*. *a*)
using *X* **by** (*auto simp*: *contractible_space_def*)
have *homotopic_with* (λ*x*. *True*) *Y Y id* (λ*x*. *f a*)
proof (*rule homotopic_with_eq*)
show *homotopic_with* (λ*x*. *True*) *Y Y* (*f* ∘ *id* ∘ *g*) (*f* ∘ (λ*x*. *a*) ∘ *g*)
using *f g a homotopic_with_compose_continuous_map_left homotopic_with_compose_continuous_map_right* **by** *metis*
qed (*use fg* **in** *auto*)
then show *?thesis*
unfolding *contractible_space_def* **by** *blast*
qed

lemma *contractible_space_prod_topology*:
contractible_space(*prod_topology X Y*) ⟷
topspace X = {} ∨ *topspace Y* = {} ∨ *contractible_space X* ∧ *contractible_space Y*
proof (*cases topspace X* = {} ∨ *topspace Y* = {})
case *True*
then have *topspace* (*prod_topology X Y*) = {}
by *simp*
then show *?thesis*
by (*auto simp*: *contractible_space_empty*)
next
case *False*
have *contractible_space*(*prod_topology X Y*) ⟷ *contractible_space X* ∧ *contractible_space Y*
proof *safe*
assume *XY*: *contractible_space* (*prod_topology X Y*)
with *False* **have** *retraction_map* (*prod_topology X Y*) *X fst*
by (*auto simp*: *contractible_space False retraction_map_fst*)
then show *contractible_space X*
by (*rule contractible_space_retraction_map_image* [*OF _ XY*])
have *retraction_map* (*prod_topology X Y*) *Y snd*
using *False XY* **by** (*auto simp*: *contractible_space False retraction_map_snd*)
then show *contractible_space Y*

```
    by (rule contractible_space_retraction_map_image [OF _ XY])
  next
    assume contractible_space X and contractible_space Y
    with False obtain a b
      where a ∈ topspace X and a: homotopic_with (λx. True) X X id (λx. a)
        and b ∈ topspace Y and b: homotopic_with (λx. True) Y Y id (λx. b)
      by (auto simp: contractible_space)
    with False show contractible_space (prod_topology X Y)
      apply (simp add: contractible_space)
      apply (rule_tac x=a in bexI)
       apply (rule_tac x=b in bexI)
      using homotopic_with_prod_topology [OF a b]
        apply (metis (no_types, lifting) case_prod_Pair case_prod_beta' eq_id_iff)
       apply auto
      done
  qed
  with False show ?thesis
    by auto
qed


lemma contractible_space_product_topology:
  contractible_space(product_topology X I) ⟷
    topspace (product_topology X I) = {} ∨ (∀ i ∈ I. contractible_space(X i))
proof (cases topspace (product_topology X I) = {})
  case False
  have 1: contractible_space (X i)
    if XI: contractible_space (product_topology X I) and i ∈ I
    for i
  proof (rule contractible_space_retraction_map_image [OF _ XI])
    show retraction_map (product_topology X I) (X i) (λx. x i)
      using False by (simp add: retraction_map_product_projection ‹i ∈ I›)
  qed
  have 2: contractible_space (product_topology X I)
    if x ∈ topspace (product_topology X I) and cs: ∀ i∈I. contractible_space (X i)
    for x :: 'a ⇒ 'b
  proof −
   obtain f where f: ⋀i. i∈I ⟹ homotopic_with (λx. True) (X i) (X i) id (λx.
f i)
      using cs unfolding contractible_space_def by metis
    have homotopic_with (λx. True)
                    (product_topology X I) (product_topology X I) id (λx. restrict f
I)
       by (rule homotopic_with_eq [OF homotopic_with_product_topology [OF f]])
(auto)
    then show ?thesis
      by (auto simp: contractible_space_def)
  qed
  show ?thesis
```

```
    using False 1 2 by blast
qed (simp add: contractible_space_empty)


lemma contractible_space_subtopology_euclideanreal [simp]:
  contractible_space(subtopology euclideanreal S) ⟷ is_interval S
  (is ?lhs = ?rhs)
proof
  assume ?lhs
  then have path_connectedin (subtopology euclideanreal S) S
   using contractible_imp_path_connected_space path_connectedin_topspace path_connectedin_absolute
    by (simp add: contractible_imp_path_connected)
  then show ?rhs
    by (simp add: is_interval_path_connected_1)
next
  assume ?rhs
  then have convex S
    by (simp add: is_interval_convex_1)
  show ?lhs
  proof (cases S = {})
    case False
    then obtain z where z ∈ S
      by blast
    show ?thesis
      unfolding contractible_space_def homotopic_with_def
    proof (intro exI conjI allI)
      note § = convexD [OF ‹convex S›, simplified]
     show continuous_map (prod_topology (top_of_set {0..1}) (top_of_set S)) (top_of_set
S)
                        (λ(t,x). (1 − t) ∗ x + t ∗ z)
        using ‹z ∈ S›
        by (auto simp add: case_prod_unfold intro!: continuous_intros §)
    qed auto
  qed (simp add: contractible_space_empty)
qed


corollary contractible_space_euclideanreal: contractible_space euclideanreal
proof −
  have contractible_space (subtopology euclideanreal UNIV)
    using contractible_space_subtopology_euclideanreal by blast
  then show ?thesis
    by simp
qed


abbreviation homotopy_eqv :: 'a::topological_space set ⇒ 'b::topological_space set
⇒ bool
             (infix homotopy'_eqv 50)
```

where *S homotopy_eqv T* ≡ *top_of_set S homotopy_equivalent_space top_of_set T*

lemma *homeomorphic_imp_homotopy_eqv*: *S homeomorphic T* ⟹ *S homotopy_eqv T*
unfolding *homeomorphic_def homeomorphism_def homotopy_equivalent_space_def*
by (*metis continuous_map_subtopology_eu homotopic_with_id2 openin_imp_subset openin_subtopology_self topspace_euclidean_subtopology*)

lemma *homotopy_eqv_inj_linear_image*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *linear f inj f*
shows (*f ‘ S*) *homotopy_eqv S*
by (*metis assms homeomorphic_sym homeomorphic_imp_homotopy_eqv linear_homeomorphic_image*)

lemma *homotopy_eqv_translation*:
fixes *S* :: *'a::real_normed_vector set*
shows (+) *a ‘ S homotopy_eqv S*
using *homeomorphic_imp_homotopy_eqv homeomorphic_translation homeomorphic_sym* **by** *blast*

lemma *homotopy_eqv_homotopic_triviality_imp*:
fixes *S* :: *'a::real_normed_vector set*
and *T* :: *'b::real_normed_vector set*
and *U* :: *'c::real_normed_vector set*
assumes *S homotopy_eqv T*
and *f*: *continuous_on U f f ‘ U* ⊆ *T*
and *g*: *continuous_on U g g ‘ U* ⊆ *T*
and *homUS*: ⋀*f g*. ⟦*continuous_on U f*; *f ‘ U* ⊆ *S*;
continuous_on U g; *g ‘ U* ⊆ *S*⟧
⟹ *homotopic_with_canon* (λ*x*. *True*) *U S f g*
shows *homotopic_with_canon* (λ*x*. *True*) *U T f g*
proof −
obtain *h k* **where** *h*: *continuous_on S h h ‘ S* ⊆ *T*
and *k*: *continuous_on T k k ‘ T* ⊆ *S*
and *hom*: *homotopic_with_canon* (λ*x*. *True*) *S S* (*k* ∘ *h*) *id*
homotopic_with_canon (λ*x*. *True*) *T T* (*h* ∘ *k*) *id*
using *assms* **by** (*auto simp*: *homotopy_equivalent_space_def*)
have *homotopic_with_canon* (λ*f*. *True*) *U S* (*k* ∘ *f*) (*k* ∘ *g*)
proof (*rule homUS*)
show *continuous_on U* (*k* ∘ *f*) *continuous_on U* (*k* ∘ *g*)
using *continuous_on_compose continuous_on_subset f g k* **by** *blast*+
qed (*use f g k* **in** ‹(*force simp*: *o_def*)+›)
then have *homotopic_with_canon* (λ*x*. *True*) *U T* (*h* ∘ (*k* ∘ *f*)) (*h* ∘ (*k* ∘ *g*))
by (*rule homotopic_with_compose_continuous_left*; *simp add*: *h*)

moreover have *homotopic_with_canon* (λ*x*. *True*) *U T* (*h* ∘ *k* ∘ *f*) (*id* ∘ *f*)
by (*rule homotopic_with_compose_continuous_right* [**where** *X*=*T* **and** *Y*=*T*]; *simp add*: *hom f*)
moreover have *homotopic_with_canon* (λ*x*. *True*) *U T* (*h* ∘ *k* ∘ *g*) (*id* ∘ *g*)
by (*rule homotopic_with_compose_continuous_right* [**where** *X*=*T* **and** *Y*=*T*]; *simp add*: *hom g*)
ultimately show *homotopic_with_canon* (λ*x*. *True*) *U T f g*
unfolding *o_assoc*
by (*metis homotopic_with_trans homotopic_with_sym id_comp*)
qed

lemma *homotopy_eqv_homotopic_triviality*:
fixes *S* :: *'a::real_normed_vector set*
and *T* :: *'b::real_normed_vector set*
and *U* :: *'c::real_normed_vector set*
assumes *S homotopy_eqv T*
shows (∀*f g*. *continuous_on U f* ∧ *f* ‘ *U* ⊆ *S* ∧
continuous_on U g ∧ *g* ‘ *U* ⊆ *S*
⟶ *homotopic_with_canon* (λ*x*. *True*) *U S f g*) ⟷
(∀*f g*. *continuous_on U f* ∧ *f* ‘ *U* ⊆ *T* ∧
continuous_on U g ∧ *g* ‘ *U* ⊆ *T*
⟶ *homotopic_with_canon* (λ*x*. *True*) *U T f g*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*metis assms homotopy_eqv_homotopic_triviality_imp*)
next
assume *?rhs*
moreover
have *T homotopy_eqv S*
using *assms homotopy_equivalent_space_sym* **by** *blast*
ultimately show *?lhs*
by (*blast intro*: *homotopy_eqv_homotopic_triviality_imp*)
qed

lemma *homotopy_eqv_cohomotopic_triviality_null_imp*:
fixes *S* :: *'a::real_normed_vector set*
and *T* :: *'b::real_normed_vector set*
and *U* :: *'c::real_normed_vector set*
assumes *S homotopy_eqv T*
and *f*: *continuous_on T f f* ‘ *T* ⊆ *U*
and *homSU*: ⋀*f*. ⟦*continuous_on S f*; *f* ‘ *S* ⊆ *U*⟧
⟹ ∃ *c*. *homotopic_with_canon* (λ*x*. *True*) *S U f* (λ*x*. *c*)
obtains *c* **where** *homotopic_with_canon* (λ*x*. *True*) *T U f* (λ*x*. *c*)
proof −
obtain *h k* **where** *h*: *continuous_on S h h* ‘ *S* ⊆ *T*
and *k*: *continuous_on T k k* ‘ *T* ⊆ *S*

and *hom*: *homotopic_with_canon* (λ*x*. *True*) *S S* (*k* ∘ *h*) *id*
homotopic_with_canon (λ*x*. *True*) *T T* (*h* ∘ *k*) *id*
using *assms* **by** (*auto simp*: *homotopy_equivalent_space_def*)
obtain *c* **where** *homotopic_with_canon* (λ*x*. *True*) *S U* (*f* ∘ *h*) (λ*x*. *c*)
proof (*rule exE* [*OF homSU*])
show *continuous_on S* (*f* ∘ *h*)
using *continuous_on_compose continuous_on_subset f h* **by** *blast*
qed (*use f h* **in** *force*)
then have *homotopic_with_canon* (λ*x*. *True*) *T U* ((*f* ∘ *h*) ∘ *k*) ((λ*x*. *c*) ∘ *k*)
by (*rule homotopic_with_compose_continuous_right* [**where** *X*=*S*]) (*use k* **in** *auto*)
moreover have *homotopic_with_canon* (λ*x*. *True*) *T U* (*f* ∘ *id*) (*f* ∘ (*h* ∘ *k*))
by (*rule homotopic_with_compose_continuous_left* [**where** *Y*=*T*])
(*use f* **in** ‹*auto simp add*: *hom homotopic_with_symD*›)
ultimately show *?thesis*
using *that homotopic_with_trans* **by** (*fastforce simp add*: *o_def*)
qed

lemma *homotopy_eqv_cohomotopic_triviality_null*:
fixes *S* :: ′*a*::*real_normed_vector set*
and *T* :: ′*b*::*real_normed_vector set*
and *U* :: ′*c*::*real_normed_vector set*
assumes *S homotopy_eqv T*
shows (∀*f*. *continuous_on S f* ∧ *f* ‘ *S* ⊆ *U*
⟶ (∃*c*. *homotopic_with_canon* (λ*x*. *True*) *S U f* (λ*x*. *c*))) ⟷
(∀*f*. *continuous_on T f* ∧ *f* ‘ *T* ⊆ *U*
⟶ (∃*c*. *homotopic_with_canon* (λ*x*. *True*) *T U f* (λ*x*. *c*)))
by (*rule iffI*; *metis assms homotopy_eqv_cohomotopic_triviality_null_imp homotopy_equivalent_space_sym*)

Similar to the proof above

lemma *homotopy_eqv_homotopic_triviality_null_imp*:
fixes *S* :: ′*a*::*real_normed_vector set*
and *T* :: ′*b*::*real_normed_vector set*
and *U* :: ′*c*::*real_normed_vector set*
assumes *S homotopy_eqv T*
and *f*: *continuous_on U f f* ‘ *U* ⊆ *T*
and *homSU*: ⋀*f*. ⟦*continuous_on U f*; *f* ‘ *U* ⊆ *S*⟧
⟹ ∃*c*. *homotopic_with_canon* (λ*x*. *True*) *U S f* (λ*x*. *c*)
shows ∃*c*. *homotopic_with_canon* (λ*x*. *True*) *U T f* (λ*x*. *c*)
proof −
obtain *h k* **where** *h*: *continuous_on S h h* ‘ *S* ⊆ *T*
and *k*: *continuous_on T k k* ‘ *T* ⊆ *S*
and *hom*: *homotopic_with_canon* (λ*x*. *True*) *S S* (*k* ∘ *h*) *id*
homotopic_with_canon (λ*x*. *True*) *T T* (*h* ∘ *k*) *id*
using *assms* **by** (*auto simp*: *homotopy_equivalent_space_def*)
obtain *c*::′*a* **where** *homotopic_with_canon* (λ*x*. *True*) *U S* (*k* ∘ *f*) (λ*x*. *c*)
proof (*rule exE* [*OF homSU* [*of k* ∘ *f*]])
show *continuous_on U* (*k* ∘ *f*)
using *continuous_on_compose continuous_on_subset f k* **by** *blast*

```
  qed (use f k in force)
  then have homotopic_with_canon (λx. True) U T (h ∘ (k ∘ f)) (h ∘ (λx. c))
   by (rule homotopic_with_compose_continuous_left [where Y=S]) (use h in auto)
  moreover have homotopic_with_canon (λx. True) U T (id ∘ f) ((h ∘ k) ∘ f)
    by (rule homotopic_with_compose_continuous_right [where X=T])
       (use f in ‹auto simp add: hom homotopic_with_symD›)
  ultimately show ?thesis
    using homotopic_with_trans by (fastforce simp add: o_def)
qed

lemma homotopy_eqv_homotopic_triviality_null:
  fixes S :: 'a::real_normed_vector set
    and T :: 'b::real_normed_vector set
    and U :: 'c::real_normed_vector set
  assumes S homotopy_eqv T
    shows (∀f. continuous_on U f ∧ f ‘ U ⊆ S
              ⟶ (∃c. homotopic_with_canon (λx. True) U S f (λx. c))) ⟷
          (∀f. continuous_on U f ∧ f ‘ U ⊆ T
              ⟶ (∃c. homotopic_with_canon (λx. True) U T f (λx. c)))
by (rule iffI; metis assms homotopy_eqv_homotopic_triviality_null_imp homotopy_equivalent_space_sym)

lemma homotopy_eqv_contractible_sets:
  fixes S :: 'a::real_normed_vector set
    and T :: 'b::real_normed_vector set
  assumes contractible S contractible T S = {} ⟷ T = {}
    shows S homotopy_eqv T
proof (cases S = {})
  case True with assms show ?thesis
    by (simp add: homeomorphic_imp_homotopy_eqv)
next
  case False
  with assms obtain a b where a ∈ S b ∈ T
    by auto
  then show ?thesis
    unfolding homotopy_equivalent_space_def
    apply (rule_tac x=λx. b in exI, rule_tac x=λx. a in exI)
    apply (intro assms conjI continuous_on_id' homotopic_into_contractible; force)
    done
qed

lemma homotopy_eqv_empty1 [simp]:
  fixes S :: 'a::real_normed_vector set
   shows S homotopy_eqv ({}::'b::real_normed_vector set) ⟷ S = {} (is ?lhs =
?rhs)
proof
  assume ?lhs then show ?rhs
   by (metis continuous_map_subtopology_eu empty_iff equalityI homotopy_equivalent_space_def
image_subset_iff subsetI)
qed (simp add: homotopy_eqv_contractible_sets)
```

lemma *homotopy_eqv_empty2* [*simp*]:
fixes *S* :: *'a::real_normed_vector set*
shows ({}::*'b::real_normed_vector set*) *homotopy_eqv S* ⟷ *S* = {}
using *homotopy_equivalent_space_sym homotopy_eqv_empty1* **by** *blast*

lemma *homotopy_eqv_contractibility*:
fixes *S* :: *'a::real_normed_vector set* **and** *T* :: *'b::real_normed_vector set*
shows *S homotopy_eqv T* ⟹ (*contractible S* ⟷ *contractible T*)
by (*meson contractible_space_top_of_set homotopy_equivalent_space_contractibility*)

lemma *homotopy_eqv_sing*:
fixes *S* :: *'a::real_normed_vector set* **and** *a* :: *'b::real_normed_vector*
shows *S homotopy_eqv* {*a*} ⟷ *S* ≠ {} ∧ *contractible S*
proof (*cases S* = {})
case *False* **then show** *?thesis*
by (*metis contractible_sing empty_not_insert homotopy_eqv_contractibility homotopy_eqv_contractible_sets*)
qed *simp*

lemma *homeomorphic_contractible_eq*:
fixes *S* :: *'a::real_normed_vector set* **and** *T* :: *'b::real_normed_vector set*
shows *S homeomorphic T* ⟹ (*contractible S* ⟷ *contractible T*)
by (*simp add*: *homeomorphic_imp_homotopy_eqv homotopy_eqv_contractibility*)

lemma *homeomorphic_contractible*:
fixes *S* :: *'a::real_normed_vector set* **and** *T* :: *'b::real_normed_vector set*
shows ⟦*contractible S*; *S homeomorphic T*⟧ ⟹ *contractible T*
by (*metis homeomorphic_contractible_eq*)

6.18.25 Misc other results

lemma *bounded_connected_Compl_real*:
fixes *S* :: *real set*
assumes *bounded S* **and** *conn*: *connected*(− *S*)
shows *S* = {}
proof −
obtain *a b* **where** *S* ⊆ *box a b*
by (*meson assms bounded_subset_box_symmetric*)
then have *a* ∉ *S b* ∉ *S*
by *auto*
then have ∀ *x*. *a* ≤ *x* ∧ *x* ≤ *b* ⟶ *x* ∈ − *S*
by (*meson Compl_iff conn connected_iff_interval*)
then show *?thesis*
using ‹*S* ⊆ *box a b*› **by** *auto*
qed

corollary *bounded_path_connected_Compl_real*:
fixes *S* :: *real set*

assumes *bounded S path_connected*(− *S*) **shows** *S* = {}
by (*simp add*: *assms bounded_connected_Compl_real path_connected_imp_connected*)

lemma *bounded_connected_Compl_1*:
fixes *S* :: *'a*::{*euclidean_space*} *set*
assumes *bounded S* **and** *conn*: *connected*(− *S*) **and** *1*: *DIM*(*'a*) = *1*
shows *S* = {}
proof −
have *DIM*(*'a*) = *DIM*(*real*)
by (*simp add*: *1*)
then obtain *f*::*'a* ⇒ *real* **and** *g*
where *linear f* ⋀*x*. *norm*(*f x*) = *norm x* **and** *fg*: ⋀*x*. *g*(*f x*) = *x* ⋀*y*. *f*(*g y*) = *y*
by (*rule isomorphisms_UNIV_UNIV*) *blast*
with ‹*bounded S*› **have** *bounded* (*f* ‘ *S*)
using *bounded_linear_image linear_linear* **by** *blast*
have *bij f* **by** (*metis fg bijI'*)
have *connected* (*f* ‘ (−*S*))
using *connected_linear_image assms* ‹*linear f*› **by** *blast*
moreover have *f* ‘ (−*S*) = − (*f* ‘ *S*)
by (*simp add*: ‹*bij f*› *bij_image_Compl_eq*)
finally have *connected* (− (*f* ‘ *S*))
by *simp*
then have *f* ‘ *S* = {}
using ‹*bounded* (*f* ‘ *S*)› *bounded_connected_Compl_real* **by** *blast*
then show *?thesis*
by *blast*
qed

6.18.26 Some Uncountable Sets

lemma *uncountable_closed_segment*:
fixes *a* :: *'a*::*real_normed_vector*
assumes *a* ≠ *b* **shows** *uncountable* (*closed_segment a b*)
unfolding *path_image_linepath* [*symmetric*] *path_image_def*
using *inj_on_linepath* [*OF assms*] *uncountable_closed_interval* [*of 0 1*]
countable_image_inj_on **by** *auto*

lemma *uncountable_open_segment*:
fixes *a* :: *'a*::*real_normed_vector*
assumes *a* ≠ *b* **shows** *uncountable* (*open_segment a b*)
by (*simp add*: *assms open_segment_def uncountable_closed_segment uncountable_minus_countable*)

lemma *uncountable_convex*:
fixes *a* :: *'a*::*real_normed_vector*
assumes *convex S a* ∈ *S b* ∈ *S a* ≠ *b*
shows *uncountable S*
proof −
have *uncountable* (*closed_segment a b*)

    by (*simp add*: *uncountable_closed_segment assms*)
  then show *?thesis*
    by (*meson assms convex_contains_segment countable_subset*)
qed

lemma *uncountable_ball*:
  fixes *a* :: *'a::euclidean_space*
  assumes *r* > *0*
    shows *uncountable* (*ball a r*)
proof −
  have *uncountable* (*open_segment a* (*a* + *r* $*_R$ (*SOME i. i* ∈ *Basis*)))
   by (*metis Basis_zero SOME_Basis add_cancel_right_right assms less_le scale_eq_0_iff uncountable_open_segment*)
  moreover have *open_segment a* (*a* + *r* $*_R$ (*SOME i. i* ∈ *Basis*)) ⊆ *ball a r*
    using *assms* **by** (*auto simp*: *in_segment algebra_simps dist_norm SOME_Basis*)
  ultimately show *?thesis*
    by (*metis countable_subset*)
qed

lemma *ball_minus_countable_nonempty*:
  assumes *countable* (*A* :: *'a* :: *euclidean_space set*) *r* > *0*
  shows *ball z r* − *A* ≠ {}
proof
  assume ∗: *ball z r* − *A* = {}
  have *uncountable* (*ball z r* − *A*)
    by (*intro uncountable_minus_countable assms uncountable_ball*)
  thus *False* **by** (*subst* (*asm*) ∗) *auto*
qed

lemma *uncountable_cball*:
  fixes *a* :: *'a::euclidean_space*
  assumes *r* > *0*
  shows *uncountable* (*cball a r*)
  using *assms countable_subset uncountable_ball* **by** *auto*

lemma *pairwise_disjnt_countable*:
  fixes 𝒩 :: *nat set set*
  assumes *pairwise disjnt* 𝒩
    shows *countable* 𝒩
proof −
  have *inj_on* (λ*X. SOME n. n* ∈ *X*) (𝒩 − {{}})
    by (*clarsimp simp*: *inj_on_def*) (*metis assms disjnt_iff pairwiseD some_in_eq*)
  then show *?thesis*
    by (*metis countable_Diff_eq countable_def*)
qed

lemma *pairwise_disjnt_countable_Union*:
    assumes *countable* (⋃𝒩) **and** *pwd*: *pairwise disjnt* 𝒩
    shows *countable* 𝒩

proof −
  obtain *f* :: _ ⇒ *nat* **where** *f*: *inj_on f* (⋃𝒩)
    using *assms* **by** *blast*
  then have *pairwise disjnt* (⋃ *X* ∈ 𝒩. {*f* ' *X*})
    using *assms* **by** (*force simp*: *pairwise_def disjnt_inj_on_iff* [*OF f*])
  then have *countable* (⋃ *X* ∈ 𝒩. {*f* ' *X*})
    using *pairwise_disjnt_countable* **by** *blast*
  then show *?thesis*
      by (*meson pwd countable_image_inj_on disjoint_image f inj_on_image pairwise_disjnt_countable*)
qed

lemma *connected_uncountable*:
  fixes *S* :: *'a::metric_space set*
  assumes *connected S a* ∈ *S b* ∈ *S a* ≠ *b* **shows** *uncountable S*
proof −
  have *continuous_on S* (*dist a*)
    by (*intro continuous_intros*)
  then have *connected* (*dist a* ' *S*)
    by (*metis connected_continuous_image* ‹*connected S*›)
  then have *closed_segment 0* (*dist a b*) ⊆ (*dist a* ' *S*)
   by (*simp add*: *assms closed_segment_subset is_interval_connected_1 is_interval_convex*)
  then have *uncountable* (*dist a* ' *S*)
    by (*metis* ‹*a* ≠ *b*› *countable_subset dist_eq_0_iff uncountable_closed_segment*)
  then show *?thesis*
    by *blast*
qed

lemma *path_connected_uncountable*:
  fixes *S* :: *'a::metric_space set*
  assumes *path_connected S a* ∈ *S b* ∈ *S a* ≠ *b* **shows** *uncountable S*
  using *path_connected_imp_connected assms connected_uncountable* **by** *metis*

lemma *connected_finite_iff_sing*:
  fixes *S* :: *'a::metric_space set*
  assumes *connected S*
  shows *finite S* ⟷ *S* = {} ∨ (∃ *a*. *S* = {*a*}) (**is** _ = *?rhs*)
proof −
  have *uncountable S* **if** ¬ *?rhs*
    using *connected_uncountable assms that* **by** *blast*
  then show *?thesis*
    using *uncountable_infinite* **by** *auto*
qed

lemma *connected_card_eq_iff_nontrivial*:
  fixes *S* :: *'a::metric_space set*
  shows *connected S* ⟹ *uncountable S* ⟷ ¬(∃ *a*. *S* ⊆ {*a*})
  by (*metis connected_uncountable finite.emptyI finite.insertI rev_finite_subset singleton_iff subsetI uncountable_infinite*)

lemma *simple_path_image_uncountable*:
  fixes *g* :: *real* ⇒ *'a*::*metric_space*
  assumes *simple_path g*
  shows *uncountable* (*path_image g*)
proof −
  have *g 0* ∈ *path_image g g* (*1/2*) ∈ *path_image g*
    by (*simp_all add*: *path_defs*)
  moreover have *g 0* ≠ *g* (*1/2*)
    using *assms* **by** (*fastforce simp add*: *simple_path_def*)
  ultimately have ∀ *a*. ¬ *path_image g* ⊆ {*a*}
    by *blast*
  then show *?thesis*
    using *assms connected_simple_path_image connected_uncountable* **by** *blast*
qed

lemma *arc_image_uncountable*:
  fixes *g* :: *real* ⇒ *'a*::*metric_space*
  assumes *arc g*
  shows *uncountable* (*path_image g*)
  by (*simp add*: *arc_imp_simple_path assms simple_path_image_uncountable*)

6.18.27 Some simple positive connection theorems

proposition *path_connected_convex_diff_countable*:
  fixes *U* :: *'a*::*euclidean_space set*
  assumes *convex U* ¬ *collinear U countable S*
    shows *path_connected*(*U* − *S*)
proof (*clarsimp simp add*: *path_connected_def*)
  fix *a b*
  assume *a* ∈ *U a* ∉ *S b* ∈ *U b* ∉ *S*
  let *?m* = *midpoint a b*
  show ∃ *g*. *path g* ∧ *path_image g* ⊆ *U* − *S* ∧ *pathstart g* = *a* ∧ *pathfinish g* = *b*
  proof (*cases a* = *b*)
    case *True*
    then show *?thesis*
      by (*metis DiffI* ‹*a* ∈ *U*› ‹*a* ∉ *S*› *path_component_def path_component_refl*)
  next
    case *False*
    then have *a* ≠ *?m b* ≠ *?m*
      using *midpoint_eq_endpoint* **by** *fastforce*+
    have *?m* ∈ *U*
      using ‹*a* ∈ *U*› ‹*b* ∈ *U*› ‹*convex U*› *convex_contains_segment* **by** *force*
    obtain *c* **where** *c* ∈ *U* **and** *nc_abc*: ¬ *collinear* {*a*,*b*,*c*}
     by (*metis False* ‹*a* ∈ *U*› ‹*b* ∈ *U*› ‹¬ *collinear U*› *collinear_triples insert_absorb*)
    have *ncoll_mca*: ¬ *collinear* {*?m*,*c*,*a*}
        by (*metis* (*full_types*) ‹*a* ≠ *?m*› *collinear_3_trans collinear_midpoint insert_commute nc_abc*)
    have *ncoll_mcb*: ¬ *collinear* {*?m*,*c*,*b*}

by (*metis* (*full_types*) ‹*b* ≠ *?m*› *collinear_3_trans collinear_midpoint insert_commute nc_abc*)
have *c* ≠ *?m*
by (*metis collinear_midpoint insert_commute nc_abc*)
then have *closed_segment ?m c* ⊆ *U*
by (*simp add*: ‹*c* ∈ *U*› ‹*?m* ∈ *U*› ‹*convex U*› *closed_segment_subset*)
then obtain *z* **where** *z*: *z* ∈ *closed_segment ?m c*
and *disjS*: (*closed_segment a z* ∪ *closed_segment z b*) ∩ *S* = {}
proof −
have *False* **if** *closed_segment ?m c* ⊆ {*z*. (*closed_segment a z* ∪ *closed_segment z b*) ∩ *S* ≠ {}}
proof −
have *closb*: *closed_segment ?m c* ⊆
{*z* ∈ *closed_segment ?m c*. *closed_segment a z* ∩ *S* ≠ {}} ∪ {*z* ∈ *closed_segment ?m c*. *closed_segment z b* ∩ *S* ≠ {}}
using *that* **by** *blast*
have ∗: *countable* {*z* ∈ *closed_segment ?m c*. *closed_segment z u* ∩ *S* ≠ {}}
if *u* ∈ *U u* ∉ *S* **and** *ncoll*: ¬ *collinear* {*?m*, *c*, *u*} **for** *u*
proof −
have ∗∗: *False* **if** *x1*: *x1* ∈ *closed_segment ?m c* **and** *x2*: *x2* ∈ *closed_segment ?m c*
and *x1* ≠ *x2 x1* ≠ *u*
and *w*: *w* ∈ *closed_segment x1 u w* ∈ *closed_segment x2 u*
and *w* ∈ *S* **for** *x1 x2 w*
proof −
have *x1* ∈ *affine hull* {*?m*,*c*} *x2* ∈ *affine hull* {*?m*,*c*}
using *segment_as_ball x1 x2* **by** *auto*
then have *coll_x1*: *collinear* {*x1*, *?m*, *c*} **and** *coll_x2*: *collinear* {*?m*, *c*, *x2*}
by (*simp_all add*: *affine_hull_3_imp_collinear*) (*metis affine_hull_3_imp_collinear insert_commute*)
have ¬ *collinear* {*x1*, *u*, *x2*}
proof
assume *collinear* {*x1*, *u*, *x2*}
then have *collinear* {*?m*, *c*, *u*}
by (*metis* (*full_types*) ‹*c* ≠ *?m*› *coll_x1 coll_x2 collinear_3_trans insert_commute ncoll* ‹*x1* ≠ *x2*›)
with *ncoll* **show** *False* **..**
qed
then have *closed_segment x1 u* ∩ *closed_segment u x2* = {*u*}
by (*blast intro*!: *Int_closed_segment*)
then have *w* = *u*
using *closed_segment_commute w* **by** *auto*
show *?thesis*
using ‹*u* ∉ *S*› ‹*w* = *u*› *that*(*7*) **by** *auto*
qed
then have *disj*: *disjoint* ((⋃ *z*∈*closed_segment ?m c*. {*closed_segment z u* ∩ *S*}))
by (*fastforce simp*: *pairwise_def disjnt_def*)

have *cou*: *countable* $((\bigcup z \in$ *closed_segment ?m c*. {*closed_segment z u* $\cap$ *S*}) $-$ {{}})
apply (*rule pairwise_disjnt_countable_Union* [*OF* _ *pairwise_subset* [*OF disj*]])
apply (*rule countable_subset* [*OF* _ ‹*countable S*›], *auto*)
done
define *f* **where** $f \equiv \lambda X$. (*THE z*. $z \in$ *closed_segment ?m c* $\wedge$ $X =$ *closed_segment z u* $\cap$ *S*)
show *?thesis*
proof (*rule countable_subset* [*OF* _ *countable_image* [*OF cou*, **where** *f*=*f*]], *clarify*)
fix *x*
assume *x*: $x \in$ *closed_segment ?m c closed_segment x u* $\cap$ $S \neq$ {}
show $x \in f$ ‘ $((\bigcup z{\in}$*closed_segment ?m c*. {*closed_segment z u* $\cap$ *S*}) $-$ {{}})
proof (*rule_tac x*=*closed_segment x u* $\cap$ *S* **in** *image_eqI*)
show $x = f$ (*closed_segment x u* $\cap$ *S*)
unfolding *f_def*
by (*rule the_equality* [*symmetric*]) (*use x* **in** ‹*auto dest*: **›)
qed (*use x* **in** *auto*)
qed
qed
have *uncountable* (*closed_segment ?m c*)
by (*metis* ‹$c \neq$ *?m*› *uncountable_closed_segment*)
then show *False*
using *closb* $*$ [*OF* ‹$a \in U$› ‹$a \notin S$› *ncoll_mca*] $*$ [*OF* ‹$b \in U$› ‹$b \notin S$› *ncoll_mcb*]
by (*simp add*: *closed_segment_commute countable_subset*)
qed
then show *?thesis*
by (*force intro*: *that*)
qed
show *?thesis*
proof (*intro exI conjI*)
have *path_image* (*linepath a z* +++ *linepath z b*) $\subseteq U$
by (*metis* ‹$a \in U$› ‹$b \in U$› ‹*closed_segment ?m c* $\subseteq U$› *z* ‹*convex U*› *closed_segment_subset contra_subsetD path_image_linepath subset_path_image_join*)
with *disjS* **show** *path_image* (*linepath a z* +++ *linepath z b*) $\subseteq U - S$
by (*force simp*: *path_image_join*)
qed *auto*
qed
qed

corollary *connected_convex_diff_countable*:
fixes *U* :: *'a*::*euclidean_space set*
assumes *convex U* $\neg$ *collinear U countable S*
shows *connected*$(U - S)$
by (*simp add*: *assms path_connected_convex_diff_countable path_connected_imp_connected*)

```
lemma path_connected_punctured_convex:
  assumes convex S and aff: aff_dim S ≠ 1
    shows path_connected(S − {a})
proof −
  consider aff_dim S = −1 | aff_dim S = 0 | aff_dim S ≥ 2
    using assms aff_dim_geq [of S] by linarith
  then show ?thesis
  proof cases
    assume aff_dim S = −1
    then show ?thesis
      by (metis aff_dim_empty empty_Diff path_connected_empty)
  next
    assume aff_dim S = 0
    then show ?thesis
      by (metis aff_dim_eq_0 Diff_cancel Diff_empty Diff_insert0 convex_empty con-
vex_imp_path_connected path_connected_singleton singletonD)
  next
    assume ge2: aff_dim S ≥ 2
    then have ¬ collinear S
    proof (clarsimp simp add: collinear_affine_hull)
      fix u v
      assume S ⊆ affine hull {u, v}
      then have aff_dim S ≤ aff_dim {u, v}
        by (metis (no_types) aff_dim_affine_hull aff_dim_subset)
      with ge2 show False
      by (metis (no_types) aff_dim_2 antisym aff not_numeral_le_zero one_le_numeral
order_trans)
    qed
    moreover have countable {a}
      by simp
    ultimately show ?thesis
      by (metis path_connected_convex_diff_countable [OF ‹convex S›])
  qed
qed

lemma connected_punctured_convex:
  shows ⟦convex S; aff_dim S ≠ 1⟧ ⟹ connected(S − {a})
  using path_connected_imp_connected path_connected_punctured_convex by blast

lemma path_connected_complement_countable:
  fixes S :: 'a::euclidean_space set
  assumes 2 ≤ DIM('a) countable S
  shows path_connected(− S)
proof −
  have ¬ collinear (UNIV::'a set)
    using assms by (auto simp: collinear_aff_dim [of UNIV :: 'a set])
  then have path_connected(UNIV − S)
    by (simp add: ‹countable S› path_connected_convex_diff_countable)
```

then show *?thesis*
by (*simp add*: *Compl_eq_Diff_UNIV*)
qed

proposition *path_connected_openin_diff_countable*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *connected S* **and** *ope*: *openin* (*top_of_set* (*affine hull S*)) *S*
and ¬ *collinear S countable T*
shows *path_connected*(*S* − *T*)
proof (*clarsimp simp add*: *path_connected_component*)
fix *x y*
assume *xy*: *x* ∈ *S x* ∉ *T y* ∈ *S y* ∉ *T*
show *path_component* (*S* − *T*) *x y*
proof (*rule connected_equivalence_relation_gen* [*OF* ‹*connected S*›, **where** *P* = *λx. x* ∉ *T*])
show ∃ *z. z* ∈ *U* ∧ *z* ∉ *T* **if** *opeU*: *openin* (*top_of_set S*) *U* **and** *x* ∈ *U* **for** *U x*
proof −
have *openin* (*top_of_set* (*affine hull S*)) *U*
using *opeU ope openin_trans* **by** *blast*
with ‹*x* ∈ *U*› **obtain** *r* **where** *Usub*: *U* ⊆ *affine hull S* **and** *r* > *0*
and *subU*: *ball x r* ∩ *affine hull S* ⊆ *U*
by (*auto simp*: *openin_contains_ball*)
with ‹*x* ∈ *U*› **have** *x*: *x* ∈ *ball x r* ∩ *affine hull S*
by *auto*
have ¬ *S* ⊆ {*x*}
using ‹¬ *collinear S*› *collinear_subset* **by** *blast*
then obtain *x'* **where** *x'* ≠ *x x'* ∈ *S*
by *blast*
obtain *y* **where** *y*: *y* ≠ *x y* ∈ *ball x r* ∩ *affine hull S*
proof
show *x* + (*r* / *2* / *norm*(*x'* − *x*)) $*_R$ (*x'* − *x*) ≠ *x*
using ‹*x'* ≠ *x*› ‹*r* > *0*› **by** *auto*
show *x* + (*r* / *2* / *norm* (*x'* − *x*)) $*_R$ (*x'* − *x*) ∈ *ball x r* ∩ *affine hull S*
using ‹*x'* ≠ *x*› ‹*r* > *0*› ‹*x'* ∈ *S*› *x*
by (*simp add*: *dist_norm mem_affine_3_minus hull_inc*)
qed
have *convex* (*ball x r* ∩ *affine hull S*)
by (*simp add*: *affine_imp_convex convex_Int*)
with *x y subU* **have** *uncountable U*
by (*meson countable_subset uncountable_convex*)
then have ¬ *U* ⊆ *T*
using ‹*countable T*› *countable_subset* **by** *blast*
then show *?thesis* **by** *blast*
qed
show ∃ *U. openin* (*top_of_set S*) *U* ∧ *x* ∈ *U* ∧
(∀ *x*∈*U*. ∀ *y*∈*U*. *x* ∉ *T* ∧ *y* ∉ *T* ⟶ *path_component* (*S* − *T*) *x y*)
if *x* ∈ *S* **for** *x*
proof −

obtain *r* **where** *Ssub*: *S* ⊆ *affine hull S* **and** *r* > *0*
  and *subS*: *ball x r* ∩ *affine hull S* ⊆ *S*
 using *ope* ‹*x* ∈ *S*› **by** (*auto simp*: *openin_contains_ball*)
then have *conv*: *convex* (*ball x r* ∩ *affine hull S*)
 by (*simp add*: *affine_imp_convex convex_Int*)
have ¬ *aff_dim* (*affine hull S*) ≤ *1*
 using ‹¬ *collinear S*› *collinear_aff_dim* **by** *auto*
then have ¬ *aff_dim* (*ball x r* ∩ *affine hull S*) ≤ *1*
 by (*metis* (*no_types*, *hide_lams*) *aff_dim_convex_Int_open IntI open_ball* ‹*0* < *r*› *aff_dim_affine_hull affine_affine_hull affine_imp_convex centre_in_ball empty_iff hull_subset inf_commute subsetCE that*)
then have ¬ *collinear* (*ball x r* ∩ *affine hull S*)
 by (*simp add*: *collinear_aff_dim*)
then have ∗: *path_connected* ((*ball x r* ∩ *affine hull S*) − *T*)
 by (*rule path_connected_convex_diff_countable* [*OF conv* _ ‹*countable T*›])
have *ST*: *ball x r* ∩ *affine hull S* − *T* ⊆ *S* − *T*
 using *subS* **by** *auto*
show *?thesis*
proof (*intro exI conjI*)
 show *x* ∈ *ball x r* ∩ *affine hull S*
  using ‹*x* ∈ *S*› ‹*r* > *0*› **by** (*simp add*: *hull_inc*)
 have *openin* (*top_of_set* (*affine hull S*)) (*ball x r* ∩ *affine hull S*)
  by (*subst inf.commute*) (*simp add*: *openin_Int_open*)
 then show *openin* (*top_of_set S*) (*ball x r* ∩ *affine hull S*)
  by (*rule openin_subset_trans* [*OF* _ *subS Ssub*])
qed (*use* ∗ *path_component_trans* **in** ‹*auto simp*: *path_connected_component path_component_of_subset* [*OF ST*]›)
qed
qed (*use xy path_component_trans* **in** *auto*)
qed

corollary *connected_openin_diff_countable*:
 fixes *S* :: *'a::euclidean_space set*
 assumes *connected S* **and** *ope*: *openin* (*top_of_set* (*affine hull S*)) *S*
  and ¬ *collinear S countable T*
 shows *connected*(*S* − *T*)
by (*metis path_connected_imp_connected path_connected_openin_diff_countable* [*OF assms*])

corollary *path_connected_open_diff_countable*:
 fixes *S* :: *'a::euclidean_space set*
 assumes *2* ≤ *DIM*(*'a*) *open S connected S countable T*
 shows *path_connected*(*S* − *T*)
proof (*cases S* = {})
 case *True*
 then show *?thesis*
  by (*simp*)
next
 case *False*

```
  show ?thesis
  proof (rule path_connected_openin_diff_countable)
    show openin (top_of_set (affine hull S)) S
      by (simp add: assms hull_subset open_subset)
    show ¬ collinear S
      using assms False by (simp add: collinear_aff_dim aff_dim_open)
  qed (simp_all add: assms)
qed

corollary connected_open_diff_countable:
  fixes S :: 'a::euclidean_space set
  assumes 2 ≤ DIM('a) open S connected S countable T
  shows connected(S − T)
by (simp add: assms path_connected_imp_connected path_connected_open_diff_countable)
```

6.18.28 Self-homeomorphisms shuffling points about

The theorem *homeomorphism_moving_points_exists*

```
lemma homeomorphism_moving_point_1:
  fixes a :: 'a::euclidean_space
  assumes affine T a ∈ T and u: u ∈ ball a r ∩ T
  obtains f g where homeomorphism (cball a r ∩ T) (cball a r ∩ T) f g
                    f a = u ⋀x. x ∈ sphere a r ⟹ f x = x
proof −
  have nou: norm (u − a) < r and u ∈ T
    using u by (auto simp: dist_norm norm_minus_commute)
  then have 0 < r
    by (metis DiffD1 Diff_Diff_Int ball_eq_empty centre_in_ball not_le u)
  define f where f ≡ λx. (1 − norm(x − a) / r) *R (u − a) + x
  have ∗: False if eq: x + (norm y / r) *R u = y + (norm x / r) *R u
               and nou: norm u < r and yx: norm y < norm x for x y and u::'a
  proof −
    have x = y + (norm x / r − (norm y / r)) *R u
      using eq by (simp add: algebra_simps)
    then have norm x = norm (y + ((norm x − norm y) / r) *R u)
      by (metis diff_divide_distrib)
    also have ... ≤ norm y + norm(((norm x − norm y) / r) *R u)
      using norm_triangle_ineq by blast
    also have ... = norm y + (norm x − norm y) ∗ (norm u / r)
      using yx ‹r > 0›
      by (simp add: field_split_simps)
    also have ... < norm y + (norm x − norm y) ∗ 1
    proof (subst add_less_cancel_left)
      show (norm x − norm y) ∗ (norm u / r) < (norm x − norm y) ∗ 1
      proof (rule mult_strict_left_mono)
        show norm u / r < 1
          using ‹0 < r› divide_less_eq_1_pos nou by blast
      qed (simp add: yx)
    qed
```

also have ... = *norm x*
by *simp*
finally show *False* **by** *simp*
qed
have *inj f*
unfolding *f_def*
proof (*clarsimp simp*: *inj_on_def*)
fix *x y*
assume (*1* − *norm* (*x* − *a*) / *r*) $*_R$ (*u* − *a*) + *x* =
(*1* − *norm* (*y* − *a*) / *r*) $*_R$ (*u* − *a*) + *y*
then have *eq*: (*x* − *a*) + (*norm* (*y* − *a*) / *r*) $*_R$ (*u* − *a*) = (*y* − *a*) + (*norm* (*x* − *a*) / *r*) $*_R$ (*u* − *a*)
by (*auto simp*: *algebra_simps*)
show *x*=*y*
proof (*cases norm* (*x* − *a*) = *norm* (*y* − *a*))
case *True*
then show *?thesis*
using *eq* **by** *auto*
next
case *False*
then consider *norm* (*x* − *a*) < *norm* (*y* − *a*) | *norm* (*x* − *a*) > *norm* (*y* − *a*)
by *linarith*
then have *False*
proof *cases*
case *1* **show** *False*
using * [*OF* _ *nou 1*] *eq* **by** *simp*
next
case *2* **with** * [*OF eq nou*] **show** *False*
by *auto*
qed
then show *x*=*y* ..
qed
qed
then have *inj_onf*: *inj_on f* (*cball a r* ∩ *T*)
using *inj_on_Int* **by** *fastforce*
have *contf*: *continuous_on* (*cball a r* ∩ *T*) *f*
unfolding *f_def* **using** ‹*0* < *r*› **by** (*intro continuous_intros*) *blast*
have *fim*: *f* ‘ (*cball a r* ∩ *T*) = *cball a r* ∩ *T*
proof
have *: *norm* (*y* + (*1* − *norm y* / *r*) $*_R$ *u*) ≤ *r* **if** *norm y* ≤ *r norm u* < *r* **for** *y u*::′*a*
proof −
have *norm* (*y* + (*1* − *norm y* / *r*) $*_R$ *u*) ≤ *norm y* + *norm*((*1* − *norm y* / *r*) $*_R$ *u*)
using *norm_triangle_ineq* **by** *blast*
also have ... = *norm y* + *abs*(*1* − *norm y* / *r*) * *norm u*
by *simp*
also have ... ≤ *r*

```
      proof −
        have (r − norm u) ∗ (r − norm y) ≥ 0
          using that by auto
        then have r ∗ norm u + r ∗ norm y ≤ r ∗ r + norm u ∗ norm y
          by (simp add: algebra_simps)
        then show ?thesis
        using that ‹0 < r› by (simp add: abs_if field_simps)
      qed
      finally show ?thesis .
    qed
    have f ‘ (cball a r) ⊆ cball a r
      using ∗ nou
      apply (clarsimp simp: dist_norm norm_minus_commute f_def)
      by (metis diff_add_eq diff_diff_add diff_diff_eq2 norm_minus_commute)
    moreover have f ‘ T ⊆ T
      unfolding f_def using ‹affine T› ‹a ∈ T› ‹u ∈ T›
      by (force simp: add.commute mem_affine_3_minus)
    ultimately show f ‘ (cball a r ∩ T) ⊆ cball a r ∩ T
      by blast
  next
    show cball a r ∩ T ⊆ f ‘ (cball a r ∩ T)
    proof (clarsimp simp add: dist_norm norm_minus_commute)
      fix x
      assume x: norm (x − a) ≤ r and x ∈ T
      have ∃ v ∈ {0..1}. ((1 − v) ∗ r − norm ((x − a) − v ∗R (u − a))) · 1 = 0
        by (rule ivt_decreasing_component_on_1) (auto simp: x continuous_intros)
      then obtain v where 0 ≤ v v ≤ 1
        and v: (1 − v) ∗ r = norm ((x − a) − v ∗R (u − a))
        by auto
      then have n: norm (a − (x − v ∗R (u − a))) = r − r ∗ v
        by (simp add: field_simps norm_minus_commute)
      show x ∈ f ‘ (cball a r ∩ T)
      proof (rule image_eqI)
        show x = f (x − v ∗R (u − a))
          using ‹r > 0› v by (simp add: f_def) (simp add: field_simps)
        have x − v ∗R (u − a) ∈ cball a r
          using ‹r > 0›‹0 ≤ v›
          by (simp add: dist_norm n)
        moreover have x − v ∗R (u − a) ∈ T
          by (simp add: f_def ‹u ∈ T› ‹x ∈ T› assms mem_affine_3_minus2)
        ultimately show x − v ∗R (u − a) ∈ cball a r ∩ T
          by blast
      qed
    qed
  qed
  have compact (cball a r ∩ T)
    by (simp add: affine_closed compact_Int_closed ‹affine T›)
  then obtain g where homeomorphism (cball a r ∩ T) (cball a r ∩ T) f g
    by (metis homeomorphism_compact [OF _ contf fim inj_onf])
```

```
  then show thesis
    apply (rule_tac f=f in that)
    using ‹r > 0› by (simp_all add: f_def dist_norm norm_minus_commute)
qed

corollary homeomorphism_moving_point_2:
  fixes a :: 'a::euclidean_space
  assumes affine T a ∈ T and u: u ∈ ball a r ∩ T and v: v ∈ ball a r ∩ T
  obtains f g where homeomorphism (cball a r ∩ T) (cball a r ∩ T) f g
                  f u = v ⋀x. ⟦x ∈ sphere a r; x ∈ T⟧ ⟹ f x = x
proof -
  have 0 < r
    by (metis DiffD1 Diff_Diff_Int ball_eq_empty centre_in_ball not_le u)
  obtain f1 g1 where hom1: homeomorphism (cball a r ∩ T) (cball a r ∩ T) f1
g1
                and f1 a = u and f1: ⋀x. x ∈ sphere a r ⟹ f1 x = x
    using homeomorphism_moving_point_1 [OF ‹affine T› ‹a ∈ T› u] by blast
  obtain f2 g2 where hom2: homeomorphism (cball a r ∩ T) (cball a r ∩ T) f2
g2
                and f2 a = v and f2: ⋀x. x ∈ sphere a r ⟹ f2 x = x
    using homeomorphism_moving_point_1 [OF ‹affine T› ‹a ∈ T› v] by blast
  show ?thesis
  proof
    show homeomorphism (cball a r ∩ T) (cball a r ∩ T) (f2 ∘ g1) (f1 ∘ g2)
      by (metis homeomorphism_compose homeomorphism_symD hom1 hom2)
    have g1 u = a
      using ‹0 < r› ‹f1 a = u› assms hom1 homeomorphism_apply1 by fastforce
    then show (f2 ∘ g1) u = v
      by (simp add: ‹f2 a = v›)
    show ⋀x. ⟦x ∈ sphere a r; x ∈ T⟧ ⟹ (f2 ∘ g1) x = x
      using f1 f2 hom1 homeomorphism_apply1 by fastforce
  qed
qed


corollary homeomorphism_moving_point_3:
  fixes a :: 'a::euclidean_space
  assumes affine T a ∈ T and ST: ball a r ∩ T ⊆ S S ⊆ T
    and u: u ∈ ball a r ∩ T and v: v ∈ ball a r ∩ T
  obtains f g where homeomorphism S S f g
                  f u = v {x. ¬ (f x = x ∧ g x = x)} ⊆ ball a r ∩ T
proof -
  obtain f g where hom: homeomorphism (cball a r ∩ T) (cball a r ∩ T) f g
             and f u = v and fid: ⋀x. ⟦x ∈ sphere a r; x ∈ T⟧ ⟹ f x = x
    using homeomorphism_moving_point_2 [OF ‹affine T› ‹a ∈ T› u v] by blast
  have gid: ⋀x. ⟦x ∈ sphere a r; x ∈ T⟧ ⟹ g x = x
    using fid hom homeomorphism_apply1 by fastforce
  define ff where ff ≡ λx. if x ∈ ball a r ∩ T then f x else x
  define gg where gg ≡ λx. if x ∈ ball a r ∩ T then g x else x
```

show *?thesis*
proof
show *homeomorphism S S ff gg*
proof (*rule homeomorphismI*)
have *continuous_on* ((*cball a r* ∩ *T*) ∪ (*T* − *ball a r*)) *ff*
unfolding *ff_def*
using *homeomorphism_cont1* [*OF hom*]
by (*intro continuous_on_cases*) (*auto simp*: *affine_closed* ‹*affine T*› *fid*)
then show *continuous_on S ff*
by (*rule continuous_on_subset*) (*use ST* **in** *auto*)
have *continuous_on* ((*cball a r* ∩ *T*) ∪ (*T* − *ball a r*)) *gg*
unfolding *gg_def*
using *homeomorphism_cont2* [*OF hom*]
by (*intro continuous_on_cases*) (*auto simp*: *affine_closed* ‹*affine T*› *gid*)
then show *continuous_on S gg*
by (*rule continuous_on_subset*) (*use ST* **in** *auto*)
show *ff* ‘ *S* ⊆ *S*
proof (*clarsimp simp add*: *ff_def*)
fix *x*
assume *x* ∈ *S* **and** *x*: *dist a x* < *r* **and** *x* ∈ *T*
then have *f x* ∈ *cball a r* ∩ *T*
using *homeomorphism_image1* [*OF hom*] **by** *force*
then show *f x* ∈ *S*
using *ST*(*1*) ‹*x* ∈ *T*› *gid hom homeomorphism_def x* **by** *fastforce*
qed
show *gg* ‘ *S* ⊆ *S*
proof (*clarsimp simp add*: *gg_def*)
fix *x*
assume *x* ∈ *S* **and** *x*: *dist a x* < *r* **and** *x* ∈ *T*
then have *g x* ∈ *cball a r* ∩ *T*
using *homeomorphism_image2* [*OF hom*] **by** *force*
then have *g x* ∈ *ball a r*
using *homeomorphism_apply2* [*OF hom*]
by (*metis Diff_Diff_Int Diff_iff* ‹*x* ∈ *T*› *cball_def fid le_less mem_Collect_eq mem_ball mem_sphere x*)
then show *g x* ∈ *S*
using *ST*(*1*) ‹*g x* ∈ *cball a r* ∩ *T*› **by** *force*
qed
show ⋀*x*. *x* ∈ *S* ⟹ *gg* (*ff x*) = *x*
unfolding *ff_def gg_def*
using *homeomorphism_apply1* [*OF hom*] *homeomorphism_image1* [*OF hom*]
by *simp* (*metis Int_iff homeomorphism_apply1* [*OF hom*] *fid image_eqI less_eq_real_def mem_cball mem_sphere*)
show ⋀*x*. *x* ∈ *S* ⟹ *ff* (*gg x*) = *x*
unfolding *ff_def gg_def*
using *homeomorphism_apply2* [*OF hom*] *homeomorphism_image2* [*OF hom*]
by *simp* (*metis Int_iff fid image_eqI less_eq_real_def mem_cball mem_sphere*)
qed
show *ff u* = *v*

```
    using u by (auto simp: ff_def ‹f u = v›)
  show {x. ¬ (ff x = x ∧ gg x = x)} ⊆ ball a r ∩ T
    by (auto simp: ff_def gg_def)
 qed
qed


proposition homeomorphism_moving_point:
 fixes a :: 'a::euclidean_space
 assumes ope: openin (top_of_set (affine hull S)) S
    and S ⊆ T
    and TS: T ⊆ affine hull S
    and S: connected S a ∈ S b ∈ S
 obtains f g where homeomorphism T T f g f a = b
                  {x. ¬ (f x = x ∧ g x = x)} ⊆ S
                  bounded {x. ¬ (f x = x ∧ g x = x)}
proof −
 have 1: ∃ h k. homeomorphism T T h k ∧ h (f d) = d ∧
          {x. ¬ (h x = x ∧ k x = x)} ⊆ S ∧ bounded {x. ¬ (h x = x ∧ k x = x)}
      if d ∈ S f d ∈ S and homfg: homeomorphism T T f g
      and S: {x. ¬ (f x = x ∧ g x = x)} ⊆ S
      and bo: bounded {x. ¬ (f x = x ∧ g x = x)} for d f g
 proof (intro exI conjI)
  show homgf: homeomorphism T T g f
    by (metis homeomorphism_symD homfg)
  then show g (f d) = d
    by (meson ‹S ⊆ T› homeomorphism_def subsetD ‹d ∈ S›)
  show {x. ¬ (g x = x ∧ f x = x)} ⊆ S
    using S by blast
  show bounded {x. ¬ (g x = x ∧ f x = x)}
    using bo by (simp add: conj_commute)
 qed
 have 2: ∃ f g. homeomorphism T T f g ∧ f x = f2 (f1 x) ∧
             {x. ¬ (f x = x ∧ g x = x)} ⊆ S ∧ bounded {x. ¬ (f x = x ∧ g x =
x)}
          if x ∈ S f1 x ∈ S f2 (f1 x) ∈ S
            and hom: homeomorphism T T f1 g1 homeomorphism T T f2 g2
            and sub: {x. ¬ (f1 x = x ∧ g1 x = x)} ⊆ S  {x. ¬ (f2 x = x ∧ g2 x
= x)} ⊆ S
            and bo: bounded {x. ¬ (f1 x = x ∧ g1 x = x)}  bounded {x. ¬ (f2 x
= x ∧ g2 x = x)}
          for x f1 f2 g1 g2
 proof (intro exI conjI)
  show homgf: homeomorphism T T (f2 ∘ f1) (g1 ∘ g2)
    by (metis homeomorphism_compose hom)
  then show (f2 ∘ f1) x = f2 (f1 x)
    by force
  show {x. ¬ ((f2 ∘ f1) x = x ∧ (g1 ∘ g2) x = x)} ⊆ S
    using sub by force
```

have *bounded* ({*x*. ¬(*f1 x* = *x* ∧ *g1 x* = *x*)} ∪ {*x*. ¬(*f2 x* = *x* ∧ *g2 x* = *x*)})
using *bo* **by** *simp*
then show *bounded* {*x*. ¬ ((*f2* ∘ *f1*) *x* = *x* ∧ (*g1* ∘ *g2*) *x* = *x*)}
by (*rule bounded_subset*) *auto*
qed
have *3*: ∃ *U*. *openin* (*top_of_set S*) *U* ∧
d ∈ *U* ∧
(∀ *x*∈*U*.
∃ *f g*. *homeomorphism T T f g* ∧ *f d* = *x* ∧
{*x*. ¬ (*f x* = *x* ∧ *g x* = *x*)} ⊆ *S* ∧
bounded {*x*. ¬ (*f x* = *x* ∧ *g x* = *x*)})
if *d* ∈ *S* **for** *d*
proof −
obtain *r* **where** *r* > *0* **and** *r*: *ball d r* ∩ *affine hull S* ⊆ *S*
by (*metis* ‹*d* ∈ *S*› *ope openin_contains_ball*)
have ∗: ∃ *f g*. *homeomorphism T T f g* ∧ *f d* = *e* ∧
{*x*. ¬ (*f x* = *x* ∧ *g x* = *x*)} ⊆ *S* ∧
bounded {*x*. ¬ (*f x* = *x* ∧ *g x* = *x*)} **if** *e* ∈ *S e* ∈ *ball d r* **for** *e*
apply (*rule homeomorphism_moving_point_3* [*of affine hull S d r T d e*])
using *r* ‹*S* ⊆ *T*› *TS that*
apply (*auto simp*: ‹*d* ∈ *S*› ‹*0* < *r*› *hull_inc*)
using *bounded_subset* **by** *blast*
show *?thesis*
by (*rule_tac x*=*S* ∩ *ball d r* **in** *exI*) (*fastforce simp*: *openin_open_Int* ‹*0* < *r*› *that intro*: ∗)
qed
have ∃ *f g*. *homeomorphism T T f g* ∧ *f a* = *b* ∧
{*x*. ¬ (*f x* = *x* ∧ *g x* = *x*)} ⊆ *S* ∧ *bounded* {*x*. ¬ (*f x* = *x* ∧ *g x* = *x*)}
by (*rule connected_equivalence_relation* [*OF S*]; *blast intro*: *1 2 3*)
then show *?thesis*
using *that* **by** *auto*
qed

lemma *homeomorphism_moving_points_exists_gen*:
assumes *K*: *finite K* ⋀*i*. *i* ∈ *K* ⟹ *x i* ∈ *S* ∧ *y i* ∈ *S*
pairwise (λ*i j*. (*x i* ≠ *x j*) ∧ (*y i* ≠ *y j*)) *K*
and *2* ≤ *aff_dim S*
and *ope*: *openin* (*top_of_set* (*affine hull S*)) *S*
and *S* ⊆ *T T* ⊆ *affine hull S connected S*
shows ∃ *f g*. *homeomorphism T T f g* ∧ (∀ *i* ∈ *K*. *f*(*x i*) = *y i*) ∧
{*x*. ¬ (*f x* = *x* ∧ *g x* = *x*)} ⊆ *S* ∧ *bounded* {*x*. ¬ (*f x* = *x* ∧ *g x* = *x*)}
using *assms*
proof (*induction K*)
case *empty*
then show *?case*
by (*force simp*: *homeomorphism_ident*)
next
case (*insert i K*)

then have *xney*: $\bigwedge j.\ [\![j \in K; j \neq i]\!] \Longrightarrow x\ i \neq x\ j \wedge y\ i \neq y\ j$
and *pw*: *pairwise* $(\lambda i\ j.\ x\ i \neq x\ j \wedge y\ i \neq y\ j)\ K$
and $x\ i \in S\ y\ i \in S$
and *xyS*: $\bigwedge i.\ i \in K \Longrightarrow x\ i \in S \wedge y\ i \in S$
by (*simp_all add*: *pairwise_insert*)
obtain $f\ g$ **where** *homfg*: *homeomorphism* $T\ T\ f\ g$ **and** *feq*: $\bigwedge i.\ i \in K \Longrightarrow f(x\ i) = y\ i$
and *fg_sub*: $\{x.\ \neg\ (f\ x = x \wedge g\ x = x)\} \subseteq S$
and *bo_fg*: *bounded* $\{x.\ \neg\ (f\ x = x \wedge g\ x = x)\}$
using *insert.IH* [*OF xyS pw*] *insert.prems* **by** (*blast intro*: *that*)
then have $\exists f\ g.$ *homeomorphism* $T\ T\ f\ g \wedge (\forall i \in K.\ f(x\ i) = y\ i) \wedge$
$\{x.\ \neg\ (f\ x = x \wedge g\ x = x)\} \subseteq S \wedge$ *bounded* $\{x.\ \neg\ (f\ x = x \wedge g\ x = x)\}$
using *insert* **by** *blast*
have *aff_eq*: *affine hull* $(S - y\ `\ K) =$ *affine hull* S
proof (*rule affine_hull_Diff* [*OF ope*])
show *finite* $(y\ `\ K)$
by (*simp add*: *insert.hyps*(*1*))
show $y\ `\ K \subset S$
using ‹$y\ i \in S$› *insert.hyps*(*2*) *xney xyS* **by** *fastforce*
qed
have *f_in_S*: $f\ x \in S$ **if** $x \in S$ **for** x
using *homfg fg_sub homeomorphism_apply1* ‹$S \subseteq T$›
proof −
have $(f\ (f\ x) \neq f\ x \vee g\ (f\ x) \neq f\ x) \vee f\ x \in S$
by (*metis* ‹$S \subseteq T$› *homfg subsetD homeomorphism_apply1 that*)
then show *?thesis*
using *fg_sub* **by** *force*
qed
obtain $h\ k$ **where** *homhk*: *homeomorphism* $T\ T\ h\ k$ **and** *heq*: $h\ (f\ (x\ i)) = y\ i$
and *hk_sub*: $\{x.\ \neg\ (h\ x = x \wedge k\ x = x)\} \subseteq S - y\ `\ K$
and *bo_hk*: *bounded* $\{x.\ \neg\ (h\ x = x \wedge k\ x = x)\}$
proof (*rule homeomorphism_moving_point* [*of* $S - y`K\ T\ f(x\ i)\ y\ i$])
show *openin* (*top_of_set* (*affine hull* $(S - y\ `\ K)$)) $(S - y\ `\ K)$
by (*simp add*: *aff_eq openin_diff finite_imp_closedin image_subset_iff hull_inc insert xyS*)
show $S - y\ `\ K \subseteq T$
using ‹$S \subseteq T$› **by** *auto*
show $T \subseteq$ *affine hull* $(S - y\ `\ K)$
using *insert* **by** (*simp add*: *aff_eq*)
show *connected* $(S - y\ `\ K)$
proof (*rule connected_openin_diff_countable* [*OF* ‹*connected S*› *ope*])
show ¬ *collinear S*
using *collinear_aff_dim* ‹$2 \leq$ *aff_dim S*› **by** *force*
show *countable* $(y\ `\ K)$
using *countable_finite insert.hyps*(*1*) **by** *blast*
qed
have $\bigwedge k.\ [\![f\ (x\ i) = y\ k; k \in K]\!] \Longrightarrow$ *False*
by (*metis feq homfg* ‹$x\ i \in S$› *homeomorphism_def* ‹$S \subseteq T$› ‹$i \notin K$› *subsetCE*

xney xyS)
  then show *f* (*x i*) ∈ *S* − *y* ‘ *K*
    by (*auto simp*: *f_in_S* ‹*x i* ∈ *S*›)
  show *y i* ∈ *S* − *y* ‘ *K*
    using *insert.hyps xney* **by** (*auto simp*: ‹*y i* ∈ *S*›)
 qed *blast*
 show *?case*
 proof (*intro exI conjI*)
  show *homeomorphism T T* (*h* ∘ *f*) (*g* ∘ *k*)
    using *homfg homhk homeomorphism_compose* **by** *blast*
  show ∀ *i* ∈ *insert i K*. (*h* ∘ *f*) (*x i*) = *y i*
    using *feq hk_sub* **by** (*auto simp*: *heq*)
  show {*x*. ¬ ((*h* ∘ *f*) *x* = *x* ∧ (*g* ∘ *k*) *x* = *x*)} ⊆ *S*
    using *fg_sub hk_sub* **by** *force*
  have *bounded* ({*x*. ¬(*f x* = *x* ∧ *g x* = *x*)} ∪ {*x*. ¬(*h x* = *x* ∧ *k x* = *x*)})
    using *bo_fg bo_hk bounded_Un* **by** *blast*
  then show *bounded* {*x*. ¬ ((*h* ∘ *f*) *x* = *x* ∧ (*g* ∘ *k*) *x* = *x*)}
    by (*rule bounded_subset*) *auto*
 qed
qed

proposition *homeomorphism_moving_points_exists*:
 fixes *S* :: ′*a*::*euclidean_space set*
 assumes *2*: *2* ≤ *DIM*(′*a*) *open S connected S S* ⊆ *T finite K*
   and *KS*: ⋀*i*. *i* ∈ *K* ⟹ *x i* ∈ *S* ∧ *y i* ∈ *S*
   and *pw*: *pairwise* (λ*i j*. (*x i* ≠ *x j*) ∧ (*y i* ≠ *y j*)) *K*
   and *S*: *S* ⊆ *T T* ⊆ *affine hull S connected S*
 obtains *f g* **where** *homeomorphism T T f g* ⋀*i*. *i* ∈ *K* ⟹ *f*(*x i*) = *y i*
         {*x*. ¬ (*f x* = *x* ∧ *g x* = *x*)} ⊆ *S bounded* {*x*. (¬ (*f x* = *x* ∧ *g x* =
x))}
proof (*cases S* = {})
 case *True*
 then show *?thesis*
  using *KS homeomorphism_ident that* **by** *fastforce*
next
 case *False*
 then have *affS*: *affine hull S* = *UNIV*
  by (*simp add*: *affine_hull_open* ‹*open S*›)
 then have *ope*: *openin* (*top_of_set* (*affine hull S*)) *S*
  using ‹*open S*› *open_openin* **by** *auto*
 have *2* ≤ *DIM*(′*a*) **by** (*rule 2*)
 also have … = *aff_dim* (*UNIV* :: ′*a set*)
  by *simp*
 also have … ≤ *aff_dim S*
  by (*metis aff_dim_UNIV aff_dim_affine_hull aff_dim_le_DIM affS*)
 finally have *2* ≤ *aff_dim S*
  by *linarith*
 then show *?thesis*
  using *homeomorphism_moving_points_exists_gen* [*OF* ‹*finite K*› *KS pw* _ *ope S*]

that **by** *fastforce*
qed

The theorem *homeomorphism_grouping_points_exists*

lemma *homeomorphism_grouping_point_1*:
fixes *a*::*real* **and** *c*::*real*
assumes $a < b$ $c < d$
obtains *f g* **where** *homeomorphism* (*cbox a b*) (*cbox c d*) *f g* $f\ a = c$ $f\ b = d$
proof −
define *f* **where** $f \equiv \lambda x.\ ((d - c) / (b - a)) * x + (c - a * ((d - c) / (b - a)))$
have ∃ *g*. *homeomorphism* (*cbox a b*) (*cbox c d*) *f g*
proof (*rule homeomorphism_compact*)
show *continuous_on* (*cbox a b*) *f*
unfolding *f_def* **by** (*intro continuous_intros*)
have *f* ' {*a*..*b*} = {*c*..*d*}
unfolding *f_def image_affinity_atLeastAtMost*
using *assms sum_sqs_eq* **by** (*auto simp*: *field_split_simps*)
then show *f* ' *cbox a b* = *cbox c d*
by *auto*
show *inj_on f* (*cbox a b*)
unfolding *f_def inj_on_def* **using** *assms* **by** *auto*
qed *auto*
then obtain *g* **where** *homeomorphism* (*cbox a b*) (*cbox c d*) *f g* **..**
then show *?thesis*
proof
show $f\ a = c$
by (*simp add*: *f_def*)
show $f\ b = d$
using *assms sum_sqs_eq* [*of a b*] **by** (*auto simp*: *f_def field_split_simps*)
qed
qed

lemma *homeomorphism_grouping_point_2*:
fixes *a*::*real* **and** *w*::*real*
assumes *hom_ab*: *homeomorphism* (*cbox a b*) (*cbox u v*) *f1 g1*
and *hom_bc*: *homeomorphism* (*cbox b c*) (*cbox v w*) *f2 g2*
and $b \in$ *cbox a c* $v \in$ *cbox u w*
and *eq*: $f1\ a = u$ $f1\ b = v$ $f2\ b = v$ $f2\ c = w$
obtains *f g* **where** *homeomorphism* (*cbox a c*) (*cbox u w*) *f g* $f\ a = u$ $f\ c = w$
$\bigwedge x.\ x \in cbox\ a\ b \Longrightarrow f\ x = f1\ x$ $\bigwedge x.\ x \in cbox\ b\ c \Longrightarrow f\ x = f2\ x$
proof −
have *le*: $a \le b$ $b \le c$ $u \le v$ $v \le w$
using *assms* **by** *simp_all*
then have *ac*: *cbox a c* = *cbox a b* ∪ *cbox b c* **and** *uw*: *cbox u w* = *cbox u v* ∪ *cbox v w*
by *auto*
define *f* **where** $f \equiv \lambda x.$ *if* $x \le b$ *then* *f1 x* *else* *f2 x*

have ∃ *g*. *homeomorphism* (*cbox a c*) (*cbox u w*) *f g*
proof (*rule homeomorphism_compact*)
have *cf1*: *continuous_on* (*cbox a b*) *f1*
using *hom_ab homeomorphism_cont1* **by** *blast*
have *cf2*: *continuous_on* (*cbox b c*) *f2*
using *hom_bc homeomorphism_cont1* **by** *blast*
show *continuous_on* (*cbox a c*) *f*
unfolding *f_def* **using** *le eq*
by (*force intro*: *continuous_on_cases_le* [*OF continuous_on_subset* [*OF cf1*] *continuous_on_subset* [*OF cf2*]])
have *f* ' *cbox a b* = *f1* ' *cbox a b f* ' *cbox b c* = *f2* ' *cbox b c*
unfolding *f_def* **using** *eq* **by** *force+*
then show *f* ' *cbox a c* = *cbox u w*
unfolding *ac uw image_Un* **by** (*metis hom_ab hom_bc homeomorphism_def*)
have *neq12*: *f1 x* ≠ *f2 y* **if** *x*: *a* ≤ *x x* ≤ *b* **and** *y*: *b* < *y y* ≤ *c* **for** *x y*
proof −
have *f1 x* ∈ *cbox u v*
by (*metis hom_ab homeomorphism_def image_eqI mem_box_real*(*2*) *x*)
moreover have *f2 y* ∈ *cbox v w*
by (*metis* (*full_types*) *hom_bc homeomorphism_def image_subset_iff mem_box_real*(*2*) *not_le not_less_iff_gr_or_eq order_refl y*)
moreover have *f2 y* ≠ *f2 b*
by (*metis cancel_comm_monoid_add_class.diff_cancel diff_gt_0_iff_gt hom_bc homeomorphism_def le*(*2*) *less_imp_le less_numeral_extra*(*3*) *mem_box_real*(*2*) *order_refl y*)
ultimately show *?thesis*
using *le eq* **by** *simp*
qed
have *inj_on f1* (*cbox a b*)
by (*metis* (*full_types*) *hom_ab homeomorphism_def inj_onI*)
moreover have *inj_on f2* (*cbox b c*)
by (*metis* (*full_types*) *hom_bc homeomorphism_def inj_onI*)
ultimately show *inj_on f* (*cbox a c*)
apply (*simp* (*no_asm*) *add*: *inj_on_def*)
apply (*simp add*: *f_def inj_on_eq_iff*)
using *neq12* **by** *force*
qed *auto*
then obtain *g* **where** *homeomorphism* (*cbox a c*) (*cbox u w*) *f g* **..**
then show *?thesis*
using *eq f_def le that* **by** *force*
qed

lemma *homeomorphism_grouping_point_3*:
fixes *a*::*real*
assumes *cbox_sub*: *cbox c d* ⊆ *box a b cbox u v* ⊆ *box a b*
and *box_ne*: *box c d* ≠ {} *box u v* ≠ {}
obtains *f g* **where** *homeomorphism* (*cbox a b*) (*cbox a b*) *f g f a* = *a f b* = *b*
⋀*x*. *x* ∈ *cbox c d* ⟹ *f x* ∈ *cbox u v*
proof −

have *less*: *a* < *c a* < *u d* < *b v* < *b c* < *d u* < *v cbox c d* ≠ {}
  using *assms*
  by (*simp_all add*: *cbox_sub subset_eq*)
obtain *f1 g1* **where** *1*: *homeomorphism* (*cbox a c*) (*cbox a u*) *f1 g1*
  and *f1_eq*: *f1 a* = *a f1 c* = *u*
  using *homeomorphism_grouping_point_1* [*OF* ‹*a* < *c*› ‹*a* < *u*›] **.**
obtain *f2 g2* **where** *2*: *homeomorphism* (*cbox c d*) (*cbox u v*) *f2 g2*
  and *f2_eq*: *f2 c* = *u f2 d* = *v*
  using *homeomorphism_grouping_point_1* [*OF* ‹*c* < *d*› ‹*u* < *v*›] **.**
obtain *f3 g3* **where** *3*: *homeomorphism* (*cbox d b*) (*cbox v b*) *f3 g3*
  and *f3_eq*: *f3 d* = *v f3 b* = *b*
  using *homeomorphism_grouping_point_1* [*OF* ‹*d* < *b*› ‹*v* < *b*›] **.**
obtain *f4 g4* **where** *4*: *homeomorphism* (*cbox a d*) (*cbox a v*) *f4 g4* **and** *f4 a* = *a f4 d* = *v*
  and *f4_eq*: ⋀*x*. *x* ∈ *cbox a c* ⟹ *f4 x* = *f1 x* ⋀*x*. *x* ∈ *cbox c d* ⟹ *f4 x* = *f2 x*
  using *homeomorphism_grouping_point_2* [*OF 1 2*] *less* **by** (*auto simp*: *f1_eq f2_eq*)
obtain *f g* **where** *fg*: *homeomorphism* (*cbox a b*) (*cbox a b*) *f g f a* = *a f b* = *b*
  and *f_eq*: ⋀*x*. *x* ∈ *cbox a d* ⟹ *f x* = *f4 x* ⋀*x*. *x* ∈ *cbox d b* ⟹ *f x* = *f3 x*
  using *homeomorphism_grouping_point_2* [*OF 4 3*] *less* **by** (*auto simp*: *f4_eq f3_eq f2_eq f1_eq*)
show *?thesis*
proof (*rule that* [*OF fg*])
  show *f x* ∈ *cbox u v* **if** *x* ∈ *cbox c d* **for** *x*
    using *that f4_eq f_eq homeomorphism_image1* [*OF 2*]
      by (*metis atLeastAtMost_iff box_real*(*2*) *image_eqI less*(*1*) *less_eq_real_def order_trans*)
qed
qed

lemma *homeomorphism_grouping_point_4*:
  fixes *T* :: *real set*
  assumes *open U open S connected S U* ≠ {} *finite K K* ⊆ *S U* ⊆ *S S* ⊆ *T*
  obtains *f g* **where** *homeomorphism T T f g*
    ⋀*x*. *x* ∈ *K* ⟹ *f x* ∈ *U* {*x*. (¬ (*f x* = *x* ∧ *g x* = *x*))} ⊆ *S*
    bounded {*x*. (¬ (*f x* = *x* ∧ *g x* = *x*))}
proof −
  obtain *c d* **where** *box c d* ≠ {} *cbox c d* ⊆ *U*
  proof −
    obtain *u* **where** *u* ∈ *U*
      using ‹*U* ≠ {}› **by** *blast*
    then obtain *e* **where** *e* > *0 cball u e* ⊆ *U*
      using ‹*open U*› *open_contains_cball* **by** *blast*
    then show *?thesis*
      by (*rule_tac c*=*u* **and** *d*=*u*+*e* **in** *that*) (*auto simp*: *dist_norm subset_iff*)
  qed

```
  have compact K
    by (simp add: ‹finite K› finite_imp_compact)
  obtain a b where box a b ≠ {} K ⊆ cbox a b cbox a b ⊆ S
  proof (cases K = {})
    case True then show ?thesis
      using ‹box c d ≠ {}› ‹cbox c d ⊆ U› ‹U ⊆ S› that by blast
  next
    case False
    then obtain a b where a ∈ K b ∈ K
            and a: ⋀x. x ∈ K ⟹ a ≤ x and b: ⋀x. x ∈ K ⟹ x ≤ b
      using compact_attains_inf compact_attains_sup by (metis ‹compact K›)+
    obtain e where e > 0 cball b e ⊆ S
      using ‹open S› open_contains_cball
      by (metis ‹b ∈ K› ‹K ⊆ S› subsetD)
    show ?thesis
    proof
      show box a (b + e) ≠ {}
        using ‹0 < e› ‹b ∈ K› a by force
      show K ⊆ cbox a (b + e)
        using ‹0 < e› a b by fastforce
      have a ∈ S
        using ‹a ∈ K› assms(6) by blast
      have b + e ∈ S
        using ‹0 < e› ‹cball b e ⊆ S›  by (force simp: dist_norm)
      show cbox a (b + e) ⊆ S
        using ‹a ∈ S› ‹b + e ∈ S› ‹connected S› connected_contains_Icc by auto
    qed
  qed
  obtain w z where cbox w z ⊆ S and sub_wz: cbox a b ∪ cbox c d ⊆ box w z
  proof −
    have a ∈ S b ∈ S
      using ‹box a b ≠ {}› ‹cbox a b ⊆ S› by auto
    moreover have c ∈ S d ∈ S
      using ‹box c d ≠ {}› ‹cbox c d ⊆ U› ‹U ⊆ S› by force+
    ultimately have min a c ∈ S max b d ∈ S
      by linarith+
    then obtain e1 e2 where e1 > 0 cball (min a c) e1 ⊆ S e2 > 0 cball (max
b d) e2 ⊆ S
      using ‹open S› open_contains_cball by metis
    then have *: min a c − e1 ∈ S max b d + e2 ∈ S
      by (auto simp: dist_norm)
    show ?thesis
    proof
      show cbox (min a c − e1) (max b d+ e2) ⊆ S
        using * ‹connected S› connected_contains_Icc by auto
      show cbox a b ∪ cbox c d ⊆ box (min a c − e1) (max b d + e2)
        using ‹0 < e1› ‹0 < e2› by auto
    qed
  qed
```

then
obtain *f g* **where** *hom*: *homeomorphism* (*cbox w z*) (*cbox w z*) *f g*
and *f w* = *w f z* = *z*
and *fin*: ⋀*x*. *x* ∈ *cbox a b* ⟹ *f x* ∈ *cbox c d*
using *homeomorphism_grouping_point_3* [*of a b w z c d*]
using ‹*box a b* ≠ {}› ‹*box c d* ≠ {}› **by** *blast*
have *contfg*: *continuous_on* (*cbox w z*) *f continuous_on* (*cbox w z*) *g*
using *hom homeomorphism_def* **by** *blast*+
define *f'* **where** *f'* ≡ λ*x*. *if x* ∈ *cbox w z then f x else x*
define *g'* **where** *g'* ≡ λ*x*. *if x* ∈ *cbox w z then g x else x*
show *?thesis*
proof
have *T*: *cbox w z* ∪ (*T* − *box w z*) = *T*
using ‹*cbox w z* ⊆ *S*› ‹*S* ⊆ *T*› **by** *auto*
show *homeomorphism T T f' g'*
proof
have *clo*: *closedin* (*top_of_set* (*cbox w z* ∪ (*T* − *box w z*))) (*T* − *box w z*)
by (*metis Diff_Diff_Int Diff_subset T closedin_def open_box openin_open_Int topspace_euclidean_subtopology*)
have ⋀*x*. ⟦*w* ≤ *x* ∧ *x* ≤ *z*; *w* < *x* ⟶ ¬ *x* < *z*⟧ ⟹ *f x* = *x*
using ‹*f w* = *w*› ‹*f z* = *z*› **by** *auto*
moreover have ⋀*x*. ⟦*w* ≤ *x* ∧ *x* ≤ *z*; *w* < *x* ⟶ ¬ *x* < *z*⟧ ⟹ *g x* = *x*
using ‹*f w* = *w*› ‹*f z* = *z*› *hom homeomorphism_apply1* **by** *fastforce*
ultimately
have *continuous_on* (*cbox w z* ∪ (*T* − *box w z*)) *f' continuous_on* (*cbox w z* ∪ (*T* − *box w z*)) *g'*
unfolding *f'_def g'_def*
by (*intro continuous_on_cases_local contfg continuous_on_id clo*; *auto simp*: *closed_subset*)+
then show *continuous_on T f' continuous_on T g'*
by (*simp_all only*: *T*)
show *f'* ‘ *T* ⊆ *T*
unfolding *f'_def*
by *clarsimp* (*metis* ‹*cbox w z* ⊆ *S*› ‹*S* ⊆ *T*› *subsetD hom homeomorphism_def imageI mem_box_real*(*2*))
show *g'* ‘ *T* ⊆ *T*
unfolding *g'_def*
by *clarsimp* (*metis* ‹*cbox w z* ⊆ *S*› ‹*S* ⊆ *T*› *subsetD hom homeomorphism_def imageI mem_box_real*(*2*))
show ⋀*x*. *x* ∈ *T* ⟹ *g'* (*f' x*) = *x*
unfolding *f'_def g'_def*
using *homeomorphism_apply1* [*OF hom*] *homeomorphism_image1* [*OF hom*] **by** *fastforce*
show ⋀*y*. *y* ∈ *T* ⟹ *f'* (*g' y*) = *y*
unfolding *f'_def g'_def*
using *homeomorphism_apply2* [*OF hom*] *homeomorphism_image2* [*OF hom*] **by** *fastforce*
qed
show ⋀*x*. *x* ∈ *K* ⟹ *f' x* ∈ *U*

```
    using fin sub_wz ‹K ⊆ cbox a b› ‹cbox c d ⊆ U› by (force simp: f'_def)
  show {x. ¬ (f' x = x ∧ g' x = x)} ⊆ S
    using ‹cbox w z ⊆ S› by (auto simp: f'_def g'_def)
  show bounded {x. ¬ (f' x = x ∧ g' x = x)}
  proof (rule bounded_subset [of cbox w z])
    show bounded (cbox w z)
      using bounded_cbox by blast
    show {x. ¬ (f' x = x ∧ g' x = x)} ⊆ cbox w z
      by (auto simp: f'_def g'_def)
  qed
 qed
qed

proposition homeomorphism_grouping_points_exists:
 fixes S :: 'a::euclidean_space set
 assumes open U open S connected S U ≠ {} finite K K ⊆ S U ⊆ S S ⊆ T
 obtains f g where homeomorphism T T f g {x. (¬ (f x = x ∧ g x = x))} ⊆ S
                   bounded {x. (¬ (f x = x ∧ g x = x))} ⋀x. x ∈ K ⟹ f x ∈ U
proof (cases 2 ≤ DIM('a))
 case True
 have TS: T ⊆ affine hull S
   using affine_hull_open assms by blast
 have infinite U
   using ‹open U› ‹U ≠ {}› finite_imp_not_open by blast
 then obtain P where P ⊆ U finite P card K = card P
   using infinite_arbitrarily_large by metis
 then obtain γ where γ: bij_betw γ K P
   using ‹finite K› finite_same_card_bij by blast
 obtain f g where homeomorphism T T f g ⋀i. i ∈ K ⟹ f (id i) = γ i {x. ¬
(f x = x ∧ g x = x)} ⊆ S bounded {x. ¬ (f x = x ∧ g x = x)}
 proof (rule homeomorphism_moving_points_exists [OF True ‹open S› ‹connected
S› ‹S ⊆ T› ‹finite K›])
   show ⋀i. i ∈ K ⟹ id i ∈ S ∧ γ i ∈ S
     using ‹P ⊆ U› ‹bij_betw γ K P› ‹K ⊆ S› ‹U ⊆ S› bij_betwE by blast
   show pairwise (λi j. id i ≠ id j ∧ γ i ≠ γ j) K
     using γ by (auto simp: pairwise_def bij_betw_def inj_on_def)
 qed (use affine_hull_open assms that in auto)
 then show ?thesis
   using γ ‹P ⊆ U› bij_betwE by (fastforce simp add: intro!: that)
next
 case False
 with DIM_positive have DIM('a) = 1
   by (simp add: dual_order.antisym)
 then obtain h::'a ⇒real and j
 where linear h linear j
   and noh: ⋀x. norm(h x) = norm x and noj: ⋀y. norm(j y) = norm y
   and hj:  ⋀x. j(h x) = x ⋀y. h(j y) = y
   and ranh: surj h
   using isomorphisms_UNIV_UNIV
```

```
    by (metis (mono_tags, hide_lams) DIM_real UNIV_eq_I range_eqI)
  obtain f g where hom: homeomorphism (h ` T) (h ` T) f g
              and f: ⋀x. x ∈ h ` K ⟹ f x ∈ h ` U
              and sub: {x. ¬ (f x = x ∧ g x = x)} ⊆ h ` S
              and bou: bounded {x. ¬ (f x = x ∧ g x = x)}
    apply (rule homeomorphism_grouping_point_4 [of h ` U h ` S h ` K h ` T])
     by (simp_all add: assms image_mono ‹linear h› open_surjective_linear_image
connected_linear_image ranh)
  have jf: j (f (h x)) = x ⟷ f (h x) = h x for x
    by (metis hj)
  have jg: j (g (h x)) = x ⟷ g (h x) = h x for x
    by (metis hj)
  have cont_hj: continuous_on X h  continuous_on Y j for X Y
    by (simp_all add: ‹linear h› ‹linear j› linear_linear linear_continuous_on)
  show ?thesis
  proof
    show homeomorphism T T (j ∘ f ∘ h) (j ∘ g ∘ h)
    proof
      show continuous_on T (j ∘ f ∘ h) continuous_on T (j ∘ g ∘ h)
        using hom homeomorphism_def
        by (blast intro: continuous_on_compose cont_hj)+
      show (j ∘ f ∘ h) ` T ⊆ T (j ∘ g ∘ h) ` T ⊆ T
          by auto (metis (mono_tags, hide_lams) hj(1) hom homeomorphism_def
imageE imageI)+
      show ⋀x. x ∈ T ⟹ (j ∘ g ∘ h) ((j ∘ f ∘ h) x) = x
        using hj hom homeomorphism_apply1 by fastforce
      show ⋀y. y ∈ T ⟹ (j ∘ f ∘ h) ((j ∘ g ∘ h) y) = y
        using hj hom homeomorphism_apply2 by fastforce
    qed
    show {x. ¬ ((j ∘ f ∘ h) x = x ∧ (j ∘ g ∘ h) x = x)} ⊆ S
    proof (clarsimp simp: jf jg hj)
      show f (h x) = h x ⟶ g (h x) ≠ h x ⟹ x ∈ S for x
        using sub [THEN subsetD, of h x] hj by simp (metis imageE)
    qed
    have bounded (j ` {x. (¬ (f x = x ∧ g x = x))})
     by (rule bounded_linear_image [OF bou]) (use ‹linear j› linear_conv_bounded_linear
in auto)
    moreover
    have *: {x. ¬((j ∘ f ∘ h) x = x ∧ (j ∘ g ∘ h) x = x)} = j ` {x. (¬ (f x = x ∧
g x = x))}
      using hj by (auto simp: jf jg image_iff, metis+)
    ultimately show bounded {x. ¬ ((j ∘ f ∘ h) x = x ∧ (j ∘ g ∘ h) x = x)}
      by metis
    show ⋀x. x ∈ K ⟹ (j ∘ f ∘ h) x ∈ U
      using f hj by fastforce
  qed
qed
```

proposition *homeomorphism_grouping_points_exists_gen*:
fixes S :: $'a$::*euclidean_space set*
assumes *opeU*: *openin* (*top_of_set* S) U
and *opeS*: *openin* (*top_of_set* (*affine hull* S)) S
and $U \neq \{\}$ *finite* K $K \subseteq S$ **and** S: $S \subseteq T$ $T \subseteq$ *affine hull* S *connected* S
obtains f g **where** *homeomorphism* T T f g $\{x.\ (\neg\ (f\ x = x \wedge g\ x = x))\} \subseteq S$
bounded $\{x.\ (\neg\ (f\ x = x \wedge g\ x = x))\}$ $\bigwedge x.\ x \in K \Longrightarrow f\ x \in U$
proof (*cases* $2 \leq$ *aff_dim* S)
case *True*
have *opeU'*: *openin* (*top_of_set* (*affine hull* S)) U
using *opeS opeU openin_trans* **by** *blast*
obtain u **where** $u \in U$ $u \in S$
using ‹$U \neq \{\}$› *opeU openin_imp_subset* **by** *fastforce+*
have *infinite* U
proof (*rule infinite_openin* [*OF opeU* ‹$u \in U$›])
show u *islimpt* S
using *True* ‹$u \in S$› *assms*(*8*) *connected_imp_perfect_aff_dim* **by** *fastforce*
qed
then obtain P **where** $P \subseteq U$ *finite* P *card* K = *card* P
using *infinite_arbitrarily_large* **by** *metis*
then obtain γ **where** γ: *bij_betw* γ K P
using ‹*finite* K› *finite_same_card_bij* **by** *blast*
have $\exists f\ g.$ *homeomorphism* T T f g $\wedge$ $(\forall i \in K.\ f(id\ i) = \gamma\ i) \wedge$
$\{x.\ \neg\ (f\ x = x \wedge g\ x = x)\} \subseteq S \wedge$ *bounded* $\{x.\ \neg\ (f\ x = x \wedge g\ x = x)\}$
proof (*rule homeomorphism_moving_points_exists_gen* [*OF* ‹*finite* K› _ _ *True*
opeS S])
show $\bigwedge i.\ i \in K \Longrightarrow id\ i \in S \wedge \gamma\ i \in S$
by (*metis id_apply opeU openin_contains_cball subsetCE* ‹$P \subseteq U$› ‹*bij_betw* γ
K P› ‹$K \subseteq S$› *bij_betwE*)
show *pairwise* $(\lambda i\ j.\ id\ i \neq id\ j \wedge \gamma\ i \neq \gamma\ j)$ K
using γ **by** (*auto simp*: *pairwise_def bij_betw_def inj_on_def*)
qed
then show *?thesis*
using γ ‹$P \subseteq U$› *bij_betwE* **by** (*fastforce simp add*: *intro*!: *that*)
next
case *False*
with *aff_dim_geq* [*of* S] **consider** *aff_dim* $S = -1$ | *aff_dim* $S = 0$ | *aff_dim* $S =$
1 **by** *linarith*
then show *?thesis*
proof *cases*
assume *aff_dim* $S = -1$
then have $S = \{\}$
using *aff_dim_empty* **by** *blast*
then have *False*
using ‹$U \neq \{\}$› ‹$K \subseteq S$› *openin_imp_subset* [*OF opeU*] **by** *blast*
then show *?thesis* **..**
next
assume *aff_dim* $S = 0$
then obtain a **where** $S = \{a\}$

using *aff_dim_eq_0* **by** *blast*
then have $K \subseteq U$
using ‹$U \neq \{\}$› ‹$K \subseteq S$› *openin_imp_subset* [*OF opeU*] **by** *blast*
show *?thesis*
using ‹$K \subseteq U$› **by** (*intro that* [*of id id*]) (*auto intro*: *homeomorphismI*)
next
assume *aff_dim* $S = 1$
then have *affine hull S homeomorphic* (*UNIV* :: *real set*)
by (*auto simp*: *homeomorphic_affine_sets*)
then obtain *h*::'*a*⇒*real* **and** *j* **where** *homhj*: *homeomorphism* (*affine hull S*) *UNIV h j*
using *homeomorphic_def* **by** *blast*
then have *h*: $\bigwedge x.\ x \in$ *affine hull S* $\Longrightarrow j(h(x)) = x$ **and** *j*: $\bigwedge y.\ j\ y \in$ *affine hull S* $\wedge\ h(j\ y) = y$
by (*auto simp*: *homeomorphism_def*)
have *connh*: *connected* (*h* ‘ *S*)
by (*meson Topological_Spaces.connected_continuous_image* ‹*connected S*› *homeomorphism_cont1 homeomorphism_of_subsets homhj hull_subset top_greatest*)
have *hUS*: *h* ‘ *U* ⊆ *h* ‘ *S*
by (*meson homeomorphism_imp_open_map homeomorphism_of_subsets homhj hull_subset opeS opeU open_UNIV openin_open_eq*)
have *opn*: *openin* (*top_of_set* (*affine hull S*)) *U* ⟹ *open* (*h* ‘ *U*) **for** *U*
using *homeomorphism_imp_open_map* [*OF homhj*] **by** *simp*
have *open* (*h* ‘ *U*) *open* (*h* ‘ *S*)
by (*auto intro*: *opeS opeU openin_trans opn*)
then obtain *f g* **where** *hom*: *homeomorphism* (*h* ‘ *T*) (*h* ‘ *T*) *f g*
and *f*: $\bigwedge x.\ x \in h$ ‘ $K \Longrightarrow f\ x \in h$ ‘ U
and *sub*: $\{x.\ \neg\ (f\ x = x \wedge g\ x = x)\} \subseteq h$ ‘ S
and *bou*: *bounded* $\{x.\ \neg\ (f\ x = x \wedge g\ x = x)\}$
apply (*rule homeomorphism_grouping_points_exists* [*of h* ‘ *U h* ‘ *S h* ‘ *K h* ‘ *T*])
using *assms* **by** (*auto simp*: *connh hUS*)
have *jf*: $\bigwedge x.\ x \in$ *affine hull S* $\Longrightarrow j\ (f\ (h\ x)) = x \longleftrightarrow f\ (h\ x) = h\ x$
by (*metis h j*)
have *jg*: $\bigwedge x.\ x \in$ *affine hull S* $\Longrightarrow j\ (g\ (h\ x)) = x \longleftrightarrow g\ (h\ x) = h\ x$
by (*metis h j*)
have *cont_hj*: *continuous_on T h* *continuous_on Y j* **for** *Y*
proof (*rule continuous_on_subset* [*OF* _ ‹*T* ⊆ *affine hull S*›])
show *continuous_on* (*affine hull S*) *h*
using *homeomorphism_def homhj* **by** *blast*
qed (*meson continuous_on_subset homeomorphism_def homhj top_greatest*)
define *f'* **where** $f' \equiv \lambda x.$ *if* $x \in$ *affine hull S then* $(j \circ f \circ h)\ x$ *else* x
define *g'* **where** $g' \equiv \lambda x.$ *if* $x \in$ *affine hull S then* $(j \circ g \circ h)\ x$ *else* x
show *?thesis*
proof
show *homeomorphism T T f' g'*
proof
have *continuous_on T* $(j \circ f \circ h)$
using *hom homeomorphism_def* **by** (*intro continuous_on_compose cont_hj*)

blast
    then show *continuous_on T f'*
      apply (*rule continuous_on_eq*)
      using ‹*T* ⊆ *affine hull S*› *f'_def* **by** *auto*
    have *continuous_on T* (*j* ∘ *g* ∘ *h*)
      using *hom homeomorphism_def* **by** (*intro continuous_on_compose cont_hj*)
blast
    then show *continuous_on T g'*
      apply (*rule continuous_on_eq*)
      using ‹*T* ⊆ *affine hull S*› *g'_def* **by** *auto*
    show *f'* ‘ *T* ⊆ *T*
    proof (*clarsimp simp*: *f'_def*)
      fix *x* **assume** *x* ∈ *T*
      then have *f* (*h x*) ∈ *h* ‘ *T*
      by (*metis* (*no_types*) *hom homeomorphism_def image_subset_iff subset_refl*)
      then show *j* (*f* (*h x*)) ∈ *T*
        using ‹*T* ⊆ *affine hull S*› *h* **by** *auto*
    qed
    show *g'* ‘ *T* ⊆ *T*
    proof (*clarsimp simp*: *g'_def*)
      fix *x* **assume** *x* ∈ *T*
      then have *g* (*h x*) ∈ *h* ‘ *T*
      by (*metis* (*no_types*) *hom homeomorphism_def image_subset_iff subset_refl*)
      then show *j* (*g* (*h x*)) ∈ *T*
        using ‹*T* ⊆ *affine hull S*› *h* **by** *auto*
    qed
    show ⋀*x*. *x* ∈ *T* ⟹ *g'* (*f' x*) = *x*
        using *h j hom homeomorphism_apply1* **by** (*fastforce simp add*: *f'_def*
g'_def)
    show ⋀*y*. *y* ∈ *T* ⟹ *f'* (*g' y*) = *y*
        using *h j hom homeomorphism_apply2* **by** (*fastforce simp add*: *f'_def*
g'_def)
  qed
next
  have §: ⋀*x y*. ⟦*x* ∈ *affine hull S*; *h x* = *h y*; *y* ∈ *S*⟧ ⟹ *x* ∈ *S*
    by (*metis h hull_inc*)
  show {*x*. ¬ (*f' x* = *x* ∧ *g' x* = *x*)} ⊆ *S*
    using *sub* **by** (*simp add*: *f'_def g'_def jf jg*) (*force elim*: §)
next
  have *compact* (*j* ‘ *closure* {*x*. ¬ (*f x* = *x* ∧ *g x* = *x*)})
    using *bou* **by** (*auto simp*: *compact_continuous_image cont_hj*)
  then have *bounded* (*j* ‘ {*x*. ¬ (*f x* = *x* ∧ *g x* = *x*)})
    by (*rule bounded_closure_image* [*OF compact_imp_bounded*])
  moreover
  have ∗: {*x* ∈ *affine hull S*. *j* (*f* (*h x*)) ≠ *x* ∨ *j* (*g* (*h x*)) ≠ *x*} = *j* ‘ {*x*. (¬ (*f*
x = *x* ∧ *g x* = *x*))}
    using *h j* **by** (*auto simp*: *image_iff*; *metis*)
  ultimately have *bounded* {*x* ∈ *affine hull S*. *j* (*f* (*h x*)) ≠ *x* ∨ *j* (*g* (*h x*))
≠ *x*}

```
        by metis
      then show bounded {x. ¬ (f' x = x ∧ g' x = x)}
        by (simp add: f'_def g'_def Collect_mono bounded_subset)
    next
      show f' x ∈ U if x ∈ K for x
      proof -
        have U ⊆ S
          using opeU openin_imp_subset by blast
        then have j (f (h x)) ∈ U
          using f h hull_subset that by fastforce
        then show f' x ∈ U
          using ‹K ⊆ S› S f'_def that by auto
      qed
    qed
  qed
qed
```

6.18.29 Nullhomotopic mappings

A mapping out of a sphere is nullhomotopic iff it extends to the ball. This even works out in the degenerate cases when the radius is ≤ 0, and we also don't need to explicitly assume continuity since it's already implicit in both sides of the equivalence.

```
lemma nullhomotopic_from_lemma:
  assumes contg: continuous_on (cball a r - {a}) g
      and fa: ⋀e. 0 < e
             ⟹ ∃d. 0 < d ∧ (∀x. x ≠ a ∧ norm(x - a) < d ⟶ norm(g x - f
a) < e)
      and r: ⋀x. x ∈ cball a r ∧ x ≠ a ⟹ f x = g x
    shows continuous_on (cball a r) f
proof (clarsimp simp: continuous_on_eq_continuous_within Ball_def)
  fix x
  assume x: dist a x ≤ r
  show continuous (at x within cball a r) f
  proof (cases x=a)
    case True
    then show ?thesis
      by (metis continuous_within_eps_delta fa dist_norm dist_self r)
  next
    case False
    show ?thesis
    proof (rule continuous_transform_within [where f=g and d = norm(x-a)])
      have ∃d>0. ∀x'∈cball a r.
                    dist x' x < d ⟶ dist (g x') (g x) < e if e>0 for e
      proof -
        obtain d where d > 0
          and d: ⋀x'. ⟦dist x' a ≤ r; x' ≠ a; dist x' x < d⟧ ⟹
                           dist (g x') (g x) < e
```

```
          using contg False x ‹e>0›
          unfolding continuous_on_iff by (fastforce simp add: dist_commute intro:
that)
        show ?thesis
          using ‹d > 0› ‹x ≠ a›
          by (rule_tac x=min d (norm(x − a)) in exI)
             (auto simp: dist_commute dist_norm [symmetric]  intro!: d)
      qed
      then show continuous (at x within cball a r) g
        using contg False by (auto simp: continuous_within_eps_delta)
      show 0 < norm (x − a)
        using False by force
      show x ∈ cball a r
        by (simp add: x)
      show ⋀x'. ⟦x' ∈ cball a r; dist x' x < norm (x − a)⟧
        ⟹ g x' = f x'
        by (metis dist_commute dist_norm less_le r)
    qed
  qed
qed

proposition nullhomotopic_from_sphere_extension:
  fixes f :: 'M::euclidean_space ⇒ 'a::real_normed_vector
  shows  (∃c. homotopic_with_canon (λx. True) (sphere a r) S f (λx. c)) ⟷
          (∃g. continuous_on (cball a r) g ∧ g ` (cball a r) ⊆ S ∧
               (∀x ∈ sphere a r. g x = f x))
         (is ?lhs = ?rhs)
proof (cases r 0::real rule: linorder_cases)
  case less
  then show ?thesis
    by (simp add: homotopic_on_emptyI)
next
  case equal
  show ?thesis
  proof
    assume L: ?lhs
    with equal have [simp]: f a ∈ S
      using homotopic_with_imp_subset1 by fastforce
    obtain h:: real × 'M ⇒ 'a
      where h: continuous_on ({0..1} × {a}) h h ` ({0..1} × {a}) ⊆ S h (0, a)
= f a
      using L equal by (auto simp: homotopic_with)
    then have continuous_on (cball a r) (λx. h (0, a)) (λx. h (0, a)) ` cball a r
⊆ S
      by (auto simp: equal)
    then show ?rhs
      using h(3) local.equal by force
  next
    assume ?rhs
```

```
    then show ?lhs
      using equal continuous_on_const by (force simp add: homotopic_with)
  qed
next
  case greater
  let ?P = continuous_on {x. norm(x − a) = r} f ∧ f ` {x. norm(x − a) = r}
⊆ S
  have ?P if ?lhs using that
  proof
    fix c
    assume c: homotopic_with_canon (λx. True) (sphere a r) S f (λx. c)
    then have contf: continuous_on (sphere a r) f
      by (metis homotopic_with_imp_continuous)
    moreover have fim: f ` sphere a r ⊆ S
     by (meson continuous_map_subtopology_eu c homotopic_with_imp_continuous_maps)
    show ?P
      using contf fim by (auto simp: sphere_def dist_norm norm_minus_commute)
  qed
  moreover have ?P if ?rhs using that
  proof
    fix g
    assume g: continuous_on (cball a r) g ∧ g ` cball a r ⊆ S ∧ (∀ xa∈sphere a r.
g xa = f xa)
    then have f ` {x. norm (x − a) = r} ⊆ S
      using sphere_cball [of a r] unfolding image_subset_iff sphere_def
      by (metis dist_commute dist_norm mem_Collect_eq subset_eq)
    with g show ?P
      by (auto simp: dist_norm norm_minus_commute elim!: continuous_on_eq [OF
continuous_on_subset])
  qed
  moreover have ?thesis if ?P
  proof
    assume ?lhs
     then obtain c where homotopic_with_canon (λx. True) (sphere a r) S (λx.
c) f
      using homotopic_with_sym by blast
    then obtain h where conth: continuous_on ({0..1::real} × sphere a r) h
                    and him: h ` ({0..1} × sphere a r) ⊆ S
                    and h: ⋀x. h(0, x) = c ⋀x. h(1, x) = f x
      by (auto simp: homotopic_with_def)
    obtain b1::'M where b1 ∈ Basis
      using SOME_Basis by auto
    have c ∈ h ` ({0..1} × sphere a r)
    proof
      show c = h (0, a + r *R b1)
        by (simp add: h)
      show (0, a + r *R b1) ∈ {0..1::real} × sphere a r
        using greater ‹b1 ∈ Basis› by (auto simp: dist_norm)
    qed
```

then have *c* ∈ *S*
using *him* **by** *blast*
have *uconth*: *uniformly_continuous_on* ({*0..1*::*real*} × (*sphere a r*)) *h*
by (*force intro*: *compact_Times conth compact_uniformly_continuous*)
let *?g* = λ*x*. *h* (*norm* (*x* − *a*)/*r*,
a + (*if x* = *a then r* ∗*R* *b1 else* (*r* / *norm*(*x* − *a*)) ∗*R* (*x* − *a*)))
let *?g′* = λ*x*. *h* (*norm* (*x* − *a*)/*r*, *a* + (*r* / *norm*(*x* − *a*)) ∗*R* (*x* − *a*))
show *?rhs*
proof (*intro exI conjI*)
have *continuous_on* (*cball a r* − {*a*}) *?g′*
using *greater*
by (*force simp*: *dist_norm norm_minus_commute intro*: *continuous_on_compose2* [*OF conth*] *continuous_intros*)
then show *continuous_on* (*cball a r*) *?g*
proof (*rule nullhomotopic_from_lemma*)
show ∃ *d*>*0*. ∀ *x*. *x* ≠ *a* ∧ *norm* (*x* − *a*) < *d* ⟶ *norm* (*?g′ x* − *?g a*) < *e* **if** *0* < *e* **for** *e*
proof −
obtain *d* **where** *0* < *d*
and *d*: ⋀*x x′*. ⟦*x* ∈ {*0..1*} × *sphere a r*; *x′* ∈ {*0..1*} × *sphere a r*; *norm* (*x′* − *x*) < *d*⟧
⟹ *norm* (*h x′* − *h x*) < *e*
using *uniformly_continuous_onE* [*OF uconth* ‹*0* < *e*›] **by** (*auto simp*: *dist_norm*)
have ∗: *norm* (*h* (*norm* (*x* − *a*) / *r*,
a + (*r* / *norm* (*x* − *a*)) ∗*R* (*x* − *a*)) − *h* (*0*, *a* + *r* ∗*R* *b1*)) < *e* (**is** *norm* (*?ha* − *?hb*) < *e*)
if *x* ≠ *a norm* (*x* − *a*) < *r norm* (*x* − *a*) < *d* ∗ *r* **for** *x*
proof −
have *norm* (*?ha* − *?hb*) = *norm* (*?ha* − *h* (*0*, *a* + (*r* / *norm* (*x* − *a*)) ∗*R* (*x* − *a*)))
by (*simp add*: *h*)
also have . . . < *e*
using *greater* ‹*0* < *d*› ‹*b1* ∈ *Basis*› *that*
by (*intro d*) (*simp_all add*: *dist_norm*, *simp add*: *field_simps*)
finally show *?thesis* **.**
qed
show *?thesis*
using *greater* ‹*0* < *d*›
by (*rule_tac x* = *min r* (*d* ∗ *r*) **in** *exI*) (*auto simp*: ∗)
qed
show ⋀*x*. *x* ∈ *cball a r* ∧ *x* ≠ *a* ⟹ *?g x* = *?g′ x*
by *auto*
qed
next
show *?g* ‘ *cball a r* ⊆ *S*
using *greater him* ‹*c* ∈ *S*›
by (*force simp*: *h dist_norm norm_minus_commute*)
next

```
    show ∀x∈sphere a r. ?g x = f x
      using greater by (auto simp: h dist_norm norm_minus_commute)
  qed
 next
  assume ?rhs
  then obtain g where contg: continuous_on (cball a r) g
                 and gim: g ‘ cball a r ⊆ S
                 and gf: ∀ x ∈ sphere a r. g x = f x
    by auto
  let ?h = λy. g (a + (fst y) *R (snd y − a))
  have continuous_on ({0..1} × sphere a r) ?h
  proof (rule continuous_on_compose2 [OF contg])
    show continuous_on ({0..1} × sphere a r) (λx. a + fst x *R (snd x − a))
      by (intro continuous_intros)
    qed (auto simp: dist_norm norm_minus_commute mult_left_le_one_le)
  moreover
  have ?h ‘ ({0..1} × sphere a r) ⊆ S
   by (auto simp: dist_norm norm_minus_commute mult_left_le_one_le gim [THEN
subsetD])
  moreover
  have ∀x∈sphere a r. ?h (0, x) = g a ∀x∈sphere a r. ?h (1, x) = f x
    by (auto simp: dist_norm norm_minus_commute mult_left_le_one_le gf)
  ultimately have homotopic_with_canon (λx. True) (sphere a r) S (λx. g a) f
    by (auto simp: homotopic_with)
  then show ?lhs
    using homotopic_with_symD by blast
 qed
 ultimately
 show ?thesis by meson
qed

end
```

6.19 Homeomorphism Theorems

```
theory Homeomorphism
imports Homotopy
begin

lemma homeomorphic_spheres':
 fixes a ::'a::euclidean_space and b ::'b::euclidean_space
 assumes 0 < δ and dimeq: DIM('a) = DIM('b)
 shows (sphere a δ) homeomorphic (sphere b δ)
proof −
 obtain f :: 'a⇒'b and g where linear f linear g
    and fg: ⋀x. norm(f x) = norm x ⋀y. norm(g y) = norm y ⋀x. g(f x) = x
⋀y. f(g y) = y
   by (blast intro: isomorphisms_UNIV_UNIV [OF dimeq])
 then have continuous_on UNIV f continuous_on UNIV g
```

```
    using linear_continuous_on linear_linear by blast+
  then show ?thesis
    unfolding homeomorphic_minimal
    apply(rule_tac x=λx. b + f(x − a) in exI)
    apply(rule_tac x=λx. a + g(x − b) in exI)
    using assms
    apply (force intro: continuous_intros
                continuous_on_compose2 [of _ f] continuous_on_compose2 [of _ g]
simp: dist_commute dist_norm fg)
    done
qed

lemma homeomorphic_spheres_gen:
    fixes a :: 'a::euclidean_space and b :: 'b::euclidean_space
  assumes 0 < r 0 < s DIM('a::euclidean_space) = DIM('b::euclidean_space)
  shows (sphere a r homeomorphic sphere b s)
 using assms homeomorphic_trans [OF homeomorphic_spheres homeomorphic_spheres']
by auto
```

6.19.1 Homeomorphism of all convex compact sets with nonempty interior

```
proposition
  fixes S :: 'a::euclidean_space set
  assumes compact S and 0: 0 ∈ rel_interior S
      and star: ⋀x. x ∈ S ⟹ open_segment 0 x ⊆ rel_interior S
    shows starlike_compact_projective1_0:
            S − rel_interior S homeomorphic sphere 0 1 ∩ affine hull S
            (is ?SMINUS homeomorphic ?SPHER)
      and starlike_compact_projective2_0:
            S homeomorphic cball 0 1 ∩ affine hull S
            (is S homeomorphic ?CBALL)
proof −
  have starI: (u *R x) ∈ rel_interior S if x ∈ S 0 ≤ u u < 1 for x u
  proof (cases x=0 ∨ u=0)
    case True with 0 show ?thesis by force
  next
    case False with that show ?thesis
      by (auto simp: in_segment intro: star [THEN subsetD])
  qed
  have 0 ∈ S  using assms rel_interior_subset by auto
  define proj where proj ≡ λx::'a. x /R norm x
  have eqI: x = y if proj x = proj y norm x = norm y for x y
    using that  by (force simp: proj_def)
  then have iff_eq: ⋀x y. (proj x = proj y ∧ norm x = norm y) ⟷ x = y
    by blast
  have projI: x ∈ affine hull S ⟹ proj x ∈ affine hull S for x
    by (metis ‹0 ∈ S› affine_hull_span_0 hull_inc span_mul proj_def)
  have nproj1 [simp]: x ≠ 0 ⟹ norm(proj x) = 1 for x
```

```
    by (simp add: proj_def)
  have proj0_iff [simp]: proj x = 0 ⟷ x = 0 for x
    by (simp add: proj_def)
  have cont_proj: continuous_on (UNIV − {0}) proj
    unfolding proj_def by (rule continuous_intros | force)+
  have proj_spherI: ⋀x. ⟦x ∈ affine hull S; x ≠ 0⟧ ⟹ proj x ∈ ?SPHER
    by (simp add: projI)
  have bounded S closed S
    using ‹compact S› compact_eq_bounded_closed by blast+
  have inj_on_proj: inj_on proj (S − rel_interior S)
  proof
    fix x y
    assume x: x ∈ S − rel_interior S
      and y: y ∈ S − rel_interior S and eq: proj x = proj y
    then have xynot: x ≠ 0 y ≠ 0 x ∈ S y ∈ S x ∉ rel_interior S y ∉ rel_interior
S
      using 0 by auto
   consider norm x = norm y | norm x < norm y | norm x > norm y by linarith
    then show x = y
    proof cases
      assume norm x = norm y
        with iff_eq eq show x = y by blast
    next
      assume ∗: norm x < norm y
      have x /R norm x = norm x ∗R (x /R norm x) /R norm (norm x ∗R (x /R
norm x))
        by force
      then have proj ((norm x / norm y) ∗R y) = proj x
        by (metis (no_types) divide_inverse local.proj_def eq scaleR_scaleR)
      then have [simp]: (norm x / norm y) ∗R y = x
        by (rule eqI) (simp add: ‹y ≠ 0›)
      have no: 0 ≤ norm x / norm y norm x / norm y < 1
        using ∗ by (auto simp: field_split_simps)
      then show x = y
        using starI [OF ‹y ∈ S› no] xynot by auto
    next
      assume ∗: norm x > norm y
      have y /R norm y = norm y ∗R (y /R norm y) /R norm (norm y ∗R (y /R
norm y))
        by force
      then have proj ((norm y / norm x) ∗R x) = proj y
        by (metis (no_types) divide_inverse local.proj_def eq scaleR_scaleR)
      then have [simp]: (norm y / norm x) ∗R x = y
        by (rule eqI) (simp add: ‹x ≠ 0›)
      have no: 0 ≤ norm y / norm x norm y / norm x < 1
        using ∗ by (auto simp: field_split_simps)
      then show x = y
        using starI [OF ‹x ∈ S› no] xynot by auto
    qed
```

```
  qed
  have ∃ surf. homeomorphism (S − rel_interior S) ?SPHER proj surf
  proof (rule homeomorphism_compact)
    show compact (S − rel_interior S)
       using ‹compact S› compact_rel_boundary by blast
    show continuous_on (S − rel_interior S) proj
      using 0 by (blast intro: continuous_on_subset [OF cont_proj])
    show proj ‘ (S − rel_interior S) = ?SPHER
    proof
      show proj ‘ (S − rel_interior S) ⊆ ?SPHER
        using 0 by (force simp: hull_inc projI intro: nproj1)
      show ?SPHER ⊆ proj ‘ (S − rel_interior S)
      proof (clarsimp simp: proj_def)
        fix x
        assume x ∈ affine hull S and nox: norm x = 1
        then have x ≠ 0 by auto
        obtain d where 0 < d and dx: (d *R x) ∈ rel_frontier S
          and ri: ⋀e. ⟦0 ≤ e; e < d⟧ ⟹ (e *R x) ∈ rel_interior S
           using ray_to_rel_frontier [OF ‹bounded S› 0] ‹x ∈ affine hull S› ‹x ≠ 0›
by auto
        show x ∈ (λx. x /R norm x) ‘ (S − rel_interior S)
        proof
          show x = d *R x /R norm (d *R x)
            using ‹0 < d› by (auto simp: nox)
          show d *R x ∈ S − rel_interior S
            using dx ‹closed S› by (auto simp: rel_frontier_def)
        qed
      qed
    qed
  qed (rule inj_on_proj)
  then obtain surf where surf: homeomorphism (S − rel_interior S) ?SPHER
proj surf
    by blast
  then have cont_surf: continuous_on (proj ‘ (S − rel_interior S)) surf
    by (auto simp: homeomorphism_def)
  have surf_nz: ⋀x. x ∈ ?SPHER ⟹ surf x ≠ 0
    by (metis 0 DiffE homeomorphism_def imageI surf)
  have cont_nosp: continuous_on (?SPHER) (λx. norm x *R ((surf o proj) x))
  proof (intro continuous_intros)
    show continuous_on (sphere 0 1 ∩ affine hull S) proj
      by (rule continuous_on_subset [OF cont_proj], force)
    show continuous_on (proj ‘ (sphere 0 1 ∩ affine hull S)) surf
       by (intro continuous_on_subset [OF cont_surf]) (force simp: homeomor-
phism_image1 [OF surf] dest: proj_spherI)
  qed
  have surfpS: ⋀x. ⟦norm x = 1; x ∈ affine hull S⟧ ⟹ surf (proj x) ∈ S
    by (metis (full_types) DiffE ‹0 ∈ S› homeomorphism_def image_eqI norm_zero
proj_spherI real_vector.scale_zero_left scaleR_one surf)
  have *: ∃ y. norm y = 1 ∧ y ∈ affine hull S ∧ x = surf (proj y)
```

```
        if x ∈ S x ∉ rel_interior S for x
  proof -
    have proj x ∈ ?SPHER
      by (metis (full_types) 0 hull_inc proj_spherI that)
    moreover have surf (proj x) = x
      by (metis Diff_iff homeomorphism_def surf that)
    ultimately show ?thesis
      by (metis ‹⋀x. x ∈ ?SPHER ⟹ surf x ≠ 0› hull_inc inverse_1 local.proj_def
norm_sgn projI scaleR_one sgn_div_norm that(1))
  qed
  have surfp_notin: ⋀x. ⟦norm x = 1; x ∈ affine hull S⟧ ⟹ surf (proj x) ∉
rel_interior S
   by (metis (full_types) DiffE one_neq_zero homeomorphism_def image_eqI norm_zero
proj_spherI surf)
  have no_sp_im: (λx. norm x *R surf (proj x)) ` (?SPHER) = S - rel_interior S
    by (auto simp: surfpS image_def Bex_def surfp_notin *)
  have inj_spher: inj_on (λx. norm x *R surf (proj x)) ?SPHER
  proof
    fix x y
    assume xy: x ∈ ?SPHER y ∈ ?SPHER
       and eq: norm x *R surf (proj x) = norm y *R surf (proj y)
    then have norm x = 1 norm y = 1 x ∈ affine hull S y ∈ affine hull S
      using 0 by auto
    with eq show x = y
      by (simp add: proj_def) (metis surf xy homeomorphism_def)
  qed
  have co01: compact ?SPHER
    by (simp add: compact_Int_closed)
  show ?SMINUS homeomorphic ?SPHER
    using homeomorphic_def surf by blast
  have proj_scaleR: ⋀a x. 0 < a ⟹ proj (a *R x) = proj x
    by (simp add: proj_def)
  have cont_sp0: continuous_on (affine hull S - {0}) (surf o proj)
  proof (rule continuous_on_compose [OF continuous_on_subset [OF cont_proj]])
    show continuous_on (proj ` (affine hull S - {0})) surf
     using homeomorphism_image1 proj_spherI surf by (intro continuous_on_subset
[OF cont_surf]) fastforce
  qed auto
  obtain B where B>0 and B: ⋀x. x ∈ S ⟹ norm x ≤ B
    by (metis compact_imp_bounded ‹compact S› bounded_pos_less less_eq_real_def)
   have cont_nosp: continuous (at x within ?CBALL) (λx. norm x *R surf (proj
x))
        if norm x ≤ 1 x ∈ affine hull S for x
  proof (cases x=0)
    case True
    have (norm ⟶ 0) (at 0 within cball 0 1 ∩ affine hull S)
      by (simp add: tendsto_norm_zero eventually_at)
    with True show ?thesis
      apply (simp add: continuous_within)
```

apply (*rule lim_null_scaleR_bounded* [**where** *B*=*B*])
using *B* ‹*0* < *B*› *local.proj_def projI surfpS* **by** (*auto simp*: *eventually_at*)
next
case *False*
then have $\forall_F$ *x in at x*. (*x* ∈ *affine hull S* − {*0*}) = (*x* ∈ *affine hull S*)
by (*force simp*: *False eventually_at*)
moreover
have *continuous* (*at x within affine hull S* − {*0*}) (λ*x*. *surf* (*proj x*))
using *cont_sp0 False that* **by** (*auto simp add*: *continuous_on_eq_continuous_within*)
ultimately have ∗: *continuous* (*at x within affine hull S*) (λ*x*. *surf* (*proj x*))
by (*simp add*: *continuous_within Lim_transform_within_set continuous_on_eq_continuous_within*)
show *?thesis*
by (*intro continuous_within_subset* [**where** *s* = *affine hull S*, *OF* _ *Int_lower2*]
continuous_intros ∗)
qed
have *cont_nosp2*: *continuous_on ?CBALL* (λ*x*. *norm x* $*_R$ ((*surf o proj*) *x*))
by (*simp add*: *continuous_on_eq_continuous_within cont_nosp*)
have *norm y* $*_R$ *surf* (*proj y*) ∈ *S* **if** *y* ∈ *cball 0 1* **and** *yaff*: *y* ∈ *affine hull S*
for *y*
proof (*cases y*=*0*)
case *True* **then show** *?thesis*
by (*simp add*: ‹*0* ∈ *S*›)
next
case *False*
then have *norm y* $*_R$ *surf* (*proj y*) = *norm y* $*_R$ *surf* (*proj* (*y* $/_R$ *norm y*))
by (*simp add*: *proj_def*)
have *norm y* ≤ *1* **using** *that* **by** *simp*
have *surf* (*proj* (*y* $/_R$ *norm y*)) ∈ *S*
using *False local.proj_def nproj1 projI surfpS yaff* **by** *blast*
then have *surf* (*proj y*) ∈ *S*
by (*simp add*: *False proj_def*)
then show *norm y* $*_R$ *surf* (*proj y*) ∈ *S*
by (*metis dual_order.antisym le_less_linear norm_ge_zero rel_interior_subset*
scaleR_one
starI subset_eq ‹*norm y* ≤ *1*›)
qed
moreover have *x* ∈ (λ*x*. *norm x* $*_R$ *surf* (*proj x*)) ‘ (*?CBALL*) **if** *x* ∈ *S* **for** *x*
proof (*cases x*=*0*)
case *True* **with** *that hull_inc* **show** *?thesis* **by** *fastforce*
next
case *False*
then have *psp*: *proj* (*surf* (*proj x*)) = *proj x*
by (*metis homeomorphism_def hull_inc proj_spherI surf that*)
have *nxx*: *norm x* $*_R$ *proj x* = *x*
by (*simp add*: *False local.proj_def*)
have *affineI*: (*1* / *norm* (*surf* (*proj x*))) $*_R$ *x* ∈ *affine hull S*
by (*metis* ‹*0* ∈ *S*› *affine_hull_span_0 hull_inc span_clauses*(*4*) *that*)
have *sproj_nz*: *surf* (*proj x*) ≠ *0*
by (*metis False proj0_iff psp*)

then have *proj x = proj (proj x)*
by (*metis False nxx proj_scaleR zero_less_norm_iff*)
moreover have *scaleproj*: $\bigwedge$*a r. r* $*_R$ *proj a* = (*r* / *norm a*) $*_R$ *a*
by (*simp add*: *divide_inverse local.proj_def*)
ultimately have (*norm* (*surf* (*proj x*)) / *norm x*) $*_R$ *x* $\notin$ *rel_interior S*
by (*metis* (*no_types*) *sproj_nz divide_self_if hull_inc norm_eq_zero nproj1 projI psp scaleR_one surfp_notin that*)
then have (*norm* (*surf* (*proj x*)) / *norm x*) $\geq$ *1*
using *starI* [*OF that*] **by** (*meson starI* [*OF that*] *le_less_linear norm_ge_zero zero_le_divide_iff*)
then have *nole*: *norm x* $\leq$ *norm* (*surf* (*proj x*))
by (*simp add*: *le_divide_eq_1*)
let *?inx* = *x* $/_R$ *norm* (*surf* (*proj x*))
show *?thesis*
proof
show *x* = *norm ?inx* $*_R$ *surf* (*proj ?inx*)
by (*simp add*: *field_simps*) (*metis inverse_eq_divide nxx positive_imp_inverse_positive proj_scaleR psp scaleproj sproj_nz zero_less_norm_iff*)
qed (*auto simp*: *field_split_simps nole affineI*)
qed
ultimately have *im_cball*: (λ*x. norm x* $*_R$ *surf* (*proj x*)) ‘ *?CBALL* = *S*
by *blast*
have *inj_cball*: *inj_on* (λ*x. norm x* $*_R$ *surf* (*proj x*)) *?CBALL*
proof
fix *x y*
assume *x* $\in$ *?CBALL y* $\in$ *?CBALL*
and *eq*: *norm x* $*_R$ *surf* (*proj x*) = *norm y* $*_R$ *surf* (*proj y*)
then have *x*: *x* $\in$ *affine hull S* **and** *y*: *y* $\in$ *affine hull S*
using *0* **by** *auto*
show *x* = *y*
proof (*cases x=0* $\vee$ *y=0*)
case *True* **then show** *x* = *y* **using** *eq proj_spherI surf_nz x y* **by** *force*
next
case *False*
with *x y* **have** *speq*: *surf* (*proj x*) = *surf* (*proj y*)
by (*metis eq homeomorphism_apply2 proj_scaleR proj_spherI surf zero_less_norm_iff*)
then have *norm x* = *norm y*
by (*metis* ‹*x* $\in$ *affine hull S*› ‹*y* $\in$ *affine hull S*› *eq proj_spherI real_vector.scale_cancel_right surf_nz*)
moreover have *proj x* = *proj y*
by (*metis* (*no_types*) *False speq homeomorphism_apply2 proj_spherI surf x y*)
ultimately show *x* = *y*
using *eq eqI* **by** *blast*
qed
qed
have *co01*: *compact ?CBALL*
by (*simp add*: *compact_Int_closed*)
show *S homeomorphic ?CBALL*
using *homeomorphic_compact* [*OF co01 cont_nosp2* [*unfolded o_def*] *im_cball*

inj_cball] *homeomorphic_sym* **by** *blast*
qed

corollary
fixes *S* :: *'a::euclidean_space set*
assumes *compact S* **and** *a*: *a* ∈ *rel_interior S*
and *star*: ⋀*x*. *x* ∈ *S* ⟹ *open_segment a x* ⊆ *rel_interior S*
shows *starlike_compact_projective1*:
S − *rel_interior S homeomorphic sphere a 1* ∩ *affine hull S*
and *starlike_compact_projective2*:
S homeomorphic cball a 1 ∩ *affine hull S*
proof −
have *1*: *compact* ((+) (−*a*) ‘ *S*) **by** (*meson assms compact_translation*)
have *2*: *0* ∈ *rel_interior* ((+) (−*a*) ‘ *S*)
using *a rel_interior_translation* [*of* − *a S*] **by** (*simp cong*: *image_cong_simp*)
have *3*: *open_segment 0 x* ⊆ *rel_interior* ((+) (−*a*) ‘ *S*) **if** *x* ∈ ((+) (−*a*) ‘ *S*)
for *x*
proof −
have *x*+*a* ∈ *S* **using** *that* **by** *auto*
then have *open_segment a* (*x*+*a*) ⊆ *rel_interior S* **by** (*metis star*)
then show *?thesis* **using** *open_segment_translation* [*of a 0 x*]
using *rel_interior_translation* [*of* − *a S*] **by** (*fastforce simp add*: *ac_simps image_iff cong*: *image_cong_simp*)
qed
have *S* − *rel_interior S homeomorphic* ((+) (−*a*) ‘ *S*) − *rel_interior* ((+) (−*a*) ‘ *S*)
by (*metis rel_interior_translation translation_diff homeomorphic_translation*)
also have ... *homeomorphic sphere 0 1* ∩ *affine hull* ((+) (−*a*) ‘ *S*)
by (*rule starlike_compact_projective1_0* [*OF 1 2 3*])
also have ... = (+) (−*a*) ‘ (*sphere a 1* ∩ *affine hull S*)
by (*metis affine_hull_translation left_minus sphere_translation translation_Int*)
also have ... *homeomorphic sphere a 1* ∩ *affine hull S*
using *homeomorphic_translation homeomorphic_sym* **by** *blast*
finally show *S* − *rel_interior S homeomorphic sphere a 1* ∩ *affine hull S* **.**

have *S homeomorphic* ((+) (−*a*) ‘ *S*)
by (*metis homeomorphic_translation*)
also have ... *homeomorphic cball 0 1* ∩ *affine hull* ((+) (−*a*) ‘ *S*)
by (*rule starlike_compact_projective2_0* [*OF 1 2 3*])
also have ... = (+) (−*a*) ‘ (*cball a 1* ∩ *affine hull S*)
by (*metis affine_hull_translation left_minus cball_translation translation_Int*)
also have ... *homeomorphic cball a 1* ∩ *affine hull S*
using *homeomorphic_translation homeomorphic_sym* **by** *blast*
finally show *S homeomorphic cball a 1* ∩ *affine hull S* **.**
qed

corollary *starlike_compact_projective_special*:
assumes *compact S*
and *cb01*: *cball* (*0::'a::euclidean_space*) *1* ⊆ *S*

```
    and scale: ⋀x u. ⟦x ∈ S; 0 ≤ u; u < 1⟧ ⟹ u *R x ∈ S − frontier S
  shows S homeomorphic (cball (0::'a::euclidean_space) 1)
proof −
  have ball 0 1 ⊆ interior S
    using cb01 interior_cball interior_mono by blast
  then have 0: 0 ∈ rel_interior S
   by (meson centre_in_ball subsetD interior_subset_rel_interior le_numeral_extra(2)
not_le)
  have [simp]: affine hull S = UNIV
    using ‹ball 0 1 ⊆ interior S› by (auto intro!: affine_hull_nonempty_interior)
  have star: open_segment 0 x ⊆ rel_interior S if x ∈ S for x
  proof
    fix p assume p ∈ open_segment 0 x
    then obtain u where x ≠ 0 and u: 0 ≤ u u < 1 and p: u *R x = p
      by (auto simp: in_segment)
    then show p ∈ rel_interior S
      using scale [OF that u] closure_subset frontier_def interior_subset_rel_interior
by fastforce
  qed
  show ?thesis
    using starlike_compact_projective2_0 [OF ‹compact S› 0 star] by simp
qed

lemma homeomorphic_convex_lemma:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes convex S compact S convex T compact T
      and affeq: aff_dim S = aff_dim T
    shows (S − rel_interior S) homeomorphic (T − rel_interior T) ∧
          S homeomorphic T
proof (cases rel_interior S = {} ∨ rel_interior T = {})
  case True
    then show ?thesis
      by (metis Diff_empty affeq ‹convex S› ‹convex T› aff_dim_empty homeomor-
phic_empty rel_interior_eq_empty aff_dim_empty)
next
  case False
  then obtain a b where a: a ∈ rel_interior S and b: b ∈ rel_interior T by auto
  have starS: ⋀x. x ∈ S ⟹ open_segment a x ⊆ rel_interior S
    using rel_interior_closure_convex_segment
          a ‹convex S› closure_subset subsetCE by blast
  have starT: ⋀x. x ∈ T ⟹ open_segment b x ⊆ rel_interior T
    using rel_interior_closure_convex_segment
          b ‹convex T› closure_subset subsetCE by blast
  let ?aS = (+) (−a) ` S and ?bT = (+) (−b) ` T
  have 0: 0 ∈ affine hull ?aS 0 ∈ affine hull ?bT
    by (metis a b subsetD hull_inc image_eqI left_minus rel_interior_subset)+
  have subs: subspace (span ?aS) subspace (span ?bT)
    by (rule subspace_span)+
  moreover
```

have *dim* (*span* ((+) (− *a*) ‘ *S*)) = *dim* (*span* ((+) (− *b*) ‘ *T*))
  by (*metis 0 aff_dim_translation_eq aff_dim_zero affeq dim_span nat_int*)
ultimately obtain *f g* **where** *linear f linear g*
    and *fim*: *f* ‘ *span ?aS* = *span ?bT*
    and *gim*: *g* ‘ *span ?bT* = *span ?aS*
    and *fno*: ⋀*x*. *x* ∈ *span ?aS* ⟹ *norm*(*f x*) = *norm x*
    and *gno*: ⋀*x*. *x* ∈ *span ?bT* ⟹ *norm*(*g x*) = *norm x*
    and *gf*: ⋀*x*. *x* ∈ *span ?aS* ⟹ *g*(*f x*) = *x*
    and *fg*: ⋀*x*. *x* ∈ *span ?bT* ⟹ *f*(*g x*) = *x*
  by (*rule isometries_subspaces*) *blast*
have [*simp*]: *continuous_on A f* **for** *A*
  using ‹*linear f*› *linear_conv_bounded_linear linear_continuous_on* **by** *blast*
have [*simp*]: *continuous_on B g* **for** *B*
  using ‹*linear g*› *linear_conv_bounded_linear linear_continuous_on* **by** *blast*
have *eqspanS*: *affine hull ?aS* = *span ?aS*
  by (*metis a affine_hull_span_0 subsetD hull_inc image_eqI left_minus rel_interior_subset*)
have *eqspanT*: *affine hull ?bT* = *span ?bT*
  by (*metis b affine_hull_span_0 subsetD hull_inc image_eqI left_minus rel_interior_subset*)
have *S homeomorphic cball a 1* ∩ *affine hull S*
  by (*rule starlike_compact_projective2* [*OF* ‹*compact S*› *a starS*])
also have ... *homeomorphic* (+) (−*a*) ‘ (*cball a 1* ∩ *affine hull S*)
  by (*metis homeomorphic_translation*)
also have ... = *cball 0 1* ∩ (+) (−*a*) ‘ (*affine hull S*)
  by (*auto simp*: *dist_norm*)
also have ... = *cball 0 1* ∩ *span ?aS*
  using *eqspanS affine_hull_translation* **by** *blast*
also have ... *homeomorphic cball 0 1* ∩ *span ?bT*
proof (*rule homeomorphicI*)
  show *fim1*: *f* ‘ (*cball 0 1* ∩ *span ?aS*) = *cball 0 1* ∩ *span ?bT*
  proof
    show *f* ‘ (*cball 0 1* ∩ *span ?aS*) ⊆ *cball 0 1* ∩ *span ?bT*
      using *fim fno* **by** *auto*
    show *cball 0 1* ∩ *span ?bT* ⊆ *f* ‘ (*cball 0 1* ∩ *span ?aS*)
      by *clarify* (*metis IntI fg gim gno image_eqI mem_cball_0*)
  qed
  show *g* ‘ (*cball 0 1* ∩ *span ?bT*) = *cball 0 1* ∩ *span ?aS*
  proof
    show *g* ‘ (*cball 0 1* ∩ *span ?bT*) ⊆ *cball 0 1* ∩ *span ?aS*
      using *gim gno* **by** *auto*
    show *cball 0 1* ∩ *span ?aS* ⊆ *g* ‘ (*cball 0 1* ∩ *span ?bT*)
      by *clarify* (*metis IntI fim1 gf image_eqI*)
  qed
qed (*auto simp*: *fg gf*)
also have ... = *cball 0 1* ∩ (+) (−*b*) ‘ (*affine hull T*)
  using *eqspanT affine_hull_translation* **by** *blast*
also have ... = (+) (−*b*) ‘ (*cball b 1* ∩ *affine hull T*)
  by (*auto simp*: *dist_norm*)
also have ... *homeomorphic* (*cball b 1* ∩ *affine hull T*)
  by (*metis homeomorphic_translation homeomorphic_sym*)

```
  also have ... homeomorphic T
     by (metis starlike_compact_projective2 [OF ‹compact T› b starT] homeomor-
phic_sym)
  finally have 1: S homeomorphic T .

  have S − rel_interior S homeomorphic sphere a 1 ∩ affine hull S
    by (rule starlike_compact_projective1 [OF ‹compact S› a starS])
  also have ... homeomorphic (+) (−a) ` (sphere a 1 ∩ affine hull S)
    by (metis homeomorphic_translation)
  also have ... = sphere 0 1 ∩ (+) (−a) ` (affine hull S)
    by (auto simp: dist_norm)
  also have ... = sphere 0 1 ∩ span ?aS
    using eqspanS affine_hull_translation by blast
  also have ... homeomorphic sphere 0 1 ∩ span ?bT
  proof (rule homeomorphicI)
    show fim1: f ` (sphere 0 1 ∩ span ?aS) = sphere 0 1 ∩ span ?bT
    proof
      show f ` (sphere 0 1 ∩ span ?aS) ⊆ sphere 0 1 ∩ span ?bT
        using fim fno by auto
      show sphere 0 1 ∩ span ?bT ⊆ f ` (sphere 0 1 ∩ span ?aS)
        by clarify (metis IntI fg gim gno image_eqI mem_sphere_0)
    qed
    show g ` (sphere 0 1 ∩ span ?bT) = sphere 0 1 ∩ span ?aS
    proof
      show g ` (sphere 0 1 ∩ span ?bT) ⊆ sphere 0 1 ∩ span ?aS
        using gim gno by auto
      show sphere 0 1 ∩ span ?aS ⊆ g ` (sphere 0 1 ∩ span ?bT)
        by clarify (metis IntI fim1 gf image_eqI)
    qed
  qed (auto simp: fg gf)
  also have ... = sphere 0 1 ∩ (+) (−b) ` (affine hull T)
    using eqspanT affine_hull_translation by blast
  also have ... = (+) (−b) ` (sphere b 1 ∩ affine hull T)
    by (auto simp: dist_norm)
  also have ... homeomorphic (sphere b 1 ∩ affine hull T)
    by (metis homeomorphic_translation homeomorphic_sym)
  also have ... homeomorphic T − rel_interior T
     by (metis starlike_compact_projective1 [OF ‹compact T› b starT] homeomor-
phic_sym)
  finally have 2: S − rel_interior S homeomorphic T − rel_interior T .
  show ?thesis
    using 1 2 by blast
qed

lemma homeomorphic_convex_compact_sets:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes convex S compact S convex T compact T
      and affeq: aff_dim S = aff_dim T
    shows S homeomorphic T
```

using *homeomorphic_convex_lemma* [*OF assms*] *assms*
by (*auto simp*: *rel_frontier_def*)

lemma *homeomorphic_rel_frontiers_convex_bounded_sets*:
  fixes *S* :: *'a::euclidean_space set* **and** *T* :: *'b::euclidean_space set*
  assumes *convex S bounded S convex T bounded T*
    and *affeq*: *aff_dim S = aff_dim T*
  shows *rel_frontier S homeomorphic rel_frontier T*
using *assms homeomorphic_convex_lemma* [*of closure S closure T*]
by (*simp add*: *rel_frontier_def convex_rel_interior_closure*)

6.19.2 Homeomorphisms between punctured spheres and affine sets

Including the famous stereoscopic projection of the 3-D sphere to the complex plane

The special case with centre 0 and radius 1

lemma *homeomorphic_punctured_affine_sphere_affine_01*:
  assumes *b* ∈ *sphere 0 1 affine T 0* ∈ *T b* ∈ *T affine p*
    and *affT*: *aff_dim T = aff_dim p + 1*
  shows (*sphere 0 1* ∩ *T*) − {*b*} *homeomorphic p*
proof −
  have [*simp*]: *norm b = 1 b•b = 1*
    using *assms* **by** (*auto simp*: *norm_eq_1*)
  have [*simp*]: *T* ∩ {*v. b•v = 0*} ≠ {}
    using ‹*0* ∈ *T*› **by** *auto*
  have [*simp*]: ¬ *T* ⊆ {*v. b•v = 0*}
    using ‹*norm b = 1*› ‹*b* ∈ *T*› **by** *auto*
  define *f* **where** *f* ≡ λ*x. 2* $*_R$ *b* + (*2* / (*1* − *b•x*)) $*_R$ (*x* − *b*)
  define *g* **where** *g* ≡ λ*y. b* + (*4* / (*norm y* ˆ *2* + *4*)) $*_R$ (*y* − *2* $*_R$ *b*)
  have *fg*[*simp*]: ⋀*x*. ⟦*x* ∈ *T*; *b•x = 0*⟧ ⟹ *f* (*g x*) = *x*
   unfolding *f_def g_def* **by** (*simp add*: *algebra_simps field_split_simps add_nonneg_eq_0_iff*)
  have *no*: (*norm* (*f x*))^2 = *4* ∗ (*1* + *b* · *x*) / (*1* − *b* · *x*)
    if *norm x = 1* **and** *b* · *x* ≠ *1* **for** *x*
    using *that sum_sqs_eq* [*of 1 b* · *x*]
    apply (*simp flip*: *dot_square_norm add*: *norm_eq_1 nonzero_eq_divide_eq*)
    apply (*simp add*: *f_def vector_add_divide_simps inner_simps*)
    apply (*auto simp add*: *field_split_simps inner_commute*)
    done
  have [*simp*]: ⋀*u::real. 8* + *u* ∗ (*u* ∗ *8*) = *u* ∗ *16* ⟷ *u=1*
    by *algebra*
  have *gf*[*simp*]: ⋀*x*. ⟦*norm x = 1*; *b* · *x* ≠ *1*⟧ ⟹ *g* (*f x*) = *x*
    unfolding *g_def no* **by** (*auto simp*: *f_def field_split_simps*)
  have *g1*: *norm* (*g x*) = *1* **if** *x* ∈ *T* **and** *b* · *x = 0* **for** *x*
    using *that*
    apply (*simp only*: *g_def*)
    apply (*rule power2_eq_imp_eq*)
   apply (*simp_all add*: *dot_square_norm* [*symmetric*] *divide_simps vector_add_divide_simps*)

apply (*simp add*: *algebra_simps inner_commute*)
done
have *ne1*: $b \cdot g\ x \neq 1$ **if** $x \in T$ **and** $b \cdot x = 0$ **for** x
using *that* **unfolding** *g_def*
apply (*simp_all add*: *dot_square_norm* [*symmetric*] *divide_simps vector_add_divide_simps add_nonneg_eq_0_iff*)
apply (*auto simp*: *algebra_simps*)
done
have *subspace T*
by (*simp add*: *assms subspace_affine*)
have *gT*: $\bigwedge x.\ [\![x \in T;\ b \cdot x = 0]\!] \Longrightarrow g\ x \in T$
unfolding *g_def*
by (*blast intro*: ‹*subspace T*› ‹$b \in T$› *subspace_add subspace_mul subspace_diff*)
have f ‘ $\{x.\ norm\ x = 1 \wedge b{\cdot}x \neq 1\} \subseteq \{x.\ b{\cdot}x = 0\}$
unfolding *f_def* **using** ‹$norm\ b = 1$› *norm_eq_1*
by (*force simp*: *field_simps inner_add_right inner_diff_right*)
moreover have f ‘ $T \subseteq T$
unfolding *f_def* **using** *assms* ‹*subspace T*›
by (*auto simp add*: *inner_add_right inner_diff_right mem_affine_3_minus subspace_mul*)
moreover have $\{x.\ b{\cdot}x = 0\} \cap T \subseteq f$ ‘ $(\{x.\ norm\ x = 1 \wedge b{\cdot}x \neq 1\} \cap T)$
by *clarify* (*metis* (*mono_tags*) *IntI ne1 fg gT g1 imageI mem_Collect_eq*)
ultimately have *imf*: f ‘ $(\{x.\ norm\ x = 1 \wedge b{\cdot}x \neq 1\} \cap T) = \{x.\ b{\cdot}x = 0\} \cap T$
by *blast*
have *no4*: $\bigwedge y.\ b{\cdot}y = 0 \Longrightarrow norm\ ((y{\cdot}y + 4) *_R b + 4 *_R (y - 2 *_R b)) = y{\cdot}y + 4$
apply (*rule power2_eq_imp_eq*)
apply (*simp_all flip*: *dot_square_norm*)
apply (*auto simp*: *power2_eq_square algebra_simps inner_commute*)
done
have [*simp*]: $\bigwedge x.\ [\![norm\ x = 1;\ b \cdot x \neq 1]\!] \Longrightarrow b \cdot f\ x = 0$
by (*simp add*: *f_def algebra_simps field_split_simps*)
have [*simp*]: $\bigwedge x.\ [\![x \in T;\ norm\ x = 1;\ b \cdot x \neq 1]\!] \Longrightarrow f\ x \in T$
unfolding *f_def*
by (*blast intro*: ‹*subspace T*› ‹$b \in T$› *subspace_add subspace_mul subspace_diff*)
have g ‘ $\{x.\ b{\cdot}x = 0\} \subseteq \{x.\ norm\ x = 1 \wedge b{\cdot}x \neq 1\}$
unfolding *g_def*
apply (*clarsimp simp*: *no4 vector_add_divide_simps divide_simps add_nonneg_eq_0_iff dot_square_norm* [*symmetric*])
apply (*auto simp*: *algebra_simps*)
done
moreover have g ‘ $T \subseteq T$
unfolding *g_def*
by (*blast intro*: ‹*subspace T*› ‹$b \in T$› *subspace_add subspace_mul subspace_diff*)
moreover have $\{x.\ norm\ x = 1 \wedge b{\cdot}x \neq 1\} \cap T \subseteq g$ ‘ $(\{x.\ b{\cdot}x = 0\} \cap T)$
by *clarify* (*metis* (*mono_tags, lifting*) *IntI gf image_iff imf mem_Collect_eq*)
ultimately have *img*: g ‘ $(\{x.\ b{\cdot}x = 0\} \cap T) = \{x.\ norm\ x = 1 \wedge b{\cdot}x \neq 1\} \cap T$

by *blast*
have *aff*: *affine* ({*x*. *b*•*x* = *0*} ∩ *T*)
by (*blast intro*: *affine_hyperplane assms*)
have *contf*: *continuous_on* ({*x*. *norm x* = *1* ∧ *b*•*x* ≠ *1*} ∩ *T*) *f*
unfolding *f_def* by (*rule continuous_intros* | *force*)+
have *contg*: *continuous_on* ({*x*. *b*•*x* = *0*} ∩ *T*) *g*
unfolding *g_def* by (*rule continuous_intros* | *force simp*: *add_nonneg_eq_0_iff*)+
have (*sphere 0 1* ∩ *T*) − {*b*} = {*x*. *norm x* = *1* ∧ (*b*•*x* ≠ *1*)} ∩ *T*
using ‹*norm b* = *1*› by (*auto simp*: *norm_eq_1*) (*metis vector_eq* ‹*b*•*b* = *1*›)
also have ... *homeomorphic* {*x*. *b*•*x* = *0*} ∩ *T*
by (*rule homeomorphicI* [*OF imf img contf contg*]) *auto*
also have ... *homeomorphic p*
proof (*rule homeomorphic_affine_sets* [*OF aff* ‹*affine p*›])
show *aff_dim* ({*x*. *b* • *x* = *0*} ∩ *T*) = *aff_dim p*
by (*simp add*: *Int_commute aff_dim_affine_Int_hyperplane* [*OF* ‹*affine T*›] *affT*)
qed
finally show *?thesis* .
qed

theorem *homeomorphic_punctured_affine_sphere_affine*:
fixes *a* :: ′*a* :: *euclidean_space*
assumes *0* < *r b* ∈ *sphere a r affine T a* ∈ *T b* ∈ *T affine p*
and *aff*: *aff_dim T* = *aff_dim p* + *1*
shows (*sphere a r* ∩ *T*) − {*b*} *homeomorphic p*
proof −
have *a* ≠ *b* using *assms* by *auto*
then have *inj*: *inj* (λ*x*::′*a*. *x* $/_R$ *norm* (*a* − *b*))
by (*simp add*: *inj_on_def*)
have ((*sphere a r* ∩ *T*) − {*b*}) *homeomorphic*
(+) (−*a*) ‘ ((*sphere a r* ∩ *T*) − {*b*})
by (*rule homeomorphic_translation*)
also have ... *homeomorphic* ($*_R$) (*inverse r*) ‘ (+) (− *a*) ‘ (*sphere a r* ∩ *T* − {*b*})
by (*metis* ‹*0* < *r*› *homeomorphic_scaling inverse_inverse_eq inverse_zero less_irrefl*)
also have ... = *sphere 0 1* ∩ (($*_R$) (*inverse r*) ‘ (+) (− *a*) ‘ *T*) − {(*b* − *a*) $/_R$ *r*}
using *assms* by (*auto simp*: *dist_norm norm_minus_commute divide_simps*)
also have ... *homeomorphic p*
using *assms affine_translation* [*symmetric*, *of* − *a*] *aff_dim_translation_eq* [*of* − *a*]
by (*intro homeomorphic_punctured_affine_sphere_affine_01*) (*auto simp*: *dist_norm norm_minus_commute affine_scaling inj*)
finally show *?thesis* .
qed

corollary *homeomorphic_punctured_sphere_affine*:
fixes *a* :: ′*a* :: *euclidean_space*
assumes *0* < *r* and *b*: *b* ∈ *sphere a r*
and *affine T* and *affS*: *aff_dim T* + *1* = *DIM*(′*a*)

shows (*sphere a r* − {*b*}) *homeomorphic T*
using *homeomorphic_punctured_affine_sphere_affine* [*of r b a UNIV T*] *assms* **by** *auto*

corollary *homeomorphic_punctured_sphere_hyperplane*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *0* < *r* **and** *b*: *b* ∈ *sphere a r*
and *c* ≠ *0*
shows (*sphere a r* − {*b*}) *homeomorphic* {*x*::*'a*. *c* · *x* = *d*}
using *assms*
by (*intro homeomorphic_punctured_sphere_affine*) (*auto simp*: *affine_hyperplane of_nat_diff*)

proposition *homeomorphic_punctured_sphere_affine_gen*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *convex S bounded S* **and** *a*: *a* ∈ *rel_frontier S*
and *affine T* **and** *affS*: *aff_dim S* = *aff_dim T* + *1*
shows *rel_frontier S* − {*a*} *homeomorphic T*
proof −
obtain *U* :: *'a set* **where** *affine U convex U* **and** *affdS*: *aff_dim U* = *aff_dim S*
using *choose_affine_subset* [*OF affine_UNIV aff_dim_geq*]
by (*meson aff_dim_affine_hull affine_affine_hull affine_imp_convex*)
have *S* ≠ {} **using** *assms* **by** *auto*
then obtain *z* **where** *z* ∈ *U*
by (*metis aff_dim_negative_iff equals0I affdS*)
then have *bne*: *ball z 1* ∩ *U* ≠ {} **by** *force*
then have [*simp*]: *aff_dim*(*ball z 1* ∩ *U*) = *aff_dim U*
using *aff_dim_convex_Int_open* [*OF* ‹*convex U*› *open_ball*]
by (*fastforce simp add*: *Int_commute*)
have *rel_frontier S homeomorphic rel_frontier* (*ball z 1* ∩ *U*)
by (*rule homeomorphic_rel_frontiers_convex_bounded_sets*)
(*auto simp*: ‹*affine U*› *affine_imp_convex convex_Int affdS assms*)
also have ... = *sphere z 1* ∩ *U*
using *convex_affine_rel_frontier_Int* [*of ball z 1 U*]
by (*simp add*: ‹*affine U*› *bne*)
finally have *rel_frontier S homeomorphic sphere z 1* ∩ *U* .
then obtain *h k* **where** *him*: *h* ‘ *rel_frontier S* = *sphere z 1* ∩ *U*
and *kim*: *k* ‘ (*sphere z 1* ∩ *U*) = *rel_frontier S*
and *hcon*: *continuous_on* (*rel_frontier S*) *h*
and *kcon*: *continuous_on* (*sphere z 1* ∩ *U*) *k*
and *kh*: $\bigwedge$*x*. *x* ∈ *rel_frontier S* ⟹ *k*(*h*(*x*)) = *x*
and *hk*: $\bigwedge$*y*. *y* ∈ *sphere z 1* ∩ *U* ⟹ *h*(*k*(*y*)) = *y*
unfolding *homeomorphic_def homeomorphism_def* **by** *auto*
have *rel_frontier S* − {*a*} *homeomorphic* (*sphere z 1* ∩ *U*) − {*h a*}
proof (*rule homeomorphicI*)
show *h*: *h* ‘ (*rel_frontier S* − {*a*}) = *sphere z 1* ∩ *U* − {*h a*}
using *him a kh* **by** *auto metis*
show *k* ‘ (*sphere z 1* ∩ *U* − {*h a*}) = *rel_frontier S* − {*a*}
by (*force simp*: *h* [*symmetric*] *image_comp o_def kh*)

qed (*auto intro*: *continuous_on_subset hcon kcon simp*: *kh hk*)
also have ... *homeomorphic T*
by (*rule homeomorphic_punctured_affine_sphere_affine*)
(*use a him* **in** ‹*auto simp*: *affS affdS* ‹*affine T*› ‹*affine U*› ‹*z* ∈ *U*››)
finally show *?thesis* **.**
qed

When dealing with AR, ANR and ANR later, it's useful to know that every set is homeomorphic to a closed subset of a convex set, and if the set is locally compact we can take the convex set to be the universe.

proposition *homeomorphic_closedin_convex*:
fixes *S* :: *'m::euclidean_space set*
assumes *aff_dim S* < *DIM('n)*
obtains *U* **and** *T* :: *'n::euclidean_space set*
where *convex U U* ≠ {} *closedin* (*top_of_set U*) *T*
S homeomorphic T
proof (*cases S* = {})
case *True* **then show** *?thesis*
by (*rule_tac U=UNIV* **and** *T*={} **in** *that*) *auto*
next
case *False*
then obtain *a* **where** *a* ∈ *S* **by** *auto*
obtain *i::'n* **where** *i*: *i* ∈ *Basis i* ≠ *0*
using *SOME_Basis Basis_zero* **by** *force*
have *0* ∈ *affine hull* ((+) (− *a*) ' *S*)
by (*simp add*: ‹*a* ∈ *S*› *hull_inc*)
then have *dim* ((+) (− *a*) ' *S*) = *aff_dim* ((+) (− *a*) ' *S*)
by (*simp add*: *aff_dim_zero*)
also have ... < *DIM('n)*
by (*simp add*: *aff_dim_translation_eq_subtract assms cong*: *image_cong_simp*)
finally have *dd*: *dim* ((+) (− *a*) ' *S*) < *DIM('n)*
by *linarith*
have *span*: *span* {*x*. *i* · *x* = *0*} = {*x*. *i* · *x* = *0*}
using *span_eq_iff* [*symmetric*, *of* {*x*. *i* · *x* = *0*}] *subspace_hyperplane* [*of i*] **by** *simp*
have *dim* ((+) (− *a*) ' *S*) ≤ *dim* {*x*. *i* · *x* = *0*}
using *dd* **by** (*simp add*: *dim_hyperplane* [*OF* ‹*i* ≠ *0*›])
then obtain *T* **where** *subspace T* **and** *Tsub*: *T* ⊆ {*x*. *i* · *x* = *0*}
and *dimT*: *dim T* = *dim* ((+) (− *a*) ' *S*)
by (*rule choose_subspace_of_subspace*) (*simp add*: *span*)
have *subspace* (*span* ((+) (− *a*) ' *S*))
using *subspace_span* **by** *blast*
then obtain *h k* **where** *linear h linear k*
and *heq*: *h* ' *span* ((+) (− *a*) ' *S*) = *T*
and *keq*:*k* ' *T* = *span* ((+) (− *a*) ' *S*)
and *hinv* [*simp*]: ⋀*x*. *x* ∈ *span* ((+) (− *a*) ' *S*) ⟹ *k*(*h x*) = *x*
and *kinv* [*simp*]: ⋀*x*. *x* ∈ *T* ⟹ *h*(*k x*) = *x*
by (*auto simp*: *dimT intro*: *isometries_subspaces* [*OF* _ ‹*subspace T*›] *dimT*)
have *hcont*: *continuous_on A h* **and** *kcont*: *continuous_on B k* **for** *A B*

using ‹*linear h*› ‹*linear k*› *linear_continuous_on linear_conv_bounded_linear* **by** *blast+*
have *ihhhh*[*simp*]: $\bigwedge x.\ x \in S \Longrightarrow i \cdot h\ (x - a) = 0$
using *Tsub* [*THEN subsetD*] *heq span_superset* **by** *fastforce*
have *sphere 0 1* − {*i*} *homeomorphic* {*x*. *i* · *x* = *0*}
proof (*rule homeomorphic_punctured_sphere_affine*)
show *affine* {*x*. *i* · *x* = *0*}
by (*auto simp*: *affine_hyperplane*)
show *aff_dim* {*x*. *i* · *x* = *0*} + *1* = *int DIM*(′*n*)
using *i* **by** *clarsimp* (*metis DIM_positive Suc_pred add.commute of_nat_Suc*)
qed (*use i* **in** *auto*)
then obtain *f g* **where** *fg*: *homeomorphism* (*sphere 0 1* − {*i*}) {*x*. *i* · *x* = *0*} *f g*
by (*force simp*: *homeomorphic_def*)
show *?thesis*
proof
have *h* ‘ (+) (− *a*) ‘ *S* ⊆ *T*
using *heq span_superset span_linear_image* **by** *blast*
then have *g* ‘ *h* ‘ (+) (− *a*) ‘ *S* ⊆ *g* ‘ {*x*. *i* · *x* = *0*}
using *Tsub* **by** (*simp add*: *image_mono*)
also have ... ⊆ *sphere 0 1* − {*i*}
by (*simp add*: *fg* [*unfolded homeomorphism_def*])
finally have *gh_sub_sph*: (*g* ∘ *h*) ‘ (+) (− *a*) ‘ *S* ⊆ *sphere 0 1* − {*i*}
by (*metis image_comp*)
then have *gh_sub_cb*: (*g* ∘ *h*) ‘ (+) (− *a*) ‘ *S* ⊆ *cball 0 1*
by (*metis Diff_subset order_trans sphere_cball*)
have [*simp*]: $\bigwedge u.\ u \in S \Longrightarrow$ *norm* (*g* (*h* (*u* − *a*))) = *1*
using *gh_sub_sph* [*THEN subsetD*] **by** (*auto simp*: *o_def*)
show *convex* (*ball 0 1* ∪ (*g* ∘ *h*) ‘ (+) (− *a*) ‘ *S*)
by (*meson ball_subset_cball convex_intermediate_ball gh_sub_cb sup.bounded_iff sup.cobounded1*)
show *closedin* (*top_of_set* (*ball 0 1* ∪ (*g* ∘ *h*) ‘ (+) (− *a*) ‘ *S*)) ((*g* ∘ *h*) ‘ (+) (− *a*) ‘ *S*)
unfolding *closedin_closed*
by (*rule_tac x=sphere 0 1* **in** *exI*) *auto*
have *ghcont*: *continuous_on* ((λ*x*. *x* − *a*) ‘ *S*) (λ*x*. *g* (*h x*))
by (*rule continuous_on_compose2* [*OF homeomorphism_cont2* [*OF fg*] *hcont*], *force*)
have *kfcont*: *continuous_on* ((λ*x*. *g* (*h* (*x* − *a*))) ‘ *S*) (λ*x*. *k* (*f x*))
proof (*rule continuous_on_compose2* [*OF kcont*])
show *continuous_on* ((λ*x*. *g* (*h* (*x* − *a*))) ‘ *S*) *f*
using *homeomorphism_cont1* [*OF fg*] *gh_sub_sph* **by** (*fastforce intro*: *continuous_on_subset*)
qed *auto*
have *S homeomorphic* (+) (− *a*) ‘ *S*
by (*fact homeomorphic_translation*)
also have ... *homeomorphic* (*g* ∘ *h*) ‘ (+) (− *a*) ‘ *S*
apply (*simp add*: *homeomorphic_def homeomorphism_def cong*: *image_cong_simp*)
apply (*rule_tac x=g* ∘ *h* **in** *exI*)

apply (*rule_tac* $x=k \circ f$ **in** *exI*)
apply (*auto simp*: *ghcont kfcont span_base homeomorphism_apply2* [*OF fg*] *image_comp cong*: *image_cong_simp*)
done
finally show *S homeomorphic* $(g \circ h)$ ‘ $(+)$ $(- a)$ ‘ *S* **.**
qed *auto*
qed

6.19.3 Locally compact sets in an open set

Locally compact sets are closed in an open set and are homeomorphic to an absolutely closed set if we have one more dimension to play with.

lemma *locally_compact_open_Int_closure*:
fixes *S* :: *'a* :: *metric_space set*
assumes *locally compact S*
obtains *T* **where** *open T S* = *T* ∩ *closure S*
proof −
have ∀ *x*∈*S*. ∃ *T v u*. *u* = *S* ∩ *T* ∧ *x* ∈ *u* ∧ *u* ⊆ *v* ∧ *v* ⊆ *S* ∧ *open T* ∧ *compact v*
by (*metis assms locally_compact openin_open*)
then obtain *t v* **where**
tv: ⋀*x*. *x* ∈ *S*
⟹ *v x* ⊆ *S* ∧ *open* (*t x*) ∧ *compact* (*v x*) ∧ (∃ *u*. *x* ∈ *u* ∧ *u* ⊆ *v x* ∧ *u* = *S* ∩ *t x*)
by *metis*
then have *o*: *open* (⋃(*t* ‘ *S*))
by *blast*
have *S* = ⋃ (*v* ‘ *S*)
using *tv* **by** *blast*
also have ... = ⋃(*t* ‘ *S*) ∩ *closure S*
proof
show ⋃(*v* ‘ *S*) ⊆ ⋃(*t* ‘ *S*) ∩ *closure S*
by *clarify* (*meson IntD2 IntI UN_I closure_subset subsetD tv*)
have *t x* ∩ *closure S* ⊆ *v x* **if** *x* ∈ *S* **for** *x*
proof −
have *t x* ∩ *closure S* ⊆ *closure* (*t x* ∩ *S*)
by (*simp add*: *open_Int_closure_subset that tv*)
also have ... ⊆ *v x*
by (*metis Int_commute closure_minimal compact_imp_closed that tv*)
finally show *?thesis* **.**
qed
then show ⋃(*t* ‘ *S*) ∩ *closure S* ⊆ ⋃(*v* ‘ *S*)
by *blast*
qed
finally have *e*: *S* = ⋃(*t* ‘ *S*) ∩ *closure S* **.**
show *?thesis*
by (*rule that* [*OF o e*])
qed

lemma *locally_compact_closedin_open*:
  fixes *S* :: *'a* :: *metric_space set*
  assumes *locally compact S*
  obtains *T* **where** *open T closedin* (*top_of_set T*) *S*
 by (*metis locally_compact_open_Int_closure* [*OF assms*] *closed_closure closedin_closed_Int*)

lemma *locally_compact_homeomorphism_projection_closed*:
 assumes *locally compact S*
 obtains *T* **and** *f* :: *'a* ⇒ *'a* :: *euclidean_space* × *'b* :: *euclidean_space*
 where *closed T homeomorphism S T f fst*
proof (*cases closed S*)
 case *True*
 show *?thesis*
 proof
  show *homeomorphism S* (*S* × {*0*}) (λ*x*. (*x*, *0*)) *fst*
   by (*auto simp*: *homeomorphism_def continuous_intros*)
 qed (*use True closed_Times* **in** *auto*)
next
 case *False*
  obtain *U* **where** *open U* **and** *US*: *U* ∩ *closure S* = *S*
   by (*metis locally_compact_open_Int_closure* [*OF assms*])
  with *False* **have** *Ucomp*: −*U* ≠ {}
   using *closure_eq* **by** *auto*
  have [*simp*]: *closure* (− *U*) = −*U*
   by (*simp add*: ‹*open U*› *closed_Compl*)
  define *f* :: *'a* ⇒ *'a* × *'b* **where** *f* ≡ λ*x*. (*x*, *One* /$_R$ *setdist* {*x*} (− *U*))
  have *continuous_on U* (λ*x*. (*x*, *One* /$_R$ *setdist* {*x*} (− *U*)))
  proof (*intro continuous_intros continuous_on_setdist*)
   show ∀ *x*∈*U*. *setdist* {*x*} (− *U*) ≠ *0*
    by (*simp add*: *Ucomp setdist_eq_0_sing_1*)
  qed
  then have *homU*: *homeomorphism U* (*f*‘*U*) *f fst*
   by (*auto simp*: *f_def homeomorphism_def image_iff continuous_intros*)
  have *cloS*: *closedin* (*top_of_set U*) *S*
   by (*metis US closed_closure closedin_closed_Int*)
  have *cont*: *isCont* ((λ*x*. *setdist* {*x*} (− *U*)) *o fst*) *z* **for** *z* :: *'a* × *'b*
   by (*rule continuous_at_compose continuous_intros continuous_at_setdist*)+
  have *setdist1D*: *setdist* {*a*} (− *U*) ∗$_R$ *b* = *One* ⟹ *setdist* {*a*} (− *U*) ≠ *0* **for**
a::*'a* **and** *b*::*'b*
   by *force*
  have ∗: *r* ∗$_R$ *b* = *One* ⟹ *b* = (*1* / *r*) ∗$_R$ *One* **for** *r* **and** *b*::*'b*
  by (*metis One_non_0 nonzero_divide_eq_eq real_vector.scale_eq_0_iff real_vector.scale_scale scaleR_one*)
  have ⋀*a b*::*'b*. *setdist* {*a*} (− *U*) ∗$_R$ *b* = *One* ⟹ (*a*,*b*) ∈ (λ*x*. (*x*, (*1* / *setdist* {*x*} (− *U*)) ∗$_R$ *One*)) ‘ *U*
  by (*metis* (*mono_tags*, *lifting*) ∗ *ComplI image_eqI setdist1D setdist_sing_in_set*)
  then have *f* ‘ *U* = (λ*z*. (*setdist* {*fst z*} (− *U*) ∗$_R$ *snd z*)) −‘ {*One*}

by (*auto simp*: *f_def setdist_eq_0_sing_1 field_simps Ucomp*)
then have *clfU*: *closed* (*f* ‘ *U*)
by (*force intro*: *continuous_intros cont* [*unfolded o_def*] *continuous_closed_vimage*)
have *closed* (*f* ‘ *S*)
by (*metis closedin_closed_trans* [*OF _ clfU*] *homeomorphism_imp_closed_map* [*OF homU cloS*])
then show *?thesis*
by (*metis US homU homeomorphism_of_subsets inf_sup_ord*(*1*) *that*)
qed

lemma *locally_compact_closed_Int_open*:
fixes *S* :: *'a* :: *euclidean_space set*
shows *locally compact S* ⟷ (∃ *U V*. *closed U* ∧ *open V* ∧ *S* = *U* ∩ *V*) (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
by (*metis closed_closure inf_commute locally_compact_open_Int_closure*)
show *?rhs* ⟹ *?lhs*
by (*meson closed_imp_locally_compact locally_compact_Int open_imp_locally_compact*)
qed

lemma *lowerdim_embeddings*:
assumes *DIM*(*'a*) < *DIM*(*'b*)
obtains *f* :: *'a*::*euclidean_space*∗*real* ⇒ *'b*::*euclidean_space*
and *g* :: *'b* ⇒ *'a*∗*real*
and *j* :: *'b*
where *linear f linear g* ⋀*z*. *g* (*f z*) = *z j* ∈ *Basis* ⋀*x*. *f*(*x*,*0*) · *j* = *0*
proof −
let *?B* = *Basis* :: (*'a*∗*real*) *set*
have *b01*: (*0*,*1*) ∈ *?B*
by (*simp add*: *Basis_prod_def*)
have *DIM*(*'a* ∗ *real*) ≤ *DIM*(*'b*)
by (*simp add*: *Suc_leI assms*)
then obtain *basf* :: *'a*∗*real* ⇒ *'b* **where** *sbf*: *basf* ‘ *?B* ⊆ *Basis* **and** *injbf*: *inj_on basf Basis*
by (*metis finite_Basis card_le_inj*)
define *basg*:: *'b* ⇒ *'a* ∗ *real* **where**
basg ≡ λ*i*. *if i* ∈ *basf* ‘ *Basis then inv_into Basis basf i else* (*0*,*1*)
have *bgf*[*simp*]: *basg* (*basf i*) = *i* **if** *i* ∈ *Basis* **for** *i*
using *inv_into_f_f injbf that* **by** (*force simp*: *basg_def*)
have *sbg*: *basg* ‘ *Basis* ⊆ *?B*
by (*force simp*: *basg_def injbf b01*)
define *f* :: *'a*∗*real* ⇒ *'b* **where** *f* ≡ λ*u*. ∑ *j*∈*Basis*. (*u* · *basg j*) $*_R$ *j*
define *g* :: *'b* ⇒ *'a*∗*real* **where** *g* ≡ λ*z*. (∑ *i*∈*Basis*. (*z* · *basf i*) $*_R$ *i*)
show *?thesis*
proof
show *linear f*
unfolding *f_def*
by (*intro linear_compose_sum linearI ballI*) (*auto simp*: *algebra_simps*)

```
    show linear g
      unfolding g_def
      by (intro linear_compose_sum linearI ballI) (auto simp: algebra_simps)
    have *: (∑ a ∈ Basis. a · basf b * (x · basg a)) = x · b if b ∈ Basis for x b
      using sbf that by auto
    show gf: g (f x) = x for x
    proof (rule euclidean_eqI)
      show ⋀b. b ∈ Basis ⟹ g (f x) · b = x · b
        using f_def g_def sbf by auto
    qed
    show basf(0,1) ∈ Basis
      using b01 sbf by auto
    then show f(x,0) · basf(0,1) = 0 for x
      unfolding f_def inner_sum_left
      using b01 inner_not_same_Basis
      by (fastforce intro: comm_monoid_add_class.sum.neutral)
  qed
qed

proposition locally_compact_homeomorphic_closed:
  fixes S :: 'a::euclidean_space set
  assumes locally compact S and dimlt: DIM('a) < DIM('b)
  obtains T :: 'b::euclidean_space set where closed T S homeomorphic T
proof –
  obtain U:: ('a*real)set and h
    where closed U and homU: homeomorphism S U h fst
    using locally_compact_homeomorphism_projection_closed assms by metis
  obtain f :: 'a*real ⇒ 'b and g :: 'b ⇒ 'a*real
    where linear f linear g and gf [simp]: ⋀z. g (f z) = z
    using lowerdim_embeddings [OF dimlt] by metis
  then have inj f
    by (metis injI)
  have gfU: g ‘ f ‘ U = U
    by (simp add: image_comp o_def)
  have S homeomorphic U
    using homU homeomorphic_def by blast
  also have ... homeomorphic f ‘ U
  proof (rule homeomorphicI [OF refl gfU])
    show continuous_on U f
     by (meson ‹inj f› ‹linear f› homeomorphism_cont2 linear_homeomorphism_image)
    show continuous_on (f ‘ U) g
      using ‹linear g› linear_continuous_on linear_conv_bounded_linear by blast
  qed (auto simp: o_def)
  finally show ?thesis
    using ‹closed U› ‹inj f› ‹linear f› closed_injective_linear_image that by blast
qed


lemma homeomorphic_convex_compact_lemma:
```

```
  fixes S :: 'a::euclidean_space set
  assumes convex S
    and compact S
    and cball 0 1 ⊆ S
  shows S homeomorphic (cball (0::'a) 1)
proof (rule starlike_compact_projective_special[OF assms(2−3)])
  fix x u
  assume x ∈ S and 0 ≤ u and u < (1::real)
  have open (ball (u *R x) (1 − u))
    by (rule open_ball)
  moreover have u *R x ∈ ball (u *R x) (1 − u)
    unfolding centre_in_ball using ‹u < 1› by simp
  moreover have ball (u *R x) (1 − u) ⊆ S
  proof
    fix y
    assume y ∈ ball (u *R x) (1 − u)
    then have dist (u *R x) y < 1 − u
      unfolding mem_ball .
    with ‹u < 1› have inverse (1 − u) *R (y − u *R x) ∈ cball 0 1
      by (simp add: dist_norm inverse_eq_divide norm_minus_commute)
    with assms(3) have inverse (1 − u) *R (y − u *R x) ∈ S ..
    with assms(1) have (1 − u) *R ((y − u *R x) /R (1 − u)) + u *R x ∈ S
      using ‹x ∈ S› ‹0 ≤ u› ‹u < 1› [THEN less_imp_le] by (rule convexD_alt)
    then show y ∈ S using ‹u < 1›
      by simp
  qed
  ultimately have u *R x ∈ interior S ..
  then show u *R x ∈ S − frontier S
    using frontier_def and interior_subset by auto
qed

proposition homeomorphic_convex_compact_cball:
  fixes e :: real
    and S :: 'a::euclidean_space set
  assumes S: convex S compact S interior S ≠ {} and e > 0
  shows S homeomorphic (cball (b::'a) e)
proof (rule homeomorphic_trans[OF _ homeomorphic_balls(2)])
  obtain a where a ∈ interior S
    using assms by auto
  then show S homeomorphic cball (0::'a) 1
    by (metis (no_types) aff_dim_cball S compact_cball convex_cball
        homeomorphic_convex_lemma interior_rel_interior_gen zero_less_one)
qed (use ‹e>0› in auto)

corollary homeomorphic_convex_compact:
  fixes S :: 'a::euclidean_space set
    and T :: 'a set
  assumes convex S compact S interior S ≠ {}
    and convex T compact T interior T ≠ {}
```

shows *S homeomorphic T*
using *assms*
by (*meson zero_less_one homeomorphic_trans homeomorphic_convex_compact_cball homeomorphic_sym*)

lemma *homeomorphic_closed_intervals*:
fixes *a* :: *'a::euclidean_space* **and** *b* **and** *c* :: *'a::euclidean_space* **and** *d*
assumes *box a b* ≠ {} **and** *box c d* ≠ {}
shows (*cbox a b*) *homeomorphic* (*cbox c d*)
by (*simp add*: *assms homeomorphic_convex_compact*)

lemma *homeomorphic_closed_intervals_real*:
fixes *a*::*real* **and** *b* **and** *c*::*real* **and** *d*
assumes *a*<*b* **and** *c*<*d*
shows {*a..b*} *homeomorphic* {*c..d*}
using *assms* **by** (*auto intro*: *homeomorphic_convex_compact*)

6.19.4 Covering spaces and lifting results for them

definition *covering_space*
:: *'a::topological_space set* ⇒ (*'a* ⇒ *'b*) ⇒ *'b::topological_space set* ⇒ *bool*
where
covering_space c p S ≡
continuous_on c p ∧ *p* ‘ *c* = *S* ∧
(∀ *x* ∈ *S*. ∃ *T*. *x* ∈ *T* ∧ *openin* (*top_of_set S*) *T* ∧
(∃ *v*. ⋃ *v* = *c* ∩ *p* −‘ *T* ∧
(∀ *u* ∈ *v*. *openin* (*top_of_set c*) *u*) ∧
pairwise disjnt v ∧
(∀ *u* ∈ *v*. ∃ *q*. *homeomorphism u T p q*)))

lemma *covering_space_imp_continuous*: *covering_space c p S* ⟹ *continuous_on c p*
by (*simp add*: *covering_space_def*)

lemma *covering_space_imp_surjective*: *covering_space c p S* ⟹ *p* ‘ *c* = *S*
by (*simp add*: *covering_space_def*)

lemma *homeomorphism_imp_covering_space*: *homeomorphism S T f g* ⟹ *covering_space S f T*
apply (*clarsimp simp add*: *homeomorphism_def covering_space_def*)
apply (*rule_tac x*=*T* **in** *exI*, *simp*)
apply (*rule_tac x*={*S*} **in** *exI*, *auto*)
done

lemma *covering_space_local_homeomorphism*:
assumes *covering_space c p S x* ∈ *c*
obtains *T u q* **where** *x* ∈ *T openin* (*top_of_set c*) *T*
p x ∈ *u openin* (*top_of_set S*) *u*
homeomorphism T u p q

using *assms*
by (*clarsimp simp add*: *covering_space_def*) (*metis IntI UnionE vimage_eq*)

lemma *covering_space_local_homeomorphism_alt*:
assumes *p*: *covering_space c p S* **and** $y \in S$
obtains *x T U q* **where** $p\ x = y$
$x \in T$ *openin* (*top_of_set c*) *T*
$y \in U$ *openin* (*top_of_set S*) *U*
homeomorphism T U p q
proof −
obtain *x* **where** $p\ x = y\ x \in c$
using *assms covering_space_imp_surjective* **by** *blast*
show *?thesis*
using *that* ‹$p\ x = y$› **by** (*auto intro*: *covering_space_local_homeomorphism* [*OF* *p* ‹$x \in c$›])
qed

proposition *covering_space_open_map*:
fixes *S* :: *'a* :: *metric_space set* **and** *T* :: *'b* :: *metric_space set*
assumes *p*: *covering_space c p S* **and** *T*: *openin* (*top_of_set c*) *T*
shows *openin* (*top_of_set S*) (*p* ‘ *T*)
proof −
have *pce*: *p* ‘ *c* = *S*
and *covs*:
$\bigwedge x.\ x \in S \Longrightarrow$
$\exists X\ VS.\ x \in X \wedge$ *openin* (*top_of_set S*) $X \wedge$
$\bigcup VS = c \cap p -` X \wedge$
($\forall u \in VS.$ *openin* (*top_of_set c*) *u*) $\wedge$
pairwise disjnt VS $\wedge$
($\forall u \in VS.\ \exists q.$ *homeomorphism u X p q*)
using *p* **by** (*auto simp*: *covering_space_def*)
have $T \subseteq c$ **by** (*metis openin_euclidean_subtopology_iff T*)
have $\exists X.$ *openin* (*top_of_set S*) $X \wedge y \in X \wedge X \subseteq p$ ‘ *T*
if $y \in p$ ‘ *T* **for** *y*
proof −
have $y \in S$ **using** ‹$T \subseteq c$› *pce that* **by** *blast*
obtain *U VS* **where** $y \in U$ **and** *U*: *openin* (*top_of_set S*) *U*
and *VS*: $\bigcup VS = c \cap p -` U$
and *openVS*: $\forall V \in VS.$ *openin* (*top_of_set c*) *V*
and *homVS*: $\bigwedge V.\ V \in VS \Longrightarrow \exists q.$ *homeomorphism V U p q*
using *covs* [*OF* ‹$y \in S$›] **by** *auto*
obtain *x* **where** $x \in c\ p\ x \in U\ x \in T\ p\ x = y$
using *T* [*unfolded openin_euclidean_subtopology_iff*] ‹$y \in U$› ‹$y \in p$ ‘ *T*› **by** *blast*
with *VS* **obtain** *V* **where** $x \in V\ V \in VS$ **by** *auto*
then obtain *q* **where** *q*: *homeomorphism V U p q* **using** *homVS* **by** *blast*
then have *ptV*: *p* ‘ $(T \cap V) = U \cap q -` (T \cap V)$
using *VS* ‹$V \in VS$› **by** (*auto simp*: *homeomorphism_def*)

```
    have ocv: openin (top_of_set c) V
      by (simp add: ‹V ∈ VS› openVS)
    have openin (top_of_set (q ‘ U)) (T ∩ V)
      using q unfolding homeomorphism_def
        by (metis T inf.absorb_iff2 ocv openin_imp_subset openin_subtopology_Int
subtopology_subtopology)
    then have openin (top_of_set U) (U ∩ q −‘ (T ∩ V))
      using continuous_on_open homeomorphism_def q by blast
    then have os: openin (top_of_set S) (U ∩ q −‘ (T ∩ V))
      using openin_trans [of U] by (simp add: Collect_conj_eq U)
    show ?thesis
    proof (intro exI conjI)
      show openin (top_of_set S) (p ‘ (T ∩ V))
        by (simp only: ptV os)
    qed (use ‹p x = y› ‹x ∈ V› ‹x ∈ T› in auto)
  qed
  with openin_subopen show ?thesis by blast
qed

lemma covering_space_lift_unique_gen:
  fixes f :: 'a::topological_space ⇒ 'b::topological_space
  fixes g1 :: 'a ⇒ 'c::real_normed_vector
  assumes cov: covering_space c p S
      and eq: g1 a = g2 a
      and f: continuous_on T f  f ‘ T ⊆ S
      and g1: continuous_on T g1  g1 ‘ T ⊆ c
      and fg1: ⋀x. x ∈ T ⟹ f x = p(g1 x)
      and g2: continuous_on T g2  g2 ‘ T ⊆ c
      and fg2: ⋀x. x ∈ T ⟹ f x = p(g2 x)
      and u_compt: U ∈ components T and a ∈ U x ∈ U
    shows g1 x = g2 x
proof −
  have U ⊆ T by (rule in_components_subset [OF u_compt])
  define G12 where G12 ≡ {x ∈ U. g1 x − g2 x = 0}
  have connected U by (rule in_components_connected [OF u_compt])
  have contu: continuous_on U g1 continuous_on U g2
       using ‹U ⊆ T› continuous_on_subset g1 g2 by blast+
  have o12: openin (top_of_set U) G12
  unfolding G12_def
  proof (subst openin_subopen, clarify)
    fix z
    assume z: z ∈ U g1 z − g2 z = 0
    obtain v w q where g1 z ∈ v and ocv: openin (top_of_set c) v
      and p (g1 z) ∈ w and osw: openin (top_of_set S) w
      and hom: homeomorphism v w p q
    proof (rule covering_space_local_homeomorphism [OF cov])
      show g1 z ∈ c
        using ‹U ⊆ T› ‹z ∈ U› g1(2) by blast
    qed auto
```

have *g2 z* ∈ *v* **using** ‹*g1 z* ∈ *v*› *z* **by** *auto*
have *gg*: *U* ∩ *g* −‘ *v* = *U* ∩ *g* −‘ (*v* ∩ *g* ‘ *U*) **for** *g*
by *auto*
have *openin* (*top_of_set* (*g1* ‘ *U*)) (*v* ∩ *g1* ‘ *U*)
using *ocv* ‹*U* ⊆ *T*› *g1* **by** (*fastforce simp add*: *openin_open*)
then have *1*: *openin* (*top_of_set U*) (*U* ∩ *g1* −‘ *v*)
unfolding *gg* **by** (*blast intro*: *contu continuous_on_open* [*THEN iffD1*, *rule_format*])
have *openin* (*top_of_set* (*g2* ‘ *U*)) (*v* ∩ *g2* ‘ *U*)
using *ocv* ‹*U* ⊆ *T*› *g2* **by** (*fastforce simp add*: *openin_open*)
then have *2*: *openin* (*top_of_set U*) (*U* ∩ *g2* −‘ *v*)
unfolding *gg* **by** (*blast intro*: *contu continuous_on_open* [*THEN iffD1*, *rule_format*])
let *?T* = (*U* ∩ *g1* −‘ *v*) ∩ (*U* ∩ *g2* −‘ *v*)
show ∃ *T*. *openin* (*top_of_set U*) *T* ∧ *z* ∈ *T* ∧ *T* ⊆ {*z* ∈ *U*. *g1 z* − *g2 z* = *0*}
proof (*intro exI conjI*)
show *openin* (*top_of_set U*) *?T*
using *1 2* **by** *blast*
show *z* ∈ *?T*
using *z* **by** (*simp add*: ‹*g1 z* ∈ *v*› ‹*g2 z* ∈ *v*›)
show *?T* ⊆ {*z* ∈ *U*. *g1 z* − *g2 z* = *0*}
using *hom*
by (*clarsimp simp*: *homeomorphism_def*) (*metis* ‹*U* ⊆ *T*› *fg1 fg2 subsetD*)
qed
qed
have *c12*: *closedin* (*top_of_set U*) *G12*
unfolding *G12_def*
by (*intro continuous_intros continuous_closedin_preimage_constant contu*)
have *G12* = {} ∨ *G12* = *U*
by (*intro connected_clopen* [*THEN iffD1*, *rule_format*] ‹*connected U*› *conjI o12 c12*)
with *eq* ‹*a* ∈ *U*› **have** ⋀*x*. *x* ∈ *U* ⟹ *g1 x* − *g2 x* = *0* **by** (*auto simp*: *G12_def*)
then show *?thesis*
using ‹*x* ∈ *U*› **by** *force*
qed

proposition *covering_space_lift_unique*:
fixes *f* :: *'a*::*topological_space* ⇒ *'b*::*topological_space*
fixes *g1* :: *'a* ⇒ *'c*::*real_normed_vector*
assumes *covering_space c p S*
g1 a = *g2 a*
continuous_on T f *f* ‘ *T* ⊆ *S*
continuous_on T g1 *g1* ‘ *T* ⊆ *c* ⋀*x*. *x* ∈ *T* ⟹ *f x* = *p*(*g1 x*)
continuous_on T g2 *g2* ‘ *T* ⊆ *c* ⋀*x*. *x* ∈ *T* ⟹ *f x* = *p*(*g2 x*)
connected T *a* ∈ *T* *x* ∈ *T*
shows *g1 x* = *g2 x*
using *covering_space_lift_unique_gen* [*of c p S*] *in_components_self assms ex_in_conv*
by *blast*

lemma *covering_space_locally*:
fixes *p* :: *'a::real_normed_vector ⇒ 'b::real_normed_vector*
assumes *loc*: *locally φ C* **and** *cov*: *covering_space C p S*
and *pim*: *⋀T. ⟦T ⊆ C; φ T⟧ ⟹ ψ(p ‘ T)*
shows *locally ψ S*
proof −
have *locally ψ (p ‘ C)*
proof (*rule locally_open_map_image* [*OF loc*])
show *continuous_on C p*
using *cov covering_space_imp_continuous* **by** *blast*
show *⋀T. openin (top_of_set C) T ⟹ openin (top_of_set (p ‘ C)) (p ‘ T)*
using *cov covering_space_imp_surjective covering_space_open_map* **by** *blast*
qed (*simp add*: *pim*)
then show *?thesis*
using *covering_space_imp_surjective* [*OF cov*] **by** *metis*
qed

proposition *covering_space_locally_eq*:
fixes *p* :: *'a::real_normed_vector ⇒ 'b::real_normed_vector*
assumes *cov*: *covering_space C p S*
and *pim*: *⋀T. ⟦T ⊆ C; φ T⟧ ⟹ ψ(p ‘ T)*
and *qim*: *⋀q U. ⟦U ⊆ S; continuous_on U q; ψ U⟧ ⟹ φ(q ‘ U)*
shows *locally ψ S ⟷ locally φ C*
(**is** *?lhs = ?rhs*)
proof
assume *L*: *?lhs*
show *?rhs*
proof (*rule locallyI*)
fix *V x*
assume *V*: *openin (top_of_set C) V* **and** *x ∈ V*
have *p x ∈ p ‘ C*
by (*metis IntE V ‹x ∈ V› imageI openin_open*)
then obtain *T 𝒱* **where** *p x ∈ T*
and *opeT*: *openin (top_of_set S) T*
and *veq*: *⋃𝒱 = C ∩ p −‘ T*
and *ope*: *∀U∈𝒱. openin (top_of_set C) U*
and *hom*: *∀U∈𝒱. ∃q. homeomorphism U T p q*
using *cov* **unfolding** *covering_space_def* **by** (*blast intro*: *that*)
have *x ∈ ⋃𝒱*
using *V veq ‹p x ∈ T› ‹x ∈ V› openin_imp_subset* **by** *fastforce*
then obtain *U* **where** *x ∈ U U ∈ 𝒱*
by *blast*
then obtain *q* **where** *opeU*: *openin (top_of_set C) U* **and** *q*: *homeomorphism U T p q*
using *ope hom* **by** *blast*
with *V* **have** *openin (top_of_set C) (U ∩ V)*
by *blast*
then have *UV*: *openin (top_of_set S) (p ‘ (U ∩ V))*

using *cov covering_space_open_map* **by** *blast*
obtain *W W′* **where** *opeW*: *openin* (*top_of_set S*) *W* **and** *ψ W′ p x ∈ W W ⊆ W′* **and** *W′sub*: *W′ ⊆ p ‘* (*U ∩ V*)
using *locallyE* [*OF L UV*] ‹*x ∈ U*› ‹*x ∈ V*› **by** *blast*
then have *W ⊆ T*
by (*metis Int_lower1 q homeomorphism_image1 image_Int_subset order_trans*)
show ∃ *U Z*. *openin* (*top_of_set C*) *U* ∧
φ Z ∧ *x ∈ U* ∧ *U ⊆ Z* ∧ *Z ⊆ V*
proof (*intro exI conjI*)
have *openin* (*top_of_set T*) *W*
by (*meson opeW opeT openin_imp_subset openin_subset_trans* ‹*W ⊆ T*›)
then have *openin* (*top_of_set U*) (*q ‘ W*)
by (*meson homeomorphism_imp_open_map homeomorphism_symD q*)
then show *openin* (*top_of_set C*) (*q ‘ W*)
using *opeU openin_trans* **by** *blast*
show *φ* (*q ‘ W′*)
by (*metis* (*mono_tags, lifting*) *Int_subset_iff UV W′sub* ‹*ψ W′*› *continuous_on_subset dual_order.trans homeomorphism_def image_Int_subset openin_imp_subset q qim*)
show *x ∈ q ‘ W*
by (*metis* ‹*p x ∈ W*› ‹*x ∈ U*› *homeomorphism_def imageI q*)
show *q ‘ W ⊆ q ‘ W′*
using ‹*W ⊆ W′*› **by** *blast*
have *W′ ⊆ p ‘ V*
using *W′sub* **by** *blast*
then show *q ‘ W′ ⊆ V*
using *W′sub homeomorphism_apply1* [*OF q*] **by** *auto*
qed
qed
next
assume *?rhs*
then show *?lhs*
using *cov covering_space_locally pim* **by** *blast*
qed

lemma *covering_space_locally_compact_eq*:
fixes *p* :: *′a::real_normed_vector ⇒ ′b::real_normed_vector*
assumes *covering_space C p S*
shows *locally compact S ⟷ locally compact C*
proof (*rule covering_space_locally_eq* [*OF assms*])
show ⋀*T*. ⟦*T ⊆ C*; *compact T*⟧ ⟹ *compact* (*p ‘ T*)
by (*meson assms compact_continuous_image continuous_on_subset covering_space_imp_continuous*)
qed (*use compact_continuous_image* **in** *blast*)

lemma *covering_space_locally_connected_eq*:
fixes *p* :: *′a::real_normed_vector ⇒ ′b::real_normed_vector*
assumes *covering_space C p S*
shows *locally connected S ⟷ locally connected C*
proof (*rule covering_space_locally_eq* [*OF assms*])

show $\bigwedge T$. ⟦$T \subseteq C$; *connected* T⟧ $\Longrightarrow$ *connected* (p ‘ T)
by (*meson connected_continuous_image assms continuous_on_subset covering_space_imp_continuous*)
qed (*use connected_continuous_image* **in** *blast*)

lemma *covering_space_locally_path_connected_eq*:
fixes p :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *covering_space C p S*
shows *locally path_connected S* ⟷ *locally path_connected C*
proof (*rule covering_space_locally_eq* [*OF assms*])
show $\bigwedge T$. ⟦$T \subseteq C$; *path_connected* T⟧ $\Longrightarrow$ *path_connected* (p ‘ T)
by (*meson path_connected_continuous_image assms continuous_on_subset covering_space_imp_continuous*)
qed (*use path_connected_continuous_image* **in** *blast*)

lemma *covering_space_locally_compact*:
fixes p :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *locally compact C covering_space C p S*
shows *locally compact S*
using *assms covering_space_locally_compact_eq* **by** *blast*

lemma *covering_space_locally_connected*:
fixes p :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *locally connected C covering_space C p S*
shows *locally connected S*
using *assms covering_space_locally_connected_eq* **by** *blast*

lemma *covering_space_locally_path_connected*:
fixes p :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *locally path_connected C covering_space C p S*
shows *locally path_connected S*
using *assms covering_space_locally_path_connected_eq* **by** *blast*

proposition *covering_space_lift_homotopy*:
fixes p :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
and h :: *real* × *'c::real_normed_vector* ⇒ *'b*
assumes *cov*: *covering_space C p S*
and *conth*: *continuous_on* ({*0..1*} × U) h
and *him*: h ‘ ({*0..1*} × U) ⊆ S
and *heq*: $\bigwedge y$. $y \in U \Longrightarrow h\ (0,y) = p(f\ y)$
and *contf*: *continuous_on U f* **and** *fim*: f ‘ $U \subseteq C$
obtains k **where** *continuous_on* ({*0..1*} × U) k
k ‘ ({*0..1*} × U) ⊆ C
$\bigwedge y$. $y \in U \Longrightarrow k(0,\ y) = f\ y$
$\bigwedge z$. $z \in$ {*0..1*} × $U \Longrightarrow h\ z = p(k\ z)$
proof −
have ∃ V k. *openin* (*top_of_set* U) V ∧ $y \in V$ ∧
continuous_on ({*0..1*} × V) k ∧ k ‘ ({*0..1*} × V) ⊆ C ∧

$(\forall z \in V.\ k(0,\ z) = f\ z) \wedge (\forall z \in \{0..1\} \times V.\ h\ z = p(k\ z))$
if $y \in U$ **for** y
proof −
obtain *UU* **where** *UU*: $\bigwedge s.\ s \in S \Longrightarrow s \in (UU\ s) \wedge$ *openin* (*top_of_set* S) ($UU\ s$) $\wedge$
$(\exists \mathcal{V}.\ \bigcup \mathcal{V} = C \cap p -` UU\ s \wedge$
$(\forall U \in \mathcal{V}.$ *openin* (*top_of_set* C) $U) \wedge$
pairwise disjnt $\mathcal{V} \wedge$
$(\forall U \in \mathcal{V}.\ \exists q.$ *homeomorphism* U ($UU\ s$) $p\ q$))
using *cov* **unfolding** *covering_space_def* **by** (*metis* (*mono_tags*))
then have *ope*: $\bigwedge s.\ s \in S \Longrightarrow s \in (UU\ s) \wedge$ *openin* (*top_of_set* S) ($UU\ s$)
by *blast*
have $\exists k\ n\ i.$ *open* $k \wedge$ *open* $n \wedge$
$t \in k \wedge y \in n \wedge i \in S \wedge h\ ` ((\{0..1\} \cap k) \times (U \cap n)) \subseteq UU\ i$ **if** $t \in \{0..1\}$ **for** t
proof −
have *hinS*: $h\ (t,\ y) \in S$
using ‹$y \in U$› *him that* **by** *blast*
then have $(t,y) \in (\{0..1\} \times U) \cap h -` UU(h(t,\ y))$
using ‹$y \in U$› ‹$t \in \{0..1\}$› **by** (*auto simp*: *ope*)
moreover have *ope_01U*: *openin* (*top_of_set* $(\{0..1\} \times U)$) $((\{0..1\} \times U) \cap h -` UU(h(t,\ y)))$
using *hinS ope continuous_on_open_gen* [*OF him*] *conth* **by** *blast*
ultimately obtain $V\ W$ **where** *opeV*: *open* V **and** $t \in \{0..1\} \cap V\ t \in \{0..1\} \cap V$
and *opeW*: *open* W **and** $y \in U\ y \in W$
and *VW*: $(\{0..1\} \cap V) \times (U \cap W) \subseteq ((\{0..1\} \times U) \cap h -` UU(h(t,\ y)))$
by (*rule Times_in_interior_subtopology*) (*auto simp*: *openin_open*)
then show *?thesis*
using *hinS* **by** *blast*
qed
then obtain *K NN X* **where**
K: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow$ *open* $(K\ t)$
and *NN*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow$ *open* $(NN\ t)$
and *inUS*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow t \in K\ t \wedge y \in NN\ t \wedge X\ t \in S$
and *him*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow h\ ` ((\{0..1\} \cap K\ t) \times (U \cap NN\ t)) \subseteq UU\ (X\ t)$
by (*metis* (*mono_tags*))
obtain $\mathcal{T}$ **where** $\mathcal{T} \subseteq ((\lambda i.\ K\ i \times NN\ i))\ ` \{0..1\}$ *finite* $\mathcal{T}$ $\{0::real..1\} \times \{y\} \subseteq \bigcup \mathcal{T}$
proof (*rule compactE*)
show *compact* $(\{0::real..1\} \times \{y\})$
by (*simp add*: *compact_Times*)
show $\{0..1\} \times \{y\} \subseteq (\bigcup i \in \{0..1\}.\ K\ i \times NN\ i)$
using *K inUS* **by** *auto*
show $\bigwedge B.\ B \in (\lambda i.\ K\ i \times NN\ i)\ ` \{0..1\} \Longrightarrow$ *open* B
using *K NN* **by** (*auto simp*: *open_Times*)
qed *blast*

```
    then obtain tk where tk ⊆ {0..1} finite tk
                  and tk: {0::real..1} × {y} ⊆ (⋃i ∈ tk. K i × NN i)
      by (metis (no_types, lifting) finite_subset_image)
    then have tk ≠ {}
      by auto
    define n where n = ⋂(NN ‘ tk)
    have y ∈ n open n
      using inUS NN ‹tk ⊆ {0..1}› ‹finite tk›
      by (auto simp: n_def open_INT subset_iff)
    obtain δ where 0 < δ and δ: ⋀T. ⟦T ⊆ {0..1}; diameter T < δ⟧ ⟹ ∃B∈K
‘ tk. T ⊆ B
    proof (rule Lebesgue_number_lemma [of {0..1} K ‘ tk])
      show K ‘ tk ≠ {}
        using ‹tk ≠ {}› by auto
      show {0..1} ⊆ ⋃(K ‘ tk)
        using tk by auto
      show ⋀B. B ∈ K ‘ tk ⟹ open B
        using ‹tk ⊆ {0..1}› K by auto
    qed auto
    obtain N::nat where N: N > 1 / δ
      using reals_Archimedean2 by blast
    then have N > 0
      using ‹0 < δ› order.asym by force
    have *: ∃V k. openin (top_of_set U) V ∧ y ∈ V ∧
                continuous_on ({0..of_nat n / N} × V) k ∧
                k ‘ ({0..of_nat n / N} × V) ⊆ C ∧
                (∀z∈V. k (0, z) = f z) ∧
                (∀z∈{0..of_nat n / N} × V. h z = p (k z)) if n ≤ N for n
      using that
    proof (induction n)
      case 0
      show ?case
        apply (rule_tac x=U in exI)
        apply (rule_tac x=f ∘ snd in exI)
          apply (intro conjI ‹y ∈ U› continuous_intros continuous_on_subset [OF
contf])
        using fim  apply (auto simp: heq)
        done
    next
      case (Suc n)
      then obtain V k where opeUV: openin (top_of_set U) V
                    and y ∈ V
                    and contk: continuous_on ({0..n/N} × V) k
                    and kim: k ‘ ({0..n/N} × V) ⊆ C
                    and keq: ⋀z. z ∈ V ⟹ k (0, z) = f z
                    and heq: ⋀z. z ∈ {0..n/N} × V ⟹ h z = p (k z)
        using Suc_leD by auto
      have n ≤ N
        using Suc.prems by auto
```

obtain *t* **where** *t* ∈ *tk* **and** *t*: {*n*/*N* .. (*1* + *real n*) / *N*} ⊆ *K t*
proof (*rule bexE* [*OF δ*])
  show {*n*/*N* .. (*1* + *real n*) / *N*} ⊆ {*0*..*1*}
    using *Suc.prems* **by** (*auto simp*: *field_split_simps*)
  show *diameter_less*: *diameter* {*n*/*N* .. (*1* + *real n*) / *N*} < *δ*
    using ‹*0* < *δ*› *N* **by** (*auto simp*: *field_split_simps*)
qed *blast*
have *t01*: *t* ∈ {*0*..*1*}
  using ‹*t* ∈ *tk*› ‹*tk* ⊆ {*0*..*1*}› **by** *blast*
obtain 𝒱 **where** 𝒱: ⋃𝒱 = *C* ∩ *p* −‘ *UU* (*X t*)
      and *opeC*: ⋀*U*. *U* ∈ 𝒱 ⟹ *openin* (*top_of_set C*) *U*
      and *pairwise disjnt* 𝒱
      and *homuu*: ⋀*U*. *U* ∈ 𝒱 ⟹ ∃ *q*. *homeomorphism U* (*UU* (*X t*)) *p q*
  using *inUS* [*OF t01*] *UU* **by** *meson*
have *n_div_N_in*: *n*/*N* ∈ {*n*/*N* .. (*1* + *real n*) / *N*}
  using *N* **by** (*auto simp*: *field_split_simps*)
with *t* **have** *nN_in_kkt*: *n*/*N* ∈ *K t*
  by *blast*
have *k* (*n*/*N*, *y*) ∈ *C* ∩ *p* −‘ *UU* (*X t*)
proof (*simp*, *rule conjI*)
  show *k* (*n*/*N*, *y*) ∈ *C*
    using ‹*y* ∈ *V*› *kim keq* **by** *force*
  have *p* (*k* (*n*/*N*, *y*)) = *h* (*n*/*N*, *y*)
    by (*simp add*: ‹*y* ∈ *V*› *heq*)
  also have ... ∈ *h* ‘ (({*0*..*1*} ∩ *K t*) × (*U* ∩ *NN t*))
     using ‹*y* ∈ *V*› *t01* ‹*n* ≤ *N*›
     by (*simp add*: *nN_in_kkt* ‹*y* ∈ *U*› *inUS field_split_simps*)
  also have ... ⊆ *UU* (*X t*)
    using *him t01* **by** *blast*
  finally show *p* (*k* (*n*/*N*, *y*)) ∈ *UU* (*X t*) **.**
qed
with 𝒱 **have** *k* (*n*/*N*, *y*) ∈ ⋃𝒱
  by *blast*
then obtain *W* **where** *W*: *k* (*n*/*N*, *y*) ∈ *W* **and** *W* ∈ 𝒱
  by *blast*
then obtain *p′* **where** *opeC′*: *openin* (*top_of_set C*) *W*
      and *hom′*: *homeomorphism W* (*UU* (*X t*)) *p p′*
  using *homuu opeC* **by** *blast*
then have *W* ⊆ *C*
  using *openin_imp_subset* **by** *blast*
define *W′* **where** *W′* = *UU*(*X t*)
have *opeVW*: *openin* (*top_of_set V*) (*V* ∩ (*k* ∘ *Pair* (*n* / *N*)) −‘ *W*)
proof (*rule continuous_openin_preimage* [*OF* _ _ *opeC′*])
  show *continuous_on V* (*k* ∘ *Pair* (*n*/*N*))
    by (*intro continuous_intros continuous_on_subset* [*OF contk*], *auto*)
  show (*k* ∘ *Pair* (*n*/*N*)) ‘ *V* ⊆ *C*
    using *kim* **by** (*auto simp*: ‹*y* ∈ *V*› *W*)
qed
obtain *N′* **where** *opeUN′*: *openin* (*top_of_set U*) *N′*

```
            and y ∈ N' and kimw: k ` ({(n/N)} × N') ⊆ W
      proof
        show openin (top_of_set U) (V ∩ (k ∘ Pair (n/N)) -` W)
          using opeUV opeVW openin_trans by blast
      qed (use ‹y ∈ V› W in ‹force+›)
      obtain Q Q' where opeUQ: openin (top_of_set U) Q
                and cloUQ': closedin (top_of_set U) Q'
                and y ∈ Q Q ⊆ Q'
                and Q': Q' ⊆ (U ∩ NN(t)) ∩ N' ∩ V
      proof -
         obtain VO VX where open VO open VX and VO: V = U ∩ VO and
VX: N' = U ∩ VX
          using opeUV opeUN' by (auto simp: openin_open)
        then have open (NN(t) ∩ VO ∩ VX)
          using NN t01 by blast
        then obtain e where e > 0 and e: cball y e ⊆ NN(t) ∩ VO ∩ VX
         by (metis Int_iff ‹N' = U ∩ VX› ‹V = U ∩ VO› ‹y ∈ N'› ‹y ∈ V› inUS
open_contains_cball t01)
        show ?thesis
        proof
          show openin (top_of_set U) (U ∩ ball y e)
            by blast
          show closedin (top_of_set U) (U ∩ cball y e)
            using e by (auto simp: closedin_closed)
        qed (use ‹y ∈ U› ‹e > 0› VO VX e in auto)
      qed
      then have y ∈ Q' Q ⊆ (U ∩ NN(t)) ∩ N' ∩ V
        by blast+
      have neq: {0..n/N} ∪ {n/N..(1 + real n) / N} = {0..(1 + real n) / N}
        apply (auto simp: field_split_simps)
       by (metis not_less of_nat_0_le_iff of_nat_0_less_iff order_trans zero_le_mult_iff)
     then have neqQ': {0..n/N} × Q' ∪ {n/N..(1 + real n) / N} × Q' = {0..(1
+ real n) / N} × Q'
        by blast
     have cont: continuous_on ({0..(1 + real n) / N} × Q') (λx. if x ∈ {0..n/N}
× Q' then k x else (p' ∘ h) x)
        unfolding neqQ' [symmetric]
      proof (rule continuous_on_cases_local, simp_all add: neqQ' del: comp_apply)
        have ∃T. closed T ∧ {0..n/N} × Q' = {0..(1+n)/N} × Q' ∩ T
          using n_div_N_in
          by (rule_tac x={0 .. n/N} × UNIV in exI) (auto simp: closed_Times)
         then show closedin (top_of_set ({0..(1 + real n) / N} × Q')) ({0..n/N}
× Q')
          by (simp add: closedin_closed)
        have ∃T. closed T ∧ {n/N..(1+n)/N} × Q' = {0..(1+n)/N} × Q' ∩ T
        by (rule_tac x={n/N..(1+n)/N} × UNIV in exI) (auto simp: closed_Times
order_trans [rotated])
         then show closedin (top_of_set ({0..(1 + real n) / N} × Q')) ({n/N..(1
+ real n) / N} × Q')
```

```
        by (simp add: closedin_closed)
      show continuous_on ({0..n/N} × Q') k
        using Q' by (auto intro: continuous_on_subset [OF contk])
      have continuous_on ({n/N..(1 + real n) / N} × Q') h
      proof (rule continuous_on_subset [OF conth])
        show {n/N..(1 + real n) / N} × Q' ⊆ {0..1} × U
        proof (clarsimp, intro conjI)
          fix a b
          assume b ∈ Q' and a: n/N ≤ a a ≤ (1 + real n) / N
          have 0 ≤ n/N (1 + real n) / N ≤ 1
            using a Suc.prems by (auto simp: divide_simps)
          with a show 0 ≤ a  a ≤ 1
            by linarith+
          show b ∈ U
            using ‹b ∈ Q'› cloUQ' closedin_imp_subset by blast
        qed
      qed
      moreover have continuous_on (h ` ({n/N..(1 + real n) / N} × Q')) p'
      proof (rule continuous_on_subset [OF homeomorphism_cont2 [OF hom']])
        have h ` ({n/N..(1 + real n) / N} × Q') ⊆ h ` (({0..1} ∩ K t) × (U ∩
NN t))
        proof (rule image_mono)
          show {n/N..(1 + real n) / N} × Q' ⊆ ({0..1} ∩ K t) × (U ∩ NN t)
          proof (clarsimp, intro conjI)
            fix a::real and b
            assume b ∈ Q' n/N ≤ a a ≤ (1 + real n) / N
            show 0 ≤ a
            by (meson ‹n/N ≤ a› divide_nonneg_nonneg of_nat_0_le_iff order_trans)
            show a ≤ 1
              using Suc.prems ‹a ≤ (1 + real n) / N› order_trans by force
            show a ∈ K t
              using ‹a ≤ (1 + real n) / N› ‹n/N ≤ a› t by auto
            show b ∈ U
              using ‹b ∈ Q'› cloUQ' closedin_imp_subset by blast
            show b ∈ NN t
              using Q' ‹b ∈ Q'› by auto
          qed
        qed
        with him show h ` ({n/N..(1 + real n) / N} × Q') ⊆ UU (X t)
          using t01 by blast
      qed
      ultimately show continuous_on ({n/N..(1 + real n) / N} × Q') (p' ∘ h)
        by (rule continuous_on_compose)
      have k (n/N, b) = p' (h (n/N, b)) if b ∈ Q' for b
      proof −
        have k (n/N, b) ∈ W
          using that Q' kimw  by force
        then have k (n/N, b) = p' (p (k (n/N, b)))
          by (simp add:  homeomorphism_apply1 [OF hom'])
```

```
        then show ?thesis
          using Q' that by (force simp: heq)
      qed
      then show ⋀x. x ∈ {n/N..(1 + real n) / N} × Q' ∧
                x ∈ {0..n/N} × Q' ⟹ k x = (p' ∘ h) x
        by auto
    qed
    have h_in_UU: h (x, y) ∈ UU (X t) if y ∈ Q ¬ x ≤ n/N 0 ≤ x x ≤ (1 +
real n) / N for x y
    proof −
      have x ≤ 1
        using Suc.prems that order_trans by force
      moreover have x ∈ K t
        by (meson atLeastAtMost_iff le_less not_le subset_eq t that)
      moreover have y ∈ U
        using ‹y ∈ Q› opeUQ openin_imp_subset by blast
      moreover have y ∈ NN t
        using Q' ‹Q ⊆ Q'› ‹y ∈ Q› by auto
      ultimately have (x, y) ∈ (({0..1} ∩ K t) × (U ∩ NN t))
        using that by auto
      then have h (x, y) ∈ h ` (({0..1} ∩ K t) × (U ∩ NN t))
        by blast
      also have ... ⊆ UU (X t)
        by (metis him t01)
      finally show ?thesis .
    qed
    let ?k = (λx. if x ∈ {0..n/N} × Q' then k x else (p' ∘ h) x)
    show ?case
    proof (intro exI conjI)
      show continuous_on ({0..real (Suc n) / N} × Q) ?k
        using ‹Q ⊆ Q'› by (auto intro: continuous_on_subset [OF cont])
      have ⋀x y. ⟦x ≤ n/N; y ∈ Q'; 0 ≤ x⟧ ⟹ k (x, y) ∈ C
        using kim Q' by force
      moreover have p' (h (x, y)) ∈ C if y ∈ Q ¬ x ≤ n/N 0 ≤ x x ≤ (1 +
real n) / N for x y
      proof (rule ‹W ⊆ C› [THEN subsetD])
        show p' (h (x, y)) ∈ W
          using homeomorphism_image2 [OF hom', symmetric] h_in_UU Q' ‹Q
⊆ Q'› ‹W ⊆ C› that by auto
      qed
      ultimately show ?k ` ({0..real (Suc n) / N} × Q) ⊆ C
        using Q' ‹Q ⊆ Q'› by force
      show ∀z∈Q. ?k (0, z) = f z
        using Q' keq ‹Q ⊆ Q'› by auto
      show ∀z ∈ {0..real (Suc n) / N} × Q. h z = p(?k z)
        using ‹Q ⊆ U ∩ NN t ∩ N' ∩ V› heq Q' ‹Q ⊆ Q'›
          by (auto simp: homeomorphism_apply2 [OF hom'] dest: h_in_UU)
      qed (auto simp: ‹y ∈ Q› opeUQ)
  qed
```

show *?thesis*
  using ∗[*OF order_refl*] *N* ‹*0* < δ› **by** (*simp add*: *split*: *if_split_asm*)
qed
then obtain *V fs* **where** *opeV*: ⋀*y*. *y* ∈ *U* ⟹ *openin* (*top_of_set U*) (*V y*)
    and *V*: ⋀*y*. *y* ∈ *U* ⟹ *y* ∈ *V y*
    and *contfs*: ⋀*y*. *y* ∈ *U* ⟹ *continuous_on* ({*0..1*} × *V y*) (*fs y*)
    and ∗: ⋀*y*. *y* ∈ *U* ⟹ (*fs y*) ‘ ({*0..1*} × *V y*) ⊆ *C* ∧
        (∀ *z* ∈ *V y*. *fs y* (*0*, *z*) = *f z*) ∧
        (∀ *z* ∈ {*0..1*} × *V y*. *h z* = *p*(*fs y z*))
  by (*metis* (*mono_tags*))
then have *VU*: ⋀*y*. *y* ∈ *U* ⟹ *V y* ⊆ *U*
  by (*meson openin_imp_subset*)
obtain *k* **where** *contk*: *continuous_on* ({*0..1*} × *U*) *k*
    and *k*: ⋀*x i*. ⟦*i* ∈ *U*; *x* ∈ {*0..1*} × *U* ∩ {*0..1*} × *V i*⟧ ⟹ *k x* = *fs i x*
proof (*rule pasting_lemma_exists*)
  let *?X* = *top_of_set* ({*0..1*::*real*} × *U*)
  show *topspace ?X* ⊆ (⋃*i*∈*U*. {*0..1*} × *V i*)
    using *V* **by** *force*
  show ⋀*i*. *i* ∈ *U* ⟹ *openin* (*top_of_set* ({*0..1*} × *U*)) ({*0..1*} × *V i*)
    by (*simp add*: *Abstract_Topology.openin_Times opeV*)
  show ⋀*i*. *i* ∈ *U* ⟹ *continuous_map*
      (*subtopology* (*top_of_set* ({*0..1*} × *U*)) ({*0..1*} × *V i*)) *euclidean* (*fs i*)
  by (*metis contfs subtopology_subtopology continuous_map_iff_continuous Times_Int_Times VU inf.absorb_iff2 inf.idem*)
  show *fs i x* = *fs j x* **if** *i* ∈ *U j* ∈ *U* **and** *x*: *x* ∈ *topspace ?X* ∩ {*0..1*} × *V i* ∩ {*0..1*} × *V j*
      for *i j x*
  proof −
    obtain *u y* **where** *x* = (*u*, *y*) *y* ∈ *V i y* ∈ *V j 0* ≤ *u u* ≤ *1*
      using *x* **by** *auto*
    show *?thesis*
    proof (*rule covering_space_lift_unique* [*OF cov*, *of* _ (*0*,*y*) _ {*0..1*} × {*y*} *h*])
      show *fs i* (*0*, *y*) = *fs j* (*0*, *y*)
        using∗ *V* **by** (*simp add*: ‹*y* ∈ *V i*› ‹*y* ∈ *V j*› *that*)
      show *conth_y*: *continuous_on* ({*0..1*} × {*y*}) *h*
        using *VU* ‹*y* ∈ *V j*› *that* **by** (*auto intro*: *continuous_on_subset* [*OF conth*])
      show *h* ‘ ({*0..1*} × {*y*}) ⊆ *S*
        using ‹*y* ∈ *V i*› *assms*(*3*) *VU that* **by** *fastforce*
      show *continuous_on* ({*0..1*} × {*y*}) (*fs i*)
        using *continuous_on_subset* [*OF contfs*] ‹*i* ∈ *U*›
        by (*simp add*: ‹*y* ∈ *V i*› *subset_iff*)
      show *fs i* ‘ ({*0..1*} × {*y*}) ⊆ *C*
        using ∗ ‹*y* ∈ *V i*› ‹*i* ∈ *U*› **by** *fastforce*
      show ⋀*x*. *x* ∈ {*0..1*} × {*y*} ⟹ *h x* = *p* (*fs i x*)
        using ∗ ‹*y* ∈ *V i*› ‹*i* ∈ *U*› **by** *blast*
      show *continuous_on* ({*0..1*} × {*y*}) (*fs j*)
        using *continuous_on_subset* [*OF contfs*] ‹*j* ∈ *U*›
        by (*simp add*: ‹*y* ∈ *V j*› *subset_iff*)
      show *fs j* ‘ ({*0..1*} × {*y*}) ⊆ *C*

```
        using ∗ ‹y ∈ V j› ‹j ∈ U› by fastforce
      show ⋀x. x ∈ {0..1} × {y} ⟹ h x = p (fs j x)
        using ∗ ‹y ∈ V j› ‹j ∈ U› by blast
      show connected ({0..1::real} × {y})
        using connected_Icc connected_Times connected_sing by blast
      show (0, y) ∈ {0..1::real} × {y}
        by force
      show x ∈ {0..1} × {y}
        using ‹x = (u, y)› x by blast
    qed
  qed
qed force
show ?thesis
proof
  show k ` ({0..1} × U) ⊆ C
    using V∗k VU by fastforce
  show ⋀y. y ∈ U ⟹ k (0, y) = f y
    by (simp add: V∗k)
  show ⋀z. z ∈ {0..1} × U ⟹ h z = p (k z)
    using V∗k by auto
qed (auto simp: contk)
qed

corollary covering_space_lift_homotopy_alt:
  fixes p :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
    and h :: 'c::real_normed_vector × real ⇒ 'b
  assumes cov: covering_space C p S
    and conth: continuous_on (U × {0..1}) h
    and him: h ` (U × {0..1}) ⊆ S
    and heq: ⋀y. y ∈ U ⟹ h (y,0) = p(f y)
    and contf: continuous_on U f and fim: f ` U ⊆ C
  obtains k where continuous_on (U × {0..1}) k
                k ` (U × {0..1}) ⊆ C
                ⋀y. y ∈ U ⟹ k(y, 0) = f y
                ⋀z. z ∈ U × {0..1} ⟹ h z = p(k z)
proof −
  have continuous_on ({0..1} × U) (h ∘ (λz. (snd z, fst z)))
    by (intro continuous_intros continuous_on_subset [OF conth]) auto
  then obtain k where contk: continuous_on ({0..1} × U) k
                and kim: k ` ({0..1} × U) ⊆ C
                and k0: ⋀y. y ∈ U ⟹ k(0, y) = f y
               and heqp: ⋀z. z ∈ {0..1} × U ⟹ (h ∘ (λz. Pair (snd z) (fst z)))
z = p(k z)
    apply (rule covering_space_lift_homotopy [OF cov _ _ _ contf fim])
    using him by (auto simp: contf heq)
  show ?thesis
  proof
    show continuous_on (U × {0..1}) (k ∘ (λz. (snd z, fst z)))
      by (intro continuous_intros continuous_on_subset [OF contk]) auto
```

qed (*use kim heqp* **in** ‹*auto simp*: *k0*›)
qed

corollary *covering_space_lift_homotopic_function*:
fixes *p* :: *'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector* **and** *g*:: *'c*::*real_normed_vector* ⇒ *'a*
assumes *cov*: *covering_space C p S*
and *contg*: *continuous_on U g*
and *gim*: *g ' U* ⊆ *C*
and *pgeq*: ⋀*y*. *y* ∈ *U* ⟹ *p*(*g y*) = *f y*
and *hom*: *homotopic_with_canon* (λ*x*. *True*) *U S f f'*
obtains *g'* **where** *continuous_on U g' image g' U* ⊆ *C* ⋀*y*. *y* ∈ *U* ⟹ *p*(*g' y*) = *f' y*
proof −
obtain *h* **where** *conth*: *continuous_on* ({*0..1*::*real*} × *U*) *h*
and *him*: *h* ' ({*0..1*} × *U*) ⊆ *S*
and *h0*: ⋀*x*. *h*(*0*, *x*) = *f x*
and *h1*: ⋀*x*. *h*(*1*, *x*) = *f' x*
using *hom* **by** (*auto simp*: *homotopic_with_def*)
have ⋀*y*. *y* ∈ *U* ⟹ *h* (*0*, *y*) = *p* (*g y*)
by (*simp add*: *h0 pgeq*)
then obtain *k* **where** *contk*: *continuous_on* ({*0..1*} × *U*) *k*
and *kim*: *k* ' ({*0..1*} × *U*) ⊆ *C*
and *k0*: ⋀*y*. *y* ∈ *U* ⟹ *k*(*0*, *y*) = *g y*
and *heq*: ⋀*z*. *z* ∈ {*0..1*} × *U* ⟹ *h z* = *p*(*k z*)
using *covering_space_lift_homotopy* [*OF cov conth him* _ *contg gim*] **by** *metis*
show *?thesis*
proof
show *continuous_on U* (*k* ∘ *Pair 1*)
by (*meson contk atLeastAtMost_iff continuous_on_o_Pair order_refl zero_le_one*)
show (*k* ∘ *Pair 1*) ' *U* ⊆ *C*
using *kim* **by** *auto*
show ⋀*y*. *y* ∈ *U* ⟹ *p* ((*k* ∘ *Pair 1*) *y*) = *f' y*
by (*auto simp*: *h1 heq* [*symmetric*])
qed
qed

corollary *covering_space_lift_inessential_function*:
fixes *p* :: *'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector* **and** *U* :: *'c*::*real_normed_vector set*
assumes *cov*: *covering_space C p S*
and *hom*: *homotopic_with_canon* (λ*x*. *True*) *U S f* (λ*x*. *a*)
obtains *g* **where** *continuous_on U g g* ' *U* ⊆ *C* ⋀*y*. *y* ∈ *U* ⟹ *p*(*g y*) = *f y*
proof (*cases U* = {})
case *True*
then show *?thesis*
using *that continuous_on_empty* **by** *blast*
next
case *False*

```
  then obtain b where b: b ∈ C p b = a
    using covering_space_imp_surjective [OF cov] homotopic_with_imp_subset2 [OF hom]
    by auto
  then have gim: (λy. b) ` U ⊆ C
    by blast
  show ?thesis
  proof (rule covering_space_lift_homotopic_function [OF cov continuous_on_const gim])
    show ⋀y. y ∈ U ⟹ p b = a
      using b by auto
  qed (use that homotopic_with_symD [OF hom] in auto)
qed
```

6.19.5 Lifting of general functions to covering space

```
proposition covering_space_lift_path_strong:
  fixes p :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
    and f :: 'c::real_normed_vector ⇒ 'b
  assumes cov: covering_space C p S and a ∈ C
      and path g and pag: path_image g ⊆ S and pas: pathstart g = p a
    obtains h where path h path_image h ⊆ C pathstart h = a
                and ⋀t. t ∈ {0..1} ⟹ p(h t) = g t
proof –
  obtain k:: real × 'c ⇒ 'a
    where contk: continuous_on ({0..1} × {undefined}) k
      and kim: k ` ({0..1} × {undefined}) ⊆ C
      and k0:  k (0, undefined) = a
      and pk: ⋀z. z ∈ {0..1} × {undefined} ⟹ p(k z) = (g ∘ fst) z
  proof (rule covering_space_lift_homotopy [OF cov, of {undefined} g ∘ fst])
    show continuous_on ({0..1::real} × {undefined::'c}) (g ∘ fst)
      using ‹path g› by (intro continuous_intros) (simp add: path_def)
    show (g ∘ fst) ` ({0..1} × {undefined}) ⊆ S
      using pag by (auto simp: path_image_def)
    show (g ∘ fst) (0, y) = p a if y ∈ {undefined} for y::'c
      by (metis comp_def fst_conv pas pathstart_def)
  qed (use assms in auto)
  show ?thesis
  proof
    show path (k ∘ (λt. Pair t undefined))
      unfolding path_def
      by (intro continuous_on_compose continuous_intros continuous_on_subset [OF contk]) auto
    show path_image (k ∘ (λt. (t, undefined))) ⊆ C
      using kim by (auto simp: path_image_def)
    show pathstart (k ∘ (λt. (t, undefined))) = a
      by (auto simp: pathstart_def k0)
    show ⋀t. t ∈ {0..1} ⟹ p ((k ∘ (λt. (t, undefined))) t) = g t
      by (auto simp: pk)
```

qed
qed

corollary *covering_space_lift_path*:
fixes *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *cov*: *covering_space C p S* **and** *path g* **and** *pig*: *path_image g* ⊆ *S*
obtains *h* **where** *path h path_image h* ⊆ *C* ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h t*) = *g t*
proof −
obtain *a* **where** *a* ∈ *C pathstart g* = *p a*
by (*metis pig cov covering_space_imp_surjective imageE pathstart_in_path_image subsetCE*)
show *?thesis*
using *covering_space_lift_path_strong* [*OF cov* ‹*a* ∈ *C*› ‹*path g*› *pig*]
by (*metis* ‹*pathstart g* = *p a*› *that*)
qed

proposition *covering_space_lift_homotopic_paths*:
fixes *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *cov*: *covering_space C p S*
and *path g1* **and** *pig1*: *path_image g1* ⊆ *S*
and *path g2* **and** *pig2*: *path_image g2* ⊆ *S*
and *hom*: *homotopic_paths S g1 g2*
and *path h1* **and** *pih1*: *path_image h1* ⊆ *C* **and** *ph1*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h1 t*) = *g1 t*
and *path h2* **and** *pih2*: *path_image h2* ⊆ *C* **and** *ph2*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h2 t*) = *g2 t*
and *h1h2*: *pathstart h1* = *pathstart h2*
shows *homotopic_paths C h1 h2*
proof −
obtain *h* :: *real* × *real* ⇒ *'b*
where *conth*: *continuous_on* ({*0..1*} × {*0..1*}) *h*
and *him*: *h* ‘ ({*0..1*} × {*0..1*}) ⊆ *S*
and *h0*: ⋀*x*. *h* (*0*, *x*) = *g1 x* **and** *h1*: ⋀*x*. *h* (*1*, *x*) = *g2 x*
and *heq0*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *h* (*t*, *0*) = *g1 0*
and *heq1*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *h* (*t*, *1*) = *g1 1*
using *hom* **by** (*auto simp*: *homotopic_paths_def homotopic_with_def pathstart_def pathfinish_def*)
obtain *k* **where** *contk*: *continuous_on* ({*0..1*} × {*0..1*}) *k*
and *kim*: *k* ‘ ({*0..1*} × {*0..1*}) ⊆ *C*
and *kh2*: ⋀*y*. *y* ∈ {*0..1*} ⟹ *k* (*y*, *0*) = *h2 0*
and *hpk*: ⋀*z*. *z* ∈ {*0..1*} × {*0..1*} ⟹ *h z* = *p* (*k z*)
proof (*rule covering_space_lift_homotopy_alt* [*OF cov conth him*])
show ⋀*y*. *y* ∈ {*0..1*} ⟹ *h* (*y*, *0*) = *p* (*h2 0*)
by (*metis atLeastAtMost_iff h1h2 heq0 order_refl pathstart_def ph1 zero_le_one*)
qed (*use path_image_def pih2* **in** ‹*fastforce+*›)
have *contg1*: *continuous_on* {*0..1*} *g1* **and** *contg2*: *continuous_on* {*0..1*} *g2*
using ‹*path g1*› ‹*path g2*› *path_def* **by** *blast+*
have *g1im*: *g1* ‘ {*0..1*} ⊆ *S* **and** *g2im*: *g2* ‘ {*0..1*} ⊆ *S*

```
    using path_image_def pig1 pig2 by auto
  have conth1: continuous_on {0..1} h1 and conth2: continuous_on {0..1} h2
    using ‹path h1› ‹path h2› path_def by blast+
  have h1im: h1 ‘ {0..1} ⊆ C and h2im: h2 ‘ {0..1} ⊆ C
    using path_image_def pih1 pih2 by auto
  show ?thesis
    unfolding homotopic_paths pathstart_def pathfinish_def
  proof (intro exI conjI ballI)
    show keqh1: k(0, x) = h1 x if x ∈ {0..1} for x
    proof (rule covering_space_lift_unique [OF cov _ contg1 g1im])
      show k (0,0) = h1 0
        by (metis atLeastAtMost_iff h1h2 kh2 order_refl pathstart_def zero_le_one)
      show continuous_on {0..1} (λa. k (0, a))
        by (intro continuous_intros continuous_on_compose2 [OF contk]) auto
      show ⋀x. x ∈ {0..1} =⇒ g1 x = p (k (0, x))
        by (metis atLeastAtMost_iff h0 hpk zero_le_one mem_Sigma_iff order_refl)
    qed (use conth1 h1im kim that in ‹auto simp: ph1›)
    show k(1, x) = h2 x if x ∈ {0..1} for x
    proof (rule covering_space_lift_unique [OF cov _ contg2 g2im])
      show k (1,0) = h2 0
        by (metis atLeastAtMost_iff kh2 order_refl zero_le_one)
      show continuous_on {0..1} (λa. k (1, a))
        by (intro continuous_intros continuous_on_compose2 [OF contk]) auto
      show ⋀x. x ∈ {0..1} =⇒ g2 x = p (k (1, x))
        by (metis atLeastAtMost_iff h1 hpk mem_Sigma_iff order_refl zero_le_one)
    qed (use conth2 h2im kim that in ‹auto simp: ph2›)
    show ⋀t. t ∈ {0..1} =⇒ (k ◦ Pair t) 0 = h1 0
      by (metis comp_apply h1h2 kh2 pathstart_def)
    show (k ◦ Pair t) 1 = h1 1 if t ∈ {0..1} for t
    proof (rule covering_space_lift_unique
           [OF cov, of λa. (k ◦ Pair a) 1 0 λa. h1 1 {0..1}  λx. g1 1])
      show (k ◦ Pair 0) 1 = h1 1
        using keqh1 by auto
      show continuous_on {0..1} (λa. (k ◦ Pair a) 1)
        by (auto intro!: continuous_intros continuous_on_compose2 [OF contk])
      show ⋀x. x ∈ {0..1} =⇒ g1 1 = p ((k ◦ Pair x) 1)
        using heq1 hpk by auto
    qed (use contk kim g1im h1im that in ‹auto simp: ph1›)
  qed (use contk kim in auto)
qed


corollary covering_space_monodromy:
  fixes p :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
  assumes cov: covering_space C p S
      and path g1 and pig1: path_image g1 ⊆ S
      and path g2 and pig2: path_image g2 ⊆ S
      and hom: homotopic_paths S g1 g2
       and path h1 and pih1: path_image h1 ⊆ C and ph1: ⋀t. t ∈ {0..1} =⇒
```

p(*h1 t*) = *g1 t*
and *path h2* **and** *pih2*: *path_image h2* ⊆ *C* **and** *ph2*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h2 t*) = *g2 t*
and *h1h2*: *pathstart h1* = *pathstart h2*
shows *pathfinish h1* = *pathfinish h2*
using *covering_space_lift_homotopic_paths* [*OF assms*] *homotopic_paths_imp_pathfinish*
by *blast*

corollary *covering_space_lift_homotopic_path*:
fixes *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *cov*: *covering_space C p S*
and *hom*: *homotopic_paths S f f'*
and *path g* **and** *pig*: *path_image g* ⊆ *C*
and *a*: *pathstart g* = *a* **and** *b*: *pathfinish g* = *b*
and *pgeq*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*g t*) = *f t*
obtains *g'* **where** *path g' path_image g'* ⊆ *C*
pathstart g' = *a pathfinish g'* = *b* ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*g' t*) = *f' t*
proof (*rule covering_space_lift_path_strong* [*OF cov*, *of a f'*])
show *a* ∈ *C*
using *a pig* **by** *auto*
show *path f' path_image f'* ⊆ *S*
using *hom homotopic_paths_imp_path homotopic_paths_imp_subset* **by** *blast+*
show *pathstart f'* = *p a*
by (*metis a atLeastAtMost_iff hom homotopic_paths_imp_pathstart order_refl pathstart_def pgeq zero_le_one*)
qed (*metis* (*mono_tags*, *lifting*) *assms cov covering_space_monodromy hom homotopic_paths_imp_path homotopic_paths_imp_subset pgeq pig*)

proposition *covering_space_lift_general*:
fixes *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
and *f* :: *'c::real_normed_vector* ⇒ *'b*
assumes *cov*: *covering_space C p S* **and** *a* ∈ *C z* ∈ *U*
and *U*: *path_connected U locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: *f* ‘ *U* ⊆ *S*
and *feq*: *f z* = *p a*
and *hom*: ⋀*r*. ⟦*path r*; *path_image r* ⊆ *U*; *pathstart r* = *z*; *pathfinish r* = *z*⟧
⟹ ∃ *q*. *path q* ∧ *path_image q* ⊆ *C* ∧
pathstart q = *a* ∧ *pathfinish q* = *a* ∧
homotopic_paths S (*f* ∘ *r*) (*p* ∘ *q*)
obtains *g* **where** *continuous_on U g g* ‘ *U* ⊆ *C g z* = *a* ⋀*y*. *y* ∈ *U* ⟹ *p*(*g y*) = *f y*
proof −
have ∗: ∃ *g h*. *path g* ∧ *path_image g* ⊆ *U* ∧
pathstart g = *z* ∧ *pathfinish g* = *y* ∧
path h ∧ *path_image h* ⊆ *C* ∧ *pathstart h* = *a* ∧
(∀ *t* ∈ {*0..1*}. *p*(*h t*) = *f*(*g t*))
if *y* ∈ *U* **for** *y*

proof −
obtain *g* **where** *path g path_image g* ⊆ *U* **and** *pastg*: *pathstart g* = *z*
and *pafig*: *pathfinish g* = *y*
using *U* ‹*z* ∈ *U*› ‹*y* ∈ *U*› **by** (*force simp*: *path_connected_def*)
obtain *h* **where** *path h path_image h* ⊆ *C pathstart h* = *a*
and ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h t*) = (*f* ∘ *g*) *t*
proof (*rule covering_space_lift_path_strong* [*OF cov* ‹*a* ∈ *C*›])
show *path* (*f* ∘ *g*)
using ‹*path g*› ‹*path_image g* ⊆ *U*› *contf continuous_on_subset path_continuous_image*
by *blast*
show *path_image* (*f* ∘ *g*) ⊆ *S*
by (*metis* ‹*path_image g* ⊆ *U*› *fim image_mono path_image_compose subset_trans*)
show *pathstart* (*f* ∘ *g*) = *p a*
by (*simp add*: *feq pastg pathstart_compose*)
qed *auto*
then show *?thesis*
by (*metis* ‹*path g*› ‹*path_image g* ⊆ *U*› *comp_apply pafig pastg*)
qed
have ∃ *l*. ∀ *g h*. *path g* ∧ *path_image g* ⊆ *U* ∧ *pathstart g* = *z* ∧ *pathfinish g* = *y* ∧
path h ∧ *path_image h* ⊆ *C* ∧ *pathstart h* = *a* ∧
(∀ *t* ∈ {*0..1*}. *p*(*h t*) = *f*(*g t*)) ⟶ *pathfinish h* = *l* **for** *y*
proof −
have *pathfinish h* = *pathfinish h'*
if *g*: *path g path_image g* ⊆ *U pathstart g* = *z pathfinish g* = *y*
and *h*: *path h path_image h* ⊆ *C pathstart h* = *a*
and *phg*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h t*) = *f*(*g t*)
and *g'*: *path g' path_image g'* ⊆ *U pathstart g'* = *z pathfinish g'* = *y*
and *h'*: *path h' path_image h'* ⊆ *C pathstart h'* = *a*
and *phg'*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h' t*) = *f*(*g' t*)
for *g h g' h'*
proof −
obtain *q* **where** *path q* **and** *piq*: *path_image q* ⊆ *C* **and** *pastq*: *pathstart q* = *a* **and** *pafiq*: *pathfinish q* = *a*
and *homS*: *homotopic_paths S* (*f* ∘ *g* +++ *reversepath g'*) (*p* ∘ *q*)
using *g g' hom* [*of g* +++ *reversepath g'*] **by** (*auto simp*: *subset_path_image_join*)
have *papq*: *path* (*p* ∘ *q*)
using *homS homotopic_paths_imp_path* **by** *blast*
have *pipq*: *path_image* (*p* ∘ *q*) ⊆ *S*
using *homS homotopic_paths_imp_subset* **by** *blast*
obtain *q'* **where** *path q' path_image q'* ⊆ *C*
and *pathstart q'* = *pathstart q pathfinish q'* = *pathfinish q*
and *pq'_eq*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p* (*q' t*) = (*f* ∘ *g* +++ *reversepath g'*) *t*
using *covering_space_lift_homotopic_path* [*OF cov homotopic_paths_sym* [*OF homS*] ‹*path q*› *piq refl refl*]
by *auto*
have *q' t* = (*h* ∘ ($*_R$) *2*) *t* **if** *0* ≤ *t t* ≤ *1*/*2* **for** *t*

proof (*rule covering_space_lift_unique* [*OF cov*, *of q′ 0 h* ∘ (∗$_R$) *2* {*0..1/2*} *f* ∘ *g* ∘ (∗$_R$) *2 t*])
show *q′ 0* = (*h* ∘ (∗$_R$) *2*) *0*
by (*metis* ‹*pathstart q′* = *pathstart q*› *comp_def h*(*3*) *pastq pathstart_def pth_4*(*2*))
show *continuous_on* {*0..1/2*} (*f* ∘ *g* ∘ (∗$_R$) *2*)
proof (*intro continuous_intros continuous_on_path* [*OF* ‹*path g*›] *continuous_on_subset* [*OF contf*])
show *g* ‘ (∗$_R$) *2* ‘ {*0..1/2*} ⊆ *U*
using *g path_image_def* **by** *fastforce*
qed *auto*
show (*f* ∘ *g* ∘ (∗$_R$) *2*) ‘ {*0..1/2*} ⊆ *S*
using *g*(*2*) *path_image_def fim* **by** *fastforce*
show (*h* ∘ (∗$_R$) *2*) ‘ {*0..1/2*} ⊆ *C*
using *h path_image_def* **by** *fastforce*
show *q′* ‘ {*0..1/2*} ⊆ *C*
using ‹*path_image q′* ⊆ *C*› *path_image_def* **by** *fastforce*
show ⋀*x*. *x* ∈ {*0..1/2*} ⟹ (*f* ∘ *g* ∘ (∗$_R$) *2*) *x* = *p* (*q′ x*)
by (*auto simp*: *joinpaths_def pq′_eq*)
show ⋀*x*. *x* ∈ {*0..1/2*} ⟹ (*f* ∘ *g* ∘ (∗$_R$) *2*) *x* = *p* ((*h* ∘ (∗$_R$) *2*) *x*)
by (*simp add*: *phg*)
show *continuous_on* {*0..1/2*} *q′*
by (*simp add*: *continuous_on_path* ‹*path q′*›)
show *continuous_on* {*0..1/2*} (*h* ∘ (∗$_R$) *2*)
by (*intro continuous_intros continuous_on_path* [*OF* ‹*path h*›]) *auto*
qed (*use that* **in** *auto*)
moreover have *q′ t* = (*reversepath h′* ∘ (λ*t*. *2* ∗$_R$ *t* − *1*)) *t* **if** *1/2* < *t t* ≤ *1* **for** *t*
proof (*rule covering_space_lift_unique* [*OF cov*, *of q′ 1 reversepath h′* ∘ (λ*t*. *2* ∗$_R$ *t* − *1*) {*1/2<..1*} *f* ∘ *reversepath g′* ∘ (λ*t*. *2* ∗$_R$ *t* − *1*) *t*])
show *q′ 1* = (*reversepath h′* ∘ (λ*t*. *2* ∗$_R$ *t* − *1*)) *1*
using *h′* ‹*pathfinish q′* = *pathfinish q*› *pafiq*
by (*simp add*: *pathstart_def pathfinish_def reversepath_def*)
show *continuous_on* {*1/2<..1*} (*f* ∘ *reversepath g′* ∘ (λ*t*. *2* ∗$_R$ *t* − *1*))
proof (*intro continuous_intros continuous_on_path* ‹*path g′*› *continuous_on_subset* [*OF contf*])
show *reversepath g′* ‘ (λ*t*. *2* ∗$_R$ *t* − *1*) ‘ {*1/2<..1*} ⊆ *U*
using *g′* **by** (*auto simp*: *path_image_def reversepath_def*)
qed (*use g′* **in** *auto*)
show (*f* ∘ *reversepath g′* ∘ (λ*t*. *2* ∗$_R$ *t* − *1*)) ‘ {*1/2<..1*} ⊆ *S*
using *g′*(*2*) *path_image_def fim* **by** (*auto simp*: *image_subset_iff path_image_def reversepath_def*)
show *q′* ‘ {*1/2<..1*} ⊆ *C*
using ‹*path_image q′* ⊆ *C*› *path_image_def* **by** *fastforce*
show (*reversepath h′* ∘ (λ*t*. *2* ∗$_R$ *t* − *1*)) ‘ {*1/2<..1*} ⊆ *C*
using *h′* **by** (*simp add*: *path_image_def reversepath_def subset_eq*)
show ⋀*x*. *x* ∈ {*1/2<..1*} ⟹ (*f* ∘ *reversepath g′* ∘ (λ*t*. *2* ∗$_R$ *t* − *1*)) *x* = *p* (*q′ x*)
by (*auto simp*: *joinpaths_def pq′_eq*)

```
      show ⋀x. x ∈ {1/2<..1} ⟹
               (f ∘ reversepath g' ∘ (λt. 2 *R t − 1)) x = p ((reversepath h' ∘ (λt.
2 *R t − 1)) x)
        by (simp add: phg' reversepath_def)
      show continuous_on {1/2<..1} q'
        by (auto intro: continuous_on_path [OF ‹path q'›])
      show continuous_on {1/2<..1} (reversepath h' ∘ (λt. 2 *R t − 1))
        by (intro continuous_intros continuous_on_path ‹path h'›) (use h' in auto)
    qed (use that in auto)
    ultimately have q' t = (h +++ reversepath h') t if 0 ≤ t t ≤ 1 for t
      using that by (simp add: joinpaths_def)
    then have path(h +++ reversepath h')
      by (auto intro: path_eq [OF ‹path q'›])
    then show ?thesis
      by (auto simp: ‹path h› ‹path h'›)
  qed
  then show ?thesis by metis
qed
then obtain l :: 'c ⇒ 'a
      where l: ⋀y g h. ⟦path g; path_image g ⊆ U; pathstart g = z; pathfinish g
= y;
                        path h; path_image h ⊆ C; pathstart h = a;
                        ⋀t. t ∈ {0..1} ⟹ p(h t) = f(g t)⟧ ⟹ pathfinish h = l y
  by metis
show ?thesis
proof
  show pleq: p (l y) = f y if y ∈ U for y
     using*[OF ‹y ∈ U›] by (metis l atLeastAtMost_iff order_refl pathfinish_def
zero_le_one)
  show l z = a
    using l [of linepath z z z linepath a a] by (auto simp: assms)
  show LC: l ` U ⊆ C
    by (clarify dest!: *) (metis (full_types) l pathfinish_in_path_image subsetCE)
  have ∃T. openin (top_of_set U) T ∧ y ∈ T ∧ T ⊆ U ∩ l −` X
      if X: openin (top_of_set C) X and y ∈ U l y ∈ X for X y
  proof −
    have X ⊆ C
      using X openin_euclidean_subtopology_iff by blast
    have f y ∈ S
      using fim ‹y ∈ U› by blast
    then obtain W 𝒱
          where WV: f y ∈ W ∧ openin (top_of_set S) W ∧
                    (⋃𝒱 = C ∩ p −` W ∧
                     (∀U ∈ 𝒱. openin (top_of_set C) U) ∧
                     pairwise disjnt 𝒱 ∧
                     (∀U ∈ 𝒱. ∃q. homeomorphism U W p q))
      using cov by (force simp: covering_space_def)
    then have l y ∈ ⋃𝒱
      using ‹X ⊆ C› pleq that by auto
```

then obtain W' **where** $l\ y \in W'$ **and** $W' \in \mathcal{V}$
by *blast*
with *WV* **obtain** p' **where** *opeCW'*: *openin* (*top_of_set* C) W'
and *homUW'*: *homeomorphism* W' W p p'
by *blast*
then have *contp'*: *continuous_on* W p' **and** *p'im*: p' ‘ $W \subseteq W'$
using *homUW' homeomorphism_image2 homeomorphism_cont2* **by** *fastforce+*
obtain V **where** $y \in V$ $y \in U$ **and** *fimW*: f ‘ $V \subseteq W$ $V \subseteq U$
and *path_connected* V **and** *opeUV*: *openin* (*top_of_set* U) V
proof −
have *openin* (*top_of_set* U) ($U \cap f$ −‘ W)
using *WV contf continuous_on_open_gen fim* **by** *auto*
then obtain *UO* **where** *openin* (*top_of_set* U) *UO* ∧ *path_connected UO* ∧ $y \in$ *UO* ∧ *UO* $\subseteq U \cap f$ −‘ W
using U *WV* ‹$y \in U$› **unfolding** *locally_path_connected* **by** (*meson IntI vimage_eq*)
then show *?thesis*
by (*meson* ‹$y \in U$› *image_subset_iff_subset_vimage le_inf_iff that*)
qed
have $W' \subseteq C$ $W \subseteq S$
using *opeCW' WV openin_imp_subset* **by** *auto*
have *p'im*: p' ‘ $W \subseteq W'$
using *homUW' homeomorphism_image2* **by** *fastforce*
show *?thesis*
proof (*intro exI conjI*)
have *openin* (*top_of_set* S) ($W \cap p'$ −‘ ($W' \cap X$))
proof (*rule openin_trans*)
show *openin* (*top_of_set* W) ($W \cap p'$ −‘ ($W' \cap X$))
using X ‹$W' \subseteq C$› **by** (*intro continuous_openin_preimage* [*OF contp' p'im*]) (*auto simp*: *openin_open*)
show *openin* (*top_of_set* S) W
using *WV* **by** *blast*
qed
then show *openin* (*top_of_set* U) ($V \cap (U \cap (f$ −‘ ($W \cap (p'$ −‘ ($W' \cap X$))))))
by (*blast intro*: *opeUV openin_subtopology_self continuous_openin_preimage* [*OF contf fim*])
have p' ($f\ y$) $\in X$
using ‹$l\ y \in W'$› *homeomorphism_apply1* [*OF homUW'*] *pleq* ‹$y \in U$› ‹$l\ y \in X$› **by** *fastforce*
then show $y \in V \cap (U \cap f$ −‘ ($W \cap p'$ −‘ ($W' \cap X$)))
using ‹$y \in U$› ‹$y \in V$› *WV p'im* **by** *auto*
show $V \cap (U \cap f$ −‘ ($W \cap p'$ −‘ ($W' \cap X$))) $\subseteq U \cap l$ −‘ X
proof (*intro subsetI IntI*; *clarify*)
fix y'
assume y': $y' \in V$ $y' \in U$ $f\ y' \in W$ p' ($f\ y'$) $\in W'$ p' ($f\ y'$) $\in X$
then obtain γ **where** *path* γ *path_image* $\gamma \subseteq V$ *pathstart* $\gamma = y$ *pathfinish* $\gamma = y'$

```
          by (meson ‹path_connected V› ‹y ∈ V› path_connected_def)
        obtain pp qq where pp: path pp path_image pp ⊆ U pathstart pp = z
pathfinish pp = y
                   and qq: path qq path_image qq ⊆ C pathstart qq = a
                   and pqqeq: ⋀t. t ∈ {0..1} ⟹ p(qq t) = f(pp t)
          using*[OF ‹y ∈ U›] by blast
        have finW: ⋀x. ⟦0 ≤ x; x ≤ 1⟧ ⟹ f (γ x) ∈ W
         using ‹path_image γ ⊆ V› by (auto simp: image_subset_iff path_image_def
fimW [THEN subsetD])
        have pathfinish (qq +++ (p' ∘ f ∘ γ)) = l y'
        proof (rule l [of pp +++ γ y' qq +++ (p' ∘ f ∘ γ)])
          show path (pp +++ γ)
            by (simp add: ‹path γ› ‹path pp› ‹pathfinish pp = y› ‹pathstart γ = y›)
          show path_image (pp +++ γ) ⊆ U
        using ‹V ⊆ U› ‹path_image γ ⊆ V› ‹path_image pp ⊆ U› not_in_path_image_join
by blast
          show pathstart (pp +++ γ) = z
            by (simp add: ‹pathstart pp = z›)
          show pathfinish (pp +++ γ) = y'
            by (simp add: ‹pathfinish γ = y'›)
          have pathfinish qq = l y
             using ‹path pp› ‹path qq› ‹path_image pp ⊆ U› ‹path_image qq ⊆ C›
‹pathfinish pp = y› ‹pathstart pp = z› ‹pathstart qq = a› l pqqeq by blast
          also have ... = p' (f y)
              using ‹l y ∈ W'› homUW' homeomorphism_apply1 pleq that(2) by
fastforce
          finally have pathfinish qq = p' (f y) .
          then have paqq: pathfinish qq = pathstart (p' ∘ f ∘ γ)
            by (simp add: ‹pathstart γ = y› pathstart_compose)
          have continuous_on (path_image γ) (p' ∘ f)
          proof (rule continuous_on_compose)
            show continuous_on (path_image γ) f
               using ‹path_image γ ⊆ V› ‹V ⊆ U› contf continuous_on_subset by
blast
            show continuous_on (f ` path_image γ) p'
            proof (rule continuous_on_subset [OF contp'])
              show f ` path_image γ ⊆ W
                by (auto simp: path_image_def pathfinish_def pathstart_def finW)
            qed
          qed
          then show path (qq +++ (p' ∘ f ∘ γ))
            using ‹path γ› ‹path qq› paqq path_continuous_image path_join_imp by
blast
          show path_image (qq +++ (p' ∘ f ∘ γ)) ⊆ C
          proof (rule subset_path_image_join)
            show path_image qq ⊆ C
              by (simp add: ‹path_image qq ⊆ C›)
            show path_image (p' ∘ f ∘ γ) ⊆ C
              by (metis ‹W' ⊆ C› ‹path_image γ ⊆ V› dual_order.trans fimW(1)
```

image_comp image_mono p'im path_image_compose)
qed
show *pathstart* (*qq* +++ (*p'* ∘ *f* ∘ γ)) = *a*
by (*simp add*: ‹*pathstart qq* = *a*›)
show *p* ((*qq* +++ (*p'* ∘ *f* ∘ γ)) ξ) = *f* ((*pp* +++ γ) ξ) **if** ξ: ξ ∈ {*0..1*}
for ξ
proof (*simp add*: *joinpaths_def*, *safe*)
show *p* (*qq* (*2*∗ξ)) = *f* (*pp* (*2*∗ξ)) **if** ξ∗*2* ≤ *1*
using ‹ξ ∈ {*0..1*}› *pqqeq that* **by** *auto*
show *p* (*p'* (*f* (γ (*2*∗ξ − *1*)))) = *f* (γ (*2*∗ξ − *1*)) **if** ¬ ξ∗*2* ≤ *1*
using *that* ξ **by** (*auto intro*: *homeomorphism_apply2* [*OF homUW'*
finW])
qed
qed
with ‹*pathfinish* γ = *y'*› ‹*p'* (*f y'*) ∈ *X*› **show** *y'* ∈ *l* −‘ *X*
unfolding *pathfinish_join* **by** (*simp add*: *pathfinish_def*)
qed
qed
qed
then show *continuous_on U l*
by (*metis IntD1 IntD2 vimage_eq openin_subopen continuous_on_open_gen* [*OF*
LC])
qed
qed

corollary *covering_space_lift_stronger*:
fixes *p* :: *'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector*
and *f* :: *'c*::*real_normed_vector* ⇒ *'b*
assumes *cov*: *covering_space C p S a* ∈ *C z* ∈ *U*
and *U*: *path_connected U locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: *f* ‘ *U* ⊆ *S*
and *feq*: *f z* = *p a*
and *hom*: ⋀*r*. ⟦*path r*; *path_image r* ⊆ *U*; *pathstart r* = *z*; *pathfinish r* = *z*⟧
⟹ ∃ *b*. *homotopic_paths S* (*f* ∘ *r*) (*linepath b b*)
obtains *g* **where** *continuous_on U g g* ‘ *U* ⊆ *C g z* = *a* ⋀*y*. *y* ∈ *U* ⟹ *p*(*g y*)
= *f y*
proof (*rule covering_space_lift_general* [*OF cov U contf fim feq*])
fix *r*
assume *path r path_image r* ⊆ *U pathstart r* = *z pathfinish r* = *z*
then obtain *b* **where** *b*: *homotopic_paths S* (*f* ∘ *r*) (*linepath b b*)
using *hom* **by** *blast*
then have *f* (*pathstart r*) = *b*
by (*metis homotopic_paths_imp_pathstart pathstart_compose pathstart_linepath*)
then have *homotopic_paths S* (*f* ∘ *r*) (*linepath* (*f z*) (*f z*))
by (*simp add*: *b* ‹*pathstart r* = *z*›)
then have *homotopic_paths S* (*f* ∘ *r*) (*p* ∘ *linepath a a*)
by (*simp add*: *o_def feq linepath_def*)
then show ∃ *q*. *path q* ∧
path_image q ⊆ *C* ∧

pathstart q = a ∧ pathfinish q = a ∧ homotopic_paths S (f ∘ r) (p ∘ q)
by (*force simp*: ‹*a* ∈ *C*›)
qed *auto*

corollary *covering_space_lift_strong*:
fixes *p* :: *'a::real_normed_vector ⇒ 'b::real_normed_vector*
and *f* :: *'c::real_normed_vector ⇒ 'b*
assumes *cov*: *covering_space C p S a ∈ C z ∈ U*
and *scU*: *simply_connected U* **and** *lpcU*: *locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: *f ' U ⊆ S*
and *feq*: *f z = p a*
obtains *g* **where** *continuous_on U g g ' U ⊆ C g z = a ⋀y. y ∈ U ⟹ p(g y) = f y*
proof (*rule covering_space_lift_stronger* [*OF cov _ lpcU contf fim feq*])
show *path_connected U*
using *scU simply_connected_eq_contractible_loop_some* **by** *blast*
fix *r*
assume *r*: *path r path_image r ⊆ U pathstart r = z pathfinish r = z*
have *linepath (f z) (f z) = f ∘ linepath z z*
by (*simp add*: *o_def linepath_def*)
then have *homotopic_paths S (f ∘ r) (linepath (f z) (f z))*
by (*metis r contf fim homotopic_paths_continuous_image scU simply_connected_eq_contractible_path*)
then show ∃ *b. homotopic_paths S (f ∘ r) (linepath b b)*
by *blast*
qed *blast*

corollary *covering_space_lift*:
fixes *p* :: *'a::real_normed_vector ⇒ 'b::real_normed_vector*
and *f* :: *'c::real_normed_vector ⇒ 'b*
assumes *cov*: *covering_space C p S*
and *U*: *simply_connected U locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: *f ' U ⊆ S*
obtains *g* **where** *continuous_on U g g ' U ⊆ C ⋀y. y ∈ U ⟹ p(g y) = f y*
proof (*cases U* = {})
case *True*
with *that* **show** *?thesis* **by** *auto*
next
case *False*
then obtain *z* **where** *z ∈ U* **by** *blast*
then obtain *a* **where** *a ∈ C f z = p a*
by (*metis cov covering_space_imp_surjective fim image_iff image_subset_iff*)
then show *?thesis*
by (*metis that covering_space_lift_strong* [*OF cov _* ‹*z* ∈ *U*› *U contf fim*])
qed

6.19.6 Homeomorphisms of arc images

lemma *homeomorphism_arc*:

fixes *g* :: *real* ⇒ *'a*::*t2_space*
assumes *arc g*
obtains *h* **where** *homeomorphism* {*0..1*} (*path_image g*) *g h*
using *assms* **by** (*force simp*: *arc_def homeomorphism_compact path_def path_image_def*)

lemma *homeomorphic_arc_image_interval*:
fixes *g* :: *real* ⇒ *'a*::*t2_space* **and** *a*::*real*
assumes *arc g a* < *b*
shows (*path_image g*) *homeomorphic* {*a..b*}
proof −
have (*path_image g*) *homeomorphic* {*0..1*::*real*}
by (*meson assms*(*1*) *homeomorphic_def homeomorphic_sym homeomorphism_arc*)
also have ... *homeomorphic* {*a..b*}
using *assms* **by** (*force intro*: *homeomorphic_closed_intervals_real*)
finally show *?thesis* **.**
qed

lemma *homeomorphic_arc_images*:
fixes *g* :: *real* ⇒ *'a*::*t2_space* **and** *h* :: *real* ⇒ *'b*::*t2_space*
assumes *arc g arc h*
shows (*path_image g*) *homeomorphic* (*path_image h*)
proof −
have (*path_image g*) *homeomorphic* {*0..1*::*real*}
by (*meson assms homeomorphic_def homeomorphic_sym homeomorphism_arc*)
also have ... *homeomorphic* (*path_image h*)
by (*meson assms homeomorphic_def homeomorphism_arc*)
finally show *?thesis* **.**
qed

end

theory *Equivalence_Lebesgue_Henstock_Integration*
imports
Lebesgue_Measure
Henstock_Kurzweil_Integration
Complete_Measure
Set_Integral
Homeomorphism
Cartesian_Euclidean_Space
begin

lemma *LIMSEQ_if_less*: $(\lambda k.\ if\ i < k\ then\ a\ else\ b) \longrightarrow a$
by (*rule_tac k*=*Suc i* **in** *LIMSEQ_offset*) *auto*

Note that the rhs is an implication. This lemma plays a specific role in one proof.

lemma *le_left_mono*: $x \le y \Longrightarrow y \le a \longrightarrow x \le (a::'a::preorder)$
by (*auto intro*: *order_trans*)

lemma *ball_trans*:
  assumes $y \in$ *ball z q r* + *q* $\leq$ *s* **shows** *ball y r* $\subseteq$ *ball z s*
  using *assms* **by** *metric*

lemma *has_integral_implies_lebesgue_measurable_cbox*:
  fixes *f* :: *'a* :: *euclidean_space* ⇒ *real*
  assumes *f*: (*f has_integral I*) (*cbox x y*)
  shows *f* ∈ *lebesgue_on* (*cbox x y*) $\rightarrow_M$ *borel*
proof (*rule cld_measure.borel_measurable_cld*)
  let *?L* = *lebesgue_on* (*cbox x y*)
  let *?μ* = *emeasure ?L*
  let *?μ'* = *outer_measure_of ?L*
  interpret *L*: *finite_measure ?L*
  proof
    show *?μ* (*space ?L*) ≠ ∞
    by (*simp add*: *emeasure_restrict_space space_restrict_space emeasure_lborel_cbox_eq*)
  qed

  show *cld_measure ?L*
  proof
    fix *B A* **assume** *B* ⊆ *A A* ∈ *null_sets ?L*
    then show *B* ∈ *sets ?L*
      using *null_sets_completion_subset*[*OF* ‹*B* ⊆ *A*›, *of lborel*]
      by (*auto simp add*: *null_sets_restrict_space sets_restrict_space_iff intro*:)
  next
    fix *A* **assume** *A* ⊆ *space ?L* ⋀*B*. *B* ∈ *sets ?L* ⟹ *?μ B* < ∞ ⟹ *A* ∩ *B* ∈ *sets ?L*
    from *this*(*1*) *this*(*2*)[*of space ?L*] **show** *A* ∈ *sets ?L*
      by (*auto simp*: *Int_absorb2 less_top*[*symmetric*])
  qed *auto*
  then interpret *cld_measure ?L*
    .

  have *content_eq_L*: *A* ∈ *sets borel* ⟹ *A* ⊆ *cbox x y* ⟹ *content A* = *measure ?L A* **for** *A*
    by (*subst measure_restrict_space*) (*auto simp*: *measure_def*)

  fix *E* **and** *a b* :: *real* **assume** *E* ∈ *sets ?L a* < *b 0* < *?μ E ?μ E* < ∞
  then obtain *M* :: *real* **where** *?μ E* = *M 0* < *M*
    by (*cases ?μ E*) *auto*
  define *e* **where** *e* = *M* / (*4* + *2* / (*b* − *a*))
  from ‹*a* < *b*› ‹*0*<*M*› **have** *0* < *e*
    by (*auto intro*!: *divide_pos_pos simp*: *field_simps e_def*)

  have *e* < *M* / (*3* + *2* / (*b* − *a*))
    using ‹*a* < *b*› ‹*0* < *M*›
   unfolding *e_def* **by** (*intro divide_strict_left_mono add_strict_right_mono mult_pos_pos*) (*auto simp*: *field_simps*)

then have $2 * e < (b - a) * (M - e * 3)$
using ‹$0<M$› ‹$0 < e$› ‹$a < b$› **by** (*simp add*: *field_simps*)

have *e_less_M*: $e < M / 1$
unfolding *e_def* **using** ‹$a < b$› ‹$0<M$› **by** (*intro divide_strict_left_mono*) (*auto simp*: *field_simps*)

obtain *d*
where *gauge d*
and *integral_f*: $\forall p.\ p$ *tagged_division_of cbox x y* $\wedge$ *d fine p* $\longrightarrow$
norm $((\sum (x,k) \in p.$ *content* $k *_R f\, x) - I) < e$
using ‹$0<e$› *f* **unfolding** *has_integral* **by** *auto*

define *C* **where** $C\ X\ m = X \cap \{x.$ *ball* $x\ (1/Suc\ m) \subseteq d\ x\}$ **for** *X m*
have *incseq* (*C X*) **for** *X*
unfolding *C_def* [*abs_def*]
by (*intro monoI Collect_mono conj_mono imp_refl le_left_mono subset_ball divide_left_mono Int_mono*) *auto*

{ fix *X* **assume** $X \subseteq$ *space ?L* **and** *eq*: $?\mu'\ X = ?\mu\ E$
have (*SUP m. outer_measure_of ?L* (*C X m*)) = *outer_measure_of ?L* ($\bigcup m.$ *C X m*)
using ‹$X \subseteq$ *space ?L*› **by** (*intro SUP_outer_measure_of_incseq* ‹*incseq* (*C X*)›) (*auto simp*: *C_def*)
also have ($\bigcup m.$ *C X m*) = *X*
proof −
{ fix *x*
obtain *e* **where** $0 < e$ *ball x e* $\subseteq$ *d x*
using *gaugeD*[*OF* ‹*gauge d*›, *of x*] **unfolding** *open_contains_ball* **by** *auto*
moreover
obtain *n* **where** $1 / (1 + real\ n) < e$
using *reals_Archimedean*[*OF* ‹$0<e$›] **by** (*auto simp*: *inverse_eq_divide*)
then have *ball x* $(1 / (1 + real\ n)) \subseteq$ *ball x e*
by (*intro subset_ball*) *auto*
ultimately have $\exists n.$ *ball x* $(1 / (1 + real\ n)) \subseteq d\ x$
by *blast* **}**
then show *?thesis*
by (*auto simp*: *C_def*)
qed
finally have (*SUP m. outer_measure_of ?L* (*C X m*)) = $?\mu\ E$
using *eq* **by** *auto*
also have ... $> M - e$
using ‹$0 < M$› ‹$?\mu\ E = M$› ‹$0<e$› **by** (*auto intro*!: *ennreal_lessI*)
finally have $\exists m.\ M - e <$ *outer_measure_of ?L* (*C X m*)
unfolding *less_SUP_iff* **by** *auto* **}**
note *C* = *this*

let $?E = \{x{\in}E.\ f\ x \le a\}$ **and** $?F = \{x{\in}E.\ b \le f\ x\}$

```
  have ¬ (?μ' ?E = ?μ E ∧ ?μ' ?F = ?μ E)
  proof
    assume eq: ?μ' ?E = ?μ E ∧ ?μ' ?F = ?μ E
    with C[of ?E] C[of ?F] ‹E ∈ sets ?L›[THEN sets.sets_into_space] obtain ma
mb
      where M − e < outer_measure_of ?L (C ?E ma) M − e < outer_measure_of
?L (C ?F mb)
      by auto
    moreover define m where m = max ma mb
    ultimately have M_minus_e: M − e < outer_measure_of ?L (C ?E m) M −
e < outer_measure_of ?L (C ?F m)
      using
        incseqD[OF ‹incseq (C ?E)›, of ma m, THEN outer_measure_of_mono]
        incseqD[OF ‹incseq (C ?F)›, of mb m, THEN outer_measure_of_mono]
      by (auto intro: less_le_trans)
    define d' where d' x = d x ∩ ball x (1 / (3 ∗ Suc m)) for x
    have gauge d'
      unfolding d'_def by (intro gauge_Int ‹gauge d› gauge_ball) auto
    then obtain p where p: p tagged_division_of cbox x y d' fine p
      by (rule fine_division_exists)
    then have d fine p
      unfolding d'_def[abs_def] fine_def by auto

    define s where s = {(x::'a, k). k ∩ (C ?E m) ≠ {} ∧ k ∩ (C ?F m) ≠ {}}
    define T where T E k = (SOME x. x ∈ k ∩ C E m) for E k
    let ?A = (λ(x, k). (T ?E k, k)) ‘ (p ∩ s) ∪ (p − s)
    let ?B = (λ(x, k). (T ?F k, k)) ‘ (p ∩ s) ∪ (p − s)

    { fix X assume X_eq: X = ?E ∨ X = ?F
      let ?T = (λ(x, k). (T X k, k))
      let ?p = ?T ‘ (p ∩ s) ∪ (p − s)

      have in_s: (x, k) ∈ s =⇒ T X k ∈ k ∩ C X m for x k
          using someI_ex[of λx. x ∈ k ∩ C X m] X_eq unfolding ex_in_conv by
(auto simp: T_def s_def)

      { fix x k assume (x, k) ∈ p (x, k) ∈ s
        have k: k ⊆ ball x (1 / (3 ∗ Suc m))
          using ‹d' fine p›[THEN fineD, OF ‹(x, k) ∈ p›] by (auto simp: d'_def)
        then have x ∈ ball (T X k) (1 / (3 ∗ Suc m))
          using in_s[OF ‹(x, k) ∈ s›] by (auto simp: C_def subset_eq dist_commute)
        then have ball x (1 / (3 ∗ Suc m)) ⊆ ball (T X k) (1 / Suc m)
          by (rule ball_trans) (auto simp: field_split_simps)
        with k in_s[OF ‹(x, k) ∈ s›] have k ⊆ d (T X k)
          by (auto simp: C_def) }
      then have d fine ?p
        using ‹d fine p› by (auto intro!: fineI)
      moreover
      have ?p tagged_division_of cbox x y
```

proof (*rule tagged_division_ofI*)
  show *finite ?p*
    using *p*(*1*) **by** *auto*
next
  fix *z k* **assume** ∗: (*z*, *k*) ∈ *?p*
  then consider (*z*, *k*) ∈ *p* (*z*, *k*) ∉ *s*
    | *x′* **where** (*x′*, *k*) ∈ *p* (*x′*, *k*) ∈ *s z* = *T X k*
    by (*auto simp*: *T_def*)
  then have *z* ∈ *k* ∧ *k* ⊆ *cbox x y* ∧ (∃ *a b*. *k* = *cbox a b*)
    using *p*(*1*) **by** *cases* (*auto dest*: *in_s*)
  then show *z* ∈ *k k* ⊆ *cbox x y* ∃ *a b*. *k* = *cbox a b*
    by *auto*
next
  fix *z k z′ k′* **assume** (*z*, *k*) ∈ *?p* (*z′*, *k′*) ∈ *?p* (*z*, *k*) ≠ (*z′*, *k′*)
  with *tagged_division_ofD*(*5*)[*OF p*(*1*), *of* _ *k* _ *k′*]
  show *interior k* ∩ *interior k′* = {}
    by (*auto simp*: *T_def dest*: *in_s*)
next
  have {*k*. ∃ *x*. (*x*, *k*) ∈ *?p*} = {*k*. ∃ *x*. (*x*, *k*) ∈ *p*}
    by (*auto simp*: *T_def image_iff Bex_def*)
  then show ⋃{*k*. ∃ *x*. (*x*, *k*) ∈ *?p*} = *cbox x y*
    using *p*(*1*) **by** *auto*
qed
ultimately have *I*: *norm* ((∑(*x*,*k*) ∈ *?p*. *content k* $*_R$ *f x*) − *I*) < *e*
  using *integral_f* **by** *auto*

have (∑(*x*,*k*) ∈ *?p*. *content k* $*_R$ *f x*) =
  (∑(*x*,*k*) ∈ *?T* ‘ (*p* ∩ *s*). *content k* $*_R$ *f x*) + (∑(*x*,*k*) ∈ *p* − *s*. *content k* $*_R$ *f x*)
  using *p*(*1*)[*THEN tagged_division_ofD*(*1*)]
  by (*safe intro*!: *sum.union_inter_neutral*) (*auto simp*: *s_def T_def*)
also have (∑(*x*,*k*) ∈ *?T* ‘ (*p* ∩ *s*). *content k* $*_R$ *f x*) = (∑(*x*,*k*) ∈ *p* ∩ *s*. *content k* $*_R$ *f* (*T X k*))
proof (*subst sum.reindex_nontrivial*, *safe*)
  fix *x1 x2 k* **assume** *1*: (*x1*, *k*) ∈ *p* (*x1*, *k*) ∈ *s* **and** *2*: (*x2*, *k*) ∈ *p* (*x2*, *k*) ∈ *s*
    and *eq*: *content k* $*_R$ *f* (*T X k*) ≠ *0*
  with *tagged_division_ofD*(*5*)[*OF p*(*1*), *of x1 k x2 k*] *tagged_division_ofD*(*4*)[*OF p*(*1*), *of x1 k*]
  show *x1* = *x2*
    by (*auto simp*: *content_eq_0_interior*)
qed (*use p* **in** ‹*auto intro*!: *sum.cong*›)
finally have *eq*: (∑(*x*,*k*) ∈ *?p*. *content k* $*_R$ *f x*) =
  (∑(*x*,*k*) ∈ *p* ∩ *s*. *content k* $*_R$ *f* (*T X k*)) + (∑(*x*,*k*) ∈ *p* − *s*. *content k* $*_R$ *f x*) **.**

have *in_T*: (*x*, *k*) ∈ *s* ⟹ *T X k* ∈ *X* **for** *x k*
  using *in_s*[*of x k*] **by** (*auto simp*: *C_def*)

```
      note I eq in_T }
    note parts = this

    have p_in_L: (x, k) ∈ p ⟹ k ∈ sets ?L for x k
     using tagged_division_ofD(3, 4)[OF p(1), of x k] by (auto simp: sets_restrict_space)

    have [simp]: finite p
      using tagged_division_ofD(1)[OF p(1)] .

    have (M − 3*e) * (b − a) ≤ (∑(x,k) ∈ p ∩ s. content k) * (b − a)
    proof (intro mult_right_mono)
      have fin: ?μ (E ∩ ⋃{k∈snd`p. k ∩ C X m = {}}) < ∞ for X
      using ‹?μ E < ∞› by (rule le_less_trans[rotated]) (auto intro!: emeasure_mono
‹E ∈ sets ?L›)
      have sets: (E ∩ ⋃{k∈snd`p. k ∩ C X m = {}}) ∈ sets ?L for X
          using tagged_division_ofD(1)[OF p(1)] by (intro sets.Diff ‹E ∈ sets ?L›
sets.finite_Union sets.Int) (auto intro: p_in_L)
      { fix X assume X ⊆ E M − e < ?μ' (C X m)
        have M − e ≤ ?μ' (C X m)
          by (rule less_imp_le) fact
        also have ... ≤ ?μ' (E − (E ∩ ⋃{k∈snd`p. k ∩ C X m = {}}))
        proof (intro outer_measure_of_mono subsetI)
          fix v assume v ∈ C X m
          then have v ∈ cbox x y v ∈ E
           using ‹E ⊆ space ?L› ‹X ⊆ E› by (auto simp: space_restrict_space C_def)
          then obtain z k where (z, k) ∈ p v ∈ k
            using tagged_division_ofD(6)[OF p(1), symmetric] by auto
          then show v ∈ E − E ∩ (⋃{k∈snd`p. k ∩ C X m = {}})
            using ‹v ∈ C X m› ‹v ∈ E› by auto
        qed
        also have ... = ?μ E − ?μ (E ∩ ⋃{k∈snd`p. k ∩ C X m = {}})
          using ‹E ∈ sets ?L› fin[of X] sets[of X] by (auto intro!: emeasure_Diff)
        finally have ?μ (E ∩ ⋃{k∈snd`p. k ∩ C X m = {}}) ≤ e
          using ‹0 < e› e_less_M
           by (cases ?μ (E ∩ ⋃{k∈snd`p. k ∩ C X m = {}})) (auto simp add: ‹?μ
E = M› ennreal_minus ennreal_le_iff2)
        note this }
      note upper_bound = this

      have ?μ (E ∩ ⋃(snd`(p − s))) =
         ?μ ((E ∩ ⋃{k∈snd`p. k ∩ C ?E m = {}}) ∪ (E ∩ ⋃{k∈snd`p. k ∩ C ?F
m = {}}))
        by (intro arg_cong[where f=?μ]) (auto simp: s_def image_def Bex_def)
       also have ... ≤ ?μ (E ∩ ⋃{k∈snd`p. k ∩ C ?E m = {}}) + ?μ (E ∩
⋃{k∈snd`p. k ∩ C ?F m = {}})
         using sets[of ?E] sets[of ?F] M_minus_e by (intro emeasure_subadditive)
auto
      also have ... ≤ e + ennreal e
      using upper_bound[of ?E] upper_bound[of ?F] M_minus_e by (intro add_mono)
```

auto
finally have *?μ E − 2∗e ≤ ?μ (E − (E ∩ ⋃(snd'(p − s))))*
using ‹*0 < e*› ‹*E ∈ sets ?L*› *tagged_division_ofD(1)[OF p(1)]*
by (*subst emeasure_Diff*)
(*auto simp*: *top_unique simp flip*: *ennreal_plus*
intro!: *sets.Int sets.finite_UN ennreal_mono_minus intro*: *p_in_L*)
also have ... ≤ *?μ* (⋃*x*∈*p* ∩ *s. snd x*)
proof (*safe intro*!: *emeasure_mono subsetI*)
fix *v* **assume** *v ∈ E* **and** *not*: *v ∉* (⋃*x*∈*p* ∩ *s. snd x*)
then have *v ∈ cbox x y*
using ‹*E ⊆ space ?L*› **by** (*auto simp*: *space_restrict_space*)
then obtain *z k* **where** (*z*, *k*) ∈ *p v* ∈ *k*
using *tagged_division_ofD(6)[OF p(1), symmetric]* **by** *auto*
with *not* **show** *v* ∈ ⋃(*snd* ' (*p* − *s*))
by (*auto intro*!: *bexI[of _ (z, k)] elim*: *ballE[of _ _ (z, k)]*)
qed (*auto intro*!: *sets.Int sets.finite_UN ennreal_mono_minus intro*: *p_in_L*)
also have ... = *measure ?L* (⋃*x*∈*p* ∩ *s. snd x*)
by (*auto intro*!: *emeasure_eq_ennreal_measure*)
finally have *M − 2 ∗ e ≤ measure ?L* (⋃*x*∈*p* ∩ *s. snd x*)
unfolding ‹*?μ E = M*› **using** ‹*0 < e*› **by** (*simp add*: *ennreal_minus*)
also have *measure ?L* (⋃*x*∈*p* ∩ *s. snd x*) = *content* (⋃*x*∈*p* ∩ *s. snd x*)
using *tagged_division_ofD(1,3,4) [OF p(1)]*
by (*intro content_eq_L[symmetric]*)
(*fastforce intro*!: *sets.finite_UN UN_least del*: *subsetI*)+
also have *content* (⋃*x*∈*p* ∩ *s. snd x*) ≤ (∑*k*∈*p* ∩ *s. content* (*snd k*))
using *p(1)* **by** (*auto simp*: *emeasure_lborel_cbox_eq intro*!: *measure_subadditive_finite*
dest!: *p(1)[THEN tagged_division_ofD(4)]*)
finally show *M − 3 ∗ e ≤* (∑(*x*, *y*)∈*p* ∩ *s. content y*)
using ‹*0 < e*› **by** (*simp add*: *split_beta*)
qed (*use* ‹*a < b*› **in** *auto*)
also have ... = (∑(*x*,*k*) ∈ *p* ∩ *s. content k ∗ (b − a)*)
by (*simp add*: *sum_distrib_right split_beta'*)
also have ... ≤ (∑(*x*,*k*) ∈ *p* ∩ *s. content k ∗ (f (T ?F k) − f (T ?E k))*)
using *parts(3)* **by** (*auto intro*!: *sum_mono mult_left_mono diff_mono*)
also have ... = (∑(*x*,*k*) ∈ *p* ∩ *s. content k ∗ f (T ?F k)*) − (∑(*x*,*k*) ∈ *p* ∩ *s. content k ∗ f (T ?E k)*)
by (*auto intro*!: *sum.cong simp*: *field_simps sum_subtractf[symmetric]*)
also have ... = (∑(*x*,*k*) ∈ *?B. content k* $*_R$ *f x*) − (∑(*x*,*k*) ∈ *?A. content k* $*_R$ *f x*)
by (*subst (1 2) parts*) *auto*
also have ... ≤ *norm* ((∑(*x*,*k*) ∈ *?B. content k* $*_R$ *f x*) − (∑(*x*,*k*) ∈ *?A. content k* $*_R$ *f x*))
by *auto*
also have ... ≤ *e + e*
using *parts(1)[of ?E] parts(1)[of ?F]* **by** (*intro norm_diff_triangle_le[of _ I]*) *auto*
finally show *False*
using ‹*2 ∗ e < (b − a) ∗ (M − e ∗ 3)*› **by** (*auto simp*: *field_simps*)
qed

moreover have *?μ' ?E ≤ ?μ E ?μ' ?F ≤ ?μ E*
unfolding *outer_measure_of_eq*[*OF* ‹*E* ∈ *sets ?L*›, *symmetric*] **by** (*auto intro*!: *outer_measure_of_mono*)
ultimately show *min* (*?μ' ?E*) (*?μ' ?F*) < *?μ E*
unfolding *min_less_iff_disj* **by** (*auto simp*: *less_le*)
qed

lemma *has_integral_implies_lebesgue_measurable_real*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *real*
assumes *f*: (*f has_integral I*) Ω
shows (λ*x*. *f x* ∗ *indicator* Ω *x*) ∈ *lebesgue* $\rightarrow_M$ *borel*
proof −
define *B* :: *nat* ⇒ *'a set* **where** *B n* = *cbox* (− *real n* $*_R$ *One*) (*real n* $*_R$ *One*)
for *n*
show (λ*x*. *f x* ∗ *indicator* Ω *x*) ∈ *lebesgue* $\rightarrow_M$ *borel*
proof (*rule measurable_piecewise_restrict*)
have (⋃*n*. *box* (− *real n* $*_R$ *One*) (*real n* $*_R$ *One*)) ⊆ ⋃(*B* ‘ *UNIV*)
unfolding *B_def* **by** (*intro UN_mono box_subset_cbox order_refl*)
then show *countable* (*range B*) *space lebesgue* ⊆ ⋃(*B* ‘ *UNIV*)
by (*auto simp*: *B_def UN_box_eq_UNIV*)
next
fix Ω' **assume** Ω' ∈ *range B*
then obtain *n* **where** Ω': Ω' = *B n* **by** *auto*
then show Ω' ∩ *space lebesgue* ∈ *sets lebesgue*
by (*auto simp*: *B_def*)

have *f integrable_on* Ω
using *f* **by** *auto*
then have (λ*x*. *f x* ∗ *indicator* Ω *x*) *integrable_on* Ω
by (*auto simp*: *integrable_on_def cong*: *has_integral_cong*)
then have (λ*x*. *f x* ∗ *indicator* Ω *x*) *integrable_on* (Ω ∪ *B n*)
by (*rule integrable_on_superset*) *auto*
then have (λ*x*. *f x* ∗ *indicator* Ω *x*) *integrable_on B n*
unfolding *B_def* **by** (*rule integrable_on_subcbox*) *auto*
then show (λ*x*. *f x* ∗ *indicator* Ω *x*) ∈ *lebesgue_on* Ω' $\rightarrow_M$ *borel*
unfolding *B_def* Ω' **by** (*auto intro*: *has_integral_implies_lebesgue_measurable_cbox simp*: *integrable_on_def*)
qed
qed

lemma *has_integral_implies_lebesgue_measurable*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *euclidean_space*
assumes *f*: (*f has_integral I*) Ω
shows (λ*x*. *indicator* Ω *x* $*_R$ *f x*) ∈ *lebesgue* $\rightarrow_M$ *borel*
proof (*intro borel_measurable_euclidean_space*[**where** *'c*=*'b*, *THEN iffD2*] *ballI*)
fix *i* :: *'b* **assume** *i* ∈ *Basis*
have (λ*x*. (*f x* · *i*) ∗ *indicator* Ω *x*) ∈ *borel_measurable* (*completion lborel*)
using *has_integral_linear*[*OF f bounded_linear_inner_left*, *of i*]
by (*intro has_integral_implies_lebesgue_measurable_real*) (*auto simp*: *comp_def*)

```
  then show (λx. indicator Ω x *_R f x · i) ∈ borel_measurable (completion lborel)
    by (simp add: ac_simps)
qed
```

6.19.7 Equivalence Lebesgue integral on *lborel* and HK-integral

```
lemma has_integral_measure_lborel:
  fixes A :: 'a::euclidean_space set
  assumes A[measurable]: A ∈ sets borel and finite: emeasure lborel A < ∞
  shows ((λx. 1) has_integral measure lborel A) A
proof -
  { fix l u :: 'a
    have ((λx. 1) has_integral measure lborel (box l u)) (box l u)
    proof cases
      assume ∀ b∈Basis. l · b ≤ u · b
      then show ?thesis
        using has_integral_const[of 1::real l u]
      by (simp flip: has_integral_restrict[OF box_subset_cbox] add: has_integral_spike_interior)
    next
      assume ¬ (∀ b∈Basis. l · b ≤ u · b)
      then have box l u = {}
        unfolding box_eq_empty by (auto simp: not_le intro: less_imp_le)
      then show ?thesis
        by simp
    qed }
  note has_integral_box = this

  { fix a b :: 'a let ?M = λA. measure lborel (A ∩ box a b)
    have Int_stable (range (λ(a, b). box a b))
      by (auto simp: Int_stable_def box_Int_box)
    moreover have (range (λ(a, b). box a b)) ⊆ Pow UNIV
      by auto
    moreover have A ∈ sigma_sets UNIV (range (λ(a, b). box a b))
       using A unfolding borel_eq_box by simp
    ultimately have ((λx. 1) has_integral ?M A) (A ∩ box a b)
    proof (induction rule: sigma_sets_induct_disjoint)
      case (basic A) then show ?case
        by (auto simp: box_Int_box has_integral_box)
    next
      case empty then show ?case
        by simp
    next
      case (compl A)
      then have [measurable]: A ∈ sets borel
        by (simp add: borel_eq_box)

      have ((λx. 1) has_integral ?M (box a b)) (box a b)
        by (simp add: has_integral_box)
      moreover have ((λx. if x ∈ A ∩ box a b then 1 else 0) has_integral ?M A)
```

(*box a b*)
    by (*subst has_integral_restrict*) (*auto intro*: *compl*)
  ultimately have ((λ*x*. *1* − (*if x* ∈ *A* ∩ *box a b then 1 else 0*)) *has_integral* *?M* (*box a b*) − *?M A*) (*box a b*)
    by (*rule has_integral_diff*)
  then have ((λ*x*. (*if x* ∈ (*UNIV* − *A*) ∩ *box a b then 1 else 0*)) *has_integral* *?M* (*box a b*) − *?M A*) (*box a b*)
    by (*rule has_integral_cong*[*THEN iffD1*, *rotated 1*]) *auto*
  then have ((λ*x*. *1*) *has_integral ?M* (*box a b*) − *?M A*) ((*UNIV* − *A*) ∩ *box a b*)
    by (*subst* (*asm*) *has_integral_restrict*) *auto*
  also have *?M* (*box a b*) − *?M A* = *?M* (*UNIV* − *A*)
    by (*subst measure_Diff*[*symmetric*]) (*auto simp*: *emeasure_lborel_box_eq Diff_Int_distrib2*)
  finally show *?case* **.**
 next
  case (*union F*)
  then have [*measurable*]: ⋀*i*. *F i* ∈ *sets borel*
    by (*simp add*: *borel_eq_box subset_eq*)
  have ((λ*x*. *if x* ∈ ⋃(*F* ‘ *UNIV*) ∩ *box a b then 1 else 0*) *has_integral ?M* (⋃*i*. *F i*)) (*box a b*)
  proof (*rule has_integral_monotone_convergence_increasing*)
    let *?f* = λ*k x*. ∑*i*<*k*. *if x* ∈ *F i* ∩ *box a b then 1 else 0* :: *real*
    show ⋀*k*. (*?f k has_integral* (∑*i*<*k*. *?M* (*F i*))) (*box a b*)
      using *union.IH* **by** (*auto intro*!: *has_integral_sum simp del*: *Int_iff*)
    show ⋀*k x*. *?f k x* ≤ *?f* (*Suc k*) *x*
      by (*intro sum_mono2*) *auto*
    from *union*(*1*) **have** ∗: ⋀*x i j*. *x* ∈ *F i* ⟹ *x* ∈ *F j* ⟷ *j* = *i*
      by (*auto simp add*: *disjoint_family_on_def*)
    show (λ*k*. *?f k x*) ⟶ (*if x* ∈ ⋃(*F* ‘ *UNIV*) ∩ *box a b then 1 else 0*) **for** *x*
      by (*auto simp*: ∗ *sum.If_cases Iio_Int_singleton if_distrib LIMSEQ_if_less cong*: *if_cong*)
    have ∗: *emeasure lborel* ((⋃*x*. *F x*) ∩ *box a b*) ≤ *emeasure lborel* (*box a b*)
      by (*intro emeasure_mono*) *auto*

    with *union*(*1*) **show** (λ*k*. ∑*i*<*k*. *?M* (*F i*)) ⟶ *?M* (⋃*i*. *F i*)
      unfolding *sums_def*[*symmetric*] *UN_extend_simps*
        by (*intro measure_UNION*) (*auto simp*: *disjoint_family_on_def emeasure_lborel_box_eq top_unique*)
  qed
  then show *?case*
    by (*subst* (*asm*) *has_integral_restrict*) *auto*
 qed }
note ∗ = *this*

show *?thesis*
proof (*rule has_integral_monotone_convergence_increasing*)
 let *?B* = λ*n*::*nat*. *box* (− *real n* $*_R$ *One*) (*real n* $*_R$ *One*) :: ′*a set*

let *?f* = *λn::nat. λx. if x ∈ A ∩ ?B n then 1 else 0 :: real*
let *?M* = *λn. measure lborel (A ∩ ?B n)*

show *⋀n::nat. (?f n has_integral ?M n) A*
using ∗ **by** (*subst has_integral_restrict*) *simp_all*
show *⋀k x. ?f k x ≤ ?f (Suc k) x*
by (*auto simp*: *box_def*)
{ fix *x* **assume** *x ∈ A*
moreover have (*λk. indicator (A ∩ ?B k) x :: real*) ⟶ *indicator* (⋃*k::nat. A ∩ ?B k*) *x*
by (*intro LIMSEQ_indicator_incseq*) (*auto simp*: *incseq_def box_def*)
ultimately show (*λk. if x ∈ A ∩ ?B k then 1 else 0::real*) ⟶ *1*
by (*simp add*: *indicator_def UN_box_eq_UNIV*) **}**

have (*λn. emeasure lborel (A ∩ ?B n)*) ⟶ *emeasure lborel* (⋃*n::nat. A ∩ ?B n*)
by (*intro Lim_emeasure_incseq*) (*auto simp*: *incseq_def box_def*)
also have (*λn. emeasure lborel (A ∩ ?B n)*) = (*λn. measure lborel (A ∩ ?B n)*)
proof (*intro ext emeasure_eq_ennreal_measure*)
fix *n* **have** *emeasure lborel (A ∩ ?B n) ≤ emeasure lborel (?B n)*
by (*intro emeasure_mono*) *auto*
then show *emeasure lborel (A ∩ ?B n) ≠ top*
by (*auto simp*: *top_unique*)
qed
finally show (*λn. measure lborel (A ∩ ?B n)*) ⟶ *measure lborel A*
using *emeasure_eq_ennreal_measure*[*of lborel A*] *finite*
by (*simp add*: *UN_box_eq_UNIV less_top*)
qed
qed

lemma *nn_integral_has_integral*:
fixes *f::'a::euclidean_space ⇒ real*
assumes *f*: *f ∈ borel_measurable borel ⋀x. 0 ≤ f x* (*∫⁺x. f x ∂lborel*) = *ennreal r 0 ≤ r*
shows (*f has_integral r*) *UNIV*
using *f* **proof** (*induct f arbitrary*: *r rule*: *borel_measurable_induct_real*)
case (*set A*)
then have ((*λx. 1*) *has_integral measure lborel A*) *A*
by (*intro has_integral_measure_lborel*) (*auto simp*: *ennreal_indicator*)
with *set* **show** *?case*
by (*simp add*: *ennreal_indicator measure_def*) (*simp add*: *indicator_def*)
next
case (*mult g c*)
then have *ennreal c ∗* (*∫⁺ x. g x ∂lborel*) = *ennreal r*
by (*subst nn_integral_cmult*[*symmetric*]) (*auto simp*: *ennreal_mult*)
with ‹*0 ≤ r*› ‹*0 ≤ c*›
obtain *r'* **where** (*c = 0 ∧ r = 0*) ∨ (*0 ≤ r' ∧* (*∫⁺ x. ennreal (g x) ∂lborel*) = *ennreal r' ∧ r = c ∗ r'*)

by (*cases* $\int^+$ *x. ennreal* (*g x*) ∂*lborel rule*: *ennreal_cases*)
(*auto split*: *if_split_asm simp*: *ennreal_mult_top ennreal_mult*[*symmetric*])
with *mult* **show** *?case*
by (*auto intro*!: *has_integral_cmult_real*)
next
case (*add g h*)
then have ($\int^+$ *x. h x* + *g x* ∂*lborel*) = ($\int^+$ *x. h x* ∂*lborel*) + ($\int^+$ *x. g x* ∂*lborel*)
by (*simp add*: *nn_integral_add*)
with *add* **obtain** *a b* **where** *0* ≤ *a 0* ≤ *b* ($\int^+$ *x. h x* ∂*lborel*) = *ennreal a* ($\int^+$ *x. g x* ∂*lborel*) = *ennreal b r* = *a* + *b*
by (*cases* $\int^+$ *x. h x* ∂*lborel* $\int^+$ *x. g x* ∂*lborel rule*: *ennreal2_cases*)
(*auto simp*: *add_top nn_integral_add top_add simp flip*: *ennreal_plus*)
with *add* **show** *?case*
by (*auto intro*!: *has_integral_add*)
next
case (*seq U*)
note *seq*(*1*)[*measurable*] **and** *f*[*measurable*]

have *U_le_f*: *U i x* ≤ *f x* **for** *i x*
by (*metis* (*no_types*) *LIMSEQ_le_const UNIV_I incseq_def le_fun_def seq.hyps*(*4*) *seq.hyps*(*5*) *space_borel*)

{ **fix** *i*
have ($\int^+$*x. U i x* ∂*lborel*) ≤ ($\int^+$*x. f x* ∂*lborel*)
using *seq*(*2*) *f*(*2*) *U_le_f* **by** (*intro nn_integral_mono*) *simp*
then obtain *p* **where** ($\int^+$*x. U i x* ∂*lborel*) = *ennreal p p* ≤ *r 0* ≤ *p*
using *seq*(*6*) ‹*0*≤*r*› **by** (*cases* $\int^+$*x. U i x* ∂*lborel rule*: *ennreal_cases*) (*auto simp*: *top_unique*)
moreover note *seq*
ultimately have ∃*p.* ($\int^+$*x. U i x* ∂*lborel*) = *ennreal p* ∧ *0* ≤ *p* ∧ *p* ≤ *r* ∧ (*U i has_integral p*) *UNIV*
by *auto* }
then obtain *p* **where** *p*: ⋀*i.* ($\int^+$*x. ennreal* (*U i x*) ∂*lborel*) = *ennreal* (*p i*)
and *bnd*: ⋀*i. p i* ≤ *r* ⋀*i. 0* ≤ *p i*
and *U_int*: ⋀*i.*(*U i has_integral* (*p i*)) *UNIV* **by** *metis*

have *int_eq*: ⋀*i. integral UNIV* (*U i*) = *p i* **using** *U_int* **by** (*rule integral_unique*)

have ∗: *f integrable_on UNIV* ∧ (λ*k. integral UNIV* (*U k*)) ⟶ *integral UNIV f*
proof (*rule monotone_convergence_increasing*)
show ⋀*k. U k integrable_on UNIV* **using** *U_int* **by** *auto*
show ⋀*k x. x*∈*UNIV* ⟹ *U k x* ≤ *U* (*Suc k*) *x* **using** ‹*incseq U*› **by** (*auto simp*: *incseq_def le_fun_def*)
then show *bounded* (*range* (λ*k. integral UNIV* (*U k*)))
using *bnd int_eq* **by** (*auto simp*: *bounded_real intro*!: *exI*[*of _ r*])
show ⋀*x. x*∈*UNIV* ⟹ (λ*k. U k x*) ⟶ *f x*
using *seq* **by** *auto*

qed
moreover have $(\lambda i. (\int^+ x. U\ i\ x\ \partial lborel)) \longrightarrow (\int^+ x. f\ x\ \partial lborel)$
using *seq f(2) U_le_f* **by** (*intro nn_integral_dominated_convergence*[**where** *w=f*]) *auto*
ultimately have *integral UNIV f = r*
by (*auto simp add*: *bnd int_eq p seq intro*: *LIMSEQ_unique*)
with ∗ **show** *?case*
by (*simp add*: *has_integral_integral*)
qed

lemma *nn_integral_lborel_eq_integral*:
fixes *f*::′*a*::*euclidean_space* ⇒ *real*
assumes *f*: $f \in borel\_measurable\ borel\ \bigwedge x.\ 0 \le f\ x\ (\int^+ x.\ f\ x\ \partial lborel) < \infty$
shows $(\int^+ x.\ f\ x\ \partial lborel) = integral\ UNIV\ f$
proof −
from *f(3)* **obtain** *r* **where** *r*: $(\int^+ x.\ f\ x\ \partial lborel) = ennreal\ r\ 0 \le r$
by (*cases* $\int^+ x.\ f\ x\ \partial lborel$ *rule*: *ennreal_cases*) *auto*
then show *?thesis*
using *nn_integral_has_integral*[*OF f(1,2) r*] **by** (*simp add*: *integral_unique*)
qed

lemma *nn_integral_integrable_on*:
fixes *f*::′*a*::*euclidean_space* ⇒ *real*
assumes *f*: $f \in borel\_measurable\ borel\ \bigwedge x.\ 0 \le f\ x\ (\int^+ x.\ f\ x\ \partial lborel) < \infty$
shows *f integrable_on UNIV*
proof −
from *f(3)* **obtain** *r* **where** *r*: $(\int^+ x.\ f\ x\ \partial lborel) = ennreal\ r\ 0 \le r$
by (*cases* $\int^+ x.\ f\ x\ \partial lborel$ *rule*: *ennreal_cases*) *auto*
then show *?thesis*
by (*intro has_integral_integrable*[**where** *i=r*] *nn_integral_has_integral*[**where** *r=r*] *f*)
qed

lemma *nn_integral_has_integral_lborel*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ *real*
assumes *f_borel*: *f* ∈ *borel_measurable borel* **and** *nonneg*: $\bigwedge x.\ 0 \le f\ x$
assumes *I*: (*f has_integral I*) *UNIV*
shows $integral^N\ lborel\ f = I$
proof −
from *f_borel* **have** $(\lambda x.\ ennreal\ (f\ x)) \in borel\_measurable\ lborel$ **by** *auto*
from *borel_measurable_implies_simple_function_sequence*′[*OF this*]
obtain *F* **where** *F*: $\bigwedge i.$ *simple_function lborel* (*F i*) *incseq F*
$\bigwedge i\ x.\ F\ i\ x < top\ \bigwedge x.\ (SUP\ i.\ F\ i\ x) = ennreal\ (f\ x)$
by *blast*
then have [*measurable*]: $\bigwedge i.\ F\ i \in borel\_measurable\ lborel$
by (*metis borel_measurable_simple_function*)
let $?B = \lambda i{::}nat.\ box\ (-\ (real\ i *_R One))\ (real\ i *_R One) :: 'a\ set$

have $0 \le I$

```
    using I by (rule has_integral_nonneg) (simp add: nonneg)

  have F_le_f: enn2real (F i x) ≤ f x for i x
    using F(3,4)[where x=x] nonneg SUP_upper[of i UNIV λi. F i x]
    by (cases F i x rule: ennreal_cases) auto
  let ?F = λi x. F i x * indicator (?B i) x
  have (∫+ x. ennreal (f x) ∂lborel) = (SUP i. integral^N lborel (λx. ?F i x))
  proof (subst nn_integral_monotone_convergence_SUP[symmetric])
    { fix x
      obtain j where j: x ∈ ?B j
        using UN_box_eq_UNIV by auto

      have ennreal (f x) = (SUP i. F i x)
        using F(4)[of x] nonneg[of x] by (simp add: max_def)
      also have ... = (SUP i. ?F i x)
      proof (rule SUP_eq)
        fix i show ∃j∈UNIV. F i x ≤ ?F j x
          using j F(2)
          by (intro bexI[of _ max i j])
             (auto split: split_max split_indicator simp: incseq_def le_fun_def box_def)
      qed (auto intro!: F split: split_indicator)
      finally have ennreal (f x) = (SUP i. ?F i x) . }
    then show (∫+ x. ennreal (f x) ∂lborel) = (∫+ x. (SUP i. ?F i x) ∂lborel)
      by simp
  qed (insert F, auto simp: incseq_def le_fun_def box_def split: split_indicator)
  also have ... ≤ ennreal I
  proof (rule SUP_least)
    fix i :: nat
    have finite_F: (∫+ x. ennreal (enn2real (F i x) * indicator (?B i) x) ∂lborel)
< ∞
    proof (rule nn_integral_bound_simple_function)
      have emeasure lborel {x ∈ space lborel. ennreal (enn2real (F i x) * indicator
(?B i) x) ≠ 0} ≤
        emeasure lborel (?B i)
        by (intro emeasure_mono) (auto split: split_indicator)
       then show emeasure lborel {x ∈ space lborel. ennreal (enn2real (F i x) *
indicator (?B i) x) ≠ 0} < ∞
        by (auto simp: less_top[symmetric] top_unique)
    qed (auto split: split_indicator
          intro!: F simple_function_compose1[where g=enn2real] simple_function_ennreal)

    have int_F: (λx. enn2real (F i x) * indicator (?B i) x) integrable_on UNIV
      using F(4) finite_F
     by (intro nn_integral_integrable_on) (auto split: split_indicator simp: enn2real_nonneg)

    have (∫+ x. F i x * indicator (?B i) x ∂lborel) =
      (∫+ x. ennreal (enn2real (F i x) * indicator (?B i) x) ∂lborel)
      using F(3,4)
     by (intro nn_integral_cong) (auto simp: image_iff eq_commute split: split_indicator)
```

also have . . . = *ennreal* (*integral UNIV* (λ*x*. *enn2real* (*F i x*) ∗ *indicator* (*?B i*) *x*))
using *F*
by (*intro nn_integral_lborel_eq_integral*[*OF* _ _ *finite_F*])
(*auto split*: *split_indicator intro*: *enn2real_nonneg*)
also have . . . ≤ *ennreal I*
by (*auto intro*!: *has_integral_le*[*OF integrable_integral*[*OF int_F*] *I*] *nonneg F_le_f*
simp: ‹*0* ≤ *I*› *split*: *split_indicator*)
finally show (∫⁺ *x*. *F i x* ∗ *indicator* (*?B i*) *x* ∂*lborel*) ≤ *ennreal I* **.**
qed
finally have (∫⁺ *x*. *ennreal* (*f x*) ∂*lborel*) < ∞
by (*auto simp*: *less_top*[*symmetric*] *top_unique*)
from *nn_integral_lborel_eq_integral*[*OF assms*(*1*,*2*) *this*] *I* **show** *?thesis*
by (*simp add*: *integral_unique*)
qed

lemma *has_integral_iff_emeasure_lborel*:
fixes *A* :: ′*a*::*euclidean_space set*
assumes *A*[*measurable*]: *A* ∈ *sets borel* **and** [*simp*]: *0* ≤ *r*
shows ((λ*x*. *1*) *has_integral r*) *A* ⟷ *emeasure lborel A* = *ennreal r*
proof (*cases emeasure lborel A* = ∞)
case *emeasure_A*: *True*
have ¬ (λ*x*. *1*::*real*) *integrable_on A*
proof
assume *int*: (λ*x*. *1*::*real*) *integrable_on A*
then have (*indicator A*::′*a* ⇒ *real*) *integrable_on UNIV*
unfolding *indicator_def*[*abs_def*] *integrable_restrict_UNIV* **.**
then obtain *r* **where** ((*indicator A*::′*a*⇒*real*) *has_integral r*) *UNIV*
by *auto*
from *nn_integral_has_integral_lborel*[*OF* _ _ *this*] *emeasure_A* **show** *False*
by (*simp add*: *ennreal_indicator*)
qed
with *emeasure_A* **show** *?thesis*
by *auto*
next
case *False*
then have ((λ*x*. *1*) *has_integral measure lborel A*) *A*
by (*simp add*: *has_integral_measure_lborel less_top*)
with *False* **show** *?thesis*
by (*auto simp*: *emeasure_eq_ennreal_measure has_integral_unique*)
qed

lemma *ennreal_max_0*: *ennreal* (*max 0 x*) = *ennreal x*
by (*auto simp*: *max_def ennreal_neg*)

lemma *has_integral_integral_real*:
fixes *f*::′*a*::*euclidean_space* ⇒ *real*
assumes *f*: *integrable lborel f*

```
  shows (f has_integral (integral^L lborel f)) UNIV
proof -
  from integrableE[OF f] obtain r q
    where 0 ≤ r 0 ≤ q
      and r: (∫+ x. ennreal (max 0 (f x)) ∂lborel) = ennreal r
      and q: (∫+ x. ennreal (max 0 (- f x)) ∂lborel) = ennreal q
      and f: f ∈ borel_measurable lborel and eq: integral^L lborel f = r - q
    unfolding ennreal_max_0 by auto
   then have ((λx. max 0 (f x)) has_integral r) UNIV ((λx. max 0 (- f x))
has_integral q) UNIV
     using nn_integral_has_integral[OF _ _ r] nn_integral_has_integral[OF _ _ q] by
auto
  note has_integral_diff[OF this]
  moreover have (λx. max 0 (f x) - max 0 (- f x)) = f
    by auto
  ultimately show ?thesis
    by (simp add: eq)
qed

lemma has_integral_AE:
  assumes ae: AE x in lborel. x ∈ Ω ⟶ f x = g x
  shows (f has_integral x) Ω = (g has_integral x) Ω
proof -
  from ae obtain N
    where N: N ∈ sets borel emeasure lborel N = 0 {x. ¬ (x ∈ Ω ⟶ f x = g x)}
⊆ N
    by (auto elim!: AE_E)
  then have not_N: AE x in lborel. x ∉ N
    by (simp add: AE_iff_measurable)
  show ?thesis
  proof (rule has_integral_spike_eq[symmetric])
    show ⋀x. x∈Ω - N ⟹ f x = g x using N(3) by auto
    show negligible N
      unfolding negligible_def
    proof (intro allI)
      fix a b :: 'a
      let ?F = λx::'a. if x ∈ cbox a b then indicator N x else 0 :: real
      have integrable lborel ?F = integrable lborel (λx::'a. 0::real)
        using not_N N(1) by (intro integrable_cong_AE) auto
      moreover have (LINT x|lborel. ?F x) = (LINT x::'a|lborel. 0::real)
        using not_N N(1) by (intro integral_cong_AE) auto
      ultimately have (?F has_integral 0) UNIV
        using has_integral_integral_real[of ?F] by simp
      then show (indicator N has_integral (0::real)) (cbox a b)
        unfolding has_integral_restrict_UNIV .
    qed
  qed
qed
```

lemma *nn_integral_has_integral_lebesgue*:
fixes f :: *'a::euclidean_space* ⇒ *real*
assumes *nonneg*: $\bigwedge x.\ 0 \leq f\ x$ **and** *I*: (f *has_integral* I) Ω
shows *integral*^{N} *lborel* (λ*x*. *indicator* Ω $x * f\ x$) = I
proof −
from *I* **have** (λ*x*. *indicator* Ω $x *_R f\ x$) ∈ *lebesgue* $\rightarrow_M$ *borel*
by (*rule has_integral_implies_lebesgue_measurable*)
then obtain f' :: *'a* ⇒ *real*
where [*measurable*]: f' ∈ *borel* $\rightarrow_M$ *borel* **and** *eq*: *AE x in lborel*. *indicator* Ω $x * f\ x = f'\ x$
by (*auto dest*: *completion_ex_borel_measurable_real*)

from *I* **have** ((λ*x*. *abs* (*indicator* Ω $x * f\ x$)) *has_integral* I) *UNIV*
using *nonneg* **by** (*simp add*: *indicator_def if_distrib*[*of* λ*x*. $x * f\ y$ **for** *y*] *cong*: *if_cong*)
also have ((λ*x*. *abs* (*indicator* Ω $x * f\ x$)) *has_integral* I) *UNIV* ⟷ ((λ*x*. *abs* ($f'\ x$)) *has_integral* I) *UNIV*
using *eq* **by** (*intro has_integral_AE*) *auto*
finally have *integral*^{N} *lborel* (λ*x*. *abs* ($f'\ x$)) = I
by (*rule nn_integral_has_integral_lborel*[*rotated 2*]) *auto*
also have *integral*^{N} *lborel* (λ*x*. *abs* ($f'\ x$)) = *integral*^{N} *lborel* (λ*x*. *abs* (*indicator* Ω $x * f\ x$))
using *eq* **by** (*intro nn_integral_cong_AE*) *auto*
finally show *?thesis*
using *nonneg* **by** *auto*
qed

lemma *has_integral_iff_nn_integral_lebesgue*:
assumes f: $\bigwedge x.\ 0 \leq f\ x$
shows (f *has_integral* r) *UNIV* ⟷ (f ∈ *lebesgue* $\rightarrow_M$ *borel* ∧ *integral*^{N} *lebesgue* $f = r$ ∧ $0 \leq r$) (**is** *?I* = *?N*)
proof
assume *?I*
have $0 \leq r$
using *has_integral_nonneg*[*OF* ‹*?I*›] f **by** *auto*
then show *?N*
using *nn_integral_has_integral_lebesgue*[*OF f* ‹*?I*›]
has_integral_implies_lebesgue_measurable[*OF* ‹*?I*›]
by (*auto simp*: *nn_integral_completion*)
next
assume *?N*
then obtain f' **where** f': f' ∈ *borel* $\rightarrow_M$ *borel AE x in lborel*. $f\ x = f'\ x$
by (*auto dest*: *completion_ex_borel_measurable_real*)
moreover have ($\int^+ x$. *ennreal* $|f'\ x|$ ∂*lborel*) = ($\int^+ x$. *ennreal* $|f\ x|$ ∂*lborel*)
using f' **by** (*intro nn_integral_cong_AE*) *auto*
moreover have ((λ*x*. $|f'\ x|$) *has_integral* r) *UNIV* ⟷ ((λ*x*. $|f\ x|$) *has_integral* r) *UNIV*
using f' **by** (*intro has_integral_AE*) *auto*
moreover note *nn_integral_has_integral*[*of* λ*x*. $|f'\ x|$ r] ‹*?N*›

ultimately show *?I*
using *f* **by** (*auto simp*: *nn_integral_completion*)
qed

context
fixes *f*::'*a*::*euclidean_space* ⇒ '*b*::*euclidean_space*
begin

lemma *has_integral_integral_lborel*:
assumes *f*: *integrable lborel f*
shows (*f has_integral* (*integral*^{L} *lborel f*)) *UNIV*
proof −
have ((λ*x*. ∑ *b*∈*Basis*. (*f x* · *b*) $*_R$ *b*) *has_integral* (∑ *b*∈*Basis*. *integral*^{L} *lborel* (λ*x*. *f x* · *b*) $*_R$ *b*)) *UNIV*
using *f* **by** (*intro has_integral_sum finite_Basis ballI has_integral_scaleR_left has_integral_integral_real*) *auto*
also have *eq_f*: (λ*x*. ∑ *b*∈*Basis*. (*f x* · *b*) $*_R$ *b*) = *f*
by (*simp add*: *fun_eq_iff euclidean_representation*)
also have (∑ *b*∈*Basis*. *integral*^{L} *lborel* (λ*x*. *f x* · *b*) $*_R$ *b*) = *integral*^{L} *lborel f*
using *f* **by** (*subst* (*2*) *eq_f*[*symmetric*]) *simp*
finally show *?thesis* **.**
qed

lemma *integrable_on_lborel*: *integrable lborel f* ⟹ *f integrable_on UNIV*
using *has_integral_integral_lborel* **by** *auto*

lemma *integral_lborel*: *integrable lborel f* ⟹ *integral UNIV f* = (∫ *x*. *f x* ∂*lborel*)
using *has_integral_integral_lborel* **by** *auto*

end

context
begin

private lemma *has_integral_integral_lebesgue_real*:
fixes *f* :: '*a*::*euclidean_space* ⇒ *real*
assumes *f*: *integrable lebesgue f*
shows (*f has_integral* (*integral*^{L} *lebesgue f*)) *UNIV*
proof −
obtain *f'* **where** *f'*: *f'* ∈ *borel* $\rightarrow_M$ *borel AE x in lborel*. *f x* = *f' x*
using *completion_ex_borel_measurable_real*[*OF borel_measurable_integrable*[*OF f*]] **by** *auto*
moreover have (∫$^+$ *x*. *ennreal* (*norm* (*f x*)) ∂*lborel*) = (∫$^+$ *x*. *ennreal* (*norm* (*f' x*)) ∂*lborel*)
using *f'* **by** (*intro nn_integral_cong_AE*) *auto*
ultimately have *integrable lborel f'*
using *f* **by** (*auto simp*: *integrable_iff_bounded nn_integral_completion cong*: *nn_integral_cong_AE*)
note *has_integral_integral_real*[*OF this*]

moreover have *integral*^{L} *lebesgue f* = *integral*^{L} *lebesgue f'*
  using *f' f* **by** (*intro integral_cong_AE*) (*auto intro*: *AE_completion measurable_completion*)
moreover have *integral*^{L} *lebesgue f'* = *integral*^{L} *lborel f'*
  using *f'* **by** (*simp add*: *integral_completion*)
moreover have (*f' has_integral integral*^{L} *lborel f'*) *UNIV* ⟷ (*f has_integral integral*^{L} *lborel f'*) *UNIV*
  using *f'* **by** (*intro has_integral_AE*) *auto*
ultimately show *?thesis*
  by *auto*
qed

lemma *has_integral_integral_lebesgue*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *f*: *integrable lebesgue f*
shows (*f has_integral* (*integral*^{L} *lebesgue f*)) *UNIV*
proof −
have ((λ*x*. ∑ *b*∈*Basis*. (*f x* · *b*) $*_R$ *b*) *has_integral* (∑ *b*∈*Basis*. *integral*^{L} *lebesgue* (λ*x*. *f x* · *b*) $*_R$ *b*)) *UNIV*
  using *f* **by** (*intro has_integral_sum finite_Basis ballI has_integral_scaleR_left has_integral_integral_lebesgue_real*) *auto*
also have *eq_f*: (λ*x*. ∑ *b*∈*Basis*. (*f x* · *b*) $*_R$ *b*) = *f*
  by (*simp add*: *fun_eq_iff euclidean_representation*)
also have (∑ *b*∈*Basis*. *integral*^{L} *lebesgue* (λ*x*. *f x* · *b*) $*_R$ *b*) = *integral*^{L} *lebesgue f*
  using *f* **by** (*subst* (*2*) *eq_f*[*symmetric*]) *simp*
finally show *?thesis* **.**
qed

lemma *has_integral_integral_lebesgue_on*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *integrable* (*lebesgue_on S*) *f S* ∈ *sets lebesgue*
shows (*f has_integral* (*integral*^{L} (*lebesgue_on S*) *f*)) *S*
proof −
let *?f* = λ*x*. *if x* ∈ *S then f x else 0*
have *integrable lebesgue* (λ*x*. *indicat_real S x* $*_R$ *f x*)
  using *indicator_scaleR_eq_if* [*of S _ f*] *assms*
by (*metis* (*full_types*) *integrable_restrict_space sets.Int_space_eq2*)
then have *integrable lebesgue ?f*
  using *indicator_scaleR_eq_if* [*of S _ f*] *assms* **by** *auto*
then have (*?f has_integral* (*integral*^{L} *lebesgue ?f*)) *UNIV*
  by (*rule has_integral_integral_lebesgue*)
then have (*f has_integral* (*integral*^{L} *lebesgue ?f*)) *S*
  using *has_integral_restrict_UNIV* **by** *blast*
moreover
have *S* ∩ *space lebesgue* ∈ *sets lebesgue*
  by (*simp add*: *assms*)
then have (*integral*^{L} *lebesgue ?f*) = (*integral*^{L} (*lebesgue_on S*) *f*)
  by (*simp add*: *integral_restrict_space indicator_scaleR_eq_if*)

ultimately show *?thesis*
by *auto*
qed

lemma *lebesgue_integral_eq_integral*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *integrable* (*lebesgue_on* S) f S ∈ *sets lebesgue*
shows *integral*^{L} (*lebesgue_on* S) f = *integral* S f
by (*metis has_integral_integral_lebesgue_on assms integral_unique*)

lemma *integrable_on_lebesgue*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
shows *integrable lebesgue* f ⟹ f *integrable_on UNIV*
using *has_integral_integral_lebesgue* **by** *auto*

lemma *integral_lebesgue*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
shows *integrable lebesgue* f ⟹ *integral UNIV* f = (∫ x. f x ∂*lebesgue*)
using *has_integral_integral_lebesgue* **by** *auto*

end

6.19.8 Absolute integrability (this is the same as Lebesgue integrability)

translations
LBINT x. f == *CONST lebesgue_integral CONST lborel* (λx. f)

translations
LBINT x:A. f == *CONST set_lebesgue_integral CONST lborel* A (λx. f)

lemma *set_integral_reflect*:
fixes S **and** f :: *real* ⇒ *'a* :: {*banach*, *second_countable_topology*}
shows (*LBINT* x : S. f x) = (*LBINT* x : {x. − x ∈ S}. f (− x))
unfolding *set_lebesgue_integral_def*
by (*subst lborel_integral_real_affine*[**where** *c*=−*1* **and** *t*=*0*])
(*auto intro*!: *Bochner_Integration.integral_cong split*: *split_indicator*)

lemma *borel_integrable_atLeastAtMost'*:
fixes f :: *real* ⇒ *'a*::{*banach*, *second_countable_topology*}
assumes f: *continuous_on* {*a..b*} f
shows *set_integrable lborel* {*a..b*} f
unfolding *set_integrable_def*
by (*intro borel_integrable_compact compact_Icc* f)

lemma *integral_FTC_atLeastAtMost*:
fixes f :: *real* ⇒ *'a* :: *euclidean_space*
assumes $a \leq b$
and F: ⋀x. $a \leq x$ ⟹ $x \leq b$ ⟹ (F *has_vector_derivative* f x) (*at* x *within* {*a*

.. *b*})
  and *f*: *continuous_on* {*a* .. *b*} *f*
 shows *integral*^{L} *lborel* (λ*x*. *indicator* {*a* .. *b*} *x* $*_R$ *f x*) = *F b* − *F a*
proof −
 let *?f* = λ*x*. *indicator* {*a* .. *b*} *x* $*_R$ *f x*
 have (*?f has_integral* (∫ *x*. *?f x* ∂*lborel*)) *UNIV*
  using *borel_integrable_atLeastAtMost'*[*OF f*]
  unfolding *set_integrable_def* **by** (*rule has_integral_integral_lborel*)
 moreover
 have (*f has_integral F b* − *F a*) {*a* .. *b*}
  by (*intro fundamental_theorem_of_calculus ballI assms*) *auto*
 then have (*?f has_integral F b* − *F a*) {*a* .. *b*}
  by (*subst has_integral_cong*[**where** *g*=*f*]) *auto*
 then have (*?f has_integral F b* − *F a*) *UNIV*
  by (*intro has_integral_on_superset*[**where** *T*=*UNIV* **and** *S*={*a*..*b*}]) *auto*
 ultimately show *integral*^{L} *lborel ?f* = *F b* − *F a*
  by (*rule has_integral_unique*)
qed

lemma *set_borel_integral_eq_integral*:
 fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
 assumes *set_integrable lborel S f*
 shows *f integrable_on S LINT x* : *S* | *lborel*. *f x* = *integral S f*
proof −
 let *?f* = λ*x*. *indicator S x* $*_R$ *f x*
 have (*?f has_integral LINT x* : *S* | *lborel*. *f x*) *UNIV*
  using *assms has_integral_integral_lborel*
  unfolding *set_integrable_def set_lebesgue_integral_def* **by** *blast*
 hence *1*: (*f has_integral* (*set_lebesgue_integral lborel S f*)) *S*
  by (*simp add*: *indicator_scaleR_eq_if*)
 thus *f integrable_on S*
  by (*auto simp add*: *integrable_on_def*)
 with *1* **have** (*f has_integral* (*integral S f*)) *S*
  by (*intro integrable_integral*, *auto simp add*: *integrable_on_def*)
 thus *LINT x* : *S* | *lborel*. *f x* = *integral S f*
  by (*intro has_integral_unique* [*OF 1*])
qed

lemma *has_integral_set_lebesgue*:
 fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
 assumes *f*: *set_integrable lebesgue S f*
 shows (*f has_integral* (*LINT x*:*S*|*lebesgue*. *f x*)) *S*
 using *has_integral_integral_lebesgue f*
  by (*fastforce simp add*: *set_integrable_def set_lebesgue_integral_def indicator_def if_distrib*[*of* λ*x*. *x* $*_R$ *f* _] *cong*: *if_cong*)

lemma *set_lebesgue_integral_eq_integral*:
 fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
 assumes *f*: *set_integrable lebesgue S f*

shows *f integrable_on S LINT x:S | lebesgue. f x = integral S f*
using *has_integral_set_lebesgue*[*OF f*] **by** (*auto simp*: *integral_unique integrable_on_def*)

lemma *lmeasurable_iff_has_integral*:
S ∈ *lmeasurable* ⟷ ((*indicator S*) *has_integral measure lebesgue S*) *UNIV*
by (*subst has_integral_iff_nn_integral_lebesgue*)
(*auto simp*: *ennreal_indicator emeasure_eq_measure2 borel_measurable_indicator_iff intro*!: *fmeasurableI*)

abbreviation
absolutely_integrable_on :: (*'a*::*euclidean_space* ⇒ *'b*::{*banach*, *second_countable_topology*}) ⇒ *'a set* ⇒ *bool*
(**infixr** *absolutely'_integrable'_on 46*)
where *f absolutely_integrable_on s* ≡ *set_integrable lebesgue s f*

lemma *absolutely_integrable_zero* [*simp*]: (λ*x*. *0*) *absolutely_integrable_on S*
by (*simp add*: *set_integrable_def*)

lemma *absolutely_integrable_on_def*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
shows *f absolutely_integrable_on S* ⟷ *f integrable_on S* ∧ (λ*x*. *norm* (*f x*)) *integrable_on S*
proof *safe*
assume *f*: *f absolutely_integrable_on S*
then have *nf*: *integrable lebesgue* (λ*x*. *norm* (*indicator S x* $*_R$ *f x*))
using *integrable_norm set_integrable_def* **by** *blast*
show *f integrable_on S*
by (*rule set_lebesgue_integral_eq_integral*[*OF f*])
have (λ*x*. *norm* (*indicator S x* $*_R$ *f x*)) = (λ*x*. *if x* ∈ *S then norm* (*f x*) *else 0*)
by *auto*
with *integrable_on_lebesgue*[*OF nf*] **show** (λ*x*. *norm* (*f x*)) *integrable_on S*
by (*simp add*: *integrable_restrict_UNIV*)
next
assume *f*: *f integrable_on S* **and** *nf*: (λ*x*. *norm* (*f x*)) *integrable_on S*
show *f absolutely_integrable_on S*
unfolding *set_integrable_def*
proof (*rule integrableI_bounded*)
show (λ*x*. *indicator S x* $*_R$ *f x*) ∈ *borel_measurable lebesgue*
using *f has_integral_implies_lebesgue_measurable*[*of f _ S*] **by** (*auto simp*: *integrable_on_def*)
show ($\int^+$ *x*. *ennreal* (*norm* (*indicator S x* $*_R$ *f x*)) ∂*lebesgue*) < ∞
using *nf nn_integral_has_integral_lebesgue*[*of* λ*x*. *norm* (*f x*) _ *S*]
by (*auto simp*: *integrable_on_def nn_integral_completion*)
qed
qed

lemma *integrable_on_lebesgue_on*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*

assumes *f*: *integrable* (*lebesgue_on S*) *f* **and** *S*: *S* ∈ *sets lebesgue*
shows *f integrable_on S*
proof −
have *integrable lebesgue* (λ*x*. *indicator S x* $*_R$ *f x*)
using *S f inf_top.comm_neutral integrable_restrict_space* **by** *blast*
then show *?thesis*
using *absolutely_integrable_on_def set_integrable_def* **by** *blast*
qed

lemma *absolutely_integrable_imp_integrable*:
assumes *f absolutely_integrable_on S S* ∈ *sets lebesgue*
shows *integrable* (*lebesgue_on S*) *f*
by (*meson assms integrable_restrict_space set_integrable_def sets.Int sets.top*)

lemma *absolutely_integrable_on_null* [*intro*]:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
shows *content* (*cbox a b*) = *0* ⟹ *f absolutely_integrable_on* (*cbox a b*)
by (*auto simp*: *absolutely_integrable_on_def*)

lemma *absolutely_integrable_on_open_interval*:
fixes *f* :: ′*a* :: *euclidean_space* ⇒ ′*b* :: *euclidean_space*
shows *f absolutely_integrable_on box a b* ⟷
f absolutely_integrable_on cbox a b
by (*auto simp*: *integrable_on_open_interval absolutely_integrable_on_def*)

lemma *absolutely_integrable_restrict_UNIV*:
(λ*x*. *if x* ∈ *S then f x else 0*) *absolutely_integrable_on UNIV* ⟷ *f absolutely_integrable_on S*
unfolding *set_integrable_def*
by (*intro arg_cong2*[**where** *f*=*integrable*]) *auto*

lemma *absolutely_integrable_onI*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
shows *f integrable_on S* ⟹ (λ*x*. *norm* (*f x*)) *integrable_on S* ⟹ *f absolutely_integrable_on S*
unfolding *absolutely_integrable_on_def* **by** *auto*

lemma *nonnegative_absolutely_integrable_1*:
fixes *f* :: ′*a* :: *euclidean_space* ⇒ *real*
assumes *f*: *f integrable_on A* **and** ⋀*x*. *x* ∈ *A* ⟹ *0* ≤ *f x*
shows *f absolutely_integrable_on A*
by (*rule absolutely_integrable_onI* [*OF f*]) (*use assms* **in** ‹*simp add*: *integrable_eq*›)

lemma *absolutely_integrable_on_iff_nonneg*:
fixes *f* :: ′*a* :: *euclidean_space* ⇒ *real*
assumes ⋀*x*. *x* ∈ *S* ⟹ *0* ≤ *f x* **shows** *f absolutely_integrable_on S* ⟷ *f integrable_on S*
proof −
{ **assume** *f integrable_on S*

then have (*λx. if x ∈ S then f x else 0*) *integrable_on UNIV*
by (*simp add*: *integrable_restrict_UNIV*)
then have (*λx. if x ∈ S then f x else 0*) *absolutely_integrable_on UNIV*
using ‹*f integrable_on S*› *absolutely_integrable_restrict_UNIV assms nonnegative_absolutely_integrable_1* **by** *blast*
then have *f absolutely_integrable_on S*
using *absolutely_integrable_restrict_UNIV* **by** *blast*
}
then show *?thesis*
unfolding *absolutely_integrable_on_def* **by** *auto*
qed

lemma *absolutely_integrable_on_scaleR_iff*:
fixes *f* :: *'a::euclidean_space ⇒ 'b::euclidean_space*
shows
(*λx. c ∗R f x*) *absolutely_integrable_on S* ⟷
c = *0* ∨ *f absolutely_integrable_on S*
proof (*cases c=0*)
case *False*
then show *?thesis*
unfolding *absolutely_integrable_on_def*
by (*simp add*: *norm_mult*)
qed *auto*

lemma *absolutely_integrable_spike*:
fixes *f* :: *'a::euclidean_space ⇒ 'b::euclidean_space*
assumes *f absolutely_integrable_on T* **and** *S*: *negligible S* ⋀*x. x ∈ T − S* ⟹ *g x* = *f x*
shows *g absolutely_integrable_on T*
using *assms integrable_spike*
unfolding *absolutely_integrable_on_def* **by** *metis*

lemma *absolutely_integrable_negligible*:
fixes *f* :: *'a::euclidean_space ⇒ 'b::euclidean_space*
assumes *negligible S*
shows *f absolutely_integrable_on S*
using *assms* **by** (*simp add*: *absolutely_integrable_on_def integrable_negligible*)

lemma *absolutely_integrable_spike_eq*:
fixes *f* :: *'a::euclidean_space ⇒ 'b::euclidean_space*
assumes *negligible S* ⋀*x. x ∈ T − S* ⟹ *g x* = *f x*
shows (*f absolutely_integrable_on T* ⟷ *g absolutely_integrable_on T*)
using *assms* **by** (*blast intro*: *absolutely_integrable_spike sym*)

lemma *absolutely_integrable_spike_set_eq*:
fixes *f* :: *'a::euclidean_space ⇒ 'b::euclidean_space*
assumes *negligible* {*x ∈ S − T. f x ≠ 0*} *negligible* {*x ∈ T − S. f x ≠ 0*}
shows (*f absolutely_integrable_on S* ⟷ *f absolutely_integrable_on T*)
proof −

have (*λx. if x ∈ S then f x else 0*) *absolutely_integrable_on UNIV* ⟷
(*λx. if x ∈ T then f x else 0*) *absolutely_integrable_on UNIV*
proof (*rule absolutely_integrable_spike_eq*)
show *negligible* ({*x* ∈ *S* − *T*. *f x* ≠ *0*} ∪ {*x* ∈ *T* − *S*. *f x* ≠ *0*})
by (*rule negligible_Un* [*OF assms*])
qed *auto*
with *absolutely_integrable_restrict_UNIV* **show** *?thesis*
by *blast*
qed

lemma *absolutely_integrable_spike_set*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f*: *f absolutely_integrable_on S* **and** *neg*: *negligible* {*x* ∈ *S* − *T*. *f x* ≠ *0*} *negligible* {*x* ∈ *T* − *S*. *f x* ≠ *0*}
shows *f absolutely_integrable_on T*
using *absolutely_integrable_spike_set_eq f neg* **by** *blast*

lemma *absolutely_integrable_reflect*[*simp*]:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
shows (*λx. f*(−*x*)) *absolutely_integrable_on cbox* (−*b*) (−*a*) ⟷ *f absolutely_integrable_on cbox a b*
unfolding *absolutely_integrable_on_def*
by (*metis* (*mono_tags*, *lifting*) *integrable_eq integrable_reflect*)

lemma *absolutely_integrable_reflect_real*[*simp*]:
fixes *f* :: *real* ⇒ *'b*::*euclidean_space*
shows (*λx. f*(−*x*)) *absolutely_integrable_on* {−*b* .. −*a*} ⟷ *f absolutely_integrable_on* {*a*..*b*::*real*}
unfolding *box_real*[*symmetric*] **by** (*rule absolutely_integrable_reflect*)

lemma *absolutely_integrable_on_subcbox*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
shows ⟦*f absolutely_integrable_on S*; *cbox a b* ⊆ *S*⟧ ⟹ *f absolutely_integrable_on cbox a b*
by (*meson absolutely_integrable_on_def integrable_on_subcbox*)

lemma *absolutely_integrable_on_subinterval*:
fixes *f* :: *real* ⇒ *'b*::*euclidean_space*
shows ⟦*f absolutely_integrable_on S*; {*a*..*b*} ⊆ *S*⟧ ⟹ *f absolutely_integrable_on* {*a*..*b*}
using *absolutely_integrable_on_subcbox* **by** *fastforce*

lemma *integrable_subinterval*:
fixes *f* :: *real* ⇒ *'a*::*euclidean_space*
assumes *integrable* (*lebesgue_on* {*a*..*b*}) *f*
and {*c*..*d*} ⊆ {*a*..*b*}
shows *integrable* (*lebesgue_on* {*c*..*d*}) *f*
proof (*rule absolutely_integrable_imp_integrable*)
show *f absolutely_integrable_on* {*c*..*d*}

```
  proof –
    have f integrable_on {c..d}
      using assms integrable_on_lebesgue_on integrable_on_subinterval by fastforce
    moreover have (λx. norm (f x)) integrable_on {c..d}
    proof (rule integrable_on_subinterval)
      show (λx. norm (f x)) integrable_on {a..b}
        by (simp add: assms integrable_on_lebesgue_on)
    qed (use assms in auto)
    ultimately show ?thesis
      by (auto simp: absolutely_integrable_on_def)
  qed
qed auto

lemma indefinite_integral_continuous_real:
  fixes f :: real ⇒ 'a::euclidean_space
  assumes integrable (lebesgue_on {a..b}) f
  shows continuous_on {a..b} (λx. integral^L (lebesgue_on {a..x}) f)
proof –
  have f integrable_on {a..b}
    by (simp add: assms integrable_on_lebesgue_on)
  then have continuous_on {a..b} (λx. integral {a..x} f)
    using indefinite_integral_continuous_1 by blast
  moreover have integral^L (lebesgue_on {a..x}) f = integral {a..x} f if a ≤ x x
≤ b for x
  proof –
    have {a..x} ⊆ {a..b}
      using that by auto
    then have integrable (lebesgue_on {a..x}) f
      using integrable_subinterval assms by blast
    then show integral^L (lebesgue_on {a..x}) f = integral {a..x} f
      by (simp add: lebesgue_integral_eq_integral)
  qed
  ultimately show ?thesis
    by (metis (no_types, lifting) atLeastAtMost_iff continuous_on_cong)
qed

lemma lmeasurable_iff_integrable_on: S ∈ lmeasurable ⟷ (λx. 1::real) integrable_on
S
  by (subst absolutely_integrable_on_iff_nonneg[symmetric])
     (simp_all add: lmeasurable_iff_integrable set_integrable_def)

lemma lmeasure_integral_UNIV: S ∈ lmeasurable ⟹ measure lebesgue S = inte-
gral UNIV (indicator S)
  by (simp add: lmeasurable_iff_has_integral integral_unique)

lemma lmeasure_integral: S ∈ lmeasurable ⟹ measure lebesgue S = integral S
(λx. 1::real)
  by (fastforce simp add: lmeasure_integral_UNIV indicator_def[abs_def] lmeasur-
able_iff_integrable_on)
```

lemma *integrable_on_const*: $S \in lmeasurable \Longrightarrow (\lambda x.\ c)$ *integrable_on* S
unfolding *lmeasurable_iff_integrable*
by (*metis* (*mono_tags*, *lifting*) *integrable_eq integrable_on_scaleR_left lmeasurable_iff_integrable lmeasurable_iff_integrable_on scaleR_one*)

lemma *integral_indicator*:
assumes $(S \cap T) \in lmeasurable$
shows *integral* T (*indicator* S) = *measure lebesgue* $(S \cap T)$
proof −
have *integral UNIV* (*indicator* $(S \cap T)$) = *integral UNIV* ($\lambda a.$ *if* $a \in S \cap T$ *then* 1::*real else* 0)
by (*meson indicator_def*)
moreover have (*indicator* $(S \cap T)$ *has_integral measure lebesgue* $(S \cap T)$) *UNIV*
using *assms* **by** (*simp add*: *lmeasurable_iff_has_integral*)
ultimately have *integral UNIV* ($\lambda x.$ *if* $x \in S \cap T$ *then* 1 *else* 0) = *measure lebesgue* $(S \cap T)$
by (*metis* (*no_types*) *integral_unique*)
moreover have *integral* T ($\lambda a.$ *if* $a \in S$ *then* 1::*real else* 0) = *integral* $(S \cap T \cap UNIV)$ ($\lambda a.$ 1)
by (*simp add*: *Henstock_Kurzweil_Integration.integral_restrict_Int*)
moreover have *integral* T (*indicat_real* S) = *integral* T ($\lambda a.$ *if* $a \in S$ *then* 1 *else* 0)
by (*meson indicator_def*)
ultimately show *?thesis*
by (*simp add*: *assms lmeasure_integral*)
qed

lemma *measurable_integrable*:
fixes S :: *'a*::*euclidean_space set*
shows $S \in lmeasurable \longleftrightarrow$ (*indicat_real* S) *integrable_on UNIV*
by (*auto simp*: *lmeasurable_iff_integrable absolutely_integrable_on_iff_nonneg* [*symmetric*] *set_integrable_def*)

lemma *integrable_on_indicator*:
fixes S :: *'a*::*euclidean_space set*
shows *indicat_real* S *integrable_on* $T \longleftrightarrow (S \cap T) \in lmeasurable$
unfolding *measurable_integrable*
unfolding *integrable_restrict_UNIV* [*of* T, *symmetric*]
by (*fastforce simp add*: *indicator_def elim*: *integrable_eq*)

lemma
assumes $\mathcal{D}$: $\mathcal{D}$ *division_of* S
shows *lmeasurable_division*: $S \in lmeasurable$ (**is** *?l*)
and *content_division*: ($\sum k{\in}\mathcal{D}.$ *measure lebesgue* k) = *measure lebesgue* S (**is** *?m*)
proof −
{ fix *d1 d2* **assume** ∗: $d1 \in \mathcal{D}$ $d2 \in \mathcal{D}$ $d1 \neq d2$

then obtain *a b c d* **where** *d1* = *cbox a b d2* = *cbox c d*
using *division_ofD*(*4*)[*OF* $\mathcal{D}$] **by** *blast*
with *division_ofD*(*5*)[*OF* $\mathcal{D}$ ∗]
have *d1* ∈ *sets lborel d2* ∈ *sets lborel d1* ∩ *d2* ⊆ (*cbox a b* − *box a b*) ∪ (*cbox c d* − *box c d*)
by *auto*
moreover have (*cbox a b* − *box a b*) ∪ (*cbox c d* − *box c d*) ∈ *null_sets lborel*
by (*intro null_sets.Un null_sets_cbox_Diff_box*)
ultimately have *d1* ∩ *d2* ∈ *null_sets lborel*
by (*blast intro*: *null_sets_subset*) **}**
then show *?l ?m*
unfolding *division_ofD*(*6*)[*OF* $\mathcal{D}$, *symmetric*]
using *division_ofD*(*1*,*4*)[*OF* $\mathcal{D}$]
by (*auto intro*!: *measure_Union_AE*[*symmetric*] *simp*: *completion.AE_iff_null_sets Int_def*[*symmetric*] *pairwise_def null_sets_def*)
qed

lemma *has_measure_limit*:
assumes *S* ∈ *lmeasurable e* > *0*
obtains *B* **where** *B* > *0*
⋀*a b*. *ball 0 B* ⊆ *cbox a b* ⟹ |*measure lebesgue* (*S* ∩ *cbox a b*) − *measure lebesgue S*| < *e*
using *assms* **unfolding** *lmeasurable_iff_has_integral has_integral_alt′*
by (*force simp*: *integral_indicator integrable_on_indicator*)

lemma *lmeasurable_iff_indicator_has_integral*:
fixes *S* :: ′*a*::*euclidean_space set*
shows *S* ∈ *lmeasurable* ∧ *m* = *measure lebesgue S* ⟷ (*indicat_real S has_integral m*) *UNIV*
by (*metis has_integral_iff lmeasurable_iff_has_integral measurable_integrable*)

lemma *has_measure_limit_iff*:
fixes *f* :: ′*n*::*euclidean_space* ⇒ ′*a*::*banach*
shows *S* ∈ *lmeasurable* ∧ *m* = *measure lebesgue S* ⟷
(∀ *e*>*0*. ∃ *B*>*0*. ∀ *a b*. *ball 0 B* ⊆ *cbox a b* ⟶
(*S* ∩ *cbox a b*) ∈ *lmeasurable* ∧ |*measure lebesgue* (*S* ∩ *cbox a b*) − *m*| < *e*) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*meson has_measure_limit fmeasurable.Int lmeasurable_cbox*)
next
assume *RHS* [*rule_format*]: *?rhs*
then show *?lhs*
apply (*simp add*: *lmeasurable_iff_indicator_has_integral has_integral′* [**where** *i*=*m*])
by (*metis* (*full_types*) *integral_indicator integrable_integral integrable_on_indicator*)
qed

6.19.9 Applications to Negligibility

lemma *negligible_iff_null_sets*: *negligible S* ⟷ *S* ∈ *null_sets lebesgue*
proof
  assume *negligible S*
  then have (*indicator S has_integral* (*0*::*real*)) *UNIV*
    by (*auto simp*: *negligible*)
  then show *S* ∈ *null_sets lebesgue*
    by (*subst* (*asm*) *has_integral_iff_nn_integral_lebesgue*)
      (*auto simp*: *borel_measurable_indicator_iff nn_integral_0_iff_AE AE_iff_null_sets indicator_eq_0_iff*)
next
  assume *S*: *S* ∈ *null_sets lebesgue*
  show *negligible S*
    unfolding *negligible_def*
  proof (*safe intro*!: *has_integral_iff_nn_integral_lebesgue*[*THEN iffD2*]
                    has_integral_restrict_UNIV[**where** *s*=*cbox* _ _, *THEN iffD1*])
    fix *a b*
    show (λ*x*. *if x* ∈ *cbox a b then indicator S x else 0*) ∈ *lebesgue* $\rightarrow_M$ *borel*
      using *S* **by** (*auto intro*!: *measurable_If*)
    then show ($\int^+$ *x*. *ennreal* (*if x* ∈ *cbox a b then indicator S x else 0*) ∂*lebesgue*) = *ennreal 0*
      using *S*[*THEN AE_not_in*] **by** (*auto intro*!: *nn_integral_0_iff_AE*[*THEN iffD2*])
  qed *auto*
qed

corollary *eventually_ae_filter_negligible*:
  eventually P (*ae_filter lebesgue*) ⟷ (∃ *N*. *negligible N* ∧ {*x*. ¬ *P x*} ⊆ *N*)
  by (*auto simp*: *completion.AE_iff_null_sets negligible_iff_null_sets null_sets_completion_subset*)

lemma *starlike_negligible*:
  assumes *closed S*
      and *eq1*: ⋀*c x*. (*a* + *c* $*_R$ *x*) ∈ *S* ⟹ *0* ≤ *c* ⟹ *a* + *x* ∈ *S* ⟹ *c* = *1*
    shows *negligible S*
proof −
  have *negligible* ((+) (−*a*) ‘ *S*)
  proof (*subst negligible_on_intervals*, *intro allI*)
    fix *u v*
    show *negligible* ((+) (− *a*) ‘ *S* ∩ *cbox u v*)
      using ‹*closed S*› *eq1* **by** (*auto simp add*: *negligible_iff_null_sets algebra_simps*
      intro!: *closed_translation_subtract starlike_negligible_compact cong*: *image_cong_simp*)
        (*metis add_diff_eq diff_add_cancel scale_right_diff_distrib*)
  qed
  then show *?thesis*
    by (*rule negligible_translation_rev*)
qed

lemma *starlike_negligible_strong*:
  assumes *closed S*
      and *star*: ⋀*c x*. ⟦*0* ≤ *c*; *c* < *1*; *a*+*x* ∈ *S*⟧ ⟹ *a* + *c* $*_R$ *x* ∉ *S*

```
    shows negligible S
proof -
  show ?thesis
  proof (rule starlike_negligible [OF ‹closed S›, of a])
    fix c x
    assume cx: a + c *R x ∈ S 0 ≤ c a + x ∈ S
    with star have ¬ (c < 1) by auto
    moreover have ¬ (c > 1)
      using star [of 1/c c *R x] cx by force
    ultimately show c = 1 by arith
  qed
qed

lemma negligible_hyperplane:
  assumes a ≠ 0 ∨ b ≠ 0 shows negligible {x. a · x = b}
proof -
  obtain x where x: a · x ≠ b
    using assms by (metis euclidean_all_zero_iff inner_zero_right)
  moreover have c = 1 if a · (x + c *R w) = b a · (x + w) = b for c w
    using that
   by (metis (no_types, hide_lams) add_diff_eq diff_0 diff_minus_eq_add inner_diff_right
inner_scaleR_right mult_cancel_right2 right_minus_eq x)
  ultimately
  show ?thesis
    using starlike_negligible [OF closed_hyperplane, of x] by simp
qed

lemma negligible_lowdim:
  fixes S :: 'N :: euclidean_space set
  assumes dim S < DIM('N)
    shows negligible S
proof -
  obtain a where a ≠ 0 and a: span S ⊆ {x. a · x = 0}
    using lowdim_subset_hyperplane [OF assms] by blast
  have negligible (span S)
    using ‹a ≠ 0› a negligible_hyperplane by (blast intro: negligible_subset)
  then show ?thesis
    using span_base by (blast intro: negligible_subset)
qed

proposition negligible_convex_frontier:
  fixes S :: 'N :: euclidean_space set
  assumes convex S
    shows negligible(frontier S)
proof -
  have nf: negligible(frontier S) if convex S 0 ∈ S for S :: 'N set
  proof -
    obtain B where B ⊆ S and indB: independent B
           and spanB: S ⊆ span B and cardB: card B = dim S
```

```
      by (metis basis_exists)
    consider dim S < DIM('N) | dim S = DIM('N)
      using dim_subset_UNIV le_eq_less_or_eq by auto
    then show ?thesis
    proof cases
      case 1
      show ?thesis
        by (rule negligible_subset [of closure S])
           (simp_all add: frontier_def negligible_lowdim 1)
    next
      case 2
      obtain a where a ∈ interior (convex hull insert 0 B)
      proof (rule interior_simplex_nonempty [OF indB])
        show finite B
          by (simp add: indB independent_finite)
        show card B = DIM('N)
          by (simp add: cardB 2)
      qed
      then have a: a ∈ interior S
      by (metis ‹B ⊆ S› ‹0 ∈ S› ‹convex S› insert_absorb insert_subset interior_mono
subset_hull)
      show ?thesis
      proof (rule starlike_negligible_strong [where a=a])
        fix c::real and x
        have eq: a + c *R x = (a + x) − (1 − c) *R ((a + x) − a)
          by (simp add: algebra_simps)
        assume 0 ≤ c c < 1 a + x ∈ frontier S
        then show a + c *R x ∉ frontier S
          using eq mem_interior_closure_convex_shrink [OF ‹convex S› a, of _ 1−c]
          unfolding frontier_def
       by (metis Diff_iff add_diff_cancel_left' add_diff_eq diff_gt_0_iff_gt group_cancel.rule0
not_le)
      qed auto
    qed
  qed
  show ?thesis
  proof (cases S = {})
    case True then show ?thesis by auto
  next
    case False
    then obtain a where a ∈ S by auto
    show ?thesis
      using nf [of (λx. −a + x) ‘ S]
     by (metis ‹a ∈ S› add.left_inverse assms convex_translation_eq frontier_translation
              image_eqI negligible_translation_rev)
  qed
qed

corollary negligible_sphere: negligible (sphere a e)
```

using *frontier_cball negligible_convex_frontier convex_cball*
by (*blast intro*: *negligible_subset*)

lemma *non_negligible_UNIV* [*simp*]: ¬ *negligible UNIV*
unfolding *negligible_iff_null_sets* **by** (*auto simp*: *null_sets_def*)

lemma *negligible_interval*:
negligible (*cbox a b*) ⟷ *box a b* = {} *negligible* (*box a b*) ⟷ *box a b* = {}
by (*auto simp*: *negligible_iff_null_sets null_sets_def prod_nonneg inner_diff_left box_eq_empty*
not_le emeasure_lborel_cbox_eq emeasure_lborel_box_eq
intro: *eq_refl antisym less_imp_le*)

proposition *open_not_negligible*:
assumes *open S S* ≠ {}
shows ¬ *negligible S*
proof
assume *neg*: *negligible S*
obtain *a* **where** *a* ∈ *S*
using ‹*S* ≠ {}› **by** *blast*
then obtain *e* **where** *e* > *0 cball a e* ⊆ *S*
using ‹*open S*› *open_contains_cball_eq* **by** *blast*
let *?p* = *a* − (*e* / *DIM*(′*a*)) $*_R$ *One* **let** *?q* = *a* + (*e* / *DIM*(′*a*)) $*_R$ *One*
have *cbox ?p ?q* ⊆ *cball a e*
proof (*clarsimp simp*: *mem_box dist_norm*)
fix *x*
assume ∀ *i*∈*Basis*. *?p* · *i* ≤ *x* · *i* ∧ *x* · *i* ≤ *?q* · *i*
then have *ax*: |(*a* − *x*) · *i*| ≤ *e* / *real DIM*(′*a*) **if** *i* ∈ *Basis* **for** *i*
using *that* **by** (*auto simp*: *algebra_simps*)
have *norm* (*a* − *x*) ≤ (∑ *i*∈*Basis*. |(*a* − *x*) · *i*|)
by (*rule norm_le_l1*)
also have . . . ≤ *DIM*(′*a*) ∗ (*e* / *real DIM*(′*a*))
by (*intro sum_bounded_above ax*)
also have . . . = *e*
by *simp*
finally show *norm* (*a* − *x*) ≤ *e* .
qed
then have *negligible* (*cbox ?p ?q*)
by (*meson* ‹*cball a e* ⊆ *S*› *neg negligible_subset*)
with ‹*e* > *0*› **show** *False*
by (*simp add*: *negligible_interval box_eq_empty algebra_simps field_split_simps mult_le_0_iff*)
qed

lemma *negligible_convex_interior*:
convex S ⟹ (*negligible S* ⟷ *interior S* = {})
by (*metis Diff_empty closure_subset frontier_def interior_subset negligible_convex_frontier negligible_subset open_interior open_not_negligible*)

lemma *measure_eq_0_null_sets*: $S \in$ *null_sets* $M \Longrightarrow$ *measure* M $S = 0$
by (*auto simp*: *measure_def null_sets_def*)

lemma *negligible_imp_measure0*: *negligible* $S \Longrightarrow$ *measure lebesgue* $S = 0$
by (*simp add*: *measure_eq_0_null_sets negligible_iff_null_sets*)

lemma *negligible_iff_emeasure0*: $S \in$ *sets lebesgue* $\Longrightarrow$ (*negligible* $S \longleftrightarrow$ *emeasure lebesgue* $S = 0$)
by (*auto simp*: *measure_eq_0_null_sets negligible_iff_null_sets*)

lemma *negligible_iff_measure0*: $S \in$ *lmeasurable* $\Longrightarrow$ (*negligible* $S \longleftrightarrow$ *measure lebesgue* $S = 0$)
by (*metis* (*full_types*) *completion.null_sets_outer negligible_iff_null_sets negligible_imp_measure0 order_refl*)

lemma *negligible_imp_sets*: *negligible* $S \Longrightarrow S \in$ *sets lebesgue*
by (*simp add*: *negligible_iff_null_sets null_setsD2*)

lemma *negligible_imp_measurable*: *negligible* $S \Longrightarrow S \in$ *lmeasurable*
by (*simp add*: *fmeasurableI_null_sets negligible_iff_null_sets*)

lemma *negligible_iff_measure*: *negligible* $S \longleftrightarrow S \in$ *lmeasurable* $\wedge$ *measure lebesgue* $S = 0$
by (*fastforce simp*: *negligible_iff_measure0 negligible_imp_measurable dest*: *negligible_imp_measure0*)

lemma *negligible_outer*:
negligible $S \longleftrightarrow (\forall e>0.\ \exists T.\ S \subseteq T \wedge T \in$ *lmeasurable* $\wedge$ *measure lebesgue* $T < e$) (**is** _ = *?rhs*)
proof
assume *negligible* S **then show** *?rhs*
by (*metis negligible_iff_measure order_refl*)
next
assume *?rhs* **then show** *negligible* S
by (*meson completion.null_sets_outer negligible_iff_null_sets*)
qed

lemma *negligible_outer_le*:
negligible $S \longleftrightarrow (\forall e>0.\ \exists T.\ S \subseteq T \wedge T \in$ *lmeasurable* $\wedge$ *measure lebesgue* $T \le e$) (**is** _ = *?rhs*)
proof
assume *negligible* S **then show** *?rhs*
by (*metis dual_order.strict_implies_order negligible_imp_measurable negligible_imp_measure0 order_refl*)
next
assume *?rhs* **then show** *negligible* S
by (*metis le_less_trans negligible_outer field_lbound_gt_zero*)
qed

lemma *negligible_UNIV*: *negligible S* ⟷ (*indicat_real S has_integral 0*) *UNIV* (**is** _=?*rhs*)
 by (*metis lmeasurable_iff_indicator_has_integral negligible_iff_measure*)

lemma *sets_negligible_symdiff*:
 ⟦*S* ∈ *sets lebesgue*; *negligible*((*S* − *T*) ∪ (*T* − *S*))⟧ ⟹ *T* ∈ *sets lebesgue*
 by (*metis Diff_Diff_Int Int_Diff_Un inf_commute negligible_Un_eq negligible_imp_sets sets.Diff sets.Un*)

lemma *lmeasurable_negligible_symdiff*:
 ⟦*S* ∈ *lmeasurable*; *negligible*((*S* − *T*) ∪ (*T* − *S*))⟧ ⟹ *T* ∈ *lmeasurable*
 using *integrable_spike_set_eq lmeasurable_iff_integrable_on* **by** *blast*

lemma *measure_Un3_negligible*:
 assumes *meas*: *S* ∈ *lmeasurable T* ∈ *lmeasurable U* ∈ *lmeasurable*
 and *neg*: *negligible*(*S* ∩ *T*) *negligible*(*S* ∩ *U*) *negligible*(*T* ∩ *U*) **and** *V*: *S* ∪ *T* ∪ *U* = *V*
shows *measure lebesgue V* = *measure lebesgue S* + *measure lebesgue T* + *measure lebesgue U*
proof −
 have [*simp*]: *measure lebesgue* (*S* ∩ *T*) = *0*
  using *neg*(*1*) *negligible_imp_measure0* **by** *blast*
 have [*simp*]: *measure lebesgue* (*S* ∩ *U* ∪ *T* ∩ *U*) = *0*
  using *neg*(*2*) *neg*(*3*) *negligible_Un negligible_imp_measure0* **by** *blast*
 have *measure lebesgue V* = *measure lebesgue* (*S* ∪ *T* ∪ *U*)
  using *V* **by** *simp*
 also have ... = *measure lebesgue S* + *measure lebesgue T* + *measure lebesgue U*
  by (*simp add*: *measure_Un3 meas fmeasurable.Un Int_Un_distrib2*)
 finally show *?thesis* .
qed

lemma *measure_translate_add*:
 assumes *meas*: *S* ∈ *lmeasurable T* ∈ *lmeasurable*
  and *U*: *S* ∪ ((+)*a* ‘ *T*) = *U* **and** *neg*: *negligible*(*S* ∩ ((+)*a* ‘ *T*))
 shows *measure lebesgue S* + *measure lebesgue T* = *measure lebesgue U*
proof −
 have [*simp*]: *measure lebesgue* (*S* ∩ (+) *a* ‘ *T*) = *0*
  using *neg negligible_imp_measure0* **by** *blast*
 have *measure lebesgue* (*S* ∪ ((+)*a* ‘ *T*)) = *measure lebesgue S* + *measure lebesgue T*
  by (*simp add*: *measure_Un3 meas measurable_translation measure_translation fmeasurable.Un*)
 then show *?thesis*
  using *U* **by** *auto*
qed

lemma *measure_negligible_symdiff*:

assumes *S*: *S* ∈ *lmeasurable*
and *neg*: *negligible* (*S* − *T* ∪ (*T* − *S*))
shows *measure lebesgue T* = *measure lebesgue S*
proof −
have *measure lebesgue* (*S* − *T*) = *0*
using *neg negligible_Un_eq negligible_imp_measure0* **by** *blast*
then show *?thesis*
by (*metis S Un_commute add.right_neutral lmeasurable_negligible_symdiff measure_Un2 neg negligible_Un_eq negligible_imp_measure0*)
qed

lemma *measure_closure*:
assumes *bounded S* **and** *neg*: *negligible* (*frontier S*)
shows *measure lebesgue* (*closure S*) = *measure lebesgue S*
proof −
have *measure lebesgue* (*frontier S*) = *0*
by (*metis neg negligible_imp_measure0*)
then show *?thesis*
by (*metis assms lmeasurable_iff_integrable_on eq_iff_diff_eq_0 has_integral_interior integrable_on_def integral_unique lmeasurable_interior lmeasure_integral measure_frontier*)
qed

lemma *measure_interior*:
⟦*bounded S*; *negligible*(*frontier S*)⟧ ⟹ *measure lebesgue* (*interior S*) = *measure lebesgue S*
using *measure_closure measure_frontier negligible_imp_measure0* **by** *fastforce*

lemma *measurable_Jordan*:
assumes *bounded S* **and** *neg*: *negligible* (*frontier S*)
shows *S* ∈ *lmeasurable*
proof −
have *closure S* ∈ *lmeasurable*
by (*metis lmeasurable_closure* ‹*bounded S*›)
moreover have *interior S* ∈ *lmeasurable*
by (*simp add*: *lmeasurable_interior* ‹*bounded S*›)
moreover have *interior S* ⊆ *S*
by (*simp add*: *interior_subset*)
ultimately show *?thesis*
using *assms* **by** (*metis* (*full_types*) *closure_subset completion.complete_sets_sandwich_fmeasurable measure_closure measure_interior*)
qed

lemma *measurable_convex*: ⟦*convex S*; *bounded S*⟧ ⟹ *S* ∈ *lmeasurable*
by (*simp add*: *measurable_Jordan negligible_convex_frontier*)

lemma *content_cball_conv_ball*: *content* (*cball c r*) = *content* (*ball c r*)
proof −
have *ball c r* − *cball c r* ∪ (*cball c r* − *ball c r*) = *sphere c r*
by *auto*

hence *measure lebesgue* (*cball c r*) = *measure lebesgue* (*ball c r*)
  using *negligible_sphere*[*of c r*] **by** (*intro measure_negligible_symdiff*) *simp_all*
thus *?thesis* **by** *simp*
qed

6.19.10 Negligibility of image under non-injective linear map

lemma *negligible_Union_nat*:
  assumes $\bigwedge n::nat.\ negligible(S\ n)$
  shows $negligible(\bigcup n.\ S\ n)$
proof –
  have $negligible\ (\bigcup m\leq k.\ S\ m)$ **for** *k*
    using *assms* **by** *blast*
  then have *0*: *integral UNIV* (*indicat_real* ($\bigcup m\leq k.\ S\ m$)) = *0*
    and *1*: (*indicat_real* ($\bigcup m\leq k.\ S\ m$)) *integrable_on UNIV* **for** *k*
    by (*auto simp*: *negligible has_integral_iff*)
  have *2*: $\bigwedge k\ x.\ indicat\_real\ (\bigcup m\leq k.\ S\ m)\ x \leq (indicat\_real\ (\bigcup m\leq Suc\ k.\ S\ m)$
x)
    by (*simp add*: *indicator_def*)
  have *3*: $\bigwedge x.\ (\lambda k.\ indicat\_real\ (\bigcup m\leq k.\ S\ m)\ x) \longrightarrow (indicat\_real\ (\bigcup n.\ S\ n)$
x)
    by (*force simp*: *indicator_def eventually_sequentially intro*: *tendsto_eventually*)
  have *4*: *bounded* (*range* ($\lambda k.$ *integral UNIV* (*indicat_real* ($\bigcup m\leq k.\ S\ m$))))
    by (*simp add*: *0*)
  have ∗: *indicat_real* ($\bigcup n.\ S\ n$) *integrable_on UNIV* ∧
        ($\lambda k.$ *integral UNIV* (*indicat_real* ($\bigcup m\leq k.\ S\ m$))) ⟶ (*integral UNIV*
(*indicat_real* ($\bigcup n.\ S\ n$)))
    by (*intro monotone_convergence_increasing 1 2 3 4*)
  then have *integral UNIV* (*indicat_real* ($\bigcup n.\ S\ n$)) = (*0*::*real*)
    using *LIMSEQ_unique* **by** (*auto simp*: *0*)
  then show *?thesis*
    using ∗ **by** (*simp add*: *negligible_UNIV has_integral_iff*)
qed

lemma *negligible_linear_singular_image*:
  fixes *f* :: *'n*::*euclidean_space* ⇒ *'n*
  assumes *linear f* ¬ *inj f*
  shows *negligible* (*f* ‘ *S*)
proof –
  obtain *a* **where** $a \neq 0\ \bigwedge S.\ f\ `\ S \subseteq \{x.\ a \cdot x = 0\}$
    using *assms linear_singular_image_hyperplane* **by** *blast*
  then show *negligible* (*f* ‘ *S*)
    by (*metis negligible_hyperplane negligible_subset*)
qed

lemma *measure_negligible_finite_Union*:
  assumes *finite* $\mathcal{F}$
    and *meas*: $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow S \in lmeasurable$

and *djointish*: *pairwise* ($\lambda S\ T.$ *negligible* $(S \cap T)$) $\mathcal{F}$
shows *measure lebesgue* $(\bigcup \mathcal{F}) = (\sum S{\in}\mathcal{F}.$ *measure lebesgue* $S)$
using *assms*
proof (*induction*)
case *empty*
then show *?case*
by *auto*
next
case (*insert* $S\ \mathcal{F}$)
then have $S \in$ *lmeasurable* $\bigcup \mathcal{F} \in$ *lmeasurable pairwise* ($\lambda S\ T.$ *negligible* $(S \cap T)$) $\mathcal{F}$
by (*simp_all add*: *fmeasurable.finite_Union insert.hyps*(*1*) *insert.prems*(*1*) *pairwise_insert subsetI*)
then show *?case*
proof (*simp add*: *measure_Un3 insert*)
have $*$: $\bigwedge T.\ T \in (\cap)\ S$ ‘ $\mathcal{F} \Longrightarrow$ *negligible* T
using *insert* **by** (*force simp*: *pairwise_def*)
have *negligible*$(S \cap \bigcup \mathcal{F})$
unfolding *Int_Union*
by (*rule negligible_Union*) (*simp_all add*: $*$ *insert.hyps*(*1*))
then show *measure lebesgue* $(S \cap \bigcup \mathcal{F}) = 0$
using *negligible_imp_measure0* **by** *blast*
qed
qed

lemma *measure_negligible_finite_Union_image*:
assumes *finite* S
and *meas*: $\bigwedge x.\ x \in S \Longrightarrow f\,x \in$ *lmeasurable*
and *djointish*: *pairwise* ($\lambda x\ y.$ *negligible* $(f\,x \cap f\,y)$) S
shows *measure lebesgue* $(\bigcup (f$ ‘ $S)) = (\sum x{\in}S.$ *measure lebesgue* $(f\,x))$
proof −
have *measure lebesgue* $(\bigcup (f$ ‘ $S))$ = *sum* (*measure lebesgue*) (f ‘ S)
using *assms* **by** (*auto simp*: *pairwise_mono pairwise_image intro*: *measure_negligible_finite_Union*)
also have ... = *sum* (*measure lebesgue* $\circ f$) S
using *djointish* [*unfolded pairwise_def*] **by** (*metis inf.idem negligible_imp_measure0 sum.reindex_nontrivial* [*OF* ‹*finite S*›])
also have ... = $(\sum x{\in}S.$ *measure lebesgue* $(f\,x))$
by *simp*
finally show *?thesis* **.**
qed

6.19.11 Negligibility of a Lipschitz image of a negligible set

The bound will be eliminated by a sort of onion argument

lemma *locally_Lipschitz_negl_bounded*:
fixes f :: *'M::euclidean_space* $\Rightarrow$ *'N::euclidean_space*
assumes *MleN*: *DIM*(*'M*) $\leq$ *DIM*(*'N*) $0 < B$ *bounded* S *negligible* S
and *lips*: $\bigwedge x.\ x \in S$
$\Longrightarrow \exists\,T.$ *open* $T \wedge x \in T \wedge$

```
                         (∀ y ∈ S ∩ T. norm(f y − f x) ≤ B ∗ norm(y − x))
  shows negligible (f ‘ S)
  unfolding negligible_iff_null_sets
proof (clarsimp simp: completion.null_sets_outer)
  fix e::real
  assume 0 < e
  have S ∈ lmeasurable
    using ‹negligible S› by (simp add: negligible_iff_null_sets fmeasurableI_null_sets)
  then have S ∈ sets lebesgue
    by blast
  have e22: 0 < e/2 / (2 ∗ B ∗ real DIM('M)) ˆ DIM('N)
    using ‹0 < e› ‹0 < B› by (simp add: field_split_simps)
  obtain T where open T S ⊆ T (T − S) ∈ lmeasurable
              measure lebesgue (T − S) < e/2 / (2 ∗ B ∗ DIM('M)) ˆ DIM('N)
    using sets_lebesgue_outer_open [OF ‹S ∈ sets lebesgue› e22]
    by (metis emeasure_eq_measure2 ennreal_leI linorder_not_le)
  then have T: measure lebesgue T ≤ e/2 / (2 ∗ B ∗ DIM('M)) ˆ DIM('N)
    using ‹negligible S› by (simp add: measure_Diff_null_set negligible_iff_null_sets)
  have ∃ r. 0 < r ∧ r ≤ 1/2 ∧
          (x ∈ S −→ (∀ y. norm(y − x) < r
                 −→ y ∈ T ∧ (y ∈ S −→ norm(f y − f x) ≤ B ∗ norm(y − x))))
       for x
  proof (cases x ∈ S)
    case True
    obtain U where open U x ∈ U and U: ⋀y. y ∈ S ∩ U =⇒ norm(f y − f x)
≤ B ∗ norm(y − x)
      using lips [OF ‹x ∈ S›] by auto
    have x ∈ T ∩ U
      using ‹S ⊆ T› ‹x ∈ U› ‹x ∈ S› by auto
    then obtain ε where 0 < ε ball x ε ⊆ T ∩ U
      by (metis ‹open T› ‹open U› openE open_Int)
    then show ?thesis
     by (rule_tac x=min (1/2) ε in exI) (simp add: U dist_norm norm_minus_commute
subset_iff)
  next
    case False
    then show ?thesis
      by (rule_tac x=1/4 in exI) auto
  qed
  then obtain R where R12: ⋀x. 0 < R x ∧ R x ≤ 1/2
               and RT: ⋀x y. [[x ∈ S; norm(y − x) < R x]] =⇒ y ∈ T
               and RB: ⋀x y. [[x ∈ S; y ∈ S; norm(y − x) < R x]] =⇒ norm(f y
− f x) ≤ B ∗ norm(y − x)
    by metis+
  then have gaugeR: gauge (λx. ball x (R x))
    by (simp add: gauge_def)
  obtain c where c: S ⊆ cbox (−c ∗R One) (c ∗R One) box (−c ∗R One:: 'M)
(c ∗R One) ≠ {}
  proof −
```

obtain *B* **where** *B*: $\bigwedge x.\ x \in S \Longrightarrow norm\ x \le B$
using ‹*bounded S*› *bounded_iff* **by** *blast*
show *?thesis*
proof (*rule_tac c = abs B + 1* **in** *that*)
show $S \subseteq cbox\ (-\ (|B| + 1) *_R One)\ ((|B| + 1) *_R One)$
using *norm_bound_Basis_le Basis_le_norm*
by (*fastforce simp*: *mem_box dest*!: *B intro*: *order_trans*)
show $box\ (-\ (|B| + 1) *_R One)\ ((|B| + 1) *_R One) \neq \{\}$
by (*simp add*: *box_eq_empty*(*1*))
qed
qed
obtain $\mathcal{D}$ **where** *countable* $\mathcal{D}$
and *Dsub*: $\bigcup\mathcal{D} \subseteq cbox\ (-c *_R One)\ (c *_R One)$
and *cbox*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow interior\ K \neq \{\} \wedge (\exists c\ d.\ K = cbox\ c\ d)$
and *pw*: *pairwise* $(\lambda A\ B.\ interior\ A \cap interior\ B = \{\})\ \mathcal{D}$
and *Ksub*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow \exists x \in S \cap K.\ K \subseteq (\lambda x.\ ball\ x\ (R\ x))\ x$
and *exN*: $\bigwedge u\ v.\ cbox\ u\ v \in \mathcal{D} \Longrightarrow \exists n.\ \forall i \in Basis.\ v \cdot i - u \cdot i = (2{*}c)\ /$
$2\hat{}n$
and $S \subseteq \bigcup\mathcal{D}$
using *covering_lemma* [*OF c gaugeR*] **by** *force*
have $\exists u\ v\ z.\ K = cbox\ u\ v \wedge box\ u\ v \neq \{\} \wedge z \in S \wedge z \in cbox\ u\ v \wedge$
$cbox\ u\ v \subseteq ball\ z\ (R\ z)$ **if** $K \in \mathcal{D}$ **for** *K*
proof −
obtain *u v* **where** *K* = *cbox u v*
using ‹$K \in \mathcal{D}$› *cbox* **by** *blast*
with *that* **show** *?thesis*
by (*metis Int_iff interior_cbox cbox Ksub*)
qed
then obtain *uf vf zf*
where *uvz*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow$
$K = cbox\ (uf\ K)\ (vf\ K) \wedge box\ (uf\ K)\ (vf\ K) \neq \{\} \wedge zf\ K \in S \wedge$
$zf\ K \in cbox\ (uf\ K)\ (vf\ K) \wedge cbox\ (uf\ K)\ (vf\ K) \subseteq ball\ (zf\ K)\ (R\ (zf$
$K))$
by *metis*
define *prj1* **where** $prj1 \equiv \lambda x{::}'M.\ x \cdot (SOME\ i.\ i \in Basis)$
define *fbx* **where** $fbx \equiv \lambda D.\ cbox\ (f(zf\ D) - (B * DIM('M) * (prj1(vf\ D - uf$
$D))) *_R One{::}'N)$
$(f(zf\ D) + (B * DIM('M) * prj1(vf\ D - uf\ D)) *_R$
$One)$
have *vu_pos*: $0 < prj1\ (vf\ X - uf\ X)$ **if** $X \in \mathcal{D}$ **for** *X*
using *uvz* [*OF that*] **by** (*simp add*: *prj1_def box_ne_empty SOME_Basis in-ner_diff_left*)
have *prj1_idem*: $prj1\ (vf\ X - uf\ X) = (vf\ X - uf\ X) \cdot i$ **if** $X \in \mathcal{D}\ i \in Basis$
for *X i*
proof −
have $cbox\ (uf\ X)\ (vf\ X) \in \mathcal{D}$
using *uvz* ‹$X \in \mathcal{D}$› **by** *auto*
with *exN* **obtain** *n* **where** $\bigwedge i.\ i \in Basis \Longrightarrow vf\ X \cdot i - uf\ X \cdot i = (2{*}c)\ /$
$2\hat{}n$

```
      by blast
    then show ?thesis
      by (simp add: ‹i ∈ Basis› SOME_Basis inner_diff prj1_def)
  qed
  have countbl: countable (fbx ` 𝒟)
    using ‹countable 𝒟› by blast
  have (∑ k∈fbx`𝒟'. measure lebesgue k) ≤ e/2 if 𝒟' ⊆ 𝒟 finite 𝒟' for 𝒟'
  proof −
    have BM_ge0: 0 ≤ B * (DIM('M) * prj1 (vf X − uf X)) if X ∈ 𝒟' for X
      using ‹0 < B› ‹𝒟' ⊆ 𝒟› that vu_pos by fastforce
    have {} ∉ 𝒟'
      using cbox ‹𝒟' ⊆ 𝒟› interior_empty by blast
    have (∑ k∈fbx`𝒟'. measure lebesgue k) ≤ sum (measure lebesgue o fbx) 𝒟'
      by (rule sum_image_le [OF ‹finite 𝒟'›]) (force simp: fbx_def)
   also have ... ≤ (∑ X∈𝒟'. (2 * B * DIM('M)) ^ DIM('N) * measure lebesgue
X)
    proof (rule sum_mono)
      fix X assume X ∈ 𝒟'
      then have X ∈ 𝒟 using ‹𝒟' ⊆ 𝒟› by blast
      then have ufvf: cbox (uf X) (vf X) = X
        using uvz by blast
       have prj1 (vf X − uf X) ^ DIM('M) = (∏ i::'M ∈ Basis. prj1 (vf X − uf
X))
        by (rule prod_constant [symmetric])
      also have ... = (∏ i∈Basis. vf X · i − uf X · i)
        by (simp add: ‹X ∈ 𝒟› inner_diff_left prj1_idem cong: prod.cong)
      finally have prj1_eq: prj1 (vf X − uf X) ^ DIM('M) = (∏ i∈Basis. vf X ·
i − uf X · i) .
      have uf X ∈ cbox (uf X) (vf X) vf X ∈ cbox (uf X) (vf X)
        using uvz [OF ‹X ∈ 𝒟›] by (force simp: mem_box)+
      moreover have cbox (uf X) (vf X) ⊆ ball (zf X) (1/2)
        by (meson R12 order_trans subset_ball uvz [OF ‹X ∈ 𝒟›])
      ultimately have uf X ∈ ball (zf X) (1/2)  vf X ∈ ball (zf X) (1/2)
        by auto
      then have dist (vf X) (uf X) ≤ 1
        unfolding mem_ball
        by (metis dist_commute dist_triangle_half_l dual_order.order_iff_strict)
      then have 1: prj1 (vf X − uf X) ≤ 1
       unfolding prj1_def dist_norm using Basis_le_norm SOME_Basis order_trans
by fastforce
      have 0: 0 ≤ prj1 (vf X − uf X)
        using ‹X ∈ 𝒟› prj1_def vu_pos by fastforce
     have (measure lebesgue ∘ fbx) X ≤ (2 * B * DIM('M)) ^ DIM('N) * content
(cbox (uf X) (vf X))
       apply (simp add: fbx_def content_cbox_cases algebra_simps BM_ge0 ‹X ∈ 𝒟'›
‹0 < B› flip: prj1_eq)
        using MleN 0 1 uvz ‹X ∈ 𝒟›
        by (fastforce simp add: box_ne_empty power_decreasing)
      also have ... = (2 * B * DIM('M)) ^ DIM('N) * measure lebesgue X
```

```
        by (subst (3) ufvf[symmetric]) simp
      finally show (measure lebesgue ∘ fbx) X ≤ (2 ∗ B ∗ DIM('M)) ^ DIM('N)
∗ measure lebesgue X .
    qed
    also have . . . = (2 ∗ B ∗ DIM('M)) ^ DIM('N) ∗ sum (measure lebesgue) 𝒟'
      by (simp add: sum_distrib_left)
    also have . . . ≤ e/2
    proof −
      have ⋀K. K ∈ 𝒟' ⟹ ∃ a b. K = cbox a b
        using cbox that by blast
      then have div: 𝒟' division_of ⋃𝒟'
        using pairwise_subset [OF pw ‹𝒟' ⊆ 𝒟›] unfolding pairwise_def
        by (force simp: ‹finite 𝒟'› ‹{} ∉ 𝒟'› division_of_def)
      have le_meaT: measure lebesgue (⋃𝒟') ≤ measure lebesgue T
      proof (rule measure_mono_fmeasurable)
        show (⋃𝒟') ∈ sets lebesgue
          using div lmeasurable_division by auto
        have ⋃𝒟' ⊆ ⋃𝒟
          using ‹𝒟' ⊆ 𝒟› by blast
        also have ... ⊆ T
        proof (clarify)
          fix x D
          assume x ∈ D D ∈ 𝒟
          show x ∈ T
            using Ksub [OF ‹D ∈ 𝒟›]
              by (metis ‹x ∈ D› Int_iff dist_norm mem_ball norm_minus_commute
subsetD RT)
        qed
        finally show ⋃𝒟' ⊆ T .
        show T ∈ lmeasurable
         using ‹S ∈ lmeasurable› ‹S ⊆ T› ‹T − S ∈ lmeasurable› fmeasurable_Diff_D
by blast
      qed
      have sum (measure lebesgue) 𝒟' = sum content 𝒟'
        using ‹𝒟' ⊆ 𝒟› cbox by (force intro: sum.cong)
      then have (2 ∗ B ∗ DIM('M)) ^ DIM('N) ∗ sum (measure lebesgue) 𝒟' =
                (2 ∗ B ∗ real DIM('M)) ^ DIM('N) ∗ measure lebesgue (⋃𝒟')
        using content_division [OF div] by auto
      also have . . . ≤ (2 ∗ B ∗ real DIM('M)) ^ DIM('N) ∗ measure lebesgue T
        using ‹0 < B›
        by (intro mult_left_mono [OF le_meaT]) (force simp add: algebra_simps)
      also have . . . ≤ e/2
        using T ‹0 < B› by (simp add: field_simps)
      finally show ?thesis .
    qed
    finally show ?thesis .
  qed
  then have e2: sum (measure lebesgue) 𝒢 ≤ e/2 if 𝒢 ⊆ fbx ` 𝒟 finite 𝒢 for 𝒢
    by (metis finite_subset_image that)
```

```
  show ∃W∈lmeasurable. f ` S ⊆ W ∧ measure lebesgue W < e
  proof (intro bexI conjI)
    have ∃X∈𝒟. f y ∈ fbx X if y ∈ S for y
    proof –
      obtain X where y ∈ X X ∈ 𝒟
        using ‹S ⊆ ⋃𝒟› ‹y ∈ S› by auto
      then have y: y ∈ ball(zf X) (R(zf X))
        using uvz by fastforce
      have conj_le_eq: z − b ≤ y ∧ y ≤ z + b ⟷ abs(y − z) ≤ b for z y b::real
        by auto
      have yin: y ∈ cbox (uf X) (vf X) and zin: (zf X) ∈ cbox (uf X) (vf X)
        using uvz ‹X ∈ 𝒟› ‹y ∈ X› by auto
      have norm (y − zf X) ≤ (∑i∈Basis. |(y − zf X) · i|)
        by (rule norm_le_l1)
      also have ... ≤ real DIM('M) ∗ prj1 (vf X − uf X)
      proof (rule sum_bounded_above)
        fix j::'M assume j: j ∈ Basis
        show |(y − zf X) · j| ≤ prj1 (vf X − uf X)
          using yin zin j
         by (fastforce simp add: mem_box prj1_idem [OF ‹X ∈ 𝒟› j] inner_diff_left)
      qed
      finally have nole: norm (y − zf X) ≤ DIM('M) ∗ prj1 (vf X − uf X)
        by simp
     have fle: |f y · i − f(zf X) · i| ≤ B ∗ DIM('M) ∗ prj1 (vf X − uf X) if i ∈
Basis for i
      proof –
        have |f y · i − f (zf X) · i| = |(f y − f (zf X)) · i|
          by (simp add: algebra_simps)
        also have ... ≤ norm (f y − f (zf X))
          by (simp add: Basis_le_norm that)
        also have ... ≤ B ∗ norm(y − zf X)
          by (metis uvz RB ‹X ∈ 𝒟› dist_commute dist_norm mem_ball ‹y ∈ S› y)
        also have ... ≤ B ∗ real DIM('M) ∗ prj1 (vf X − uf X)
          using ‹0 < B› by (simp add: nole)
        finally show ?thesis .
      qed
      show ?thesis
        by (rule_tac x=X in bexI)
           (auto simp: fbx_def prj1_idem mem_box conj_le_eq inner_add inner_diff fle
‹X ∈ 𝒟›)
    qed
    then show f ` S ⊆ (⋃D∈𝒟. fbx D) by auto
  next
    have 1: ⋀D. D ∈ 𝒟 ⟹ fbx D ∈ lmeasurable
      by (auto simp: fbx_def)
    have 2: I' ⊆ 𝒟 ⟹ finite I' ⟹ measure lebesgue (⋃D∈I'. fbx D) ≤ e/2 for
I'
      by (rule order_trans[OF measure_Union_le e2]) (auto simp: fbx_def)
    show (⋃D∈𝒟. fbx D) ∈ lmeasurable
```

```
      by (intro fmeasurable_UN_bound[OF ‹countable 𝒟› 1 2])
    have measure lebesgue (⋃D∈𝒟. fbx D) ≤ e/2
      by (intro measure_UN_bound[OF ‹countable 𝒟› 1 2])
    then show measure lebesgue (⋃D∈𝒟. fbx D) < e
      using ‹0 < e› by linarith
  qed
qed

proposition negligible_locally_Lipschitz_image:
  fixes f :: 'M::euclidean_space ⇒ 'N::euclidean_space
  assumes MleN: DIM('M) ≤ DIM('N) negligible S
      and lips: ⋀x. x ∈ S
                   ⟹ ∃T B. open T ∧ x ∈ T ∧
                           (∀y ∈ S ∩ T. norm(f y − f x) ≤ B ∗ norm(y − x))
    shows negligible (f ` S)
proof −
  let ?S = λn. ({x ∈ S. norm x ≤ n ∧
                        (∃T. open T ∧ x ∈ T ∧
                             (∀y∈S ∩ T. norm (f y − f x) ≤ (real n + 1) ∗ norm (y
− x)))})
  have negfn: f ` ?S n ∈ null_sets lebesgue for n::nat
    unfolding negligible_iff_null_sets[symmetric]
    apply (rule_tac B = real n + 1 in locally_Lipschitz_negl_bounded)
    by (auto simp: MleN bounded_iff intro: negligible_subset [OF ‹negligible S›])
  have S = (⋃n. ?S n)
  proof (intro set_eqI iffI)
    fix x assume x ∈ S
    with lips obtain T B where T: open T x ∈ T
                        and B: ⋀y. y ∈ S ∩ T ⟹ norm(f y − f x) ≤ B ∗ norm(y
− x)
      by metis+
    have no: norm (f y − f x) ≤ (nat ⌈max B (norm x)⌉ + 1) ∗ norm (y − x) if
y ∈ S ∩ T for y
    proof −
      have B ∗ norm(y − x) ≤ (nat ⌈max B (norm x)⌉ + 1) ∗ norm (y − x)
        by (meson max.cobounded1 mult_right_mono nat_ceiling_le_eq nat_le_iff_add
norm_ge_zero order_trans)
      then show ?thesis
        using B order_trans that by blast
    qed
    have norm x ≤ real (nat ⌈max B (norm x)⌉)
      by linarith
    then have x ∈ ?S (nat (ceiling (max B (norm x))))
      using T no by (force simp: ‹x ∈ S› algebra_simps)
    then show x ∈ (⋃n. ?S n) by force
  qed auto
  then show ?thesis
    by (rule ssubst) (auto simp: image_Union negligible_iff_null_sets intro: negfn)
qed
```

```
corollary negligible_differentiable_image_negligible:
  fixes f :: 'M::euclidean_space ⇒ 'N::euclidean_space
  assumes MleN: DIM('M) ≤ DIM('N) negligible S
      and diff_f: f differentiable_on S
    shows negligible (f ` S)
proof –
  have ∃ T B. open T ∧ x ∈ T ∧ (∀ y ∈ S ∩ T. norm(f y − f x) ≤ B ∗ norm(y
− x))
        if x ∈ S for x
  proof –
    obtain f' where linear f'
       and f': ⋀e. e>0 ⟹
                    ∃ d>0. ∀ y∈S. norm (y − x) < d ⟶
                                norm (f y − f x − f' (y − x)) ≤ e ∗ norm (y − x)
      using diff_f ‹x ∈ S›
     by (auto simp: linear_linear differentiable_on_def differentiable_def has_derivative_within_alt)
    obtain B where B > 0 and B: ∀ x. norm (f' x) ≤ B ∗ norm x
      using linear_bounded_pos ‹linear f'› by blast
    obtain d where d>0
             and d: ⋀y. ⟦y ∈ S; norm (y − x) < d⟧ ⟹
                         norm (f y − f x − f' (y − x)) ≤ norm (y − x)
      using f' [of 1] by (force simp:)
    show ?thesis
    proof (intro exI conjI ballI)
      show norm (f y − f x) ≤ (B + 1) ∗ norm (y − x)
        if y ∈ S ∩ ball x d for y
      proof –
         have norm (f y − f x) − B ∗ norm (y − x) ≤ norm (f y − f x) − norm
(f' (y − x))
          by (simp add: B)
        also have . . . ≤ norm (f y − f x − f' (y − x))
          by (rule norm_triangle_ineq2)
        also have ... ≤ norm (y − x)
          by (metis IntE d dist_norm mem_ball norm_minus_commute that)
        finally show ?thesis
          by (simp add: algebra_simps)
      qed
    qed (use ‹d>0› in auto)
  qed
  with negligible_locally_Lipschitz_image assms show ?thesis by metis
qed

corollary negligible_differentiable_image_lowdim:
  fixes f :: 'M::euclidean_space ⇒ 'N::euclidean_space
  assumes MlessN: DIM('M) < DIM('N) and diff_f: f differentiable_on S
    shows negligible (f ` S)
proof –
  have x ≤ DIM('M) ⟹ x ≤ DIM('N) for x
```

using *MlessN* **by** *linarith*
obtain *lift* :: *'M* ∗ *real* ⇒ *'N* **and** *drop* :: *'N* ⇒ *'M* ∗ *real* **and** *j* :: *'N*
where *linear lift linear drop* **and** *dropl* [*simp*]: ⋀*z*. *drop* (*lift z*) = *z*
and *j* ∈ *Basis* **and** *j*: ⋀*x*. *lift*(*x*,*0*) · *j* = *0*
using *lowerdim_embeddings* [*OF MlessN*] **by** *metis*
have *negligible* ((λ*x*. *lift* (*x*, *0*)) ' *S*)
proof −
have *negligible* {*x*. *x*·*j* = *0*}
by (*metis* ‹*j* ∈ *Basis*› *negligible_standard_hyperplane*)
moreover have (λ*m*. *lift* (*m*, *0*)) ' *S* ⊆ {*n*. *n* · *j* = *0*}
using *j* **by** *force*
ultimately show *?thesis*
using *negligible_subset* **by** *auto*
qed
moreover
have *f* ∘ *fst* ∘ *drop differentiable_on* (λ*x*. *lift* (*x*, *0*)) ' *S*
using *diff_f*
apply (*clarsimp simp add*: *differentiable_on_def*)
apply (*intro differentiable_chain_within linear_imp_differentiable* [*OF* ‹*linear drop*›]
linear_imp_differentiable [*OF linear_fst*])
apply (*force simp*: *image_comp o_def*)
done
moreover
have *f* = *f* ∘ *fst* ∘ *drop* ∘ (λ*x*. *lift* (*x*, *0*))
by (*simp add*: *o_def*)
ultimately show *?thesis*
by (*metis* (*no_types*) *image_comp negligible_differentiable_image_negligible order_refl*)
qed

6.19.12 Measurability of countable unions and intersections of various kinds.

lemma
assumes *S*: ⋀*n*. *S n* ∈ *lmeasurable*
and *leB*: ⋀*n*. *measure lebesgue* (*S n*) ≤ *B*
and *nest*: ⋀*n*. *S n* ⊆ *S*(*Suc n*)
shows *measurable_nested_Union*: (⋃*n*. *S n*) ∈ *lmeasurable*
and *measure_nested_Union*: (λ*n*. *measure lebesgue* (*S n*)) ⟶ *measure lebesgue* (⋃*n*. *S n*) (**is** *?Lim*)
proof −
have *indicat_real* (⋃ (*range S*)) *integrable_on UNIV* ∧
(λ*n*. *integral UNIV* (*indicat_real* (*S n*)))
⟶ *integral UNIV* (*indicat_real* (⋃ (*range S*)))
proof (*rule monotone_convergence_increasing*)
show ⋀*n*. (*indicat_real* (*S n*)) *integrable_on UNIV*
using *S measurable_integrable* **by** *blast*
show ⋀*n x*::*'a*. *indicat_real* (*S n*) *x* ≤ (*indicat_real* (*S* (*Suc n*)) *x*)

by (*simp add*: *indicator_leI nest rev_subsetD*)
have ⋀*x*. (∃*n*. *x* ∈ *S n*) ⟶ (∀$_F$ *n in sequentially*. *x* ∈ *S n*)
by (*metis eventually_sequentiallyI lift_Suc_mono_le nest subsetCE*)
then
show ⋀*x*. (λ*n*. *indicat_real* (*S n*) *x*) ⟶ (*indicat_real* (⋃(*S* ‘ *UNIV*)) *x*)
by (*simp add*: *indicator_def tendsto_eventually*)
show *bounded* (*range* (λ*n*. *integral UNIV* (*indicat_real* (*S n*))))
using *leB* **by** (*auto simp*: *lmeasure_integral_UNIV* [*symmetric*] *S bounded_iff*)
qed
then have (⋃*n*. *S n*) ∈ *lmeasurable* ∧ *?Lim*
by (*simp add*: *lmeasure_integral_UNIV S cong*: *conj_cong*) (*simp add*: *measurable_integrable*)
then show (⋃*n*. *S n*) ∈ *lmeasurable* *?Lim*
by *auto*
qed

lemma
assumes *S*: ⋀*n*. *S n* ∈ *lmeasurable*
and *djointish*: *pairwise* (λ*m n*. *negligible* (*S m* ∩ *S n*)) *UNIV*
and *leB*: ⋀*n*. (∑*k*≤*n*. *measure lebesgue* (*S k*)) ≤ *B*
shows *measurable_countable_negligible_Union*: (⋃*n*. *S n*) ∈ *lmeasurable*
and *measure_countable_negligible_Union*: (λ*n*. (*measure lebesgue* (*S n*))) *sums measure lebesgue* (⋃*n*. *S n*) (**is** *?Sums*)
proof −
have *1*: ⋃ (*S* ‘ {..*n*}) ∈ *lmeasurable* **for** *n*
using *S* **by** *blast*
have *2*: *measure lebesgue* (⋃ (*S* ‘ {..*n*})) ≤ *B* **for** *n*
proof −
have *measure lebesgue* (⋃ (*S* ‘ {..*n*})) ≤ (∑*k*≤*n*. *measure lebesgue* (*S k*))
by (*simp add*: *S fmeasurableD measure_UNION_le*)
with *leB* **show** *?thesis*
using *order_trans* **by** *blast*
qed
have *3*: ⋀*n*. ⋃ (*S* ‘ {..*n*}) ⊆ ⋃ (*S* ‘ {..*Suc n*})
by (*simp add*: *SUP_subset_mono*)
have *eqS*: (⋃*n*. *S n*) = (⋃*n*. ⋃ (*S* ‘ {..*n*}))
using *atLeastAtMost_iff* **by** *blast*
also have (⋃*n*. ⋃ (*S* ‘ {..*n*})) ∈ *lmeasurable*
by (*intro measurable_nested_Union* [*OF 1 2*] *3*)
finally show (⋃*n*. *S n*) ∈ *lmeasurable* **.**
have *eqm*: (∑*i*≤*n*. *measure lebesgue* (*S i*)) = *measure lebesgue* (⋃ (*S* ‘ {..*n*})) **for** *n*
using *assms* **by** (*simp add*: *measure_negligible_finite_Union_image pairwise_mono*)
have (λ*n*. (*measure lebesgue* (*S n*))) *sums measure lebesgue* (⋃*n*. ⋃ (*S* ‘ {..*n*}))
by (*simp add*: *sums_def' eqm atLeast0AtMost*) (*intro measure_nested_Union* [*OF 1 2*] *3*)
then show *?Sums*
by (*simp add*: *eqS*)
qed

lemma *negligible_countable_Union* [*intro*]:
  assumes *countable* $\mathcal{F}$ **and** *meas*: $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *negligible S*
  shows *negligible* ($\bigcup \mathcal{F}$)
proof (*cases* $\mathcal{F} = \{\}$)
  case *False*
  then show *?thesis*
    by (*metis from_nat_into range_from_nat_into assms negligible_Union_nat*)
qed *simp*

lemma
  assumes *S*: $\bigwedge n.$ (*S n*) $\in$ *lmeasurable*
    and *djointish*: *pairwise* ($\lambda m\ n.$ *negligible* (*S m* $\cap$ *S n*)) *UNIV*
    and *bo*: *bounded* ($\bigcup n.$ *S n*)
  shows *measurable_countable_negligible_Union_bounded*: ($\bigcup n.$ *S n*) $\in$ *lmeasurable*
  and *measure_countable_negligible_Union_bounded*: ($\lambda n.$ (*measure lebesgue* (*S n*))) *sums measure lebesgue* ($\bigcup n.$ *S n*) (**is** *?Sums*)
proof −
  obtain *a b* **where** *ab*: ($\bigcup n.$ *S n*) $\subseteq$ *cbox a b*
    using *bo bounded_subset_cbox_symmetric* **by** *metis*
  then have *B*: ($\sum k \le n.$ *measure lebesgue* (*S k*)) $\le$ *measure lebesgue* (*cbox a b*) **for** *n*
  proof −
    have ($\sum k \le n.$ *measure lebesgue* (*S k*)) = *measure lebesgue* ($\bigcup$ (*S* ` {..*n*}))
      using *measure_negligible_finite_Union_image* [*OF _ _ pairwise_subset*] *djointish*
      by (*metis S finite_atMost subset_UNIV*)
    also have ... $\le$ *measure lebesgue* (*cbox a b*)
    proof (*rule measure_mono_fmeasurable*)
      show $\bigcup$ (*S* ` {..*n*}) $\in$ *sets lebesgue* **using** *S* **by** *blast*
    qed (*use ab* **in** *auto*)
    finally show *?thesis* **.**
  qed
  show ($\bigcup n.$ *S n*) $\in$ *lmeasurable*
    by (*rule measurable_countable_negligible_Union* [*OF S djointish B*])
  show *?Sums*
    by (*rule measure_countable_negligible_Union* [*OF S djointish B*])
qed

lemma *measure_countable_Union_approachable*:
  assumes *countable* $\mathcal{D}$ *e* > *0* **and** *measD*: $\bigwedge d.\ d \in \mathcal{D} \Longrightarrow d \in$ *lmeasurable*
      and *B*: $\bigwedge D'.\ [\![D' \subseteq \mathcal{D};$ *finite* $D']\!] \Longrightarrow$ *measure lebesgue* ($\bigcup D'$) $\le B$
    obtains D' **where** $D' \subseteq \mathcal{D}$ *finite* D' *measure lebesgue* ($\bigcup \mathcal{D}$) − *e* < *measure lebesgue* ($\bigcup D'$)
proof (*cases* $\mathcal{D} = \{\}$)
  case *True*
  then show *?thesis*
    by (*simp add*: ‹*e* > *0*› *that*)
next
  case *False*

let *?S* = $\lambda n. \bigcup k \leq n.$ *from_nat_into* $\mathcal{D}$ *k*
have ($\lambda n.$ *measure lebesgue* (*?S n*)) $\longrightarrow$ *measure lebesgue* ($\bigcup n.$ *?S n*)
proof (*rule measure_nested_Union*)
  show *?S n* $\in$ *lmeasurable* **for** *n*
    by (*simp add*: *False fmeasurable.finite_UN from_nat_into measD*)
  show *measure lebesgue* (*?S n*) $\leq$ *B* **for** *n*
    by (*metis* (*mono_tags*, *lifting*) *B False finite_atMost finite_imageI from_nat_into image_iff subsetI*)
  show *?S n* $\subseteq$ *?S* (*Suc n*) **for** *n*
    by *force*
qed
then obtain *N* **where** *N*: $\bigwedge n.\ n \geq N \Longrightarrow$ *dist* (*measure lebesgue* (*?S n*)) (*measure lebesgue* ($\bigcup n.$ *?S n*)) < *e*
  using *metric_LIMSEQ_D* ‹*e* > *0*› **by** *blast*
show *?thesis*
proof
  show *from_nat_into* $\mathcal{D}$ ‘ {..*N*} $\subseteq \mathcal{D}$
    by (*auto simp*: *False from_nat_into*)
  have *eq*: ($\bigcup n. \bigcup k{\leq}n.$ *from_nat_into* $\mathcal{D}$ *k*) = ($\bigcup\mathcal{D}$)
    using ‹*countable* $\mathcal{D}$› *False*
    by (*auto intro*: *from_nat_into dest*: *from_nat_into_surj* [*OF* ‹*countable* $\mathcal{D}$›])
  show *measure lebesgue* ($\bigcup\mathcal{D}$) − *e* < *measure lebesgue* ($\bigcup$ (*from_nat_into* $\mathcal{D}$ ‘ {..*N*}))
    using *N* [*OF order_refl*]
    by (*auto simp*: *eq algebra_simps dist_norm*)
qed *auto*
qed

6.19.13 Negligibility is a local property

lemma *locally_negligible_alt*:
  negligible S $\longleftrightarrow$ ($\forall x \in S. \exists U.$ *openin* (*top_of_set S*) *U* $\wedge$ *x* $\in$ *U* $\wedge$ *negligible U*)
  (**is** _ = *?rhs*)
proof
  assume *negligible S*
  then show *?rhs*
    using *openin_subtopology_self* **by** *blast*
next
  assume *?rhs*
  then obtain *U* **where** *ope*: $\bigwedge x.\ x \in S \Longrightarrow$ *openin* (*top_of_set S*) (*U x*)
    and *cov*: $\bigwedge x.\ x \in S \Longrightarrow x \in U\ x$
    and *neg*: $\bigwedge x.\ x \in S \Longrightarrow$ *negligible* (*U x*)
    by *metis*
  obtain $\mathcal{F}$ **where** $\mathcal{F} \subseteq$ *U* ‘ *S countable* $\mathcal{F}$ **and** *eq*: $\bigcup\mathcal{F} = \bigcup$(*U* ‘ *S*)
    using *ope* **by** (*force intro*: *Lindelof_openin* [*of U* ‘ *S S*])
  then have *negligible* ($\bigcup\mathcal{F}$)
    by (*metis imageE neg negligible_countable_Union subset_eq*)
  with *eq* **have** *negligible* ($\bigcup$(*U* ‘ *S*))

by *metis*
moreover have $S \subseteq \bigcup(U \text{ ` } S)$
using *cov* **by** *blast*
ultimately show *negligible S*
using *negligible_subset* **by** *blast*
qed

lemma *locally_negligible*: *locally negligible S* $\longleftrightarrow$ *negligible S*
unfolding *locally_def*
by (*metis locally_negligible_alt negligible_subset openin_imp_subset openin_subtopology_self*)

6.19.14 Integral bounds

lemma *set_integral_norm_bound*:
fixes f :: $\_ \Rightarrow$ *'a* :: {*banach*, *second_countable_topology*}
shows *set_integrable M k f* $\Longrightarrow$ *norm* (*LINT x*:*k*|*M*. *f x*) $\leq$ *LINT x*:*k*|*M*. *norm* (*f x*)
using *integral_norm_bound*[*of M* λx. *indicator k x* $*_R$ *f x*] **by** (*simp add*: *set_lebesgue_integral_def*)

lemma *set_integral_finite_UN_AE*:
fixes f :: $\_ \Rightarrow \_$ :: {*banach*, *second_countable_topology*}
assumes *finite I*
and *ae*: $\bigwedge i\ j.\ i \in I \Longrightarrow j \in I \Longrightarrow$ *AE x in M*. $(x \in A\ i \wedge x \in A\ j) \longrightarrow i = j$
and [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow A\ i \in$ *sets M*
and f: $\bigwedge i.\ i \in I \Longrightarrow$ *set_integrable M* (*A i*) *f*
shows *LINT x*:($\bigcup i \in I.\ A\ i$)|*M*. *f x* = ($\sum i \in I$. *LINT x*:*A i*|*M*. *f x*)
using ‹*finite I*› *order_refl*[*of I*]
proof (*induction I rule*: *finite_subset_induct'*)
case (*insert i I'*)
have *AE x in M*. ($\forall j \in I'$. $x \in A\ i \longrightarrow x \notin A\ j$)
proof (*intro AE_ball_countable*[*THEN iffD2*] *ballI*)
fix j **assume** $j \in I'$
with ‹$I' \subseteq I$› ‹$i \notin I'$› **have** $i \neq j$ $j \in I$
by *auto*
then show *AE x in M*. $x \in A\ i \longrightarrow x \notin A\ j$
using *ae*[*of i j*] ‹$i \in I$› **by** *auto*
qed (*use* ‹*finite I'*› **in** ‹*rule countable_finite*›)
then have *AE x*$\in$*A i in M*. $\forall$ *xa*$\in I'$. $x \notin A$ *xa*
by *auto*
with *insert.hyps insert.IH*[*symmetric*]
show *?case*
by (*auto intro*!: *set_integral_Un_AE sets.finite_UN f set_integrable_UN*)
qed (*simp add*: *set_lebesgue_integral_def*)

lemma *set_integrable_norm*:
fixes f :: *'a* $\Rightarrow$ *'b*::{*banach*, *second_countable_topology*}
assumes f: *set_integrable M k f* **shows** *set_integrable M k* (λx. *norm* (*f x*))
using *integrable_norm f* **by** (*force simp add*: *set_integrable_def*)

lemma *absolutely_integrable_bounded_variation*:
  fixes f :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
  assumes f: f *absolutely_integrable_on UNIV*
  obtains B **where** ∀ d. d *division_of* (⋃ d) ⟶ *sum* (λk. *norm* (*integral* k f)) d
≤ B
proof (*rule that*[*of integral UNIV* (λx. *norm* (f x))]; *safe*)
  fix d :: $'a$ *set set* **assume** d: d *division_of* ⋃ d
  have ∗: k ∈ d ⟹ f *absolutely_integrable_on* k **for** k
    using f[*THEN set_integrable_subset*, *of* k] *division_ofD*(*2*,*4*)[*OF* d, *of* k] **by**
auto
  note d' = *division_ofD*[*OF* d]
  have (∑ k∈d. *norm* (*integral* k f)) = (∑ k∈d. *norm* (*LINT* x:k|*lebesgue*. f x))
   by (*intro sum.cong refl arg_cong*[**where** f=*norm*] *set_lebesgue_integral_eq_integral*(*2*)[*symmetric*]
∗)
  also have . . . ≤ (∑ k∈d. *LINT* x:k|*lebesgue*. *norm* (f x))
    by (*intro sum_mono set_integral_norm_bound* ∗)
  also have . . . = (∑ k∈d. *integral* k (λx. *norm* (f x)))
    by (*intro sum.cong refl set_lebesgue_integral_eq_integral*(*2*) *set_integrable_norm*
∗)
  also have . . . ≤ *integral* (⋃ d) (λx. *norm* (f x))
   using *integrable_on_subdivision*[*OF* d] *assms* f **unfolding** *absolutely_integrable_on_def*
    by (*subst integral_combine_division_topdown*[*OF* _ d]) *auto*
  also have . . . ≤ *integral UNIV* (λx. *norm* (f x))
   using *integrable_on_subdivision*[*OF* d] *assms* **unfolding** *absolutely_integrable_on_def*
    by (*intro integral_subset_le*) *auto*
  finally show (∑ k∈d. *norm* (*integral* k f)) ≤ *integral UNIV* (λx. *norm* (f x)) **.**
qed

lemma *absdiff_norm_less*:
  assumes *sum* (λx. *norm* (f x − g x)) S < e
  shows |*sum* (λx. *norm*(f x)) S − *sum* (λx. *norm*(g x)) S| < e (**is** *?lhs* < e)
proof −
  have *?lhs* ≤ (∑ i∈S. |*norm* (f i) − *norm* (g i)|)
    by (*metis* (*no_types*) *sum_abs sum_subtractf*)
  also have ... ≤ (∑ x∈S. *norm* (f x − g x))
    by (*simp add*: *norm_triangle_ineq3 sum_mono*)
  also have ... < e
    using *assms*(*1*) **by** *blast*
  finally show *?thesis* **.**
qed

proposition *bounded_variation_absolutely_integrable_interval*:
  fixes f :: $'n$::*euclidean_space* ⇒ $'m$::*euclidean_space*
  assumes f: f *integrable_on cbox* a b
    and ∗: ⋀d. d *division_of* (*cbox* a b) ⟹ *sum* (λK. *norm*(*integral* K f)) d ≤ B
  shows f *absolutely_integrable_on cbox* a b
proof −
  let *?f* = λd. ∑ K∈d. *norm* (*integral* K f) **and** *?D* = {d. d *division_of* (*cbox* a
b)}

have D_1: $?D \neq \{\}$
  by (*rule elementary_interval*[*of a b*]) *auto*
have D_2: *bdd_above* (*?f‘?D*)
  by (*metis* ∗ *mem_Collect_eq bdd_aboveI2*)
note $D = D\_1\ D\_2$
let $?S = SUP\ x{\in}?D.\ ?f\ x$
have ∗: $\exists\gamma.$ *gauge* $\gamma \wedge$
    $(\forall p.\ p$ *tagged_division_of cbox a b* $\wedge$
      γ *fine p* $\longrightarrow$
      norm $((\sum(x,k) \in p.$ *content k* $*_R$ *norm* $(f\ x)) - ?S) < e)$
  if e: $e > 0$ **for** e
proof −
  have $?S - e/2 < ?S$ **using** ‹$e > 0$› **by** *simp*
  then obtain d **where** d: d *division_of* (*cbox a b*) $?S - e/2 < (\sum k{\in}d.$ *norm* (*integral k f*))
    unfolding *less_cSUP_iff*[*OF D*] **by** *auto*
  note $d' =$ *division_ofD*[*OF this*(*1*)]

  have $\exists e{>}0.\ \forall i{\in}d.\ x \notin i \longrightarrow$ *ball x e* $\cap\ i = \{\}$ **for** x
  proof −
    have $\exists d'{>}0.\ \forall x'{\in}\bigcup\{i \in d.\ x \notin i\}.\ d' \leq$ *dist x x'*
    proof (*rule separate_point_closed*)
      show *closed* $(\bigcup\{i \in d.\ x \notin i\})$
        using d' **by** *force*
      show $x \notin \bigcup\{i \in d.\ x \notin i\}$
        by *auto*
    qed
    then show *?thesis*
      by *force*
  qed
  then obtain k **where** k: $\bigwedge x.\ 0 < k\ x\ \bigwedge i\ x.\ [\![i \in d;\ x \notin i]\!] \Longrightarrow$ *ball x* $(k\ x) \cap i = \{\}$
    by *metis*
  have $e/2 > 0$
    using e **by** *auto*
  with *Henstock_lemma*[*OF f*]
  obtain γ **where** g: *gauge* γ
    $\bigwedge p.\ [\![p$ *tagged_partial_division_of cbox a b*; γ *fine p*$]\!]$
      $\Longrightarrow (\sum(x,k) \in p.$ *norm* (*content k* $*_R f\ x -$ *integral k f*)) $< e/2$
    by (*metis* (*no_types*, *lifting*))
  let $?g = \lambda x.\ \gamma\ x \cap$ *ball x* $(k\ x)$
  show *?thesis*
  proof (*intro exI conjI allI impI*)
    show *gauge ?g*
      using $g(1)\ k(1)$ **by** (*auto simp*: *gauge_def*)
  next
    fix p
    assume p *tagged_division_of* (*cbox a b*) $\wedge$ *?g fine p*
    then have p: p *tagged_division_of cbox a b* γ *fine p* ($\lambda x.$ *ball x* $(k\ x)$) *fine p*

```
        by (auto simp: fine_Int)
      note p' = tagged_division_ofD[OF p(1)]
      define p' where p' = {(x,k) | x k. ∃ i l. x ∈ i ∧ i ∈ d ∧ (x,l) ∈ p ∧ k = i
∩ l}
      have gp': γ fine p'
        using p(2) by (auto simp: p'_def fine_def)
      have p'': p' tagged_division_of (cbox a b)
      proof (rule tagged_division_ofI)
        show finite p'
        proof (rule finite_subset)
          show p' ⊆ (λ(k, x, l). (x, k ∩ l)) ` (d × p)
            by (force simp: p'_def image_iff)
          show finite ((λ(k, x, l). (x, k ∩ l)) ` (d × p))
            by (simp add: d'(1) p'(1))
        qed
      next
        fix x K
        assume (x, K) ∈ p'
        then have ∃ i l. x ∈ i ∧ i ∈ d ∧ (x, l) ∈ p ∧ K = i ∩ l
          unfolding p'_def by auto
        then obtain i l where il: x ∈ i i ∈ d (x, l) ∈ p K = i ∩ l by blast
        show x ∈ K and K ⊆ cbox a b
          using p'(2−3)[OF il(3)] il by auto
        show ∃ a b. K = cbox a b
        unfolding il using p'(4)[OF il(3)] d'(4)[OF il(2)] by (meson Int_interval)
      next
        fix x1 K1
        assume (x1, K1) ∈ p'
        then have ∃ i l. x1 ∈ i ∧ i ∈ d ∧ (x1, l) ∈ p ∧ K1 = i ∩ l
          unfolding p'_def by auto
         then obtain i1 l1 where il1: x1 ∈ i1 i1 ∈ d (x1, l1) ∈ p K1 = i1 ∩ l1
by blast
        fix x2 K2
        assume (x2,K2) ∈ p'
        then have ∃ i l. x2 ∈ i ∧ i ∈ d ∧ (x2, l) ∈ p ∧ K2 = i ∩ l
          unfolding p'_def by auto
         then obtain i2 l2 where il2: x2 ∈ i2 i2 ∈ d (x2, l2) ∈ p K2 = i2 ∩ l2
by blast
        assume (x1, K1) ≠ (x2, K2)
        then have interior i1 ∩ interior i2 = {} ∨ interior l1 ∩ interior l2 = {}
          using d'(5)[OF il1(2) il2(2)] p'(5)[OF il1(3) il2(3)]  by (auto simp: il1
il2)
        then show interior K1 ∩ interior K2 = {}
          unfolding il1 il2 by auto
      next
        have ∗: ∀ (x, X) ∈ p'. X ⊆ cbox a b
          unfolding p'_def using d' by blast
        show ⋃{K. ∃ x. (x, K) ∈ p'} = cbox a b
        proof
```

```
        show ⋃{k. ∃x. (x, k) ∈ p'} ⊆ cbox a b
          using ∗ by auto
      next
        show cbox a b ⊆ ⋃{k. ∃x. (x, k) ∈ p'}
        proof
          fix y
          assume y: y ∈ cbox a b
          obtain x L where xl: (x, L) ∈ p y ∈ L
            using y unfolding p'(6)[symmetric] by auto
          obtain I where i: I ∈ d y ∈ I
            using y unfolding d'(6)[symmetric] by auto
          have x ∈ I
            using fineD[OF p(3) xl(1)] using k(2) i xl by auto
          then show y ∈ ⋃{K. ∃x. (x, K) ∈ p'}
          proof −
            obtain x l where xl: (x, l) ∈ p y ∈ l
              using y unfolding p'(6)[symmetric] by auto
            obtain i where i: i ∈ d y ∈ i
              using y unfolding d'(6)[symmetric] by auto
            have x ∈ i
              using fineD[OF p(3) xl(1)] using k(2) i xl by auto
            then show ?thesis
             unfolding p'_def by (rule_tac X=i ∩ l in UnionI) (use i xl in auto)
          qed
        qed
      qed
    qed
    then have sum_less_e2: (∑(x,K) ∈ p'. norm (content K ∗R f x − integral
K f)) < e/2
      using g(2) gp' tagged_division_of_def by blast

    have in_p': (x, I ∩ L) ∈ p' if x: (x, L) ∈ p I ∈ d and y: y ∈ I y ∈ L
      for x I L y
    proof −
      have x ∈ I
        using fineD[OF p(3) that(1)] k(2)[OF ‹I ∈ d›] y by auto
      with x have (∃i l. x ∈ i ∧ i ∈ d ∧ (x, l) ∈ p ∧ I ∩ L = i ∩ l)
        by blast
      then have (x, I ∩ L) ∈ p'
        by (simp add: p'_def)
      with y show ?thesis by auto
    qed
    moreover
    have Ex_p_p': ∃y i l. (x, K) = (y, i ∩ l) ∧ (y, l) ∈ p ∧ i ∈ d ∧ i ∩ l ≠ {}
      if xK: (x,K) ∈ p' for x K
    proof −
      obtain i l where il: x ∈ i i ∈ d (x, l) ∈ p K = i ∩ l
        using xK unfolding p'_def by auto
      then show ?thesis
```

using *p′(2)* **by** *fastforce*
qed
ultimately have *p′alt*: $p' = \{(x, I \cap L) \mid x\ I\ L.\ (x,L) \in p \wedge I \in d \wedge I \cap L \neq \{\}\}$
by *auto*
have *sum_p′*: $(\sum (x,K) \in p'.\ norm\ (integral\ K\ f)) = (\sum k{\in}snd\ `\ p'.\ norm\ (integral\ k\ f))$
proof (*rule sum.over_tagged_division_lemma*[*OF p′*])
show $\bigwedge u\ v.\ box\ u\ v = \{\} \Longrightarrow norm\ (integral\ (cbox\ u\ v)\ f) = 0$
by (*auto intro*: *integral_null simp*: *content_eq_0_interior*)
qed
have *snd_p_div*: *snd ‘ p division_of cbox a b*
by (*rule division_of_tagged_division*[*OF p*(*1*)])
note *snd_p* = *division_ofD*[*OF snd_p_div*]
have *fin_d_sndp*: *finite* $(d \times snd\ `\ p)$
by (*simp add*: *d′*(*1*) *snd_p*(*1*))

have ∗: $\bigwedge sni\ sni'\ sf\ sf'.\ [\![|sf' - sni'| < e/2;\ ?S - e/2 < sni;\ sni' \leq ?S;\ sni \leq sni';\ sf' = sf]\!] \Longrightarrow |sf - ?S| < e$
by *arith*
show $norm\ ((\sum (x,k) \in p.\ content\ k *_R norm\ (f\ x)) - ?S) < e$
unfolding *real_norm_def*
proof (*rule* ∗)
show $|(\sum (x,K){\in}p'.\ norm\ (content\ K *_R f\ x)) - (\sum (x,k){\in}p'.\ norm\ (integral\ k\ f))| < e/2$
using *p′′ sum_less_e2* **unfolding** *split_def* **by** (*force intro*!: *absdiff_norm_less*)
show $(\sum (x,k) \in p'.\ norm\ (integral\ k\ f)) \leq ?S$
by (*auto simp*: *sum_p′ division_of_tagged_division*[*OF p′*] *D intro*!: *cSUP_upper*)
show $(\sum k{\in}d.\ norm\ (integral\ k\ f)) \leq (\sum (x,k) \in p'.\ norm\ (integral\ k\ f))$
proof −
have ∗: $\{k \cap l \mid k\ l.\ k \in d \wedge l \in snd\ `\ p\} = (\lambda(k,l).\ k \cap l)\ `\ (d \times snd\ `\ p)$
by *auto*
have $(\sum K{\in}d.\ norm\ (integral\ K\ f)) \leq (\sum i{\in}d.\ \sum l{\in}snd\ `\ p.\ norm\ (integral\ (i \cap l)\ f))$
proof (*rule sum_mono*)
fix *K* **assume** *k*: $K \in d$
from *d′*(*4*)[*OF this*] **obtain** *u v* **where** *uv*: $K = cbox\ u\ v$ **by** *metis*
define *d′* **where** $d' = \{cbox\ u\ v \cap l\ |l.\ l \in snd\ `\ p \wedge\ cbox\ u\ v \cap l \neq \{\}\}$
have *uvab*: $cbox\ u\ v \subseteq cbox\ a\ b$
using *d*(*1*) *k uv* **by** *blast*
have *d′_div*: *d′ division_of cbox u v*
unfolding *d′_def* **by** (*rule division_inter_1* [*OF snd_p_div uvab*])
moreover have $norm\ (\sum i{\in}d'.\ integral\ i\ f) \leq (\sum k{\in}d'.\ norm\ (integral\ k\ f))$
by (*simp add*: *sum_norm_le*)
moreover have *f integrable_on K*
using *f integrable_on_subcbox uv uvab* **by** *blast*
moreover have *d′ division_of K*

using *d′_div uv* **by** *blast*
ultimately have *norm* (*integral K f*) ≤ *sum* (λ*k*. *norm* (*integral k f*)) *d′*
by (*simp add*: *integral_combine_division_topdown*)
also have ... = (∑ *I*∈{*K* ∩ *L* |*L*. *L* ∈ *snd* ‘ *p*}. *norm* (*integral I f*))
proof (*rule sum.mono_neutral_left*)
show *finite* {*K* ∩ *L* |*L*. *L* ∈ *snd* ‘ *p*}
by (*simp add*: *snd_p*(*1*))
show ∀ *i*∈{*K* ∩ *L* |*L*. *L* ∈ *snd* ‘ *p*} − *d′*. *norm* (*integral i f*) = *0*
d′ ⊆ {*K* ∩ *L* |*L*. *L* ∈ *snd* ‘ *p*}
using *d′_def image_eqI uv* **by** *auto*
qed
also have ... = (∑ *l*∈*snd* ‘ *p*. *norm* (*integral* (*K* ∩ *l*) *f*))
unfolding *Setcompr_eq_image*
proof (*rule sum.reindex_nontrivial* [*unfolded o_def*])
show *finite* (*snd* ‘ *p*)
using *snd_p*(*1*) **by** *blast*
show *norm* (*integral* (*K* ∩ *l*) *f*) = *0*
if *l* ∈ *snd* ‘ *p y* ∈ *snd* ‘ *p l* ≠ *y K* ∩ *l* = *K* ∩ *y* **for** *l y*
proof −
have *interior* (*K* ∩ *l*) ⊆ *interior* (*l* ∩ *y*)
by (*metis Int_lower2 interior_mono le_inf_iff that*(*4*))
then have *interior* (*K* ∩ *l*) = {}
by (*simp add*: *snd_p*(*5*) *that*)
moreover from *d′*(*4*)[*OF k*] *snd_p*(*4*)[*OF that*(*1*)]
obtain *u1 v1 u2 v2*
where *uv*: *K* = *cbox u1 u2 l* = *cbox v1 v2* **by** *metis*
ultimately show *?thesis*
using *that integral_null*
unfolding *uv Int_interval content_eq_0_interior*
by (*metis* (*mono_tags*, *lifting*) *norm_eq_zero*)
qed
qed
finally show *norm* (*integral K f*) ≤ (∑ *l*∈*snd* ‘ *p*. *norm* (*integral* (*K* ∩ *l*) *f*)) .
qed
also have ... = (∑ (*i*,*l*) ∈ *d* × *snd* ‘ *p*. *norm* (*integral* (*i*∩*l*) *f*))
by (*simp add*: *sum.cartesian_product*)
also have ... = (∑ *x* ∈ *d* × *snd* ‘ *p*. *norm* (*integral* (*case_prod* (∩) *x*) *f*))
by (*force simp*: *split_def intro*!: *sum.cong*)
also have ... = (∑ *k*∈{*i* ∩ *l* |*i l*. *i* ∈ *d* ∧ *l* ∈ *snd* ‘ *p*}. *norm* (*integral k f*))
proof −
have *eq0*: (*integral* (*l1* ∩ *k1*) *f*) = *0*
if *l1* ∩ *k1* = *l2* ∩ *k2* (*l1*, *k1*) ≠ (*l2*, *k2*)
l1 ∈ *d* (*j1*,*k1*) ∈ *p l2* ∈ *d* (*j2*,*k2*) ∈ *p*
for *l1 l2 k1 k2 j1 j2*
proof −
obtain *u1 v1 u2 v2* **where** *uv*: *l1* = *cbox u1 u2 k1* = *cbox v1 v2*

```
        using ‹(j1, k1) ∈ p› ‹l1 ∈ d› d'(4) p'(4) by blast
      have l1 ≠ l2 ∨ k1 ≠ k2
        using that by auto
       then have interior k1 ∩ interior k2 = {} ∨ interior l1 ∩ interior l2
= {}
      by (meson d'(5) old.prod.inject p'(5) that(3) that(4) that(5) that(6))
      moreover have interior (l1 ∩ k1) = interior (l2 ∩ k2)
        by (simp add: that(1))
      ultimately have interior(l1 ∩ k1) = {}
        by auto
      then show ?thesis
        unfolding uv Int_interval content_eq_0_interior[symmetric] by auto
    qed
    show ?thesis
      unfolding ∗
     apply (rule sum.reindex_nontrivial [OF fin_d_sndp, symmetric, unfolded
o_def])
      apply clarsimp
      by (metis eq0 fst_conv snd_conv)
  qed
  also have ... = (∑(x,k) ∈ p'. norm (integral k f))
    unfolding sum_p'
  proof (rule sum.mono_neutral_right)
    show finite {i ∩ l |i l. i ∈ d ∧ l ∈ snd ‘ p}
      by (metis ∗ finite_imageI[OF fin_d_sndp])
    show snd ‘ p' ⊆ {i ∩ l |i l. i ∈ d ∧ l ∈ snd ‘ p}
      by (clarsimp simp: p'_def) (metis image_eqI snd_conv)
     show ∀i∈{i ∩ l |i l. i ∈ d ∧ l ∈ snd ‘ p} − snd ‘ p'. norm (integral i
f) = 0
          by clarsimp (metis Henstock_Kurzweil_Integration.integral_empty
disjoint_iff image_eqI in_p' snd_conv)
  qed
  finally show ?thesis .
 qed
 show (∑(x,k) ∈ p'. norm (content k ∗R f x)) = (∑(x,k) ∈ p. content k
∗R norm (f x))
 proof −
  let ?S = {(x, i ∩ l) |x i l. (x, l) ∈ p ∧ i ∈ d}
  have ∗: ?S = (λ(xl,i). (fst xl, snd xl ∩ i)) ‘ (p × d)
    by force
  have fin_pd: finite (p × d)
    using finite_cartesian_product[OF p'(1) d'(1)] by metis
  have (∑(x,k) ∈ p'. norm (content k ∗R f x)) = (∑(x,k) ∈ ?S. |content
k| ∗ norm (f x))
    unfolding norm_scaleR
  proof (rule sum.mono_neutral_left)
    show finite {(x, i ∩ l) |x i l. (x, l) ∈ p ∧ i ∈ d}
      by (simp add: ∗ fin_pd)
  qed (use p'alt in ‹force+›)
```

```
also have ... = (∑((x,l),i)∈p × d. |content (l ∩ i)| * norm (f x))
proof -
  have |content (l1 ∩ k1)| * norm (f x1) = 0
    if (x1, l1) ∈ p (x2, l2) ∈ p k1 ∈ d k2 ∈ d
      x1 = x2 l1 ∩ k1 = l2 ∩ k2 x1 ≠ x2 ∨ l1 ≠ l2 ∨ k1 ≠ k2
    for x1 l1 k1 x2 l2 k2
  proof -
    obtain u1 v1 u2 v2 where uv: k1 = cbox u1 u2 l1 = cbox v1 v2
      by (meson ‹(x1, l1) ∈ p› ‹k1 ∈ d› d(1) division_ofD(4) p'(4))
    have l1 ≠ l2 ∨ k1 ≠ k2
      using that by auto
     then have interior k1 ∩ interior k2 = {} ∨ interior l1 ∩ interior l2
= {}
      using that p'(5) d'(5) by (metis snd_conv)
    moreover have interior (l1 ∩ k1) = interior (l2 ∩ k2)
      unfolding that ..
    ultimately have interior (l1 ∩ k1) = {}
      by auto
    then show |content (l1 ∩ k1)| * norm (f x1) = 0
      unfolding uv Int_interval content_eq_0_interior[symmetric] by auto
  qed
  then show ?thesis
    unfolding *
    apply (subst sum.reindex_nontrivial [OF fin_pd])
    unfolding split_paired_all o_def split_def prod.inject
    by force+
qed
also have ... = (∑(x,k) ∈ p. content k *R norm (f x))
proof -
  have sumeq: (∑i∈d. content (l ∩ i) * norm (f x)) = content l * norm
(f x)
    if (x, l) ∈ p for x l
  proof -
    note xl = p'(2-4)[OF that]
    then obtain u v where uv: l = cbox u v by blast
    have (∑i∈d. |content (l ∩ i)|) = (∑k∈d. content (k ∩ cbox u v))
      by (simp add: Int_commute uv)
    also have ... = sum content {k ∩ cbox u v| k. k ∈ d}
    proof -
      have eq0: content (k ∩ cbox u v) = 0
        if k ∈ d y ∈ d k ≠ y and eq: k ∩ cbox u v = y ∩ cbox u v for k y
      proof -
        from d'(4)[OF that(1)] d'(4)[OF that(2)]
        obtain α β where α: k ∩ cbox u v = cbox α β
          by (meson Int_interval)
        have {} = interior ((k ∩ y) ∩ cbox u v)
          by (simp add: d'(5) that)
        also have ... = interior (y ∩ (k ∩ cbox u v))
          by auto
```

```
                  also have ... = interior (k ∩ cbox u v)
                    unfolding eq by auto
                  finally show ?thesis
                    unfolding α content_eq_0_interior ..
                qed
                then show ?thesis
                  unfolding Setcompr_eq_image
                  by (fastforce intro: sum.reindex_nontrivial [OF ‹finite d›, unfolded
o_def, symmetric])
              qed
              also have ... = sum content {cbox u v ∩ k |k. k ∈ d ∧ cbox u v ∩ k
≠ {}}
              proof (rule sum.mono_neutral_right)
                show finite {k ∩ cbox u v |k. k ∈ d}
                  by (simp add: d'(1))
              qed (fastforce simp: inf.commute)+
              finally have (∑ i∈d. |content (l ∩ i)|) = content (cbox u v)
              using additive_content_division[OF division_inter_1[OF d(1)]] uv xl(2)
by auto
              then show (∑ i∈d. content (l ∩ i) * norm (f x)) = content l * norm
(f x)
                unfolding sum_distrib_right[symmetric] using uv by auto
            qed
            show ?thesis
             by (subst sum_Sigma_product[symmetric]) (auto intro!: sumeq sum.cong
p' d')
          qed
          finally show ?thesis .
        qed
      qed (rule d)
    qed
  qed
  then show ?thesis
    using absolutely_integrable_onI [OF f has_integral_integrable] has_integral[of _
?S]
    by blast
qed


lemma bounded_variation_absolutely_integrable:
  fixes f :: 'n::euclidean_space ⇒ 'm::euclidean_space
  assumes f integrable_on UNIV
    and ∀ d. d division_of (⋃ d) ⟶ sum (λk. norm (integral k f)) d ≤ B
  shows f absolutely_integrable_on UNIV
proof (rule absolutely_integrable_onI, fact)
  let ?f = λD. ∑ k∈D. norm (integral k f) and ?D = {d. d division_of (⋃ d)}
  define SDF where SDF ≡ SUP d∈?D. ?f d
  have D_1: ?D ≠ {}
    by (rule elementary_interval) auto
```

```
  have D_2: bdd_above (?f`?D)
    using assms(2) by auto
  have f_int: ⋀a b. f absolutely_integrable_on cbox a b
    using assms integrable_on_subcbox
    by (blast intro!: bounded_variation_absolutely_integrable_interval)
  have ∃B>0. ∀a b. ball 0 B ⊆ cbox a b ⟶
                 |integral (cbox a b) (λx. norm (f x)) − SDF| < e
    if 0 < e for e
  proof −
    have ∃y ∈ ?f ` ?D. ¬ y ≤ SDF − e
    proof (rule ccontr)
      assume ¬ ?thesis
      then have SDF ≤ SDF − e
        unfolding SDF_def
        by (metis (mono_tags) D_1 cSUP_least image_eqI)
      then show False
        using that by auto
    qed
   then obtain d K where ddiv: d division_of ⋃d and K = ?f d SDF − e < K
     by (auto simp add: image_iff not_le)
    then have d: SDF − e < ?f d
      by auto
    note d'=division_ofD[OF ddiv]
    have bounded (⋃d)
      using ddiv by blast
    then obtain K where K: 0 < K ∀x∈⋃d. norm x ≤ K
      using bounded_pos by blast
    show ?thesis
    proof (intro conjI impI allI exI)
      fix a b :: 'n
      assume ab: ball 0 (K + 1) ⊆ cbox a b
      have *: ⋀s s1. ⟦SDF − e < s1; s1 ≤ s; s < SDF + e⟧ ⟹ |s − SDF| < e
        by arith
      show |integral (cbox a b) (λx. norm (f x)) − SDF| < e
        unfolding real_norm_def
      proof (rule * [OF d])
        have ?f d ≤ sum (λk. integral k (λx. norm (f x))) d
        proof (intro sum_mono)
          fix k assume k ∈ d
          with d'(4) f_int show norm (integral k f) ≤ integral k (λx. norm (f x))
           by (force simp: absolutely_integrable_on_def integral_norm_bound_integral)
        qed
        also have ... = integral (⋃d) (λx. norm (f x))
           by (metis (full_types) absolutely_integrable_on_def d'(4) ddiv f_int inte-
gral_combine_division_bottomup)
        also have ... ≤ integral (cbox a b) (λx. norm (f x))
        proof −
          have ⋃d ⊆ cbox a b
            using K(2) ab by fastforce
```

```
        then show ?thesis
          using integrable_on_subdivision[OF ddiv] f_int[of a b] unfolding abso-
lutely_integrable_on_def
          by (auto intro!: integral_subset_le)
      qed
      finally show ?f d ≤ integral (cbox a b) (λx. norm (f x)) .
    next
      have e/2>0
        using ‹e > 0› by auto
      moreover
      have f: f integrable_on cbox a b (λx. norm (f x)) integrable_on cbox a b
        using f_int by (auto simp: absolutely_integrable_on_def)
      ultimately obtain d1 where gauge d1
         and d1: ⋀p. ⟦p tagged_division_of (cbox a b); d1 fine p⟧ ⟹
          norm ((∑(x,k) ∈ p. content k *R norm (f x)) − integral (cbox a b) (λx.
norm (f x))) < e/2
        unfolding has_integral_integral has_integral by meson
      obtain d2 where gauge d2
        and d2: ⋀p. ⟦p tagged_partial_division_of (cbox a b); d2 fine p⟧ ⟹
          (∑(x,k) ∈ p. norm (content k *R f x − integral k f)) < e/2
        by (blast intro: Henstock_lemma [OF f(1) ‹e/2>0›])
      obtain p where
        p: p tagged_division_of (cbox a b) d1 fine p d2 fine p
        by (rule fine_division_exists [OF gauge_Int [OF ‹gauge d1› ‹gauge d2›], of
a b])
          (auto simp add: fine_Int)
      have *: ⋀sf sf' si di. ⟦sf' = sf; si ≤ SDF; |sf − si| < e/2;
                    |sf' − di| < e/2⟧ ⟹ di < SDF + e
        by arith
      have integral (cbox a b) (λx. norm (f x)) < SDF + e
      proof (rule *)
        show |(∑(x,k)∈p. norm (content k *R f x)) − (∑(x,k)∈p. norm (integral
k f))| < e/2
          unfolding split_def
        proof (rule absdiff_norm_less)
          show (∑p∈p. norm (content (snd p) *R f (fst p) − integral (snd p) f))
< e/2
            using d2[of p] p(1,3) by (auto simp: tagged_division_of_def split_def)
        qed
        show |(∑(x,k) ∈ p. content k *R norm (f x)) − integral (cbox a b) (λx.
norm(f x))| < e/2
          using d1[OF p(1,2)] by (simp only: real_norm_def)
          show (∑(x,k) ∈ p. content k *R norm (f x)) = (∑(x,k) ∈ p. norm
(content k *R f x))
          by (auto simp: split_paired_all sum.cong [OF refl])
        have (∑(x,k) ∈ p. norm (integral k f)) = (∑k∈snd ‘ p. norm (integral
k f))
          apply (rule sum.over_tagged_division_lemma[OF p(1)])
        by (metis Henstock_Kurzweil_Integration.integral_empty integral_open_interval
```

norm_zero)
        also have ... ≤ *SDF*
          using *partial_division_of_tagged_division*[*of p cbox a b*] *p*(*1*)
         by (*auto simp*: *SDF_def tagged_partial_division_of_def intro*!: *cSUP_upper2 D_1 D_2*)
        finally show $(\sum(x,k) \in p.\ norm\ (integral\ k\ f)) \leq SDF$ **.**
      qed
      then show *integral* (*cbox a b*) (λ*x*. *norm* (*f x*)) < *SDF* + *e*
        by *simp*
    qed
  qed (*use K* **in** *auto*)
  qed
  moreover have ⋀*a b*. (λ*x*. *norm* (*f x*)) *integrable_on cbox a b*
    using *absolutely_integrable_on_def f_int* **by** *auto*
  ultimately
  have ((λ*x*. *norm* (*f x*)) *has_integral SDF*) *UNIV*
    by (*auto simp*: *has_integral_alt*′)
  then show (λ*x*. *norm* (*f x*)) *integrable_on UNIV*
    by *blast*
qed

6.19.15 Outer and inner approximation of measurable sets by well-behaved sets.

proposition *measurable_outer_intervals_bounded*:
  assumes *S* ∈ *lmeasurable S* ⊆ *cbox a b e* > *0*
  obtains $\mathcal{D}$
  where *countable* $\mathcal{D}$
      $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow K \subseteq cbox\ a\ b \wedge K \neq \{\} \wedge (\exists c\ d.\ K = cbox\ c\ d)$
      pairwise (λ*A B*. *interior A* ∩ *interior B* = {}) $\mathcal{D}$
      $\bigwedge u\ v.\ cbox\ u\ v \in \mathcal{D} \Longrightarrow \exists n.\ \forall i \in Basis.\ v \cdot i - u \cdot i = (b \cdot i - a \cdot i)/2\hat{\ }n$
      $\bigwedge K.\ [\![K \in \mathcal{D};\ box\ a\ b \neq \{\}]\!] \Longrightarrow interior\ K \neq \{\}$
      $S \subseteq \bigcup\mathcal{D}\ \bigcup\mathcal{D} \in lmeasurable\ measure\ lebesgue\ (\bigcup\mathcal{D}) \leq measure\ lebesgue\ S + e$
proof (*cases box a b* = {})
  case *True*
  show *?thesis*
  proof (*cases cbox a b* = {})
    case *True*
    with *assms* **have** [*simp*]: *S* = {}
      by *auto*
    show *?thesis*
    proof
      show *countable* {}
        by *simp*
    qed (*use* ‹*e* > *0*› **in** *auto*)
  next
    case *False*
    show *?thesis*

proof
show *countable* {*cbox a b*}
by *simp*
show $\bigwedge u\ v.\ cbox\ u\ v \in \{cbox\ a\ b\} \Longrightarrow \exists\ n.\ \forall\ i{\in}Basis.\ v \cdot i - u \cdot i = (b \cdot i - a \cdot i)/2\ \hat{}\ n$
using *False* **by** (*force simp*: *eq_cbox intro*: *exI* [**where** *x=0*])
show *measure lebesgue* ($\bigcup$ {*cbox a b*}) ≤ *measure lebesgue S* + *e*
using *assms* **by** (*simp add*: *sum_content.box_empty_imp* [*OF True*])
qed (*use assms* ‹*cbox a b* ≠ {}› **in** *auto*)
qed
next
case *False*
let *?μ* = *measure lebesgue*
have *S* ∩ *cbox a b* ∈ *lmeasurable*
using ‹*S* ∈ *lmeasurable*› **by** *blast*
then have *indS_int*: (*indicator S has_integral* (*?μ S*)) (*cbox a b*)
by (*metis integral_indicator* ‹*S* ⊆ *cbox a b*› *has_integral_integrable_integral inf.orderE integrable_on_indicator*)
with ‹*e* > *0*› **obtain** γ **where** *gauge* γ **and** γ:
$\bigwedge\mathcal{D}.\ [\![\mathcal{D}\ tagged\_division\_of\ (cbox\ a\ b);\ \gamma\ fine\ \mathcal{D}]\!] \Longrightarrow norm\ ((\sum(x,K){\in}\mathcal{D}.\ content(K) *_R indicator\ S\ x) - ?\mu\ S) < e$
by (*force simp*: *has_integral*)
have *inteq*: *integral* (*cbox a b*) (*indicat_real S*) = *integral UNIV* (*indicator S*)
using *assms* **by** (*metis has_integral_iff indS_int lmeasure_integral_UNIV*)
obtain 𝒟 **where** 𝒟: *countable* 𝒟 $\bigcup\mathcal{D}$ ⊆ *cbox a b*
and *cbox*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow interior\ K \neq \{\} \wedge (\exists\ c\ d.\ K = cbox\ c\ d)$
and *djointish*: *pairwise* (λ*A B*. *interior A* ∩ *interior B* = {}) 𝒟
and *covered*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow \exists\ x \in S \cap K.\ K \subseteq \gamma\ x$
and *close*: $\bigwedge u\ v.\ cbox\ u\ v \in \mathcal{D} \Longrightarrow \exists\ n.\ \forall\ i \in Basis.\ v{\cdot}i - u{\cdot}i = (b{\cdot}i - a{\cdot}i)/2\hat{}n$
and *covers*: $S \subseteq \bigcup\mathcal{D}$
using *covering_lemma* [*of S a b* γ] ‹*gauge* γ› ‹*box a b* ≠ {}› *assms* **by** *force*
show *?thesis*
proof
show $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow K \subseteq cbox\ a\ b \wedge K \neq \{\} \wedge (\exists\ c\ d.\ K = cbox\ c\ d)$
by (*meson Sup_le_iff* 𝒟(*2*) *cbox interior_empty*)
have *negl_int*: *negligible*(*K* ∩ *L*) **if** *K* ∈ 𝒟 *L* ∈ 𝒟 *K* ≠ *L* **for** *K L*
proof −
have *interior K* ∩ *interior L* = {}
using *djointish pairwiseD that* **by** *fastforce*
moreover obtain *u v x y* **where** *K* = *cbox u v L* = *cbox x y*
using *cbox* ‹*K* ∈ 𝒟› ‹*L* ∈ 𝒟› **by** *blast*
ultimately show *?thesis*
by (*simp add*: *Int_interval box_Int_box negligible_interval*(*1*))
qed
have *fincase*: $\bigcup\mathcal{F} \in lmeasurable \wedge ?\mu\ (\bigcup\mathcal{F}) \leq ?\mu\ S + e$ **if** *finite* ℱ ℱ ⊆ 𝒟 **for** ℱ
proof −
obtain *t* **where** *t*: $\bigwedge K.\ K \in \mathcal{F} \Longrightarrow t\ K \in S \cap K \wedge K \subseteq \gamma(t\ K)$

using *covered* ‹$\mathcal{F} \subseteq \mathcal{D}$› *subsetD* **by** *metis*
have $\forall K \in \mathcal{F}. \forall L \in \mathcal{F}. K \neq L \longrightarrow$ *interior* $K \cap$ *interior* $L = \{\}$
using *that djointish* **by** (*simp add*: *pairwise_def*) (*metis subsetD*)
with *cbox that* $\mathcal{D}$ **have** $\mathcal{F}$*div*: $\mathcal{F}$ *division_of* ($\bigcup \mathcal{F}$)
by (*fastforce simp*: *division_of_def dest*: *cbox*)
then have *1*: $\bigcup \mathcal{F} \in$ *lmeasurable*
by *blast*
have *norme*: $\bigwedge p$. ⟦*p tagged_division_of cbox a b*; γ *fine p*⟧
$\Longrightarrow$ *norm* (($\sum$(*x*,*K*)$\in$*p*. *content K* $*$ *indicator S x*) $-$ *integral* (*cbox a b*) (*indicator S*)) $<$ *e*
by (*auto simp*: *lmeasure_integral_UNIV assms inteq dest*: γ)
have $\forall x\ K\ y\ L.$ (*x*,*K*) $\in$ (λK. (*t K*,*K*)) ‘ $\mathcal{F} \wedge$ (*y*,*L*) $\in$ (λK. (*t K*,*K*)) ‘ $\mathcal{F} \wedge$ (*x*,*K*) $\neq$ (*y*,*L*) $\longrightarrow$ *interior K* $\cap$ *interior L* $= \{\}$
using *that djointish* **by** (*clarsimp simp*: *pairwise_def*) (*metis subsetD*)
with *that* $\mathcal{D}$ **have** *tagged*: (λK. (*t K*, *K*)) ‘ $\mathcal{F}$ *tagged_partial_division_of cbox a b*
by (*auto simp*: *tagged_partial_division_of_def dest*: *t cbox*)
have *fine*: γ *fine* (λK. (*t K*, *K*)) ‘ $\mathcal{F}$
using *t* **by** (*auto simp*: *fine_def*)
have $*$: $y \leq$ *?μ S* $\Longrightarrow |x - y| \leq e \Longrightarrow x \leq$ *?μ S* $+ e$ **for** *x y*
by *arith*
have *?μ* ($\bigcup \mathcal{F}$) $\leq$ *?μ S* $+ e$
proof (*rule* $*$)
have ($\sum K\in\mathcal{F}$. *?μ* ($K \cap S$)) $=$ *?μ* ($\bigcup C\in\mathcal{F}$. $C \cap S$)
proof (*rule measure_negligible_finite_Union_image* [*OF* ‹*finite* $\mathcal{F}$›, *symmetric*])
show $\bigwedge K$. $K \in \mathcal{F} \Longrightarrow K \cap S \in$ *lmeasurable*
using $\mathcal{F}$*div* ‹$S \in$ *lmeasurable*› **by** *blast*
show *pairwise* ($\lambda K\ y$. *negligible* ($K \cap S \cap (y \cap S)$)) $\mathcal{F}$
unfolding *pairwise_def*
by (*metis inf.commute inf_sup_aci*(*3*) *negligible_Int subsetCE negl_int* ‹$\mathcal{F} \subseteq \mathcal{D}$›)
qed
also have ... $=$ *?μ* ($\bigcup \mathcal{F} \cap S$)
by *simp*
also have ... $\leq$ *?μ S*
by (*simp add*: *1* ‹$S \in$ *lmeasurable*› *fmeasurableD measure_mono_fmeasurable sets.Int*)
finally show ($\sum K\in\mathcal{F}$. *?μ* ($K \cap S$)) $\leq$ *?μ S* **.**
next
have *?μ* ($\bigcup \mathcal{F}$) $=$ *sum ?μ* $\mathcal{F}$
by (*metis* $\mathcal{F}$*div content_division*)
also have ... $=$ ($\sum K\in\mathcal{F}$. *content K*)
using $\mathcal{F}$*div* **by** (*force intro*: *sum.cong*)
also have ... $=$ ($\sum x\in\mathcal{F}$. *content x* $*$ *indicator S* (*t x*))
using *t* **by** *auto*
finally have *eq1*: *?μ* ($\bigcup \mathcal{F}$) $=$ ($\sum x\in\mathcal{F}$. *content x* $*$ *indicator S* (*t x*)) **.**
have *eq2*: ($\sum K\in\mathcal{F}$. *?μ* ($K \cap S$)) $=$ ($\sum K\in\mathcal{F}$. *integral K* (*indicator S*))
apply (*rule sum.cong* [*OF refl*])
by (*metis integral_indicator* $\mathcal{F}$*div* ‹$S \in$ *lmeasurable*› *division_ofD*(*4*)

```
fmeasurable.Int inf.commute lmeasurable_cbox)
        have |∑(x,K)∈(λK. (t K, K)) ` ℱ. content K * indicator S x − integral
K (indicator S)| ≤ e
         using Henstock_lemma_part1 [of indicator S::'a⇒real, OF _ ‹e > 0› ‹gauge
γ› _ tagged fine]
            indS_int norme by auto
        then show |?μ (⋃ℱ) − (∑K∈ℱ. ?μ (K ∩ S))| ≤ e
            by (simp add: eq1 eq2 comm_monoid_add_class.sum.reindex inj_on_def
sum_subtractf)
      qed
      with 1 show ?thesis by blast
    qed
    have ⋃𝒟 ∈ lmeasurable ∧ ?μ (⋃𝒟) ≤ ?μ S + e
    proof (cases finite 𝒟)
      case True
      with fincase show ?thesis
        by blast
    next
      case False
      let ?T = from_nat_into 𝒟
      have T: bij_betw ?T UNIV 𝒟
        by (simp add: False 𝒟(1) bij_betw_from_nat_into)
      have TM: ⋀n. ?T n ∈ lmeasurable
        by (metis False cbox finite.emptyI from_nat_into lmeasurable_cbox)
      have TN: ⋀m n. m ≠ n ⟹ negligible (?T m ∩ ?T n)
        by (simp add: False 𝒟(1) from_nat_into infinite_imp_nonempty negl_int)
      have TB: (∑k≤n. ?μ (?T k)) ≤ ?μ S + e for n
      proof −
        have (∑k≤n. ?μ (?T k)) = ?μ (⋃ (?T ` {..n}))
         by (simp add: pairwise_def TM TN measure_negligible_finite_Union_image)
        also have ?μ (⋃ (?T ` {..n})) ≤ ?μ S + e
          using fincase [of ?T ` {..n}] T by (auto simp: bij_betw_def)
        finally show ?thesis .
      qed
      have ⋃𝒟 ∈ lmeasurable
           by (metis lmeasurable_compact T 𝒟(2) bij_betw_def cbox compact_cbox
countable_Un_Int(1) fmeasurableD fmeasurableI2 rangeI)
      moreover
      have ?μ (⋃x. from_nat_into 𝒟 x) ≤ ?μ S + e
      proof (rule measure_countable_Union_le [OF TM])
        show ?μ (⋃x≤n. from_nat_into 𝒟 x) ≤ ?μ S + e for n
            by (metis (mono_tags, lifting) False fincase finite.emptyI finite_atMost
finite_imageI from_nat_into imageE subsetI)
      qed
      ultimately show ?thesis by (metis T bij_betw_def)
    qed
    then show ⋃𝒟 ∈ lmeasurable measure lebesgue (⋃𝒟) ≤ ?μ S + e by blast+
  qed (use 𝒟 cbox djointish close covers in auto)
qed
```

6.19.16 Transformation of measure by linear maps

lemma *emeasure_lebesgue_ball_conv_unit_ball*:
  fixes *c* :: *'a* :: *euclidean_space*
  assumes $r \geq 0$
  shows *emeasure lebesgue* (*ball c r*) =
      ennreal (*r* ^ *DIM*(*'a*)) ∗ *emeasure lebesgue* (*ball* (*0* :: *'a*) *1*)
proof (*cases r* = *0*)
  case *False*
  with *assms* **have** *r*: *r* > *0* **by** *auto*
  have *emeasure lebesgue* ((λ*x*. *c* + *x*) ' (λ*x*. *r* $*_R$ *x*) ' *ball* (*0* :: *'a*) *1*) =
     r ^ *DIM*(*'a*) ∗ *emeasure lebesgue* (*ball* (*0* :: *'a*) *1*)
   unfolding *image_image* **using** *emeasure_lebesgue_affine*[*of r c ball 0 1*] *assms*
   by (*simp add*: *add_ac*)
  also have (λ*x*. *r* $*_R$ *x*) ' *ball 0 1* = *ball* (*0* :: *'a*) *r*
   using *r* **by** (*subst ball_scale*) *auto*
  also have (λ*x*. *c* + *x*) ' ... = *ball c r*
   by (*subst image_add_ball*) (*simp_all add*: *algebra_simps*)
  finally show *?thesis* **by** *simp*
qed *auto*

lemma *content_ball_conv_unit_ball*:
  fixes *c* :: *'a* :: *euclidean_space*
  assumes $r \geq 0$
  shows *content* (*ball c r*) = *r* ^ *DIM*(*'a*) ∗ *content* (*ball* (*0* :: *'a*) *1*)
proof −
  have *ennreal* (*content* (*ball c r*)) = *emeasure lebesgue* (*ball c r*)
   using *emeasure_lborel_ball_finite*[*of c r*] **by** (*subst emeasure_eq_ennreal_measure*)
auto
  also have ... = *ennreal* (*r* ^ *DIM*(*'a*)) ∗ *emeasure lebesgue* (*ball* (*0* :: *'a*) *1*)
   using *assms* **by** (*intro emeasure_lebesgue_ball_conv_unit_ball*) *auto*
  also have ... = *ennreal* (*r* ^ *DIM*(*'a*) ∗ *content* (*ball* (*0*::*'a*) *1*))
   using *emeasure_lborel_ball_finite*[*of 0*::*'a 1*] *assms*
   by (*subst emeasure_eq_ennreal_measure*) (*auto simp*: *ennreal_mult'*)
  finally show *?thesis*
   using *assms* **by** (*subst* (*asm*) *ennreal_inj*) *auto*
qed

lemma *measurable_linear_image_interval*:
  linear f ⟹ *f* ' (*cbox a b*) ∈ *lmeasurable*
 by (*metis bounded_linear_image linear_linear bounded_cbox closure_bounded_linear_image closure_cbox compact_closure lmeasurable_compact*)

proposition *measure_linear_sufficient*:
  fixes *f* :: *'n*::*euclidean_space* ⇒ *'n*
  assumes *linear f* **and** *S*: *S* ∈ *lmeasurable*
   and *im*: ⋀*a b*. *measure lebesgue* (*f* ' (*cbox a b*)) = *m* ∗ *measure lebesgue* (*cbox a b*)
  shows *f* ' *S* ∈ *lmeasurable* ∧ *m* ∗ *measure lebesgue S* = *measure lebesgue* (*f* ' *S*)
  using *le_less_linear* [*of 0 m*]

```
proof
  assume m < 0
  then show ?thesis
    using im [of 0 One] by auto
next
  assume m ≥ 0
  let ?μ = measure lebesgue
  show ?thesis
  proof (cases inj f)
    case False
    then have ?μ (f ‘ S) = 0
       using ‹linear f› negligible_imp_measure0 negligible_linear_singular_image by
blast
    then have m ∗ ?μ (cbox 0 (One)) = 0
        by (metis False ‹linear f› cbox_borel content_unit im measure_completion
negligible_imp_measure0 negligible_linear_singular_image sets_lborel)
    then show ?thesis
      using ‹linear f› negligible_linear_singular_image negligible_imp_measure0 False
      by (auto simp: lmeasurable_iff_has_integral negligible_UNIV)
  next
    case True
    then obtain h where linear h and hf: ⋀x. h (f x) = x and fh: ⋀x. f (h x)
= x
      using ‹linear f› linear_injective_isomorphism by blast
    have fBS: (f ‘ S) ∈ lmeasurable ∧ m ∗ ?μ S = ?μ (f ‘ S)
      if bounded S S ∈ lmeasurable for S
    proof −
      obtain a b where S ⊆ cbox a b
        using ‹bounded S› bounded_subset_cbox_symmetric by metis
      have fUD: (f ‘ ⋃𝒟) ∈ lmeasurable ∧ ?μ (f ‘ ⋃𝒟) = (m ∗ ?μ (⋃𝒟))
        if countable 𝒟
          and cbox: ⋀K. K ∈ 𝒟 ⟹ K ⊆ cbox a b ∧ K ≠ {} ∧ (∃ c d. K = cbox
c d)
          and intint: pairwise (λA B. interior A ∩ interior B = {}) 𝒟
        for 𝒟
      proof −
        have conv: ⋀K. K ∈ 𝒟 ⟹ convex K
          using cbox convex_box(1) by blast
        have neg: negligible (g ‘ K ∩ g ‘ L) if linear g K ∈ 𝒟 L ∈ 𝒟 K ≠ L
          for K L and g :: 'n⇒'n
        proof (cases inj g)
          case True
          have negligible (frontier(g ‘ K ∩ g ‘ L) ∪ interior(g ‘ K ∩ g ‘ L))
          proof (rule negligible_Un)
            show negligible (frontier (g ‘ K ∩ g ‘ L))
          by (simp add: negligible_convex_frontier convex_Int conv convex_linear_image
that)
          next
            have ∀ p N. pairwise p N = (∀ Na. (Na::'n set) ∈ N ⟶ (∀ Nb. Nb ∈ N
```

$\wedge$ *Na* $\neq$ *Nb* $\longrightarrow$ *p Na Nb*))
by (*metis pairwise_def*)
then have *interior K* $\cap$ *interior L* = {}
using *intint that*(*2*) *that*(*3*) *that*(*4*) **by** *presburger*
then show *negligible* (*interior* (*g* ' *K* $\cap$ *g* ' *L*))
by (*metis True empty_imp_negligible image_Int image_empty interior_Int interior_injective_linear_image that*(*1*))
qed
moreover have *g* ' *K* $\cap$ *g* ' *L* $\subseteq$ *frontier* (*g* ' *K* $\cap$ *g* ' *L*) $\cup$ *interior* (*g* ' *K* $\cap$ *g* ' *L*)
by (*metis Diff_partition Int_commute calculation closure_Un_frontier frontier_def inf.absorb_iff2 inf_bot_right inf_sup_absorb negligible_Un_eq open_interior open_not_negligible sup_commute*)
ultimately show *?thesis*
by (*rule negligible_subset*)
next
case *False*
then show *?thesis*
by (*simp add*: *negligible_Int negligible_linear_singular_image* ‹*linear g*›)
qed
have *negf*: *negligible* ((*f* ' *K*) $\cap$ (*f* ' *L*))
and *negid*: *negligible* (*K* $\cap$ *L*) **if** *K* $\in$ $\mathcal{D}$ *L* $\in$ $\mathcal{D}$ *K* $\neq$ *L* **for** *K L*
using *neg* [*OF* ‹*linear f*›] *neg* [*OF linear_id*] *that* **by** *auto*
show *?thesis*
proof (*cases finite* $\mathcal{D}$)
case *True*
then have *?μ* ($\bigcup$ *x*∈$\mathcal{D}$. *f* ' *x*) = ($\sum$ *x*∈$\mathcal{D}$. *?μ* (*f* ' *x*))
using ‹*linear f*› *cbox measurable_linear_image_interval negf*
by (*blast intro*: *measure_negligible_finite_Union_image* [*unfolded pairwise_def*])
also have ... = ($\sum$ *k*∈$\mathcal{D}$. *m* ∗ *?μ k*)
by (*metis* (*no_types*, *lifting*) *cbox im sum.cong*)
also have ... = *m* ∗ *?μ* ($\bigcup$ $\mathcal{D}$)
unfolding *sum_distrib_left* [*symmetric*]
by (*metis True cbox lmeasurable_cbox measure_negligible_finite_Union* [*unfolded pairwise_def*] *negid*)
finally show *?thesis*
by (*metis True* ‹*linear f*› *cbox image_Union fmeasurable.finite_UN measurable_linear_image_interval*)
next
case *False*
with ‹*countable* $\mathcal{D}$› **obtain** *X* :: *nat* ⇒ *'n set* **where** *S*: *bij_betw X UNIV* $\mathcal{D}$
using *bij_betw_from_nat_into* **by** *blast*
then have *eq*: ($\bigcup$ $\mathcal{D}$) = ($\bigcup$ *n*. *X n*) (*f* ' $\bigcup$ $\mathcal{D}$) = ($\bigcup$ *n*. *f* ' *X n*)
by (*auto simp*: *bij_betw_def*)
have *meas*: $\bigwedge$*K*. *K* $\in$ $\mathcal{D}$ $\Longrightarrow$ *K* $\in$ *lmeasurable*
using *cbox* **by** *blast*
with *S* **have** *1*: $\bigwedge$*n*. *X n* $\in$ *lmeasurable*

by (*auto simp*: *bij_betw_def*)
have *2*: *pairwise* (λ*m n*. *negligible* (*X m* ∩ *X n*)) *UNIV*
using *S* **unfolding** *bij_betw_def pairwise_def* **by** (*metis injD negid range_eqI*)
have *bounded* (⋃𝒟)
by (*meson Sup_least bounded_cbox bounded_subset cbox*)
then have *3*: *bounded* (⋃*n*. *X n*)
using *S* **unfolding** *bij_betw_def* **by** *blast*
have (⋃*n*. *X n*) ∈ *lmeasurable*
by (*rule measurable_countable_negligible_Union_bounded* [*OF 1 2 3*])
with *S* **have** *f1*: ⋀*n*. *f* ‘ (*X n*) ∈ *lmeasurable*
unfolding *bij_betw_def* **by** (*metis assms*(*1*) *cbox measurable_linear_image_interval rangeI*)
have *f2*: *pairwise* (λ*m n*. *negligible* (*f* ‘ (*X m*) ∩ *f* ‘ (*X n*))) *UNIV*
using *S* **unfolding** *bij_betw_def pairwise_def* **by** (*metis injD negf rangeI*)
have *bounded* (⋃𝒟)
by (*meson Sup_least bounded_cbox bounded_subset cbox*)
then have *f3*: *bounded* (⋃*n*. *f* ‘ *X n*)
using *S* **unfolding** *bij_betw_def*
by (*metis bounded_linear_image linear_linear assms*(*1*) *image_Union range_composition*)
have (λ*n*. *?μ* (*X n*)) *sums ?μ* (⋃*n*. *X n*)
by (*rule measure_countable_negligible_Union_bounded* [*OF 1 2 3*])
have *meq*: *?μ* (⋃*n*. *f* ‘ *X n*) = *m* ∗ *?μ* (⋃(*X* ‘ *UNIV*))
proof (*rule sums_unique2* [*OF measure_countable_negligible_Union_bounded* [*OF f1 f2 f3*]])
have *m*: ⋀*n*. *?μ* (*f* ‘ *X n*) = (*m* ∗ *?μ* (*X n*))
using *S* **unfolding** *bij_betw_def* **by** (*metis cbox im rangeI*)
show (λ*n*. *?μ* (*f* ‘ *X n*)) *sums* (*m* ∗ *?μ* (⋃(*X* ‘ *UNIV*)))
unfolding *m*
using *measure_countable_negligible_Union_bounded* [*OF 1 2 3*] *sums_mult* **by** *blast*
qed
show *?thesis*
using *measurable_countable_negligible_Union_bounded* [*OF f1 f2 f3*] *meq*
by (*auto simp*: *eq* [*symmetric*])
qed
qed
show *?thesis*
unfolding *completion.fmeasurable_measure_inner_outer_le*
proof (*intro conjI allI impI*)
fix *e* :: *real*
assume *e* > *0*
have *1*: *cbox a b* − *S* ∈ *lmeasurable*
by (*simp add*: *fmeasurable.Diff that*)
have *2*: *0* < *e* / (*1* + |*m*|)
using ‹*e* > *0*› **by** (*simp add*: *field_split_simps abs_add_one_gt_zero*)
obtain 𝒟
where *countable* 𝒟

and *cbox*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow K \subseteq cbox\ a\ b \wedge K \neq \{\} \wedge (\exists c\ d.\ K = cbox\ c\ d)$
and *intdisj*: *pairwise* $(\lambda A\ B.\ interior\ A \cap interior\ B = \{\})\ \mathcal{D}$
and *DD*: $cbox\ a\ b - S \subseteq \bigcup\mathcal{D}\ \bigcup\mathcal{D} \in lmeasurable$
and *le*: $?\mu\ (\bigcup\mathcal{D}) \leq ?\mu\ (cbox\ a\ b - S) + e/(1 + |m|)$
by (*rule measurable_outer_intervals_bounded* [*of cbox a b* − *S a b e*/(*1* + |*m*|)]; *use 1 2 pairwise_def* **in** *force*)
show $\exists T \in lmeasurable.\ T \subseteq f\ `\ S \wedge m * ?\mu\ S - e \leq ?\mu\ T$
proof (*intro bexI conjI*)
show $f\ `\ (cbox\ a\ b) - f\ `\ (\bigcup\mathcal{D}) \subseteq f\ `\ S$
using ‹$cbox\ a\ b - S \subseteq \bigcup\mathcal{D}$› **by** *force*
have $m * ?\mu\ S - e \leq m * (?\mu\ S - e\ /\ (1 + |m|))$
using ‹$m \geq 0$› ‹$e > 0$› **by** (*simp add*: *field_simps*)
also have ... $\leq ?\mu\ (f\ `\ cbox\ a\ b) - ?\mu\ (f\ `\ (\bigcup\mathcal{D}))$
proof −
have $?\mu\ (cbox\ a\ b - S) = ?\mu\ (cbox\ a\ b) - ?\mu\ S$
by (*simp add*: *measurable_measure_Diff* ‹$S \subseteq cbox\ a\ b$› *fmeasurableD that*(*2*))
then have $(?\mu\ S - e\ /\ (1 + m)) \leq (content\ (cbox\ a\ b) - ?\mu\ (\bigcup\ \mathcal{D}))$
using ‹$m \geq 0$› *le* **by** *auto*
then show *?thesis*
using ‹$m \geq 0$› ‹$e > 0$›
by (*simp add*: *mult_left_mono im fUD* [*OF* ‹*countable* $\mathcal{D}$› *cbox intdisj*] *flip*: *right_diff_distrib*)
qed
also have ... $= ?\mu\ (f\ `\ cbox\ a\ b - f\ `\ \bigcup\mathcal{D})$
proof (*rule measurable_measure_Diff* [*symmetric*])
show $f\ `\ cbox\ a\ b \in lmeasurable$
by (*simp add*: *assms*(*1*) *measurable_linear_image_interval*)
show $f\ `\ \bigcup\ \mathcal{D} \in sets\ lebesgue$
by (*simp add*: ‹*countable* $\mathcal{D}$› *cbox fUD fmeasurableD intdisj*)
show $f\ `\ \bigcup\ \mathcal{D} \subseteq f\ `\ cbox\ a\ b$
by (*simp add*: *Sup_le_iff cbox image_mono*)
qed
finally show $m * ?\mu\ S - e \leq ?\mu\ (f\ `\ cbox\ a\ b - f\ `\ \bigcup\mathcal{D})$.
show $f\ `\ cbox\ a\ b - f\ `\ \bigcup\mathcal{D} \in lmeasurable$
by (*simp add*: *fUD* ‹*countable* $\mathcal{D}$› ‹*linear f*› *cbox fmeasurable.Diff intdisj measurable_linear_image_interval*)
qed
next
fix *e* :: *real*
assume $e > 0$
have *em*: $0 < e\ /\ (1 + |m|)$
using ‹$e > 0$› **by** (*simp add*: *field_split_simps abs_add_one_gt_zero*)
obtain $\mathcal{D}$
where *countable* $\mathcal{D}$
and *cbox*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow K \subseteq cbox\ a\ b \wedge K \neq \{\} \wedge (\exists c\ d.\ K = cbox\ c\ d)$
and *intdisj*: *pairwise* $(\lambda A\ B.\ interior\ A \cap interior\ B = \{\})\ \mathcal{D}$

```
          and DD: S ⊆ ⋃𝒟 ⋃𝒟 ∈ lmeasurable
          and le: ?μ (⋃𝒟) ≤ ?μ S + e/(1 + |m|)
        by (rule measurable_outer_intervals_bounded [of S a b e/(1 + |m|)]; use ‹S
∈ lmeasurable› ‹S ⊆ cbox a b› em in force)
      show ∃ U ∈ lmeasurable. f ‘ S ⊆ U ∧ ?μ U ≤ m ∗ ?μ S + e
      proof (intro bexI conjI)
        show f ‘ S ⊆ f ‘ (⋃𝒟)
          by (simp add: DD(1) image_mono)
        have ?μ (f ‘ ⋃𝒟) ≤ m ∗ (?μ S + e / (1 + |m|))
          using ‹m ≥ 0› le mult_left_mono
          by (auto simp: fUD ‹countable 𝒟› ‹linear f› cbox fmeasurable.Diff intdisj
measurable_linear_image_interval)
        also have . . . ≤ m ∗ ?μ S + e
            using ‹m ≥ 0› ‹e > 0› by (simp add: fUD [OF ‹countable 𝒟› cbox
intdisj] field_simps)
        finally show ?μ (f ‘ ⋃𝒟) ≤ m ∗ ?μ S + e .
        show f ‘ ⋃𝒟 ∈ lmeasurable
          by (simp add: ‹countable 𝒟› cbox fUD intdisj)
      qed
    qed
  qed
  show ?thesis
    unfolding has_measure_limit_iff
  proof (intro allI impI)
    fix e :: real
    assume e > 0
    obtain B where B > 0 and B:
      ⋀a b. ball 0 B ⊆ cbox a b =⇒ |?μ (S ∩ cbox a b) − ?μ S| < e / (1 + |m|)
        using has_measure_limit [OF S] ‹e > 0› by (metis abs_add_one_gt_zero
zero_less_divide_iff)
    obtain c d::'n where cd: ball 0 B ⊆ cbox c d
      by (metis bounded_subset_cbox_symmetric bounded_ball)
    with B have less: |?μ (S ∩ cbox c d) − ?μ S| < e / (1 + |m|) .
    obtain D where D > 0 and D: cbox c d ⊆ ball 0 D
      by (metis bounded_cbox bounded_subset_ballD)
    obtain C where C > 0 and C: ⋀x. norm (f x) ≤ C ∗ norm x
      using linear_bounded_pos ‹linear f› by blast
    have f ‘ S ∩ cbox a b ∈ lmeasurable ∧
           |?μ (f ‘ S ∩ cbox a b) − m ∗ ?μ S| < e
      if ball 0 (D∗C) ⊆ cbox a b for a b
    proof −
      have bounded (S ∩ h ‘ cbox a b)
        by (simp add: bounded_linear_image linear_linear ‹linear h› bounded_Int)
      moreover have Shab: S ∩ h ‘ cbox a b ∈ lmeasurable
      by (simp add: S ‹linear h› fmeasurable.Int measurable_linear_image_interval)
      moreover have fim: f ‘ (S ∩ h ‘ (cbox a b)) = (f ‘ S) ∩ cbox a b
        by (auto simp: hf rev_image_eqI fh)
      ultimately have 1: (f ‘ S) ∩ cbox a b ∈ lmeasurable
          and 2: ?μ ((f ‘ S) ∩ cbox a b) = m ∗ ?μ (S ∩ h ‘ cbox a b)
```

```
          using fBS [of S ∩ (h ` (cbox a b))] by auto
        have *: ⟦|z - m| < e; z ≤ w; w ≤ m⟧ ⟹ |w - m| ≤ e
          for w z m and e::real by auto
        have meas_adiff: |?μ (S ∩ h ` cbox a b) - ?μ S| ≤ e / (1 + |m|)
        proof (rule * [OF less])
          show ?μ (S ∩ cbox c d) ≤ ?μ (S ∩ h ` cbox a b)
          proof (rule measure_mono_fmeasurable [OF _ _ Shab])
            have f ` ball 0 D ⊆ ball 0 (C * D)
              using C ‹C > 0›
              apply (clarsimp simp: algebra_simps)
          by (meson le_less_trans linordered_comm_semiring_strict_class.comm_mult_strict_left_mono)
            then have f ` ball 0 D ⊆ cbox a b
              by (metis mult.commute order_trans that)
            have ball 0 D ⊆ h ` cbox a b
              by (metis ‹f ` ball 0 D ⊆ cbox a b› hf image_subset_iff subsetI)
            then show S ∩ cbox c d ⊆ S ∩ h ` cbox a b
              using D by blast
          next
            show S ∩ cbox c d ∈ sets lebesgue
              using S fmeasurable_cbox by blast
          qed
        next
          show ?μ (S ∩ h ` cbox a b) ≤ ?μ S
            by (simp add: S Shab fmeasurableD measure_mono_fmeasurable)
        qed
        have |?μ (f ` S ∩ cbox a b) - m * ?μ S| ≤ |?μ S - ?μ (S ∩ h ` cbox a b)|
* m
            by (metis 2 ‹m ≥ 0› abs_minus_commute abs_mult_pos mult.commute
order_refl right_diff_distrib')
        also have ... ≤ e / (1 + m) * m
        by (metis ‹m ≥ 0› abs_minus_commute abs_of_nonneg meas_adiff mult.commute
mult_left_mono)
        also have ... < e
          using ‹e > 0› ‹m ≥ 0› by (simp add: field_simps)
        finally have |?μ (f ` S ∩ cbox a b) - m * ?μ S| < e .
        with 1 show ?thesis by auto
      qed
      then show ∃B>0. ∀a b. ball 0 B ⊆ cbox a b ⟶
                        f ` S ∩ cbox a b ∈ lmeasurable ∧
                        |?μ (f ` S ∩ cbox a b) - m * ?μ S| < e
        using ‹C>0› ‹D>0› by (metis mult_zero_left mult_less_iff1)
    qed
  qed
qed
```

6.19.17 Lemmas about absolute integrability

```
lemma absolutely_integrable_linear:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
```

and *h* :: *'n::euclidean_space* ⇒ *'p::euclidean_space*
shows *f absolutely_integrable_on s* ⟹ *bounded_linear h* ⟹ (*h* ∘ *f*) *absolutely_integrable_on s*
using *integrable_bounded_linear*[*of h lebesgue* λ*x. indicator s x* $*_R$ *f x*]
by (*simp add*: *linear_simps*[*of h*] *set_integrable_def*)

lemma *absolutely_integrable_sum*:
fixes *f* :: *'a* ⇒ *'n::euclidean_space* ⇒ *'m::euclidean_space*
assumes *finite T* **and** ⋀*a*. *a* ∈ *T* ⟹ (*f a*) *absolutely_integrable_on S*
shows (λ*x*. *sum* (λ*a*. *f a x*) *T*) *absolutely_integrable_on S*
using *assms* **by** *induction auto*

lemma *absolutely_integrable_integrable_bound*:
fixes *f* :: *'n::euclidean_space* ⇒ *'m::euclidean_space*
assumes *le*: ⋀*x*. *x*∈*S* ⟹ *norm* (*f x*) ≤ *g x* **and** *f*: *f integrable_on S* **and** *g*: *g integrable_on S*
shows *f absolutely_integrable_on S*
unfolding *set_integrable_def*
proof (*rule Bochner_Integration.integrable_bound*)
have *g absolutely_integrable_on S*
unfolding *absolutely_integrable_on_def*
proof
show (λ*x*. *norm* (*g x*)) *integrable_on S*
using *le norm_ge_zero*[*of f _*]
by (*intro integrable_spike_finite*[*OF _ _ g*, *of* {}])
(*auto intro*!: *abs_of_nonneg intro*: *order_trans simp del*: *norm_ge_zero*)
qed *fact*
then show *integrable lebesgue* (λ*x*. *indicat_real S x* $*_R$ *g x*)
by (*simp add*: *set_integrable_def*)
show (λ*x*. *indicat_real S x* $*_R$ *f x*) ∈ *borel_measurable lebesgue*
using *f* **by** (*auto intro*: *has_integral_implies_lebesgue_measurable simp*: *integrable_on_def*)
qed (*use le* **in** ‹*force intro*!: *always_eventually split*: *split_indicator*›)

corollary *absolutely_integrable_on_const* [*simp*]:
fixes *c* :: *'a::euclidean_space*
assumes *S* ∈ *lmeasurable*
shows (λ*x*. *c*) *absolutely_integrable_on S*
by (*metis* (*full_types*) *assms absolutely_integrable_integrable_bound integrable_on_const order_refl*)

lemma *absolutely_integrable_continuous*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
shows *continuous_on* (*cbox a b*) *f* ⟹ *f absolutely_integrable_on cbox a b*
using *absolutely_integrable_integrable_bound*
by (*simp add*: *absolutely_integrable_on_def continuous_on_norm integrable_continuous*)

lemma *absolutely_integrable_continuous_real*:
fixes *f* :: *real* ⇒ *'b::euclidean_space*

shows *continuous_on* {*a..b*} *f* ⟹ *f absolutely_integrable_on* {*a..b*}
by (*metis absolutely_integrable_continuous box_real*(*2*))

lemma *continuous_imp_integrable*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *continuous_on* (*cbox a b*) *f*
shows *integrable* (*lebesgue_on* (*cbox a b*)) *f*
proof −
have *f absolutely_integrable_on cbox a b*
by (*simp add*: *absolutely_integrable_continuous assms*)
then show *?thesis*
by (*simp add*: *integrable_restrict_space set_integrable_def*)
qed

lemma *continuous_imp_integrable_real*:
fixes *f* :: *real* ⇒ *'b::euclidean_space*
assumes *continuous_on* {*a..b*} *f*
shows *integrable* (*lebesgue_on* {*a..b*}) *f*
by (*metis assms continuous_imp_integrable interval_cbox*)

6.19.18 Componentwise

proposition *absolutely_integrable_componentwise_iff*:
shows *f absolutely_integrable_on A* ⟷ (∀ *b*∈*Basis*. (λ*x*. *f x* · *b*) *absolutely_integrable_on A*)
proof −
have ∗: (λ*x*. *norm* (*f x*)) *integrable_on A* ⟷ (∀ *b*∈*Basis*. (λ*x*. *norm* (*f x* · *b*)) *integrable_on A*) (**is** *?lhs* = *?rhs*)
if *f integrable_on A*
proof
assume *?lhs*
then show *?rhs*
by (*metis absolutely_integrable_on_def Topology_Euclidean_Space.norm_nth_le absolutely_integrable_integrable_bound integrable_component that*)
next
assume *R*: *?rhs*
have *f absolutely_integrable_on A*
proof (*rule absolutely_integrable_integrable_bound*)
show (λ*x*. ∑ *i*∈*Basis*. *norm* (*f x* · *i*)) *integrable_on A*
using *R* **by** (*force intro*: *integrable_sum*)
qed (*use that norm_le_l1* **in** *auto*)
then show *?lhs*
using *absolutely_integrable_on_def* **by** *auto*
qed
show *?thesis*
unfolding *absolutely_integrable_on_def*
by (*simp add*: *integrable_componentwise_iff* [*symmetric*] *ball_conj_distrib* ∗ *cong*: *conj_cong*)
qed

lemma *absolutely_integrable_componentwise*:
shows ($\bigwedge b.\ b \in Basis \Longrightarrow (\lambda x.\ f\ x \cdot b)$ *absolutely_integrable_on A*) $\Longrightarrow$ *f absolutely_integrable_on A*
using *absolutely_integrable_componentwise_iff* **by** *blast*

lemma *absolutely_integrable_component*:
f absolutely_integrable_on A $\Longrightarrow$ ($\lambda x.\ f\ x \cdot (b :: 'b :: euclidean\_space)$) *absolutely_integrable_on A*
by (*drule absolutely_integrable_linear*[*OF _ bounded_linear_inner_left*[*of b*]]) (*simp add*: *o_def*)

lemma *absolutely_integrable_scaleR_left*:
fixes *f* :: *'n::euclidean_space* ⇒ *'m::euclidean_space*
assumes *f absolutely_integrable_on S*
shows ($\lambda x.\ c *_R f\ x$) *absolutely_integrable_on S*
proof –
have ($\lambda x.\ c *_R x$) *o f absolutely_integrable_on S*
by (*simp add*: *absolutely_integrable_linear assms bounded_linear_scaleR_right*)
then show *?thesis*
using *assms* **by** *blast*
qed

lemma *absolutely_integrable_scaleR_right*:
assumes *f absolutely_integrable_on S*
shows ($\lambda x.\ f\ x *_R c$) *absolutely_integrable_on S*
using *assms* **by** *blast*

lemma *absolutely_integrable_norm*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *euclidean_space*
assumes *f absolutely_integrable_on S*
shows (*norm o f*) *absolutely_integrable_on S*
using *assms* **by** (*simp add*: *absolutely_integrable_on_def o_def*)

lemma *absolutely_integrable_abs*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *euclidean_space*
assumes *f absolutely_integrable_on S*
shows ($\lambda x.\ \sum i{\in}Basis.\ |f\ x \cdot i| *_R i$) *absolutely_integrable_on S*
(**is** *?g absolutely_integrable_on S*)
proof –
have ∗: ($\lambda y.\ \sum j{\in}Basis.$ *if* $j = i$ *then* $y *_R j$ *else 0*) ∘
($\lambda x.$ *norm* ($\sum j{\in}Basis.$ *if* $j = i$ *then* $(x \cdot i) *_R j$ *else 0*)) ∘ *f*
absolutely_integrable_on S
if $i \in Basis$ **for** *i*
proof –
have *bounded_linear* ($\lambda y.\ \sum j{\in}Basis.$ *if* $j = i$ *then* $y *_R j$ *else 0*)
by (*simp add*: *linear_linear algebra_simps linearI*)
moreover have ($\lambda x.$ *norm* ($\sum j{\in}Basis.$ *if* $j = i$ *then* $(x \cdot i) *_R j$ *else 0*)) ∘ *f*

absolutely_integrable_on S
using *assms* ‹$i \in Basis$›
unfolding *o_def*
by (*intro absolutely_integrable_norm* [*unfolded o_def*])
(*auto simp*: *algebra_simps dest*: *absolutely_integrable_component*)
ultimately show *?thesis*
by (*subst comp_assoc*) (*blast intro*: *absolutely_integrable_linear*)
qed
have *eq*: *?g* =
($\lambda x.\ \sum i \in Basis.\ ((\lambda y.\ \sum j \in Basis.\ if\ j = i\ then\ y *_R j\ else\ 0) \circ$
$(\lambda x.\ norm(\sum j \in Basis.\ if\ j = i\ then\ (x \cdot i) *_R j\ else\ 0)) \circ f)\ x)$
by (*simp*)
show *?thesis*
unfolding *eq*
by (*rule absolutely_integrable_sum*) (*force simp*: *intro*!: ∗)+
qed

lemma *abs_absolutely_integrableI_1*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *real*
assumes *f*: *f integrable_on A* **and** ($\lambda x.\ |f\ x|$) *integrable_on A*
shows *f absolutely_integrable_on A*
by (*rule absolutely_integrable_integrable_bound* [*OF _ assms*]) *auto*

lemma *abs_absolutely_integrableI*:
assumes *f*: *f integrable_on S* **and** *fcomp*: ($\lambda x.\ \sum i \in Basis.\ |f\ x \cdot i| *_R i$) *integrable_on S*
shows *f absolutely_integrable_on S*
proof −
have ($\lambda x.\ (f\ x \cdot i) *_R i$) *absolutely_integrable_on S* **if** $i \in Basis$ **for** *i*
proof −
have ($\lambda x.\ |f\ x \cdot i|$) *integrable_on S*
using *assms integrable_component* [*OF fcomp*, **where** *y=i*] *that* **by** *simp*
then have ($\lambda x.\ f\ x \cdot i$) *absolutely_integrable_on S*
using *abs_absolutely_integrableI_1 f integrable_component* **by** *blast*
then show *?thesis*
by (*rule absolutely_integrable_scaleR_right*)
qed
then have ($\lambda x.\ \sum i \in Basis.\ (f\ x \cdot i) *_R i$) *absolutely_integrable_on S*
by (*simp add*: *absolutely_integrable_sum*)
then show *?thesis*
by (*simp add*: *euclidean_representation*)
qed

lemma *absolutely_integrable_abs_iff*:
f absolutely_integrable_on S ⟷
f integrable_on S ∧ ($\lambda x.\ \sum i \in Basis.\ |f\ x \cdot i| *_R i$) *integrable_on S*
(**is** *?lhs* = *?rhs*)

```
proof
  assume ?lhs then show ?rhs
    using absolutely_integrable_abs absolutely_integrable_on_def by blast
next
  assume ?rhs
  moreover
  have (λx. if x ∈ S then ∑i∈Basis. |f x · i| *R i else 0) = (λx. ∑i∈Basis. |(if
x ∈ S then f x else 0) · i| *R i)
    by force
  ultimately show ?lhs
      by (simp only: absolutely_integrable_restrict_UNIV [of S, symmetric] inte-
grable_restrict_UNIV [of S, symmetric] abs_absolutely_integrableI)
qed

lemma absolutely_integrable_max:
  fixes f :: 'n::euclidean_space ⇒ 'm::euclidean_space
  assumes f absolutely_integrable_on S g absolutely_integrable_on S
   shows (λx. ∑i∈Basis. max (f x · i) (g x · i) *R i)
           absolutely_integrable_on S
proof −
  have (λx. ∑i∈Basis. max (f x · i) (g x · i) *R i) =
        (λx. (1/2) *R (f x + g x + (∑i∈Basis. |f x · i − g x · i| *R i)))
  proof (rule ext)
    fix x
    have (∑i∈Basis. max (f x · i) (g x · i) *R i) = (∑i∈Basis. ((f x · i + g x
· i + |f x · i − g x · i|) / 2) *R i)
      by (force intro: sum.cong)
    also have ... = (1 / 2) *R (∑i∈Basis. (f x · i + g x · i + |f x · i − g x · i|)
*R i)
      by (simp add: scaleR_right.sum)
    also have ... = (1 / 2) *R (f x + g x + (∑i∈Basis. |f x · i − g x · i| *R i))
      by (simp add: sum.distrib algebra_simps euclidean_representation)
    finally
    show (∑i∈Basis. max (f x · i) (g x · i) *R i) =
         (1 / 2) *R (f x + g x + (∑i∈Basis. |f x · i − g x · i| *R i)) .
  qed
  moreover have (λx. (1 / 2) *R (f x + g x + (∑i∈Basis. |f x · i − g x · i| *R
i)))
                 absolutely_integrable_on S
    using absolutely_integrable_abs [OF set_integral_diff(1) [OF assms]]
     by (intro set_integral_add absolutely_integrable_scaleR_left assms) (simp add:
algebra_simps)
  ultimately show ?thesis by metis
qed

corollary absolutely_integrable_max_1:
  fixes f :: 'n::euclidean_space ⇒ real
  assumes f absolutely_integrable_on S g absolutely_integrable_on S
   shows (λx. max (f x) (g x)) absolutely_integrable_on S
```

using *absolutely_integrable_max* [*OF assms*] **by** *simp*

lemma *absolutely_integrable_min*:
fixes f :: $'n$::*euclidean_space* ⇒ $'m$::*euclidean_space*
assumes f *absolutely_integrable_on S g absolutely_integrable_on S*
shows (λx. $\sum$ i∈*Basis*. *min* (f x · i) (g x · i) $*_R$ i)
absolutely_integrable_on S
proof −
have (λx. $\sum$ i∈*Basis*. *min* (f x · i) (g x · i) $*_R$ i) =
(λx. (*1/2*) $*_R$ (f x + g x − ($\sum$ i∈*Basis*. |f x · i − g x · i| $*_R$ i)))
proof (*rule ext*)
fix x
have ($\sum$ i∈*Basis*. *min* (f x · i) (g x · i) $*_R$ i) = ($\sum$ i∈*Basis*. ((f x · i + g x · i − |f x · i − g x · i|) / *2*) $*_R$ i)
by (*force intro*: *sum.cong*)
also have ... = (*1* / *2*) $*_R$ ($\sum$ i∈*Basis*. (f x · i + g x · i − |f x · i − g x · i|) $*_R$ i)
by (*simp add*: *scaleR_right.sum*)
also have ... = (*1* / *2*) $*_R$ (f x + g x − ($\sum$ i∈*Basis*. |f x · i − g x · i| $*_R$ i))
by (*simp add*: *sum.distrib sum_subtractf algebra_simps euclidean_representation*)
finally
show ($\sum$ i∈*Basis*. *min* (f x · i) (g x · i) $*_R$ i) =
(*1* / *2*) $*_R$ (f x + g x − ($\sum$ i∈*Basis*. |f x · i − g x · i| $*_R$ i)) .
qed
moreover have (λx. (*1* / *2*) $*_R$ (f x + g x − ($\sum$ i∈*Basis*. |f x · i − g x · i| $*_R$ i)))
absolutely_integrable_on S
using *absolutely_integrable_abs* [*OF set_integral_diff*(*1*) [*OF assms*]]
by (*intro set_integral_add set_integral_diff absolutely_integrable_scaleR_left assms*)
(*simp add*: *algebra_simps*)
ultimately show *?thesis* **by** *metis*
qed

corollary *absolutely_integrable_min_1*:
fixes f :: $'n$::*euclidean_space* ⇒ *real*
assumes f *absolutely_integrable_on S g absolutely_integrable_on S*
shows (λx. *min* (f x) (g x)) *absolutely_integrable_on S*
using *absolutely_integrable_min* [*OF assms*] **by** *simp*

lemma *nonnegative_absolutely_integrable*:
fixes f :: $'a$ :: *euclidean_space* ⇒ $'b$ :: *euclidean_space*
assumes f *integrable_on A* **and** *comp*: $\bigwedge x\ b$. ⟦x ∈ A; b ∈ *Basis*⟧ ⟹ *0* ≤ f x · b
shows f *absolutely_integrable_on A*
proof −
have (λx. (f x · i) $*_R$ i) *absolutely_integrable_on A* **if** i ∈ *Basis* **for** i
proof −
have (λx. f x · i) *integrable_on A*
by (*simp add*: *assms*(*1*) *integrable_component*)
then have (λx. f x · i) *absolutely_integrable_on A*

```
      by (metis that comp nonnegative_absolutely_integrable_1)
    then show ?thesis
      by (rule absolutely_integrable_scaleR_right)
  qed
  then have (λx. ∑i∈Basis. (f x · i) *R i) absolutely_integrable_on A
    by (simp add: absolutely_integrable_sum)
  then show ?thesis
    by (simp add: euclidean_representation)
qed

lemma absolutely_integrable_component_ubound:
  fixes f :: 'a :: euclidean_space ⇒ 'b :: euclidean_space
  assumes f: f integrable_on A and g: g absolutely_integrable_on A
      and comp: ⋀x b. ⟦x ∈ A; b ∈ Basis⟧ ⟹ f x · b ≤ g x · b
  shows f absolutely_integrable_on A
proof -
  have (λx. g x − (g x − f x)) absolutely_integrable_on A
  proof (rule set_integral_diff [OF g nonnegative_absolutely_integrable])
    show (λx. g x − f x) integrable_on A
     using Henstock_Kurzweil_Integration.integrable_diff absolutely_integrable_on_def
f g by blast
  qed (simp add: comp inner_diff_left)
  then show ?thesis
    by simp
qed

lemma absolutely_integrable_component_lbound:
  fixes f :: 'a :: euclidean_space ⇒ 'b :: euclidean_space
  assumes f: f absolutely_integrable_on A and g: g integrable_on A
      and comp: ⋀x b. ⟦x ∈ A; b ∈ Basis⟧ ⟹ f x · b ≤ g x · b
  shows g absolutely_integrable_on A
proof -
  have (λx. f x + (g x − f x)) absolutely_integrable_on A
  proof (rule set_integral_add [OF f nonnegative_absolutely_integrable])
    show (λx. g x − f x) integrable_on A
     using Henstock_Kurzweil_Integration.integrable_diff absolutely_integrable_on_def
f g by blast
  qed (simp add: comp inner_diff_left)
  then show ?thesis
    by simp
qed

lemma integrable_on_1_iff:
  fixes f :: 'a::euclidean_space ⇒ real^1
  shows f integrable_on S ⟷ (λx. f x $ 1) integrable_on S
 by (auto simp: integrable_componentwise_iff [of f] Basis_vec_def cart_eq_inner_axis)

lemma integral_on_1_eq:
  fixes f :: 'a::euclidean_space ⇒ real^1
```

shows *integral S f* = *vec* (*integral S* (λ*x*. *f x* $ *1*))
by (*cases f integrable_on S*) (*simp_all add*: *integrable_on_1_iff vec_eq_iff not_integrable_integral*)

lemma *absolutely_integrable_on_1_iff*:
fixes *f* :: *'a::euclidean_space* ⇒ *real^1*
shows *f absolutely_integrable_on S* ⟷ (λ*x*. *f x* $ *1*) *absolutely_integrable_on S*
unfolding *absolutely_integrable_on_def*
by (*auto simp*: *integrable_on_1_iff norm_real*)

lemma *absolutely_integrable_absolutely_integrable_lbound*:
fixes *f* :: *'m::euclidean_space* ⇒ *real*
assumes *f*: *f integrable_on S* **and** *g*: *g absolutely_integrable_on S*
and ∗: ⋀*x*. *x* ∈ *S* ⟹ *g x* ≤ *f x*
shows *f absolutely_integrable_on S*
by (*rule absolutely_integrable_component_lbound* [*OF g f*]) (*simp add*: ∗)

lemma *absolutely_integrable_absolutely_integrable_ubound*:
fixes *f* :: *'m::euclidean_space* ⇒ *real*
assumes *fg*: *f integrable_on S g absolutely_integrable_on S*
and ∗: ⋀*x*. *x* ∈ *S* ⟹ *f x* ≤ *g x*
shows *f absolutely_integrable_on S*
by (*rule absolutely_integrable_component_ubound* [*OF fg*]) (*simp add*: ∗)

lemma *has_integral_vec1_I_cbox*:
fixes *f* :: *real^1* ⇒ *'a::real_normed_vector*
assumes (*f has_integral y*) (*cbox a b*)
shows ((*f* ∘ *vec*) *has_integral y*) {*a*$*1*..*b*$*1*}
proof −
have ((λ*x*. *f*(*vec x*)) *has_integral* (*1* / *1*) $*_R$ *y*) ((λ*x*. *x* $ *1*) ` *cbox a b*)
proof (*rule has_integral_twiddle*)
show ∃ *w z*::*real^1*. *vec* ` *cbox u v* = *cbox w z*
content (*vec* ` *cbox u v* :: (*real^1*) *set*) = *1* ∗ *content* (*cbox u v*) **for** *u v*
unfolding *vec_cbox_1_eq*
by (*auto simp*: *content_cbox_if_cart interval_eq_empty_cart*)
show ∃ *w z*. (λ*x*. *x* $ *1*) ` *cbox u v* = *cbox w z* **for** *u v* :: *real^1*
using *vec_nth_cbox_1_eq* **by** *blast*
qed (*auto simp*: *continuous_vec assms*)
then show *?thesis*
by (*simp add*: *o_def*)
qed

lemma *has_integral_vec1_I*:
fixes *f* :: *real^1* ⇒ *'a::real_normed_vector*
assumes (*f has_integral y*) *S*
shows (*f* ∘ *vec has_integral y*) ((λ*x*. *x* $ *1*) ` *S*)
proof −
have ∗: ∃ *z*. ((λ*x*. *if x* ∈ (λ*x*. *x* $ *1*) ` *S then* (*f* ∘ *vec*) *x else 0*) *has_integral z*)
{*a*..*b*} ∧ *norm* (*z* − *y*) < *e*
if *int*: ⋀*a b*. *ball 0 B* ⊆ *cbox a b* ⟹

```
                    (∃z. ((λx. if x ∈ S then f x else 0) has_integral z) (cbox a b) ∧
norm (z − y) < e)
     and B: ball 0 B ⊆ {a..b} for e B a b
  proof −
    have [simp]: (∃y∈S. x = y $ 1) ⟷ vec x ∈ S for x
      by force
    have B': ball (0::real^1) B ⊆ cbox (vec a) (vec b)
     using B by (simp add: Basis_vec_def cart_eq_inner_axis [symmetric] mem_box
norm_real subset_iff)
    show ?thesis
     using int [OF B'] by (auto simp: image_iff o_def cong: if_cong dest!: has_integral_vec1_I_cbox)
  qed
  show ?thesis
    using assms
    apply (subst has_integral_alt)
    apply (subst (asm) has_integral_alt)
    apply (simp add: has_integral_vec1_I_cbox split: if_split_asm)
    subgoal by (metis vector_one_nth)
    subgoal
      apply (erule all_forward imp_forward ex_forward asm_rl)+
      by (blast intro!: ∗)+
    done
qed

lemma has_integral_vec1_nth_cbox:
  fixes f :: real ⇒ 'a::real_normed_vector
  assumes (f has_integral y) {a..b}
  shows ((λx::real^1. f(x$1)) has_integral y) (cbox (vec a) (vec b))
proof −
  have ((λx::real^1. f(x$1)) has_integral (1 / 1) ∗R y) (vec ` cbox a b)
  proof (rule has_integral_twiddle)
    show ∃w z::real. (λx. x $ 1) ` cbox u v = cbox w z
         content ((λx. x $ 1) ` cbox u v) = 1 ∗ content (cbox u v) for u v::real^1
     unfolding vec_cbox_1_eq by (auto simp: content_cbox_if_cart interval_eq_empty_cart)
    show ∃w z::real^1. vec ` cbox u v = cbox w z for u v :: real
      using vec_cbox_1_eq by auto
  qed (auto simp: continuous_vec assms)
  then show ?thesis
    using vec_cbox_1_eq by auto
qed

lemma has_integral_vec1_D_cbox:
  fixes f :: real^1 ⇒ 'a::real_normed_vector
  assumes ((f ∘ vec) has_integral y) {a$1..b$1}
  shows (f has_integral y) (cbox a b)
 by (metis (mono_tags, lifting) assms comp_apply has_integral_eq has_integral_vec1_nth_cbox
vector_one_nth)

lemma has_integral_vec1_D:
```

fixes *f* :: *real^1* ⇒ *'a*::*real_normed_vector*
assumes ((*f* ∘ *vec*) *has_integral y*) ((λ*x*. *x* $ *1*) ‘ *S*)
shows (*f has_integral y*) *S*
proof −
have ∗: ∃ *z*. ((λ*x*. *if x* ∈ *S then f x else 0*) *has_integral z*) (*cbox a b*) ∧ *norm* (*z* − *y*) < *e*
if *int*: ⋀*a b*. *ball 0 B* ⊆ {*a*..*b*} ⟹
(∃ *z*. ((λ*x*. *if x* ∈ (λ*x*. *x* $ *1*) ‘ *S then* (*f* ∘ *vec*) *x else 0*) *has_integral z*) {*a*..*b*} ∧ *norm* (*z* − *y*) < *e*)
and *B*: *ball 0 B* ⊆ *cbox a b* **for** *e B* **and** *a b*::*real^1*
proof −
have *B'*: *ball 0 B* ⊆ {*a*$*1*..*b*$*1*}
proof (*clarsimp*)
fix *t*
assume |*t*| < *B* **then show** *a* $ *1* ≤ *t* ∧ *t* ≤ *b* $ *1*
using *subsetD* [*OF B*]
by (*metis* (*mono_tags*, *hide_lams*) *mem_ball_0 mem_box_cart*(*2*) *norm_real vec_component*)
qed
have *eq*: (λ*x*. *if vec x* ∈ *S then f* (*vec x*) *else 0*) = (λ*x*. *if x* ∈ *S then f x else 0*) ∘ *vec*
by *force*
have [*simp*]: (∃ *y*∈*S*. *x* = *y* $ *1*) ⟷ *vec x* ∈ *S* **for** *x*
by *force*
show *?thesis*
using *int* [*OF B'*] **by** (*auto simp*: *image_iff eq cong*: *if_cong dest*!: *has_integral_vec1_D_cbox*)
qed
show *?thesis*
using *assms*
apply (*subst has_integral_alt*)
apply (*subst* (*asm*) *has_integral_alt*)
apply (*simp add*: *has_integral_vec1_D_cbox eq_cbox split*: *if_split_asm*, *blast*)
apply (*intro conjI impI*)
subgoal by (*metis vector_one_nth*)
apply (*erule thin_rl*)
apply (*erule all_forward ex_forward conj_forward*)+
by (*blast intro*!: ∗)+
qed

lemma *integral_vec1_eq*:
fixes *f* :: *real^1* ⇒ *'a*::*real_normed_vector*
shows *integral S f* = *integral* ((λ*x*. *x* $ *1*) ‘ *S*) (*f* ∘ *vec*)
using *has_integral_vec1_I* [*of f*] *has_integral_vec1_D* [*of f*]
by (*metis has_integral_iff not_integrable_integral*)

lemma *absolutely_integrable_drop*:
fixes *f* :: *real^1* ⇒ *'b*::*euclidean_space*
shows *f absolutely_integrable_on S* ⟷ (*f* ∘ *vec*) *absolutely_integrable_on* (λ*x*. *x*

$ 1) ‘ S
  unfolding *absolutely_integrable_on_def integrable_on_def*
proof *safe*
  fix *y r*
  assume (*f has_integral y*) *S* ((λ*x*. *norm* (*f x*)) *has_integral r*) *S*
  then show ∃ *y*. (*f* ∘ *vec has_integral y*) ((λ*x*. *x* $ *1*) ‘ *S*)
      ∃ *y*. ((λ*x*. *norm* ((*f* ∘ *vec*) *x*)) *has_integral y*) ((λ*x*. *x* $ *1*) ‘ *S*)
    by (*force simp*: *o_def dest*!: *has_integral_vec1_I*)+
next
  fix *y* :: '*b* **and** *r* :: *real*
  assume (*f* ∘ *vec has_integral y*) ((λ*x*. *x* $ *1*) ‘ *S*)
    ((λ*x*. *norm* ((*f* ∘ *vec*) *x*)) *has_integral r*) ((λ*x*. *x* $ *1*) ‘ *S*)
  then show ∃ *y*. (*f has_integral y*) *S* ∃ *y*. ((λ*x*. *norm* (*f x*)) *has_integral y*) *S*
    by (*force simp*: *o_def intro*: *has_integral_vec1_D*)+
qed

6.19.19 Dominated convergence

lemma *dominated_convergence*:
  fixes *f* :: *nat* ⇒ '*n*::*euclidean_space* ⇒ '*m*::*euclidean_space*
  assumes *f*: ⋀*k*. (*f k*) *integrable_on S* **and** *h*: *h integrable_on S*
    and *le*: ⋀*k x*. *x* ∈ *S* ⟹ *norm* (*f k x*) ≤ *h x*
    and *conv*: ⋀*x*. *x* ∈ *S* ⟹ (λ*k*. *f k x*) ⟶ *g x*
  shows *g integrable_on S* (λ*k*. *integral S* (*f k*)) ⟶ *integral S g*
proof −
  have *3*: *h absolutely_integrable_on S*
    unfolding *absolutely_integrable_on_def*
  proof
    show (λ*x*. *norm* (*h x*)) *integrable_on S*
    proof (*intro integrable_spike_finite*[*OF* _ _ *h*, *of* {}] *ballI*)
      fix *x* **assume** *x* ∈ *S* − {} **then show** *norm* (*h x*) = *h x*
        by (*metis Diff_empty abs_of_nonneg bot_set_def le norm_ge_zero order_trans real_norm_def*)
    qed *auto*
  qed *fact*
  have *2*: *set_borel_measurable lebesgue S* (*f k*) **for** *k*
    unfolding *set_borel_measurable_def*
     using *f* **by** (*auto intro*: *has_integral_implies_lebesgue_measurable simp*: *integrable_on_def*)
  then have *1*: *set_borel_measurable lebesgue S g*
    unfolding *set_borel_measurable_def*
   by (*rule borel_measurable_LIMSEQ_metric*) (*use conv* **in** ‹*auto split*: *split_indicator*›)
  have *4*: *AE x in lebesgue*. (λ*i*. *indicator S x* $*_R$ *f i x*) ⟶ *indicator S x* $*_R$ *g x*
    AE x in lebesgue. *norm* (*indicator S x* $*_R$ *f k x*) ≤ *indicator S x* $*_R$ *h x* **for** *k*
    using *conv le* **by** (*auto intro*!: *always_eventually split*: *split_indicator*)
  have *g*: *g absolutely_integrable_on S*
    using *1 2 3 4* **unfolding** *set_borel_measurable_def set_integrable_def*
    by (*rule integrable_dominated_convergence*)

then show *g integrable_on S*
by (*auto simp*: *absolutely_integrable_on_def*)
have (λk. (*LINT x*:*S*|*lebesgue*. *f k x*)) $\longrightarrow$ (*LINT x*:*S*|*lebesgue*. *g x*)
unfolding *set_borel_measurable_def set_lebesgue_integral_def*
using *1 2 3 4* **unfolding** *set_borel_measurable_def set_lebesgue_integral_def set_integrable_def*
by (*rule integral_dominated_convergence*)
then show (λk. *integral S* (*f k*)) $\longrightarrow$ *integral S g*
using *g absolutely_integrable_integrable_bound*[*OF le f h*]
by (*subst* (*asm*) (*1 2*) *set_lebesgue_integral_eq_integral*) *auto*
qed

lemma *has_integral_dominated_convergence*:
fixes *f* :: *nat* ⇒ *'n*::*euclidean_space* ⇒ *'m*::*euclidean_space*
assumes $\bigwedge$*k*. (*f k has_integral y k*) *S h integrable_on S*
$\bigwedge$*k*. ∀ *x*∈*S*. *norm* (*f k x*) ≤ *h x* ∀ *x*∈*S*. (λk. *f k x*) $\longrightarrow$ *g x*
and *x*: *y* $\longrightarrow$ *x*
shows (*g has_integral x*) *S*
proof −
have *int_f*: $\bigwedge$*k*. (*f k*) *integrable_on S*
using *assms* **by** (*auto simp*: *integrable_on_def*)
have (*g has_integral* (*integral S g*)) *S*
by (*metis assms*(*2*−*4*) *dominated_convergence*(*1*) *has_integral_integral int_f*)
moreover have *integral S g* = *x*
proof (*rule LIMSEQ_unique*)
show (λi. *integral S* (*f i*)) $\longrightarrow$ *x*
using *integral_unique*[*OF assms*(*1*)] *x* **by** *simp*
show (λi. *integral S* (*f i*)) $\longrightarrow$ *integral S g*
by (*metis assms*(*2*) *assms*(*3*) *assms*(*4*) *dominated_convergence*(*2*) *int_f*)
qed
ultimately show *?thesis*
by *simp*
qed

lemma *dominated_convergence_integrable_1*:
fixes *f* :: *nat* ⇒ *'n*::*euclidean_space* ⇒ *real*
assumes *f*: $\bigwedge$*k*. *f k absolutely_integrable_on S*
and *h*: *h integrable_on S*
and *normg*: $\bigwedge$*x*. *x* ∈ *S* ⟹ *norm*(*g x*) ≤ (*h x*)
and *lim*: $\bigwedge$*x*. *x* ∈ *S* ⟹ (λk. *f k x*) $\longrightarrow$ *g x*
shows *g integrable_on S*
proof −
have *habs*: *h absolutely_integrable_on S*
using *h normg nonnegative_absolutely_integrable_1 norm_ge_zero order_trans* **by** *blast*
let *?f* = λ*n x*. (*min* (*max* (− *h x*) (*f n x*)) (*h x*))
have *h0*: *h x* ≥ *0* **if** *x* ∈ *S* **for** *x*
using *normg that* **by** *force*
have *leh*: *norm* (*?f k x*) ≤ *h x* **if** *x* ∈ *S* **for** *k x*

```
    using h0 that by force
  have limf: (λk. ?f k x) ⟶ g x if x ∈ S for x
  proof -
    have ⋀e y. |f y x − g x| < e ⟹ |min (max (− h x) (f y x)) (h x) − g x| < e
      using h0 [OF that] normg [OF that] by simp
    then show ?thesis
      using lim [OF that] by (auto simp add: tendsto_iff dist_norm elim!: eventually_mono)
  qed
  show ?thesis
  proof (rule dominated_convergence [of ?f S h g])
    have (λx. − h x) absolutely_integrable_on S
      using habs unfolding set_integrable_def by auto
    then show ?f k integrable_on S for k
      by (intro set_lebesgue_integral_eq_integral absolutely_integrable_min_1 absolutely_integrable_max_1 f habs)
  qed (use assms leh limf in auto)
qed

lemma dominated_convergence_integrable:
  fixes f :: nat ⇒ 'n::euclidean_space ⇒ 'm::euclidean_space
  assumes f: ⋀k. f k absolutely_integrable_on S
    and h: h integrable_on S
    and normg: ⋀x. x ∈ S ⟹ norm(g x) ≤ (h x)
    and lim: ⋀x. x ∈ S ⟹ (λk. f k x) ⟶ g x
  shows g integrable_on S
  using f
 unfolding integrable_componentwise_iff [of g] absolutely_integrable_componentwise_iff
[where f = f k for k]
proof clarify
  fix b :: 'm
  assume fb [rule_format]: ⋀k. ∀ b∈Basis. (λx. f k x • b) absolutely_integrable_on
S and b: b ∈ Basis
  show (λx. g x • b) integrable_on S
  proof (rule dominated_convergence_integrable_1 [OF fb h])
    fix x
    assume x ∈ S
    show norm (g x • b) ≤ h x
      using norm_nth_le ‹x ∈ S› b normg order.trans by blast
    show (λk. f k x • b) ⟶ g x • b
      using ‹x ∈ S› b lim tendsto_componentwise_iff by fastforce
  qed (use b in auto)
qed

lemma dominated_convergence_absolutely_integrable:
  fixes f :: nat ⇒ 'n::euclidean_space ⇒ 'm::euclidean_space
  assumes f: ⋀k. f k absolutely_integrable_on S
    and h: h integrable_on S
    and normg: ⋀x. x ∈ S ⟹ norm(g x) ≤ (h x)
```

and *lim*: $\bigwedge x.\ x \in S \Longrightarrow (\lambda k.\ f\ k\ x) \longlongrightarrow g\ x$
shows *g absolutely_integrable_on S*
proof –
have *g integrable_on S*
by (*rule dominated_convergence_integrable* [*OF assms*])
with *assms* **show** *?thesis*
by (*blast intro*: *absolutely_integrable_integrable_bound* [**where** *g=h*])
qed

proposition *integral_countable_UN*:
fixes *f* :: *real^'m ⇒ real^'n*
assumes *f*: *f absolutely_integrable_on* ($\bigcup$(*range s*))
and *s*: $\bigwedge m.\ s\ m \in$ *sets lebesgue*
shows $\bigwedge n.$ *f absolutely_integrable_on* ($\bigcup m \le n.\ s\ m$)
and ($\lambda n.$ *integral* ($\bigcup m \le n.\ s\ m$) *f*) $\longrightarrow$ *integral* ($\bigcup$(*s* ` *UNIV*)) *f* (**is** *?F* $\longrightarrow$ *?I*)
proof –
show *fU*: *f absolutely_integrable_on* ($\bigcup m \le n.\ s\ m$) **for** *n*
using *assms* **by** (*blast intro*: *set_integrable_subset* [*OF f*])
have *fint*: *f integrable_on* ($\bigcup$ (*range s*))
using *absolutely_integrable_on_def f* **by** *blast*
let *?h* = $\lambda x.$ *if* $x \in \bigcup$(*s* ` *UNIV*) *then norm*(*f x*) *else 0*
have ($\lambda n.$ *integral UNIV* ($\lambda x.$ *if* $x \in$ ($\bigcup m \le n.\ s\ m$) *then f x else 0*))
$\longrightarrow$ *integral UNIV* ($\lambda x.$ *if* $x \in \bigcup$(*s* ` *UNIV*) *then f x else 0*)
proof (*rule dominated_convergence*)
show ($\lambda x.$ *if* $x \in$ ($\bigcup m \le n.\ s\ m$) *then f x else 0*) *integrable_on UNIV* **for** *n*
unfolding *integrable_restrict_UNIV*
using *fU absolutely_integrable_on_def* **by** *blast*
show ($\lambda x.$ *if* $x \in \bigcup$(*s* ` *UNIV*) *then norm*(*f x*) *else 0*) *integrable_on UNIV*
by (*metis* (*no_types*) *absolutely_integrable_on_def f integrable_restrict_UNIV*)
show $\bigwedge x.$ ($\lambda n.$ *if* $x \in$ ($\bigcup m \le n.\ s\ m$) *then f x else 0*)
$\longrightarrow$ (*if* $x \in \bigcup$(*s* ` *UNIV*) *then f x else 0*)
by (*force intro*: *tendsto_eventually eventually_sequentiallyI*)
qed *auto*
then show *?F* $\longrightarrow$ *?I*
by (*simp only*: *integral_restrict_UNIV*)
qed

6.19.20 Fundamental Theorem of Calculus for the Lebesgue integral

For the positive integral we replace continuity with Borel-measurability.

lemma
fixes *f* :: *real ⇒ real*
assumes [*measurable*]: *f* ∈ *borel_measurable borel*
assumes *f*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow$ *DERIV F x* :> *f x* $\bigwedge x.\ x \in \{a..b\} \Longrightarrow 0 \le f\ x$
and $a \le b$

shows *nn_integral_FTC_Icc*: $(\int^+ x.\ ennreal\ (f\ x) * indicator\ \{a\ ..\ b\}\ x\ \partial lborel) = F\ b - F\ a$ (**is** *?nn*)
and *has_bochner_integral_FTC_Icc_nonneg*:
has_bochner_integral lborel $(\lambda x.\ f\ x * indicator\ \{a\ ..\ b\}\ x)\ (F\ b - F\ a)$ (**is** *?has*)
and *integral_FTC_Icc_nonneg*: $(\int x.\ f\ x * indicator\ \{a\ ..\ b\}\ x\ \partial lborel) = F\ b - F\ a$ (**is** *?eq*)
and *integrable_FTC_Icc_nonneg*: *integrable lborel* $(\lambda x.\ f\ x * indicator\ \{a\ ..\ b\}\ x)$ (**is** *?int*)
proof −
have $*$: $(\lambda x.\ f\ x * indicator\ \{a..b\}\ x) \in borel\_measurable\ borel\ \bigwedge x.\ 0 \le f\ x * indicator\ \{a..b\}\ x$
using *f(2)* **by** (*auto split*: *split_indicator*)

have *F_mono*: $a \le x \Longrightarrow x \le y \Longrightarrow y \le b \Longrightarrow F\ x \le F\ y$ **for** *x y*
using *f* **by** (*intro DERIV_nonneg_imp_nondecreasing*[*of x y F*]) (*auto intro*: *order_trans*)

have (*f has_integral F b* − *F a*) {*a..b*}
by (*intro fundamental_theorem_of_calculus*)
(*auto simp*: *has_field_derivative_iff_has_vector_derivative*[*symmetric*]
intro: *has_field_derivative_subset*[*OF f(1)*] ‹*a* ≤ *b*›)
then have *i*: $((\lambda x.\ f\ x * indicator\ \{a\ ..\ b\}\ x)$ *has_integral F b* − *F a*) *UNIV*
unfolding *indicator_def if_distrib*[**where** *f*=λ*x*. *a* ∗ *x* **for** *a*]
by (*simp cong del*: *if_weak_cong del*: *atLeastAtMost_iff*)
then have *nn*: $(\int^+ x.\ f\ x * indicator\ \{a\ ..\ b\}\ x\ \partial lborel) = F\ b - F\ a$
by (*rule nn_integral_has_integral_lborel*[*OF* ∗])
then show *?has*
by (*rule has_bochner_integral_nn_integral*[*rotated 3*]) (*simp_all add*: ∗ *F_mono* ‹*a* ≤ *b*›)
then show *?eq ?int*
unfolding *has_bochner_integral_iff* **by** *auto*
show *?nn*
by (*subst nn*[*symmetric*])
(*auto intro*!: *nn_integral_cong simp add*: *ennreal_mult f split*: *split_indicator*)
qed

lemma
fixes *f* :: *real* ⇒ *'a* :: *euclidean_space*
assumes *a* ≤ *b*
assumes $\bigwedge x.\ a \le x \Longrightarrow x \le b \Longrightarrow$ (*F has_vector_derivative f x*) (*at x within* {*a* .. *b*})
assumes *cont*: *continuous_on* {*a* .. *b*} *f*
shows *has_bochner_integral_FTC_Icc*:
has_bochner_integral lborel $(\lambda x.\ indicator\ \{a\ ..\ b\}\ x *_R f\ x)\ (F\ b - F\ a)$ (**is** *?has*)
and *integral_FTC_Icc*: $(\int x.\ indicator\ \{a\ ..\ b\}\ x *_R f\ x\ \partial lborel) = F\ b - F\ a$ (**is** *?eq*)
proof −

let *?f* = $\lambda x.$ *indicator* $\{a\ ..\ b\}$ $x *_R f\ x$
have *int*: *integrable lborel ?f*
using *borel_integrable_compact*[*OF _ cont*] **by** *auto*
have (*f has_integral F b* − *F a*) $\{a..b\}$
using *assms*(*1*,*2*) **by** (*intro fundamental_theorem_of_calculus*) *auto*
moreover
have (*f has_integral integral*^{L} *lborel ?f*) $\{a..b\}$
using *has_integral_integral_lborel*[*OF int*]
unfolding *indicator_def if_distrib*[**where** $f{=}\lambda x.\ x *_R a$ **for** *a*]
by (*simp cong del*: *if_weak_cong del*: *atLeastAtMost_iff*)
ultimately show *?eq*
by (*auto dest*: *has_integral_unique*)
then show *?has*
using *int* **by** (*auto simp*: *has_bochner_integral_iff*)
qed

lemma
fixes *f* :: *real* ⇒ *real*
assumes $a \leq b$
assumes *deriv*: $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow$ *DERIV F x* :> *f x*
assumes *cont*: $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow$ *isCont f x*
shows *has_bochner_integral_FTC_Icc_real*:
has_bochner_integral lborel ($\lambda x.$ *f x* ∗ *indicator* $\{a\ ..\ b\}$ *x*) (*F b* − *F a*) (**is** *?has*)
and *integral_FTC_Icc_real*: ($\int x.$ *f x* ∗ *indicator* $\{a\ ..\ b\}$ *x* ∂*lborel*) = *F b* − *F a* (**is** *?eq*)
proof −
have *1*: $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow$ (*F has_vector_derivative f x*) (*at x within* $\{a\ ..\ b\}$)
unfolding *has_field_derivative_iff_has_vector_derivative*[*symmetric*]
using *deriv* **by** (*auto intro*: *DERIV_subset*)
have *2*: *continuous_on* $\{a\ ..\ b\}$ *f*
using *cont* **by** (*intro continuous_at_imp_continuous_on*) *auto*
show *?has ?eq*
using *has_bochner_integral_FTC_Icc*[*OF* ‹$a \leq b$› *1 2*] *integral_FTC_Icc*[*OF* ‹$a \leq b$› *1 2*]
by (*auto simp*: *mult.commute*)
qed

lemma *nn_integral_FTC_atLeast*:
fixes *f* :: *real* ⇒ *real*
assumes *f_borel*: *f* ∈ *borel_measurable borel*
assumes *f*: $\bigwedge x.\ a \leq x \Longrightarrow$ *DERIV F x* :> *f x*
assumes *nonneg*: $\bigwedge x.\ a \leq x \Longrightarrow 0 \leq$ *f x*
assumes *lim*: (*F* ⟶ *T*) *at_top*
shows ($\int^+ x.$ *ennreal* (*f x*) ∗ *indicator* $\{a\ ..\}$ *x* ∂*lborel*) = *T* − *F a*
proof −
let *?f* = λ(*i*::*nat*) (*x*::*real*). *ennreal* (*f x*) ∗ *indicator* $\{a..a + real\ i\}$ *x*
let *?fR* = $\lambda x.$ *ennreal* (*f x*) ∗ *indicator* $\{a\ ..\}$ *x*

have *F_mono*: $a \leq x \Longrightarrow x \leq y \Longrightarrow F\ x \leq F\ y$ **for** $x\ y$
using *f nonneg* **by** (*intro DERIV_nonneg_imp_nondecreasing*[*of x y F*]) (*auto intro*: *order_trans*)
then have *F_le_T*: $a \leq x \Longrightarrow F\ x \leq T$ **for** x
by (*intro tendsto_lowerbound*[*OF lim*])
(*auto simp*: *eventually_at_top_linorder*)

have (*SUP i. ?f i x*) = *?fR x* **for** x
proof (*rule LIMSEQ_unique*[*OF LIMSEQ_SUP*])
obtain n **where** $x - a <$ *real n*
using *reals_Archimedean2*[*of* $x - a$] **..**
then have *eventually* (λ*n. ?f n x* = *?fR x*) *sequentially*
by (*auto intro*!: *eventually_sequentiallyI*[**where** *c*=*n*] *split*: *split_indicator*)
then show (λ*n. ?f n x*) ⟶ *?fR x*
by (*rule tendsto_eventually*)
qed (*auto simp*: *nonneg incseq_def le_fun_def split*: *split_indicator*)
then have *integral*^{N} *lborel ?fR* = (∫$^+$ *x*. (*SUP i. ?f i x*) ∂*lborel*)
by *simp*
also have ... = (*SUP i*. (∫$^+$ *x. ?f i x* ∂*lborel*))
proof (*rule nn_integral_monotone_convergence_SUP*)
show *incseq ?f*
using *nonneg* **by** (*auto simp*: *incseq_def le_fun_def split*: *split_indicator*)
show ⋀*i*. (*?f i*) ∈ *borel_measurable lborel*
using *f_borel* **by** *auto*
qed
also have ... = (*SUP i. ennreal* (*F* (*a* + *real i*) − *F a*))
by (*subst nn_integral_FTC_Icc*[*OF f_borel f nonneg*]) *auto*
also have ... = *T* − *F a*
proof (*rule LIMSEQ_unique*[*OF LIMSEQ_SUP*])
have (λ*x. F* (*a* + *real x*)) ⟶ *T*
by (*auto intro*: *filterlim_compose*[*OF lim filterlim_tendsto_add_at_top*] *filterlim_real_sequentially*)
then show (λ*n. ennreal* (*F* (*a* + *real n*) − *F a*)) ⟶ *ennreal* (*T* − *F a*)
by (*simp add*: *F_mono F_le_T tendsto_diff*)
qed (*auto simp*: *incseq_def intro*!: *ennreal_le_iff*[*THEN iffD2*] *F_mono*)
finally show *?thesis* **.**
qed

lemma *integral_power*:
$a \leq b \Longrightarrow$ (∫ *x. x*ˆ*k* ∗ *indicator* {*a..b*} *x* ∂*lborel*) = (*b*ˆ*Suc k* − *a*ˆ*Suc k*) / *Suc k*
proof (*subst integral_FTC_Icc_real*)
fix x **show** *DERIV* (λ*x. x*ˆ*Suc k* / *Suc k*) *x* :> *x*ˆ*k*
by (*intro derivative_eq_intros*) *auto*
qed (*auto simp*: *field_simps simp del*: *of_nat_Suc*)

6.19.21 Integration by parts

lemma *integral_by_parts_integrable*:
fixes *f g F G*::*real* ⇒ *real*
assumes $a \le b$
assumes *cont_f* [*intro*]: !!*x*. $a \le x \Longrightarrow x \le b \Longrightarrow$ *isCont f x*
assumes *cont_g* [*intro*]: !!*x*. $a \le x \Longrightarrow x \le b \Longrightarrow$ *isCont g x*
assumes [*intro*]: !!*x*. *DERIV F x* :> *f x*
assumes [*intro*]: !!*x*. *DERIV G x* :> *g x*
shows *integrable lborel* (λ*x*.((*F x*) ∗ (*g x*) + (*f x*) ∗ (*G x*)) ∗ *indicator* {*a* .. *b*} *x*)
by (*auto intro*!: *borel_integrable_atLeastAtMost continuous_intros*) (*auto intro*!: *DERIV_isCont*)

lemma *integral_by_parts*:
fixes *f g F G*::*real* ⇒ *real*
assumes [*arith*]: $a \le b$
assumes *cont_f* [*intro*]: !!*x*. $a \le x \Longrightarrow x \le b \Longrightarrow$ *isCont f x*
assumes *cont_g* [*intro*]: !!*x*. $a \le x \Longrightarrow x \le b \Longrightarrow$ *isCont g x*
assumes [*intro*]: !!*x*. *DERIV F x* :> *f x*
assumes [*intro*]: !!*x*. *DERIV G x* :> *g x*
shows (∫ *x*. (*F x* ∗ *g x*) ∗ *indicator* {*a* .. *b*} *x* ∂*lborel*)
= *F b* ∗ *G b* − *F a* ∗ *G a* − ∫ *x*. (*f x* ∗ *G x*) ∗ *indicator* {*a* .. *b*} *x* ∂*lborel*
proof−
have (∫ *x*. (*F x* ∗ *g x* + *f x* ∗ *G x*) ∗ *indicator* {*a* .. *b*} *x* ∂*lborel*)
= *LBINT x*. *F x* ∗ *g x* ∗ *indicat_real* {*a*..*b*} *x* + *f x* ∗ *G x* ∗ *indicat_real* {*a*..*b*} *x*
by (*meson vector_space_over_itself*.*scale_left_distrib*)
also have ... = (∫ *x*. (*F x* ∗ *g x*) ∗ *indicator* {*a* .. *b*} *x* ∂*lborel*) + ∫ *x*. (*f x* ∗ *G x*) ∗ *indicator* {*a* .. *b*} *x* ∂*lborel*
proof (*intro Bochner_Integration*.*integral_add borel_integrable_atLeastAtMost cont_f cont_g continuous_intros*)
show ⋀*x*. ⟦$a \le x$; $x \le b$⟧ ⟹ *isCont F x* ⋀*x*. ⟦$a \le x$; $x \le b$⟧ ⟹ *isCont G x*
using *DERIV_isCont* **by** *blast*+
qed
finally have (∫ *x*. (*F x* ∗ *g x* + *f x* ∗ *G x*) ∗ *indicator* {*a* .. *b*} *x* ∂*lborel*) =
(∫ *x*. (*F x* ∗ *g x*) ∗ *indicator* {*a* .. *b*} *x* ∂*lborel*) + ∫ *x*. (*f x* ∗ *G x*) ∗ *indicator* {*a* .. *b*} *x* ∂*lborel* **.**
moreover have (∫ *x*. (*F x* ∗ *g x* + *f x* ∗ *G x*) ∗ *indicator* {*a* .. *b*} *x* ∂*lborel*) = *F b* ∗ *G b* − *F a* ∗ *G a*
proof (*intro integral_FTC_Icc_real derivative_eq_intros cont_f cont_g continuous_intros*)
show ⋀*x*. ⟦$a \le x$; $x \le b$⟧ ⟹ *isCont F x* ⋀*x*. ⟦$a \le x$; $x \le b$⟧ ⟹ *isCont G x*
using *DERIV_isCont* **by** *blast*+
qed *auto*
ultimately show *?thesis* **by** *auto*
qed

lemma *integral_by_parts*′:
fixes *f g F G*::*real* ⇒ *real*

```
  assumes a ≤ b
  assumes !!x. a ≤x ⟹ x≤b ⟹ isCont f x
  assumes !!x. a ≤x ⟹ x≤b ⟹ isCont g x
  assumes !!x. DERIV F x :> f x
  assumes !!x. DERIV G x :> g x
  shows (∫ x. indicator {a .. b} x *R (F x * g x) ∂lborel)
           =  F b * G b − F a * G a − ∫ x. indicator {a .. b} x *R (f x * G x)
∂lborel
  using integral_by_parts[OF assms] by (simp add: ac_simps)

lemma has_bochner_integral_even_function:
  fixes f :: real ⇒ 'a :: {banach, second_countable_topology}
  assumes f: has_bochner_integral lborel (λx. indicator {0..} x *R f x) x
  assumes even: ⋀x. f (− x) = f x
  shows has_bochner_integral lborel f (2 *R x)
proof −
  have indicator: ⋀x::real. indicator {..0} (− x) = indicator {0..} x
    by (auto split: split_indicator)
  have has_bochner_integral lborel (λx. indicator {.. 0} x *R f x) x
    by (subst lborel_has_bochner_integral_real_affine_iff[where c=−1 and t=0])
      (auto simp: indicator even f)
  with f have has_bochner_integral lborel (λx. indicator {0..} x *R f x + indicator
{.. 0} x *R f x) (x + x)
    by (rule has_bochner_integral_add)
  then have has_bochner_integral lborel f (x + x)
    by (rule has_bochner_integral_discrete_difference[where X={0}, THEN iffD1,
rotated 4])
      (auto split: split_indicator)
  then show ?thesis
    by (simp add: scaleR_2)
qed

lemma has_bochner_integral_odd_function:
  fixes f :: real ⇒ 'a :: {banach, second_countable_topology}
  assumes f: has_bochner_integral lborel (λx. indicator {0..} x *R f x) x
  assumes odd: ⋀x. f (− x) = − f x
  shows has_bochner_integral lborel f 0
proof −
  have indicator: ⋀x::real. indicator {..0} (− x) = indicator {0..} x
    by (auto split: split_indicator)
  have has_bochner_integral lborel (λx. − indicator {.. 0} x *R f x) x
    by (subst lborel_has_bochner_integral_real_affine_iff[where c=−1 and t=0])
      (auto simp: indicator odd f)
  from has_bochner_integral_minus[OF this]
  have has_bochner_integral lborel (λx. indicator {.. 0} x *R f x) (− x)
    by simp
  with f have has_bochner_integral lborel (λx. indicator {0..} x *R f x + indicator
{.. 0} x *R f x) (x + − x)
    by (rule has_bochner_integral_add)
```

then have *has_bochner_integral lborel f* $(x + - x)$
  by (*rule has_bochner_integral_discrete_difference*[**where** $X=\{0\}$, *THEN iffD1*, *rotated 4*])
    (*auto split*: *split_indicator*)
  then show *?thesis*
    by *simp*
qed

lemma *has_integral_0_closure_imp_0*:
  fixes f :: *'a::euclidean_space* ⇒ *real*
  assumes f: *continuous_on* (*closure S*) f
    and *nonneg_interior*: $\bigwedge x.\ x \in S \Longrightarrow 0 \le f\,x$
    and *pos*: $0 <$ *emeasure lborel S*
    and *finite*: *emeasure lborel S* $< \infty$
    and *regular*: *emeasure lborel* (*closure S*) = *emeasure lborel S*
    and *opn*: *open S*
  assumes *int*: (f *has_integral 0*) (*closure S*)
  assumes x: $x \in$ *closure S*
  shows $f\,x = 0$
proof −
  have *zero*: *emeasure lborel* (*frontier S*) = *0*
    using *finite closure_subset regular*
    unfolding *frontier_def*
    by (*subst emeasure_Diff*) (*auto simp*: *frontier_def interior_open* ‹*open S*›)
  have *nonneg*: $0 \le f\,x$ **if** $x \in$ *closure S* **for** x
    using *continuous_ge_on_closure*[*OF f that nonneg_interior*] **by** *simp*
  have *0* = *integral* (*closure S*) f
    by (*blast intro*: *int sym*)
  also
  note *intl* = *has_integral_integrable*[*OF int*]
  have *af*: *f absolutely_integrable_on* (*closure S*)
    using *nonneg*
    by (*intro absolutely_integrable_onI intl integrable_eq*[*OF intl*]) *simp*
  then have *integral* (*closure S*) f = *set_lebesgue_integral lebesgue* (*closure S*) f
    by (*intro set_lebesgue_integral_eq_integral*(*2*)[*symmetric*])
  also have ... = *0* ⟷ (*AE x in lebesgue*. *indicator* (*closure S*) $x *_R f\,x = 0$)
    unfolding *set_lebesgue_integral_def*
  proof (*rule integral_nonneg_eq_0_iff_AE*)
    show *integrable lebesgue* (λx. *indicat_real* (*closure S*) $x *_R f\,x$)
      by (*metis af set_integrable_def*)
  qed (*use nonneg* **in** ‹*auto simp*: *indicator_def*›)
  also have ... ⟷ (*AE x in lebesgue*. $x \in \{x.\ x \in$ *closure S* $\longrightarrow f\,x = 0\}$)
    by (*auto simp*: *indicator_def*)
  finally have (*AE x in lebesgue*. $x \in \{x.\ x \in$ *closure S* $\longrightarrow f\,x = 0\}$) **by** *simp*
  moreover have (*AE x in lebesgue*. $x \in -$ *frontier S*)
    using *zero*
    by (*auto simp*: *eventually_ae_filter null_sets_def intro*!: *exI*[**where** x=*frontier S*])
  ultimately have *ae*: *AE* $x \in S$ *in lebesgue*. $x \in \{x \in$ *closure S*. $f\,x = 0\}$ (**is**

```
?th)
    by eventually_elim (use closure_subset in ‹auto simp: ›)
  have closed {0::real} by simp
  with continuous_on_closed_vimage[OF closed_closure, of S f] f
  have closed (f -` {0} ∩ closure S) by blast
  then have closed {x ∈ closure S. f x = 0} by (auto simp: vimage_def Int_def
conj_commute)
  with ‹open S› have x ∈ {x ∈ closure S. f x = 0} if x ∈ S for x using ae that
    by (rule mem_closed_if_AE_lebesgue_open)
  then have f x = 0 if x ∈ S for x using that by auto
  from continuous_constant_on_closure[OF f this ‹x ∈ closure S›]
  show f x = 0 .
qed

lemma has_integral_0_cbox_imp_0:
  fixes f :: 'a::euclidean_space ⇒ real
  assumes f: continuous_on (cbox a b) f
    and nonneg_interior: ⋀x. x ∈ box a b ⟹ 0 ≤ f x
  assumes int: (f has_integral 0) (cbox a b)
  assumes ne: box a b ≠ {}
  assumes x: x ∈ cbox a b
  shows f x = 0
proof -
  have 0 < emeasure lborel (box a b)
    using ne x unfolding emeasure_lborel_box_eq
    by (force intro!: prod_pos simp: mem_box algebra_simps)
  then show ?thesis using assms
    by (intro has_integral_0_closure_imp_0[of box a b f x])
     (auto simp: emeasure_lborel_box_eq emeasure_lborel_cbox_eq algebra_simps mem_box)
qed
```

6.19.22 Various common equivalent forms of function measurability

```
lemma indicator_sum_eq:
  fixes m::real and f :: 'a ⇒ real
  assumes |m| ≤ 2 ^ (2*n) m/2^n ≤ f x f x < (m+1)/2^n m ∈ ℤ
  shows  (∑ k::real | k ∈ ℤ ∧ |k| ≤ 2 ^ (2*n).
          k/2^n * indicator {y. k/2^n ≤ f y ∧ f y < (k+1)/2^n} x) = m/2^n
        (is sum ?f ?S = _)
proof -
  have sum ?f ?S = sum (λk. k/2^n * indicator {y. k/2^n ≤ f y ∧ f y <
(k+1)/2^n} x) {m}
  proof (rule comm_monoid_add_class.sum.mono_neutral_right)
    show finite ?S
      by (rule finite_abs_int_segment)
    show {m} ⊆ {k ∈ ℤ. |k| ≤ 2 ^ (2*n)}
      using assms by auto
    show ∀ i∈{k ∈ ℤ. |k| ≤ 2 ^ (2*n)} - {m}. ?f i = 0
```

using *assms* **by** (*auto simp*: *indicator_def Ints_def abs_le_iff field_split_simps*)
qed
also have ... = *m/2^n*
using *assms* **by** (*auto simp*: *indicator_def not_less*)
finally show *?thesis* **.**
qed

lemma *measurable_on_sf_limit_lemma1*:
fixes *f* :: *'a::euclidean_space* ⇒ *real*
assumes $\bigwedge a\ b.\ \{x \in S.\ a \le f\ x \wedge f\ x < b\} \in sets\ (lebesgue\_on\ S)$
obtains *g* **where** $\bigwedge n.\ g\ n \in borel\_measurable\ (lebesgue\_on\ S)$
$\bigwedge n.\ finite(range\ (g\ n))$
$\bigwedge x.\ (\lambda n.\ g\ n\ x) \longrightarrow f\ x$
proof
show $(\lambda x.\ sum\ (\lambda k::real.\ k/2\hat{}n * indicator\ \{y.\ k/2\hat{}n \le f\ y \wedge f\ y < (k+1)/2\hat{}n\}\ x)$
$\{k \in \mathbb{Z}.\ |k| \le 2\ \hat{}\ (2*n)\}) \in borel\_measurable\ (lebesgue\_on\ S)$
(**is** *?g* ∈ _) **for** *n*
proof −
have $\bigwedge k.\ [\![k \in \mathbb{Z};\ |k| \le 2\ \hat{}\ (2*n)]\!]$
$\Longrightarrow Measurable.pred\ (lebesgue\_on\ S)\ (\lambda x.\ k\ /\ (2\hat{}n) \le f\ x \wedge f\ x < (k+1)\ /\ (2\hat{}n))$
using *assms* **by** (*force simp*: *pred_def space_restrict_space*)
then show *?thesis*
by (*simp add*: *field_class.field_divide_inverse*)
qed
show *finite* (*range* (*?g n*)) **for** *n*
proof −
have $range\ (?g\ n) \subseteq (\lambda k.\ k/2\hat{}n)\ `\ \{k \in \mathbb{Z}.\ |k| \le 2\ \hat{}\ (2*n)\}$
proof *clarify*
fix *x*
show $?g\ n\ x \in (\lambda k.\ k/2\hat{}n)\ `\ \{k \in \mathbb{Z}.\ |k| \le 2\ \hat{}\ (2*n)\}$
proof $(cases\ \exists k::real.\ k \in \mathbb{Z} \wedge |k| \le 2\ \hat{}\ (2*n) \wedge k/2\hat{}n \le (f\ x) \wedge (f\ x) < (k+1)/2\hat{}n)$
case *True*
then show *?thesis*
apply *clarify*
by (*subst indicator_sum_eq*) *auto*
next
case *False*
then have *?g n x* = *0* **by** *auto*
then show *?thesis* **by** *force*
qed
qed
moreover have $finite\ ((\lambda k::real.\ (k/2\hat{}n))\ `\ \{k \in \mathbb{Z}.\ |k| \le 2\ \hat{}\ (2*n)\})$
by (*simp add*: *finite_abs_int_segment*)
ultimately show *?thesis*
using *finite_subset* **by** *blast*
qed

```
  show (λn. ?g n x) ⟶ f x for x
  proof (rule LIMSEQ_I)
    fix e::real
    assume e > 0
    obtain N1 where N1: |f x| < 2 ^ N1
      using real_arch_pow by fastforce
    obtain N2 where N2: (1/2) ^ N2 < e
      using real_arch_pow_inv ‹e > 0› by force
    have norm (?g n x − f x) < e if n: n ≥ max N1 N2 for n
    proof −
      define m where m ≡ floor(2^n ∗ (f x))
      have 1 ≤ |2^n| ∗ e
         using n N2 ‹e > 0› less_eq_real_def less_le_trans by (fastforce simp add:
field_split_simps)
      then have ∗: [[x ≤ y; y < x + 1]] ⟹ abs(x − y) < |2^n| ∗ e for x y::real
        by linarith
      have |2^n| ∗ |m/2^n − f x| = |2^n ∗ (m/2^n − f x)|
        by (simp add: abs_mult)
      also have ... = |real_of_int ⌊2^n ∗ f x⌋ − f x ∗ 2^n|
        by (simp add: algebra_simps m_def)
      also have ... < |2^n| ∗ e
        by (rule ∗; simp add: mult.commute)
      finally have |2^n| ∗ |m/2^n − f x| < |2^n| ∗ e .
      then have me: |m/2^n − f x| < e
        by simp
      have |real_of_int m| ≤ 2 ^ (2∗n)
      proof (cases f x < 0)
        case True
        then have −⌊f x⌋ ≤ ⌊(2::real) ^ N1⌋
          using N1 le_floor_iff minus_le_iff by fastforce
        with n True have |real_of_int⌊f x⌋| ≤ 2 ^ N1
          by linarith
        also have ... ≤ 2^n
          using n by (simp add: m_def)
        finally have |real_of_int ⌊f x⌋| ∗ 2^n ≤ 2^n ∗ 2^n
          by simp
        moreover
        have |real_of_int ⌊2^n ∗ f x⌋| ≤ |real_of_int ⌊f x⌋| ∗ 2^n
        proof −
          have |real_of_int ⌊2^n ∗ f x⌋| = − (real_of_int ⌊2^n ∗ f x⌋)
            using True by (simp add: abs_if mult_less_0_iff)
          also have ... ≤ − (real_of_int (⌊(2::real) ^ n⌋ ∗ ⌊f x⌋))
            using le_mult_floor_Ints [of (2::real)^n] by simp
          also have ... ≤ |real_of_int ⌊f x⌋| ∗ 2^n
            using True
            by simp
          finally show ?thesis .
        qed
        ultimately show ?thesis
```

by (*metis* (*no_types*, *hide_lams*) *m_def order_trans power2_eq_square power_even_eq*)
next
case *False*
with *n N1* **have** *f x* ≤ *2^n*
by (*simp add*: *not_less*) (*meson less_eq_real_def one_le_numeral order_trans power_increasing*)
moreover have *0* ≤ *m*
using *False m_def* **by** *force*
ultimately show *?thesis*
by (*metis abs_of_nonneg floor_mono le_floor_iff m_def of_int_0_le_iff power2_eq_square power_mult mult_le_cancel_iff1 zero_less_numeral mult.commute zero_less_power*)
qed
then have *?g n x* = *m*/*2^n*
by (*rule indicator_sum_eq*) (*auto simp add*: *m_def field_split_simps*, *linarith*)
then have *norm* (*?g n x* − *f x*) = *norm* (*m*/*2^n* − *f x*)
by *simp*
also have ... < *e*
by (*simp add*: *me*)
finally show *?thesis* **.**
qed
then show ∃ *no*. ∀ *n*≥*no*. *norm* (*?g n x* − *f x*) < *e*
by *blast*
qed
qed

lemma *borel_measurable_simple_function_limit*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
shows *f* ∈ *borel_measurable* (*lebesgue_on S*) ⟷
(∃ *g*. (∀ *n*. (*g n*) ∈ *borel_measurable* (*lebesgue_on S*)) ∧
(∀ *n*. *finite* (*range* (*g n*))) ∧ (∀ *x*. (λ*n*. *g n x*) ⟶ *f x*))
proof −
have ∃ *g*. (∀ *n*. (*g n*) ∈ *borel_measurable* (*lebesgue_on S*)) ∧
(∀ *n*. *finite* (*range* (*g n*))) ∧ (∀ *x*. (λ*n*. *g n x*) ⟶ *f x*)
if *f*: ⋀*a i*. *i* ∈ *Basis* ⟹ {*x* ∈ *S*. *f x* · *i* < *a*} ∈ *sets* (*lebesgue_on S*)
proof −
have ∃ *g*. (∀ *n*. (*g n*) ∈ *borel_measurable* (*lebesgue_on S*)) ∧
(∀ *n*. *finite*(*image* (*g n*) *UNIV*)) ∧
(∀ *x*. ((λ*n*. *g n x*) ⟶ *f x* · *i*)) **if** *i* ∈ *Basis* **for** *i*
proof (*rule measurable_on_sf_limit_lemma1* [*of S* λ*x*. *f x* · *i*])
show {*x* ∈ *S*. *a* ≤ *f x* · *i* ∧ *f x* · *i* < *b*} ∈ *sets* (*lebesgue_on S*) **for** *a b*
proof −
have {*x* ∈ *S*. *a* ≤ *f x* · *i* ∧ *f x* · *i* < *b*} = {*x* ∈ *S*. *f x* · *i* < *b*} − {*x* ∈ *S*. *a* > *f x* · *i*}
by *auto*
also have ... ∈ *sets* (*lebesgue_on S*)
using *f that* **by** *blast*
finally show *?thesis* **.**

```
      qed
    qed blast
    then obtain g where g:
        ⋀i n. i ∈ Basis ⟹ g i n ∈ borel_measurable (lebesgue_on S)
        ⋀i n. i ∈ Basis ⟹ finite(range (g i n))
        ⋀i x. i ∈ Basis ⟹ ((λn. g i n x) ⟶ f x · i)
      by metis
    show ?thesis
    proof (intro conjI allI exI)
      show (λx. ∑i∈Basis. g i n x *R i) ∈ borel_measurable (lebesgue_on S) for n
        by (intro borel_measurable_sum borel_measurable_scaleR) (auto intro: g)
      show finite (range (λx. ∑i∈Basis. g i n x *R i)) for n
      proof −
        have range (λx. ∑i∈Basis. g i n x *R i) ⊆ (λh. ∑i∈Basis. h i *R i) `
PiE Basis (λi. range (g i n))
        proof clarify
          fix x
          show (∑i∈Basis. g i n x *R i) ∈ (λh. ∑i∈Basis. h i *R i) ` (Π_E i∈Basis.
range (g i n))
            by (rule_tac x=λi∈Basis. g i n x in image_eqI) auto
        qed
        moreover have finite(PiE Basis (λi. range (g i n)))
          by (simp add: g finite_PiE)
        ultimately show ?thesis
          by (metis (mono_tags, lifting) finite_surj)
      qed
      show (λn. ∑i∈Basis. g i n x *R i) ⟶ f x for x
      proof −
        have (λn. ∑i∈Basis. g i n x *R i) ⟶ (∑i∈Basis. (f x · i) *R i)
          by (auto intro!: tendsto_sum tendsto_scaleR g)
        moreover have (∑i∈Basis. (f x · i) *R i) = f x
          using euclidean_representation by blast
        ultimately show ?thesis
          by metis
      qed
    qed
  qed
  moreover have f ∈ borel_measurable (lebesgue_on S)
            if meas_g: ⋀n. g n ∈ borel_measurable (lebesgue_on S)
              and fin: ⋀n. finite (range (g n))
              and to_f: ⋀x. (λn. g n x) ⟶ f x for g
    by (rule borel_measurable_LIMSEQ_metric [OF meas_g to_f])
  ultimately show ?thesis
    using borel_measurable_vimage_halfspace_component_lt by blast
qed
```

6.19.23 Lebesgue sets and continuous images

proposition *lebesgue_regular_inner*:

assumes $S \in$ *sets lebesgue*
obtains *K C* **where** *negligible K* $\bigwedge$*n::nat. compact*(*C n*) $S = (\bigcup n.\ C\ n) \cup K$
proof −
 have $\exists\, T.$ *closed T* $\wedge$ $T \subseteq S$ $\wedge$ $(S - T) \in$ *lmeasurable* $\wedge$ *emeasure lebesgue* (*S* − *T*) < *ennreal* ((*1*/*2*) ^*n*) **for** *n*
  using *sets_lebesgue_inner_closed assms*
  by (*metis sets_lebesgue_inner_closed zero_less_divide_1_iff zero_less_numeral zero_less_power*)
 then obtain *C* **where** *clo*: $\bigwedge$*n. closed* (*C n*) **and** *subS*: $\bigwedge$*n. C n* $\subseteq$ *S*
  and *mea*: $\bigwedge$*n.* (*S* − *C n*) $\in$ *lmeasurable*
  and *less*: $\bigwedge$*n. emeasure lebesgue* (*S* − *C n*) < *ennreal* ((*1*/*2*) ^*n*)
  by *metis*
 have $\exists\, F.$ ($\forall$ *n::nat. compact*(*F n*)) $\wedge$ ($\bigcup$*n. F n*) = *C m* **for** *m::nat*
  by (*metis clo closed_Union_compact_subsets*)
 then obtain *D* :: [*nat*,*nat*] ⇒ ′*a set* **where** *D*: $\bigwedge$*m n. compact*(*D m n*) $\bigwedge$*m.* ($\bigcup$*n. D m n*) = *C m*
  by *metis*
 let *?C* = *from_nat_into* ($\bigcup$*m. range* (*D m*))
 have *countable* ($\bigcup$*m. range* (*D m*))
  by *blast*
 have *range* (*from_nat_into* ($\bigcup$*m. range* (*D m*))) = ($\bigcup$*m. range* (*D m*))
  using *range_from_nat_into* **by** *simp*
 then have *CD*: $\exists$ *m n. ?C k* = *D m n* **for** *k*
  by (*metis* (*mono_tags*, *lifting*) *UN_iff rangeE range_eqI*)
 show *thesis*
 proof
  show *negligible* (*S* − ($\bigcup$*n. C n*))
  proof (*clarsimp simp*: *negligible_outer_le*)
   fix *e* :: *real*
   assume *e* > *0*
   then obtain *n* **where** *n*: (*1*/*2*) ^*n* < *e*
    using *real_arch_pow_inv* [*of e 1*/*2*] **by** *auto*
   show $\exists\, T.$ *S* − ($\bigcup$*n. C n*) $\subseteq$ *T* $\wedge$ *T* $\in$ *lmeasurable* $\wedge$ *measure lebesgue T* $\leq$ *e*
   proof (*intro exI conjI*)
    show *S* − ($\bigcup$*n. C n*) $\subseteq$ *S* − *C n*
     by *blast*
    show *S* − *C n* $\in$ *lmeasurable*
     by (*simp add*: *mea*)
    show *measure lebesgue* (*S* − *C n*) $\leq$ *e*
     using *less* [*of n*] *n*
     by (*simp add*: *emeasure_eq_measure2 less_le mea*)
   qed
  qed
  show *compact* (*?C n*) **for** *n*
   using *CD D* **by** *metis*
  show *S* = ($\bigcup$*n. ?C n*) $\cup$ (*S* − ($\bigcup$*n. C n*)) (**is** _ = *?rhs*)
  proof
   show *S* $\subseteq$ *?rhs*
    using *D* **by** *fastforce*

```
      show ?rhs ⊆ S
        using subS D CD by auto (metis Sup_upper range_eqI subsetCE)
    qed
  qed
qed

lemma sets_lebesgue_continuous_image:
  assumes T: T ∈ sets lebesgue and contf: continuous_on S f
    and negim: ⋀T. ⟦negligible T; T ⊆ S⟧ ⟹ negligible(f ` T) and T ⊆ S
 shows f ` T ∈ sets lebesgue
proof −
  obtain K C where negligible K and com: ⋀n::nat. compact(C n) and Teq: T
= (⋃n. C n) ∪ K
    using lebesgue_regular_inner [OF T] by metis
  then have comf: ⋀n::nat. compact(f ` C n)
    by (metis Un_subset_iff Union_upper ‹T ⊆ S› compact_continuous_image contf
continuous_on_subset rangeI)
  have ((⋃n. f ` C n) ∪ f ` K) ∈ sets lebesgue
  proof (rule sets.Un)
    have K ⊆ S
      using Teq ‹T ⊆ S› by blast
    show (⋃n. f ` C n) ∈ sets lebesgue
    proof (rule sets.countable_Union)
      show range (λn. f ` C n) ⊆ sets lebesgue
        using borel_compact comf by (auto simp: borel_compact)
    qed auto
    show f ` K ∈ sets lebesgue
      by (simp add: ‹K ⊆ S› ‹negligible K› negim negligible_imp_sets)
  qed
  then show ?thesis
    by (simp add: Teq image_Un image_Union)
qed

lemma differentiable_image_in_sets_lebesgue:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes S: S ∈ sets lebesgue and dim: DIM('m) ≤ DIM('n) and f: f differ-
entiable_on S
  shows f`S ∈ sets lebesgue
proof (rule sets_lebesgue_continuous_image [OF S])
  show continuous_on S f
    by (meson differentiable_imp_continuous_on f)
  show ⋀T. ⟦negligible T; T ⊆ S⟧ ⟹ negligible (f ` T)
    using differentiable_on_subset f
    by (auto simp: intro!: negligible_differentiable_image_negligible [OF dim])
qed auto

lemma sets_lebesgue_on_continuous_image:
  assumes S: S ∈ sets lebesgue and X: X ∈ sets (lebesgue_on S) and contf:
continuous_on S f
```

and *negim*: $\bigwedge T$. ⟦*negligible* T; $T \subseteq S$⟧ $\Longrightarrow$ *negligible*(f ‘ T)
shows f ‘ $X \in$ *sets* (*lebesgue_on* (f ‘ S))
proof −
have $X \subseteq S$
by (*metis S X sets.Int_space_eq2 sets_restrict_space_iff*)
moreover have f ‘ $S \in$ *sets lebesgue*
using *S contf negim sets_lebesgue_continuous_image* **by** *blast*
moreover have f ‘ $X \in$ *sets lebesgue*
by (*metis S X contf negim sets_lebesgue_continuous_image sets_restrict_space_iff space_restrict_space space_restrict_space2*)
ultimately show *?thesis*
by (*auto simp*: *sets_restrict_space_iff*)
qed

lemma *differentiable_image_in_sets_lebesgue_on*:
fixes f :: $'m$::*euclidean_space* $\Rightarrow$ $'n$::*euclidean_space*
assumes S: $S \in$ *sets lebesgue* **and** X: $X \in$ *sets* (*lebesgue_on* S) **and** *dim*: *DIM*($'m$) $\leq$ *DIM*($'n$)
and f: f *differentiable_on* S
shows f ‘ $X \in$ *sets* (*lebesgue_on* (f‘S))
proof (*rule sets_lebesgue_on_continuous_image* [*OF S X*])
show *continuous_on S f*
by (*meson differentiable_imp_continuous_on f*)
show $\bigwedge T$. ⟦*negligible* T; $T \subseteq S$⟧ $\Longrightarrow$ *negligible* (f ‘ T)
using *differentiable_on_subset f*
by (*auto simp*: *intro*!: *negligible_differentiable_image_negligible* [*OF dim*])
qed

6.19.24 Affine lemmas

lemma *borel_measurable_affine*:
fixes f :: $'a$ :: *euclidean_space* $\Rightarrow$ $'b$ :: *euclidean_space*
assumes f: $f \in$ *borel_measurable lebesgue* **and** $c \neq 0$
shows (λx. $f(t + c *_R x)$) $\in$ *borel_measurable lebesgue*
proof −
{ fix *a b*
have {x. $f\,x \in$ *cbox a b*} $\in$ *sets lebesgue*
using *f cbox_borel lebesgue_measurable_vimage_borel* **by** *blast*
then have (λx. $(x - t)$ $/_R$ c) ‘ {x. $f\,x \in$ *cbox a b*} $\in$ *sets lebesgue*
proof (*rule differentiable_image_in_sets_lebesgue*)
show (λx. $(x - t)$ $/_R$ c) *differentiable_on* {x. $f\,x \in$ *cbox a b*}
unfolding *differentiable_on_def differentiable_def*
by (*rule* ‹$c \neq 0$› *derivative_eq_intros strip exI* | *simp*)+
qed *auto*
moreover
have {x. $f(t + c *_R x) \in$ *cbox a b*} = (λx. $(x-t)$ $/_R$ c) ‘ {x. $f\,x \in$ *cbox a b*}
using ‹$c \neq 0$› **by** (*auto simp*: *image_def*)
ultimately have {x. $f(t + c *_R x) \in$ *cbox a b*} $\in$ *sets lebesgue*
by (*auto simp*: *borel_measurable_vimage_closed_interval*) **}**

then show *?thesis*
by (*subst lebesgue_on_UNIV_eq* [*symmetric*]; *auto simp*: *borel_measurable_vimage_closed_interval*)
qed

lemma *lebesgue_integrable_real_affine*:
fixes *f* :: *real* ⇒ *'a* :: *euclidean_space*
assumes *f*: *integrable lebesgue f* **and** *c* ≠ *0*
shows *integrable lebesgue* (λ*x*. *f*(*t* + *c* ∗ *x*))
proof −
have (λ*x*. *norm* (*f x*)) ∈ *borel_measurable lebesgue*
by (*simp add*: *borel_measurable_integrable f*)
then show *?thesis*
using *assms borel_measurable_affine* [*of f c*]
unfolding *integrable_iff_bounded*
by (*subst* (*asm*) *nn_integral_real_affine_lebesgue*[**where** *c*=*c* **and** *t*=*t*]) (*auto simp*: *ennreal_mult_less_top*)
qed

lemma *lebesgue_integrable_real_affine_iff*:
fixes *f* :: *real* ⇒ *'a* :: *euclidean_space*
shows *c* ≠ *0* ⟹ *integrable lebesgue* (λ*x*. *f*(*t* + *c* ∗ *x*)) ⟷ *integrable lebesgue f*
using *lebesgue_integrable_real_affine*[*of f c t*]
lebesgue_integrable_real_affine[*of* λ*x*. *f*(*t* + *c* ∗ *x*) *1*/*c* −*t*/*c*]
by (*auto simp*: *field_simps*)

lemma *lebesgue_integral_real_affine*:
fixes *f* :: *real* ⇒ *'a* :: *euclidean_space* **and** *c* :: *real*
assumes *c*: *c* ≠ *0* **shows** (∫ *x*. *f x* ∂ *lebesgue*) = |*c*| $*_R$ (∫ *x*. *f*(*t* + *c* ∗ *x*) ∂*lebesgue*)
proof *cases*
have (λ*x*. *t* + *c* ∗ *x*) ∈ *lebesgue* $\rightarrow_M$ *lebesgue*
using *lebesgue_affine_measurable*[**where** *c*= λ*x*::*real*. *c*] ‹*c* ≠ *0*› **by** *simp*
moreover
assume *integrable lebesgue f*
ultimately show *?thesis*
by (*subst lebesgue_real_affine*[*OF c*, *of t*]) (*auto simp*: *integral_density integral_distr*)
next
assume ¬ *integrable lebesgue f* **with** *c* **show** *?thesis*
by (*simp add*: *lebesgue_integrable_real_affine_iff not_integrable_integral_eq*)
qed

lemma *has_bochner_integral_lebesgue_real_affine_iff*:
fixes *i* :: *'a* :: *euclidean_space*
shows *c* ≠ *0* ⟹
has_bochner_integral lebesgue f i ⟷
has_bochner_integral lebesgue (λ*x*. *f*(*t* + *c* ∗ *x*)) (*i* $/_R$ |*c*|)
unfolding *has_bochner_integral_iff lebesgue_integrable_real_affine_iff*
by (*simp_all add*: *lebesgue_integral_real_affine*[*symmetric*] *divideR_right cong*: *conj_cong*)

lemma *has_bochner_integral_reflect_real_lemma*[*intro*]:
fixes f :: *real* ⇒ *'a*::*euclidean_space*
assumes *has_bochner_integral* (*lebesgue_on* {*a*..*b*}) *f i*
shows *has_bochner_integral* (*lebesgue_on* {−*b*..−*a*}) (λ*x*. *f*(−*x*)) *i*
proof −
have *eq*: *indicat_real* {*a*..*b*} (− *x*) $*_R$ *f*(− *x*) = *indicat_real* {− *b*..− *a*} *x* $*_R$ *f*(− *x*) **for** *x*
by (*auto simp*: *indicator_def*)
have *i*: *has_bochner_integral lebesgue* (λ*x*. *indicator* {*a*..*b*} *x* $*_R$ *f x*) *i*
using *assms* **by** (*auto simp*: *has_bochner_integral_restrict_space*)
then have *has_bochner_integral lebesgue* (λ*x*. *indicator* {−*b*..−*a*} *x* $*_R$ *f*(−*x*)) *i*
using *has_bochner_integral_lebesgue_real_affine_iff* [*of* −*1* (λ*x*. *indicator* {*a*..*b*} *x* $*_R$ *f x*) *i 0*]
by (*auto simp*: *eq*)
then show *?thesis*
by (*auto simp*: *has_bochner_integral_restrict_space*)
qed

lemma *has_bochner_integral_reflect_real*[*simp*]:
fixes f :: *real* ⇒ *'a*::*euclidean_space*
shows *has_bochner_integral* (*lebesgue_on* {−*b*..−*a*}) (λ*x*. *f*(−*x*)) *i* ⟷ *has_bochner_integral* (*lebesgue_on* {*a*..*b*}) *f i*
by (*auto simp*: *dest*: *has_bochner_integral_reflect_real_lemma*)

lemma *integrable_reflect_real*[*simp*]:
fixes f :: *real* ⇒ *'a*::*euclidean_space*
shows *integrable* (*lebesgue_on* {−*b*..−*a*}) (λ*x*. *f*(−*x*)) ⟷ *integrable* (*lebesgue_on* {*a*..*b*}) *f*
by (*metis has_bochner_integral_iff has_bochner_integral_reflect_real*)

lemma *integral_reflect_real*[*simp*]:
fixes f :: *real* ⇒ *'a*::*euclidean_space*
shows $integral^L$ (*lebesgue_on* {−*b* .. −*a*}) (λ*x*. *f*(−*x*)) = $integral^L$ (*lebesgue_on* {*a*..*b*::*real*}) *f*
using *has_bochner_integral_reflect_real* [*of b a f*]
by (*metis has_bochner_integral_iff not_integrable_integral_eq*)

6.19.25 More results on integrability

lemma *integrable_on_all_intervals_UNIV*:
fixes f :: *'a*::*euclidean_space* ⇒ *'b*::*banach*
assumes *intf*: ⋀*a b*. *f integrable_on cbox a b*
and *normf*: ⋀*x*. *norm*(*f x*) ≤ *g x* **and** *g*: *g integrable_on UNIV*
shows *f integrable_on UNIV*
proof −
have *intg*: (∀ *a b*. *g integrable_on cbox a b*)
and *gle_e*: ∀ *e*>*0*. ∃ *B*>*0*. ∀ *a b c d*.
ball 0 B ⊆ *cbox a b* ∧ *cbox a b* ⊆ *cbox c d* ⟶

|*integral* (*cbox a b*) *g* − *integral* (*cbox c d*) *g*|
< *e*
using *g*
by (*auto simp*: *integrable_alt_subset* [*of* _ *UNIV*] *intf*)
have *le*: *norm* (*integral* (*cbox a b*) *f* − *integral* (*cbox c d*) *f*) ≤ |*integral* (*cbox a b*) *g* − *integral* (*cbox c d*) *g*|
if *cbox a b* ⊆ *cbox c d* **for** *a b c d*
proof −
have *norm* (*integral* (*cbox a b*) *f* − *integral* (*cbox c d*) *f*) = *norm* (*integral* (*cbox c d* − *cbox a b*) *f*)
using *intf that* **by** (*simp add*: *norm_minus_commute integral_setdiff*)
also have ... ≤ *integral* (*cbox c d* − *cbox a b*) *g*
proof (*rule integral_norm_bound_integral* [*OF* _ _ *normf*])
show *f integrable_on cbox c d* − *cbox a b g integrable_on cbox c d* − *cbox a b*
by (*meson integrable_integral integrable_setdiff intf intg negligible_setdiff that*)+
qed
also have ... = *integral* (*cbox c d*) *g* − *integral* (*cbox a b*) *g*
using *intg that* **by** (*simp add*: *integral_setdiff*)
also have ... ≤ |*integral* (*cbox a b*) *g* − *integral* (*cbox c d*) *g*|
by *simp*
finally show *?thesis* .
qed
show *?thesis*
using *gle_e*
apply (*simp add*: *integrable_alt_subset* [*of* _ *UNIV*] *intf*)
apply (*erule imp_forward all_forward ex_forward asm_rl*)+
by (*meson not_less order_trans le*)
qed

lemma *integrable_on_all_intervals_integrable_bound*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::banach*
assumes *intf*: ⋀*a b*. (λ*x*. *if x* ∈ *S then f x else 0*) *integrable_on cbox a b*
and *normf*: ⋀*x*. *x* ∈ *S* ⟹ *norm*(*f x*) ≤ *g x* **and** *g*: *g integrable_on S*
shows *f integrable_on S*
using *integrable_on_all_intervals_UNIV* [*OF intf*, *of* (λ*x*. *if x* ∈ *S then g x else 0*)]
by (*simp add*: *g integrable_restrict_UNIV normf*)

lemma *measurable_bounded_lemma*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *f*: *f* ∈ *borel_measurable lebesgue* **and** *g*: *g integrable_on cbox a b*
and *normf*: ⋀*x*. *x* ∈ *cbox a b* ⟹ *norm*(*f x*) ≤ *g x*
shows *f integrable_on cbox a b*
proof −
have *g absolutely_integrable_on cbox a b*
by (*metis* (*full_types*) *add_increasing g le_add_same_cancel1 nonnegative_absolutely_integrable_1 norm_ge_zero normf*)
then have *integrable* (*lebesgue_on* (*cbox a b*)) *g*

by (*simp add*: *integrable_restrict_space set_integrable_def*)
then have *integrable* (*lebesgue_on* (*cbox a b*)) *f*
proof (*rule Bochner_Integration.integrable_bound*)
show *AE x in lebesgue_on* (*cbox a b*). *norm* (*f x*) ≤ *norm* (*g x*)
by (*rule AE_I2*) (*auto intro*: *normf order_trans*)
qed (*simp add*: *f measurable_restrict_space1*)
then show *?thesis*
by (*simp add*: *integrable_on_lebesgue_on*)
qed

proposition *measurable_bounded_by_integrable_imp_integrable*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f*: *f* ∈ *borel_measurable* (*lebesgue_on S*) **and** *g*: *g integrable_on S*
and *normf*: ⋀*x*. *x* ∈ *S* ⟹ *norm*(*f x*) ≤ *g x* **and** *S*: *S* ∈ *sets lebesgue*
shows *f integrable_on S*
proof (*rule integrable_on_all_intervals_integrable_bound* [*OF* _ *normf g*])
show (λ*x*. *if x* ∈ *S then f x else 0*) *integrable_on cbox a b* **for** *a b*
proof (*rule measurable_bounded_lemma*)
show (λ*x*. *if x* ∈ *S then f x else 0*) ∈ *borel_measurable lebesgue*
by (*simp add*: *S borel_measurable_if f*)
show (λ*x*. *if x* ∈ *S then g x else 0*) *integrable_on cbox a b*
by (*simp add*: *g integrable_altD*(*1*))
show *norm* (*if x* ∈ *S then f x else 0*) ≤ (*if x* ∈ *S then g x else 0*) **for** *x*
using *normf* **by** *simp*
qed
qed

lemma *measurable_bounded_by_integrable_imp_lebesgue_integrable*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f*: *f* ∈ *borel_measurable* (*lebesgue_on S*) **and** *g*: *integrable* (*lebesgue_on S*) *g*
and *normf*: ⋀*x*. *x* ∈ *S* ⟹ *norm*(*f x*) ≤ *g x* **and** *S*: *S* ∈ *sets lebesgue*
shows *integrable* (*lebesgue_on S*) *f*
proof −
have *f absolutely_integrable_on S*
by (*metis* (*no_types*) *S absolutely_integrable_integrable_bound f g integrable_on_lebesgue_on measurable_bounded_by_integrable_imp_integrable normf*)
then show *?thesis*
by (*simp add*: *S integrable_restrict_space set_integrable_def*)
qed

lemma *measurable_bounded_by_integrable_imp_integrable_real*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *real*
assumes *f* ∈ *borel_measurable* (*lebesgue_on S*) *g integrable_on S* ⋀*x*. *x* ∈ *S* ⟹ *abs*(*f x*) ≤ *g x S* ∈ *sets lebesgue*
shows *f integrable_on S*
using *measurable_bounded_by_integrable_imp_integrable* [*of f S g*] *assms* **by** *simp*

6.19.26 Relation between Borel measurability and integrability.

lemma *integrable_imp_measurable_weak*:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes $S \in$ *sets lebesgue f integrable_on S*
  shows $f \in$ *borel_measurable* (*lebesgue_on S*)
   by (*metis* (*mono_tags*, *lifting*) *assms has_integral_implies_lebesgue_measurable borel_measurable_restrict_space_iff integrable_on_def sets.Int_space_eq2*)

lemma *integrable_imp_measurable*:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes *f integrable_on S*
  shows $f \in$ *borel_measurable* (*lebesgue_on S*)
proof –
  have (*UNIV*::'a set) ∈ *sets lborel*
    by *simp*
  then show *?thesis*
   by (*metis* (*mono_tags*, *lifting*) *assms borel_measurable_if_D integrable_imp_measurable_weak integrable_restrict_UNIV lebesgue_on_UNIV_eq sets_lebesgue_on_refl*)
qed

lemma *integrable_iff_integrable_on*:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes $S \in$ *sets lebesgue* ($\int^+ x.$ *ennreal* (*norm* (*f x*)) ∂*lebesgue_on S*) < ∞
  shows *integrable* (*lebesgue_on S*) *f* ⟷ *f integrable_on S*
 using *assms integrable_iff_bounded integrable_imp_measurable integrable_on_lebesgue_on*
by *blast*

lemma *absolutely_integrable_measurable*:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes $S \in$ *sets lebesgue*
  shows *f absolutely_integrable_on S* ⟷ $f \in$ *borel_measurable* (*lebesgue_on S*) ∧ *integrable* (*lebesgue_on S*) (*norm* ∘ *f*)
    (**is** *?lhs* = *?rhs*)
proof
  assume *L*: *?lhs*
  then have $f \in$ *borel_measurable* (*lebesgue_on S*)
    by (*simp add*: *absolutely_integrable_on_def integrable_imp_measurable*)
  then show *?rhs*
    using *assms set_integrable_norm* [*of lebesgue S f*] *L*
    by (*simp add*: *integrable_restrict_space set_integrable_def*)
next
  assume *?rhs* **then show** *?lhs*
    using *assms integrable_on_lebesgue_on*
   by (*metis absolutely_integrable_integrable_bound comp_def eq_iff measurable_bounded_by_integrable_imp_integrable*)
qed

lemma *absolutely_integrable_measurable_real*:
  fixes f :: 'a::euclidean_space ⇒ *real*

assumes $S \in$ *sets lebesgue*
shows *f absolutely_integrable_on S* $\longleftrightarrow$
$f \in$ *borel_measurable* (*lebesgue_on S*) $\wedge$ *integrable* (*lebesgue_on S*) ($\lambda x.\ |f\ x|$)
by (*simp add*: *absolutely_integrable_measurable assms o_def*)

lemma *absolutely_integrable_measurable_real'*:
fixes *f* :: *'a::euclidean_space* $\Rightarrow$ *real*
assumes $S \in$ *sets lebesgue*
shows *f absolutely_integrable_on S* $\longleftrightarrow$ $f \in$ *borel_measurable* (*lebesgue_on S*) $\wedge$ ($\lambda x.\ |f\ x|$) *integrable_on S*
by (*metis abs_absolutely_integrableI_1 absolutely_integrable_measurable_real assms*

measurable_bounded_by_integrable_imp_integrable order_refl real_norm_def set_integrable_abs set_lebesgue_integral_eq_integral(1))

lemma *absolutely_integrable_imp_borel_measurable*:
fixes *f* :: *'a::euclidean_space* $\Rightarrow$ *'b::euclidean_space*
assumes *f absolutely_integrable_on S S* $\in$ *sets lebesgue*
shows $f \in$ *borel_measurable* (*lebesgue_on S*)
using *absolutely_integrable_measurable assms* **by** *blast*

lemma *measurable_bounded_by_integrable_imp_absolutely_integrable*:
fixes *f* :: *'a::euclidean_space* $\Rightarrow$ *'b::euclidean_space*
assumes $f \in$ *borel_measurable* (*lebesgue_on S*) $S \in$ *sets lebesgue*
and *g integrable_on S* **and** $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) \le (g\ x)$
shows *f absolutely_integrable_on S*
using *assms absolutely_integrable_integrable_bound measurable_bounded_by_integrable_imp_integrable* **by** *blast*

proposition *negligible_differentiable_vimage*:
fixes *f* :: *'a* $\Rightarrow$ *'a::euclidean_space*
assumes *negligible T*
and *f'*: $\bigwedge x.\ x \in S \Longrightarrow inj(f'\ x)$
and *derf*: $\bigwedge x.\ x \in S \Longrightarrow$ (*f has_derivative f' x*) (*at x within S*)
shows *negligible* $\{x \in S.\ f\ x \in T\}$
proof −
define *U* **where**
$U \equiv \lambda n::nat.\ \{x \in S.\ \forall y.\ y \in S \wedge norm(y - x) < 1/n$
$\longrightarrow norm(y - x) \le n * norm(f\ y - f\ x)\}$
have *negligible* $\{x \in U\ n.\ f\ x \in T\}$ **if** $n > 0$ **for** *n*
proof (*subst locally_negligible_alt*, *clarify*)
fix *a*
assume *a*: $a \in U\ n$ **and** *fa*: $f\ a \in T$
define *V* **where** $V \equiv \{x.\ x \in U\ n \wedge f\ x \in T\} \cap ball\ a\ (1\ /\ n\ /\ 2)$
show $\exists V.$ *openin* (*top_of_set* $\{x \in U\ n.\ f\ x \in T\}$) $V \wedge a \in V \wedge$ *negligible V*
proof (*intro exI conjI*)
have *noxy*: $norm(x - y) \le n * norm(f\ x - f\ y)$ **if** $x \in V$ $y \in V$ **for** *x y*
using *that* **unfolding** *U_def V_def mem_Collect_eq Int_iff mem_ball dist_norm*
by (*meson norm_triangle_half_r*)

```
      then have inj_on f V
        by (force simp: inj_on_def)
      then obtain g where g: ⋀x. x ∈ V ⟹ g(f x) = x
        by (metis inv_into_f_f)
      have ∃T' B. open T' ∧ f x ∈ T' ∧
                  (∀y∈f ` V ∩ T ∩ T'. norm (g y − g (f x)) ≤ B ∗ norm (y − f x))
        if f x ∈ T x ∈ V for x
        using that noxy
        by (rule_tac x = ball (f x) 1 in exI) (force simp: g)
      then have negligible (g ` (f ` V ∩ T))
       by (force simp: ‹negligible T› negligible_Int intro!: negligible_locally_Lipschitz_image)
      moreover have V ⊆ g ` (f ` V ∩ T)
        by (force simp: g image_iff V_def)
      ultimately show negligible V
        by (rule negligible_subset)
    qed (use a fa V_def that in auto)
  qed
  with negligible_countable_Union have negligible (⋃n ∈ {0<..}. {x. x ∈ U n ∧ f
x ∈ T})
    by auto
  moreover have {x ∈ S. f x ∈ T} ⊆ (⋃n ∈ {0<..}. {x. x ∈ U n ∧ f x ∈ T})
  proof clarsimp
    fix x
    assume x ∈ S and f x ∈ T
    then obtain inj: inj(f' x) and der: (f has_derivative f' x) (at x within S)
      using assms by metis
    moreover have linear(f' x)
      and eps: ⋀ε. ε > 0 ⟹ ∃δ>0. ∀y∈S. norm (y − x) < δ ⟶
                      norm (f y − f x − f' x (y − x)) ≤ ε ∗ norm (y − x)
      using der by (auto simp: has_derivative_within_alt linear_linear)
    ultimately obtain g where linear g and g: g ∘ f' x = id
      using linear_injective_left_inverse by metis
    then obtain B where B > 0 and B: ⋀z. B ∗ norm z ≤ norm(f' x z)
      using linear_invertible_bounded_below_pos ‹linear (f' x)› by blast
    then obtain i where i ≠ 0 and i: 1 / real i < B
      by (metis (full_types) inverse_eq_divide real_arch_invD)
    then obtain δ where δ > 0
        and δ: ⋀y. ⟦y∈S; norm (y − x) < δ⟧ ⟹
                norm (f y − f x − f' x (y − x)) ≤ (B − 1 / real i) ∗ norm (y − x)
      using eps [of B − 1/i] by auto
    then obtain j where j ≠ 0 and j: inverse (real j) < δ
      using real_arch_inverse by blast
    have norm (y − x)/(max i j) ≤ norm (f y − f x)
      if y ∈ S and less: norm (y − x) < 1 / (max i j) for y
    proof −
      have 1 / real (max i j) < δ
        using j ‹j ≠ 0› ‹0 < δ›
        by (auto simp: field_split_simps max_mult_distrib_left of_nat_max)
    then have norm (y − x) < δ
```

```
      using less by linarith
    with δ ‹y ∈ S› have le: norm (f y − f x − f′ x (y − x)) ≤ B ∗ norm (y −
x) − norm (y − x)/i
      by (auto simp: algebra_simps)
    have norm (y − x) / real (max i j) ≤ norm (y − x) / real i
      using ‹i ≠ 0› ‹j ≠ 0› by (simp add: field_split_simps max_mult_distrib_left
of_nat_max less_max_iff_disj)
    also have ... ≤ norm (f y − f x)
      using B [of y−x] le norm_triangle_ineq3 [of f y − f x f′ x (y − x)]
      by linarith
    finally show ?thesis .
  qed
  with ‹x ∈ S› ‹i ≠ 0› ‹j ≠ 0› show ∃ n∈{0<..}. x ∈ U n
   by (rule_tac x=max i j in bexI) (auto simp: field_simps U_def less_max_iff_disj)
qed
  ultimately show ?thesis
    by (rule negligible_subset)
qed

lemma absolutely_integrable_Un:
  fixes f :: ′a::euclidean_space ⇒ ′b::euclidean_space
  assumes S: f absolutely_integrable_on S and T: f absolutely_integrable_on T
  shows f absolutely_integrable_on (S ∪ T)
proof −
  have [simp]: {x. (if x ∈ A then f x else 0) ≠ 0} = {x ∈ A. f x ≠ 0} for A
    by auto
  let ?ST = {x ∈ S. f x ≠ 0} ∩ {x ∈ T. f x ≠ 0}
  have ?ST ∈ sets lebesgue
  proof (rule Sigma_Algebra.sets.Int)
    have f integrable_on S
      using S absolutely_integrable_on_def by blast
    then have (λx. if x ∈ S then f x else 0) integrable_on UNIV
      by (simp add: integrable_restrict_UNIV)
    then have borel: (λx. if x ∈ S then f x else 0) ∈ borel_measurable (lebesgue_on
UNIV)
      using integrable_imp_measurable lebesgue_on_UNIV_eq by blast
    then show {x ∈ S. f x ≠ 0} ∈ sets lebesgue
      unfolding borel_measurable_vimage_open
      by (rule allE [where x = −{0}]) auto
  next
    have f integrable_on T
      using T absolutely_integrable_on_def by blast
    then have (λx. if x ∈ T then f x else 0) integrable_on UNIV
      by (simp add: integrable_restrict_UNIV)
    then have borel: (λx. if x ∈ T then f x else 0) ∈ borel_measurable (lebesgue_on
UNIV)
      using integrable_imp_measurable lebesgue_on_UNIV_eq by blast
    then show {x ∈ T. f x ≠ 0} ∈ sets lebesgue
      unfolding borel_measurable_vimage_open
```

```
      by (rule allE [where x = −{0}]) auto
  qed
  then have f absolutely_integrable_on ?ST
    by (rule set_integrable_subset [OF S]) auto
  then have Int: (λx. if x ∈ ?ST then f x else 0) absolutely_integrable_on UNIV
    using absolutely_integrable_restrict_UNIV by blast
  have (λx. if x ∈ S then f x else 0) absolutely_integrable_on UNIV
       (λx. if x ∈ T then f x else 0) absolutely_integrable_on UNIV
    using S T absolutely_integrable_restrict_UNIV by blast+
  then have (λx. (if x ∈ S then f x else 0) + (if x ∈ T then f x else 0)) absolutely_integrable_on UNIV
    by (rule set_integral_add)
  then have (λx. ((if x ∈ S then f x else 0) + (if x ∈ T then f x else 0)) − (if x ∈ ?ST then f x else 0)) absolutely_integrable_on UNIV
    using Int by (rule set_integral_diff)
  then have (λx. if x ∈ S ∪ T then f x else 0) absolutely_integrable_on UNIV
    by (rule absolutely_integrable_spike) (auto intro: empty_imp_negligible)
  then show ?thesis
    unfolding absolutely_integrable_restrict_UNIV .
qed

lemma absolutely_integrable_on_combine:
  fixes f :: real ⇒ 'a::euclidean_space
  assumes f absolutely_integrable_on {a..c}
    and f absolutely_integrable_on {c..b}
    and a ≤ c
    and c ≤ b
  shows f absolutely_integrable_on {a..b}
  by (metis absolutely_integrable_Un assms ivl_disj_un_two_touch(4))

lemma uniform_limit_set_lebesgue_integral_at_top:
  fixes f :: 'a ⇒ real ⇒ 'b::{banach, second_countable_topology}
    and g :: real ⇒ real
  assumes bound: ⋀x y. x ∈ A ⟹ y ≥ a ⟹ norm (f x y) ≤ g y
  assumes integrable: set_integrable M {a..} g
  assumes measurable: ⋀x. x ∈ A ⟹ set_borel_measurable M {a..} (f x)
  assumes sets borel ⊆ sets M
  shows   uniform_limit A (λb x. LINT y:{a..b}|M. f x y) (λx. LINT y:{a..}|M. f x y) at_top
proof (cases A = {})
  case False
  then obtain x where x: x ∈ A by auto
  have g_nonneg: g y ≥ 0 if y ≥ a for y
  proof −
    have 0 ≤ norm (f x y) by simp
    also have ... ≤ g y using bound[OF x that] by simp
    finally show ?thesis .
  qed
```

have *integrable'*: *set_integrable M* {*a..*} (λy. *f x y*) **if** $x \in A$ **for** *x*
unfolding *set_integrable_def*
proof (*rule Bochner_Integration.integrable_bound*)
show *integrable M* (λx. *indicator* {*a..*} *x* $*$ *g x*)
using *integrable* **by** (*simp add*: *set_integrable_def*)
show (λy. *indicat_real* {*a..*} *y* $*_R$ *f x y*) $\in$ *borel_measurable M* **using** *measurable*[*OF that*]
by (*simp add*: *set_borel_measurable_def*)
show *AE y in M*. *norm* (*indicat_real* {*a..*} *y* $*_R$ *f x y*) $\leq$ *norm* (*indicat_real* {*a..*} *y* $*$ *g y*)
using *bound*[*OF that*] **by** (*intro AE_I2*) (*auto simp*: *indicator_def g_nonneg*)
qed

show *?thesis*
proof (*rule uniform_limitI*)
fix *e* :: *real* **assume** *e*: $e > 0$
have *sets* [*intro*]: $A \in$ *sets M* **if** $A \in$ *sets borel* **for** *A*
using *that assms* **by** *blast*

have ((λb. *LINT y*:{*a..b*}|*M*. *g y*) $\longrightarrow$ (*LINT y*:{*a..*}|*M*. *g y*)) *at_top*
by (*intro tendsto_set_lebesgue_integral_at_top assms sets*) *auto*
with *e* **obtain** *b0* :: *real* **where** *b0*: $\forall b \geq b0$. |(*LINT y*:{*a..*}|*M*. *g y*) $-$ (*LINT y*:{*a..b*}|*M*. *g y*)| $<$ *e*
by (*auto simp*: *tendsto_iff eventually_at_top_linorder dist_real_def abs_minus_commute*)
define *b* **where** $b = max\ a\ b0$
have $a \leq b$ **by** (*simp add*: *b_def*)
from *b0* **have** |(*LINT y*:{*a..*}|*M*. *g y*) $-$ (*LINT y*:{*a..b*}|*M*. *g y*)| $<$ *e*
by (*auto simp*: *b_def*)
also have {*a..*} = {*a..b*} $\cup$ {*b<..*} **by** (*auto simp*: *b_def*)
also have |(*LINT y*:. . . |*M*. *g y*) $-$ (*LINT y*:{*a..b*}|*M*. *g y*)| = |(*LINT y*:{*b<..*}|*M*. *g y*)|
using ‹$a \leq b$› **by** (*subst set_integral_Un*) (*auto intro!*: *set_integrable_subset*[*OF integrable*])
also have (*LINT y*:{*b<..*}|*M*. *g y*) $\geq$ *0*
using *g_nonneg* ‹$a \leq b$› **unfolding** *set_lebesgue_integral_def*
by (*intro Bochner_Integration.integral_nonneg*) (*auto simp*: *indicator_def*)
hence |(*LINT y*:{*b<..*}|*M*. *g y*)| = (*LINT y*:{*b<..*}|*M*. *g y*) **by** *simp*
finally have *less*: (*LINT y*:{*b<..*}|*M*. *g y*) $<$ *e* **.**

have *eventually* (λb. $b \geq b0$) *at_top* **by** (*rule eventually_ge_at_top*)
moreover have *eventually* (λb. $b \geq a$) *at_top* **by** (*rule eventually_ge_at_top*)
ultimately show *eventually* (λb. $\forall x \in A$.
dist (*LINT y*:{*a..b*}|*M*. *f x y*) (*LINT y*:{*a..*}|*M*. *f x y*) $<$ *e*) *at_top*
proof *eventually_elim*
case (*elim b*)
show *?case*
proof
fix *x* **assume** *x*: $x \in A$

```
        have dist (LINT y:{a..b}|M. f x y) (LINT y:{a..}|M. f x y) =
                norm ((LINT y:{a..}|M. f x y) − (LINT y:{a..b}|M. f x y))
          by (simp add: dist_norm norm_minus_commute)
        also have {a..} = {a..b} ∪ {b<..} using elim by auto
         also have (LINT y:...|M. f x y) − (LINT y:{a..b}|M. f x y) = (LINT
y:{b<..}|M. f x y)
          using elim x
         by (subst set_integral_Un) (auto intro!: set_integrable_subset[OF integrable'])
        also have norm ... ≤ (LINT y:{b<..}|M. norm (f x y)) using elim x
         by (intro set_integral_norm_bound set_integrable_subset[OF integrable']) auto
        also have ... ≤ (LINT y:{b<..}|M. g y) using elim x bound g_nonneg
            by (intro set_integral_mono set_integrable_norm set_integrable_subset[OF
integrable']
                  set_integrable_subset[OF integrable]) auto
        also have (LINT y:{b<..}|M. g y) ≥ 0
          using g_nonneg ‹a ≤ b› unfolding set_lebesgue_integral_def
          by (intro Bochner_Integration.integral_nonneg) (auto simp: indicator_def)
        hence (LINT y:{b<..}|M. g y) = |(LINT y:{b<..}|M. g y)| by simp
        also have ... = |(LINT y:{a..b} ∪ {b<..}|M. g y) − (LINT y:{a..b}|M. g
y)|
         using elim by (subst set_integral_Un) (auto intro!: set_integrable_subset[OF
integrable])
        also have {a..b} ∪ {b<..} = {a..} using elim by auto
        also have |(LINT y:{a..}|M. g y) − (LINT y:{a..b}|M. g y)| < e
          using b0 elim by blast
        finally show dist (LINT y:{a..b}|M. f x y) (LINT y:{a..}|M. f x y) < e .
      qed
    qed
  qed
qed auto
```

Differentiability of inverse function (most basic form)

```
proposition has_derivative_inverse_within:
  fixes f :: 'a::real_normed_vector ⇒ 'b::euclidean_space
  assumes der_f: (f has_derivative f') (at a within S)
    and cont_g: continuous (at (f a) within f ` S) g
    and a ∈ S linear g' and id: g' ∘ f' = id
    and gf: ⋀x. x ∈ S ⟹ g(f x) = x
  shows (g has_derivative g') (at (f a) within f ` S)
proof −
  have [simp]: g' (f' x) = x for x
    by (simp add: local.id pointfree_idE)
  have bounded_linear f'
    and f': ⋀e. e>0 ⟹ ∃d>0. ∀y∈S. norm (y − a) < d ⟶
                        norm (f y − f a − f' (y − a)) ≤ e ∗ norm (y − a)
    using der_f by (auto simp: has_derivative_within_alt)
  obtain C where C > 0 and C: ⋀x. norm (g' x) ≤ C ∗ norm x
    using linear_bounded_pos [OF ‹linear g'›] by metis
```

obtain *B k* **where** *B* > *0 k* > *0*
and *Bk*: ⋀*x*. ⟦*x* ∈ *S*; *norm*(*f x* − *f a*) < *k*⟧ ⟹ *norm*(*x* − *a*) ≤ *B* ∗ *norm*(*f x* − *f a*)
proof −
obtain *B* **where** *B* > *0* **and** *B*: ⋀*x*. *B* ∗ *norm x* ≤ *norm* (*f′ x*)
using *linear_inj_bounded_below_pos* [*of f′*] ‹*linear g′*› *id der_f has_derivative_linear linear_invertible_bounded_below_pos* **by** *blast*
then obtain *d* **where** *d*>*0*
and *d*: ⋀*y*. ⟦*y* ∈ *S*; *norm* (*y* − *a*) < *d*⟧ ⟹
norm (*f y* − *f a* − *f′* (*y* − *a*)) ≤ *B* / *2* ∗ *norm* (*y* − *a*)
using *f′* [*of B*/*2*] **by** *auto*
then obtain *e* **where** *e* > *0*
and *e*: ⋀*x*. ⟦*x* ∈ *S*; *norm* (*f x* − *f a*) < *e*⟧ ⟹ *norm* (*g* (*f x*) − *g* (*f a*)) < *d*
using *cont_g* **by** (*auto simp*: *continuous_within_eps_delta dist_norm*)
show *thesis*
proof
show *2*/*B* > *0*
using ‹*B* > *0*› **by** *simp*
show *norm* (*x* − *a*) ≤ *2* / *B* ∗ *norm* (*f x* − *f a*)
if *x* ∈ *S norm* (*f x* − *f a*) < *e* **for** *x*
proof −
have *xa*: *norm* (*x* − *a*) < *d*
using *e* [*OF that*] *gf* **by** (*simp add*: ‹*a* ∈ *S*› *that*)
have ∗: ⟦*norm*(*y* − *f′*) ≤ *B* / *2* ∗ *norm x*; *B* ∗ *norm x* ≤ *norm f′*⟧
⟹ *norm y* ≥ *B* / *2* ∗ *norm x* **for** *y f′*::*′b* **and** *x*::*′a*
using *norm_triangle_ineq3* [*of y f′*] **by** *linarith*
show *?thesis*
using ∗ [*OF d* [*OF* ‹*x* ∈ *S*› *xa*] *B*] ‹*B* > *0*› **by** (*simp add*: *field_simps*)
qed
qed (*use* ‹*e* > *0*› **in** *auto*)
qed
show *?thesis*
unfolding *has_derivative_within_alt*
proof (*intro conjI impI allI*)
show *bounded_linear g′*
using ‹*linear g′*› **by** (*simp add*: *linear_linear*)
next
fix *e* :: *real*
assume *e* > *0*
then obtain *d* **where** *d*>*0*
and *d*: ⋀*y*. ⟦*y* ∈ *S*; *norm* (*y* − *a*) < *d*⟧ ⟹
norm (*f y* − *f a* − *f′* (*y* − *a*)) ≤ *e* / (*B* ∗ *C*) ∗ *norm* (*y* − *a*)
using *f′* [*of e* / (*B* ∗ *C*)] ‹*B* > *0*› ‹*C* > *0*› **by** *auto*
have *norm* (*x* − *a* − *g′* (*f x* − *f a*)) ≤ *e* ∗ *norm* (*f x* − *f a*)
if *x* ∈ *S* **and** *lt_k*: *norm* (*f x* − *f a*) < *k* **and** *lt_dB*: *norm* (*f x* − *f a*) < *d*/*B* **for** *x*
proof −
have *norm* (*x* − *a*) ≤ *B* ∗ *norm*(*f x* − *f a*)
using *Bk lt_k* ‹*x* ∈ *S*› **by** *blast*

```
      also have ... < d
        by (metis ‹0 < B› lt_dB mult.commute pos_less_divide_eq)
      finally have lt_d: norm (x − a) < d .
      have norm (x − a − g' (f x − f a)) ≤ norm(g'(f x − f a − (f' (x − a))))
        by (simp add: linear_diff [OF ‹linear g'›] norm_minus_commute)
      also have ... ≤ C ∗ norm (f x − f a − f' (x − a))
        using C by blast
      also have ... ≤ e ∗ norm (f x − f a)
      proof −
        have norm (f x − f a − f' (x − a)) ≤ e / (B ∗ C) ∗ norm (x − a)
          using d [OF ‹x ∈ S› lt_d] .
        also have ... ≤ (norm (f x − f a) ∗ e) / C
         using ‹B > 0› ‹C > 0› ‹e > 0› by (simp add: field_simps Bk lt_k ‹x ∈ S›)
        finally show ?thesis
          using ‹C > 0› by (simp add: field_simps)
      qed
    finally show ?thesis .
    qed
    with ‹k > 0› ‹B > 0› ‹d > 0› ‹a ∈ S›
    show ∃ d>0. ∀ y∈f ‘ S.
               norm (y − f a) < d ⟶
               norm (g y − g (f a) − g' (y − f a)) ≤ e ∗ norm (y − f a)
      by (rule_tac x=min k (d / B) in exI) (auto simp: gf)
  qed
qed

end
```

6.20 Complex Analysis Basics

Definitions of analytic and holomorphic functions, limit theorems, complex differentiation

```
theory Complex_Analysis_Basics
  imports Derivative HOL−Library.Nonpos_Ints
begin
```

6.20.1 General lemmas

```
lemma nonneg_Reals_cmod_eq_Re: z ∈ ℝ≥0 ⟹ norm z = Re z
  by (simp add: complex_nonneg_Reals_iff cmod_eq_Re)

lemma fact_cancel:
  fixes c :: 'a::real_field
  shows of_nat (Suc n) ∗ c / (fact (Suc n)) = c / (fact n)
  using of_nat_neq_0 by force

lemma vector_derivative_cnj_within:
  assumes at x within A ≠ bot and f differentiable at x within A
```

shows *vector_derivative* (λ*z*. *cnj* (*f z*)) (*at x within A*) =
cnj (*vector_derivative f* (*at x within A*)) (**is** _ = *cnj ?D*)
proof −
let *?D* = *vector_derivative f* (*at x within A*)
from *assms* **have** (*f has_vector_derivative ?D*) (*at x within A*)
by (*subst* (*asm*) *vector_derivative_works*)
hence ((λ*x*. *cnj* (*f x*)) *has_vector_derivative cnj ?D*) (*at x within A*)
by (*rule has_vector_derivative_cnj*)
thus *?thesis* **using** *assms* **by** (*auto dest*: *vector_derivative_within*)
qed

lemma *vector_derivative_cnj*:
assumes *f differentiable at x*
shows *vector_derivative* (λ*z*. *cnj* (*f z*)) (*at x*) = *cnj* (*vector_derivative f* (*at x*))
using *assms* **by** (*intro vector_derivative_cnj_within*) *auto*

lemma
shows *open_halfspace_Re_lt*: *open* {*z*. *Re*(*z*) < *b*}
and *open_halfspace_Re_gt*: *open* {*z*. *Re*(*z*) > *b*}
and *closed_halfspace_Re_ge*: *closed* {*z*. *Re*(*z*) ≥ *b*}
and *closed_halfspace_Re_le*: *closed* {*z*. *Re*(*z*) ≤ *b*}
and *closed_halfspace_Re_eq*: *closed* {*z*. *Re*(*z*) = *b*}
and *open_halfspace_Im_lt*: *open* {*z*. *Im*(*z*) < *b*}
and *open_halfspace_Im_gt*: *open* {*z*. *Im*(*z*) > *b*}
and *closed_halfspace_Im_ge*: *closed* {*z*. *Im*(*z*) ≥ *b*}
and *closed_halfspace_Im_le*: *closed* {*z*. *Im*(*z*) ≤ *b*}
and *closed_halfspace_Im_eq*: *closed* {*z*. *Im*(*z*) = *b*}
by (*intro open_Collect_less closed_Collect_le closed_Collect_eq continuous_on_Re*
continuous_on_Im continuous_on_id continuous_on_const)+

lemma *closed_complex_Reals*: *closed* (ℝ :: *complex set*)
proof −
have (ℝ :: *complex set*) = {*z*. *Im z* = *0*}
by (*auto simp*: *complex_is_Real_iff*)
then show *?thesis*
by (*metis closed_halfspace_Im_eq*)
qed

lemma *closed_Real_halfspace_Re_le*: *closed* (ℝ ∩ {*w*. *Re w* ≤ *x*})
by (*simp add*: *closed_Int closed_complex_Reals closed_halfspace_Re_le*)

lemma *closed_nonpos_Reals_complex* [*simp*]: *closed* ($\mathbb{R}_{\leq 0}$ :: *complex set*)
proof −
have $\mathbb{R}_{\leq 0}$ = ℝ ∩ {*z*. *Re*(*z*) ≤ *0*}
using *complex_nonpos_Reals_iff complex_is_Real_iff* **by** *auto*
then show *?thesis*
by (*metis closed_Real_halfspace_Re_le*)
qed

lemma *closed_Real_halfspace_Re_ge*: *closed* ($\mathbb{R}$ ∩ {*w*. *x* ≤ *Re*(*w*)})
  using *closed_halfspace_Re_ge*
  by (*simp add*: *closed_Int closed_complex_Reals*)

lemma *closed_nonneg_Reals_complex* [*simp*]: *closed* ($\mathbb{R}_{\geq 0}$ :: *complex set*)
proof –
  have $\mathbb{R}_{\geq 0}$ = $\mathbb{R}$ ∩ {*z*. *Re*(*z*) ≥ *0*}
    using *complex_nonneg_Reals_iff complex_is_Real_iff* **by** *auto*
  then show *?thesis*
    by (*metis closed_Real_halfspace_Re_ge*)
qed

lemma *closed_real_abs_le*: *closed* {*w* ∈ $\mathbb{R}$. |*Re w*| ≤ *r*}
proof –
  have {*w* ∈ $\mathbb{R}$. |*Re w*| ≤ *r*} = ($\mathbb{R}$ ∩ {*w*. *Re w* ≤ *r*}) ∩ ($\mathbb{R}$ ∩ {*w*. *Re w* ≥ −*r*})
    by *auto*
  then show *closed* {*w* ∈ $\mathbb{R}$. |*Re w*| ≤ *r*}
   by (*simp add*: *closed_Int closed_Real_halfspace_Re_ge closed_Real_halfspace_Re_le*)
qed

lemma *real_lim*:
  fixes *l*::*complex*
  assumes (*f* ⟶ *l*) *F* **and** ¬ *trivial_limit F* **and** *eventually P F* **and** ⋀*a*. *P a* ⟹ *f a* ∈ $\mathbb{R}$
  shows *l* ∈ $\mathbb{R}$
proof (*rule Lim_in_closed_set*[*OF closed_complex_Reals _ assms*(*2*,*1*)])
  show *eventually* (λ*x*. *f x* ∈ $\mathbb{R}$) *F*
    using *assms*(*3*, *4*) **by** (*auto intro*: *eventually_mono*)
qed

lemma *real_lim_sequentially*:
  fixes *l*::*complex*
  shows (*f* ⟶ *l*) *sequentially* ⟹ (∃ *N*. ∀ *n*≥*N*. *f n* ∈ $\mathbb{R}$) ⟹ *l* ∈ $\mathbb{R}$
by (*rule real_lim* [**where** *F*=*sequentially*]) (*auto simp*: *eventually_sequentially*)

lemma *real_series*:
  fixes *l*::*complex*
  shows *f sums l* ⟹ (⋀*n*. *f n* ∈ $\mathbb{R}$) ⟹ *l* ∈ $\mathbb{R}$
unfolding *sums_def*
by (*metis real_lim_sequentially sum_in_Reals*)

lemma *Lim_null_comparison_Re*:
  assumes *eventually* (λ*x*. *norm*(*f x*) ≤ *Re*(*g x*)) *F* (*g* ⟶ *0*) *F* **shows** (*f* ⟶ *0*) *F*
  by (*rule Lim_null_comparison*[*OF assms*(*1*)] *tendsto_eq_intros assms*(*2*))+ *simp*

6.20.2 Holomorphic functions

definition *holomorphic_on* :: [*complex* ⇒ *complex*, *complex set*] ⇒ *bool*

(**infixl** (*holomorphic′_on*) *50*)
where *f holomorphic_on s* ≡ ∀ *x*∈*s*. *f field_differentiable* (*at x within s*)

named_theorems *holomorphic_intros structural introduction rules for holomorphic_on*

lemma *holomorphic_onI* [*intro?*]: (⋀*x*. *x* ∈ *s* ⟹ *f field_differentiable* (*at x within s*)) ⟹ *f holomorphic_on s*
by (*simp add*: *holomorphic_on_def*)

lemma *holomorphic_onD* [*dest?*]: ⟦*f holomorphic_on s*; *x* ∈ *s*⟧ ⟹ *f field_differentiable* (*at x within s*)
by (*simp add*: *holomorphic_on_def*)

lemma *holomorphic_on_imp_differentiable_on*:
f holomorphic_on s ⟹ *f differentiable_on s*
unfolding *holomorphic_on_def differentiable_on_def*
by (*simp add*: *field_differentiable_imp_differentiable*)

lemma *holomorphic_on_imp_differentiable_at*:
⟦*f holomorphic_on s*; *open s*; *x* ∈ *s*⟧ ⟹ *f field_differentiable* (*at x*)
using *at_within_open holomorphic_on_def* **by** *fastforce*

lemma *holomorphic_on_empty* [*holomorphic_intros*]: *f holomorphic_on* {}
by (*simp add*: *holomorphic_on_def*)

lemma *holomorphic_on_open*:
open s ⟹ *f holomorphic_on s* ⟷ (∀ *x* ∈ *s*. ∃ *f* ′. *DERIV f x* :> *f* ′)
by (*auto simp*: *holomorphic_on_def field_differentiable_def has_field_derivative_def at_within_open* [*of _ s*])

lemma *holomorphic_on_imp_continuous_on*:
f holomorphic_on s ⟹ *continuous_on s f*
by (*metis field_differentiable_imp_continuous_at continuous_on_eq_continuous_within holomorphic_on_def*)

lemma *holomorphic_on_subset* [*elim*]:
f holomorphic_on s ⟹ *t* ⊆ *s* ⟹ *f holomorphic_on t*
unfolding *holomorphic_on_def*
by (*metis field_differentiable_within_subset subsetD*)

lemma *holomorphic_transform*: ⟦*f holomorphic_on s*; ⋀*x*. *x* ∈ *s* ⟹ *f x* = *g x*⟧ ⟹ *g holomorphic_on s*
by (*metis field_differentiable_transform_within linordered_field_no_ub holomorphic_on_def*)

lemma *holomorphic_cong*: *s* = *t* ==> (⋀*x*. *x* ∈ *s* ⟹ *f x* = *g x*) ⟹ *f holomorphic_on s* ⟷ *g holomorphic_on t*
by (*metis holomorphic_transform*)

lemma *holomorphic_on_linear* [*simp*, *holomorphic_intros*]: ((∗) *c*) *holomorphic_on s*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_linear*)

lemma *holomorphic_on_const* [*simp*, *holomorphic_intros*]: (λ*z*. *c*) *holomorphic_on s*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_const*)

lemma *holomorphic_on_ident* [*simp*, *holomorphic_intros*]: (λ*x*. *x*) *holomorphic_on s*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_ident*)

lemma *holomorphic_on_id* [*simp*, *holomorphic_intros*]: *id holomorphic_on s*
unfolding *id_def* **by** (*rule holomorphic_on_ident*)

lemma *holomorphic_on_compose*:
f holomorphic_on s ⟹ *g holomorphic_on* (*f* ‘ *s*) ⟹ (*g o f*) *holomorphic_on s*
using *field_differentiable_compose_within*[*of f _ s g*]
by (*auto simp*: *holomorphic_on_def*)

lemma *holomorphic_on_compose_gen*:
f holomorphic_on s ⟹ *g holomorphic_on t* ⟹ *f* ‘ *s* ⊆ *t* ⟹ (*g o f*) *holomorphic_on s*
by (*metis holomorphic_on_compose holomorphic_on_subset*)

lemma *holomorphic_on_balls_imp_entire*:
assumes ¬*bdd_above A* ⋀*r*. *r* ∈ *A* ⟹ *f holomorphic_on ball c r*
shows *f holomorphic_on B*
proof (*rule holomorphic_on_subset*)
show *f holomorphic_on UNIV* **unfolding** *holomorphic_on_def*
proof
fix *z* :: *complex*
from ‹¬*bdd_above A*› **obtain** *r* **where** *r*: *r* ∈ *A r* > *norm* (*z* − *c*)
by (*meson bdd_aboveI not_le*)
with *assms*(*2*) **have** *f holomorphic_on ball c r* **by** *blast*
moreover from *r* **have** *z* ∈ *ball c r* **by** (*auto simp*: *dist_norm norm_minus_commute*)
ultimately show *f field_differentiable at z*
by (*auto simp*: *holomorphic_on_def at_within_open*[*of _ ball c r*])
qed
qed *auto*

lemma *holomorphic_on_balls_imp_entire'*:
assumes ⋀*r*. *r* > *0* ⟹ *f holomorphic_on ball c r*
shows *f holomorphic_on B*
proof (*rule holomorphic_on_balls_imp_entire*)
{
fix *M* :: *real*
have ∃*x*. *x* > *max M 0* **by** (*intro gt_ex*)
hence ∃*x*>*0*. *x* > *M* **by** *auto*

}
thus ¬*bdd_above* {(*0*::*real*)<..} **unfolding** *bdd_above_def*
by (*auto simp*: *not_le*)
qed (*insert assms*, *auto*)

lemma *holomorphic_on_minus* [*holomorphic_intros*]: *f holomorphic_on s* ⟹ (λ*z*. −(*f z*)) *holomorphic_on s*
by (*metis field_differentiable_minus holomorphic_on_def*)

lemma *holomorphic_on_add* [*holomorphic_intros*]:
⟦*f holomorphic_on s*; *g holomorphic_on s*⟧ ⟹ (λ*z*. *f z* + *g z*) *holomorphic_on s*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_add*)

lemma *holomorphic_on_diff* [*holomorphic_intros*]:
⟦*f holomorphic_on s*; *g holomorphic_on s*⟧ ⟹ (λ*z*. *f z* − *g z*) *holomorphic_on s*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_diff*)

lemma *holomorphic_on_mult* [*holomorphic_intros*]:
⟦*f holomorphic_on s*; *g holomorphic_on s*⟧ ⟹ (λ*z*. *f z* ∗ *g z*) *holomorphic_on s*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_mult*)

lemma *holomorphic_on_inverse* [*holomorphic_intros*]:
⟦*f holomorphic_on s*; ⋀*z*. *z* ∈ *s* ⟹ *f z* ≠ *0*⟧ ⟹ (λ*z*. *inverse* (*f z*)) *holomorphic_on s*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_inverse*)

lemma *holomorphic_on_divide* [*holomorphic_intros*]:
⟦*f holomorphic_on s*; *g holomorphic_on s*; ⋀*z*. *z* ∈ *s* ⟹ *g z* ≠ *0*⟧ ⟹ (λ*z*. *f z* / *g z*) *holomorphic_on s*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_divide*)

lemma *holomorphic_on_power* [*holomorphic_intros*]:
f holomorphic_on s ⟹ (λ*z*. (*f z*)ˆ*n*) *holomorphic_on s*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_power*)

lemma *holomorphic_on_sum* [*holomorphic_intros*]:
(⋀*i*. *i* ∈ *I* ⟹ (*f i*) *holomorphic_on s*) ⟹ (λ*x*. *sum* (λ*i*. *f i x*) *I*) *holomorphic_on s*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_sum*)

lemma *holomorphic_on_prod* [*holomorphic_intros*]:
(⋀*i*. *i* ∈ *I* ⟹ (*f i*) *holomorphic_on s*) ⟹ (λ*x*. *prod* (λ*i*. *f i x*) *I*) *holomorphic_on s*
by (*induction I rule*: *infinite_finite_induct*) (*auto intro*: *holomorphic_intros*)

lemma *holomorphic_pochhammer* [*holomorphic_intros*]:
f holomorphic_on A ⟹ (λ*s*. *pochhammer* (*f s*) *n*) *holomorphic_on A*
by (*induction n*) (*auto intro*!: *holomorphic_intros simp*: *pochhammer_Suc*)

lemma *holomorphic_on_scaleR* [*holomorphic_intros*]:
f holomorphic_on A ⟹ (λ*x*. *c* $*_R$ *f x*) *holomorphic_on A*
by (*auto simp*: *scaleR_conv_of_real intro*!: *holomorphic_intros*)

lemma *holomorphic_on_Un* [*holomorphic_intros*]:
assumes *f holomorphic_on A f holomorphic_on B open A open B*
shows *f holomorphic_on* (*A* ∪ *B*)
using *assms* **by** (*auto simp*: *holomorphic_on_def at_within_open*[*of _ A*]
at_within_open[*of _ B*] *at_within_open*[*of _ A* ∪ *B*] *open_Un*)

lemma *holomorphic_on_If_Un* [*holomorphic_intros*]:
assumes *f holomorphic_on A g holomorphic_on B open A open B*
assumes ⋀*z*. *z* ∈ *A* ⟹ *z* ∈ *B* ⟹ *f z* = *g z*
shows (λ*z*. *if z* ∈ *A then f z else g z*) *holomorphic_on* (*A* ∪ *B*) (**is** *?h holomorphic_on _*)
proof (*intro holomorphic_on_Un*)
note ‹*f holomorphic_on A*›
also have *f holomorphic_on A* ⟷ *?h holomorphic_on A*
by (*intro holomorphic_cong*) *auto*
finally show ... .
next
note ‹*g holomorphic_on B*›
also have *g holomorphic_on B* ⟷ *?h holomorphic_on B*
using *assms* **by** (*intro holomorphic_cong*) *auto*
finally show ... .
qed (*insert assms*, *auto*)

lemma *holomorphic_derivI*:
⟦*f holomorphic_on S*; *open S*; *x* ∈ *S*⟧
⟹ (*f has_field_derivative deriv f x*) (*at x within T*)
by (*metis DERIV_deriv_iff_field_differentiable at_within_open holomorphic_on_def has_field_derivative_at_within*)

lemma *complex_derivative_transform_within_open*:
⟦*f holomorphic_on s*; *g holomorphic_on s*; *open s*; *z* ∈ *s*; ⋀*w*. *w* ∈ *s* ⟹ *f w* = *g w*⟧
⟹ *deriv f z* = *deriv g z*
unfolding *holomorphic_on_def*
by (*rule DERIV_imp_deriv*)
(*metis DERIV_deriv_iff_field_differentiable has_field_derivative_transform_within_open at_within_open*)

lemma *holomorphic_nonconstant*:
assumes *holf*: *f holomorphic_on S* **and** *open S* ξ ∈ *S deriv f* ξ ≠ *0*
shows ¬ *f constant_on S*
by (*rule nonzero_deriv_nonconstant* [*of f deriv f* ξ ξ *S*])
(*use assms* **in** ‹*auto simp*: *holomorphic_derivI*›)

6.20.3 Analyticity on a set

definition *analytic_on* (**infixl** (*analytic′_on*) *50*)
 where *f analytic_on S* ≡ ∀ *x* ∈ *S*. ∃ *e*. *0* < *e* ∧ *f holomorphic_on* (*ball x e*)

named_theorems *analytic_intros introduction rules for proving analyticity*

lemma *analytic_imp_holomorphic*: *f analytic_on S* ⟹ *f holomorphic_on S*
 by (*simp add*: *at_within_open* [*OF _ open_ball*] *analytic_on_def holomorphic_on_def*)
  (*metis centre_in_ball field_differentiable_at_within*)

lemma *analytic_on_open*: *open S* ⟹ *f analytic_on S* ⟷ *f holomorphic_on S*
apply (*auto simp*: *analytic_imp_holomorphic*)
apply (*auto simp*: *analytic_on_def holomorphic_on_def*)
by (*metis holomorphic_on_def holomorphic_on_subset open_contains_ball*)

lemma *analytic_on_imp_differentiable_at*:
 f analytic_on S ⟹ *x* ∈ *S* ⟹ *f field_differentiable* (*at x*)
 apply (*auto simp*: *analytic_on_def holomorphic_on_def*)
by (*metis open_ball centre_in_ball field_differentiable_within_open*)

lemma *analytic_on_subset*: *f analytic_on S* ⟹ *T* ⊆ *S* ⟹ *f analytic_on T*
 by (*auto simp*: *analytic_on_def*)

lemma *analytic_on_Un*: *f analytic_on* (*S* ∪ *T*) ⟷ *f analytic_on S* ∧ *f analytic_on T*
 by (*auto simp*: *analytic_on_def*)

lemma *analytic_on_Union*: *f analytic_on* (⋃𝒯) ⟷ (∀ *T* ∈ 𝒯. *f analytic_on T*)
 by (*auto simp*: *analytic_on_def*)

lemma *analytic_on_UN*: *f analytic_on* (⋃ *i*∈*I*. *S i*) ⟷ (∀ *i*∈*I*. *f analytic_on* (*S i*))
 by (*auto simp*: *analytic_on_def*)

lemma *analytic_on_holomorphic*:
 f analytic_on S ⟷ (∃ *T*. *open T* ∧ *S* ⊆ *T* ∧ *f holomorphic_on T*)
 (**is** *?lhs* = *?rhs*)
proof −
 have *?lhs* ⟷ (∃ *T*. *open T* ∧ *S* ⊆ *T* ∧ *f analytic_on T*)
 proof *safe*
  assume *f analytic_on S*
  then show ∃ *T*. *open T* ∧ *S* ⊆ *T* ∧ *f analytic_on T*
   apply (*simp add*: *analytic_on_def*)
   apply (*rule exI* [**where** *x*=⋃{*U*. *open U* ∧ *f analytic_on U*}], *auto*)
   apply (*metis open_ball analytic_on_open centre_in_ball*)
   by (*metis analytic_on_def*)
 next
  fix *T*
  assume *open T S* ⊆ *T f analytic_on T*

```
    then show f analytic_on S
      by (metis analytic_on_subset)
  qed
  also have ... ⟷ ?rhs
    by (auto simp: analytic_on_open)
  finally show ?thesis .
qed

lemma analytic_on_linear [analytic_intros,simp]: ((*) c) analytic_on S
  by (auto simp add: analytic_on_holomorphic)

lemma analytic_on_const [analytic_intros,simp]: (λz. c) analytic_on S
  by (metis analytic_on_def holomorphic_on_const zero_less_one)

lemma analytic_on_ident [analytic_intros,simp]: (λx. x) analytic_on S
  by (simp add: analytic_on_def gt_ex)

lemma analytic_on_id [analytic_intros]: id analytic_on S
  unfolding id_def by (rule analytic_on_ident)

lemma analytic_on_compose:
  assumes f: f analytic_on S
      and g: g analytic_on (f ` S)
    shows (g o f) analytic_on S
unfolding analytic_on_def
proof (intro ballI)
  fix x
  assume x: x ∈ S
  then obtain e where e: 0 < e and fh: f holomorphic_on ball x e using f
    by (metis analytic_on_def)
  obtain e' where e': 0 < e' and gh: g holomorphic_on ball (f x) e' using g
    by (metis analytic_on_def g image_eqI x)
  have isCont f x
   by (metis analytic_on_imp_differentiable_at field_differentiable_imp_continuous_at
f x)
  with e' obtain d where d: 0 < d and fd: f ` ball x d ⊆ ball (f x) e'
     by (auto simp: continuous_at_ball)
  have g ∘ f holomorphic_on ball x (min d e)
    apply (rule holomorphic_on_compose)
    apply (metis fh holomorphic_on_subset min.bounded_iff order_refl subset_ball)
   by (metis fd gh holomorphic_on_subset image_mono min.cobounded1 subset_ball)
  then show ∃ e>0. g ∘ f holomorphic_on ball x e
    by (metis d e min_less_iff_conj)
qed

lemma analytic_on_compose_gen:
  f analytic_on S ⟹ g analytic_on T ⟹ (⋀z. z ∈ S ⟹ f z ∈ T)
           ⟹ g o f analytic_on S
by (metis analytic_on_compose analytic_on_subset image_subset_iff)
```

```
lemma analytic_on_neg [analytic_intros]:
  f analytic_on S ⟹ (λz. −(f z)) analytic_on S
by (metis analytic_on_holomorphic holomorphic_on_minus)

lemma analytic_on_add [analytic_intros]:
  assumes f: f analytic_on S
      and g: g analytic_on S
    shows (λz. f z + g z) analytic_on S
unfolding analytic_on_def
proof (intro ballI)
  fix z
  assume z: z ∈ S
  then obtain e where e: 0 < e and fh: f holomorphic_on ball z e using f
    by (metis analytic_on_def)
  obtain e' where e': 0 < e' and gh: g holomorphic_on ball z e' using g
    by (metis analytic_on_def g z)
  have (λz. f z + g z) holomorphic_on ball z (min e e')
    apply (rule holomorphic_on_add)
    apply (metis fh holomorphic_on_subset min.bounded_iff order_refl subset_ball)
    by (metis gh holomorphic_on_subset min.bounded_iff order_refl subset_ball)
  then show ∃ e>0. (λz. f z + g z) holomorphic_on ball z e
    by (metis e e' min_less_iff_conj)
qed

lemma analytic_on_diff [analytic_intros]:
  assumes f: f analytic_on S
      and g: g analytic_on S
    shows (λz. f z − g z) analytic_on S
unfolding analytic_on_def
proof (intro ballI)
  fix z
  assume z: z ∈ S
  then obtain e where e: 0 < e and fh: f holomorphic_on ball z e using f
    by (metis analytic_on_def)
  obtain e' where e': 0 < e' and gh: g holomorphic_on ball z e' using g
    by (metis analytic_on_def g z)
  have (λz. f z − g z) holomorphic_on ball z (min e e')
    apply (rule holomorphic_on_diff)
    apply (metis fh holomorphic_on_subset min.bounded_iff order_refl subset_ball)
    by (metis gh holomorphic_on_subset min.bounded_iff order_refl subset_ball)
  then show ∃ e>0. (λz. f z − g z) holomorphic_on ball z e
    by (metis e e' min_less_iff_conj)
qed

lemma analytic_on_mult [analytic_intros]:
  assumes f: f analytic_on S
      and g: g analytic_on S
    shows (λz. f z ∗ g z) analytic_on S
```

unfolding *analytic_on_def*
proof (*intro ballI*)
  fix *z*
  assume *z*: *z* ∈ *S*
  then obtain *e* **where** *e*: *0* < *e* **and** *fh*: *f holomorphic_on ball z e* **using** *f*
    by (*metis analytic_on_def*)
  obtain *e'* **where** *e'*: *0* < *e'* **and** *gh*: *g holomorphic_on ball z e'* **using** *g*
    by (*metis analytic_on_def g z*)
  have (λ*z*. *f z* ∗ *g z*) *holomorphic_on ball z* (*min e e'*)
    apply (*rule holomorphic_on_mult*)
    apply (*metis fh holomorphic_on_subset min.bounded_iff order_refl subset_ball*)
    by (*metis gh holomorphic_on_subset min.bounded_iff order_refl subset_ball*)
  then show ∃ *e*>*0*. (λ*z*. *f z* ∗ *g z*) *holomorphic_on ball z e*
    by (*metis e e' min_less_iff_conj*)
qed

lemma *analytic_on_inverse* [*analytic_intros*]:
  assumes *f*: *f analytic_on S*
    and *nz*: (⋀*z*. *z* ∈ *S* ⟹ *f z* ≠ *0*)
  shows (λ*z*. *inverse* (*f z*)) *analytic_on S*
unfolding *analytic_on_def*
proof (*intro ballI*)
  fix *z*
  assume *z*: *z* ∈ *S*
  then obtain *e* **where** *e*: *0* < *e* **and** *fh*: *f holomorphic_on ball z e* **using** *f*
    by (*metis analytic_on_def*)
  have *continuous_on* (*ball z e*) *f*
    by (*metis fh holomorphic_on_imp_continuous_on*)
  then obtain *e'* **where** *e'*: *0* < *e'* **and** *nz'*: ⋀*y*. *dist z y* < *e'* ⟹ *f y* ≠ *0*
    by (*metis open_ball centre_in_ball continuous_on_open_avoid e z nz*)
  have (λ*z*. *inverse* (*f z*)) *holomorphic_on ball z* (*min e e'*)
    apply (*rule holomorphic_on_inverse*)
      apply (*metis fh holomorphic_on_subset min.cobounded2 min.commute subset_ball*)
    by (*metis nz' mem_ball min_less_iff_conj*)
  then show ∃ *e*>*0*. (λ*z*. *inverse* (*f z*)) *holomorphic_on ball z e*
    by (*metis e e' min_less_iff_conj*)
qed

lemma *analytic_on_divide* [*analytic_intros*]:
  assumes *f*: *f analytic_on S*
    and *g*: *g analytic_on S*
    and *nz*: (⋀*z*. *z* ∈ *S* ⟹ *g z* ≠ *0*)
  shows (λ*z*. *f z* / *g z*) *analytic_on S*
unfolding *divide_inverse*
by (*metis analytic_on_inverse analytic_on_mult f g nz*)

lemma *analytic_on_power* [*analytic_intros*]:
  f analytic_on S ⟹ (λ*z*. (*f z*) ^ *n*) *analytic_on S*

by (*induct n*) (*auto simp*: *analytic_on_mult*)

lemma *analytic_on_sum* [*analytic_intros*]:
 ($\bigwedge$*i*. *i* ∈ *I* ⟹ (*f i*) *analytic_on S*) ⟹ (λ*x*. *sum* (λ*i*. *f i x*) *I*) *analytic_on S*
 by (*induct I rule*: *infinite_finite_induct*) (*auto simp*: *analytic_on_add*)

lemma *deriv_left_inverse*:
 assumes *f holomorphic_on S* **and** *g holomorphic_on T*
  and *open S* **and** *open T*
  and *f* ‘ *S* ⊆ *T*
  and [*simp*]: $\bigwedge$*z*. *z* ∈ *S* ⟹ *g* (*f z*) = *z*
  and *w* ∈ *S*
 shows *deriv f w* ∗ *deriv g* (*f w*) = *1*
proof −
 have *deriv f w* ∗ *deriv g* (*f w*) = *deriv g* (*f w*) ∗ *deriv f w*
  by (*simp add*: *algebra_simps*)
 also have ... = *deriv* (*g o f*) *w*
  using *assms*
  by (*metis analytic_on_imp_differentiable_at analytic_on_open deriv_chain image_subset_iff*)
 also have ... = *deriv id w*
 proof (*rule complex_derivative_transform_within_open* [**where** *s*=*S*])
  show *g* ∘ *f holomorphic_on S*
   by (*rule assms holomorphic_on_compose_gen holomorphic_intros*)+
 qed (*use assms* **in** *auto*)
 also have ... = *1*
  by *simp*
 finally show *?thesis* **.**
qed

6.20.4 Analyticity at a point

lemma *analytic_at_ball*:
 f analytic_on {*z*} ⟷ (∃ *e*. *0*<*e* ∧ *f holomorphic_on ball z e*)
by (*metis analytic_on_def singleton_iff*)

lemma *analytic_at*:
 f analytic_on {*z*} ⟷ (∃ *s*. *open s* ∧ *z* ∈ *s* ∧ *f holomorphic_on s*)
by (*metis analytic_on_holomorphic empty_subsetI insert_subset*)

lemma *analytic_on_analytic_at*:
 f analytic_on s ⟷ (∀ *z* ∈ *s*. *f analytic_on* {*z*})
by (*metis analytic_at_ball analytic_on_def*)

lemma *analytic_at_two*:
 f analytic_on {*z*} ∧ *g analytic_on* {*z*} ⟷
  (∃ *s*. *open s* ∧ *z* ∈ *s* ∧ *f holomorphic_on s* ∧ *g holomorphic_on s*)
 (**is** *?lhs* = *?rhs*)
proof

```
  assume ?lhs
  then obtain s t
    where st: open s z ∈ s f holomorphic_on s
              open t z ∈ t g holomorphic_on t
    by (auto simp: analytic_at)
  show ?rhs
    apply (rule_tac x=s ∩ t in exI)
    using st
    apply (auto simp: holomorphic_on_subset)
    done
next
  assume ?rhs
  then show ?lhs
    by (force simp add: analytic_at)
qed
```

6.20.5 Combining theorems for derivative with "analytic at" hypotheses

```
lemma
  assumes f analytic_on {z} g analytic_on {z}
  shows complex_derivative_add_at: deriv (λw. f w + g w) z = deriv f z + deriv g
z
    and complex_derivative_diff_at: deriv (λw. f w − g w) z = deriv f z − deriv g z
    and complex_derivative_mult_at: deriv (λw. f w ∗ g w) z =
           f z ∗ deriv g z + deriv f z ∗ g z
proof −
  obtain s where s: open s z ∈ s f holomorphic_on s g holomorphic_on s
    using assms by (metis analytic_at_two)
  show deriv (λw. f w + g w) z = deriv f z + deriv g z
    apply (rule DERIV_imp_deriv [OF DERIV_add])
    using s
   apply (auto simp: holomorphic_on_open field_differentiable_def DERIV_deriv_iff_field_differentiable)
    done
  show deriv (λw. f w − g w) z = deriv f z − deriv g z
    apply (rule DERIV_imp_deriv [OF DERIV_diff])
    using s
   apply (auto simp: holomorphic_on_open field_differentiable_def DERIV_deriv_iff_field_differentiable)
    done
  show deriv (λw. f w ∗ g w) z = f z ∗ deriv g z + deriv f z ∗ g z
    apply (rule DERIV_imp_deriv [OF DERIV_mult'])
    using s
   apply (auto simp: holomorphic_on_open field_differentiable_def DERIV_deriv_iff_field_differentiable)
    done
qed

lemma deriv_cmult_at:
  f analytic_on {z} =⇒ deriv (λw. c ∗ f w) z = c ∗ deriv f z
by (auto simp: complex_derivative_mult_at)
```

lemma *deriv_cmult_right_at*:
f analytic_on $\{z\} \Longrightarrow$ *deriv* $(\lambda w.\ f\ w * c)\ z = deriv\ f\ z * c$
by (*auto simp*: *complex_derivative_mult_at*)

6.20.6 Complex differentiation of sequences and series

lemma *has_complex_derivative_sequence*:
fixes S :: *complex set*
assumes *cvs*: *convex* S
and *df*: $\bigwedge n\ x.\ x \in S \Longrightarrow (f\ n$ *has_field_derivative* $f'\ n\ x)$ (*at* x *within* S)
and *conv*: $\bigwedge e.\ 0 < e \Longrightarrow \exists N.\ \forall n\ x.\ n \geq N \longrightarrow x \in S \longrightarrow norm\ (f'\ n\ x - g'\ x) \leq e$
and $\exists x\ l.\ x \in S \wedge ((\lambda n.\ f\ n\ x) \longrightarrow l)$ *sequentially*
shows $\exists g.\ \forall x \in S.\ ((\lambda n.\ f\ n\ x) \longrightarrow g\ x)$ *sequentially* $\wedge$
$(g$ *has_field_derivative* $(g'\ x))$ (*at* x *within* S)
proof −
from *assms* **obtain** $x\ l$ **where** x: $x \in S$ **and** *tf*: $((\lambda n.\ f\ n\ x) \longrightarrow l)$ *sequentially*
by *blast*
{ **fix** *e*::*real* **assume** *e*: $e > 0$
then obtain N **where** N: $\forall n \geq N.\ \forall x.\ x \in S \longrightarrow cmod\ (f'\ n\ x - g'\ x) \leq e$
by (*metis conv*)
have $\exists N.\ \forall n \geq N.\ \forall x \in S.\ \forall h.\ cmod\ (f'\ n\ x * h - g'\ x * h) \leq e * cmod\ h$
proof (*rule exI* [*of* _ *N*], *clarify*)
fix $n\ y\ h$
assume $N \leq n\ y \in S$
then have $cmod\ (f'\ n\ y - g'\ y) \leq e$
by (*metis N*)
then have $cmod\ h * cmod\ (f'\ n\ y - g'\ y) \leq cmod\ h * e$
by (*auto simp*: *antisym_conv2 mult_le_cancel_left norm_triangle_ineq2*)
then show $cmod\ (f'\ n\ y * h - g'\ y * h) \leq e * cmod\ h$
by (*simp add*: *norm_mult* [*symmetric*] *field_simps*)
qed
} **note** $** = this$
show *?thesis*
unfolding *has_field_derivative_def*
proof (*rule has_derivative_sequence* [*OF cvs* _ _ *x*])
show $(\lambda n.\ f\ n\ x) \longrightarrow l$
by (*rule tf*)
next show $\bigwedge e.\ e > 0 \Longrightarrow \forall_F\ n$ *in sequentially*. $\forall x \in S.\ \forall h.\ cmod\ (f'\ n\ x * h - g'\ x * h) \leq e * cmod\ h$
unfolding *eventually_sequentially* **by** (*blast intro*: $**$)
qed (*metis has_field_derivative_def df*)
qed

lemma *has_complex_derivative_series*:
fixes S :: *complex set*
assumes *cvs*: *convex* S
and *df*: $\bigwedge n\ x.\ x \in S \Longrightarrow (f\ n$ *has_field_derivative* $f'\ n\ x)$ (*at* x *within* S)

```
    and conv: ⋀e. 0 < e ⟹ ∃N. ∀n x. n ≥ N ⟶ x ∈ S
          ⟶ cmod ((∑i<n. f' i x) − g' x) ≤ e
    and ∃x l. x ∈ S ∧ ((λn. f n x) sums l)
  shows ∃g. ∀x ∈ S. ((λn. f n x) sums g x) ∧ ((g has_field_derivative g' x) (at
x within S))
proof −
 from assms obtain x l where x: x ∈ S and sf: ((λn. f n x) sums l)
   by blast
 { fix e::real assume e: e > 0
   then obtain N where N: ∀n x. n ≥ N ⟶ x ∈ S
         ⟶ cmod ((∑i<n. f' i x) − g' x) ≤ e
     by (metis conv)
   have ∃N. ∀n≥N. ∀x∈S. ∀h. cmod ((∑i<n. h ∗ f' i x) − g' x ∗ h) ≤ e ∗
cmod h
   proof (rule exI [of _ N], clarify)
     fix n y h
     assume N ≤ n y ∈ S
     then have cmod ((∑i<n. f' i y) − g' y) ≤ e
       by (metis N)
     then have cmod h ∗ cmod ((∑i<n. f' i y) − g' y) ≤ cmod h ∗ e
       by (auto simp: antisym_conv2 mult_le_cancel_left norm_triangle_ineq2)
     then show cmod ((∑i<n. h ∗ f' i y) − g' y ∗ h) ≤ e ∗ cmod h
       by (simp add: norm_mult [symmetric] field_simps sum_distrib_left)
   qed
 } note ∗∗ = this
 show ?thesis
 unfolding has_field_derivative_def
 proof (rule has_derivative_series [OF cvs _ _ x])
   fix n x
   assume x ∈ S
   then show ((f n) has_derivative (λz. z ∗ f' n x)) (at x within S)
     by (metis df has_field_derivative_def mult_commute_abs)
 next show ((λn. f n x) sums l)
   by (rule sf)
 next show ⋀e. e>0 ⟹ ∀F n in sequentially. ∀x∈S. ∀h. cmod ((∑i<n. h ∗
f' i x) − g' x ∗ h) ≤ e ∗ cmod h
     unfolding eventually_sequentially by (blast intro: ∗∗)
 qed
qed
```

6.20.7 Taylor on Complex Numbers

```
lemma sum_Suc_reindex:
 fixes f :: nat ⇒ 'a::ab_group_add
   shows sum f {0..n} = f 0 − f (Suc n) + sum (λi. f (Suc i)) {0..n}
by (induct n) auto

lemma field_Taylor:
 assumes S: convex S
```

and *f*: $\bigwedge i\ x.\ x \in S \Longrightarrow i \le n \Longrightarrow$ (*f i has_field_derivative f* (*Suc i*) *x*) (*at x within S*)
and *B*: $\bigwedge x.\ x \in S \Longrightarrow$ *norm* (*f* (*Suc n*) *x*) $\le B$
and *w*: $w \in S$
and *z*: $z \in S$
shows *norm*(*f 0 z* − ($\sum$ *i*≤*n*. *f i w* ∗ (*z*−*w*) ˆ *i* / (*fact i*)))
≤ *B* ∗ *norm*(*z* − *w*) ˆ(*Suc n*) / *fact n*
proof −
have *wzs*: *closed_segment w z* ⊆ *S* **using** *assms*
by (*metis convex_contains_segment*)
{ fix *u*
assume *u* ∈ *closed_segment w z*
then have *u* ∈ *S*
by (*metis wzs subsetD*)
have ($\sum$ *i*≤*n*. *f i u* ∗ (− *of_nat i* ∗ (*z*−*u*) ˆ(*i* − *1*)) / (*fact i*) +
f (*Suc i*) *u* ∗ (*z*−*u*) ˆ*i* / (*fact i*)) =
f (*Suc n*) *u* ∗ (*z*−*u*) ˆ *n* / (*fact n*)
proof (*induction n*)
case *0* **show** *?case* **by** *simp*
next
case (*Suc n*)
have ($\sum$ *i*≤*Suc n*. *f i u* ∗ (− *of_nat i* ∗ (*z*−*u*) ˆ (*i* − *1*)) / (*fact i*) +
f (*Suc i*) *u* ∗ (*z*−*u*) ˆ *i* / (*fact i*)) =
f (*Suc n*) *u* ∗ (*z*−*u*) ˆ *n* / (*fact n*) +
f (*Suc* (*Suc n*)) *u* ∗ ((*z*−*u*) ∗ (*z*−*u*) ˆ *n*) / (*fact* (*Suc n*)) −
f (*Suc n*) *u* ∗ ((*1* + *of_nat n*) ∗ (*z*−*u*) ˆ *n*) / (*fact* (*Suc n*))
using *Suc* **by** *simp*
also have ... = *f* (*Suc* (*Suc n*)) *u* ∗ (*z*−*u*) ˆ *Suc n* / (*fact* (*Suc n*))
proof −
have (*fact*(*Suc n*)) ∗
(*f*(*Suc n*) *u* ∗(*z*−*u*) ˆ *n* / (*fact n*) +
f(*Suc*(*Suc n*)) *u* ∗((*z*−*u*) ∗(*z*−*u*) ˆ *n*) / (*fact*(*Suc n*)) −
f(*Suc n*) *u* ∗((*1* + *of_nat n*) ∗(*z*−*u*) ˆ *n*) / (*fact*(*Suc n*))) =
((*fact*(*Suc n*)) ∗(*f*(*Suc n*) *u* ∗(*z*−*u*) ˆ *n*)) / (*fact n*) +
((*fact*(*Suc n*)) ∗(*f*(*Suc*(*Suc n*)) *u* ∗((*z*−*u*) ∗(*z*−*u*) ˆ *n*)) / (*fact*(*Suc n*)))
−
((*fact*(*Suc n*)) ∗(*f*(*Suc n*) *u* ∗(*of_nat*(*Suc n*) ∗(*z*−*u*) ˆ *n*))) / (*fact*(*Suc n*))
by (*simp add*: *algebra_simps del*: *fact_Suc*)
also have ... = ((*fact* (*Suc n*)) ∗ (*f* (*Suc n*) *u* ∗ (*z*−*u*) ˆ *n*)) / (*fact n*) +
(*f* (*Suc* (*Suc n*)) *u* ∗ ((*z*−*u*) ∗ (*z*−*u*) ˆ *n*)) −
(*f* (*Suc n*) *u* ∗ ((*1* + *of_nat n*) ∗ (*z*−*u*) ˆ *n*))
by (*simp del*: *fact_Suc*)
also have ... = (*of_nat* (*Suc n*) ∗ (*f* (*Suc n*) *u* ∗ (*z*−*u*) ˆ *n*)) +
(*f* (*Suc* (*Suc n*)) *u* ∗ ((*z*−*u*) ∗ (*z*−*u*) ˆ *n*)) −
(*f* (*Suc n*) *u* ∗ ((*1* + *of_nat n*) ∗ (*z*−*u*) ˆ *n*))
by (*simp only*: *fact_Suc of_nat_mult ac_simps*) *simp*
also have ... = *f* (*Suc* (*Suc n*)) *u* ∗ ((*z*−*u*) ∗ (*z*−*u*) ˆ *n*)
by (*simp add*: *algebra_simps*)

```
      finally show ?thesis
        by (simp add: mult_left_cancel [where c = (fact (Suc n)), THEN iffD1]
del: fact_Suc)
    qed
    finally show ?case .
  qed
  then have ((λv. (∑i≤n. f i v * (z - v)^i / (fact i)))
              has_field_derivative f (Suc n) u * (z-u) ^ n / (fact n))
             (at u within S)
    apply (intro derivative_eq_intros)
    apply (blast intro: assms ‹u ∈ S›)
    apply (rule refl)+
    apply (auto simp: field_simps)
    done
} note sum_deriv = this
{ fix u
  assume u: u ∈ closed_segment w z
  then have us: u ∈ S
    by (metis wzs subsetD)
  have norm (f (Suc n) u) * norm (z - u) ^ n ≤ norm (f (Suc n) u) * norm
(u - z) ^ n
    by (metis norm_minus_commute order_refl)
  also have ... ≤ norm (f (Suc n) u) * norm (z - w) ^ n
    by (metis mult_left_mono norm_ge_zero power_mono segment_bound [OF u])
  also have ... ≤ B * norm (z - w) ^ n
    by (metis norm_ge_zero zero_le_power mult_right_mono  B [OF us])
  finally have norm (f (Suc n) u) * norm (z - u) ^ n ≤ B * norm (z - w) ^
n .
} note cmod_bound = this
have (∑i≤n. f i z * (z - z) ^ i / (fact i)) = (∑i≤n. (f i z / (fact i)) * 0 ^ i)
  by simp
also have ... = f 0 z / (fact 0)
  by (subst sum_zero_power) simp
finally have norm (f 0 z - (∑i≤n. f i w * (z - w) ^ i / (fact i)))
               ≤ norm ((∑i≤n. f i w * (z - w) ^ i / (fact i)) -
                      (∑i≤n. f i z * (z - z) ^ i / (fact i)))
  by (simp add: norm_minus_commute)
also have ... ≤ B * norm (z - w) ^ n / (fact n) * norm (w - z)
  apply (rule field_differentiable_bound
    [where f' = λw. f (Suc n) w * (z - w)^n / (fact n)
      and S = closed_segment w z, OF convex_closed_segment])
  apply (auto simp: DERIV_subset [OF sum_deriv wzs]
               norm_divide norm_mult norm_power divide_le_cancel cmod_bound)
  done
also have ...  ≤ B * norm (z - w) ^ Suc n / (fact n)
  by (simp add: algebra_simps norm_minus_commute)
finally show ?thesis .
qed
```

lemma *complex_Taylor*:
assumes *S*: *convex S*
and *f*: $\bigwedge i\ x.\ x \in S \Longrightarrow i \le n \Longrightarrow$ (*f i has_field_derivative f* (*Suc i*) *x*) (*at x within S*)
and *B*: $\bigwedge x.\ x \in S \Longrightarrow$ *cmod* (*f* (*Suc n*) *x*) $\le B$
and *w*: $w \in S$
and *z*: $z \in S$
shows *cmod*(*f 0 z* − ($\sum i \le n$. *f i w* ∗ (*z*−*w*) ˆ *i* / (*fact i*)))
≤ *B* ∗ *cmod*(*z* − *w*) ˆ(*Suc n*) / *fact n*
using *assms* **by** (*rule field_Taylor*)

Something more like the traditional MVT for real components

lemma *complex_mvt_line*:
assumes $\bigwedge u.\ u \in$ *closed_segment w z* $\Longrightarrow$ (*f has_field_derivative f'*(*u*)) (*at u*)
shows ∃ *u*. *u* ∈ *closed_segment w z* ∧ *Re*(*f z*) − *Re*(*f w*) = *Re*(*f'*(*u*) ∗ (*z* − *w*))
proof −
have *twz*: $\bigwedge t.$ (*1* − *t*) $*_R$ *w* + *t* $*_R$ *z* = *w* + *t* $*_R$ (*z* − *w*)
by (*simp add*: *real_vector.scale_left_diff_distrib real_vector.scale_right_diff_distrib*)
note *assms*[*unfolded has_field_derivative_def*, *derivative_intros*]
show *?thesis*
apply (*cut_tac mvt_simple*
[*of 0 1 Re o f o* (λ*t*. (*1* − *t*) $*_R$ *w* + *t* $*_R$ *z*)
λ*u*. *Re o* (λ*h*. *f'*((*1* − *u*) $*_R$ *w* + *u* $*_R$ *z*) ∗ *h*) *o* (λ*t*. *t* $*_R$ (*z* − *w*))])
apply *auto*
apply (*rule_tac x*=(*1* − *x*) $*_R$ *w* + *x* $*_R$ *z* **in** *exI*)
apply (*auto simp*: *closed_segment_def twz*) []
apply (*intro derivative_eq_intros has_derivative_at_withinI*, *simp_all*)
apply (*simp add*: *fun_eq_iff real_vector.scale_right_diff_distrib*)
apply (*force simp*: *twz closed_segment_def*)
done
qed

lemma *complex_Taylor_mvt*:
assumes $\bigwedge i\ x.$ ⟦*x* ∈ *closed_segment w z*; *i* ≤ *n*⟧ ⟹ ((*f i*) *has_field_derivative f* (*Suc i*) *x*) (*at x*)
shows ∃ *u*. *u* ∈ *closed_segment w z* ∧
Re (*f 0 z*) =
Re (($\sum$ *i* = *0*..*n*. *f i w* ∗ (*z* − *w*) ˆ *i* / (*fact i*)) +
(*f* (*Suc n*) *u* ∗ (*z*−*u*) ˆ*n* / (*fact n*)) ∗ (*z* − *w*))
proof −
{ **fix** *u*
assume *u*: *u* ∈ *closed_segment w z*
have ($\sum$ *i* = *0*..*n*.
(*f* (*Suc i*) *u* ∗ (*z*−*u*) ˆ *i* − *of_nat i* ∗ (*f i u* ∗ (*z*−*u*) ˆ (*i* − *Suc 0*))) /
(*fact i*)) =
f (*Suc 0*) *u* −
(*f* (*Suc* (*Suc n*)) *u* ∗ ((*z*−*u*) ˆ *Suc n*) − (*of_nat* (*Suc n*)) ∗ (*z*−*u*) ˆ *n*

```
* f (Suc n) u) /
            (fact (Suc n)) +
            (∑ i = 0..n.
               (f (Suc (Suc i)) u * ((z−u) ^ Suc i) − of_nat (Suc i) * (f (Suc i) u
* (z−u) ^ i)) /
                (fact (Suc i)))
      by (subst sum_Suc_reindex) simp
    also have ... = f (Suc 0) u −
            (f (Suc (Suc n)) u * ((z−u) ^ Suc n) − (of_nat (Suc n)) * (z−u) ^ n
* f (Suc n) u) /
            (fact (Suc n)) +
            (∑ i = 0..n.
                f (Suc (Suc i)) u * ((z−u) ^ Suc i) / (fact (Suc i)) −
                f (Suc i) u * (z−u) ^ i / (fact i))
      by (simp only: diff_divide_distrib fact_cancel ac_simps)
    also have ... = f (Suc 0) u −
            (f (Suc (Suc n)) u * (z−u) ^ Suc n − of_nat (Suc n) * (z−u) ^ n * f
(Suc n) u) /
            (fact (Suc n)) +
            f (Suc (Suc n)) u * (z−u) ^ Suc n / (fact (Suc n)) − f (Suc 0) u
      by (subst sum_Suc_diff) auto
    also have ... = f (Suc n) u * (z−u) ^ n / (fact n)
      by (simp only: algebra_simps diff_divide_distrib fact_cancel)
    finally have (∑ i = 0..n. (f (Suc i) u * (z − u) ^ i
                        − of_nat i * (f i u * (z−u) ^ (i − Suc 0))) / (fact i)) =
                f (Suc n) u * (z − u) ^ n / (fact n) .
    then have ((λu. ∑ i = 0..n. f i u * (z − u) ^ i / (fact i)) has_field_derivative
               f (Suc n) u * (z − u) ^ n / (fact n)) (at u)
      apply (intro derivative_eq_intros)+
      apply (force intro: u assms)
      apply (rule refl)+
      apply (auto simp: ac_simps)
      done
  }
  then show ?thesis
    apply (cut_tac complex_mvt_line [of w z λu. ∑ i = 0..n. f i u * (z−u) ^ i /
(fact i)
              λu. (f (Suc n) u * (z−u)^n / (fact n))])
    apply (auto simp add: intro: open_closed_segment)
    done
qed


end
```

6.21 Complex Transcendental Functions

By John Harrison et al. Ported from HOL Light by L C Paulson (2015)

theory *Complex_Transcendental*
imports
  Complex_Analysis_Basics Summation_Tests HOL−Library.Periodic_Fun
begin

6.21.1 Mbius transformations

definition *moebius a b c d* ≡ (λ*z*. (*a*∗*z*+*b*) / (*c*∗*z*+*d* :: ′*a* :: *field*))

theorem *moebius_inverse*:
  assumes *a* ∗ *d* ≠ *b* ∗ *c c* ∗ *z* + *d* ≠ *0*
  shows *moebius d* (−*b*) (−*c*) *a* (*moebius a b c d z*) = *z*
proof −
  from *assms* **have** (−*c*) ∗ *moebius a b c d z* + *a* ≠ *0* **unfolding** *moebius_def*
    by (*simp add*: *field_simps*)
  with *assms* **show** *?thesis*
    unfolding *moebius_def* **by** (*simp add*: *moebius_def divide_simps*) (*simp add*: *algebra_simps*)*?*
qed

lemma *moebius_inverse*′:
  assumes *a* ∗ *d* ≠ *b* ∗ *c c* ∗ *z* − *a* ≠ *0*
  shows *moebius a b c d* (*moebius d* (−*b*) (−*c*) *a z*) = *z*
  using *assms moebius_inverse*[*of d a* −*b* −*c z*]
  by (*auto simp*: *algebra_simps*)

lemma *cmod_add_real_less*:
  assumes *Im z* ≠ *0 r*≠*0*
    shows *cmod* (*z* + *r*) < *cmod z* + |*r*|
proof (*cases z*)
  case (*Complex x y*)
  then have *0* < *y* ∗ *y*
    using *assms mult_neg_neg* **by** *force*
  with *assms* **have** *r* ∗ *x* / |*r*| < *sqrt* (*x*∗*x* + *y*∗*y*)
    by (*simp add*: *real_less_rsqrt power2_eq_square*)
  then show *?thesis* **using** *assms Complex*
    apply (*simp add*: *cmod_def*)
    apply (*rule power2_less_imp_less*, *auto*)
    apply (*simp add*: *power2_eq_square field_simps*)
    done
qed

lemma *cmod_diff_real_less*: *Im z* ≠ *0* =⇒ *x*≠*0* =⇒ *cmod* (*z* − *x*) < *cmod z* + |*x*|
  using *cmod_add_real_less* [*of z* −*x*]
  by *simp*

lemma *cmod_square_less_1_plus*:
  assumes *Im z* = *0* =⇒ |*Re z*| < *1*
    shows $(cmod\ z)^2$ < *1* + *cmod* $(1 - z^2)$

proof (*cases Im z = 0* ∨ *Re z = 0*)
  case *True*
  with *assms abs_square_less_1* **show** *?thesis*
    by (*force simp add*: *Re_power2 Im_power2 cmod_def*)
next
  case *False*
  with *cmod_diff_real_less* [*of 1* − z^2 *1*] **show** *?thesis*
    by (*simp add*: *norm_power Im_power2*)
qed

6.21.2 The Exponential Function

lemma *norm_exp_i_times* [*simp*]: *norm* (*exp*(i ∗ *of_real y*)) = *1*
  by *simp*

lemma *norm_exp_imaginary*: *norm*(*exp z*) = *1* ⟹ *Re z* = *0*
  by *simp*

lemma *field_differentiable_within_exp*: *exp field_differentiable* (*at z within s*)
  using *DERIV_exp field_differentiable_at_within field_differentiable_def* **by** *blast*

lemma *continuous_within_exp*:
  fixes *z*::*'a*::{*real_normed_field*,*banach*}
  shows *continuous* (*at z within s*) *exp*
by (*simp add*: *continuous_at_imp_continuous_within*)

lemma *holomorphic_on_exp* [*holomorphic_intros*]: *exp holomorphic_on s*
  by (*simp add*: *field_differentiable_within_exp holomorphic_on_def*)

lemma *holomorphic_on_exp'* [*holomorphic_intros*]:
  f holomorphic_on s ⟹ (λ*x*. *exp* (*f x*)) *holomorphic_on s*
  using *holomorphic_on_compose*[*OF* _ *holomorphic_on_exp*] **by** (*simp add*: *o_def*)

6.21.3 Euler and de Moivre formulas

The sine series times i

lemma *sin_i_eq*: (λ*n*. (i ∗ *sin_coeff n*) ∗ *z*ˆ*n*) *sums* (i ∗ *sin z*)
proof −
  have (λ*n*. i ∗ *sin_coeff n* $*_R$ *z*ˆ*n*) *sums* (i ∗ *sin z*)
    using *sin_converges sums_mult* **by** *blast*
  then show *?thesis*
    by (*simp add*: *scaleR_conv_of_real field_simps*)
qed

theorem *exp_Euler*: *exp*(i ∗ *z*) = *cos*(*z*) + i ∗ *sin*(*z*)
proof −
  have (λ*n*. (*cos_coeff n* + i ∗ *sin_coeff n*) ∗ *z*ˆ*n*) = (λ*n*. (i ∗ *z*) ˆ *n* $/_R$ (*fact n*))
  proof
    fix *n*

show (*cos_coeff n* + i ∗ *sin_coeff n*) ∗ *z*ˆ*n* = (i ∗ *z*) ˆ *n* /$_R$ (*fact n*)
by (*auto simp*: *cos_coeff_def sin_coeff_def scaleR_conv_of_real field_simps elim*!: *evenE oddE*)
qed
also have ... *sums* (*exp* (i ∗ *z*))
by (*rule exp_converges*)
finally have (λ*n*. (*cos_coeff n* + i ∗ *sin_coeff n*) ∗ *z*ˆ*n*) *sums* (*exp* (i ∗ *z*)) **.**
moreover have (λ*n*. (*cos_coeff n* + i ∗ *sin_coeff n*) ∗ *z*ˆ*n*) *sums* (*cos z* + i ∗ *sin z*)
using *sums_add* [*OF cos_converges* [*of z*] *sin_i_eq* [*of z*]]
by (*simp add*: *field_simps scaleR_conv_of_real*)
ultimately show *?thesis*
using *sums_unique2* **by** *blast*
qed

corollary *exp_minus_Euler*: *exp*(−(i ∗ *z*)) = *cos*(*z*) − i ∗ *sin*(*z*)
using *exp_Euler* [*of* −*z*]
by *simp*

lemma *sin_exp_eq*: *sin z* = (*exp*(i ∗ *z*) − *exp*(−(i ∗ *z*))) / (*2*∗i)
by (*simp add*: *exp_Euler exp_minus_Euler*)

lemma *sin_exp_eq'*: *sin z* = i ∗ (*exp*(−(i ∗ *z*)) − *exp*(i ∗ *z*)) / *2*
by (*simp add*: *exp_Euler exp_minus_Euler*)

lemma *cos_exp_eq*: *cos z* = (*exp*(i ∗ *z*) + *exp*(−(i ∗ *z*))) / *2*
by (*simp add*: *exp_Euler exp_minus_Euler*)

theorem *Euler*: *exp*(*z*) = *of_real*(*exp*(*Re z*)) ∗
(*of_real*(*cos*(*Im z*)) + i ∗ *of_real*(*sin*(*Im z*)))
by (*cases z*) (*simp add*: *exp_add exp_Euler cos_of_real exp_of_real sin_of_real Complex_eq*)

lemma *Re_sin*: *Re*(*sin z*) = *sin*(*Re z*) ∗ (*exp*(*Im z*) + *exp*(−(*Im z*))) / *2*
by (*simp add*: *sin_exp_eq field_simps Re_divide Im_exp*)

lemma *Im_sin*: *Im*(*sin z*) = *cos*(*Re z*) ∗ (*exp*(*Im z*) − *exp*(−(*Im z*))) / *2*
by (*simp add*: *sin_exp_eq field_simps Im_divide Re_exp*)

lemma *Re_cos*: *Re*(*cos z*) = *cos*(*Re z*) ∗ (*exp*(*Im z*) + *exp*(−(*Im z*))) / *2*
by (*simp add*: *cos_exp_eq field_simps Re_divide Re_exp*)

lemma *Im_cos*: *Im*(*cos z*) = *sin*(*Re z*) ∗ (*exp*(−(*Im z*)) − *exp*(*Im z*)) / *2*
by (*simp add*: *cos_exp_eq field_simps Im_divide Im_exp*)

lemma *Re_sin_pos*: *0* < *Re z* ⟹ *Re z* < *pi* ⟹ *Re* (*sin z*) > *0*
by (*auto simp*: *Re_sin Im_sin add_pos_pos sin_gt_zero*)

lemma *Im_sin_nonneg*: *Re z* = *0* ⟹ *0* ≤ *Im z* ⟹ *0* ≤ *Im* (*sin z*)

by (*simp add*: *Re_sin Im_sin algebra_simps*)

lemma *Im_sin_nonneg2*: *Re z* = *pi* ⟹ *Im z* ≤ *0* ⟹ *0* ≤ *Im* (*sin z*)
by (*simp add*: *Re_sin Im_sin algebra_simps*)

6.21.4 Relationships between real and complex trigonometric and hyperbolic functions

lemma *real_sin_eq* [*simp*]: *Re*(*sin*(*of_real x*)) = *sin x*
by (*simp add*: *sin_of_real*)

lemma *real_cos_eq* [*simp*]: *Re*(*cos*(*of_real x*)) = *cos x*
by (*simp add*: *cos_of_real*)

lemma *DeMoivre*: (*cos z* + i ∗ *sin z*) ^ *n* = *cos*(*n* ∗ *z*) + i ∗ *sin*(*n* ∗ *z*)
by (*metis exp_Euler* [*symmetric*] *exp_of_nat_mult mult.left_commute*)

lemma *exp_cnj*: *cnj* (*exp z*) = *exp* (*cnj z*)
proof −
have (λ*n*. *cnj* (*z* ^ *n* $/_R$ (*fact n*))) = (λ*n*. (*cnj z*)^*n* $/_R$ (*fact n*))
by *auto*
also have ... *sums* (*exp* (*cnj z*))
by (*rule exp_converges*)
finally have (λ*n*. *cnj* (*z* ^ *n* $/_R$ (*fact n*))) *sums* (*exp* (*cnj z*)) **.**
moreover have (λ*n*. *cnj* (*z* ^ *n* $/_R$ (*fact n*))) *sums* (*cnj* (*exp z*))
by (*metis exp_converges sums_cnj*)
ultimately show *?thesis*
using *sums_unique2*
by *blast*
qed

lemma *cnj_sin*: *cnj*(*sin z*) = *sin*(*cnj z*)
by (*simp add*: *sin_exp_eq exp_cnj field_simps*)

lemma *cnj_cos*: *cnj*(*cos z*) = *cos*(*cnj z*)
by (*simp add*: *cos_exp_eq exp_cnj field_simps*)

lemma *field_differentiable_at_sin*: *sin field_differentiable at z*
using *DERIV_sin field_differentiable_def* **by** *blast*

lemma *field_differentiable_within_sin*: *sin field_differentiable* (*at z within S*)
by (*simp add*: *field_differentiable_at_sin field_differentiable_at_within*)

lemma *field_differentiable_at_cos*: *cos field_differentiable at z*
using *DERIV_cos field_differentiable_def* **by** *blast*

lemma *field_differentiable_within_cos*: *cos field_differentiable* (*at z within S*)
by (*simp add*: *field_differentiable_at_cos field_differentiable_at_within*)

lemma *holomorphic_on_sin*: *sin holomorphic_on S*
by (*simp add*: *field_differentiable_within_sin holomorphic_on_def*)

lemma *holomorphic_on_cos*: *cos holomorphic_on S*
by (*simp add*: *field_differentiable_within_cos holomorphic_on_def*)

lemma *holomorphic_on_sin'* [*holomorphic_intros*]:
assumes *f holomorphic_on A*
shows ($\lambda x.$ *sin* (*f x*)) *holomorphic_on A*
using *holomorphic_on_compose*[*OF assms holomorphic_on_sin*] **by** (*simp add*: *o_def*)

lemma *holomorphic_on_cos'* [*holomorphic_intros*]:
assumes *f holomorphic_on A*
shows ($\lambda x.$ *cos* (*f x*)) *holomorphic_on A*
using *holomorphic_on_compose*[*OF assms holomorphic_on_cos*] **by** (*simp add*: *o_def*)

6.21.5 More on the Polar Representation of Complex Numbers

lemma *exp_Complex*: *exp*(*Complex r t*) = *of_real*(*exp r*) ∗ *Complex* (*cos t*) (*sin t*)
by (*simp add*: *Complex_eq exp_add exp_Euler exp_of_real sin_of_real cos_of_real*)

lemma *exp_eq_1*: *exp z* = *1* ⟷ *Re*(*z*) = *0* ∧ (∃ *n*::*int*. *Im*(*z*) = *of_int* (*2* ∗ *n*) ∗ *pi*)
(**is** *?lhs* = *?rhs*)
proof
assume *exp z* = *1*
then have *Re z* = *0*
by (*metis exp_eq_one_iff norm_exp_eq_Re norm_one*)
with ‹*?lhs*› **show** *?rhs*
by (*metis Re_exp complex_Re_of_int cos_one_2pi_int exp_zero mult.commute mult_numeral_1 numeral_One of_int_mult of_int_numeral*)
next
assume *?rhs* **then show** *?lhs*
using *Im_exp Re_exp complex_eq_iff*
by (*simp add*: *cos_one_2pi_int cos_one_sin_zero mult.commute*)
qed

lemma *exp_eq*: *exp w* = *exp z* ⟷ (∃ *n*::*int*. *w* = *z* + (*of_int* (*2* ∗ *n*) ∗ *pi*) ∗ i)
(**is** *?lhs* = *?rhs*)
proof −
have *exp w* = *exp z* ⟷ *exp* (*w*−*z*) = *1*
by (*simp add*: *exp_diff*)
also have ... ⟷ (*Re w* = *Re z* ∧ (∃ *n*::*int*. *Im w* − *Im z* = *of_int* (*2* ∗ *n*) ∗ *pi*))
by (*simp add*: *exp_eq_1*)
also have ... ⟷ *?rhs*

```
    by (auto simp: algebra_simps intro!: complex_eqI)
  finally show ?thesis .
qed

lemma exp_complex_eqI: |Im w − Im z| < 2*pi ⟹ exp w = exp z ⟹ w = z
  by (auto simp: exp_eq abs_mult)

lemma exp_integer_2pi:
  assumes n ∈ ℤ
  shows exp((2 * n * pi) * i) = 1
proof −
  have exp((2 * n * pi) * i) = exp 0
    using assms unfolding Ints_def exp_eq by auto
  also have ... = 1
    by simp
  finally show ?thesis .
qed

lemma exp_plus_2pin [simp]: exp (z + i * (of_int n * (of_real pi * 2))) = exp z
  by (simp add: exp_eq)

lemma exp_integer_2pi_plus1:
  assumes n ∈ ℤ
  shows exp(((2 * n + 1) * pi) * i) = − 1
proof −
  from assms obtain n' where [simp]: n = of_int n'
    by (auto simp: Ints_def)
  have exp(((2 * n + 1) * pi) * i) = exp (pi * i)
    using assms by (subst exp_eq) (auto intro!: exI[of _ n'] simp: algebra_simps)
  also have ... = − 1
    by simp
  finally show ?thesis .
qed

lemma inj_on_exp_pi:
  fixes z::complex shows inj_on exp (ball z pi)
proof (clarsimp simp: inj_on_def exp_eq)
  fix y n
  assume dist z (y + 2 * of_int n * of_real pi * i) < pi
         dist z y < pi
  then have dist y (y + 2 * of_int n * of_real pi * i) < pi+pi
    using dist_commute_lessI dist_triangle_less_add by blast
  then have norm (2 * of_int n * of_real pi * i) < 2*pi
    by (simp add: dist_norm)
  then show n = 0
    by (auto simp: norm_mult)
qed

lemma cmod_add_squared:
```

fixes *r1 r2*::*real*
assumes *r1* ≥ *0 r2* ≥ *0*
shows (*cmod* (*r1* ∗ *exp* (i ∗ *ϑ1*) + *r2* ∗ *exp* (i ∗ *ϑ2*)))^2 = *r1*^{2} + *r2*^{2} + *2* ∗ *r1* ∗ *r2* ∗ *cos* (*ϑ1* − *ϑ2*) (**is** (*cmod* (*?z1* + *?z2*))^2 = *?rhs*)
proof −
have (*cmod* (*?z1* + *?z2*))^2 = (*?z1* + *?z2*) ∗ *cnj* (*?z1* + *?z2*)
by (*rule complex_norm_square*)
also have ... = (*?z1* ∗ *cnj ?z1* + *?z2* ∗ *cnj ?z2*) + (*?z1* ∗ *cnj ?z2* + *cnj ?z1* ∗ *?z2*)
by (*simp add*: *algebra_simps*)
also have ... = (*norm ?z1*)^2 + (*norm ?z2*)^2 + *2* ∗ *Re* (*?z1* ∗ *cnj ?z2*)
unfolding *complex_norm_square* [*symmetric*] *cnj_add_mult_eq_Re* **by** *simp*
also have ... = *?rhs*
by (*simp add*: *norm_mult*) (*simp add*: *exp_Euler complex_is_Real_iff* [*THEN iffD1*] *cos_diff algebra_simps*)
finally show *?thesis*
using *of_real_eq_iff* **by** *blast*
qed

lemma *cmod_diff_squared*:
fixes *r1 r2*::*real*
assumes *r1* ≥ *0 r2* ≥ *0*
shows (*cmod* (*r1* ∗ *exp* (i ∗ *ϑ1*) − *r2* ∗ *exp* (i ∗ *ϑ2*)))^2 = *r1*^{2} + *r2*^{2} − *2*∗*r1*∗*r2*∗*cos* (*ϑ1* − *ϑ2*) (**is** (*cmod* (*?z1* − *?z2*))^2 = *?rhs*)
proof −
have *exp* (i ∗ (*ϑ2* + *pi*)) = − *exp* (i ∗ *ϑ2*)
by (*simp add*: *exp_Euler cos_plus_pi sin_plus_pi*)
then have (*cmod* (*?z1* − *?z2*))^2 = *cmod* (*?z1* + *r2* ∗ *exp* (i ∗ (*ϑ2* + *pi*))) ˆ*2*
by *simp*
also have ... = *r1*^{2} + *r2*^{2} + *2*∗*r1*∗*r2*∗*cos* (*ϑ1* − (*ϑ2* + *pi*))
using *assms cmod_add_squared* **by** *blast*
also have ... = *?rhs*
by (*simp add*: *add.commute diff_add_eq_diff_diff_swap*)
finally show *?thesis* **.**
qed

lemma *polar_convergence*:
fixes *R*::*real*
assumes ⋀*j*. *r j* > *0 R* > *0*
shows ((λ*j*. *r j* ∗ *exp* (i ∗ *ϑ j*)) ⟶ (*R* ∗ *exp* (i ∗ Θ))) ⟷
(*r* ⟶ *R*) ∧ (∃ *k*. (λ*j*. *ϑ j* − *of_int* (*k j*) ∗ (*2* ∗ *pi*)) ⟶ Θ) (**is** (*?z* ⟶ *?Z*) = *?rhs*)
proof
assume *L*: *?z* ⟶ *?Z*
have *rR*: *r* ⟶ *R*
using *tendsto_norm* [*OF L*] *assms* **by** (*auto simp*: *norm_mult abs_of_pos*)
moreover obtain *k* **where** (λ*j*. *ϑ j* − *of_int* (*k j*) ∗ (*2* ∗ *pi*)) ⟶ Θ
proof −
have *cos* (*ϑ j* − Θ) = ((*r j*)^2 + *R*^{2} − (*norm*(*?z j* − *?Z*))^2) / (*2* ∗ *R* ∗ *r j*)

for *j*
  apply (*subst cmod_diff_squared*)
  using *assms* **by** (*auto simp*: *field_split_simps less_le*)
 moreover have (λ*j*. ((*r j*)^2 + R^2 − (*norm*(*?z j* − *?Z*))^2) / (*2* ∗ *R* ∗ *r j*))
⟶ ((R^2 + R^2 − (*norm*(*?Z* − *?Z*))^2) / (*2* ∗ *R* ∗ *R*))
  by (*intro L rR tendsto_intros*) (*use* ‹*R* > *0*› **in** *force*)
 moreover have ((R^2 + R^2 − (*norm*(*?Z* − *?Z*))^2) / (*2* ∗ *R* ∗ *R*)) = *1*
  using ‹*R* > *0*› **by** (*simp add*: *power2_eq_square field_split_simps*)
 ultimately have (λ*j*. *cos* (ϑ *j* − Θ)) ⟶ *1*
  by *auto*
 then show *?thesis*
  using *that cos_diff_limit_1* **by** *blast*
qed
ultimately show *?rhs*
 by *metis*
next
 assume *R*: *?rhs*
 show *?z* ⟶ *?Z*
 proof (*rule tendsto_mult*)
  show (λ*x*. *complex_of_real* (*r x*)) ⟶ *of_real R*
   using *R* **by** (*auto simp*: *tendsto_of_real_iff*)
  obtain *k* **where** (λ*j*. ϑ *j* − *of_int* (*k j*) ∗ (*2* ∗ *pi*)) ⟶ Θ
   using *R* **by** *metis*
  then have (λ*j*. *complex_of_real* (ϑ *j* − *of_int* (*k j*) ∗ (*2* ∗ *pi*))) ⟶ *of_real*
Θ
   using *tendsto_of_real_iff* **by** *force*
  then have (λ*j*. *exp* (i ∗ *of_real* (ϑ *j* − *of_int* (*k j*) ∗ (*2* ∗ *pi*)))) ⟶ *exp* (i
∗ Θ)
   using *tendsto_mult* [*OF tendsto_const*] *isCont_exp isCont_tendsto_compose* **by**
blast
  moreover have *exp* (i ∗ *of_real* (ϑ *j* − *of_int* (*k j*) ∗ (*2* ∗ *pi*))) = *exp* (i ∗ ϑ
j) **for** *j*
   unfolding *exp_eq*
   by (*rule_tac x*=− *k j* **in** *exI*) (*auto simp*: *algebra_simps*)
  ultimately show (λ*j*. *exp* (i ∗ ϑ *j*)) ⟶ *exp* (i ∗ Θ)
   by *auto*
 qed
qed

lemma *sin_cos_eq_iff*: *sin y* = *sin x* ∧ *cos y* = *cos x* ⟷ (∃ *n*::*int*. *y* = *x* + *2* ∗
pi ∗ *n*)
proof −
 { **assume** *sin y* = *sin x cos y* = *cos x*
  then have *cos* (*y*−*x*) = *1*
   using *cos_add* [*of y* −*x*] **by** *simp*
  then have ∃ *n*::*int*. *y*−*x* = *2* ∗ *pi* ∗ *n*
   using *cos_one_2pi_int* **by** *auto* }
 then show *?thesis*
 apply (*auto simp*: *sin_add cos_add*)

apply (*metis add.commute diff_add_cancel*)
done
qed

lemma *exp_i_ne_1*:
assumes *0 < x x < 2∗pi*
shows *exp*(i ∗ *of_real x*) ≠ *1*
proof
assume *exp* (i ∗ *of_real x*) = *1*
then have *exp* (i ∗ *of_real x*) = *exp 0*
by *simp*
then obtain *n* **where** i ∗ *of_real x* = (*of_int* (*2* ∗ *n*) ∗ *pi*) ∗ i
by (*simp only*: *Ints_def exp_eq*) *auto*
then have *of_real x* = (*of_int* (*2* ∗ *n*) ∗ *pi*)
by (*metis complex_i_not_zero mult.commute mult_cancel_left of_real_eq_iff real_scaleR_def scaleR_conv_of_real*)
then have *x* = (*of_int* (*2* ∗ *n*) ∗ *pi*)
by *simp*
then show *False* **using** *assms*
by (*cases n*) (*auto simp*: *zero_less_mult_iff mult_less_0_iff*)
qed

lemma *sin_eq_0*:
fixes *z*::*complex*
shows *sin z* = *0* ⟷ (∃ *n*::*int*. *z* = *of_real*(*n* ∗ *pi*))
by (*simp add*: *sin_exp_eq exp_eq*)

lemma *cos_eq_0*:
fixes *z*::*complex*
shows *cos z* = *0* ⟷ (∃ *n*::*int*. *z* = *of_real*(*n* ∗ *pi*) + *of_real pi*/*2*)
using *sin_eq_0* [*of z* − *of_real pi*/*2*]
by (*simp add*: *sin_diff algebra_simps*)

lemma *cos_eq_1*:
fixes *z*::*complex*
shows *cos z* = *1* ⟷ (∃ *n*::*int*. *z* = *of_real*(*2* ∗ *n* ∗ *pi*))
proof −
have *cos z* = *cos* (*2*∗(*z*/*2*))
by *simp*
also have ... = *1* − *2* ∗ *sin* (*z*/*2*) ˆ *2*
by (*simp only*: *cos_double_sin*)
finally have [*simp*]: *cos z* = *1* ⟷ *sin* (*z*/*2*) = *0*
by *simp*
show *?thesis*
by (*auto simp*: *sin_eq_0*)
qed

lemma *csin_eq_1*:
fixes *z*::*complex*

shows *sin z = 1* ⟷ (∃ *n::int. z = of_real(2 ∗ n ∗ pi) + of_real pi/2*)
using *cos_eq_1* [*of z − of_real pi/2*]
by (*simp add*: *cos_diff algebra_simps*)

lemma *csin_eq_minus1*:
fixes *z::complex*
shows *sin z = −1* ⟷ (∃ *n::int. z = of_real(2 ∗ n ∗ pi) + 3/2∗pi*)
(**is** _ = *?rhs*)
proof −
have *sin z = −1* ⟷ *sin* (−*z*) = *1*
by (*simp add*: *equation_minus_iff*)
also have ... ⟷ (∃ *n::int.* −*z = of_real(2 ∗ n ∗ pi) + of_real pi/2*)
by (*simp only*: *csin_eq_1*)
also have ... ⟷ (∃ *n::int. z = − of_real(2 ∗ n ∗ pi) − of_real pi/2*)
by (*rule iff_exI*) (*metis add.inverse_inverse add_uminus_conv_diff minus_add_distrib*)
also have ... = *?rhs*
apply *safe*
apply (*rule_tac* [*2*] *x*=−(*x*+*1*) **in** *exI*)
apply (*rule_tac x*=−(*x*+*1*) **in** *exI*)
apply (*simp_all add*: *algebra_simps*)
done
finally show *?thesis* **.**
qed

lemma *ccos_eq_minus1*:
fixes *z::complex*
shows *cos z = −1* ⟷ (∃ *n::int. z = of_real(2 ∗ n ∗ pi) + pi*)
using *csin_eq_1* [*of z − of_real pi/2*]
by (*simp add*: *sin_diff algebra_simps equation_minus_iff*)

lemma *sin_eq_1*: *sin x = 1* ⟷ (∃ *n::int. x* = (*2 ∗ n + 1 / 2*) ∗ *pi*)
(**is** _ = *?rhs*)
proof −
have *sin x = 1* ⟷ *sin* (*complex_of_real x*) = *1*
by (*metis of_real_1 one_complex.simps*(*1*) *real_sin_eq sin_of_real*)
also have ... ⟷ (∃ *n::int. complex_of_real x = of_real(2 ∗ n ∗ pi) + of_real pi/2*)
by (*simp only*: *csin_eq_1*)
also have ... ⟷ (∃ *n::int. x = of_real(2 ∗ n ∗ pi) + of_real pi/2*)
by (*rule iff_exI*) (*auto simp*: *algebra_simps intro*: *injD* [*OF inj_of_real* [**where** *'a = complex*]])
also have ... = *?rhs*
by (*auto simp*: *algebra_simps*)
finally show *?thesis* **.**
qed

lemma *sin_eq_minus1*: *sin x = −1* ⟷ (∃ *n::int. x* = (*2∗n + 3/2*) ∗ *pi*) (**is** _ = *?rhs*)
proof −

have *sin x* = −*1* ⟷ *sin* (*complex_of_real x*) = −*1*
by (*metis Re_complex_of_real of_real_def scaleR_minus1_left sin_of_real*)
also have ... ⟷ (∃ *n*::*int*. *complex_of_real x* = *of_real*(*2* ∗ *n* ∗ *pi*) + *3*/*2*∗*pi*)
by (*simp only*: *csin_eq_minus1*)
also have ... ⟷ (∃ *n*::*int*. *x* = *of_real*(*2* ∗ *n* ∗ *pi*) + *3*/*2*∗*pi*)
by (*rule iff_exI*) (*auto simp*: *algebra_simps intro*: *injD* [*OF inj_of_real* [**where** ′*a* = *complex*]])
also have ... = *?rhs*
by (*auto simp*: *algebra_simps*)
finally show *?thesis* **.**
qed

lemma *cos_eq_minus1*: *cos x* = −*1* ⟷ (∃ *n*::*int*. *x* = (*2*∗*n* + *1*) ∗ *pi*)
(**is** _ = *?rhs*)
proof −
have *cos x* = −*1* ⟷ *cos* (*complex_of_real x*) = −*1*
by (*metis Re_complex_of_real of_real_def scaleR_minus1_left cos_of_real*)
also have ... ⟷ (∃ *n*::*int*. *complex_of_real x* = *of_real*(*2* ∗ *n* ∗ *pi*) + *pi*)
by (*simp only*: *ccos_eq_minus1*)
also have ... ⟷ (∃ *n*::*int*. *x* = *of_real*(*2* ∗ *n* ∗ *pi*) + *pi*)
by (*rule iff_exI*) (*auto simp*: *algebra_simps intro*: *injD* [*OF inj_of_real* [**where** ′*a* = *complex*]])
also have ... = *?rhs*
by (*auto simp*: *algebra_simps*)
finally show *?thesis* **.**
qed

lemma *dist_exp_i_1*: *norm*(*exp*(i ∗ *of_real t*) − *1*) = *2* ∗ |*sin*(*t* / *2*)|
proof −
have *sqrt* (*2* − *cos t* ∗ *2*) = *2* ∗ |*sin* (*t* / *2*)|
using *cos_double_sin* [*of t*/*2*] **by** (*simp add*: *real_sqrt_mult*)
then show *?thesis*
by (*simp add*: *exp_Euler cmod_def power2_diff sin_of_real cos_of_real algebra_simps*)
qed

lemma *sin_cx_2pi* [*simp*]: ⟦*z* = *of_int m*; *even m*⟧ ⟹ *sin* (*z* ∗ *complex_of_real pi*) = *0*
by (*simp add*: *sin_eq_0*)

lemma *cos_cx_2pi* [*simp*]: ⟦*z* = *of_int m*; *even m*⟧ ⟹ *cos* (*z* ∗ *complex_of_real pi*) = *1*
using *cos_eq_1* **by** *auto*

lemma *complex_sin_eq*:
fixes *w* :: *complex*
shows *sin w* = *sin z* ⟷ (∃ *n* ∈ ℤ. *w* = *z* + *of_real*(*2*∗*n*∗*pi*) ∨ *w* = −*z* + *of_real*((*2*∗*n* + *1*)∗*pi*))
(**is** *?lhs* = *?rhs*)
proof

```
  assume ?lhs
  then have sin w − sin z = 0
    by (auto simp: algebra_simps)
  then have sin ((w − z) / 2)∗cos ((w + z) / 2) = 0
    by (auto simp: sin_diff_sin)
  then consider sin ((w − z) / 2) = 0 | cos ((w + z) / 2) = 0
    using mult_eq_0_iff by blast
  then show ?rhs
  proof cases
    case 1
    then show ?thesis
      by (simp add: sin_eq_0 algebra_simps) (metis Ints_of_int of_real_of_int_eq)
  next
    case 2
    then show ?thesis
      by (simp add: cos_eq_0 algebra_simps) (metis Ints_of_int of_real_of_int_eq)
  qed
next
  assume ?rhs
  then consider n::int where w = z + of_real (2 ∗ of_int n ∗ pi)
             | n::int where   w = −z + of_real ((2 ∗ of_int n + 1) ∗ pi)
    using Ints_cases by blast
  then show ?lhs
  proof cases
    case 1
    then show ?thesis
      using Periodic_Fun.sin.plus_of_int [of z n]
      by (auto simp: algebra_simps)
  next
    case 2
    then show ?thesis
      using Periodic_Fun.sin.plus_of_int [of −z n]
      apply (simp add: algebra_simps)
     by (metis add.commute add.inverse_inverse add_diff_cancel_left diff_add_cancel
sin_plus_pi)
  qed
qed

lemma complex_cos_eq:
  fixes w :: complex
  shows cos w = cos z ⟷ (∃ n ∈ ℤ. w = z + of_real(2∗n∗pi) ∨ w = −z +
of_real(2∗n∗pi))
        (is ?lhs = ?rhs)
proof
  assume ?lhs
  then have cos w − cos z = 0
    by (auto simp: algebra_simps)
  then have sin ((w + z) / 2) ∗ sin ((z − w) / 2) = 0
    by (auto simp: cos_diff_cos)
```

```
  then consider sin ((w + z) / 2) = 0 | sin ((z − w) / 2) = 0
    using mult_eq_0_iff by blast
  then show ?rhs
  proof cases
    case 1
    then obtain n where w + z = of_int n ∗ (complex_of_real pi ∗ 2)
      by (auto simp: sin_eq_0 algebra_simps)
    then have w = −z + of_real(2 ∗ of_int n ∗ pi)
      by (auto simp: algebra_simps)
    then show ?thesis
      using Ints_of_int by blast
  next
    case 2
    then obtain n where z = w + of_int n ∗ (complex_of_real pi ∗ 2)
      by (auto simp: sin_eq_0 algebra_simps)
    then have w = z + complex_of_real (2 ∗ of_int(−n) ∗ pi)
      by (auto simp: algebra_simps)
    then show ?thesis
      using Ints_of_int by blast
  qed
next
  assume ?rhs
  then obtain n::int where w: w = z + of_real (2∗ of_int n∗pi) ∨
                           w = −z + of_real(2∗n∗pi)
    using Ints_cases by (metis of_int_mult of_int_numeral)
  then show ?lhs
    using Periodic_Fun.cos.plus_of_int [of z n]
    apply (simp add: algebra_simps)
    by (metis cos.plus_of_int cos_minus minus_add_cancel mult.commute)
qed

lemma sin_eq:
   sin x = sin y ⟷ (∃ n ∈ ℤ. x = y + 2∗n∗pi ∨ x = −y + (2∗n + 1)∗pi)
  using complex_sin_eq [of x y]
 by (simp only: sin_of_real Re_complex_of_real of_real_add [symmetric] of_real_minus
[symmetric] of_real_mult [symmetric] of_real_eq_iff)

lemma cos_eq:
   cos x = cos y ⟷ (∃ n ∈ ℤ. x = y + 2∗n∗pi ∨ x = −y + 2∗n∗pi)
  using complex_cos_eq [of x y]
 by (simp only: cos_of_real Re_complex_of_real of_real_add [symmetric] of_real_minus
[symmetric] of_real_mult [symmetric] of_real_eq_iff)

lemma sinh_complex:
  fixes z :: complex
  shows (exp z − inverse (exp z)) / 2 = −i ∗ sin(i ∗ z)
  by (simp add: sin_exp_eq field_split_simps exp_minus)

lemma sin_i_times:
```

fixes *z* :: *complex*
shows *sin*(i ∗ *z*) = i ∗ ((*exp z* − *inverse* (*exp z*)) / *2*)
using *sinh_complex* **by** *auto*

lemma *sinh_real*:
fixes *x* :: *real*
shows *of_real*((*exp x* − *inverse* (*exp x*)) / *2*) = −i ∗ *sin*(i ∗ *of_real x*)
by (*simp add*: *exp_of_real sin_i_times*)

lemma *cosh_complex*:
fixes *z* :: *complex*
shows (*exp z* + *inverse* (*exp z*)) / *2* = *cos*(i ∗ *z*)
by (*simp add*: *cos_exp_eq field_split_simps exp_minus exp_of_real*)

lemma *cosh_real*:
fixes *x* :: *real*
shows *of_real*((*exp x* + *inverse* (*exp x*)) / *2*) = *cos*(i ∗ *of_real x*)
by (*simp add*: *cos_exp_eq field_split_simps exp_minus exp_of_real*)

lemmas *cos_i_times* = *cosh_complex* [*symmetric*]

lemma *norm_cos_squared*:
norm(*cos z*) ˆ *2* = *cos*(*Re z*) ˆ *2* + (*exp*(*Im z*) − *inverse*(*exp*(*Im z*))) ˆ *2* / *4*
proof (*cases z*)
case (*Complex x1 x2*)
then show *?thesis*
apply (*simp only*: *cos_add cmod_power2 cos_of_real sin_of_real Complex_eq*)
apply (*simp add*: *cos_exp_eq sin_exp_eq exp_minus exp_of_real Re_divide Im_divide power_divide*)
apply (*simp only*: *left_diff_distrib* [*symmetric*] *power_mult_distrib sin_squared_eq*)
apply (*simp add*: *power2_eq_square field_split_simps*)
done
qed

lemma *norm_sin_squared*:
norm(*sin z*) ˆ *2* = (*exp*(*2* ∗ *Im z*) + *inverse*(*exp*(*2* ∗ *Im z*)) − *2* ∗ *cos*(*2* ∗ *Re z*)) / *4*
proof (*cases z*)
case (*Complex x1 x2*)
then show *?thesis*
apply (*simp only*: *sin_add cmod_power2 cos_of_real sin_of_real cos_double_cos exp_double Complex_eq*)
apply (*simp add*: *cos_exp_eq sin_exp_eq exp_minus exp_of_real Re_divide Im_divide power_divide*)
apply (*simp only*: *left_diff_distrib* [*symmetric*] *power_mult_distrib cos_squared_eq*)
apply (*simp add*: *power2_eq_square field_split_simps*)
done
qed

lemma *exp_uminus_Im*: *exp* (− *Im z*) ≤ *exp* (*cmod z*)
using *abs_Im_le_cmod linear order_trans* **by** *fastforce*

lemma *norm_cos_le*:
fixes *z*::*complex*
shows *norm*(*cos z*) ≤ *exp*(*norm z*)
proof −
have *Im z* ≤ *cmod z*
using *abs_Im_le_cmod abs_le_D1* **by** *auto*
then have *exp* (− *Im z*) + *exp* (*Im z*) ≤ *exp* (*cmod z*) ∗ *2*
by (*metis exp_uminus_Im add_mono exp_le_cancel_iff mult_2_right*)
then show *?thesis*
by (*force simp add*: *cos_exp_eq norm_divide intro*: *order_trans* [*OF norm_triangle_ineq*])
qed

lemma *norm_cos_plus1_le*:
fixes *z*::*complex*
shows *norm*(*1* + *cos z*) ≤ *2* ∗ *exp*(*norm z*)
proof −
have *mono*: ⋀*u w z*::*real*. (*1* ≤ *w* | *1* ≤ *z*) ⟹ (*w* ≤ *u* & *z* ≤ *u*) ⟹ *2* + *w* + *z* ≤ *4* ∗ *u*
by *arith*
have ∗: *Im z* ≤ *cmod z*
using *abs_Im_le_cmod abs_le_D1* **by** *auto*
have *triangle3*: ⋀*x y z*. *norm*(*x* + *y* + *z*) ≤ *norm*(*x*) + *norm*(*y*) + *norm*(*z*)
by (*simp add*: *norm_add_rule_thm*)
have *norm*(*1* + *cos z*) = *cmod* (*1* + (*exp* (i ∗ *z*) + *exp* (− (i ∗ *z*))) / *2*)
by (*simp add*: *cos_exp_eq*)
also have ... = *cmod* ((*2* + *exp* (i ∗ *z*) + *exp* (− (i ∗ *z*))) / *2*)
by (*simp add*: *field_simps*)
also have ... = *cmod* (*2* + *exp* (i ∗ *z*) + *exp* (− (i ∗ *z*))) / *2*
by (*simp add*: *norm_divide*)
finally show *?thesis*
by (*metis exp_eq_one_iff exp_le_cancel_iff mult_2 norm_cos_le norm_ge_zero norm_one norm_triangle_mono*)
qed

lemma *sinh_conv_sin*: *sinh z* = −i ∗ *sin* (i∗*z*)
by (*simp add*: *sinh_field_def sin_i_times exp_minus*)

lemma *cosh_conv_cos*: *cosh z* = *cos* (i∗*z*)
by (*simp add*: *cosh_field_def cos_i_times exp_minus*)

lemma *tanh_conv_tan*: *tanh z* = −i ∗ *tan* (i∗*z*)
by (*simp add*: *tanh_def sinh_conv_sin cosh_conv_cos tan_def*)

lemma *sin_conv_sinh*: *sin z* = −i ∗ *sinh* (i∗*z*)
by (*simp add*: *sinh_conv_sin*)

lemma *cos_conv_cosh*: *cos z* = *cosh* (i∗*z*)
  by (*simp add*: *cosh_conv_cos*)

lemma *tan_conv_tanh*: *tan z* = −i ∗ *tanh* (i∗*z*)
  by (*simp add*: *tan_def sin_conv_sinh cos_conv_cosh tanh_def*)

lemma *sinh_complex_eq_iff*:
  sinh (*z* :: *complex*) = *sinh w* ⟷
    (∃ *n*∈ℤ. *z* = *w* − *2* ∗ i ∗ *of_real n* ∗ *of_real pi* ∨
        z = −(*2* ∗ *complex_of_real n* + *1*) ∗ i ∗ *complex_of_real pi* − *w*) (**is** _
= *?rhs*)
proof −
  have *sinh z* = *sinh w* ⟷ *sin* (i ∗ *z*) = *sin* (i ∗ *w*)
    by (*simp add*: *sinh_conv_sin*)
  also have ... ⟷ *?rhs*
    by (*subst complex_sin_eq*) (*force simp*: *field_simps complex_eq_iff*)
  finally show *?thesis* **.**
qed

6.21.6 Taylor series for complex exponential, sine and cosine

declare *power_Suc* [*simp del*]

lemma *Taylor_exp_field*:
  fixes *z*::′*a*::{*banach*,*real_normed_field*}
  shows *norm* (*exp z* − (∑ *i*≤*n*. *z* ˆ *i* / *fact i*)) ≤ *exp* (*norm z*) ∗ (*norm z* ˆ *Suc*
n) / *fact n*
proof (*rule field_Taylor*[*of* _ *n* λ*k*. *exp exp* (*norm z*) *0 z*, *simplified*])
  show *convex* (*closed_segment 0 z*)
    by (*rule convex_closed_segment* [*of 0 z*])
next
  fix *k x*
  assume *x* ∈ *closed_segment 0 z k* ≤ *n*
  show (*exp has_field_derivative exp x*) (*at x within closed_segment 0 z*)
    using *DERIV_exp DERIV_subset* **by** *blast*
next
  fix *x*
  assume *x*: *x* ∈ *closed_segment 0 z*
  have *norm* (*exp x*) ≤ *exp* (*norm x*)
    by (*rule norm_exp*)
  also have *norm x* ≤ *norm z*
    using *x* **by** (*auto simp*: *closed_segment_def intro*!: *mult_left_le_one_le*)
  finally show *norm* (*exp x*) ≤ *exp* (*norm z*)
    by *simp*
qed *auto*

lemma *Taylor_exp*:
  norm(*exp z* − (∑ *k*≤*n*. *z* ˆ *k* / (*fact k*))) ≤ *exp*|*Re z*| ∗ (*norm z*) ˆ (*Suc n*) /
(*fact n*)

proof (*rule complex_Taylor* [*of* _ *n* λ*k*. *exp exp*|*Re z*| *0 z*, *simplified*])
  show *convex* (*closed_segment 0 z*)
    by (*rule convex_closed_segment* [*of 0 z*])
next
  fix *k x*
  assume *x* ∈ *closed_segment 0 z k* ≤ *n*
  show (*exp has_field_derivative exp x*) (*at x within closed_segment 0 z*)
    using *DERIV_exp DERIV_subset* **by** *blast*
next
  fix *x*
  assume *x* ∈ *closed_segment 0 z*
  then obtain *u* **where** *u*: *x* = *complex_of_real u* ∗ *z 0* ≤ *u u* ≤ *1*
    by (*auto simp*: *closed_segment_def scaleR_conv_of_real*)
  then have *u* ∗ *Re z* ≤ |*Re z*|
    by (*metis abs_ge_self abs_ge_zero mult.commute mult.right_neutral mult_mono*)
  then show *Re x* ≤ |*Re z*|
    by (*simp add*: *u*)
qed *auto*

lemma
  assumes *0* ≤ *u u* ≤ *1*
  shows *cmod_sin_le_exp*: *cmod* (*sin* (*u* $*_R$ *z*)) ≤ *exp* |*Im z*|
    and *cmod_cos_le_exp*: *cmod* (*cos* (*u* $*_R$ *z*)) ≤ *exp* |*Im z*|
proof −
  have *mono*: ⋀*u w z*::*real*. *w* ≤ *u* ⟹ *z* ≤ *u* ⟹ (*w* + *z*)/*2* ≤ *u*
    by *simp*
  have ∗: (*cmod* (*exp* (i ∗ (*u* ∗ *z*))) + *cmod* (*exp* (− (i ∗ (*u* ∗ *z*))))) / *2* ≤ *exp* |*Im z*|
  proof (*rule mono*)
    show *cmod* (*exp* (i ∗ (*u* ∗ *z*))) ≤ *exp* |*Im z*|
      using *assms*
      by (*auto simp*: *abs_if mult_left_le_one_le not_less intro*: *order_trans* [*of* _ *0*])
    show *cmod* (*exp* (− (i ∗ (*u* ∗ *z*)))) ≤ *exp* |*Im z*|
      using *assms*
      by (*auto simp*: *abs_if mult_left_le_one_le mult_nonneg_nonpos intro*: *order_trans* [*of* _ *0*])
  qed
  have *cmod* (*sin* (*u* $*_R$ *z*)) = *cmod* (*exp* (i ∗ (*u* ∗ *z*)) − *exp* (− (i ∗ (*u* ∗ *z*)))) / *2*
    by (*auto simp*: *scaleR_conv_of_real norm_mult norm_power sin_exp_eq norm_divide*)
  also have ... ≤ (*cmod* (*exp* (i ∗ (*u* ∗ *z*))) + *cmod* (*exp* (− (i ∗ (*u* ∗ *z*))))) / *2*
    by (*intro divide_right_mono norm_triangle_ineq4*) *simp*
  also have ... ≤ *exp* |*Im z*|
    by (*rule* ∗)
  finally show *cmod* (*sin* (*u* $*_R$ *z*)) ≤ *exp* |*Im z*| **.**
  have *cmod* (*cos* (*u* $*_R$ *z*)) = *cmod* (*exp* (i ∗ (*u* ∗ *z*)) + *exp* (− (i ∗ (*u* ∗ *z*)))) / *2*
    by (*auto simp*: *scaleR_conv_of_real norm_mult norm_power cos_exp_eq norm_divide*)
  also have ... ≤ (*cmod* (*exp* (i ∗ (*u* ∗ *z*))) + *cmod* (*exp* (− (i ∗ (*u* ∗ *z*))))) / *2*

```
    by (intro divide_right_mono norm_triangle_ineq) simp
  also have ... ≤ exp |Im z|
    by (rule *)
  finally show cmod (cos (u *R z)) ≤ exp |Im z| .
qed

lemma Taylor_sin:
  norm(sin z − (∑ k≤n. complex_of_real (sin_coeff k) * z ^ k))
   ≤ exp|Im z| * (norm z) ^ (Suc n) / (fact n)
proof −
  have mono: ⋀u w z::real. w ≤ u ⟹ z ≤ u ⟹ w + z ≤ u*2
      by arith
  have *: cmod (sin z −
                 (∑ i≤n. (−1) ^ (i div 2) * (if even i then sin 0 else cos 0) * z ^ i /
(fact i)))
           ≤ exp |Im z| * cmod z ^ Suc n / (fact n)
  proof (rule complex_Taylor [of closed_segment 0 z n
                                  λk x. (−1)^(k div 2) * (if even k then sin x else cos x)
                                  exp|Im z| 0 z,  simplified])
    fix k x
    show ((λx. (− 1) ^ (k div 2) * (if even k then sin x else cos x)) has_field_derivative
            (− 1) ^ (Suc k div 2) * (if odd k then sin x else cos x))
            (at x within closed_segment 0 z)
      apply (auto simp: power_Suc)
      apply (intro derivative_eq_intros | simp)+
      done
  next
    fix x
    assume x ∈ closed_segment 0 z
    then show cmod ((− 1) ^ (Suc n div 2) * (if odd n then sin x else cos x)) ≤
exp |Im z|
      by (auto simp: closed_segment_def norm_mult norm_power cmod_sin_le_exp
cmod_cos_le_exp)
  qed
  have **: ⋀k. complex_of_real (sin_coeff k) * z ^ k
            = (−1)^(k div 2) * (if even k then sin 0 else cos 0) * z^k / of_nat (fact
k)
    by (auto simp: sin_coeff_def elim!: oddE)
  show ?thesis
    by (simp add: ** order_trans [OF _ *])
qed

lemma Taylor_cos:
  norm(cos z − (∑ k≤n. complex_of_real (cos_coeff k) * z ^ k))
   ≤ exp|Im z| * (norm z) ^ Suc n / (fact n)
proof −
  have mono: ⋀u w z::real. w ≤ u ⟹ z ≤ u ⟹ w + z ≤ u*2
      by arith
  have *: cmod (cos z −
```

($\sum i \le n$. (−*1*) ˆ (*Suc i div 2*) ∗ (*if even i then cos 0 else sin 0*) ∗ *z* ˆ *i* / (*fact i*)))
≤ *exp* |*Im z*| ∗ *cmod z* ˆ *Suc n* / (*fact n*)
proof (*rule complex_Taylor* [*of closed_segment 0 z n* λ*k x*. (−*1*) ˆ(*Suc k div 2*) ∗ (*if even k then cos x else sin x*) *exp*|*Im z*| *0 z*, *simplified*])
fix *k x*
assume *x* ∈ *closed_segment 0 z k* ≤ *n*
show ((λ*x*. (− *1*) ˆ (*Suc k div 2*) ∗ (*if even k then cos x else sin x*)) *has_field_derivative*
(− *1*) ˆ *Suc* (*k div 2*) ∗ (*if odd k then cos x else sin x*))
(*at x within closed_segment 0 z*)
apply (*auto simp*: *power_Suc*)
apply (*intro derivative_eq_intros* | *simp*)+
done
next
fix *x*
assume *x* ∈ *closed_segment 0 z*
then show *cmod* ((− *1*) ˆ *Suc* (*n div 2*) ∗ (*if odd n then cos x else sin x*)) ≤ *exp* |*Im z*|
by (*auto simp*: *closed_segment_def norm_mult norm_power cmod_sin_le_exp cmod_cos_le_exp*)
qed
have ∗∗: ⋀*k*. *complex_of_real* (*cos_coeff k*) ∗ *z* ˆ *k*
= (−*1*) ˆ(*Suc k div 2*) ∗ (*if even k then cos 0 else sin 0*) ∗ *z*ˆ*k* / *of_nat* (*fact k*)
by (*auto simp*: *cos_coeff_def elim*!: *evenE*)
show *?thesis*
by (*simp add*: ∗∗ *order_trans* [*OF* _ ∗])
qed

declare *power_Suc* [*simp*]

32-bit Approximation to e

lemma *e_approx_32*: |*exp*(*1*) − *5837465777* / *2147483648*| ≤ (*inverse*(*2* ˆ *32*)::*real*)
using *Taylor_exp* [*of 1 14*] *exp_le*
apply (*simp add*: *sum_distrib_right in_Reals_norm Re_exp atMost_nat_numeral fact_numeral*)
apply (*simp only*: *pos_le_divide_eq* [*symmetric*])
done

lemma *e_less_272*: *exp 1* < (*272*/*100*::*real*)
using *e_approx_32*
by (*simp add*: *abs_if split*: *if_split_asm*)

lemma *ln_272_gt_1*: *ln* (*272*/*100*) > (*1*::*real*)
by (*metis e_less_272 exp_less_cancel_iff exp_ln_iff less_trans ln_exp*)

Apparently redundant. But many arguments involve integers.

lemma *ln3_gt_1*: *ln 3* > (*1*::*real*)
  by (*simp add*: *less_trans* [*OF ln_272_gt_1*])

6.21.7 The argument of a complex number (HOL Light version)

definition *is_Arg* :: [*complex*,*real*] ⇒ *bool*
  where *is_Arg z r* ≡ *z* = *of_real*(*norm z*) ∗ *exp*(i ∗ *of_real r*)

definition *Arg2pi* :: *complex* ⇒ *real*
  where *Arg2pi z* ≡ *if z* = *0 then 0 else THE t*. *0* ≤ *t* ∧ *t* < *2*∗*pi* ∧ *is_Arg z t*

lemma *is_Arg_2pi_iff*: *is_Arg z* (*r* + *of_int k* ∗ (*2* ∗ *pi*)) ⟷ *is_Arg z r*
  by (*simp add*: *algebra_simps is_Arg_def*)

lemma *is_Arg_eqI*:
  assumes *r*: *is_Arg z r* **and** *s*: *is_Arg z s* **and** *rs*: *abs* (*r*−*s*) < *2*∗*pi* **and** *z* ≠ *0*
  shows *r* = *s*
proof −
  have *zr*: *z* = (*cmod z*) ∗ *exp* (i ∗ *r*) **and** *zs*: *z* = (*cmod z*) ∗ *exp* (i ∗ *s*)
    using *r s* **by** (*auto simp*: *is_Arg_def*)
  with ‹*z* ≠ *0*› **have** *eq*: *exp* (i ∗ *r*) = *exp* (i ∗ *s*)
    by (*metis mult_eq_0_iff mult_left_cancel*)
  have i ∗ *r* = i ∗ *s*
    by (*rule exp_complex_eqI*) (*use rs* **in** ‹*auto simp*: *eq exp_complex_eqI*›)
  then show *?thesis*
    by *simp*
qed

This function returns the angle of a complex number from its representation in polar coordinates. Due to periodicity, its range is arbitrary. *Arg2pi* follows HOL Light in adopting the interval $[0,2\pi)$. But we have the same periodicity issue with logarithms, and it is usual to adopt the same interval for the complex logarithm and argument functions. Further on down, we shall define both functions for the interval $(-\pi,\pi]$. The present version is provided for compatibility.

lemma *Arg2pi_0* [*simp*]: *Arg2pi*(*0*) = *0*
  by (*simp add*: *Arg2pi_def*)

lemma *Arg2pi_unique_lemma*:
  assumes *z*: *is_Arg z t*
    and *z'*: *is_Arg z t'*
    and *t*: *0* ≤ *t* *t* < *2*∗*pi*
    and *t'*: *0* ≤ *t'* *t'* < *2*∗*pi*
    and *nz*: *z* ≠ *0*
  shows *t'* = *t*
proof −
  have [*dest*]: ⋀*x y z*::*real*. *x*≥*0* ⟹ *x*+*y* < *z* ⟹ *y*<*z*

by *arith*
have *of_real* (*cmod z*) ∗ *exp* (i ∗ *of_real t′*) = *of_real* (*cmod z*) ∗ *exp* (i ∗ *of_real t*)
by (*metis z z′ is_Arg_def*)
then have *exp* (i ∗ *of_real t′*) = *exp* (i ∗ *of_real t*)
by (*metis nz mult_left_cancel mult_zero_left z is_Arg_def*)
then have *sin t′* = *sin t* ∧ *cos t′* = *cos t*
by (*metis cis.simps cis_conv_exp*)
then obtain *n*::*int* **where** *n*: *t′* = *t* + *2* ∗ *n* ∗ *pi*
by (*auto simp*: *sin_cos_eq_iff*)
then have *n=0*
by (*cases n*) (*use t t′* **in** ‹*auto simp*: *mult_less_0_iff algebra_simps*›)
then show *t′* = *t*
by (*simp add*: *n*)
qed

lemma *Arg2pi*: *0* ≤ *Arg2pi z* ∧ *Arg2pi z* < *2*∗*pi* ∧ *is_Arg z* (*Arg2pi z*)
proof (*cases z=0*)
case *True* **then show** *?thesis*
by (*simp add*: *Arg2pi_def is_Arg_def*)
next
case *False*
obtain *t* **where** *t*: *0* ≤ *t t* < *2*∗*pi*
and *ReIm*: *Re z* / *cmod z* = *cos t Im z* / *cmod z* = *sin t*
using *sincos_total_2pi* [*OF complex_unit_circle* [*OF False*]]
by *blast*
have *z*: *is_Arg z t*
unfolding *is_Arg_def*
using *t False ReIm*
by (*intro complex_eqI*) (*auto simp*: *exp_Euler sin_of_real cos_of_real field_split_simps*)
show *?thesis*
apply (*simp add*: *Arg2pi_def False*)
apply (*rule theI* [**where** *a=t*])
using *t z False*
apply (*auto intro*: *Arg2pi_unique_lemma*)
done
qed

corollary
shows *Arg2pi_ge_0*: *0* ≤ *Arg2pi z*
and *Arg2pi_lt_2pi*: *Arg2pi z* < *2*∗*pi*
and *Arg2pi_eq*: *z* = *of_real*(*norm z*) ∗ *exp*(i ∗ *of_real*(*Arg2pi z*))
using *Arg2pi is_Arg_def* **by** *auto*

lemma *complex_norm_eq_1_exp*: *norm z* = *1* ⟷ *exp*(i ∗ *of_real* (*Arg2pi z*)) = *z*
by (*metis Arg2pi_eq cis_conv_exp mult.left_neutral norm_cis of_real_1*)

lemma *Arg2pi_unique*: ⟦*of_real r* ∗ *exp*(i ∗ *of_real a*) = *z*; *0* < *r*; *0* ≤ *a*; *a* < *2*∗*pi*⟧
⟹ *Arg2pi z* = *a*

by (*rule Arg2pi_unique_lemma* [*unfolded is_Arg_def*, *OF* _ *Arg2pi_eq*]) (*use Arg2pi* [*of z*] **in** ‹*auto simp*: *norm_mult*›)

lemma *cos_Arg2pi*: *cmod z* ∗ *cos* (*Arg2pi z*) = *Re z* **and** *sin_Arg2pi*: *cmod z* ∗ *sin* (*Arg2pi z*) = *Im z*
using *Arg2pi_eq* [*of z*] *cis_conv_exp Re_rcis Im_rcis* **unfolding** *rcis_def* **by** *metis+*

lemma *Arg2pi_minus*:
assumes *z* ≠ *0* **shows** *Arg2pi* (−*z*) = (*if Arg2pi z* < *pi then Arg2pi z* + *pi else Arg2pi z* − *pi*)
apply (*rule Arg2pi_unique* [*of norm z*, *OF complex_eqI*])
using *cos_Arg2pi sin_Arg2pi Arg2pi_ge_0 Arg2pi_lt_2pi* [*of z*] *assms*
by (*auto simp*: *Re_exp Im_exp*)

lemma *Arg2pi_times_of_real* [*simp*]:
assumes *0* < *r* **shows** *Arg2pi* (*of_real r* ∗ *z*) = *Arg2pi z*
proof (*cases z*=*0*)
case *False*
show *?thesis*
by (*rule Arg2pi_unique* [*of r* ∗ *norm z*]) (*use Arg2pi False assms is_Arg_def* **in** *auto*)
qed *auto*

lemma *Arg2pi_times_of_real2* [*simp*]: *0* < *r* ⟹ *Arg2pi* (*z* ∗ *of_real r*) = *Arg2pi z*
by (*metis Arg2pi_times_of_real mult.commute*)

lemma *Arg2pi_divide_of_real* [*simp*]: *0* < *r* ⟹ *Arg2pi* (*z* / *of_real r*) = *Arg2pi z*
by (*metis Arg2pi_times_of_real2 less_numeral_extra*(*3*) *nonzero_eq_divide_eq of_real_eq_0_iff*)

lemma *Arg2pi_le_pi*: *Arg2pi z* ≤ *pi* ⟷ *0* ≤ *Im z*
proof (*cases z*=*0*)
case *False*
have *0* ≤ *Im z* ⟷ *0* ≤ *Im* (*of_real* (*cmod z*) ∗ *exp* (i ∗ *complex_of_real* (*Arg2pi z*)))
by (*metis Arg2pi_eq*)
also have ... = (*0* ≤ *Im* (*exp* (i ∗ *complex_of_real* (*Arg2pi z*))))
using *False* **by** (*simp add*: *zero_le_mult_iff*)
also have ... ⟷ *Arg2pi z* ≤ *pi*
by (*simp add*: *Im_exp*) (*metis Arg2pi_ge_0 Arg2pi_lt_2pi sin_lt_zero sin_ge_zero not_le*)
finally show *?thesis*
by *blast*
qed *auto*

lemma *Arg2pi_lt_pi*: *0* < *Arg2pi z* ∧ *Arg2pi z* < *pi* ⟷ *0* < *Im z*
proof (*cases z*=*0*)
case *False*
have *0* < *Im z* ⟷ *0* < *Im* (*of_real* (*cmod z*) ∗ *exp* (i ∗ *complex_of_real* (*Arg2pi z*)))

by (*metis Arg2pi_eq*)
also have ... = (*0* < *Im* (*exp* (i ∗ *complex_of_real* (*Arg2pi z*))))
using *False* **by** (*simp add*: *zero_less_mult_iff*)
also have ... ⟷ *0* < *Arg2pi z* ∧ *Arg2pi z* < *pi* (**is** _ = *?rhs*)
proof −
have *0* < *sin* (*Arg2pi z*) ⟹ *?rhs*
by (*meson Arg2pi_ge_0 Arg2pi_lt_2pi less_le_trans not_le sin_le_zero sin_x_le_x*)
then show *?thesis*
by (*auto simp*: *Im_exp sin_gt_zero*)
qed
finally show *?thesis*
by *blast*
qed *auto*

lemma *Arg2pi_eq_0*: *Arg2pi z* = *0* ⟷ *z* ∈ ℝ ∧ *0* ≤ *Re z*
proof (*cases z*=*0*)
case *False*
have *z* ∈ ℝ ∧ *0* ≤ *Re z* ⟷ *z* ∈ ℝ ∧ *0* ≤ *Re* (*of_real* (*cmod z*) ∗ *exp* (i ∗ *complex_of_real* (*Arg2pi z*)))
by (*metis Arg2pi_eq*)
also have ... ⟷ *z* ∈ ℝ ∧ *0* ≤ *Re* (*exp* (i ∗ *complex_of_real* (*Arg2pi z*)))
using *False* **by** (*simp add*: *zero_le_mult_iff*)
also have ... ⟷ *Arg2pi z* = *0*
proof −
have [*simp*]: *Arg2pi z* = *0* ⟹ *z* ∈ ℝ
using *Arg2pi_eq* [*of z*] **by** (*auto simp*: *Reals_def*)
moreover have ⟦*z* ∈ ℝ; *0* ≤ *cos* (*Arg2pi z*)⟧ ⟹ *Arg2pi z* = *0*
by (*metis Arg2pi_lt_pi Arg2pi_ge_0 Arg2pi_le_pi cos_pi complex_is_Real_iff leD less_linear less_minus_one_simps*(*2*) *minus_minus neg_less_eq_nonneg order_refl*)
ultimately show *?thesis*
by (*auto simp*: *Re_exp*)
qed
finally show *?thesis*
by *blast*
qed *auto*

corollary *Arg2pi_gt_0*:
assumes *z* ∉ $\mathbb{R}_{\geq 0}$
shows *Arg2pi z* > *0*
using *Arg2pi_eq_0 Arg2pi_ge_0 assms dual_order.strict_iff_order*
unfolding *nonneg_Reals_def* **by** *fastforce*

lemma *Arg2pi_eq_pi*: *Arg2pi z* = *pi* ⟷ *z* ∈ ℝ ∧ *Re z* < *0*
using *Arg2pi_le_pi* [*of z*] *Arg2pi_lt_pi* [*of z*] *Arg2pi_eq_0* [*of z*] *Arg2pi_ge_0* [*of z*]
by (*fastforce simp*: *complex_is_Real_iff*)

lemma *Arg2pi_eq_0_pi*: *Arg2pi z* = *0* ∨ *Arg2pi z* = *pi* ⟷ *z* ∈ ℝ
using *Arg2pi_eq_0 Arg2pi_eq_pi not_le* **by** *auto*

lemma *Arg2pi_of_real*: *Arg2pi* (*of_real r*) = (*if r<0 then pi else 0*)
  using *Arg2pi_eq_0_pi Arg2pi_eq_pi* **by** *fastforce*

lemma *Arg2pi_real*: *z* ∈ ℝ ⟹ *Arg2pi z* = (*if 0* ≤ *Re z then 0 else pi*)
  using *Arg2pi_eq_0 Arg2pi_eq_0_pi* **by** *auto*

lemma *Arg2pi_inverse*: *Arg2pi*(*inverse z*) = (*if z* ∈ ℝ *then Arg2pi z else 2*∗*pi* − *Arg2pi z*)
proof (*cases z=0*)
  case *False*
  show *?thesis*
    apply (*rule Arg2pi_unique* [*of inverse* (*norm z*)])
    using *Arg2pi_eq False Arg2pi_ge_0* [*of z*] *Arg2pi_lt_2pi* [*of z*] *Arg2pi_eq_0* [*of z*]
    by (*auto simp*: *Arg2pi_real in_Reals_norm exp_diff field_simps*)
qed *auto*

lemma *Arg2pi_eq_iff*:
  assumes *w* ≠ *0 z* ≠ *0*
    shows *Arg2pi w* = *Arg2pi z* ⟷ (∃ *x*. *0* < *x* & *w* = *of_real x* ∗ *z*)
  using *assms Arg2pi_eq* [*of z*] *Arg2pi_eq* [*of w*]
  apply *auto*
  apply (*rule_tac x=norm w* / *norm z* **in** *exI*)
  apply (*simp add*: *field_split_simps*)
  by (*metis mult.commute mult.left_commute*)

lemma *Arg2pi_inverse_eq_0*: *Arg2pi*(*inverse z*) = *0* ⟷ *Arg2pi z* = *0*
  by (*metis Arg2pi_eq_0 Arg2pi_inverse inverse_inverse_eq*)

lemma *Arg2pi_divide*:
  assumes *w* ≠ *0 z* ≠ *0 Arg2pi w* ≤ *Arg2pi z*
    shows *Arg2pi*(*z* / *w*) = *Arg2pi z* − *Arg2pi w*
  apply (*rule Arg2pi_unique* [*of norm*(*z* / *w*)])
  using *assms Arg2pi_eq Arg2pi_ge_0* [*of w*] *Arg2pi_lt_2pi* [*of z*]
  apply (*auto simp*: *exp_diff norm_divide field_simps*)
  done

lemma *Arg2pi_le_div_sum*:
  assumes *w* ≠ *0 z* ≠ *0 Arg2pi w* ≤ *Arg2pi z*
    shows *Arg2pi z* = *Arg2pi w* + *Arg2pi*(*z* / *w*)
  by (*simp add*: *Arg2pi_divide assms*)

lemma *Arg2pi_le_div_sum_eq*:
  assumes *w* ≠ *0 z* ≠ *0*
    shows *Arg2pi w* ≤ *Arg2pi z* ⟷ *Arg2pi z* = *Arg2pi w* + *Arg2pi*(*z* / *w*)
  using *assms* **by** (*auto simp*: *Arg2pi_ge_0 intro*: *Arg2pi_le_div_sum*)

lemma *Arg2pi_diff*:
  assumes *w* ≠ *0 z* ≠ *0*

shows *Arg2pi w − Arg2pi z = (if Arg2pi z ≤ Arg2pi w then Arg2pi(w / z) else Arg2pi(w/z) − 2∗pi)*
using *assms Arg2pi_divide Arg2pi_inverse [of w/z] Arg2pi_eq_0_pi*
by *(force simp add: Arg2pi_ge_0 Arg2pi_divide not_le split: if_split_asm)*

lemma *Arg2pi_add*:
assumes *w ≠ 0 z ≠ 0*
shows *Arg2pi w + Arg2pi z = (if Arg2pi w + Arg2pi z < 2∗pi then Arg2pi(w ∗ z) else Arg2pi(w ∗ z) + 2∗pi)*
using *assms Arg2pi_diff [of w∗z z] Arg2pi_le_div_sum_eq [of z w∗z]*
apply *(auto simp: Arg2pi_ge_0 Arg2pi_divide not_le)*
apply *(metis Arg2pi_lt_2pi add.commute)*
apply *(metis (no_types) Arg2pi add.commute diff_0 diff_add_cancel diff_less_eq diff_minus_eq_add not_less)*
done

lemma *Arg2pi_times*:
assumes *w ≠ 0 z ≠ 0*
shows *Arg2pi (w ∗ z) = (if Arg2pi w + Arg2pi z < 2∗pi then Arg2pi w + Arg2pi z*
else (Arg2pi w + Arg2pi z) − 2∗pi)
using *Arg2pi_add [OF assms]*
by *auto*

lemma *Arg2pi_cnj_eq_inverse*: *z≠0 ⟹ Arg2pi (cnj z) = Arg2pi (inverse z)*
apply *(simp add: Arg2pi_eq_iff field_split_simps complex_norm_square [symmetric])*
by *(metis of_real_power zero_less_norm_iff zero_less_power)*

lemma *Arg2pi_cnj*: *Arg2pi(cnj z) = (if z ∈ ℝ then Arg2pi z else 2∗pi − Arg2pi z)*
proof *(cases z=0)*
case *False*
then show *?thesis*
by *(simp add: Arg2pi_cnj_eq_inverse Arg2pi_inverse)*
qed *auto*

lemma *Arg2pi_exp*: *0 ≤ Im z ⟹ Im z < 2∗pi ⟹ Arg2pi(exp z) = Im z*
by *(rule Arg2pi_unique [of exp(Re z)]) (auto simp: exp_eq_polar)*

lemma *complex_split_polar*:
obtains *r a::real* **where** *z = complex_of_real r ∗ (cos a + i ∗ sin a) 0 ≤ r 0 ≤ a a < 2∗pi*
using *Arg2pi cis.ctr cis_conv_exp* **unfolding** *Complex_eq is_Arg_def* **by** *fastforce*

lemma *Re_Im_le_cmod*: *Im w ∗ sin φ + Re w ∗ cos φ ≤ cmod w*
proof *(cases w rule: complex_split_polar)*
case *(1 r a)* **with** *sin_cos_le1 [of a φ]* **show** *?thesis*
apply *(simp add: norm_mult cmod_unit_one)*
by *(metis (no_types, hide_lams) abs_le_D1 distrib_left mult.commute mult.left_commute*

mult_left_le)
qed

6.21.8 Analytic properties of tangent function

lemma *cnj_tan*: *cnj*(*tan z*) = *tan*(*cnj z*)
  by (*simp add*: *cnj_cos cnj_sin tan_def*)

lemma *field_differentiable_at_tan*: *cos z* ≠ *0* ⟹ *tan field_differentiable at z*
  unfolding *field_differentiable_def*
  using *DERIV_tan* **by** *blast*

lemma *field_differentiable_within_tan*: *cos z* ≠ *0*
    ⟹ *tan field_differentiable* (*at z within s*)
  using *field_differentiable_at_tan field_differentiable_at_within* **by** *blast*

lemma *continuous_within_tan*: *cos z* ≠ *0* ⟹ *continuous* (*at z within s*) *tan*
  using *continuous_at_imp_continuous_within isCont_tan* **by** *blast*

lemma *continuous_on_tan* [*continuous_intros*]: (⋀*z*. *z* ∈ *s* ⟹ *cos z* ≠ *0*) ⟹ *continuous_on s tan*
  by (*simp add*: *continuous_at_imp_continuous_on*)

lemma *holomorphic_on_tan*: (⋀*z*. *z* ∈ *s* ⟹ *cos z* ≠ *0*) ⟹ *tan holomorphic_on s*
  by (*simp add*: *field_differentiable_within_tan holomorphic_on_def*)

6.21.9 The principal branch of the Complex logarithm

instantiation *complex* :: *ln*
begin

definition *ln_complex* :: *complex* ⇒ *complex*
  where *ln_complex* ≡ λ*z*. *THE w*. *exp w* = *z* & −*pi* < *Im*(*w*) & *Im*(*w*) ≤ *pi*

NOTE: within this scope, the constant Ln is not yet available!

lemma
  assumes *z* ≠ *0*
    shows *exp_Ln* [*simp*]: *exp*(*ln z*) = *z*
      and *mpi_less_Im_Ln*: −*pi* < *Im*(*ln z*)
      and *Im_Ln_le_pi*: *Im*(*ln z*) ≤ *pi*
proof −
  obtain ψ **where** *z*: *z* / (*cmod z*) = *Complex* (*cos* ψ) (*sin* ψ)
    using *complex_unimodular_polar* [*of z* / (*norm z*)] *assms*
    by (*auto simp*: *norm_divide field_split_simps*)
  obtain φ **where** φ: − *pi* < φ φ ≤ *pi sin* φ = *sin* ψ *cos* φ = *cos* ψ
    using *sincos_principal_value* [*of* ψ] *assms*
    by (*auto simp*: *norm_divide field_split_simps*)
  have *exp*(*ln z*) = *z* & −*pi* < *Im*(*ln z*) & *Im*(*ln z*) ≤ *pi* **unfolding** *ln_complex_def*
    apply (*rule theI* [**where** *a* = *Complex* (*ln*(*norm z*)) φ])

using *z assms φ*
apply (*auto simp*: *field_simps exp_complex_eqI exp_eq_polar cis.code*)
done
then show *exp*(*ln z*) = *z* −*pi* < *Im*(*ln z*) *Im*(*ln z*) ≤ *pi*
by *auto*
qed

lemma *Ln_exp* [*simp*]:
assumes −*pi* < *Im*(*z*) *Im*(*z*) ≤ *pi*
shows *ln*(*exp z*) = *z*
proof (*rule exp_complex_eqI*)
show |*Im* (*ln* (*exp z*)) − *Im z*| < *2* ∗ *pi*
using *assms mpi_less_Im_Ln* [*of exp z*] *Im_Ln_le_pi* [*of exp z*] **by** *auto*
qed *auto*

6.21.10 Relation to Real Logarithm

lemma *Ln_of_real*:
assumes *0* < *z*
shows *ln*(*of_real z*::*complex*) = *of_real*(*ln z*)
proof −
have *ln*(*of_real* (*exp* (*ln z*))::*complex*) = *ln* (*exp* (*of_real* (*ln z*)))
by (*simp add*: *exp_of_real*)
also have ... = *of_real*(*ln z*)
using *assms* **by** (*subst Ln_exp*) *auto*
finally show *?thesis*
using *assms* **by** *simp*
qed

corollary *Ln_in_Reals* [*simp*]: *z* ∈ ℝ ⟹ *Re z* > *0* ⟹ *ln z* ∈ ℝ
by (*auto simp*: *Ln_of_real elim*: *Reals_cases*)

corollary *Im_Ln_of_real* [*simp*]: *r* > *0* ⟹ *Im* (*ln* (*of_real r*)) = *0*
by (*simp add*: *Ln_of_real*)

lemma *cmod_Ln_Reals* [*simp*]: *z* ∈ ℝ ⟹ *0* < *Re z* ⟹ *cmod* (*ln z*) = *norm* (*ln* (*Re z*))
using *Ln_of_real* **by** *force*

lemma *Ln_Reals_eq*: ⟦*x* ∈ ℝ; *Re x* > *0*⟧ ⟹ *ln x* = *of_real* (*ln* (*Re x*))
using *Ln_of_real* **by** *force*

lemma *Ln_1* [*simp*]: *ln 1* = (*0*::*complex*)
proof −
have *ln* (*exp 0*) = (*0*::*complex*)
by (*simp add*: *del*: *exp_zero*)
then show *?thesis*
by *simp*
qed

lemma *Ln_eq_zero_iff* [*simp*]: $x \notin \mathbb{R}_{\leq 0} \Longrightarrow ln\ x = 0 \longleftrightarrow x = 1$ **for** *x::complex*
by *auto* (*metis exp_Ln exp_zero nonpos_Reals_zero_I*)

instance
by *intro_classes* (*rule ln_complex_def Ln_1*)

end

abbreviation *Ln* :: *complex* ⇒ *complex*
where *Ln* ≡ *ln*

lemma *Ln_eq_iff*: $w \neq 0 \Longrightarrow z \neq 0 \Longrightarrow (Ln\ w = Ln\ z \longleftrightarrow w = z)$
by (*metis exp_Ln*)

lemma *Ln_unique*: $exp(z) = w \Longrightarrow -pi < Im(z) \Longrightarrow Im(z) \leq pi \Longrightarrow Ln\ w = z$
using *Ln_exp* **by** *blast*

lemma *Re_Ln* [*simp*]: $z \neq 0 \Longrightarrow Re(Ln\ z) = ln(norm\ z)$
by (*metis exp_Ln ln_exp norm_exp_eq_Re*)

corollary *ln_cmod_le*:
assumes *z*: $z \neq 0$
shows $ln\ (cmod\ z) \leq cmod\ (Ln\ z)$
using *norm_exp* [*of Ln z*, *simplified exp_Ln* [*OF z*]]
by (*metis Re_Ln complex_Re_le_cmod z*)

proposition *exists_complex_root*:
fixes *z* :: *complex*
assumes $n \neq 0$ **obtains** *w* **where** $z = w\ \hat{}\ n$
proof (*cases z=0*)
case *False*
then show *?thesis*
by (*rule_tac w = exp(Ln z / n)* **in** *that*) (*simp add*: *assms exp_of_nat_mult* [*symmetric*])
qed (*use assms* **in** *auto*)

corollary *exists_complex_root_nonzero*:
fixes *z::complex*
assumes $z \neq 0$ $n \neq 0$
obtains *w* **where** $w \neq 0$ $z = w\ \hat{}\ n$
by (*metis exists_complex_root* [*of n z*] *assms power_0_left*)

6.21.11 Derivative of Ln away from the branch cut

lemma
assumes $z \notin \mathbb{R}_{\leq 0}$
shows *has_field_derivative_Ln*: (*Ln has_field_derivative inverse*(*z*)) (*at z*)

and *Im_Ln_less_pi*: *Im* (*Ln z*) < *pi*
proof −
have *znz* [*simp*]: *z* ≠ *0*
using *assms* **by** *auto*
then have *Im* (*Ln z*) ≠ *pi*
by (*metis* (*no_types*) *Im_exp Ln_in_Reals assms complex_nonpos_Reals_iff complex_is_Real_iff exp_Ln mult_zero_right not_less pi_neq_zero sin_pi znz*)
then show ∗: *Im* (*Ln z*) < *pi* **using** *assms Im_Ln_le_pi*
by (*simp add*: *le_neq_trans*)
let *?U* = {*w*. −*pi* < *Im*(*w*) ∧ *Im*(*w*) < *pi*}
have *1*: *open ?U*
by (*simp add*: *open_Collect_conj open_halfspace_Im_gt open_halfspace_Im_lt*)
have *2*: ⋀*x*. *x* ∈ *?U* ⟹ (*exp has_derivative blinfun_apply*(*Blinfun* ((∗) (*exp x*)))) (*at x*)
by (*simp add*: *bounded_linear_Blinfun_apply bounded_linear_mult_right has_field_derivative_imp_has_deri*

have *3*: *continuous_on ?U* (λ*x*. *Blinfun* ((∗) (*exp x*)))
unfolding *blinfun_mult_right*.*abs_eq* [*symmetric*] **by** (*intro continuous_intros*)
have *4*: *Ln z* ∈ *?U*
by (*auto simp*: *mpi_less_Im_Ln* ∗)
have *5*: *Blinfun* ((∗) (*inverse z*)) o_L *Blinfun* ((∗) (*exp* (*Ln z*))) = *id_blinfun*
by (*rule blinfun_eqI*) (*simp add*: *bounded_linear_mult_right bounded_linear_Blinfun_apply*)
obtain *U′ V g g′* **where** *open U′* **and** *sub*: *U′* ⊆ *?U*
and *Ln z* ∈ *U′ open V z* ∈ *V*
and *hom*: *homeomorphism U′ V exp g*
and *g*: ⋀*y*. *y* ∈ *V* ⟹ (*g has_derivative* (*g′ y*)) (*at y*)
and *g′*: ⋀*y*. *y* ∈ *V* ⟹ *g′ y* = *inv* ((∗) (*exp* (*g y*)))
and *bij*: ⋀*y*. *y* ∈ *V* ⟹ *bij* ((∗) (*exp* (*g y*)))
using *inverse_function_theorem* [*OF 1 2 3 4 5*]
by (*simp add*: *bounded_linear_Blinfun_apply bounded_linear_mult_right*) *blast*
show (*Ln has_field_derivative inverse*(*z*)) (*at z*)
unfolding *has_field_derivative_def*
proof (*rule has_derivative_transform_within_open*)
show *g_eq_Ln*: *g y* = *Ln y* **if** *y* ∈ *V* **for** *y*
proof −
obtain *x* **where** *y* = *exp x x* ∈ *U′*
using *hom homeomorphism_image1 that* ‹*y* ∈ *V*› **by** *blast*
then show *?thesis*
using *sub hom homeomorphism_apply1* **by** *fastforce*
qed
have *0* ∉ *V*
by (*meson exp_not_eq_zero hom homeomorphism_def*)
then have ⋀*y*. *y* ∈ *V* ⟹ *g′ y* = *inv* ((∗) *y*)
by (*metis exp_Ln g′ g_eq_Ln*)
then have *g′*: *g′ z* = (λ*x*. *x*/*z*)
by (*metis* (*no_types*, *hide_lams*) *bij* ‹*z* ∈ *V*› *bij_inv_eq_iff exp_Ln g_eq_Ln nonzero_mult_div_cancel_left znz*)
show (*g has_derivative* (∗) (*inverse z*)) (*at z*)
using *g* [*OF* ‹*z* ∈ *V*›] *g′*

by (*simp add*: ‹$z \in V$› *field_class.field_divide_inverse has_derivative_imp_has_field_derivative has_field_derivative_imp_has_derivative*)
qed (*auto simp*: ‹$z \in V$› ‹*open V*›)
qed

declare *has_field_derivative_Ln* [*derivative_intros*]
declare *has_field_derivative_Ln* [*THEN DERIV_chain2*, *derivative_intros*]

lemma *field_differentiable_at_Ln*: $z \notin \mathbb{R}_{\leq 0} \Longrightarrow$ *Ln field_differentiable at z*
using *field_differentiable_def has_field_derivative_Ln* **by** *blast*

lemma *field_differentiable_within_Ln*: $z \notin \mathbb{R}_{\leq 0}$
$\Longrightarrow$ *Ln field_differentiable* (*at z within S*)
using *field_differentiable_at_Ln field_differentiable_within_subset* **by** *blast*

lemma *continuous_at_Ln*: $z \notin \mathbb{R}_{\leq 0} \Longrightarrow$ *continuous* (*at z*) *Ln*
by (*simp add*: *field_differentiable_imp_continuous_at field_differentiable_within_Ln*)

lemma *isCont_Ln'* [*simp*,*continuous_intros*]:
$[\![$*isCont f z*; $f\ z \notin \mathbb{R}_{\leq 0}]\!] \Longrightarrow$ *isCont* (λx. *Ln* (*f x*)) *z*
by (*blast intro*: *isCont_o2* [*OF _ continuous_at_Ln*])

lemma *continuous_within_Ln* [*continuous_intros*]: $z \notin \mathbb{R}_{\leq 0} \Longrightarrow$ *continuous* (*at z within S*) *Ln*
using *continuous_at_Ln continuous_at_imp_continuous_within* **by** *blast*

lemma *continuous_on_Ln* [*continuous_intros*]: ($\bigwedge z$. $z \in S \Longrightarrow z \notin \mathbb{R}_{\leq 0}$) $\Longrightarrow$ *continuous_on S Ln*
by (*simp add*: *continuous_at_imp_continuous_on continuous_within_Ln*)

lemma *continuous_on_Ln'* [*continuous_intros*]:
continuous_on S f $\Longrightarrow$ ($\bigwedge z$. $z \in S \Longrightarrow f\ z \notin \mathbb{R}_{\leq 0}$) $\Longrightarrow$ *continuous_on S* (λx. *Ln* (*f x*))
by (*rule continuous_on_compose2*[*OF continuous_on_Ln*, *of UNIV* − *nonpos_Reals S f*]) *auto*

lemma *holomorphic_on_Ln* [*holomorphic_intros*]: ($\bigwedge z$. $z \in S \Longrightarrow z \notin \mathbb{R}_{\leq 0}$) $\Longrightarrow$ *Ln holomorphic_on S*
by (*simp add*: *field_differentiable_within_Ln holomorphic_on_def*)

lemma *holomorphic_on_Ln'* [*holomorphic_intros*]:
($\bigwedge z$. $z \in A \Longrightarrow f\ z \notin \mathbb{R}_{\leq 0}$) $\Longrightarrow$ *f holomorphic_on A* $\Longrightarrow$ (λz. *Ln* (*f z*)) *holomorphic_on A*
using *holomorphic_on_compose_gen*[*OF _ holomorphic_on_Ln*, *of f A* − $\mathbb{R}_{\leq 0}$]
by (*auto simp*: *o_def*)

lemma *tendsto_Ln* [*tendsto_intros*]:
fixes *L F*
assumes ($f \longrightarrow L$) *F* $L \notin \mathbb{R}_{\leq 0}$

shows $((\lambda x.\ Ln\ (f\ x)) \longrightarrow Ln\ L)\ F$
proof −
have *nhds L* ≥ *filtermap f F*
using *assms(1)* **by** (*simp add*: *filterlim_def*)
moreover have $\forall_F$ *y in nhds L*. $y \in -\ \mathbb{R}_{\leq 0}$
using *eventually_nhds_in_open*[*of* $-\ \mathbb{R}_{\leq 0}$ *L*] *assms* **by** (*auto simp*: *open_Compl*)
ultimately have $\forall_F$ *y in filtermap f F*. $y \in -\ \mathbb{R}_{\leq 0}$ **by** (*rule filter_leD*)
moreover have *continuous_on* $(-\mathbb{R}_{\leq 0})$ *Ln* **by** (*rule continuous_on_Ln*) *auto*
ultimately show *?thesis* **using** *continuous_on_tendsto_compose*[*of* $-\ \mathbb{R}_{\leq 0}$ *Ln f L F*] *assms*
by (*simp add*: *eventually_filtermap*)
qed

lemma *divide_ln_mono*:
fixes *x y*::*real*
assumes $3 \leq x$ $x \leq y$
shows $x\ /\ ln\ x \leq y\ /\ ln\ y$
proof (*rule exE* [*OF complex_mvt_line* [*of x y* $\lambda z.\ z\ /\ Ln\ z$ $\lambda z.\ 1/(Ln\ z) - 1/(Ln\ z)\hat{\ }2$]];
clarsimp simp add: *closed_segment_Reals closed_segment_eq_real_ivl assms*)
show $\bigwedge u.\ [\![x \leq u;\ u \leq y]\!] \Longrightarrow ((\lambda z.\ z\ /\ Ln\ z)$ *has_field_derivative* $1\ /\ Ln\ u - 1\ /\ (Ln\ u)^2)$ (*at u*)
using ‹$3 \leq x$› **by** (*force intro*!: *derivative_eq_intros simp*: *field_simps power_eq_if*)
show $x\ /\ ln\ x \leq y\ /\ ln\ y$
if $Re\ (y\ /\ Ln\ y) - Re\ (x\ /\ Ln\ x) = (Re\ (1\ /\ Ln\ u) - Re\ (1\ /\ (Ln\ u)^2)) * (y - x)$
and *x*: $x \leq u$ $u \leq y$ **for** *u*
proof −
have *eq*: $y\ /\ ln\ y = (1\ /\ ln\ u - 1\ /\ (ln\ u)^2) * (y - x) + x\ /\ ln\ x$
using *that* ‹$3 \leq x$› **by** (*auto simp*: *Ln_Reals_eq in_Reals_norm group_add_class.diff_eq_eq*)
show *?thesis*
using *exp_le* ‹$3 \leq x$› *x* **by** (*simp add*: *eq*) (*simp add*: *power_eq_if divide_simps ln_ge_iff*)
qed
qed

theorem *Ln_series*:
fixes *z* :: *complex*
assumes *norm z* < *1*
shows $(\lambda n.\ (-1)\hat{\ }Suc\ n\ /\ of\_nat\ n * z\hat{\ }n)$ *sums ln* $(1 + z)$ (**is** $(\lambda n.\ ?f\ n * z\hat{\ }n)$ *sums* _)
proof −
let $?F = \lambda z.\ \sum n.\ ?f\ n * z\hat{\ }n$ **and** $?F' = \lambda z.\ \sum n.\ diffs\ ?f\ n * z\hat{\ }n$
have *r*: *conv_radius ?f* = *1*
by (*intro conv_radius_ratio_limit_nonzero*[*of* _ *1*])
(*simp_all add*: *norm_divide LIMSEQ_Suc_n_over_n del*: *of_nat_Suc*)

have $\exists c.\ \forall z \in ball\ 0\ 1.\ ln\ (1 + z) - ?F\ z = c$
proof (*rule has_field_derivative_zero_constant*)

fix *z* :: *complex* **assume** *z'*: *z* ∈ *ball 0 1*
hence *z*: *norm z* < *1* **by** *simp*
define *t* :: *complex* **where** *t* = *of_real* (*1* + *norm z*) / *2*
from *z* **have** *t*: *norm z* < *norm t norm t* < *1* **unfolding** *t_def*
by (*simp_all add*: *field_simps norm_divide del*: *of_real_add*)

have *Re* (−*z*) ≤ *norm* (−*z*) **by** (*rule complex_Re_le_cmod*)
also from *z* **have** ... < *1* **by** *simp*
finally have ((λ*z*. *ln* (*1* + *z*)) *has_field_derivative inverse* (*1*+*z*)) (*at z*)
by (*auto intro*!: *derivative_eq_intros simp*: *complex_nonpos_Reals_iff*)
moreover have (*?F has_field_derivative ?F' z*) (*at z*) **using** *t r*
by (*intro termdiffs_strong*[*of _ t*] *summable_in_conv_radius*) *simp_all*
ultimately have ((λ*z*. *ln* (*1* + *z*) − *?F z*) *has_field_derivative* (*inverse* (*1* + *z*) − *?F' z*))
(*at z within ball 0 1*)
by (*intro derivative_intros*) (*simp_all add*: *at_within_open*[*OF z'*])
also have (λ*n*. *of_nat n* ∗ *?f n* ∗ *z* ^ (*n* − *Suc 0*)) *sums ?F' z* **using** *t r*
by (*intro diffs_equiv termdiff_converges*[*OF t*(*1*)] *summable_in_conv_radius*) *simp_all*
from *sums_split_initial_segment*[*OF this*, *of 1*]
have (λ*i*. (−*z*) ^ *i*) *sums ?F' z* **by** (*simp add*: *power_minus*[*of z*] *del*: *of_nat_Suc*)
hence *?F' z* = *inverse* (*1* + *z*) **using** *z* **by** (*simp add*: *sums_iff suminf_geometric divide_inverse*)
also have *inverse* (*1* + *z*) − *inverse* (*1* + *z*) = *0* **by** *simp*
finally show ((λ*z*. *ln* (*1* + *z*) − *?F z*) *has_field_derivative 0*) (*at z within ball 0 1*) **.**
qed *simp_all*
then obtain *c* **where** *c*: ⋀*z*. *z* ∈ *ball 0 1* ⟹ *ln* (*1* + *z*) − *?F z* = *c* **by** *blast*
from *c*[*of 0*] **have** *c* = *0* **by** (*simp only*: *powser_zero*) *simp*
with *c*[*of z*] *assms* **have** *ln* (*1* + *z*) = *?F z* **by** *simp*
moreover have *summable* (λ*n*. *?f n* ∗ *z*^*n*) **using** *assms r*
by (*intro summable_in_conv_radius*) *simp_all*
ultimately show *?thesis* **by** (*simp add*: *sums_iff*)
qed

lemma *Ln_series'*: *cmod z* < *1* ⟹ (λ*n*. − ((−*z*)^*n*) / *of_nat n*) *sums ln* (*1* + *z*)
by (*drule Ln_series*) (*simp add*: *power_minus'*)

lemma *ln_series'*:
assumes *abs* (*x*::*real*) < *1*
shows (λ*n*. − ((−*x*)^*n*) / *of_nat n*) *sums ln* (*1* + *x*)
proof −
from *assms* **have** (λ*n*. − ((−*of_real x*)^*n*) / *of_nat n*) *sums ln* (*1* + *complex_of_real x*)
by (*intro Ln_series'*) *simp_all*
also have (λ*n*. − ((−*of_real x*)^*n*) / *of_nat n*) = (λ*n*. *complex_of_real* (− ((−*x*)^*n*) / *of_nat n*))
by (*rule ext*) *simp*

also from *assms* **have** *ln (1 + complex_of_real x) = of_real (ln (1 + x))*
by *(subst Ln_of_real [symmetric]) simp_all*
finally show *?thesis* **by** *(subst (asm) sums_of_real_iff)*
qed

lemma *Ln_approx_linear*:
fixes *z :: complex*
assumes *norm z < 1*
shows *norm (ln (1 + z) − z) ≤ norm z^2 / (1 − norm z)*
proof −
let *?f = λn. (−1)^Suc n / of_nat n*
from *assms* **have** *(λn. ?f n ∗ z^n) sums ln (1 + z)* **using** *Ln_series* **by** *simp*
moreover have *(λn. (if n = 1 then 1 else 0) ∗ z^n) sums z* **using** *powser_sums_if[of 1]* **by** *simp*
ultimately have *(λn. (?f n − (if n = 1 then 1 else 0)) ∗ z^n) sums (ln (1 + z) − z)*
by *(subst left_diff_distrib, intro sums_diff) simp_all*
from *sums_split_initial_segment[OF this, of Suc 1]*
have *(λi. (−(z^2)) ∗ inverse (2 + of_nat i) ∗ (− z)^i) sums (Ln (1 + z) − z)*
by *(simp add: power2_eq_square mult_ac power_minus[of z] divide_inverse)*
hence *(Ln (1 + z) − z) = (∑ i. (−(z^2)) ∗ inverse (of_nat (i+2)) ∗ (−z)^i)*
by *(simp add: sums_iff)*
also have *A*: *summable (λn. norm z^2 ∗ (inverse (real_of_nat (Suc (Suc n))) ∗ cmod z ^ n))*
by *(rule summable_mult, rule summable_comparison_test_ev[OF _ summable_geometric[of norm z]])*
(auto simp: assms field_simps intro!: always_eventually)
hence *norm (∑ i. (−(z^2)) ∗ inverse (of_nat (i+2)) ∗ (−z)^i)*
≤ (∑ i. norm (−(z^2) ∗ inverse (of_nat (i+2)) ∗ (−z)^i))
by *(intro summable_norm)*
(auto simp: norm_power norm_inverse norm_mult mult_ac simp del: of_nat_add of_nat_Suc)
also have *norm ((−z)^2 ∗ (−z)^i) ∗ inverse (of_nat (i+2)) ≤ norm ((−z)^2 ∗ (−z)^i) ∗ 1* **for** *i*
by *(intro mult_left_mono) (simp_all add: field_split_simps)*
hence *(∑ i. norm (−(z^2) ∗ inverse (of_nat (i+2)) ∗ (−z)^i))*
≤ (∑ i. norm (−(z^2) ∗ (−z)^i))
using *A assms*
unfolding *norm_power norm_inverse norm_divide norm_mult*
apply *(intro suminf_le summable_mult summable_geometric)*
apply *(auto simp: norm_power field_simps simp del: of_nat_add of_nat_Suc)*
done
also have ... *= norm z^2 ∗ (∑ i. norm z^i)* **using** *assms*
by *(subst suminf_mult [symmetric]) (auto intro!: summable_geometric simp: norm_mult norm_power)*
also have *(∑ i. norm z^i) = inverse (1 − norm z)* **using** *assms*
by *(subst suminf_geometric) (simp_all add: divide_inverse)*
also have *norm z^2 ∗ ... = norm z^2 / (1 − norm z)* **by** *(simp add: divide_inverse)*

```
  finally show ?thesis .
qed
```

6.21.12 Quadrant-type results for Ln

```
lemma cos_lt_zero_pi: pi/2 < x ⟹ x < 3*pi/2 ⟹ cos x < 0
  using cos_minus_pi cos_gt_zero_pi [of x−pi]
  by simp

lemma Re_Ln_pos_lt:
  assumes z ≠ 0
    shows |Im(Ln z)| < pi/2 ⟷ 0 < Re(z)
proof −
  { fix w
    assume w = Ln z
    then have w: Im w ≤ pi − pi < Im w
      using Im_Ln_le_pi [of z]  mpi_less_Im_Ln [of z]  assms
      by auto
    have |Im w| < pi/2 ⟷ 0 < Re(exp w)
    proof
      assume |Im w| < pi/2 then show 0 < Re(exp w)
        by (auto simp: Re_exp cos_gt_zero_pi split: if_split_asm)
    next
      assume R: 0 < Re(exp w) then
      have |Im w| ≠ pi/2
        by (metis cos_minus cos_pi_half mult_eq_0_iff Re_exp abs_if order_less_irrefl)
      then show |Im w| < pi/2
        using cos_lt_zero_pi [of −(Im w)] cos_lt_zero_pi [of (Im w)] w R
        by (force simp: Re_exp zero_less_mult_iff abs_if not_less_iff_gr_or_eq)
    qed
  }
  then show ?thesis using assms
    by auto
qed

lemma Re_Ln_pos_le:
  assumes z ≠ 0
    shows |Im(Ln z)| ≤ pi/2 ⟷ 0 ≤ Re(z)
proof −
  { fix w
    assume w = Ln z
    then have w: Im w ≤ pi − pi < Im w
      using Im_Ln_le_pi [of z]  mpi_less_Im_Ln [of z]  assms
      by auto
    then have |Im w| ≤ pi/2 ⟷ 0 ≤ Re(exp w)
      using cos_lt_zero_pi [of − (Im w)] cos_lt_zero_pi [of (Im w)] not_le
      by (auto simp: Re_exp zero_le_mult_iff abs_if intro: cos_ge_zero)
  }
  then show ?thesis using assms
```

by *auto*
qed

lemma *Im_Ln_pos_lt*:
assumes $z \neq 0$
shows $0 < Im(Ln\ z) \wedge Im(Ln\ z) < pi \longleftrightarrow 0 < Im(z)$
proof −
{ **fix** *w*
assume $w = Ln\ z$
then have *w*: $Im\ w \leq pi - pi < Im\ w$
using *Im_Ln_le_pi* [*of z*] *mpi_less_Im_Ln* [*of z*] *assms*
by *auto*
then have $0 < Im\ w \wedge Im\ w < pi \longleftrightarrow 0 < Im(exp\ w)$
using *sin_gt_zero* [*of* − (*Im w*)] *sin_gt_zero* [*of* (*Im w*)] *less_linear*
by (*fastforce simp add*: *Im_exp zero_less_mult_iff*)
}
then show *?thesis* **using** *assms*
by *auto*
qed

lemma *Im_Ln_pos_le*:
assumes $z \neq 0$
shows $0 \leq Im(Ln\ z) \wedge Im(Ln\ z) \leq pi \longleftrightarrow 0 \leq Im(z)$
proof −
{ **fix** *w*
assume $w = Ln\ z$
then have *w*: $Im\ w \leq pi - pi < Im\ w$
using *Im_Ln_le_pi* [*of z*] *mpi_less_Im_Ln* [*of z*] *assms*
by *auto*
then have $0 \leq Im\ w \wedge Im\ w \leq pi \longleftrightarrow 0 \leq Im(exp\ w)$
using *sin_ge_zero* [*of* − (*Im w*)] *sin_ge_zero* [*of abs*(*Im w*)] *sin_zero_pi_iff* [*of Im w*]
by (*force simp*: *Im_exp zero_le_mult_iff sin_ge_zero*) }
then show *?thesis* **using** *assms*
by *auto*
qed

lemma *Re_Ln_pos_lt_imp*: $0 < Re(z) \Longrightarrow |Im(Ln\ z)| < pi/2$
by (*metis Re_Ln_pos_lt less_irrefl zero_complex.simps*(*1*))

lemma *Im_Ln_pos_lt_imp*: $0 < Im(z) \Longrightarrow 0 < Im(Ln\ z) \wedge Im(Ln\ z) < pi$
by (*metis Im_Ln_pos_lt not_le order_refl zero_complex.simps*(*2*))

A reference to the set of positive real numbers

lemma *Im_Ln_eq_0*: $z \neq 0 \Longrightarrow (Im(Ln\ z) = 0 \longleftrightarrow 0 < Re(z) \wedge Im(z) = 0)$
by (*metis Im_complex_of_real Im_exp Ln_in_Reals Re_Ln_pos_lt Re_Ln_pos_lt_imp Re_complex_of_real complex_is_Real_iff exp_Ln exp_of_real pi_gt_zero*)

lemma *Im_Ln_eq_pi*: $z \neq 0 \Longrightarrow (Im(Ln\ z) = pi \longleftrightarrow Re(z) < 0 \wedge Im(z) = 0)$
by (*metis Im_Ln_eq_0 Im_Ln_pos_le Im_Ln_pos_lt add.left_neutral complex_eq less_eq_real_def mult_zero_right not_less_iff_gr_or_eq pi_ge_zero pi_neq_zero rcis_zero_arg rcis_zero_mod*)

6.21.13 More Properties of Ln

lemma *cnj_Ln*: **assumes** $z \notin \mathbb{R}_{\leq 0}$ **shows** $cnj(Ln\ z) = Ln(cnj\ z)$
proof (*cases z=0*)
  case *False*
  show *?thesis*
  proof (*rule exp_complex_eqI*)
    have $|Im\ (cnj\ (Ln\ z)) - Im\ (Ln\ (cnj\ z))| \leq |Im\ (cnj\ (Ln\ z))| + |Im\ (Ln\ (cnj\ z))|$
      by (*rule abs_triangle_ineq4*)
    also have $\ldots < pi + pi$
    proof –
      have $|Im\ (cnj\ (Ln\ z))| < pi$
       by (*simp add: False Im_Ln_less_pi abs_if assms minus_less_iff mpi_less_Im_Ln*)
      moreover have $|Im\ (Ln\ (cnj\ z))| \leq pi$
        by (*meson abs_le_iff complex_cnj_zero_iff less_eq_real_def minus_less_iff False Im_Ln_le_pi mpi_less_Im_Ln*)
      ultimately show *?thesis*
        by *simp*
    qed
    finally show $|Im\ (cnj\ (Ln\ z)) - Im\ (Ln\ (cnj\ z))| < 2 * pi$
      by *simp*
    show $exp\ (cnj\ (Ln\ z)) = exp\ (Ln\ (cnj\ z))$
      by (*metis False complex_cnj_zero_iff exp_Ln exp_cnj*)
  qed
qed (*use assms* **in** *auto*)

lemma *Ln_inverse*: **assumes** $z \notin \mathbb{R}_{\leq 0}$ **shows** $Ln(inverse\ z) = -(Ln\ z)$
proof (*cases z=0*)
  case *False*
  show *?thesis*
  proof (*rule exp_complex_eqI*)
    have $|Im\ (Ln\ (inverse\ z)) - Im\ (-\ Ln\ z)| \leq |Im\ (Ln\ (inverse\ z))| + |Im\ (-\ Ln\ z)|$
      by (*rule abs_triangle_ineq4*)
    also have $\ldots < pi + pi$
    proof –
      have $|Im\ (Ln\ (inverse\ z))| < pi$
       by (*simp add: False Im_Ln_less_pi abs_if assms minus_less_iff mpi_less_Im_Ln*)
      moreover have $|Im\ (-\ Ln\ z)| \leq pi$
        using *False Im_Ln_le_pi mpi_less_Im_Ln* **by** *fastforce*
      ultimately show *?thesis*
        by *simp*
    qed

finally show $|Im\ (Ln\ (inverse\ z)) - Im\ (-\ Ln\ z)| < 2 * pi$
by *simp*
show $exp\ (Ln\ (inverse\ z)) = exp\ (-\ Ln\ z)$
by (*simp add*: *False exp_minus*)
qed
qed (*use assms* **in** *auto*)

lemma *Ln_minus1* [*simp*]: $Ln(-1) = \mathrm{i} * pi$
proof (*rule exp_complex_eqI*)
show $|Im\ (Ln\ (-\ 1)) - Im\ (\mathrm{i} * pi)| < 2 * pi$
using *Im_Ln_le_pi* [*of* −*1*] *mpi_less_Im_Ln* [*of* −*1*] **by** *auto*
qed *auto*

lemma *Ln_ii* [*simp*]: $Ln\ \mathrm{i} = \mathrm{i} * of\_real\ pi/2$
using *Ln_exp* [*of* i ∗ (*of_real pi*/*2*)]
unfolding *exp_Euler*
by *simp*

lemma *Ln_minus_ii* [*simp*]: $Ln(-\mathrm{i}) = -\ (\mathrm{i} * pi/2)$
proof −
have $Ln(-\mathrm{i}) = Ln(inverse\ \mathrm{i})$ **by** *simp*
also have ... $= -\ (Ln\ \mathrm{i})$ **using** *Ln_inverse* **by** *blast*
also have ... $= -\ (\mathrm{i} * pi/2)$ **by** *simp*
finally show *?thesis* **.**
qed

lemma *Ln_times*:
assumes $w \neq 0\ z \neq 0$
shows $Ln(w * z) =$
$(if\ Im(Ln\ w + Ln\ z) \leq -pi\ then\ (Ln(w) + Ln(z)) + \mathrm{i} * of\_real(2*pi)$
$else\ if\ Im(Ln\ w + Ln\ z) > pi\ then\ (Ln(w) + Ln(z)) - \mathrm{i} * of\_real(2*pi)$
$else\ Ln(w) + Ln(z))$
using *pi_ge_zero Im_Ln_le_pi* [*of w*] *Im_Ln_le_pi* [*of z*]
using *assms mpi_less_Im_Ln* [*of w*] *mpi_less_Im_Ln* [*of z*]
by (*auto simp*: *exp_add exp_diff sin_double cos_double exp_Euler intro*!: *Ln_unique*)

corollary *Ln_times_simple*:
$[\![w \neq 0;\ z \neq 0;\ -pi < Im(Ln\ w) + Im(Ln\ z);\ Im(Ln\ w) + Im(Ln\ z) \leq pi]\!]$
$\Longrightarrow Ln(w * z) = Ln(w) + Ln(z)$
by (*simp add*: *Ln_times*)

corollary *Ln_times_of_real*:
$[\![r > 0;\ z \neq 0]\!] \Longrightarrow Ln(of\_real\ r * z) = ln\ r + Ln(z)$
using *mpi_less_Im_Ln Im_Ln_le_pi*
by (*force simp*: *Ln_times*)

corollary *Ln_times_Reals*:
$[\![r \in Reals;\ Re\ r > 0;\ z \neq 0]\!] \Longrightarrow Ln(r * z) = ln\ (Re\ r) + Ln(z)$
using *Ln_Reals_eq Ln_times_of_real* **by** *fastforce*

corollary *Ln_divide_of_real*:
  $[\![r > 0;\ z \neq 0]\!] \Longrightarrow Ln(z\ /\ of\_real\ r) = Ln(z) - ln\ r$
using *Ln_times_of_real* [*of inverse r z*]
by (*simp add*: *ln_inverse Ln_of_real mult.commute divide_inverse of_real_inverse* [*symmetric*]
      del: *of_real_inverse*)

corollary *Ln_prod*:
  fixes $f :: 'a \Rightarrow complex$
  assumes *finite A* $\bigwedge x.\ x \in A \Longrightarrow f\ x \neq 0$
  shows $\exists n.\ Ln\ (prod\ f\ A) = (\sum x \in A.\ Ln\ (f\ x) + (of\_int\ (n\ x) * (2 * pi)) * \mathrm{i})$
  using *assms*
proof (*induction A*)
  case (*insert x A*)
  then obtain *n* **where** *n*: $Ln\ (prod\ f\ A) = (\sum x{\in}A.\ Ln\ (f\ x) + of\_real\ (of\_int\ (n\ x) * (2 * pi)) * \mathrm{i})$
    by *auto*
  define *D* **where** $D \equiv Im\ (Ln\ (f\ x)) + Im\ (Ln\ (prod\ f\ A))$
  define *q*::*int* **where** $q \equiv (if\ D \leq -pi\ then\ 1\ else\ if\ D > pi\ then\ -1\ else\ 0)$
  have $prod\ f\ A \neq 0\ f\ x \neq 0$
    by (*auto simp*: *insert.hyps insert.prems*)
  with *insert.hyps pi_ge_zero* **show** *?case*
    by (*rule_tac x=n(x:=q)* **in** *exI*) (*force simp*: *Ln_times q_def D_def n intro*!: *sum.cong*)
qed *auto*

lemma *Ln_minus*:
  assumes $z \neq 0$
    shows $Ln(-z) = (if\ Im(z) \leq 0 \wedge \neg(Re(z) < 0 \wedge Im(z) = 0)$
                 $then\ Ln(z) + \mathrm{i} * pi$
                 $else\ Ln(z) - \mathrm{i} * pi)$ (**is** _ = *?rhs*)
  using *Im_Ln_le_pi* [*of z*] *mpi_less_Im_Ln* [*of z*] *assms*
      Im_Ln_eq_pi [*of z*] *Im_Ln_pos_lt* [*of z*]
    by (*fastforce simp*: *exp_add exp_diff exp_Euler intro*!: *Ln_unique*)

lemma *Ln_inverse_if*:
  assumes $z \neq 0$
    shows $Ln\ (inverse\ z) = (if\ z \in \mathbb{R}_{\leq 0}\ then\ -(Ln\ z) + \mathrm{i} * 2 * complex\_of\_real\ pi\ else\ -(Ln\ z))$
proof ($cases\ z \in \mathbb{R}_{\leq 0}$)
  case *False* **then show** *?thesis*
    by (*simp add*: *Ln_inverse*)
next
  case *True*
  then have *z*: $Im\ z = 0\ Re\ z < 0\ - z \notin \mathbb{R}_{\leq 0}$
    using *assms complex_eq_iff complex_nonpos_Reals_iff* **by** *auto*
  have $Ln(inverse\ z) = Ln(-\ (inverse\ (-z)))$
    by *simp*

```
  also have ... = Ln (inverse (−z)) + i ∗ complex_of_real pi
    using assms z by (simp add: Ln_minus divide_less_0_iff)
  also have ... = − Ln (− z) + i ∗ complex_of_real pi
    using z Ln_inverse by presburger
  also have ... = − (Ln z) + i ∗ 2 ∗ complex_of_real pi
    using Ln_minus assms z by auto
  finally show ?thesis by (simp add: True)
qed

lemma Ln_times_ii:
  assumes z ≠ 0
    shows  Ln(i ∗ z) = (if 0 ≤ Re(z) | Im(z) < 0
                        then Ln(z) + i ∗ of_real pi/2
                        else Ln(z) − i ∗ of_real(3 ∗ pi/2))
  using Im_Ln_le_pi [of z] mpi_less_Im_Ln [of z] assms
        Im_Ln_eq_pi [of z] Im_Ln_pos_lt [of z] Re_Ln_pos_le [of z]
  by (simp add: Ln_times) auto

lemma Ln_of_nat [simp]: 0 < n =⇒ Ln (of_nat n) = of_real (ln (of_nat n))
  by (subst of_real_of_nat_eq[symmetric], subst Ln_of_real[symmetric]) simp_all

lemma Ln_of_nat_over_of_nat:
  assumes m > 0 n > 0
  shows   Ln (of_nat m / of_nat n) = of_real (ln (of_nat m) − ln (of_nat n))
proof −
  have of_nat m / of_nat n = (of_real (of_nat m / of_nat n) :: complex) by simp
  also from assms have Ln ... = of_real (ln (of_nat m / of_nat n))
    by (simp add: Ln_of_real[symmetric])
  also from assms have ... = of_real (ln (of_nat m) − ln (of_nat n))
    by (simp add: ln_div)
  finally show ?thesis .
qed
```

6.21.14 The Argument of a Complex Number

Finally: it's is defined for the same interval as the complex logarithm: $(-\pi,\pi]$.

```
definition Arg :: complex ⇒ real where Arg z ≡ (if z = 0 then 0 else Im (Ln z))

lemma Arg_of_real: Arg (of_real r) = (if r<0 then pi else 0)
  by (simp add: Im_Ln_eq_pi Arg_def)

lemma mpi_less_Arg: −pi < Arg z
    and Arg_le_pi: Arg z ≤ pi
  by (auto simp: Arg_def mpi_less_Im_Ln Im_Ln_le_pi)

lemma
  assumes z ≠ 0
  shows Arg_eq: z = of_real(norm z) ∗ exp(i ∗ Arg z)
  using assms exp_Ln exp_eq_polar
```

```
  by (auto simp:  Arg_def)

lemma is_Arg_Arg: z ≠ 0 ⟹ is_Arg z (Arg z)
  by (simp add: Arg_eq is_Arg_def)

lemma Argument_exists:
  assumes z ≠ 0 and R: R = {r−pi<..r+pi}
  obtains s where is_Arg z s s∈R
proof −
  let ?rp = r − Arg z + pi
  define k where k ≡ ⌊?rp / (2 ∗ pi)⌋
  have (Arg z + of_int k ∗ (2 ∗ pi)) ∈ R
    using floor_divide_lower [of 2∗pi ?rp] floor_divide_upper [of 2∗pi ?rp]
    by (auto simp: k_def algebra_simps R)
  then show ?thesis
    using Arg_eq ‹z ≠ 0› is_Arg_2pi_iff is_Arg_def that by blast
qed

lemma Argument_exists_unique:
  assumes z ≠ 0 and R: R = {r−pi<..r+pi}
  obtains s where is_Arg z s s∈R ⋀t. ⟦is_Arg z t; t∈R⟧ ⟹ s=t
proof −
  obtain s where s: is_Arg z s s∈R
    using Argument_exists [OF assms] .
  moreover have ⋀t. ⟦is_Arg z t; t∈R⟧ ⟹ s=t
    using assms s  by (auto simp: is_Arg_eqI)
  ultimately show thesis
    using that by blast
qed

lemma Argument_Ex1:
  assumes z ≠ 0 and R: R = {r−pi<..r+pi}
  shows ∃!s. is_Arg z s ∧ s ∈ R
  using Argument_exists_unique [OF assms]  by metis

lemma Arg_divide:
  assumes w ≠ 0 z ≠ 0
  shows is_Arg (z / w) (Arg z − Arg w)
  using Arg_eq [of z] Arg_eq [of w] Arg_eq [of norm(z / w)] assms
  by (auto simp: is_Arg_def norm_divide field_simps exp_diff Arg_of_real)

lemma Arg_unique_lemma:
  assumes z:  is_Arg z t
      and z': is_Arg z t'
      and t:  − pi < t  t ≤ pi
      and t': − pi < t' t' ≤ pi
      and nz: z ≠ 0
    shows t' = t
  using Arg2pi_unique_lemma [of − (inverse z)]
```

proof −
  have *pi* − *t'* = *pi* − *t*
  proof (*rule Arg2pi_unique_lemma* [*of* − (*inverse z*)])
    have − (*inverse z*) = − (*inverse* (*of_real*(*norm z*) ∗ *exp*(i ∗ *t*)))
      by (*metis is_Arg_def z*)
    also have ... = (*cmod* (− *inverse z*)) ∗ *exp* (i ∗ (*pi* − *t*))
      by (*auto simp*: *field_simps exp_diff norm_divide*)
    finally show *is_Arg* (− *inverse z*) (*pi* − *t*)
      unfolding *is_Arg_def* **.**
    have − (*inverse z*) = − (*inverse* (*of_real*(*norm z*) ∗ *exp*(i ∗ *t'*)))
      by (*metis is_Arg_def z'*)
    also have ... = (*cmod* (− *inverse z*)) ∗ *exp* (i ∗ (*pi* − *t'*))
      by (*auto simp*: *field_simps exp_diff norm_divide*)
    finally show *is_Arg* (− *inverse z*) (*pi* − *t'*)
      unfolding *is_Arg_def* **.**
  qed (*use assms* **in** *auto*)
  then show *?thesis*
    by *simp*
qed

lemma *complex_norm_eq_1_exp_eq*: *norm z* = *1* ⟷ *exp*(i ∗ (*Arg z*)) = *z*
  by (*metis Arg_eq exp_not_eq_zero exp_zero mult.left_neutral norm_zero of_real_1 norm_exp_i_times*)

lemma *Arg_unique*: ⟦*of_real r* ∗ *exp*(i ∗ *a*) = *z*; *0* < *r*; −*pi* < *a*; *a* ≤ *pi*⟧ ⟹ *Arg z* = *a*
  by (*rule Arg_unique_lemma* [*unfolded is_Arg_def*, *OF* _ *Arg_eq*])
    (*use mpi_less_Arg Arg_le_pi* **in** ‹*auto simp*: *norm_mult*›)

lemma *Arg_minus*:
  assumes *z* ≠ *0*
  shows *Arg* (−*z*) = (*if Arg z* ≤ *0 then Arg z* + *pi else Arg z* − *pi*)
proof −
  have [*simp*]: *cmod z* ∗ *cos* (*Arg z*) = *Re z*
    using *assms Arg_eq* [*of z*] **by** (*metis Re_exp exp_Ln norm_exp_eq_Re Arg_def*)
  have [*simp*]: *cmod z* ∗ *sin* (*Arg z*) = *Im z*
    using *assms Arg_eq* [*of z*] **by** (*metis Im_exp exp_Ln norm_exp_eq_Re Arg_def*)
  show *?thesis*
    apply (*rule Arg_unique* [*of norm z*, *OF complex_eqI*])
    using *mpi_less_Arg* [*of z*] *Arg_le_pi* [*of z*] *assms*
    by (*auto simp*: *Re_exp Im_exp*)
qed

lemma *Arg_times_of_real* [*simp*]:
  assumes *0* < *r* **shows** *Arg* (*of_real r* ∗ *z*) = *Arg z*
proof (*cases z*=*0*)
  case *True*
  then show *?thesis*
    by (*simp add*: *Arg_def*)

next
  case *False*
  with *Arg_eq assms* **show** *?thesis*
  by (*auto simp*: *mpi_less_Arg Arg_le_pi intro*!: *Arg_unique* [*of r* ∗ *norm z*])
qed

lemma *Arg_times_of_real2* [*simp*]: *0* < *r* ⟹ *Arg* (*z* ∗ *of_real r*) = *Arg z*
  by (*metis Arg_times_of_real mult.commute*)

lemma *Arg_divide_of_real* [*simp*]: *0* < *r* ⟹ *Arg* (*z* / *of_real r*) = *Arg z*
  by (*metis Arg_times_of_real2 less_numeral_extra*(*3*) *nonzero_eq_divide_eq of_real_eq_0_iff*)

lemma *Arg_less_0*: *0* ≤ *Arg z* ⟷ *0* ≤ *Im z*
  using *Im_Ln_le_pi Im_Ln_pos_le*
  by (*simp add*: *Arg_def*)

lemma *Arg_eq_pi*: *Arg z* = *pi* ⟷ *Re z* < *0* ∧ *Im z* = *0*
  by (*auto simp*: *Arg_def Im_Ln_eq_pi*)

lemma *Arg_lt_pi*: *0* < *Arg z* ∧ *Arg z* < *pi* ⟷ *0* < *Im z*
  using *Arg_less_0* [*of z*] *Im_Ln_pos_lt*
  by (*auto simp*: *order.order_iff_strict Arg_def*)

lemma *Arg_eq_0*: *Arg z* = *0* ⟷ *z* ∈ ℝ ∧ *0* ≤ *Re z*
  using *complex_is_Real_iff*
  by (*simp add*: *Arg_def Im_Ln_eq_0*) (*metis less_eq_real_def of_real_Re of_real_def scale_zero_left*)

corollary *Arg_ne_0*: **assumes** $z \notin \mathbb{R}_{\geq 0}$ **shows** *Arg z* ≠ *0*
  using *assms* **by** (*auto simp*: *nonneg_Reals_def Arg_eq_0*)

lemma *Arg_eq_pi_iff*: *Arg z* = *pi* ⟷ *z* ∈ ℝ ∧ *Re z* < *0*
proof (*cases z*=*0*)
  case *False*
  then show *?thesis*
    using *Arg_eq_0* [*of* −*z*] *Arg_eq_pi complex_is_Real_iff* **by** *blast*
qed (*simp add*: *Arg_def*)

lemma *Arg_eq_0_pi*: *Arg z* = *0* ∨ *Arg z* = *pi* ⟷ *z* ∈ ℝ
  using *Arg_eq_pi_iff Arg_eq_0* **by** *force*

lemma *Arg_real*: *z* ∈ ℝ ⟹ *Arg z* = (*if 0* ≤ *Re z then 0 else pi*)
  using *Arg_eq_0 Arg_eq_0_pi* **by** *auto*

lemma *Arg_inverse*: *Arg*(*inverse z*) = (*if z* ∈ ℝ *then Arg z else* − *Arg z*)
proof (*cases z* ∈ ℝ)
  case *True*
  then show *?thesis*
    by *simp* (*metis Arg2pi_inverse Arg2pi_real Arg_real Reals_inverse*)

next
  case *False*
  then have *z*: *Arg z* < *pi z* ≠ *0*
    using *Arg_eq_0_pi Arg_le_pi* **by** (*auto simp*: *less_eq_real_def*)
  show *?thesis*
    apply (*rule Arg_unique* [*of inverse* (*norm z*)])
    using *False z mpi_less_Arg* [*of z*] *Arg_eq* [*of z*]
    by (*auto simp*: *exp_minus field_simps*)
qed

lemma *Arg_eq_iff*:
  assumes *w* ≠ *0 z* ≠ *0*
  shows *Arg w* = *Arg z* ⟷ (∃ *x*. *0* < *x* ∧ *w* = *of_real x* ∗ *z*) (**is** *?lhs* = *?rhs*)
proof
  assume *?lhs*
  then have *w* = *complex_of_real* (*cmod w* / *cmod z*) ∗ *z*
   by (*metis Arg_eq assms divide_divide_eq_right eq_divide_eq exp_not_eq_zero of_real_divide*)
  then show *?rhs*
    using *assms divide_pos_pos zero_less_norm_iff* **by** *blast*
qed *auto*

lemma *Arg_inverse_eq_0*: *Arg*(*inverse z*) = *0* ⟷ *Arg z* = *0*
  by (*metis Arg_eq_0 Arg_inverse inverse_inverse_eq*)

lemma *Arg_cnj_eq_inverse*: *z*≠*0* ⟹ *Arg* (*cnj z*) = *Arg* (*inverse z*)
  using *Arg2pi_cnj_eq_inverse Arg2pi_eq_iff Arg_eq_iff* **by** *auto*

lemma *Arg_cnj*: *Arg*(*cnj z*) = (*if z* ∈ ℝ *then Arg z else* − *Arg z*)
  by (*metis Arg_cnj_eq_inverse Arg_inverse Reals_0 complex_cnj_zero*)

lemma *Arg_exp*: −*pi* < *Im z* ⟹ *Im z* ≤ *pi* ⟹ *Arg*(*exp z*) = *Im z*
  by (*rule Arg_unique* [*of exp*(*Re z*)]) (*auto simp*: *exp_eq_polar*)

lemma *Ln_Arg*: *z*≠*0* ⟹ *Ln*(*z*) = *ln*(*norm z*) + i ∗ *Arg*(*z*)
  by (*metis Arg_def Re_Ln complex_eq*)

lemma *continuous_at_Arg*:
  assumes *z* ∉ $\mathbb{R}_{\leq 0}$
    shows *continuous* (*at z*) *Arg*
proof −
  have [*simp*]: (λ*z*. *Im* (*Ln z*)) −*z*→ *Arg z*
    using *Arg_def assms continuous_at* **by** *fastforce*
  show *?thesis*
    unfolding *continuous_at*
  proof (*rule Lim_transform_within_open*)
    show ⋀*w*. ⟦*w* ∈ − $\mathbb{R}_{\leq 0}$; *w* ≠ *z*⟧ ⟹ *Im* (*Ln w*) = *Arg w*
      by (*metis Arg_def Compl_iff nonpos_Reals_zero_I*)
  qed (*use assms* **in** *auto*)
qed

lemma *continuous_within_Arg*: $z \notin \mathbb{R}_{\leq 0} \Longrightarrow$ *continuous* (*at z within S*) *Arg*
  using *continuous_at_Arg continuous_at_imp_continuous_within* **by** *blast*

6.21.15 The Unwinding Number and the Ln product Formula

Note that in this special case the unwinding number is -1, 0 or 1. But it's always an integer.

lemma *is_Arg_exp_Im*: *is_Arg* (*exp z*) (*Im z*)
  using *exp_eq_polar is_Arg_def norm_exp_eq_Re* **by** *auto*

lemma *is_Arg_exp_diff_2pi*:
  assumes *is_Arg* (*exp z*) ϑ
  shows $\exists k$. *Im z* − *of_int k* ∗ (*2* ∗ *pi*) = ϑ
proof (*intro exI is_Arg_eqI*)
  let *?k* = ⌊(*Im z* − ϑ) / (*2* ∗ *pi*)⌋
  show *is_Arg* (*exp z*) (*Im z* − *real_of_int ?k* ∗ (*2* ∗ *pi*))
    by (*metis diff_add_cancel is_Arg_2pi_iff is_Arg_exp_Im*)
  show |*Im z* − *real_of_int ?k* ∗ (*2* ∗ *pi*) − ϑ| < *2* ∗ *pi*
    using *floor_divide_upper* [*of 2*∗*pi Im z* − ϑ] *floor_divide_lower* [*of 2*∗*pi Im z* − ϑ]
    by (*auto simp*: *algebra_simps abs_if*)
qed (*auto simp*: *is_Arg_exp_Im assms*)

lemma *Arg_exp_diff_2pi*: $\exists k$. *Im z* − *of_int k* ∗ (*2* ∗ *pi*) = *Arg* (*exp z*)
  using *is_Arg_exp_diff_2pi* [*OF is_Arg_Arg*] **by** *auto*

lemma *unwinding_in_Ints*: (*z* − *Ln*(*exp z*)) / (*of_real*(*2*∗*pi*) ∗ i) ∈ ℤ
  using *Arg_exp_diff_2pi* [*of z*]
  by (*force simp*: *Ints_def image_def field_simps Arg_def intro*!: *complex_eqI*)

definition *unwinding* :: *complex* ⇒ *int* **where**
  unwinding z ≡ *THE k*. *of_int k* = (*z* − *Ln*(*exp z*)) / (*of_real*(*2*∗*pi*) ∗ i)

lemma *unwinding*: *of_int* (*unwinding z*) = (*z* − *Ln*(*exp z*)) / (*of_real*(*2*∗*pi*) ∗ i)
  using *unwinding_in_Ints* [*of z*]
  unfolding *unwinding_def Ints_def* **by** *force*

lemma *unwinding_2pi*: (*2*∗*pi*) ∗ i ∗ *unwinding*(*z*) = *z* − *Ln*(*exp z*)
  by (*simp add*: *unwinding*)

lemma *Ln_times_unwinding*:
  w ≠ *0* ⟹ *z* ≠ *0* ⟹ *Ln*(*w* ∗ *z*) = *Ln*(*w*) + *Ln*(*z*) − (*2*∗*pi*) ∗ i ∗ *unwinding*(*Ln w* + *Ln z*)
  using *unwinding_2pi* **by** (*simp add*: *exp_add*)

6.21.16 Relation between Ln and Arg2pi, and hence continuity of Arg2pi

lemma *Arg2pi_Ln*:
  assumes *0 < Arg2pi z* **shows** *Arg2pi z = Im(Ln(−z)) + pi*
proof (*cases z = 0*)
  case *True*
  with *assms* **show** *?thesis*
    by *simp*
next
  case *False*
  then have *z / of_real(norm z) = exp*(i ∗ *of_real*(*Arg2pi z*))
    using *Arg2pi* [*of z*]
    by (*metis is_Arg_def abs_norm_cancel nonzero_mult_div_cancel_left norm_of_real zero_less_norm_iff*)
  then have − *z / of_real*(*norm z*) = *exp* (i ∗ (*of_real* (*Arg2pi z*) − *pi*))
    using *cis_conv_exp cis_pi*
    by (*auto simp*: *exp_diff algebra_simps*)
  then have *ln* (− *z / of_real*(*norm z*)) = *ln* (*exp* (i ∗ (*of_real* (*Arg2pi z*) − *pi*)))
    by *simp*
  also have ... = i ∗ (*of_real*(*Arg2pi z*) − *pi*)
    using *Arg2pi* [*of z*] *assms pi_not_less_zero*
    by *auto*
  finally have *Arg2pi z* = *Im* (*Ln* (− *z / of_real* (*cmod z*))) + *pi*
    by *simp*
  also have ... = *Im* (*Ln* (−*z*) − *ln* (*cmod z*)) + *pi*
    by (*metis diff_0_right minus_diff_eq zero_less_norm_iff Ln_divide_of_real False*)
  also have ... = *Im* (*Ln* (−*z*)) + *pi*
    by *simp*
  finally show *?thesis* **.**
qed

lemma *continuous_at_Arg2pi*:
  assumes $z \notin \mathbb{R}_{\geq 0}$
    shows *continuous* (*at z*) *Arg2pi*
proof −
  have ∗: *isCont* (λ*z*. *Im* (*Ln* (− *z*)) + *pi*) *z*
    by (*rule Complex.isCont_Im isCont_Ln′ continuous_intros* | *simp add*: *assms complex_is_Real_iff*)+
  have [*simp*]: *Im x* ≠ *0* ⟹ *Im* (*Ln* (− *x*)) + *pi* = *Arg2pi x* **for** *x*
    using *Arg2pi_Ln* **by** (*simp add*: *Arg2pi_gt_0 complex_nonneg_Reals_iff*)
  consider *Re z* < *0* | *Im z* ≠ *0* **using** *assms*
    using *complex_nonneg_Reals_iff not_le* **by** *blast*
  then have [*simp*]: (λ*z*. *Im* (*Ln* (− *z*)) + *pi*) −*z*→ *Arg2pi z*
    using ∗ **by** (*simp add*: *Arg2pi_Ln Arg2pi_gt_0 assms continuous_within*)
  show *?thesis*
    unfolding *continuous_at*
  proof (*rule Lim_transform_within_open*)
    show $\bigwedge x.$ ⟦$x \in - \mathbb{R}_{\geq 0}$; *x* ≠ *z*⟧ ⟹ *Im* (*Ln* (− *x*)) + *pi* = *Arg2pi x*
      by (*auto simp add*: *Arg2pi_Ln* [*OF Arg2pi_gt_0*] *complex_nonneg_Reals_iff*)

```
  qed (use assms in auto)
qed
```

Relation between Arg2pi and arctangent in upper halfplane

```
lemma Arg2pi_arctan_upperhalf:
  assumes 0 < Im z
    shows Arg2pi z = pi/2 − arctan(Re z / Im z)
proof (cases z = 0)
  case False
  show ?thesis
  proof (rule Arg2pi_unique [of norm z])
    show (cmod z) ∗ exp (i ∗ (pi / 2 − arctan (Re z / Im z))) = z
      apply (rule complex_eqI)
      using assms norm_complex_def [of z, symmetric]
      unfolding exp_Euler cos_diff sin_diff sin_of_real cos_of_real
      by (simp_all add: field_simps real_sqrt_divide sin_arctan cos_arctan)
  qed (use False arctan [of Re z / Im z] in auto)
qed (use assms in auto)

lemma Arg2pi_eq_Im_Ln:
  assumes 0 ≤ Im z 0 < Re z
    shows Arg2pi z = Im (Ln z)
proof (cases Im z = 0)
  case True then show ?thesis
    using Arg2pi_eq_0 Ln_in_Reals assms(2) complex_is_Real_iff by auto
next
  case False
  then have ∗: Arg2pi z > 0
    using Arg2pi_gt_0 complex_is_Real_iff by blast
  then have z ≠ 0
    by auto
  with ∗ assms False show ?thesis
    by (subst Arg2pi_Ln) (auto simp: Ln_minus)
qed

lemma continuous_within_upperhalf_Arg2pi:
  assumes z ≠ 0
    shows continuous (at z within {z. 0 ≤ Im z}) Arg2pi
proof (cases z ∈ ℝ≥0)
  case False then show ?thesis
    using continuous_at_Arg2pi continuous_at_imp_continuous_within by auto
next
  case True
  then have z: z ∈ ℝ 0 < Re z
    using assms by (auto simp: complex_nonneg_Reals_iff complex_is_Real_iff com-
plex_neq_0)
  then have [simp]: Arg2pi z = 0 Im (Ln z) = 0
    by (auto simp: Arg2pi_eq_0 Im_Ln_eq_0 assms complex_is_Real_iff)
  show ?thesis
```

proof (*clarsimp simp add*: *continuous_within Lim_within dist_norm*)
fix *e*::*real*
assume *0* < *e*
moreover have *continuous* (*at z*) (λ*x*. *Im* (*Ln x*))
using *z* **by** (*simp add*: *continuous_at_Ln complex_nonpos_Reals_iff*)
ultimately
obtain *d* **where** *d*: *d*>*0* ⋀*x*. *x* ≠ *z* ⟹ *cmod* (*x* − *z*) < *d* ⟹ |*Im* (*Ln x*)| < *e*
by (*auto simp*: *continuous_within Lim_within dist_norm*)
{ fix *x*
assume *cmod* (*x* − *z*) < *Re z* / *2*
then have |*Re x* − *Re z*| < *Re z* / *2*
by (*metis le_less_trans abs_Re_le_cmod minus_complex.simps*(*1*))
then have *0* < *Re x*
using *z* **by** *linarith*
}
then show ∃ *d*>*0*. ∀ *x*. *0* ≤ *Im x* ⟶ *x* ≠ *z* ∧ *cmod* (*x* − *z*) < *d* ⟶ |*Arg2pi x*| < *e*
apply (*rule_tac x*=*min d* (*Re z* / *2*) **in** *exI*)
using *z d* **by** (*auto simp*: *Arg2pi_eq_Im_Ln*)
qed
qed

lemma *continuous_on_upperhalf_Arg2pi*: *continuous_on* ({*z*. *0* ≤ *Im z*} − {*0*}) *Arg2pi*
unfolding *continuous_on_eq_continuous_within*
by (*metis DiffE Diff_subset continuous_within_subset continuous_within_upperhalf_Arg2pi insertCI*)

lemma *open_Arg2pi2pi_less_Int*:
assumes *0* ≤ *s t* ≤ *2*∗*pi*
shows *open* ({*y*. *s* < *Arg2pi y*} ∩ {*y*. *Arg2pi y* < *t*})
proof −
have *1*: *continuous_on* (*UNIV* − $\mathbb{R}_{\geq 0}$) *Arg2pi*
using *continuous_at_Arg2pi continuous_at_imp_continuous_within*
by (*auto simp*: *continuous_on_eq_continuous_within*)
have *2*: *open* (*UNIV* − $\mathbb{R}_{\geq 0}$ :: *complex set*) **by** (*simp add*: *open_Diff*)
have *open* ({*z*. *s* < *z*} ∩ {*z*. *z* < *t*})
using *open_lessThan* [*of t*] *open_greaterThan* [*of s*]
by (*metis greaterThan_def lessThan_def open_Int*)
moreover have {*y*. *s* < *Arg2pi y*} ∩ {*y*. *Arg2pi y* < *t*} ⊆ − $\mathbb{R}_{\geq 0}$
using *assms* **by** (*auto simp*: *Arg2pi_real complex_nonneg_Reals_iff complex_is_Real_iff*)
ultimately show *?thesis*
using *continuous_imp_open_vimage* [*OF 1 2*, *of* {*z*. *Re z* > *s*} ∩ {*z*. *Re z* < *t*}]
by *auto*
qed

lemma *open_Arg2pi2pi_gt*: *open* {*z*. *t* < *Arg2pi z*}

```
proof (cases t < 0)
  case True then have {z. t < Arg2pi z} = UNIV
    using Arg2pi_ge_0 less_le_trans by auto
  then show ?thesis
    by simp
next
  case False then show ?thesis
    using open_Arg2pi2pi_less_Int [of t 2*pi] Arg2pi_lt_2pi
    by auto
qed

lemma closed_Arg2pi2pi_le: closed {z. Arg2pi z ≤ t}
  using open_Arg2pi2pi_gt [of t]
  by (simp add: closed_def Set.Collect_neg_eq [symmetric] not_le)
```

6.21.17 Complex Powers

```
lemma powr_to_1 [simp]: z powr 1 = (z::complex)
  by (simp add: powr_def)

lemma powr_nat:
  fixes n::nat and z::complex shows z powr n = (if z = 0 then 0 else z^n)
  by (simp add: exp_of_nat_mult powr_def)

lemma norm_powr_real: w ∈ ℝ ⟹ 0 < Re w ⟹ norm(w powr z) = exp(Re z
* ln(Re w))
 using Ln_Reals_eq norm_exp_eq_Re by (auto simp: Im_Ln_eq_0 powr_def norm_complex_def)

lemma powr_complexpow [simp]:
  fixes x::complex shows x ≠ 0 ⟹ x powr (of_nat n) = x^n
  by (induct n) (auto simp: ac_simps powr_add)

lemma powr_complexnumeral [simp]:
  fixes x::complex shows x ≠ 0 ⟹ x powr (numeral n) = x ^ (numeral n)
  by (metis of_nat_numeral powr_complexpow)

lemma cnj_powr:
  assumes Im a = 0 ⟹ Re a ≥ 0
  shows   cnj (a powr b) = cnj a powr cnj b
proof (cases a = 0)
  case False
 with assms have a ∉ ℝ≤0 by (auto simp: complex_eq_iff complex_nonpos_Reals_iff)
  with False show ?thesis by (simp add: powr_def exp_cnj cnj_Ln)
qed simp

lemma powr_real_real:
  assumes w ∈ ℝ z ∈ ℝ 0 < Re w
  shows w powr z = exp(Re z * ln(Re w))
proof –
```

have *w* ≠ *0*
  using *assms* **by** *auto*
with *assms* **show** *?thesis*
  by (*simp add*: *powr_def Ln_Reals_eq of_real_exp*)
qed

lemma *powr_of_real*:
  fixes *x*::*real* **and** *y*::*real*
  shows *0* ≤ *x* ⟹ *of_real x powr* (*of_real y*::*complex*) = *of_real* (*x powr y*)
  by (*simp_all add*: *powr_def exp_eq_polar*)

lemma *powr_of_int*:
  fixes *z*::*complex* **and** *n*::*int*
  assumes *z*≠(*0*::*complex*)
  shows *z powr of_int n* = (*if n*≥*0 then z*ˆ*nat n else inverse* (*z*ˆ*nat* (−*n*)))
  by (*metis assms not_le of_int_of_nat powr_complexpow powr_minus*)

lemma *powr_Reals_eq*: ⟦*x* ∈ ℝ; *y* ∈ ℝ; *Re x* ≥ *0*⟧ ⟹ *x powr y* = *of_real* (*Re x powr Re y*)
  by (*metis of_real_Re powr_of_real*)

lemma *norm_powr_real_mono*:
  ⟦*w* ∈ ℝ; *1* < *Re w*⟧
  ⟹ *cmod*(*w powr z1*) ≤ *cmod*(*w powr z2*) ⟷ *Re z1* ≤ *Re z2*
  by (*auto simp*: *powr_def algebra_simps Reals_def Ln_of_real*)

lemma *powr_times_real*:
  ⟦*x* ∈ ℝ; *y* ∈ ℝ; *0* ≤ *Re x*; *0* ≤ *Re y*⟧
    ⟹ (*x* ∗ *y*) *powr z* = *x powr z* ∗ *y powr z*
  by (*auto simp*: *Reals_def powr_def Ln_times exp_add algebra_simps less_eq_real_def Ln_of_real*)

lemma *Re_powr_le*: *r* ∈ $\mathbb{R}_{\geq 0}$ ⟹ *Re* (*r powr z*) ≤ *Re r powr Re z*
  by (*auto simp*: *powr_def nonneg_Reals_def order_trans* [*OF complex_Re_le_cmod*])

lemma
  fixes *w*::*complex*
  shows *Reals_powr* [*simp*]: ⟦*w* ∈ $\mathbb{R}_{\geq 0}$; *z* ∈ ℝ⟧ ⟹ *w powr z* ∈ ℝ
  and *nonneg_Reals_powr* [*simp*]: ⟦*w* ∈ $\mathbb{R}_{\geq 0}$; *z* ∈ ℝ⟧ ⟹ *w powr z* ∈ $\mathbb{R}_{\geq 0}$
  by (*auto simp*: *nonneg_Reals_def Reals_def powr_of_real*)

lemma *powr_neg_real_complex*:
  (− *of_real x*) *powr a* = (−*1*) *powr* (*of_real* (*sgn x*) ∗ *a*) ∗ *of_real x powr* (*a* :: *complex*)
proof (*cases x* = *0*)
  assume *x*: *x* ≠ *0*
  hence (−*x*) *powr a* = *exp* (*a* ∗ *ln* (−*of_real x*)) **by** (*simp add*: *powr_def*)
  also from *x* **have** *ln* (−*of_real x*) = *Ln* (*of_real x*) + *of_real* (*sgn x*) ∗ *pi* ∗ i
    by (*simp add*: *Ln_minus Ln_of_real*)

also from *x* **have** *exp* (*a* ∗ ...) = *cis pi powr* (*of_real* (*sgn x*) ∗ *a*) ∗ *of_real x powr a*
by (*simp add*: *powr_def exp_add algebra_simps Ln_of_real cis_conv_exp*)
also note *cis_pi*
finally show *?thesis* **by** *simp*
qed *simp_all*

lemma *has_field_derivative_powr*:
fixes *z* :: *complex*
assumes $z \notin \mathbb{R}_{\leq 0}$
shows ((λ*z*. *z powr s*) *has_field_derivative* (*s* ∗ *z powr* (*s* − *1*))) (*at z*)
proof (*cases z*=*0*)
case *False*
then have §: *exp* (*s* ∗ *Ln z*) ∗ *inverse z* = *exp* ((*s* − *1*) ∗ *Ln z*)
by (*simp add*: *divide_complex_def exp_diff left_diff_distrib'*)
show *?thesis*
unfolding *powr_def*
proof (*rule has_field_derivative_transform_within*)
show ((λ*z*. *exp* (*s* ∗ *Ln z*)) *has_field_derivative s* ∗ (*if z* = *0 then 0 else exp* ((*s* − *1*) ∗ *Ln z*)))
(*at z*)
by (*intro derivative_eq_intros* | *simp add*: *assms False* §)+
qed (*use False* **in** *auto*)
qed (*use assms* **in** *auto*)

declare *has_field_derivative_powr*[*THEN DERIV_chain2*, *derivative_intros*]

lemma *has_field_derivative_powr_of_int*:
fixes *z* :: *complex*
assumes *gderiv*:(*g has_field_derivative gd*) (*at z within S*) **and** *g z*≠*0*
shows ((λ*z*. *g z powr of_int n*) *has_field_derivative* (*n* ∗ *g z powr* (*of_int n* − *1*) ∗ *gd*)) (*at z within S*)
proof −
define *dd* **where** *dd* = *of_int n* ∗ *g z powr* (*of_int* (*n* − *1*)) ∗ *gd*
obtain *e* **where** *e*>*0* **and** *e_dist*:∀ *y*∈*S*. *dist z y* < *e* ⟶ *g y* ≠ *0*
using *DERIV_continuous*[*OF gderiv*,*THEN continuous_within_avoid*] ‹*g z*≠*0*› **by** *auto*
have *?thesis* **when** *n*≥*0*
proof −
define *dd'* **where** *dd'* = *of_int n* ∗ *g z* ^ (*nat n* − *1*) ∗ *gd*
have *dd*=*dd'*
proof (*cases n*=*0*)
case *False*
then have *n*−*1* ≥*0* **using** ‹*n*≥*0*› **by** *auto*
then have *g z powr* (*of_int* (*n* − *1*)) = *g z* ^ (*nat n* − *1*)
using *powr_of_int*[*OF* ‹*g z*≠*0*›,*of n*−*1*] **by** (*simp add*: *nat_diff_distrib'*)
then show *?thesis* **unfolding** *dd_def dd'_def* **by** *simp*
qed (*simp add*:*dd_def dd'_def*)
then have ((λ*z*. *g z powr of_int n*) *has_field_derivative dd*) (*at z within S*)

$\longleftrightarrow$ ((λz. g z powr of_int n) has_field_derivative dd′) (at z within S)
by simp
also have ... $\longleftrightarrow$ ((λz. g z ^ nat n) has_field_derivative dd′) (at z within S)
proof (rule has_field_derivative_cong_eventually)
show ∀$_F$ x in at z within S. g x powr of_int n = g x ^ nat n
unfolding eventually_at
apply (rule exI[where x=e])
using powr_of_int that ‹e>0› e_dist by (simp add: dist_commute)
qed (use powr_of_int ‹g z≠0› that in simp)
also have ... unfolding dd′_def using gderiv that
by (auto intro!: derivative_eq_intros)
finally have ((λz. g z powr of_int n) has_field_derivative dd) (at z within S) .
then show ?thesis unfolding dd_def by simp
qed
moreover have ?thesis when n<0
proof −
define dd′ where dd′ = of_int n / g z ^ (nat (1 − n)) * gd
have dd=dd′
proof −
have g z powr of_int (n − 1) = inverse (g z ^ nat (1−n))
using powr_of_int[OF ‹g z≠0›,of n−1] that by auto
then show ?thesis
unfolding dd_def dd′_def by (simp add: divide_inverse)
qed
then have ((λz. g z powr of_int n) has_field_derivative dd) (at z within S)
$\longleftrightarrow$ ((λz. g z powr of_int n) has_field_derivative dd′) (at z within S)
by simp
also have ... $\longleftrightarrow$ ((λz. inverse (g z ^ nat (−n))) has_field_derivative dd′) (at z within S)
proof (rule has_field_derivative_cong_eventually)
show ∀$_F$ x in at z within S. g x powr of_int n = inverse (g x ^ nat (− n))
unfolding eventually_at
apply (rule exI[where x=e])
using powr_of_int that ‹e>0› e_dist by (simp add: dist_commute)
qed (use powr_of_int ‹g z≠0› that in simp)
also have ...
proof −
have nat (− n) + nat (1 − n) − Suc 0 = nat (− n) + nat (− n)
by auto
then show ?thesis
unfolding dd′_def using gderiv that ‹g z≠0›
by (auto intro!: derivative_eq_intros simp add:field_split_simps power_add[symmetric])
qed
finally have ((λz. g z powr of_int n) has_field_derivative dd) (at z within S) .
then show ?thesis unfolding dd_def by simp
qed
ultimately show ?thesis by force
qed

lemma *field_differentiable_powr_of_int*:
  fixes *z* :: *complex*
  assumes *gderiv*: *g field_differentiable* (*at z within S*) **and** $g\ z \neq 0$
  shows (λ*z*. *g z powr of_int n*) *field_differentiable* (*at z within S*)
using *has_field_derivative_powr_of_int assms*(*2*) *field_differentiable_def gderiv* **by** *blast*

lemma *holomorphic_on_powr_of_int* [*holomorphic_intros*]:
  assumes *holf*: *f holomorphic_on S* **and** *0*: ⋀*z*. *z*∈*S* ⟹ $f\ z \neq 0$
  shows (λ*z*. (*f z*) *powr of_int n*) *holomorphic_on S*
proof (*cases n*≥*0*)
  case *True*
  then have *?thesis* ⟷ (λ*z*. (*f z*) ^ *nat n*) *holomorphic_on S*
    by (*metis* (*no_types*, *lifting*) *0 holomorphic_cong powr_of_int*)
  moreover have (λ*z*. (*f z*) ^ *nat n*) *holomorphic_on S*
    using *holf* **by** (*auto intro*: *holomorphic_intros*)
  ultimately show *?thesis* **by** *auto*
next
  case *False*
  then have *?thesis* ⟷ (λ*z*. *inverse* (*f z*) ^ *nat* (−*n*)) *holomorphic_on S*
    by (*metis* (*no_types*, *lifting*) *0 holomorphic_cong power_inverse powr_of_int*)
  moreover have (λ*z*. *inverse* (*f z*) ^ *nat* (−*n*)) *holomorphic_on S*
    using *assms* **by** (*auto intro*!:*holomorphic_intros*)
  ultimately show *?thesis* **by** *auto*
qed

lemma *has_field_derivative_powr_right* [*derivative_intros*]:
    $w \neq 0$ ⟹ ((λ*z*. *w powr z*) *has_field_derivative Ln w* ∗ *w powr z*) (*at z*)
  unfolding *powr_def* **by** (*intro derivative_eq_intros* | *simp*)+

lemma *field_differentiable_powr_right* [*derivative_intros*]:
  fixes *w*::*complex*
  shows $w \neq 0$ ⟹ (λ*z*. *w powr z*) *field_differentiable* (*at z*)
using *field_differentiable_def has_field_derivative_powr_right* **by** *blast*

lemma *holomorphic_on_powr_right* [*holomorphic_intros*]:
  assumes *f holomorphic_on s*
  shows (λ*z*. *w powr* (*f z*)) *holomorphic_on s*
proof (*cases w* = *0*)
  case *False*
  with *assms* **show** *?thesis*
    unfolding *holomorphic_on_def field_differentiable_def*
    by (*metis* (*full_types*) *DERIV_chain′ has_field_derivative_powr_right*)
qed *simp*

lemma *holomorphic_on_divide_gen* [*holomorphic_intros*]:
  assumes *f*: *f holomorphic_on s* **and** *g*: *g holomorphic_on s* **and** *0*: ⋀*z z′*. ⟦*z* ∈ *s*; *z′* ∈ *s*⟧ ⟹ *g z* = *0* ⟷ *g z′* = *0*
shows (λ*z*. *f z* / *g z*) *holomorphic_on s*

proof (*cases* ∃ *z*∈*s*. *g z* = *0*)
  case *True*
  with *0* **have** *g z* = *0* **if** *z* ∈ *s* **for** *z*
    using *that* **by** *blast*
  then show *?thesis*
    using *g holomorphic_transform* **by** *auto*
next
  case *False*
  with *0* **have** *g z* ≠ *0* **if** *z* ∈ *s* **for** *z*
    using *that* **by** *blast*
  with *holomorphic_on_divide* **show** *?thesis*
    using *f g* **by** *blast*
qed

lemma *norm_powr_real_powr*:
  w ∈ ℝ ⟹ *0* ≤ *Re w* ⟹ *cmod* (*w powr z*) = *Re w powr Re z*
  by (*metis dual_order.order_iff_strict norm_powr_real norm_zero of_real_0 of_real_Re powr_def*)

lemma *tendsto_powr_complex*:
  fixes *f g* :: _ ⇒ *complex*
  assumes *a*: *a* ∉ $\mathbb{R}_{\leq 0}$
  assumes *f*: (*f* ⟶ *a*) *F* **and** *g*: (*g* ⟶ *b*) *F*
  shows ((λ*z*. *f z powr g z*) ⟶ *a powr b*) *F*
proof −
  from *a* **have** [*simp*]: *a* ≠ *0* **by** *auto*
  from *f g a* **have** ((λ*z*. *exp* (*g z* ∗ *ln* (*f z*))) ⟶ *a powr b*) *F* (**is** *?P*)
    by (*auto intro*!: *tendsto_intros simp*: *powr_def*)
  also {
    have *eventually* (λ*z*. *z* ≠ *0*) (*nhds a*)
      by (*intro t1_space_nhds*) *simp_all*
    with *f* **have** *eventually* (λ*z*. *f z* ≠ *0*) *F* **using** *filterlim_iff* **by** *blast*
  }
  hence *?P* ⟷ ((λ*z*. *f z powr g z*) ⟶ *a powr b*) *F*
    by (*intro tendsto_cong refl*) (*simp_all add*: *powr_def mult_ac*)
  finally show *?thesis* **.**
qed

lemma *tendsto_powr_complex_0*:
  fixes *f g* :: *'a* ⇒ *complex*
  assumes *f*: (*f* ⟶ *0*) *F* **and** *g*: (*g* ⟶ *b*) *F* **and** *b*: *Re b* > *0*
  shows ((λ*z*. *f z powr g z*) ⟶ *0*) *F*
proof (*rule tendsto_norm_zero_cancel*)
  define *h* **where**
    h = (λ*z*. *if f z* = *0 then 0 else exp* (*Re* (*g z*) ∗ *ln* (*cmod* (*f z*)) + *abs* (*Im* (*g z*)) ∗ *pi*))
  {
    fix *z* :: *'a* **assume** *z*: *f z* ≠ *0*
    define *c* **where** *c* = *abs* (*Im* (*g z*)) ∗ *pi*

from *mpi_less_Im_Ln*[*OF z*] *Im_Ln_le_pi*[*OF z*]
have *abs* (*Im* (*Ln* (*f z*))) ≤ *pi* **by** *simp*
from *mult_left_mono*[*OF this*, *of abs* (*Im* (*g z*))]
have *abs* (*Im* (*g z*) ∗ *Im* (*ln* (*f z*))) ≤ *c* **by** (*simp add*: *abs_mult c_def*)
hence −*Im* (*g z*) ∗ *Im* (*ln* (*f z*)) ≤ *c* **by** *simp*
hence *norm* (*f z powr g z*) ≤ *h z* **by** (*simp add*: *powr_def field_simps h_def c_def*)
}
hence *le*: *norm* (*f z powr g z*) ≤ *h z* **for** *z* **by** (*cases f x* = *0*) (*simp_all add*: *h_def*)

have *g'*: (*g* ⟶ *b*) (*inf F* (*principal* {*z*. *f z* ≠ *0*}))
by (*rule tendsto_mono*[*OF _ g*]) *simp_all*
have ((λ*x*. *norm* (*f x*)) ⟶ *0*) (*inf F* (*principal* {*z*. *f z* ≠ *0*}))
by (*subst tendsto_norm_zero_iff*, *rule tendsto_mono*[*OF _ f*]) *simp_all*
moreover {
have *filterlim* (λ*x*. *norm* (*f x*)) (*principal* {*0*<..}) (*principal* {*z*. *f z* ≠ *0*})
by (*auto simp*: *filterlim_def*)
hence *filterlim* (λ*x*. *norm* (*f x*)) (*principal* {*0*<..})
(*inf F* (*principal* {*z*. *f z* ≠ *0*}))
by (*rule filterlim_mono*) *simp_all*
}
ultimately have *norm*: *filterlim* (λ*x*. *norm* (*f x*)) (*at_right 0*) (*inf F* (*principal* {*z*. *f z* ≠ *0*}))
by (*simp add*: *filterlim_inf at_within_def*)

have *A*: *LIM x inf F* (*principal* {*z*. *f z* ≠ *0*}). *Re* (*g x*) ∗ −*ln* (*cmod* (*f x*)) :> *at_top*
by (*rule filterlim_tendsto_pos_mult_at_top tendsto_intros g' b*
filterlim_compose[*OF filterlim_uminus_at_top_at_bot*] *filterlim_compose*[*OF ln_at_0*] *norm*)+
have *B*: *LIM x inf F* (*principal* {*z*. *f z* ≠ *0*}).
−|*Im* (*g x*)| ∗ *pi* + −(*Re* (*g x*) ∗ *ln* (*cmod* (*f x*))) :> *at_top*
by (*rule filterlim_tendsto_add_at_top tendsto_intros g'*)+ (*insert A*, *simp_all*)
have *C*: (*h* ⟶ *0*) *F* **unfolding** *h_def*
by (*intro filterlim_If tendsto_const filterlim_compose*[*OF exp_at_bot*])
(*insert B*, *auto simp*: *filterlim_uminus_at_bot algebra_simps*)
show ((λ*x*. *norm* (*f x powr g x*)) ⟶ *0*) *F*
by (*rule Lim_null_comparison*[*OF always_eventually C*]) (*insert le*, *auto*)
qed

lemma *tendsto_powr_complex'* [*tendsto_intros*]:
fixes *f g* :: _ ⇒ *complex*
assumes *a* ∉ $\mathbb{R}_{\leq 0}$ ∨ (*a* = *0* ∧ *Re b* > *0*) **and** (*f* ⟶ *a*) *F* (*g* ⟶ *b*) *F*
shows ((λ*z*. *f z powr g z*) ⟶ *a powr b*) *F*
using *assms tendsto_powr_complex tendsto_powr_complex_0* **by** *fastforce*

lemma *tendsto_neg_powr_complex_of_real*:
assumes *filterlim f at_top F* **and** *Re s* < *0*

shows $((\lambda x.\ \mathit{complex\_of\_real}\ (f\ x)\ \mathit{powr}\ s) \longrightarrow \mathit{0})\ F$
proof −
have $((\lambda x.\ \mathit{norm}\ (\mathit{complex\_of\_real}\ (f\ x)\ \mathit{powr}\ s)) \longrightarrow \mathit{0})\ F$
proof (*rule Lim_transform_eventually*)
from *assms*(*1*) **have** $\mathit{eventually}\ (\lambda x.\ f\ x \geq \mathit{0})\ F$
by (*auto simp*: *filterlim_at_top*)
thus $\mathit{eventually}\ (\lambda x.\ f\ x\ \mathit{powr}\ \mathit{Re}\ s = \mathit{norm}\ (\mathit{of\_real}\ (f\ x)\ \mathit{powr}\ s))\ F$
by *eventually_elim* (*simp add*: *norm_powr_real_powr*)
from *assms* **show** $((\lambda x.\ f\ x\ \mathit{powr}\ \mathit{Re}\ s) \longrightarrow \mathit{0})\ F$
by (*intro tendsto_neg_powr*)
qed
thus *?thesis* **by** (*simp add*: *tendsto_norm_zero_iff*)
qed

lemma *tendsto_neg_powr_complex_of_nat*:
assumes *filterlim f at_top F* **and** $\mathit{Re}\ s < \mathit{0}$
shows $((\lambda x.\ \mathit{of\_nat}\ (f\ x)\ \mathit{powr}\ s) \longrightarrow \mathit{0})\ F$
proof −
have $((\lambda x.\ \mathit{of\_real}\ (\mathit{real}\ (f\ x))\ \mathit{powr}\ s) \longrightarrow \mathit{0})\ F$ **using** *assms*(*2*)
by (*intro filterlim_compose*[*OF _ tendsto_neg_powr_complex_of_real*]
filterlim_compose[*OF _ assms*(*1*)] *filterlim_real_sequentially filterlim_ident*)
auto
thus *?thesis* **by** *simp*
qed

lemma *continuous_powr_complex*:
assumes $f\ (\mathit{netlimit}\ F) \notin \mathbb{R}_{\leq 0}$ *continuous F f continuous F g*
shows $\mathit{continuous}\ F\ (\lambda z.\ f\ z\ \mathit{powr}\ g\ z :: \mathit{complex})$
using *assms* **unfolding** *continuous_def* **by** (*intro tendsto_powr_complex*) *simp_all*

lemma *isCont_powr_complex* [*continuous_intros*]:
assumes $f\ z \notin \mathbb{R}_{\leq 0}$ *isCont f z isCont g z*
shows $\mathit{isCont}\ (\lambda z.\ f\ z\ \mathit{powr}\ g\ z :: \mathit{complex})\ z$
using *assms* **unfolding** *isCont_def* **by** (*intro tendsto_powr_complex*) *simp_all*

lemma *continuous_on_powr_complex* [*continuous_intros*]:
assumes $A \subseteq \{z.\ \mathit{Re}\ (f\ z) \geq \mathit{0} \lor \mathit{Im}\ (f\ z) \neq \mathit{0}\}$
assumes $\bigwedge z.\ z \in A \Longrightarrow f\ z = \mathit{0} \Longrightarrow \mathit{Re}\ (g\ z) > \mathit{0}$
assumes *continuous_on A f continuous_on A g*
shows $\mathit{continuous\_on}\ A\ (\lambda z.\ f\ z\ \mathit{powr}\ g\ z)$
unfolding *continuous_on_def*
proof
fix *z* **assume** z: $z \in A$
show $((\lambda z.\ f\ z\ \mathit{powr}\ g\ z) \longrightarrow f\ z\ \mathit{powr}\ g\ z)\ (\mathit{at}\ z\ \mathit{within}\ A)$
proof (*cases* $f\ z = \mathit{0}$)
case *False*
from *assms*(*1*,*2*) *z* **have** $\mathit{Re}\ (f\ z) \geq \mathit{0} \lor \mathit{Im}\ (f\ z) \neq \mathit{0}\ f\ z = \mathit{0} \longrightarrow \mathit{Re}\ (g\ z)$
$> \mathit{0}$ **by** *auto*
with *assms*(*3*,*4*) *z* **show** *?thesis*

```
    by (intro tendsto_powr_complex')
      (auto elim!: nonpos_Reals_cases simp: complex_eq_iff continuous_on_def)
  next
    case True
    with assms z show ?thesis
      by (auto intro!: tendsto_powr_complex_0 simp: continuous_on_def)
  qed
qed
```

6.21.18 Some Limits involving Logarithms

```
lemma lim_Ln_over_power:
  fixes s::complex
  assumes 0 < Re s
    shows (λn. Ln (of_nat n) / of_nat n powr s) ⟶ 0
proof (simp add: lim_sequentially dist_norm, clarify)
  fix e::real
  assume e: 0 < e
  have ∃xo>0. ∀x≥xo. 0 < e * 2 + (e * Re s * 2 − 2) * x + e * (Re s)² * x²
  proof (rule_tac x=2/(e * (Re s)²) in exI, safe)
    show 0 < 2 / (e * (Re s)²)
      using e assms by (simp add: field_simps)
  next
    fix x::real
    assume x: 2 / (e * (Re s)²) ≤ x
    have 2 / (e * (Re s)²) > 0
      using e assms by simp
    with x have x > 0
      by linarith
    then have x * 2 ≤ e * (x² * (Re s)²)
      using e assms x by (auto simp: power2_eq_square field_simps)
    also have ... < e * (2 + (x * (Re s * 2) + x² * (Re s)²))
      using e assms ‹x > 0›
      by (auto simp: power2_eq_square field_simps add_pos_pos)
    finally show 0 < e * 2 + (e * Re s * 2 − 2) * x + e * (Re s)² * x²
      by (auto simp: algebra_simps)
  qed
  then have ∃xo>0. ∀x≥xo. x / e < 1 + (Re s * x) + (1/2) * (Re s * x)^2
    using e by (simp add: field_simps)
  then have ∃xo>0. ∀x≥xo. x / e < exp (Re s * x)
    using assms
    by (force intro: less_le_trans [OF _ exp_lower_Taylor_quadratic])
  then obtain xo where xo > 0 and xo: ⋀x. x ≥ xo ⟹ x < e * exp (Re s * x)
    using e by (auto simp: field_simps)
  have norm (Ln (of_nat n) / of_nat n powr s) < e if n ≥ nat ⌈exp xo⌉ for n
  proof -
    have ln (real n) ≥ xo
      using that exp_gt_zero ln_ge_iff [of n] nat_ceiling_le_eq by fastforce
    then show ?thesis
```

using *e xo* [*of ln n*] **by** (*auto simp*: *norm_divide norm_powr_real field_split_simps*)
qed
then show ∃ *no*. ∀ *n*≥*no*. *norm* (*Ln* (*of_nat n*) / *of_nat n powr s*) < *e*
by *blast*
qed

lemma *lim_Ln_over_n*: ((λ*n*. *Ln*(*of_nat n*) / *of_nat n*) ⟶ *0*) *sequentially*
using *lim_Ln_over_power* [*of 1*] **by** *simp*

lemma *lim_ln_over_power*:
fixes *s* :: *real*
assumes *0* < *s*
shows ((λ*n*. *ln n* / (*n powr s*)) ⟶ *0*) *sequentially*
proof −
have (λ*n*. *ln* (*Suc n*) / (*Suc n*) *powr s*) ⟶ *0*
using *lim_Ln_over_power* [*of of_real s*, *THEN filterlim_sequentially_Suc* [*THEN iffD2*]] *assms*
by (*simp add*: *lim_sequentially dist_norm Ln_Reals_eq norm_powr_real_powr norm_divide*)
then show *?thesis*
using *filterlim_sequentially_Suc*[*of* λ*n*::*nat*. *ln n* / *n powr s*] **by** *auto*
qed

lemma *lim_ln_over_n* [*tendsto_intros*]: ((λ*n*. *ln*(*real_of_nat n*) / *of_nat n*) ⟶ *0*) *sequentially*
using *lim_ln_over_power* [*of 1*] **by** *auto*

lemma *lim_log_over_n* [*tendsto_intros*]:
(λ*n*. *log k n*/*n*) ⟶ *0*
proof −
have ∗: *log k n*/*n* = (*1*/*ln k*) ∗ (*ln n* / *n*) **for** *n*
unfolding *log_def* **by** *auto*
have (λ*n*. (*1*/*ln k*) ∗ (*ln n* / *n*)) ⟶ (*1*/*ln k*) ∗ *0*
by (*intro tendsto_intros*)
then show *?thesis*
unfolding ∗ **by** *auto*
qed

lemma *lim_1_over_complex_power*:
assumes *0* < *Re s*
shows (λ*n*. *1* / *of_nat n powr s*) ⟶ *0*
proof (*rule Lim_null_comparison*)
have ∀ *n*>*0*. *3* ≤ *n* ⟶ *1* ≤ *ln* (*real_of_nat n*)
using *ln_272_gt_1*
by (*force intro*: *order_trans* [*of _ ln* (*272*/*100*)])
then show $\forall_F$ *x in sequentially*. *cmod* (*1* / *of_nat x powr s*) ≤ *cmod* (*Ln* (*of_nat x*) / *of_nat x powr s*)
by (*auto simp*: *norm_divide field_split_simps eventually_sequentially*)
show (λ*n*. *cmod* (*Ln* (*of_nat n*) / *of_nat n powr s*)) ⟶ *0*

```
    using lim_Ln_over_power [OF assms] by (metis tendsto_norm_zero_iff)
qed

lemma lim_1_over_real_power:
  fixes s :: real
  assumes 0 < s
  shows ((λn. 1 / (of_nat n powr s)) ⟶ 0) sequentially
   using lim_1_over_complex_power [of of_real s, THEN filterlim_sequentially_Suc
[THEN iffD2]] assms
  apply (subst filterlim_sequentially_Suc [symmetric])
 by (simp add: lim_sequentially dist_norm Ln_Reals_eq norm_powr_real_powr norm_divide)

lemma lim_1_over_Ln: ((λn. 1 / Ln(of_nat n)) ⟶ 0) sequentially
proof (clarsimp simp add: lim_sequentially dist_norm norm_divide field_split_simps)
  fix r::real
  assume 0 < r
  have ir: inverse (exp (inverse r)) > 0
    by simp
  obtain n where n: 1 < of_nat n * inverse (exp (inverse r))
    using ex_less_of_nat_mult [of _ 1, OF ir]
    by auto
  then have exp (inverse r) < of_nat n
    by (simp add: field_split_simps)
  then have ln (exp (inverse r)) < ln (of_nat n)
    by (metis exp_gt_zero less_trans ln_exp ln_less_cancel_iff)
  with ‹0 < r› have 1 < r * ln (real_of_nat n)
    by (simp add: field_simps)
  moreover have n > 0 using n
    using neq0_conv by fastforce
  ultimately show ∃no. ∀k. Ln (of_nat k) ≠ 0 ⟶ no ≤ k ⟶ 1 < r * cmod
(Ln (of_nat k))
    using n ‹0 < r›
    by (rule_tac x=n in exI) (force simp: field_split_simps intro: less_le_trans)
qed

lemma lim_1_over_ln: ((λn. 1 / ln(real_of_nat n)) ⟶ 0) sequentially
  using lim_1_over_Ln [THEN filterlim_sequentially_Suc [THEN iffD2]]
  apply (subst filterlim_sequentially_Suc [symmetric])
 by (simp add: lim_sequentially dist_norm Ln_Reals_eq norm_powr_real_powr norm_divide)

lemma lim_ln1_over_ln: (λn. ln(Suc n) / ln n) ⟶ 1
proof (rule Lim_transform_eventually)
  have (λn. ln(1 + 1/n) / ln n) ⟶ 0
  proof (rule Lim_transform_bound)
    show (inverse o real) ⟶ 0
      by (metis comp_def lim_inverse_n lim_explicit)
    show ∀F n in sequentially. norm (ln (1 + 1 / n) / ln n) ≤ norm ((inverse ∘
real) n)
    proof
```

fix *n::nat*
assume *n*: $3 \le n$
then have $ln\ 3 \le ln\ n$ **and** *ln0*: $0 \le ln\ n$
by *auto*
with *ln3_gt_1* **have** $1/\ ln\ n \le 1$
by (*simp add*: *field_split_simps*)
moreover have $ln\ (1 + 1\ /\ real\ n) \le 1/n$
by (*simp add*: *ln_add_one_self_le_self*)
ultimately have $ln\ (1 + 1\ /\ real\ n) * (1\ /\ ln\ n) \le (1/n) * 1$
by (*intro mult_mono*) (*use n* **in** *auto*)
then show $norm\ (ln\ (1 + 1\ /\ n)\ /\ ln\ n) \le norm\ ((inverse \circ real)\ n)$
by (*simp add*: *field_simps ln0*)
qed
qed
then show $(\lambda n.\ 1 + ln(1 + 1/n)\ /\ ln\ n) \longrightarrow 1$
by (*metis* (*full_types*) *add.right_neutral tendsto_add_const_iff*)
show $\forall_F\ k\ in\ sequentially.\ 1 + ln\ (1 + 1\ /\ k)\ /\ ln\ k = ln(Suc\ k)\ /\ ln\ k$
by (*simp add*: *field_split_simps ln_div eventually_sequentiallyI* [*of 2*])
qed

lemma *lim_ln_over_ln1*: $(\lambda n.\ ln\ n\ /\ ln(Suc\ n)) \longrightarrow 1$
proof −
have $(\lambda n.\ inverse\ (ln(Suc\ n)\ /\ ln\ n)) \longrightarrow inverse\ 1$
by (*rule tendsto_inverse* [*OF lim_ln1_over_ln*]) *auto*
then show *?thesis*
by *simp*
qed

6.21.19 Relation between Square Root and exp/ln, hence its derivative

lemma *csqrt_exp_Ln*:
assumes $z \ne 0$
shows $csqrt\ z = exp(Ln(z)\ /\ 2)$
proof −
have $(exp\ (Ln\ z\ /\ 2))^2 = (exp\ (Ln\ z))$
by (*metis exp_double nonzero_mult_div_cancel_left times_divide_eq_right zero_neq_numeral*)
also have ... = *z*
using *assms exp_Ln* **by** *blast*
finally have $csqrt\ z = csqrt\ ((exp\ (Ln\ z\ /\ 2))^2)$
by *simp*
also have ... = $exp\ (Ln\ z\ /\ 2)$
apply (*rule csqrt_square*)
using *cos_gt_zero_pi* [*of* (*Im* (*Ln z*) / *2*)] *Im_Ln_le_pi mpi_less_Im_Ln assms*
by (*fastforce simp*: *Re_exp Im_exp*)
finally show *?thesis* **using** *assms csqrt_square*
by *simp*
qed

lemma *csqrt_inverse*:
  assumes $z \notin \mathbb{R}_{\leq 0}$
    shows *csqrt* (*inverse z*) = *inverse* (*csqrt z*)
proof (*cases z=0*)
  case *False*
  then show *?thesis*
    using *assms csqrt_exp_Ln Ln_inverse exp_minus*
    by (*simp add*: *csqrt_exp_Ln Ln_inverse exp_minus*)
qed *auto*

lemma *cnj_csqrt*:
  assumes $z \notin \mathbb{R}_{\leq 0}$
    shows *cnj*(*csqrt z*) = *csqrt*(*cnj z*)
proof (*cases z=0*)
  case *False*
  then show *?thesis*
     by (*simp add*: *assms cnj_Ln csqrt_exp_Ln exp_cnj*)
qed *auto*

lemma *has_field_derivative_csqrt*:
  assumes $z \notin \mathbb{R}_{\leq 0}$
    shows (*csqrt has_field_derivative inverse*(2 ∗ *csqrt z*)) (*at z*)
proof −
  have *z*: $z \neq 0$
    using *assms* **by** *auto*
  then have ∗: *inverse z* = *inverse* (2∗*z*) ∗ 2
    by (*simp add*: *field_split_simps*)
  have [*simp*]: *exp* (*Ln z* / 2) ∗ *inverse z* = *inverse* (*csqrt z*)
     by (*simp add*: *z field_simps csqrt_exp_Ln* [*symmetric*]) (*metis power2_csqrt power2_eq_square*)
  have *Im z* = 0 ⟹ 0 < *Re z*
    using *assms complex_nonpos_Reals_iff not_less* **by** *blast*
  with *z* **have** ((λ*z*. *exp* (*Ln z* / 2)) *has_field_derivative inverse* (2 ∗ *csqrt z*)) (*at z*)
    by (*force intro*: *derivative_eq_intros* ∗ *simp add*: *assms*)
  then show *?thesis*
  proof (*rule has_field_derivative_transform_within*)
    show ⋀*x*. *dist x z* < *cmod z* ⟹ *exp* (*Ln x* / 2) = *csqrt x*
     by (*metis csqrt_exp_Ln dist_0_norm less_irrefl*)
  qed (*use z* **in** *auto*)
qed

lemma *field_differentiable_at_csqrt*:
    $z \notin \mathbb{R}_{\leq 0}$ ⟹ *csqrt field_differentiable at z*
  using *field_differentiable_def has_field_derivative_csqrt* **by** *blast*

lemma *field_differentiable_within_csqrt*:
    $z \notin \mathbb{R}_{\leq 0}$ ⟹ *csqrt field_differentiable* (*at z within s*)
  using *field_differentiable_at_csqrt field_differentiable_within_subset* **by** *blast*

lemma *continuous_at_csqrt*:
$z \notin \mathbb{R}_{\leq 0} \Longrightarrow$ *continuous* (*at z*) *csqrt*
by (*simp add*: *field_differentiable_within_csqrt field_differentiable_imp_continuous_at*)

corollary *isCont_csqrt'* [*simp*]:
⟦*isCont f z*; *f z* $\notin \mathbb{R}_{\leq 0}$⟧ $\Longrightarrow$ *isCont* (λ*x*. *csqrt* (*f x*)) *z*
by (*blast intro*: *isCont_o2* [*OF* _ *continuous_at_csqrt*])

lemma *continuous_within_csqrt*:
$z \notin \mathbb{R}_{\leq 0} \Longrightarrow$ *continuous* (*at z within s*) *csqrt*
by (*simp add*: *field_differentiable_imp_continuous_at field_differentiable_within_csqrt*)

lemma *continuous_on_csqrt* [*continuous_intros*]:
(⋀*z*. *z* ∈ *s* ⟹ *z* $\notin \mathbb{R}_{\leq 0}$) ⟹ *continuous_on s csqrt*
by (*simp add*: *continuous_at_imp_continuous_on continuous_within_csqrt*)

lemma *holomorphic_on_csqrt*:
(⋀*z*. *z* ∈ *s* ⟹ *z* $\notin \mathbb{R}_{\leq 0}$) ⟹ *csqrt holomorphic_on s*
by (*simp add*: *field_differentiable_within_csqrt holomorphic_on_def*)

lemma *continuous_within_closed_nontrivial*:
closed s ⟹ *a* ∉ *s* ==> *continuous* (*at a within s*) *f*
using *open_Compl*
by (*force simp add*: *continuous_def eventually_at_topological filterlim_iff open_Collect_neg*)

lemma *continuous_within_csqrt_posreal*:
continuous (*at z within* (ℝ ∩ {*w*. *0* ≤ *Re*(*w*)})) *csqrt*
proof (*cases z* $\in \mathbb{R}_{\leq 0}$)
case *True*
have [*simp*]: *Im z* = *0* **and** *0*: *Re z* < *0* ∨ *z* = *0*
using *True cnj.code complex_cnj_zero_iff* **by** (*auto simp*: *Complex_eq complex_nonpos_Reals_iff*) *fastforce*
show *?thesis*
using *0*
proof
assume *Re z* < *0*
then show *?thesis*
by (*auto simp*: *continuous_within_closed_nontrivial* [*OF closed_Real_halfspace_Re_ge*])
next
assume *z* = *0*
moreover
have ⋀*e*. *0* < *e*
⟹ ∀ *x'*∈ℝ ∩ {*w*. *0* ≤ *Re w*}. *cmod x'* < *e*ˆ*2* ⟶ *cmod* (*csqrt x'*) < *e*
by (*auto simp*: *Reals_def real_less_lsqrt*)
ultimately show *?thesis*
using *zero_less_power* **by** (*fastforce simp*: *continuous_within_eps_delta*)
qed
qed (*blast intro*: *continuous_within_csqrt*)

6.21.20 Complex arctangent

The branch cut gives standard bounds in the real case.

definition *Arctan* :: *complex* ⇒ *complex* **where**
  Arctan ≡ λ*z*. (i/*2*) ∗ *Ln*((*1* − i∗*z*) / (*1* + i∗*z*))

lemma *Arctan_def_moebius*: *Arctan z* = i/*2* ∗ *Ln* (*moebius* (−i) *1* i *1 z*)
 by (*simp add*: *Arctan_def moebius_def add_ac*)

lemma *Ln_conv_Arctan*:
 assumes *z* ≠ −*1*
 shows *Ln z* = −*2*∗i ∗ *Arctan* (*moebius 1* (− *1*) (− i) (− i) *z*)
proof −
 have *Arctan* (*moebius 1* (− *1*) (− i) (− i) *z*) =
   i/*2* ∗ *Ln* (*moebius* (− i) *1* i *1* (*moebius 1* (− *1*) (− i) (− i) *z*))
  by (*simp add*: *Arctan_def_moebius*)
 also from *assms* **have** i ∗ *z* ≠ i ∗ (−*1*) **by** (*subst mult_left_cancel*) *simp*
 hence i ∗ *z* − −i ≠ *0* **by** (*simp add*: *eq_neg_iff_add_eq_0*)
 from *moebius_inverse'*[*OF _ this*, *of 1 1*]
  have *moebius* (− i) *1* i *1* (*moebius 1* (− *1*) (− i) (− i) *z*) = *z* **by** *simp*
 finally show *?thesis* **by** (*simp add*: *field_simps*)
qed

lemma *Arctan_0* [*simp*]: *Arctan 0* = *0*
 by (*simp add*: *Arctan_def*)

lemma *Im_complex_div_lemma*: *Im*((*1* − i∗*z*) / (*1* + i∗*z*)) = *0* ⟷ *Re z* = *0*
 by (*auto simp*: *Im_complex_div_eq_0 algebra_simps*)

lemma *Re_complex_div_lemma*: *0* < *Re*((*1* − i∗*z*) / (*1* + i∗*z*)) ⟷ *norm z* < *1*
 by (*simp add*: *Re_complex_div_gt_0 algebra_simps cmod_def power2_eq_square*)

lemma *tan_Arctan*:
 assumes z^2 ≠ −*1*
  shows [*simp*]:*tan*(*Arctan z*) = *z*
proof −
 have *1* + i∗*z* ≠ *0*
  by (*metis assms complex_i_mult_minus i_squared minus_unique power2_eq_square power2_minus*)
 moreover
 have *1* − i∗*z* ≠ *0*
  by (*metis assms complex_i_mult_minus i_squared power2_eq_square power2_minus right_minus_eq*)
 ultimately
 show *?thesis*
  by (*simp add*: *Arctan_def tan_def sin_exp_eq cos_exp_eq exp_minus csqrt_exp_Ln* [*symmetric*]
    divide_simps power2_eq_square [*symmetric*])
qed

lemma *Arctan_tan* [*simp*]:
  assumes |*Re z*| < *pi*/*2*
    shows *Arctan*(*tan z*) = *z*
proof −
  have *Ln* ((*1* − i ∗ *tan z*) / (*1* + i ∗ *tan z*)) = *2* ∗ *z* / i
  proof (*rule Ln_unique*)
    have *ge_pi2*: ⋀*n*::*int*. |*of_int* (*2*∗*n* + *1*) ∗ *pi*/*2*| ≥ *pi*/*2*
      by (*case_tac n rule*: *int_cases*) (*auto simp*: *abs_mult*)
    have *exp* (i∗*z*)∗*exp* (i∗*z*) = −*1* ⟷ *exp* (*2*∗i∗*z*) = −*1*
      by (*metis distrib_right exp_add mult_2*)
    also have ... ⟷ *exp* (*2*∗i∗*z*) = *exp* (i∗*pi*)
      using *cis_conv_exp cis_pi* **by** *auto*
    also have ... ⟷ *exp* (*2*∗i∗*z* − i∗*pi*) = *1*
      by (*metis* (*no_types*) *diff_add_cancel diff_minus_eq_add exp_add exp_minus_inverse mult.commute*)
    also have ... ⟷ *Re*(i∗*2*∗*z* − i∗*pi*) = *0* ∧ (∃ *n*::*int*. *Im*(i∗*2*∗*z* − i∗*pi*) = *of_int* (*2* ∗ *n*) ∗ *pi*)
      by (*simp add*: *exp_eq_1*)
    also have ... ⟷ *Im z* = *0* ∧ (∃ *n*::*int*. *2* ∗ *Re z* = *of_int* (*2*∗*n* + *1*) ∗ *pi*)
      by (*simp add*: *algebra_simps*)
    also have ... ⟷ *False*
      using *assms ge_pi2*
      apply (*auto simp*: *algebra_simps*)
      by (*metis abs_mult_pos not_less of_nat_less_0_iff of_nat_numeral*)
    finally have *exp* (i∗*z*)∗*exp* (i∗*z*) + *1* ≠ *0*
      by (*auto simp*: *add.commute minus_unique*)
    then show *exp* (*2* ∗ *z* / i) = (*1* − i ∗ *tan z*) / (*1* + i ∗ *tan z*)
      apply (*simp add*: *tan_def sin_exp_eq cos_exp_eq exp_minus divide_simps*)
      by (*simp add*: *algebra_simps flip*: *power2_eq_square exp_double*)
  qed (*use assms* **in** *auto*)
  then show *?thesis*
    by (*auto simp*: *Arctan_def*)
qed

lemma
  assumes *Re z* = *0* ⟹ |*Im z*| < *1*
  shows *Re_Arctan_bounds*: |*Re*(*Arctan z*)| < *pi*/*2*
    and *has_field_derivative_Arctan*: (*Arctan has_field_derivative inverse*(*1* + z^2)) (*at z*)
proof −
  have *nz0*: *1* + i∗*z* ≠ *0*
    using *assms*
    by (*metis abs_one add_diff_cancel_left′ complex_i_mult_minus diff_0 i_squared imaginary_unit.simps*
        less_asym neg_equal_iff_equal)
  have *z* ≠ −i **using** *assms*
    by *auto*
  then have *zz*: *1* + *z* ∗ *z* ≠ *0*

```
  by (metis abs_one assms i_squared imaginary_unit.simps less_irrefl minus_unique
square_eq_iff)
 have nz1: 1 − i∗z ≠ 0
  using assms by (force simp add: i_times_eq_iff)
 have nz2: inverse (1 + i∗z) ≠ 0
  using assms
  by (metis Im_complex_div_lemma Re_complex_div_lemma cmod_eq_Im divide_complex_def
        less_irrefl mult_zero_right zero_complex.simps(1) zero_complex.simps(2))
 have nzi: ((1 − i∗z) ∗ inverse (1 + i∗z)) ≠ 0
  using nz1 nz2 by auto
 have Im ((1 − i∗z) / (1 + i∗z)) = 0 =⇒ 0 < Re ((1 − i∗z) / (1 + i∗z))
  apply (simp add: divide_complex_def)
  apply (simp add: divide_simps split: if_split_asm)
  using assms
  apply (auto simp: algebra_simps abs_square_less_1 [unfolded power2_eq_square])
  done
 then have ∗: ((1 − i∗z) / (1 + i∗z)) ∉ ℝ≤0
  by (auto simp add: complex_nonpos_Reals_iff)
 show |Re(Arctan z)| < pi/2
  unfolding Arctan_def divide_complex_def
  using mpi_less_Im_Ln [OF nzi]
  by (auto simp: abs_if intro!: Im_Ln_less_pi ∗ [unfolded divide_complex_def])
 show (Arctan has_field_derivative inverse(1 + z²)) (at z)
  unfolding Arctan_def scaleR_conv_of_real
  apply (intro derivative_eq_intros | simp add: nz0 ∗)+
  using nz1 zz
  apply (simp add: field_split_simps power2_eq_square)
  apply algebra
  done
qed

lemma field_differentiable_at_Arctan: (Re z = 0 =⇒ |Im z| < 1) =⇒ Arctan
field_differentiable at z
 using has_field_derivative_Arctan
 by (auto simp: field_differentiable_def)

lemma field_differentiable_within_Arctan:
  (Re z = 0 =⇒ |Im z| < 1) =⇒ Arctan field_differentiable (at z within s)
 using field_differentiable_at_Arctan field_differentiable_at_within by blast

declare has_field_derivative_Arctan [derivative_intros]
declare has_field_derivative_Arctan [THEN DERIV_chain2, derivative_intros]

lemma continuous_at_Arctan:
  (Re z = 0 =⇒ |Im z| < 1) =⇒ continuous (at z) Arctan
 by (simp add: field_differentiable_imp_continuous_at field_differentiable_within_Arctan)

lemma continuous_within_Arctan:
  (Re z = 0 =⇒ |Im z| < 1) =⇒ continuous (at z within s) Arctan
```

using *continuous_at_Arctan continuous_at_imp_continuous_within* **by** *blast*

lemma *continuous_on_Arctan* [*continuous_intros*]:
($\bigwedge$*z*. *z* ∈ *s* ⟹ *Re z* = *0* ⟹ |*Im z*| < *1*) ⟹ *continuous_on s Arctan*
by (*auto simp*: *continuous_at_imp_continuous_on continuous_within_Arctan*)

lemma *holomorphic_on_Arctan*:
($\bigwedge$*z*. *z* ∈ *s* ⟹ *Re z* = *0* ⟹ |*Im z*| < *1*) ⟹ *Arctan holomorphic_on s*
by (*simp add*: *field_differentiable_within_Arctan holomorphic_on_def*)

theorem *Arctan_series*:
assumes *z*: *norm* (*z* :: *complex*) < *1*
defines *g* ≡ λ*n*. *if odd n then* −i∗i ˆ*n* / *n else 0*
defines *h* ≡ λ*z n*. (−*1*) ˆ*n* / *of_nat* (*2*∗*n*+*1*) ∗ (*z*::*complex*) ˆ(*2*∗*n*+*1*)
shows (λ*n*. *g n* ∗ *z*ˆ*n*) *sums Arctan z*
and *h z sums Arctan z*
proof −
define *G* **where** [*abs_def*]: *G z* = ($\sum$*n*. *g n* ∗ *z*ˆ*n*) **for** *z*
have *summable*: *summable* (λ*n*. *g n* ∗ *u*ˆ*n*) **if** *norm u* < *1* **for** *u*
proof (*cases u* = *0*)
assume *u*: *u* ≠ *0*
have (λ*n*. *ereal* (*norm* (*h u n*) / *norm* (*h u* (*Suc n*)))) = (λ*n*. *ereal* (*inverse* (*norm u*) ˆ*2*) ∗
ereal ((*2* + *inverse* (*real* (*Suc n*))) / (*2* − *inverse* (*real* (*Suc n*)))))
proof
fix *n*
have *ereal* (*norm* (*h u n*) / *norm* (*h u* (*Suc n*))) =
ereal (*inverse* (*norm u*) ˆ*2*) ∗ *ereal* (((*2*∗*Suc n*+*1*) / (*Suc n*)) /
((*2*∗*Suc n*−*1*) / (*Suc n*)))
by (*simp add*: *h_def norm_mult norm_power norm_divide field_split_simps*
power2_eq_square eval_nat_numeral del: *of_nat_add of_nat_Suc*)
also have *of_nat* (*2*∗*Suc n*+*1*) / *of_nat* (*Suc n*) = (*2*::*real*) + *inverse* (*real* (*Suc n*))
by (*auto simp*: *field_split_simps simp del*: *of_nat_Suc*) *simp_all?*
also have *of_nat* (*2*∗*Suc n*−*1*) / *of_nat* (*Suc n*) = (*2*::*real*) − *inverse* (*real* (*Suc n*))
by (*auto simp*: *field_split_simps simp del*: *of_nat_Suc*) *simp_all?*
finally show *ereal* (*norm* (*h u n*) / *norm* (*h u* (*Suc n*))) = *ereal* (*inverse* (*norm u*) ˆ*2*) ∗
ereal ((*2* + *inverse* (*real* (*Suc n*))) / (*2* − *inverse* (*real* (*Suc n*)))) **.**
qed
also have ... ⟶ *ereal* (*inverse* (*norm u*) ˆ*2*) ∗ *ereal* ((*2* + *0*) / (*2* − *0*))
by (*intro tendsto_intros LIMSEQ_inverse_real_of_nat*) *simp_all*
finally have *liminf* (λ*n*. *ereal* (*cmod* (*h u n*) / *cmod* (*h u* (*Suc n*)))) = *inverse* (*norm u*) ˆ*2*
by (*intro lim_imp_Liminf*) *simp_all*
moreover from *power_strict_mono*[*OF that*, *of 2*] *u* **have** *inverse* (*norm u*) ˆ*2* > *1*
by (*simp add*: *field_split_simps*)

ultimately have *A*: *liminf* (λ*n*. *ereal* (*cmod* (*h u n*) / *cmod* (*h u* (*Suc n*)))) > *1* **by** *simp*
from *u* **have** *summable* (*h u*)
by (*intro summable_norm_cancel*[*OF ratio_test_convergence*[*OF* _ *A*]])
(*auto simp*: *h_def norm_divide norm_mult norm_power simp del*: *of_nat_Suc*
intro!: *mult_pos_pos divide_pos_pos always_eventually*)
thus *summable* (λ*n*. *g n* ∗ *u*ˆ*n*)
by (*subst summable_mono_reindex*[*of* λ*n*. *2*∗*n*+*1*, *symmetric*])
(*auto simp*: *power_mult strict_mono_def g_def h_def elim*!: *oddE*)
qed (*simp add*: *h_def*)

have ∃ *c*. ∀ *u*∈*ball 0 1*. *Arctan u* − *G u* = *c*
proof (*rule has_field_derivative_zero_constant*)
fix *u* :: *complex* **assume** *u* ∈ *ball 0 1*
hence *u*: *norm u* < *1* **by** (*simp*)
define *K* **where** *K* = (*norm u* + *1*) / *2*
from *u* **and** *abs_Im_le_cmod*[*of u*] **have** *Im_u*: |*Im u*| < *1* **by** *linarith*
from *u* **have** *K*: *0* ≤ *K norm u* < *K K* < *1* **by** (*simp_all add*: *K_def*)
hence (*G has_field_derivative* (∑ *n*. *diffs g n* ∗ *u* ˆ *n*)) (*at u*) **unfolding** *G_def*
by (*intro termdiffs_strong*[*of* _ *of_real K*] *summable*) *simp_all*
also have (λ*n*. *diffs g n* ∗ *u*ˆ*n*) = (λ*n*. *if even n then* (i∗*u*)ˆ*n else 0*)
by (*intro ext*) (*simp_all del*: *of_nat_Suc add*: *g_def diffs_def power_mult_distrib*)
also have *suminf* . . . = (∑ *n*. (−(*u*ˆ*2*))ˆ*n*)
by (*subst suminf_mono_reindex*[*of* λ*n*. *2*∗*n*, *symmetric*])
(*auto elim*!: *evenE simp*: *strict_mono_def power_mult power_mult_distrib*)
also from *u* **have** *norm u*ˆ*2* < *1*ˆ*2* **by** (*intro power_strict_mono*) *simp_all*
hence (∑ *n*. (−(*u*ˆ*2*))ˆ*n*) = *inverse* (*1* + *u*ˆ*2*)
by (*subst suminf_geometric*) (*simp_all add*: *norm_power inverse_eq_divide*)
finally have (*G has_field_derivative inverse* (*1* + u^2)) (*at u*) .
from *DERIV_diff*[*OF has_field_derivative_Arctan this*] *Im_u u*
show ((λ*u*. *Arctan u* − *G u*) *has_field_derivative 0*) (*at u within ball 0 1*)
by (*simp_all add*: *at_within_open*[*OF* _ *open_ball*])
qed *simp_all*
then obtain *c* **where** *c*: ⋀*u*. *norm u* < *1* =⇒ *Arctan u* − *G u* = *c* **by** *auto*
from *this*[*of 0*] **have** *c* = *0* **by** (*simp add*: *G_def g_def*)
with *c z* **have** *Arctan z* = *G z* **by** *simp*
with *summable*[*OF z*] **show** (λ*n*. *g n* ∗ *z*ˆ*n*) *sums Arctan z* **unfolding** *G_def* **by** (*simp add*: *sums_iff*)
thus *h z sums Arctan z* **by** (*subst* (*asm*) *sums_mono_reindex*[*of* λ*n*. *2*∗*n*+*1*, *symmetric*])
(*auto elim*!: *oddE simp*: *strict_mono_def power_mult g_def h_def*)
qed

A quickly-converging series for the logarithm, based on the arctangent.

theorem *ln_series_quadratic*:
assumes *x*: *x* > (*0*::*real*)
shows (λ*n*. (*2*∗((*x* − *1*) / (*x* + *1*)) ˆ (*2*∗*n*+*1*) / *of_nat* (*2*∗*n*+*1*))) *sums ln x*
proof −

define *y* :: *complex* **where** *y* = *of_real* ((*x*−*1*)/(*x*+*1*))
from *x* **have** *x'*: *complex_of_real x* ≠ *of_real* (−*1*) **by** (*subst of_real_eq_iff*) *auto*
from *x* **have** |*x* − *1*| < |*x* + *1*| **by** *linarith*
hence *norm* (*complex_of_real* (*x* − *1*) / *complex_of_real* (*x* + *1*)) < *1*
by (*simp add*: *norm_divide del*: *of_real_add of_real_diff*)
hence *norm* (i ∗ *y*) < *1* **unfolding** *y_def* **by** (*subst norm_mult*) *simp*
hence (λ*n*. (−*2*∗i) ∗ ((−*1*) ˆ*n* / *of_nat* (*2*∗*n*+*1*) ∗ (i∗*y*) ˆ(*2*∗*n*+*1*))) *sums* ((−*2*∗i) ∗ *Arctan* (i∗*y*))
by (*intro Arctan_series sums_mult*) *simp_all*
also have (λ*n*. (−*2*∗i) ∗ ((−*1*) ˆ*n* / *of_nat* (*2*∗*n*+*1*) ∗ (i∗*y*) ˆ(*2*∗*n*+*1*))) =
(λ*n*. (−*2*∗i) ∗ ((−*1*) ˆ*n* ∗ (i∗*y*∗(−*y*^{2}) ˆ*n*)/*of_nat* (*2*∗*n*+*1*)))
by (*intro ext*) (*simp_all add*: *power_mult power_mult_distrib*)
also have . . . = (λ*n*. *2*∗*y*∗ ((−*1*) ∗ (−*y*^{2})) ˆ*n*/*of_nat* (*2*∗*n*+*1*))
by (*intro ext*, *subst power_mult_distrib*) (*simp add*: *algebra_simps power_mult*)
also have . . . = (λ*n*. *2*∗*y* ˆ(*2*∗*n*+*1*) / *of_nat* (*2*∗*n*+*1*))
by (*subst power_add*, *subst power_mult*) (*simp add*: *mult_ac*)
also have . . . = (λ*n*. *of_real* (*2*∗((*x*−*1*)/(*x*+*1*)) ˆ(*2*∗*n*+*1*) / *of_nat* (*2*∗*n*+*1*)))
by (*intro ext*) (*simp add*: *y_def*)
also have i ∗ *y* = (*of_real x* − *1*) / (−i ∗ (*of_real x* + *1*))
by (*subst divide_divide_eq_left* [*symmetric*]) (*simp add*: *y_def*)
also have . . . = *moebius 1* (−*1*) (−i) (−i) (*of_real x*) **by** (*simp add*: *moebius_def algebra_simps*)
also from *x'* **have** −*2*∗i∗*Arctan* . . . = *Ln* (*of_real x*) **by** (*intro Ln_conv_Arctan* [*symmetric*]) *simp_all*
also from *x* **have** . . . = *ln x* **by** (*rule Ln_of_real*)
finally show *?thesis* **by** (*subst* (*asm*) *sums_of_real_iff*)
qed

6.21.21 Real arctangent

lemma *Im_Arctan_of_real* [*simp*]: *Im* (*Arctan* (*of_real x*)) = *0*
proof −
have *ne*: *1* + *x*^{2} ≠ *0*
by (*metis power_one sum_power2_eq_zero_iff zero_neq_one*)
have *ne1*: *1* + i ∗ *complex_of_real x* ≠ *0*
using *Complex_eq complex_eq_cancel_iff2* **by** *fastforce*
have *Re* (*Ln* ((*1* − i ∗ *x*) ∗ *inverse* (*1* + i ∗ *x*))) = *0*
apply (*rule norm_exp_imaginary*)
using *ne*
apply (*simp add*: *ne1 cmod_def*)
apply (*auto simp*: *field_split_simps*)
apply *algebra*
done
then show *?thesis*
unfolding *Arctan_def divide_complex_def* **by** (*simp add*: *complex_eq_iff*)
qed

lemma *arctan_eq_Re_Arctan*: *arctan x* = *Re* (*Arctan* (*of_real x*))
proof (*rule arctan_unique*)

have $(1 - i * x) / (1 + i * x) \notin \mathbb{R}_{\leq 0}$
  by (*auto simp*: *Im_complex_div_lemma complex_nonpos_Reals_iff*)
then show $-(pi / 2) < Re (Arctan (complex\_of\_real\ x))$
  by (*simp add*: *Arctan_def Im_Ln_less_pi*)
next
have $*$: $(1 - i*x) / (1 + i*x) \neq 0$
  by (*simp add*: *field_split_simps*) (*simp add*: *complex_eq_iff*)
show $Re (Arctan (complex\_of\_real\ x)) < pi / 2$
  using *mpi_less_Im_Ln* [*OF* ∗]
  by (*simp add*: *Arctan_def*)
next
have $tan (Re (Arctan (of\_real\ x))) = Re (tan (Arctan (of\_real\ x)))$
  by (*auto simp*: *tan_def Complex.Re_divide Re_sin Re_cos Im_sin Im_cos field_simps power2_eq_square*)
also have ... $= x$
proof −
  have $(complex\_of\_real\ x)^2 \neq -1$
    by (*metis diff_0_right minus_diff_eq mult_zero_left not_le of_real_1 of_real_eq_iff of_real_minus of_real_power power2_eq_square real_minus_mult_self_le zero_less_one*)
  then show *?thesis*
    by *simp*
qed
finally show $tan (Re (Arctan (complex\_of\_real\ x))) = x$ **.**
qed

lemma *Arctan_of_real*: $Arctan (of\_real\ x) = of\_real (arctan\ x)$
  unfolding *arctan_eq_Re_Arctan divide_complex_def*
  by (*simp add*: *complex_eq_iff*)

lemma *Arctan_in_Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow Arctan\ z \in \mathbb{R}$
  by (*metis Reals_cases Reals_of_real Arctan_of_real*)

declare *arctan_one* [*simp*]

lemma *arctan_less_pi4_pos*: $x < 1 \Longrightarrow arctan\ x < pi/4$
  by (*metis arctan_less_iff arctan_one*)

lemma *arctan_less_pi4_neg*: $-1 < x \Longrightarrow -(pi/4) < arctan\ x$
  by (*metis arctan_less_iff arctan_minus arctan_one*)

lemma *arctan_less_pi4*: $|x| < 1 \Longrightarrow |arctan\ x| < pi/4$
  by (*metis abs_less_iff arctan_less_pi4_pos arctan_minus*)

lemma *arctan_le_pi4*: $|x| \leq 1 \Longrightarrow |arctan\ x| \leq pi/4$
  by (*metis abs_le_iff arctan_le_iff arctan_minus arctan_one*)

lemma *abs_arctan*: $|arctan\ x| = arctan\ |x|$
  by (*simp add*: *abs_if arctan_minus*)

```
lemma arctan_add_raw:
  assumes |arctan x + arctan y| < pi/2
    shows arctan x + arctan y = arctan((x + y) / (1 − x ∗ y))
proof (rule arctan_unique [symmetric])
  show 12: − (pi / 2) < arctan x + arctan y arctan x + arctan y < pi / 2
    using assms by linarith+
  show tan (arctan x + arctan y) = (x + y) / (1 − x ∗ y)
    using cos_gt_zero_pi [OF 12]
    by (simp add: arctan tan_add)
qed

lemma arctan_inverse:
  assumes 0 < x
    shows arctan(inverse x) = pi/2 − arctan x
proof −
  have arctan(inverse x) = arctan(inverse(tan(arctan x)))
    by (simp add: arctan)
  also have ... = arctan (tan (pi / 2 − arctan x))
    by (simp add: tan_cot)
  also have ... = pi/2 − arctan x
  proof −
    have 0 < pi − arctan x
    using arctan_ubound [of x] pi_gt_zero by linarith
    with assms show ?thesis
      by (simp add: Transcendental.arctan_tan)
  qed
  finally show ?thesis .
qed

lemma arctan_add_small:
  assumes |x ∗ y| < 1
    shows (arctan x + arctan y = arctan((x + y) / (1 − x ∗ y)))
proof (cases x = 0 ∨ y = 0)
  case False
  with assms have |x| < inverse |y|
    by (simp add: field_split_simps abs_mult)
  with False have |arctan x| < pi / 2 − |arctan y| using assms
    by (auto simp add: abs_arctan arctan_inverse [symmetric] arctan_less_iff)
  then show ?thesis
    by (intro arctan_add_raw) linarith
qed auto

lemma abs_arctan_le:
  fixes x::real shows |arctan x| ≤ |x|
proof −
  have 1: ⋀x. x ∈ ℝ ⟹ cmod (inverse (1 + x²)) ≤ 1
   by (simp add: norm_divide divide_simps in_Reals_norm complex_is_Real_iff power2_eq_square)
  have cmod (Arctan w − Arctan z) ≤ 1 ∗ cmod (w−z) if w ∈ ℝ z ∈ ℝ for w z
    apply (rule field_differentiable_bound [OF convex_Reals, of Arctan _ 1])
```

apply (*rule has_field_derivative_at_within* [*OF has_field_derivative_Arctan*])
using *1 that* **by** (*auto simp*: *Reals_def*)
then have *cmod* (*Arctan* (*of_real x*) − *Arctan 0*) ≤ *1* ∗ *cmod* (*of_real x* − *0*)
using *Reals_0 Reals_of_real* **by** *blast*
then show *?thesis*
by (*simp add*: *Arctan_of_real*)
qed

lemma *arctan_le_self*: *0* ≤ *x* ⟹ *arctan x* ≤ *x*
by (*metis abs_arctan_le abs_of_nonneg zero_le_arctan_iff*)

lemma *abs_tan_ge*: |*x*| < *pi*/*2* ⟹ |*x*| ≤ |*tan x*|
by (*metis abs_arctan_le abs_less_iff arctan_tan minus_less_iff*)

lemma *arctan_bounds*:
assumes *0* ≤ *x x* < *1*
shows *arctan_lower_bound*:
(∑*k*<*2* ∗ *n*. (− *1*) ^ *k* ∗ (*1* / *real* (*k* ∗ *2* + *1*) ∗ *x* ^ (*k* ∗ *2* + *1*))) ≤ *arctan x*
(**is** (∑*k*<_. (− *1*) ^ *k* ∗ *?a k*) ≤ _)
and *arctan_upper_bound*:
arctan x ≤ (∑*k*<*2* ∗ *n* + *1*. (− *1*) ^ *k* ∗ (*1* / *real* (*k* ∗ *2* + *1*) ∗ *x* ^ (*k* ∗ *2* + *1*)))
proof −
have *tendsto_zero*: *?a* ⟶ *0*
proof (*rule tendsto_eq_rhs*)
show (λ*k*. *1* / *real* (*k* ∗ *2* + *1*) ∗ *x* ^ (*k* ∗ *2* + *1*)) ⟶ *0* ∗ *0*
using *assms*
by (*intro tendsto_mult real_tendsto_divide_at_top*)
(*auto simp*: *filterlim_real_sequentially filterlim_sequentially_iff_filterlim_real*
intro!: *real_tendsto_divide_at_top tendsto_power_zero filterlim_real_sequentially*
tendsto_eq_intros filterlim_at_top_mult_tendsto_pos filterlim_tendsto_add_at_top)
qed *simp*
have *nonneg*: *0* ≤ *?a n* **for** *n*
by (*force intro*!: *divide_nonneg_nonneg mult_nonneg_nonneg zero_le_power assms*)
have *le*: *?a* (*Suc n*) ≤ *?a n* **for** *n*
by (*rule mult_mono*[*OF _ power_decreasing*]) (*auto simp*: *field_split_simps assms less_imp_le*)
from *summable_Leibniz'*(*4*)[*of ?a*, *OF tendsto_zero nonneg le*, *of n*]
summable_Leibniz'(*2*)[*of ?a*, *OF tendsto_zero nonneg le*, *of n*]
assms
show (∑*k*<*2*∗*n*. (− *1*) ^ *k* ∗ *?a k*) ≤ *arctan x arctan x* ≤ (∑*k*<*2* ∗ *n* + *1*. (− *1*) ^ *k* ∗ *?a k*)
by (*auto simp*: *arctan_series*)
qed

6.21.22 Bounds on pi using real arctangent

lemma *pi_machin*: *pi* = *16* ∗ *arctan* (*1* / *5*) − *4* ∗ *arctan* (*1* / *239*)
using *machin* **by** *simp*

lemma *pi_approx*: *3.141592653588* ≤ *pi pi* ≤ *3.1415926535899*
  unfolding *pi_machin*
  using *arctan_bounds*[*of 1/5 4*]
    arctan_bounds[*of 1/239 4*]
  by (*simp_all add*: *eval_nat_numeral*)

lemma *pi_gt3*: *pi* > *3*
  using *pi_approx* **by** *simp*

6.21.23 Inverse Sine

definition *Arcsin* :: *complex* ⇒ *complex* **where**
  Arcsin ≡ λ*z*. −i ∗ *Ln*(i ∗ *z* + *csqrt*(*1* − z^2))

lemma *Arcsin_body_lemma*: i ∗ *z* + *csqrt*(*1* − z^2) ≠ *0*
  using *power2_csqrt* [*of 1* − z^2]
 by (*metis add.inverse_inverse complex_i_mult_minus diff_0 diff_add_cancel diff_minus_eq_add mult.assoc mult.commute numeral_One power2_eq_square zero_neq_numeral*)

lemma *Arcsin_range_lemma*: |*Re z*| < *1* ⟹ *0* < *Re*(i ∗ *z* + *csqrt*(*1* − z^2))
  using *Complex.cmod_power2* [*of z*, *symmetric*]
  by (*simp add*: *real_less_rsqrt algebra_simps Re_power2 cmod_square_less_1_plus*)

lemma *Re_Arcsin*: *Re*(*Arcsin z*) = *Im* (*Ln* (i ∗ *z* + *csqrt*(*1* − z^2)))
  by (*simp add*: *Arcsin_def*)

lemma *Im_Arcsin*: *Im*(*Arcsin z*) = − *ln* (*cmod* (i ∗ *z* + *csqrt* (*1* − z^2)))
  by (*simp add*: *Arcsin_def Arcsin_body_lemma*)

lemma *one_minus_z2_notin_nonpos_Reals*:
  assumes *Im z* = *0* ⟹ |*Re z*| < *1*
  shows *1* − z^2 ∉ $\mathbb{R}_{\leq 0}$
proof (*cases Im z* = *0*)
  case *True*
  with *assms* **show** *?thesis*
    by (*simp add*: *complex_nonpos_Reals_iff flip*: *abs_square_less_1*)
next
  case *False*
  have ¬ (*Im z*)^2 ≤ − *1*
    using *False power2_less_eq_zero_iff* **by** *fastforce*
  with *False* **show** *?thesis*
    by (*auto simp add*: *complex_nonpos_Reals_iff Re_power2 Im_power2*)
qed

lemma *isCont_Arcsin_lemma*:
  assumes *le0*: *Re* (i ∗ *z* + *csqrt* (*1* − z^2)) ≤ *0* **and** (*Im z* = *0* ⟹ |*Re z*| < *1*)
    shows *False*
proof (*cases Im z* = *0*)

case *True*
then show *?thesis*
using *assms* **by** (*fastforce simp*: *cmod_def abs_square_less_1* [*symmetric*])
next
case *False*
have *leim*: (*cmod* (*1* − z^2) + (*1* − *Re* (z^2))) / *2* ≤ (*Im* $z)^2$
using *le0 sqrt_le_D* **by** *fastforce*
have *neq*: (*cmod* $z)^2$ ≠ *1* + *cmod* (*1* − z^2)
proof (*clarsimp simp add*: *cmod_def*)
assume (*Re* $z)^2$ + (*Im* $z)^2$ = *1* + *sqrt* ((*1* − *Re* (z^2))^2 + (*Im* (z^2))^2)
then have ((*Re* $z)^2$ + (*Im* $z)^2$ − *1*)^2 = ((*1* − *Re* (z^2))^2 + (*Im* (z^2))^2)
by *simp*
then show *False* **using** *False*
by (*simp add*: *power2_eq_square algebra_simps*)
qed
moreover have *2*: (*Im* $z)^2$ = (*1* + ((*Im* $z)^2$ + *cmod* (*1* − z^2)) − (*Re* $z)^2$) / *2*
using *leim cmod_power2* [*of z*] *norm_triangle_ineq2* [*of z^2 1*]
by (*simp add*: *norm_power Re_power2 norm_minus_commute* [*of 1*])
ultimately show *False*
by (*simp add*: *Re_power2 Im_power2 cmod_power2*)
qed

lemma *isCont_Arcsin*:
assumes (*Im z* = *0* ⟹ |*Re z*| < *1*)
shows *isCont Arcsin z*
proof −
have *1*: i ∗ *z* + *csqrt* (*1* − z^2) ∉ $\mathbb{R}_{\leq 0}$
by (*metis isCont_Arcsin_lemma assms complex_nonpos_Reals_iff*)
have *2*: *1* − z^2 ∉ $\mathbb{R}_{\leq 0}$
by (*simp add*: *one_minus_z2_notin_nonpos_Reals assms*)
show *?thesis*
using *assms* **unfolding** *Arcsin_def* **by** (*intro isCont_Ln′ isCont_csqrt′ continuous_intros 1 2*)
qed

lemma *isCont_Arcsin′* [*simp*]:
shows *isCont f z* ⟹ (*Im* (*f z*) = *0* ⟹ |*Re* (*f z*)| < *1*) ⟹ *isCont* (λ*x*. *Arcsin* (*f x*)) *z*
by (*blast intro*: *isCont_o2* [*OF _ isCont_Arcsin*])

lemma *sin_Arcsin* [*simp*]: *sin*(*Arcsin z*) = *z*
proof −
have i∗*z*∗*2* + *csqrt* (*1* − z^2)∗*2* = *0* ⟷ (i∗*z*)∗*2* + *csqrt* (*1* − z^2)∗*2* = *0*
by (*simp add*: *algebra_simps*) — Cancelling a factor of 2
moreover have ... ⟷ (i∗*z*) + *csqrt* (*1* − z^2) = *0*
by (*metis Arcsin_body_lemma distrib_right no_zero_divisors zero_neq_numeral*)
ultimately show *?thesis*
apply (*simp add*: *sin_exp_eq Arcsin_def Arcsin_body_lemma exp_minus divide_simps*)
apply (*simp add*: *algebra_simps*)

apply (*simp add*: *power2_eq_square* [*symmetric*] *algebra_simps*)
done
qed

lemma *Re_eq_pihalf_lemma*:
|*Re z*| = *pi*/*2* ⟹ *Im z* = *0* ⟹
Re ((*exp* (i∗*z*) + *inverse* (*exp* (i∗*z*))) / *2*) = *0* ∧ *0* ≤ *Im* ((*exp* (i∗*z*) + *inverse* (*exp* (i∗*z*))) / *2*)
apply (*simp add*: *cos_i_times* [*symmetric*] *Re_cos Im_cos abs_if del*: *eq_divide_eq_numeral1*)
by (*metis cos_minus cos_pi_half*)

lemma *Re_less_pihalf_lemma*:
assumes |*Re z*| < *pi* / *2*
shows *0* < *Re* ((*exp* (i∗*z*) + *inverse* (*exp* (i∗*z*))) / *2*)
proof −
have *0* < *cos* (*Re z*) **using** *assms*
using *cos_gt_zero_pi* **by** *auto*
then show *?thesis*
by (*simp add*: *cos_i_times* [*symmetric*] *Re_cos Im_cos add_pos_pos*)
qed

lemma *Arcsin_sin*:
assumes |*Re z*| < *pi*/*2* ∨ (|*Re z*| = *pi*/*2* ∧ *Im z* = *0*)
shows *Arcsin*(*sin z*) = *z*
proof −
have *Arcsin*(*sin z*) = − (i ∗ *Ln* (*csqrt* (*1* − (i ∗ (*exp* (i∗*z*) − *inverse* (*exp* (i∗*z*))))^2 / *4*) − (*inverse* (*exp* (i∗*z*)) − *exp* (i∗*z*)) / *2*))
by (*simp add*: *sin_exp_eq Arcsin_def exp_minus power_divide*)
also have ... = − (i ∗ *Ln* (*csqrt* (((*exp* (i∗*z*) + *inverse* (*exp* (i∗*z*)))/*2*)^2) − (*inverse* (*exp* (i∗*z*)) − *exp* (i∗*z*)) / *2*))
by (*simp add*: *field_simps power2_eq_square*)
also have ... = − (i ∗ *Ln* (((*exp* (i∗*z*) + *inverse* (*exp* (i∗*z*)))/*2*) − (*inverse* (*exp* (i∗*z*)) − *exp* (i∗*z*)) / *2*))
apply (*subst csqrt_square*)
using *assms Re_eq_pihalf_lemma Re_less_pihalf_lemma* **by** *auto*
also have ... = − (i ∗ *Ln* (*exp* (i∗*z*)))
by (*simp add*: *field_simps power2_eq_square*)
also have ... = *z*
using *assms* **by** (*auto simp*: *abs_if simp del*: *eq_divide_eq_numeral1 split*: *if_split_asm*)
finally show *?thesis* **.**
qed

lemma *Arcsin_unique*:
⟦*sin z* = *w*; |*Re z*| < *pi*/*2* ∨ (|*Re z*| = *pi*/*2* ∧ *Im z* = *0*)⟧ ⟹ *Arcsin w* = *z*
by (*metis Arcsin_sin*)

lemma *Arcsin_0* [*simp*]: *Arcsin 0* = *0*
by (*metis Arcsin_sin norm_zero pi_half_gt_zero real_norm_def sin_zero zero_complex.simps*(*1*))

lemma *Arcsin_1* [*simp*]: *Arcsin 1 = pi/2*
by (*metis Arcsin_sin Im_complex_of_real Re_complex_of_real numeral_One of_real_numeral pi_half_ge_zero real_sqrt_abs real_sqrt_pow2 real_sqrt_power sin_of_real sin_pi_half*)

lemma *Arcsin_minus_1* [*simp*]: *Arcsin*(−*1*) = − (*pi/2*)
by (*metis Arcsin_1 Arcsin_sin Im_complex_of_real Re_complex_of_real abs_of_nonneg of_real_minus pi_half_ge_zero power2_minus real_sqrt_abs sin_Arcsin sin_minus*)

lemma *has_field_derivative_Arcsin*:
assumes *Im z = 0* ⟹ |*Re z*| < *1*
shows (*Arcsin has_field_derivative inverse*(*cos*(*Arcsin z*))) (*at z*)
proof −
have $(sin\ (Arcsin\ z))^2 \neq 1$
using *assms one_minus_z2_notin_nonpos_Reals* **by** *force*
then have *cos* (*Arcsin z*) ≠ *0*
by (*metis diff_0_right power_zero_numeral sin_squared_eq*)
then show *?thesis*
by (*rule has_field_derivative_inverse_basic* [*OF DERIV_sin _ _ open_ball* [*of z 1*]]) (*auto intro*: *isCont_Arcsin assms*)
qed

declare *has_field_derivative_Arcsin* [*derivative_intros*]
declare *has_field_derivative_Arcsin* [*THEN DERIV_chain2*, *derivative_intros*]

lemma *field_differentiable_at_Arcsin*:
(*Im z = 0* ⟹ |*Re z*| < *1*) ⟹ *Arcsin field_differentiable at z*
using *field_differentiable_def has_field_derivative_Arcsin* **by** *blast*

lemma *field_differentiable_within_Arcsin*:
(*Im z = 0* ⟹ |*Re z*| < *1*) ⟹ *Arcsin field_differentiable* (*at z within s*)
using *field_differentiable_at_Arcsin field_differentiable_within_subset* **by** *blast*

lemma *continuous_within_Arcsin*:
(*Im z = 0* ⟹ |*Re z*| < *1*) ⟹ *continuous* (*at z within s*) *Arcsin*
using *continuous_at_imp_continuous_within isCont_Arcsin* **by** *blast*

lemma *continuous_on_Arcsin* [*continuous_intros*]:
(⋀*z*. *z* ∈ *s* ⟹ *Im z = 0* ⟹ |*Re z*| < *1*) ⟹ *continuous_on s Arcsin*
by (*simp add*: *continuous_at_imp_continuous_on*)

lemma *holomorphic_on_Arcsin*: (⋀*z*. *z* ∈ *s* ⟹ *Im z = 0* ⟹ |*Re z*| < *1*) ⟹ *Arcsin holomorphic_on s*
by (*simp add*: *field_differentiable_within_Arcsin holomorphic_on_def*)

6.21.24 Inverse Cosine

definition *Arccos* :: *complex* ⇒ *complex* **where**
$Arccos \equiv \lambda z.\ -\mathrm{i} * Ln(z + \mathrm{i} * csqrt(1 - z^2))$

lemma *Arccos_range_lemma*: $|Re\ z| < 1 \Longrightarrow 0 < Im(z + i * csqrt(1 - z^2))$
using *Arcsin_range_lemma* [*of* −*z*] **by** *simp*

lemma *Arccos_body_lemma*: $z + i * csqrt(1 - z^2) \neq 0$
using *Arcsin_body_lemma* [*of z*]
by (*metis Arcsin_body_lemma complex_i_mult_minus diff_minus_eq_add power2_minus right_minus_eq*)

lemma *Re_Arccos*: $Re(Arccos\ z) = Im\ (Ln\ (z + i * csqrt(1 - z^2)))$
by (*simp add*: *Arccos_def*)

lemma *Im_Arccos*: $Im(Arccos\ z) = -\ ln\ (cmod\ (z + i * csqrt\ (1 - z^2)))$
by (*simp add*: *Arccos_def Arccos_body_lemma*)

A very tricky argument to find!

lemma *isCont_Arccos_lemma*:
assumes *eq0*: $Im\ (z + i * csqrt\ (1 - z^2)) = 0$ **and** $(Im\ z = 0 \Longrightarrow |Re\ z| < 1)$
shows *False*
proof (*cases* $Im\ z = 0$)
case *True*
then show *?thesis*
using *assms* **by** (*fastforce simp add*: *cmod_def abs_square_less_1* [*symmetric*])
next
case *False*
have *Imz*: $Im\ z = -\ sqrt\ ((1 + ((Im\ z)^2 + cmod\ (1 - z^2)) - (Re\ z)^2) / 2)$
using *eq0 abs_Re_le_cmod* [*of* $1-z^2$]
by (*simp add*: *Re_power2 algebra_simps*)
have $(cmod\ z)^2 - 1 \neq cmod\ (1 - z^2)$
proof (*clarsimp simp add*: *cmod_def*)
assume $(Re\ z)^2 + (Im\ z)^2 - 1 = sqrt\ ((1 - Re\ (z^2))^2 + (Im\ (z^2))^2)$
then have $((Re\ z)^2 + (Im\ z)^2 - 1)^2 = ((1 - Re\ (z^2))^2 + (Im\ (z^2))^2)$
by *simp*
then show *False* **using** *False*
by (*simp add*: *power2_eq_square algebra_simps*)
qed
moreover have $(Im\ z)^2 = (1 + ((Im\ z)^2 + cmod\ (1 - z^2)) - (Re\ z)^2) / 2$
using *abs_Re_le_cmod* [*of* $1-z^2$] **by** (*subst Imz*) (*simp add*: *Re_power2*)
ultimately show *False*
by (*simp add*: *cmod_power2*)
qed

lemma *isCont_Arccos*:
assumes $(Im\ z = 0 \Longrightarrow |Re\ z| < 1)$
shows *isCont Arccos z*
proof −
have $z + i * csqrt\ (1 - z^2) \notin \mathbb{R}_{\leq 0}$
by (*metis complex_nonpos_Reals_iff isCont_Arccos_lemma assms*)
with *assms* **show** *?thesis*
unfolding *Arccos_def*

by (*simp_all add*: *one_minus_z2_notin_nonpos_Reals assms*)
qed

lemma *isCont_Arccos'* [*simp*]:
isCont f z $\Longrightarrow$ (*Im* (*f z*) = *0* $\Longrightarrow$ |*Re* (*f z*)| < *1*) $\Longrightarrow$ *isCont* (λx. *Arccos* (*f x*)) *z*
by (*blast intro*: *isCont_o2* [*OF* _ *isCont_Arccos*])

lemma *cos_Arccos* [*simp*]: *cos*(*Arccos z*) = *z*
proof −
have $z*2 + i * (2 * csqrt\ (1 - z^2)) = 0 \longleftrightarrow z*2 + i * csqrt\ (1 - z^2)*2 = 0$
by (*simp add*: *algebra_simps*) — Cancelling a factor of 2
moreover have ... $\longleftrightarrow z + i * csqrt\ (1 - z^2) = 0$
by (*metis distrib_right mult_eq_0_iff zero_neq_numeral*)
ultimately show *?thesis*
by (*simp add*: *cos_exp_eq Arccos_def Arccos_body_lemma exp_minus field_simps flip*: *power2_eq_square*)
qed

lemma *Arccos_cos*:
assumes *0* < *Re z* ∧ *Re z* < *pi* ∨
Re z = *0* ∧ *0* ≤ *Im z* ∨
Re z = *pi* ∧ *Im z* ≤ *0*
shows *Arccos*(*cos z*) = *z*
proof −
have ∗: $((i - (exp\ (i * z))^2 * i) / (2 * exp\ (i * z))) = sin\ z$
by (*simp add*: *sin_exp_eq exp_minus field_simps power2_eq_square*)
have $1 - (exp\ (i * z) + inverse\ (exp\ (i * z)))^2 / 4 = ((i - (exp\ (i * z))^2 * i) / (2 * exp\ (i * z)))^2$
by (*simp add*: *field_simps power2_eq_square*)
then have $Arccos(cos\ z) = -\ (i * Ln\ ((exp\ (i * z) + inverse\ (exp\ (i * z))) / 2 +$
$i * csqrt\ (((i - (exp\ (i * z))^2 * i) / (2 * exp\ (i * z)))^2)))$
by (*simp add*: *cos_exp_eq Arccos_def exp_minus power_divide*)
also have ... $= -\ (i * Ln\ ((exp\ (i * z) + inverse\ (exp\ (i * z))) / 2 +$
$i * ((i - (exp\ (i * z))^2 * i) / (2 * exp\ (i * z)))))$
apply (*subst csqrt_square*)
using *assms Re_sin_pos* [*of z*] *Im_sin_nonneg* [*of z*] *Im_sin_nonneg2* [*of z*]
by (*auto simp*: ∗ *Re_sin Im_sin*)
also have ... $= -\ (i * Ln\ (exp\ (i*z)))$
by (*simp add*: *field_simps power2_eq_square*)
also have ... = *z*
using *assms*
by (*subst Complex_Transcendental.Ln_exp*, *auto*)
finally show *?thesis* .
qed

lemma *Arccos_unique*:
⟦*cos z* = *w*;
0 < *Re z* ∧ *Re z* < *pi* ∨

Re z = 0 ∧ 0 ≤ Im z ∨
Re z = pi ∧ Im z ≤ 0⟧ ⟹ *Arccos w = z*
using *Arccos_cos* **by** *blast*

lemma *Arccos_0* [*simp*]: *Arccos 0 = pi/2*
by (*rule Arccos_unique*) *auto*

lemma *Arccos_1* [*simp*]: *Arccos 1 = 0*
by (*rule Arccos_unique*) *auto*

lemma *Arccos_minus1*: *Arccos*(−*1*) = *pi*
by (*rule Arccos_unique*) *auto*

lemma *has_field_derivative_Arccos*:
assumes (*Im z = 0* ⟹ |*Re z*| < *1*)
shows (*Arccos has_field_derivative* − *inverse*(*sin*(*Arccos z*))) (*at z*)
proof −
have x^2 ≠ −*1* **for** *x*::*real*
by (*sos* ((*R<1* + (([~*1*] ∗ *A=0*) + (*R<1* ∗ (*R<1* ∗ [*x__*] ^*2*))))))
with *assms* **have** (*cos* (*Arccos z*))^2 ≠ *1*
by (*auto simp*: *complex_eq_iff Re_power2 Im_power2 abs_square_eq_1*)
then have − *sin* (*Arccos z*) ≠ *0*
by (*metis cos_squared_eq diff_0_right mult_zero_left neg_0_equal_iff_equal power2_eq_square*)
then have (*Arccos has_field_derivative inverse*(− *sin*(*Arccos z*))) (*at z*)
by (*rule has_field_derivative_inverse_basic* [*OF DERIV_cos _ _ open_ball* [*of z*
1]])
(*auto intro*: *isCont_Arccos assms*)
then show *?thesis*
by *simp*
qed

declare *has_field_derivative_Arcsin* [*derivative_intros*]
declare *has_field_derivative_Arcsin* [*THEN DERIV_chain2*, *derivative_intros*]

lemma *field_differentiable_at_Arccos*:
(*Im z = 0* ⟹ |*Re z*| < *1*) ⟹ *Arccos field_differentiable at z*
using *field_differentiable_def has_field_derivative_Arccos* **by** *blast*

lemma *field_differentiable_within_Arccos*:
(*Im z = 0* ⟹ |*Re z*| < *1*) ⟹ *Arccos field_differentiable* (*at z within s*)
using *field_differentiable_at_Arccos field_differentiable_within_subset* **by** *blast*

lemma *continuous_within_Arccos*:
(*Im z = 0* ⟹ |*Re z*| < *1*) ⟹ *continuous* (*at z within s*) *Arccos*
using *continuous_at_imp_continuous_within isCont_Arccos* **by** *blast*

lemma *continuous_on_Arccos* [*continuous_intros*]:
(⋀*z*. *z* ∈ *s* ⟹ *Im z = 0* ⟹ |*Re z*| < *1*) ⟹ *continuous_on s Arccos*
by (*simp add*: *continuous_at_imp_continuous_on*)

lemma *holomorphic_on_Arccos*: ($\bigwedge z.\ z \in s \Longrightarrow Im\ z = 0 \Longrightarrow |Re\ z| < 1) \Longrightarrow$ *Arccos holomorphic_on s*
by (*simp add*: *field_differentiable_within_Arccos holomorphic_on_def*)

6.21.25 Upper and Lower Bounds for Inverse Sine and Cosine

lemma *Arcsin_bounds*: $|Re\ z| < 1 \Longrightarrow |Re(Arcsin\ z)| < pi/2$
unfolding *Re_Arcsin*
by (*blast intro*: *Re_Ln_pos_lt_imp Arcsin_range_lemma*)

lemma *Arccos_bounds*: $|Re\ z| < 1 \Longrightarrow 0 < Re(Arccos\ z) \wedge Re(Arccos\ z) < pi$
unfolding *Re_Arccos*
by (*blast intro*!: *Im_Ln_pos_lt_imp Arccos_range_lemma*)

lemma *Re_Arccos_bounds*: $-pi < Re(Arccos\ z) \wedge Re(Arccos\ z) \le pi$
unfolding *Re_Arccos*
by (*blast intro*!: *mpi_less_Im_Ln Im_Ln_le_pi Arccos_body_lemma*)

lemma *Re_Arccos_bound*: $|Re(Arccos\ z)| \le pi$
by (*meson Re_Arccos_bounds abs_le_iff less_eq_real_def minus_less_iff*)

lemma *Im_Arccos_bound*: $|Im\ (Arccos\ w)| \le cmod\ w$
proof −
have $(Im\ (Arccos\ w))^2 \le (cmod\ (cos\ (Arccos\ w)))^2 - (cos\ (Re\ (Arccos\ w)))^2$
using *norm_cos_squared* [*of Arccos w*] *real_le_abs_sinh* [*of Im* (*Arccos w*)]
by (*simp only*: *abs_le_square_iff*) (*simp add*: *field_split_simps*)
also have ... $\le (cmod\ w)^2$
by (*auto simp*: *cmod_power2*)
finally show *?thesis*
using *abs_le_square_iff* **by** *force*
qed

lemma *Re_Arcsin_bounds*: $-pi < Re(Arcsin\ z)\ \&\ Re(Arcsin\ z) \le pi$
unfolding *Re_Arcsin*
by (*blast intro*!: *mpi_less_Im_Ln Im_Ln_le_pi Arcsin_body_lemma*)

lemma *Re_Arcsin_bound*: $|Re(Arcsin\ z)| \le pi$
by (*meson Re_Arcsin_bounds abs_le_iff less_eq_real_def minus_less_iff*)

lemma *norm_Arccos_bounded*:
fixes *w* :: *complex*
shows *norm* (*Arccos w*) $\le$ *pi* + *norm w*
proof −
have *Re*: $(Re\ (Arccos\ w))^2 \le pi^2\ (Im\ (Arccos\ w))^2 \le (cmod\ w)^2$
using *Re_Arccos_bound* [*of w*] *Im_Arccos_bound* [*of w*] *abs_le_square_iff* **by** *force+*
have $Arccos\ w \cdot Arccos\ w \le pi^2 + (cmod\ w)^2$

using *Re* **by** (*simp add*: *dot_square_norm cmod_power2* [*of Arccos w*])
then have *cmod* (*Arccos w*) ≤ *pi* + *cmod* (*cos* (*Arccos w*))
apply (*simp add*: *norm_le_square*)
by (*metis dot_square_norm norm_ge_zero norm_le_square pi_ge_zero triangle_lemma*)
then show *cmod* (*Arccos w*) ≤ *pi* + *cmod w*
by *auto*
qed

6.21.26 Interrelations between Arcsin and Arccos

lemma *cos_Arcsin_nonzero*:
assumes $z^2 \neq 1$ **shows** *cos*(*Arcsin z*) ≠ *0*
proof −
have *eq*: $(\mathrm{i} * z * (csqrt\ (1 - z^2)))^2 = z^2 * (z^2 - 1)$
by (*simp add*: *algebra_simps*)
have $\mathrm{i} * z * (csqrt\ (1 - z^2)) \neq z^2 - 1$
proof
assume $\mathrm{i} * z * (csqrt\ (1 - z^2)) = z^2 - 1$
then have $(\mathrm{i} * z * (csqrt\ (1 - z^2)))^2 = (z^2 - 1)^2$
by *simp*
then have $z^2 * (z^2 - 1) = (z^2 - 1)*(z^2 - 1)$
using *eq power2_eq_square* **by** *auto*
then show *False*
using *assms* **by** *simp*
qed
then have $1 + \mathrm{i} * z * (csqrt\ (1 - z * z)) \neq z^2$
by (*metis add_minus_cancel power2_eq_square uminus_add_conv_diff*)
then have $2*(1 + \mathrm{i} * z * (csqrt\ (1 - z * z))) \neq 2*z^2$
by (*metis mult_cancel_left zero_neq_numeral*)
then have $(\mathrm{i} * z + csqrt\ (1 - z^2))^2 \neq -1$
using *assms*
apply (*simp add*: *power2_sum*)
apply (*simp add*: *power2_eq_square algebra_simps*)
done
then show *?thesis*
apply (*simp add*: *cos_exp_eq Arcsin_def exp_minus*)
apply (*simp add*: *divide_simps Arcsin_body_lemma*)
apply (*metis add.commute minus_unique power2_eq_square*)
done
qed

lemma *sin_Arccos_nonzero*:
assumes $z^2 \neq 1$ **shows** *sin*(*Arccos z*) ≠ *0*
proof −
have *eq*: $(\mathrm{i} * z * (csqrt\ (1 - z^2)))^2 = -(z^2) * (1 - z^2)$
by (*simp add*: *algebra_simps*)
have $\mathrm{i} * z * (csqrt\ (1 - z^2)) \neq 1 - z^2$
proof
assume $\mathrm{i} * z * (csqrt\ (1 - z^2)) = 1 - z^2$

then have (i ∗ z ∗ (*csqrt* (*1* − z^2)))^2 = (*1* − z^2)^2
by *simp*
then have −(z^2) ∗ (*1* − z^2) = (*1* − z^2)∗(*1* − z^2)
using *eq power2_eq_square* **by** *auto*
then have −(z^2) = (*1* − z^2)
using *assms*
by (*metis add.commute add.right_neutral diff_add_cancel mult_right_cancel*)
then show *False*
using *assms* **by** *simp*
qed
then have z^2 + i ∗ z ∗ (*csqrt* (*1* − z^2)) ≠ *1*
by (*simp add*: *algebra_simps*)
then have *2*∗(z^2 + i ∗ z ∗ (*csqrt* (*1* − z^2))) ≠ *2*∗*1*
by (*metis mult_cancel_left2 zero_neq_numeral*)
then have (z + i ∗ *csqrt* (*1* − z^2))^2 ≠ *1*
using *assms*
by (*metis Arccos_def add.commute add.left_neutral cancel_comm_monoid_add_class.diff_cancel cos_Arccos csqrt_0 mult_zero_right*)
then show *?thesis*
apply (*simp add*: *sin_exp_eq Arccos_def exp_minus*)
apply (*simp add*: *divide_simps Arccos_body_lemma*)
apply (*simp add*: *power2_eq_square*)
done
qed

lemma *cos_sin_csqrt*:
assumes *0* < *cos*(*Re z*) ∨ *cos*(*Re z*) = *0* ∧ *Im z* ∗ *sin*(*Re z*) ≤ *0*
shows *cos z* = *csqrt*(*1* − (*sin z*)^2)
proof (*rule csqrt_unique* [*THEN sym*])
show (*cos z*)^2 = *1* − (*sin z*)^2
by (*simp add*: *cos_squared_eq*)
qed (*use assms* **in** ‹*auto simp*: *Re_cos Im_cos add_pos_pos mult_le_0_iff zero_le_mult_iff*›)

lemma *sin_cos_csqrt*:
assumes *0* < *sin*(*Re z*) ∨ *sin*(*Re z*) = *0* ∧ *0* ≤ *Im z* ∗ *cos*(*Re z*)
shows *sin z* = *csqrt*(*1* − (*cos z*)^2)
proof (*rule csqrt_unique* [*THEN sym*])
show (*sin z*)^2 = *1* − (*cos z*)^2
by (*simp add*: *sin_squared_eq*)
qed (*use assms* **in** ‹*auto simp*: *Re_sin Im_sin add_pos_pos mult_le_0_iff zero_le_mult_iff*›)

lemma *Arcsin_Arccos_csqrt_pos*:
(*0* < *Re z* | *Re z* = *0* & *0* ≤ *Im z*) ⟹ *Arcsin z* = *Arccos*(*csqrt*(*1* − z^2))
by (*simp add*: *Arcsin_def Arccos_def Complex.csqrt_square add.commute*)

lemma *Arccos_Arcsin_csqrt_pos*:
(*0* < *Re z* | *Re z* = *0* & *0* ≤ *Im z*) ⟹ *Arccos z* = *Arcsin*(*csqrt*(*1* − z^2))
by (*simp add*: *Arcsin_def Arccos_def Complex.csqrt_square add.commute*)

lemma *sin_Arccos*:
  $0 < Re\ z \mid Re\ z = 0\ \&\ 0 \leq Im\ z \Longrightarrow sin(Arccos\ z) = csqrt(1 - z^2)$
 by (*simp add*: *Arccos_Arcsin_csqrt_pos*)

lemma *cos_Arcsin*:
  $0 < Re\ z \mid Re\ z = 0\ \&\ 0 \leq Im\ z \Longrightarrow cos(Arcsin\ z) = csqrt(1 - z^2)$
 by (*simp add*: *Arcsin_Arccos_csqrt_pos*)

6.21.27 Relationship with Arcsin on the Real Numbers

lemma *Im_Arcsin_of_real*:
 assumes $|x| \leq 1$
  shows *Im* (*Arcsin* (*of_real x*)) = *0*
proof −
 have *csqrt* (*1* − (*of_real x*)^2) = (*if* x^*2* ≤ *1* *then sqrt* (*1* − x^*2*) *else* i ∗ *sqrt* (x^*2* − *1*))
  by (*simp add*: *of_real_sqrt del*: *csqrt_of_real_nonneg*)
 then have *cmod* (i ∗ *of_real x* + *csqrt* (*1* − (*of_real x*)^2))^*2* = *1*
  using *assms abs_square_le_1*
  by (*force simp add*: *Complex.cmod_power2*)
 then have *cmod* (i ∗ *of_real x* + *csqrt* (*1* − (*of_real x*)^2)) = *1*
  by (*simp add*: *norm_complex_def*)
 then show *?thesis*
  by (*simp add*: *Im_Arcsin exp_minus*)
qed

corollary *Arcsin_in_Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow |Re\ z| \leq 1 \Longrightarrow Arcsin\ z \in \mathbb{R}$
 by (*metis Im_Arcsin_of_real Re_complex_of_real Reals_cases complex_is_Real_iff*)

lemma *arcsin_eq_Re_Arcsin*:
 assumes $|x| \leq 1$
  shows *arcsin x* = *Re* (*Arcsin* (*of_real x*))
unfolding *arcsin_def*
proof (*rule the_equality*, *safe*)
 show − (*pi* / *2*) ≤ *Re* (*Arcsin* (*complex_of_real x*))
  using *Re_Ln_pos_le* [*OF Arcsin_body_lemma*, *of of_real x*]
  by (*auto simp*: *Complex.in_Reals_norm Re_Arcsin*)
next
 show *Re* (*Arcsin* (*complex_of_real x*)) ≤ *pi* / *2*
  using *Re_Ln_pos_le* [*OF Arcsin_body_lemma*, *of of_real x*]
  by (*auto simp*: *Complex.in_Reals_norm Re_Arcsin*)
next
 show *sin* (*Re* (*Arcsin* (*complex_of_real x*))) = *x*
  using *Re_sin* [*of Arcsin* (*of_real x*)] *Arcsin_body_lemma* [*of of_real x*]
  by (*simp add*: *Im_Arcsin_of_real assms*)
next
 fix x'
 assume − (*pi* / *2*) ≤ x' x' ≤ *pi* / *2* *x* = *sin* x'
 then show x' = *Re* (*Arcsin* (*complex_of_real* (*sin* x')))

unfolding *sin_of_real* [*symmetric*]
by (*subst Arcsin_sin*) *auto*
qed

lemma *of_real_arcsin*: $|x| \leq 1 \Longrightarrow$ *of_real*(*arcsin x*) = *Arcsin*(*of_real x*)
by (*metis Im_Arcsin_of_real add.right_neutral arcsin_eq_Re_Arcsin complex_eq mult_zero_right of_real_0*)

6.21.28 Relationship with Arccos on the Real Numbers

lemma *Im_Arccos_of_real*:
assumes $|x| \leq 1$
shows *Im* (*Arccos* (*of_real x*)) = *0*
proof −
have *csqrt* (*1* − (*of_real x*)^2) = (*if* $x\hat{~}2 \leq 1$ *then sqrt* (*1* − $x\hat{~}2$) *else* i ∗ *sqrt* ($x\hat{~}2$ − *1*))
by (*simp add*: *of_real_sqrt del*: *csqrt_of_real_nonneg*)
then have *cmod* (*of_real x* + i ∗ *csqrt* (*1* − (*of_real x*)^2))$\hat{~}2$ = *1*
using *assms abs_square_le_1*
by (*force simp add*: *Complex.cmod_power2*)
then have *cmod* (*of_real x* + i ∗ *csqrt* (*1* − (*of_real x*)^2)) = *1*
by (*simp add*: *norm_complex_def*)
then show *?thesis*
by (*simp add*: *Im_Arccos exp_minus*)
qed

corollary *Arccos_in_Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow |Re\ z| \leq 1 \Longrightarrow Arccos\ z \in \mathbb{R}$
by (*metis Im_Arccos_of_real Re_complex_of_real Reals_cases complex_is_Real_iff*)

lemma *arccos_eq_Re_Arccos*:
assumes $|x| \leq 1$
shows *arccos x* = *Re* (*Arccos* (*of_real x*))
unfolding *arccos_def*
proof (*rule the_equality*, *safe*)
show *0* ≤ *Re* (*Arccos* (*complex_of_real x*))
using *Im_Ln_pos_le* [*OF Arccos_body_lemma*, *of of_real x*]
by (*auto simp*: *Complex.in_Reals_norm Re_Arccos*)
next
show *Re* (*Arccos* (*complex_of_real x*)) ≤ *pi*
using *Im_Ln_pos_le* [*OF Arccos_body_lemma*, *of of_real x*]
by (*auto simp*: *Complex.in_Reals_norm Re_Arccos*)
next
show *cos* (*Re* (*Arccos* (*complex_of_real x*))) = *x*
using *Re_cos* [*of Arccos* (*of_real x*)] *Arccos_body_lemma* [*of of_real x*]
by (*simp add*: *Im_Arccos_of_real assms*)
next
fix x'
assume $0 \leq x'$ $x' \leq pi$ $x = cos\ x'$
then show x' = *Re* (*Arccos* (*complex_of_real* (*cos* x')))

unfolding *cos_of_real* [*symmetric*]
by (*subst Arccos_cos*) *auto*
qed

lemma *of_real_arccos*: $|x| \leq 1 \Longrightarrow$ *of_real*(*arccos x*) = *Arccos*(*of_real x*)
by (*metis Im_Arccos_of_real add.right_neutral arccos_eq_Re_Arccos complex_eq mult_zero_right of_real_0*)

6.21.29 Some interrelationships among the real inverse trig functions

lemma *arccos_arctan*:
assumes $-1 < x$ $x < 1$
shows *arccos x* = *pi*/2 − *arctan*(*x* / *sqrt*(1 − x^2))
proof −
have *arctan*(*x* / *sqrt*(1 − x^2)) − (*pi*/2 − *arccos x*) = 0
proof (*rule sin_eq_0_pi*)
show − *pi* < *arctan* (*x* / *sqrt* (1 − x^2)) − (*pi* / 2 − *arccos x*)
using *arctan_lbound* [*of x* / *sqrt*(1 − x^2)] *arccos_bounded* [*of x*] *assms*
by (*simp add*: *algebra_simps*)
next
show *arctan* (*x* / *sqrt* (1 − x^2)) − (*pi* / 2 − *arccos x*) < *pi*
using *arctan_ubound* [*of x* / *sqrt*(1 − x^2)] *arccos_bounded* [*of x*] *assms*
by (*simp add*: *algebra_simps*)
next
show *sin* (*arctan* (*x* / *sqrt* (1 − x^2)) − (*pi* / 2 − *arccos x*)) = 0
using *assms*
by (*simp add*: *algebra_simps sin_diff cos_add sin_arccos sin_arctan cos_arctan power2_eq_square square_eq_1_iff*)
qed
then show *?thesis*
by *simp*
qed

lemma *arcsin_plus_arccos*:
assumes $-1 \leq x$ $x \leq 1$
shows *arcsin x* + *arccos x* = *pi*/2
proof −
have *arcsin x* = *pi*/2 − *arccos x*
apply (*rule sin_inj_pi*)
using *assms arcsin* [*OF assms*] *arccos* [*OF assms*]
by (*auto simp*: *algebra_simps sin_diff*)
then show *?thesis*
by (*simp add*: *algebra_simps*)
qed

lemma *arcsin_arccos_eq*: $-1 \leq x \Longrightarrow x \leq 1 \Longrightarrow$ *arcsin x* = *pi*/2 − *arccos x*
using *arcsin_plus_arccos* **by** *force*

lemma *arccos_arcsin_eq*: $-1 \le x \Longrightarrow x \le 1 \Longrightarrow$ *arccos x* = *pi*/*2* − *arcsin x*
  using *arcsin_plus_arccos* **by** *force*

lemma *arcsin_arctan*: $-1 < x \Longrightarrow x < 1 \Longrightarrow$ *arcsin x* = *arctan*(*x* / *sqrt*(*1* − x^2))
  by (*simp add*: *arccos_arctan arcsin_arccos_eq*)

lemma *csqrt_1_diff_eq*: *csqrt* (*1* − (*of_real x*)^2) = (*if x^2* ≤ *1 then sqrt* (*1* − *x^2*) *else* i ∗ *sqrt* (*x^2* − *1*))
  by (*simp add*: *of_real_sqrt del*: *csqrt_of_real_nonneg*)

lemma *arcsin_arccos_sqrt_pos*: $0 \le x \Longrightarrow x \le 1 \Longrightarrow$ *arcsin x* = *arccos*(*sqrt*(*1* − x^2))
  apply (*simp add*: *abs_square_le_1 arcsin_eq_Re_Arcsin arccos_eq_Re_Arccos*)
  apply (*subst Arcsin_Arccos_csqrt_pos*)
  apply (*auto simp*: *power_le_one csqrt_1_diff_eq*)
  done

lemma *arcsin_arccos_sqrt_neg*: $-1 \le x \Longrightarrow x \le 0 \Longrightarrow$ *arcsin x* = −*arccos*(*sqrt*(*1* − x^2))
  using *arcsin_arccos_sqrt_pos* [*of* −*x*]
  by (*simp add*: *arcsin_minus*)

lemma *arccos_arcsin_sqrt_pos*: $0 \le x \Longrightarrow x \le 1 \Longrightarrow$ *arccos x* = *arcsin*(*sqrt*(*1* − x^2))
  apply (*simp add*: *abs_square_le_1 arcsin_eq_Re_Arcsin arccos_eq_Re_Arccos*)
  apply (*subst Arccos_Arcsin_csqrt_pos*)
  apply (*auto simp*: *power_le_one csqrt_1_diff_eq*)
  done

lemma *arccos_arcsin_sqrt_neg*: $-1 \le x \Longrightarrow x \le 0 \Longrightarrow$ *arccos x* = *pi* − *arcsin*(*sqrt*(*1* − x^2))
  using *arccos_arcsin_sqrt_pos* [*of* −*x*]
  by (*simp add*: *arccos_minus*)

6.21.30 Continuity results for arcsin and arccos

lemma *continuous_on_Arcsin_real* [*continuous_intros*]:
    continuous_on {*w* ∈ ℝ. |*Re w*| ≤ *1*} *Arcsin*
proof −
  have *continuous_on* {*w* ∈ ℝ. |*Re w*| ≤ *1*} (λ*x*. *complex_of_real* (*arcsin* (*Re x*))) =
      continuous_on {*w* ∈ ℝ. |*Re w*| ≤ *1*} (λ*x*. *complex_of_real* (*Re* (*Arcsin* (*of_real* (*Re x*)))))
    by (*rule continuous_on_cong* [*OF refl*]) (*simp add*: *arcsin_eq_Re_Arcsin*)
  also have ... = *?thesis*
    by (*rule continuous_on_cong* [*OF refl*]) *simp*
  finally show *?thesis*
    using *continuous_on_arcsin* [*OF continuous_on_Re* [*OF continuous_on_id*], *of*

{*w* ∈ ℝ. |*Re w*| ≤ *1*}]
continuous_on_of_real
by *fastforce*
qed

lemma *continuous_within_Arcsin_real*:
continuous (*at z within* {*w* ∈ ℝ. |*Re w*| ≤ *1*}) *Arcsin*
proof (*cases z* ∈ {*w* ∈ ℝ. |*Re w*| ≤ *1*})
case *True* **then show** *?thesis*
using *continuous_on_Arcsin_real continuous_on_eq_continuous_within*
by *blast*
next
case *False*
with *closed_real_abs_le* [*of 1*] **show** *?thesis*
by (*rule continuous_within_closed_nontrivial*)
qed

lemma *continuous_on_Arccos_real*:
continuous_on {*w* ∈ ℝ. |*Re w*| ≤ *1*} *Arccos*
proof −
have *continuous_on* {*w* ∈ ℝ. |*Re w*| ≤ *1*} (λ*x*. *complex_of_real* (*arccos* (*Re x*))) =
continuous_on {*w* ∈ ℝ. |*Re w*| ≤ *1*} (λ*x*. *complex_of_real* (*Re* (*Arccos* (*of_real* (*Re x*)))))
by (*rule continuous_on_cong* [*OF refl*]) (*simp add*: *arccos_eq_Re_Arccos*)
also have ... = *?thesis*
by (*rule continuous_on_cong* [*OF refl*]) *simp*
finally show *?thesis*
using *continuous_on_arccos* [*OF continuous_on_Re* [*OF continuous_on_id*], *of* {*w* ∈ ℝ. |*Re w*| ≤ *1*}]
continuous_on_of_real
by *fastforce*
qed

lemma *continuous_within_Arccos_real*:
continuous (*at z within* {*w* ∈ ℝ. |*Re w*| ≤ *1*}) *Arccos*
proof (*cases z* ∈ {*w* ∈ ℝ. |*Re w*| ≤ *1*})
case *True* **then show** *?thesis*
using *continuous_on_Arccos_real continuous_on_eq_continuous_within*
by *blast*
next
case *False*
with *closed_real_abs_le* [*of 1*] **show** *?thesis*
by (*rule continuous_within_closed_nontrivial*)
qed

lemma *sinh_ln_complex*: *x* ≠ *0* ⟹ *sinh* (*ln x* :: *complex*) = (*x* − *inverse x*) / *2*
by (*simp add*: *sinh_def exp_minus scaleR_conv_of_real exp_of_real*)

lemma *cosh_ln_complex*: *x* ≠ *0* ⟹ *cosh* (*ln x* :: *complex*) = (*x* + *inverse x*) / *2*
  by (*simp add*: *cosh_def exp_minus scaleR_conv_of_real*)

lemma *tanh_ln_complex*: *x* ≠ *0* ⟹ *tanh* (*ln x* :: *complex*) = (*x* ˆ *2* − *1*) / (*x* ˆ *2* + *1*)
  by (*simp add*: *tanh_def sinh_ln_complex cosh_ln_complex divide_simps power2_eq_square*)

6.21.31 Roots of unity

```
theorem complex_root_unity:
  fixes j::nat
  assumes n ≠ 0
    shows exp(2 ∗ of_real pi ∗ i ∗ of_nat j / of_nat n)ˆn = 1
proof −
  have ∗: of_nat j ∗ (complex_of_real pi ∗ 2) = complex_of_real (2 ∗ real j ∗ pi)
    by (simp)
  then show ?thesis
    apply (simp add: exp_of_nat_mult [symmetric] mult_ac exp_Euler)
    apply (simp only: ∗ cos_of_real sin_of_real)
    apply simp
    done
qed

lemma complex_root_unity_eq:
  fixes j::nat and k::nat
  assumes 1 ≤ n
    shows (exp(2 ∗ of_real pi ∗ i ∗ of_nat j / of_nat n) = exp(2 ∗ of_real pi ∗ i ∗
of_nat k / of_nat n)
           ⟷ j mod n = k mod n)
proof −
    have (∃ z::int. i ∗ (of_nat j ∗ (of_real pi ∗ 2)) =
              i ∗ (of_nat k ∗ (of_real pi ∗ 2)) + i ∗ (of_int z ∗ (of_nat n ∗ (of_real pi
∗ 2)))) ⟷
          (∃ z::int. of_nat j ∗ (i ∗ (of_real pi ∗ 2)) =
              (of_nat k + of_nat n ∗ of_int z) ∗ (i ∗ (of_real pi ∗ 2)))
      by (simp add: algebra_simps)
   also have ... ⟷ (∃ z::int. of_nat j = of_nat k + of_nat n ∗ (of_int z :: complex))
      by simp
    also have ... ⟷ (∃ z::int. of_nat j = of_nat k + of_nat n ∗ z)
     by (metis (mono_tags, hide_lams) of_int_add of_int_eq_iff of_int_mult of_int_of_nat_eq)
    also have ... ⟷ int j mod int n = int k mod int n
      by (auto simp: mod_eq_dvd_iff dvd_def algebra_simps)
    also have ... ⟷ j mod n = k mod n
      by (metis of_nat_eq_iff zmod_int)
    finally have (∃ z. i ∗ (of_nat j ∗ (of_real pi ∗ 2)) =
             i ∗ (of_nat k ∗ (of_real pi ∗ 2)) + i ∗ (of_int z ∗ (of_nat n ∗ (of_real pi
∗ 2)))) ⟷ j mod n = k mod n .
   note ∗ = this
  show ?thesis
```

using *assms*
by (*simp add*: *exp_eq field_split_simps* ∗)
qed

corollary *bij_betw_roots_unity*:
bij_betw (λ*j*. *exp*(*2* ∗ *of_real pi* ∗ i ∗ *of_nat j* / *of_nat n*))
{..<*n*} {*exp*(*2* ∗ *of_real pi* ∗ i ∗ *of_nat j* / *of_nat n*) | *j*. *j* < *n*}
by (*auto simp*: *bij_betw_def inj_on_def complex_root_unity_eq*)

lemma *complex_root_unity_eq_1*:
fixes *j*::*nat* **and** *k*::*nat*
assumes *1* ≤ *n*
shows *exp*(*2* ∗ *of_real pi* ∗ i ∗ *of_nat j* / *of_nat n*) = *1* ⟷ *n dvd j*
proof −
have *1* = *exp*(*2* ∗ *of_real pi* ∗ i ∗ (*of_nat n* / *of_nat n*))
using *assms* **by** *simp*
then have *exp*(*2* ∗ *of_real pi* ∗ i ∗ (*of_nat j* / *of_nat n*)) = *1* ⟷ *j mod n* = *n mod n*
using *complex_root_unity_eq* [*of n j n*] *assms*
by *simp*
then show *?thesis*
by *auto*
qed

lemma *finite_complex_roots_unity_explicit*:
finite {*exp*(*2* ∗ *of_real pi* ∗ i ∗ *of_nat j* / *of_nat n*) | *j*::*nat*. *j* < *n*}
by *simp*

lemma *card_complex_roots_unity_explicit*:
card {*exp*(*2* ∗ *of_real pi* ∗ i ∗ *of_nat j* / *of_nat n*) | *j*::*nat*. *j* < *n*} = *n*
by (*simp add*: *Finite_Set.bij_betw_same_card* [*OF bij_betw_roots_unity*, *symmetric*])

lemma *complex_roots_unity*:
assumes *1* ≤ *n*
shows {*z*::*complex*. *z*ˆ*n* = *1*} = {*exp*(*2* ∗ *of_real pi* ∗ i ∗ *of_nat j* / *of_nat n*) | *j*. *j* < *n*}
apply (*rule Finite_Set.card_seteq* [*symmetric*])
using *assms*
apply (*auto simp*: *card_complex_roots_unity_explicit finite_roots_unity complex_root_unity card_roots_unity*)
done

lemma *card_complex_roots_unity*: *1* ≤ *n* ⟹ *card* {*z*::*complex*. *z*ˆ*n* = *1*} = *n*
by (*simp add*: *card_complex_roots_unity_explicit complex_roots_unity*)

lemma *complex_not_root_unity*:
1 ≤ *n* ⟹ ∃ *u*::*complex*. *norm u* = *1* ∧ *u*ˆ*n* ≠ *1*
apply (*rule_tac x*=*exp* (*of_real pi* ∗ i ∗ *of_real* (*1* / *n*)) **in** *exI*)

apply (*auto simp*: *Re_complex_div_eq_0 exp_of_nat_mult* [*symmetric*] *mult_ac exp_Euler*)
done

end

6.22 Harmonic Numbers

theory *Harmonic_Numbers*
imports
Complex_Transcendental
Summation_Tests
begin

The definition of the Harmonic Numbers and the Euler-Mascheroni constant. Also provides a reasonably accurate approximation of *ln 2* and the Euler-Mascheroni constant.

6.22.1 The Harmonic numbers

definition *harm* :: *nat* ⇒ *'a* :: *real_normed_field* **where**
harm n = (∑ *k*=*1*..*n*. *inverse* (*of_nat k*))

lemma *harm_altdef*: *harm n* = (∑ *k*<*n*. *inverse* (*of_nat* (*Suc k*)))
unfolding *harm_def* **by** (*induction n*) *simp_all*

lemma *harm_Suc*: *harm* (*Suc n*) = *harm n* + *inverse* (*of_nat* (*Suc n*))
by (*simp add*: *harm_def*)

lemma *harm_nonneg*: *harm n* ≥ (*0* :: *'a* :: {*real_normed_field*,*linordered_field*})
unfolding *harm_def* **by** (*intro sum_nonneg*) *simp_all*

lemma *harm_pos*: *n* > *0* ⟹ *harm n* > (*0* :: *'a* :: {*real_normed_field*,*linordered_field*})
unfolding *harm_def* **by** (*intro sum_pos*) *simp_all*

lemma *harm_mono*: *m* ≤ *n* ⟹ *harm m* ≤ (*harm n* :: *'a* :: {*real_normed_field*,*linordered_field*})
by(*simp add*: *harm_def sum_mono2*)

lemma *of_real_harm*: *of_real* (*harm n*) = *harm n*
unfolding *harm_def* **by** *simp*

lemma *abs_harm* [*simp*]: (*abs* (*harm n*) :: *real*) = *harm n*
using *harm_nonneg*[*of n*] **by** (*rule abs_of_nonneg*)

lemma *norm_harm*: *norm* (*harm n*) = *harm n*
by (*subst of_real_harm* [*symmetric*]) (*simp add*: *harm_nonneg*)

lemma *harm_expand*:
harm 0 = *0*

harm (*Suc 0*) = *1*
harm (*numeral n*) = *harm* (*pred_numeral n*) + *inverse* (*numeral n*)
proof −
have *numeral n* = *Suc* (*pred_numeral n*) **by** *simp*
also have *harm* . . . = *harm* (*pred_numeral n*) + *inverse* (*numeral n*)
by (*subst harm_Suc*, *subst numeral_eq_Suc*[*symmetric*]) *simp*
finally show *harm* (*numeral n*) = *harm* (*pred_numeral n*) + *inverse* (*numeral n*) **.**
qed (*simp_all add*: *harm_def*)

theorem *not_convergent_harm*: ¬*convergent* (*harm* :: *nat* ⇒ *'a* :: *real_normed_field*)
proof −
have *convergent* (λ*n*. *norm* (*harm n* :: *'a*)) ⟷
convergent (*harm* :: *nat* ⇒ *real*) **by** (*simp add*: *norm_harm*)
also have . . . ⟷ *convergent* (λ*n*. ∑ *k*=*Suc 0*..*Suc n*. *inverse* (*of_nat k*) :: *real*)
unfolding *harm_def*[*abs_def*] **by** (*subst convergent_Suc_iff*) *simp_all*
also have ... ⟷ *convergent* (λ*n*. ∑ *k*≤*n*. *inverse* (*of_nat* (*Suc k*)) :: *real*)
by (*subst sum.shift_bounds_cl_Suc_ivl*) (*simp add*: *atLeast0AtMost*)
also have ... ⟷ *summable* (λ*n*. *inverse* (*of_nat n*) :: *real*)
by (*subst summable_Suc_iff* [*symmetric*]) (*simp add*: *summable_iff_convergent'*)
also have ¬... **by** (*rule not_summable_harmonic*)
finally show *?thesis* **by** (*blast dest*: *convergent_norm*)
qed

lemma *harm_pos_iff* [*simp*]: *harm n* > (*0* :: *'a* :: {*real_normed_field*,*linordered_field*}) ⟷ *n* > *0*
by (*rule iffI*, *cases n*, *simp add*: *harm_expand*, *simp*, *rule harm_pos*)

lemma *ln_diff_le_inverse*:
assumes *x* ≥ (*1*::*real*)
shows *ln* (*x* + *1*) − *ln x* < *1* / *x*
proof −
from *assms* **have** ∃ *z*>*x*. *z* < *x* + *1* ∧ *ln* (*x* + *1*) − *ln x* = (*x* + *1* − *x*) ∗ *inverse z*
by (*intro MVT2*) (*auto intro*!: *derivative_eq_intros simp*: *field_simps*)
then obtain *z* **where** *z*: *z* > *x z* < *x* + *1 ln* (*x* + *1*) − *ln x* = *inverse z* **by** *auto*
have *ln* (*x* + *1*) − *ln x* = *inverse z* **by** *fact*
also from *z*(*1*,*2*) *assms* **have** . . . < *1* / *x* **by** (*simp add*: *field_simps*)
finally show *?thesis* **.**
qed

lemma *ln_le_harm*: *ln* (*real n* + *1*) ≤ (*harm n* :: *real*)
proof (*induction n*)
fix *n* **assume** *IH*: *ln* (*real n* + *1*) ≤ *harm n*
have *ln* (*real* (*Suc n*) + *1*) = *ln* (*real n* + *1*) + (*ln* (*real n* + *2*) − *ln* (*real n* + *1*)) **by** *simp*
also have (*ln* (*real n* + *2*) − *ln* (*real n* + *1*)) ≤ *1* / *real* (*Suc n*)
using *ln_diff_le_inverse*[*of real n* + *1*] **by** (*simp add*: *add_ac*)

also note *IH*
also have *harm n + 1 / real (Suc n) = harm (Suc n)* **by** *(simp add: harm_Suc field_simps)*
finally show *ln (real (Suc n) + 1) ≤ harm (Suc n)* **by** *− simp*
qed *(simp_all add: harm_def)*

lemma *harm_at_top: filterlim (harm :: nat ⇒ real) at_top sequentially*
proof *(rule filterlim_at_top_mono)*
show *eventually (λn. harm n ≥ ln (real (Suc n))) at_top*
using *ln_le_harm* **by** *(intro always_eventually allI) (simp_all add: add_ac)*
show *filterlim (λn. ln (real (Suc n))) at_top sequentially*
by *(intro filterlim_compose[OF ln_at_top] filterlim_compose[OF filterlim_real_sequentially] filterlim_Suc)*
qed

6.22.2 The Euler-Mascheroni constant

The limit of the difference between the partial harmonic sum and the natural logarithm (approximately 0.577216). This value occurs e.g. in the definition of the Gamma function.

definition *euler_mascheroni :: 'a :: real_normed_algebra_1* **where**
euler_mascheroni = of_real (lim (λn. harm n − ln (of_nat n)))

lemma *of_real_euler_mascheroni [simp]: of_real euler_mascheroni = euler_mascheroni*
by *(simp add: euler_mascheroni_def)*

lemma *harm_ge_ln: harm n ≥ ln (real n + 1)*
proof *−*
have *ln (n + 1) = (∑j<n. ln (real (Suc j + 1)) − ln (real (j + 1)))*
by *(subst sum_lessThan_telescope) auto*
also have *. . . ≤ (∑j<n. 1 / (Suc j))*
proof *(intro sum_mono, clarify)*
fix *j* **assume** *j: j < n*
have *∃ξ. ξ > real j + 1 ∧ ξ < real j + 2 ∧ ln (real j + 2) − ln (real j + 1) = (real j + 2 − (real j + 1)) ∗ (1 / ξ)*
by *(intro MVT2) (auto intro!: derivative_eq_intros)*
then obtain *ξ :: real*
where *ξ: ξ ∈ {real j + 1..real j + 2} ln (real j + 2) − ln (real j + 1) = 1 / ξ*
by *auto*
note *ξ(2)*
also have *1 / ξ ≤ 1 / (Suc j)*
using *ξ(1)* **by** *(auto simp: field_simps)*
finally show *ln (real (Suc j + 1)) − ln (real (j + 1)) ≤ 1 / (Suc j)*
by *(simp add: add_ac)*
qed
also have *. . . = harm n*
by *(simp add: harm_altdef field_simps)*
finally show *?thesis* **by** *(simp add: add_ac)*

qed

lemma *decseq_harm_diff_ln*: *decseq* (λ*n*. *harm* (*Suc n*) − *ln* (*Suc n*))
proof (*rule decseq_SucI*)
  fix *m* :: *nat*
  define *n* **where** *n* = *Suc m*
  have *n* > *0* **by** (*simp add*: *n_def*)
  have *convex_on* {*0*<..} (λ*x* :: *real*. −*ln x*)
    by (*rule convex_on_realI*[**where** *f'* = λ*x*. −*1*/*x*])
      (*auto intro*!: *derivative_eq_intros simp*: *field_simps*)
  hence (−*1* / (*n* + *1*)) ∗ (*real n* − *real* (*n* + *1*)) ≤ (− *ln* (*real n*)) − (−*ln* (*real* (*n* + *1*)))
    using ‹*n* > *0*› **by** (*intro convex_on_imp_above_tangent*[**where** *A* = {*0*<..}])
      (*auto intro*!: *derivative_eq_intros simp*: *interior_open*)
  thus *harm* (*Suc n*) − *ln* (*Suc n*) ≤ *harm n* − *ln n*
    by (*auto simp*: *harm_Suc field_simps*)
qed

lemma *euler_mascheroni_sequence_nonneg*:
  assumes *n* > *0*
  shows *harm n* − *ln* (*real n*) ≥ (*0* :: *real*)
proof −
  have *ln* (*real n*) ≤ *ln* (*real n* + *1*)
    using *assms* **by** *simp*
  also have . . . ≤ *harm n*
    by (*rule harm_ge_ln*)
  finally show *?thesis* **by** *simp*
qed

lemma *euler_mascheroni_convergent*: *convergent* (λ*n*. *harm n* − *ln n*)
proof −
  have *harm* (*Suc n*) − *ln* (*real* (*Suc n*)) ≥ *0* **for** *n* :: *nat*
    using *euler_mascheroni_sequence_nonneg*[*of Suc n*] **by** *simp*
  hence *convergent* (λ*n*. *harm* (*Suc n*) − *ln* (*Suc n*))
    by (*intro Bseq_monoseq_convergent decseq_bounded*[*of _ 0*] *decseq_harm_diff_ln decseq_imp_monoseq*)
      auto
  thus *?thesis*
    by (*subst* (*asm*) *convergent_Suc_iff*)
qed

lemma *euler_mascheroni_sequence_decreasing*:
  m > *0* =⇒ *m* ≤ *n* =⇒ *harm n* − *ln* (*of_nat n*) ≤ *harm m* − *ln* (*of_nat m* :: *real*)
  using *decseqD*[*OF decseq_harm_diff_ln*, *of m* − *1 n* − *1*] **by** *simp*

lemma *euler_mascheroni_LIMSEQ*:
  (λ*n*. *harm n* − *ln* (*of_nat n*) :: *real*) ⟶ *euler_mascheroni*
  unfolding *euler_mascheroni_def*

by (*simp add*: *convergent_LIMSEQ_iff* [*symmetric*] *euler_mascheroni_convergent*)

lemma *euler_mascheroni_LIMSEQ_of_real*:
(λ*n*. *of_real* (*harm n* − *ln* (*of_nat n*))) ⟶
(*euler_mascheroni* :: ′*a* :: {*real_normed_algebra_1*, *topological_space*})
proof −
have (λ*n*. *of_real* (*harm n* − *ln* (*of_nat n*))) ⟶ (*of_real* (*euler_mascheroni*) :: ′*a*)
by (*intro tendsto_of_real euler_mascheroni_LIMSEQ*)
thus *?thesis* **by** *simp*
qed

lemma *euler_mascheroni_sum_real*:
(λ*n*. *inverse* (*of_nat* (*n*+*1*)) + *ln* (*of_nat* (*n*+*1*)) − *ln* (*of_nat* (*n*+*2*)) :: *real*)
sums euler_mascheroni
using *sums_add*[*OF telescope_sums*[*OF LIMSEQ_Suc*[*OF euler_mascheroni_LIMSEQ*]]
telescope_sums′[*OF LIMSEQ_inverse_real_of_nat*]]
by (*simp_all add*: *harm_def algebra_simps*)

lemma *euler_mascheroni_sum*:
(λ*n*. *inverse* (*of_nat* (*n*+*1*)) + *of_real* (*ln* (*of_nat* (*n*+*1*))) − *of_real* (*ln* (*of_nat* (*n*+*2*))))
sums (*euler_mascheroni* :: ′*a* :: {*banach*, *real_normed_field*})
proof −
have (λ*n*. *of_real* (*inverse* (*of_nat* (*n*+*1*)) + *ln* (*of_nat* (*n*+*1*)) − *ln* (*of_nat* (*n*+*2*))))
sums (*of_real euler_mascheroni* :: ′*a* :: {*banach*, *real_normed_field*})
by (*subst sums_of_real_iff*) (*rule euler_mascheroni_sum_real*)
thus *?thesis* **by** *simp*
qed

theorem *alternating_harmonic_series_sums*: (λ*k*. (−*1*) ^*k* / *real_of_nat* (*Suc k*)) *sums ln 2*
proof −
let *?f* = λ*n*. *harm n* − *ln* (*real_of_nat n*)
let *?g* = λ*n*. *if even n then 0 else* (*2*::*real*)
let *?em* = λ*n*. *harm n* − *ln* (*real_of_nat n*)
have *eventually* (λ*n*. *?em* (*2*∗*n*) − *?em n* + *ln 2* = (∑*k*<*2*∗*n*. (−*1*) ^*k* / *real_of_nat* (*Suc k*))) *at_top*
using *eventually_gt_at_top*[*of 0*::*nat*]
proof *eventually_elim*
fix *n* :: *nat* **assume** *n*: *n* > *0*
have (∑*k*<*2*∗*n*. (−*1*) ^*k* / *real_of_nat* (*Suc k*)) =
(∑*k*<*2*∗*n*. ((−*1*) ^*k* + *?g k*) / *of_nat* (*Suc k*)) − (∑*k*<*2*∗*n*. *?g k* / *of_nat* (*Suc k*))
by (*simp add*: *sum.distrib algebra_simps divide_inverse*)
also have (∑*k*<*2*∗*n*. ((−*1*) ^*k* + *?g k*) / *real_of_nat* (*Suc k*)) = *harm* (*2*∗*n*)
unfolding *harm_altdef* **by** (*intro sum.cong*) (*auto simp*: *field_simps*)
also have (∑*k*<*2*∗*n*. *?g k* / *real_of_nat* (*Suc k*)) = (∑*k*|*k*<*2*∗*n* ∧ *odd k*. *?g*

k / *of_nat* (*Suc k*))
  by (*intro sum.mono_neutral_right*) *auto*
 also have ... = ($\sum$ *k*|*k*<*2*∗*n* ∧ *odd k*. *2* / (*real_of_nat* (*Suc k*)))
  by (*intro sum.cong*) *auto*
 also have ($\sum$ *k*|*k*<*2*∗*n* ∧ *odd k*. *2* / (*real_of_nat* (*Suc k*))) = *harm n*
  unfolding *harm_altdef*
  by (*intro sum.reindex_cong*[*of* λ*n*. *2*∗*n*+*1*]) (*auto simp*: *inj_on_def field_simps elim*!: *oddE*)
 also have *harm* (*2*∗*n*) − *harm n* = *?em* (*2*∗*n*) − *?em n* + *ln 2* **using** *n*
  by (*simp_all add*: *algebra_simps ln_mult*)
 finally show *?em* (*2*∗*n*) − *?em n* + *ln 2* = ($\sum$ *k*<*2*∗*n*. (−*1*) ^*k* / *real_of_nat* (*Suc k*)) **..**
 qed
 moreover have (λ*n*. *?em* (*2*∗*n*) − *?em n* + *ln* (*2*::*real*))
   ⟶ *euler_mascheroni* − *euler_mascheroni* + *ln 2*
  by (*intro tendsto_intros euler_mascheroni_LIMSEQ filterlim_compose*[*OF euler_mascheroni_LIMSEQ*]
   filterlim_subseq) (*auto simp*: *strict_mono_def*)
 hence (λ*n*. *?em* (*2*∗*n*) − *?em n* + *ln* (*2*::*real*)) ⟶ *ln 2* **by** *simp*
 ultimately have (λ*n*. ($\sum$ *k*<*2*∗*n*. (−*1*) ^*k* / *real_of_nat* (*Suc k*))) ⟶ *ln 2*
  by (*blast intro*: *Lim_transform_eventually*)

 moreover have *summable* (λ*k*. (−*1*) ^*k* ∗ *inverse* (*real_of_nat* (*Suc k*)))
  using *LIMSEQ_inverse_real_of_nat*
  by (*intro summable_Leibniz*(*1*) *decseq_imp_monoseq decseq_SucI*) *simp_all*
 hence *A*: (λ*n*. $\sum$ *k*<*n*. (−*1*) ^*k* / *real_of_nat* (*Suc k*)) ⟶ ($\sum$ *k*. (−*1*) ^*k* / *real_of_nat* (*Suc k*))
  by (*simp add*: *summable_sums_iff divide_inverse sums_def*)
 from *filterlim_compose*[*OF this filterlim_subseq*[*of* (∗) (*2*::*nat*)]]
  have (λ*n*. $\sum$ *k*<*2*∗*n*. (−*1*) ^*k* / *real_of_nat* (*Suc k*)) ⟶ ($\sum$ *k*. (−*1*) ^*k* / *real_of_nat* (*Suc k*))
  by (*simp add*: *strict_mono_def*)
 ultimately have ($\sum$ *k*. (− *1*) ^ *k* / *real_of_nat* (*Suc k*)) = *ln 2* **by** (*intro LIMSEQ_unique*)
 with *A* **show** *?thesis* **by** (*simp add*: *sums_def*)
qed

lemma *alternating_harmonic_series_sums'*:
 (λ*k*. *inverse* (*real_of_nat* (*2*∗*k*+*1*)) − *inverse* (*real_of_nat* (*2*∗*k*+*2*))) *sums ln 2*
unfolding *sums_def*
proof (*rule Lim_transform_eventually*)
 show (λ*n*. $\sum$ *k*<*2*∗*n*. (−*1*) ^*k* / (*real_of_nat* (*Suc k*))) ⟶ *ln 2*
  using *alternating_harmonic_series_sums* **unfolding** *sums_def*
  by (*rule filterlim_compose*) (*rule mult_nat_left_at_top*, *simp*)
 show *eventually* (λ*n*. ($\sum$ *k*<*2*∗*n*. (−*1*) ^*k* / (*real_of_nat* (*Suc k*))) =
   ($\sum$ *k*<*n*. *inverse* (*real_of_nat* (*2*∗*k*+*1*)) − *inverse* (*real_of_nat* (*2*∗*k*+*2*)))) *sequentially*
 proof (*intro always_eventually allI*)
  fix *n* :: *nat*

```
    show (∑k<2*n. (−1) ^k / (real_of_nat (Suc k))) =
          (∑k<n. inverse (real_of_nat (2*k+1)) − inverse (real_of_nat (2*k+2)))
      by (induction n) (simp_all add: inverse_eq_divide)
  qed
qed
```

6.22.3 Bounds on the Euler-Mascheroni constant

```
lemma ln_inverse_approx_le:
  assumes (x::real) > 0 a > 0
  shows   ln (x + a) − ln x ≤ a * (inverse x + inverse (x + a))/2 (is _ ≤ ?A)
proof −
  define f' where f' = (inverse (x + a) − inverse x)/a
  let ?f = λt. (t − x) * f' + inverse x
  let ?F = λt. (t − x) ^2 * f' / 2 + t * inverse x

  have deriv: ∃D. ((λx. ?F x − ln x) has_field_derivative D) (at ξ) ∧ D ≥ 0
    if ξ ≥ x ξ ≤ x + a for ξ
  proof −
    from that assms have t: 0 ≤ (ξ − x) / a (ξ − x) / a ≤ 1 by simp_all
    have inverse ξ = inverse ((1 − (ξ − x) / a) *R x + ((ξ − x) / a) *R (x +
a)) (is _ = ?A)
      using assms by (simp add: field_simps)
   also from assms have convex_on {x..x+a} inverse by (intro convex_on_inverse)
auto
    from convex_onD_Icc[OF this _ t] assms
      have ?A ≤ (1 − (ξ − x) / a) * inverse x + (ξ − x) / a * inverse (x + a)
by simp
    also have ... = (ξ − x) * f' + inverse x using assms
      by (simp add: f'_def divide_simps) (simp add: field_simps)
    finally have ?f ξ − 1 / ξ ≥ 0 by (simp add: field_simps)
    moreover have ((λx. ?F x − ln x) has_field_derivative ?f ξ − 1 / ξ) (at ξ)
      using that assms by (auto intro!: derivative_eq_intros simp: field_simps)
    ultimately show ?thesis by blast
  qed
  have ?F x − ln x ≤ ?F (x + a) − ln (x + a)
   by (rule DERIV_nonneg_imp_nondecreasing[of x x + a, OF _ deriv]) (use assms
in auto)
  thus ?thesis
     using assms by (simp add: f'_def divide_simps) (simp add: algebra_simps
power2_eq_square)?
qed

lemma ln_inverse_approx_ge:
  assumes (x::real) > 0 x < y
  shows   ln y − ln x ≥ 2 * (y − x) / (x + y) (is _ ≥ ?A)
proof −
  define m where m = (x+y)/2
  define f' where f' = −inverse (m ^2)
```

from *assms* **have** *m*: *m* > *0* **by** (*simp add*: *m_def*)
let *?F* = λ*t*. (*t* − *m*)ˆ*2* ∗ *f* ′ / *2* + *t* / *m*
let *?f* = λ*t*. (*t* − *m*) ∗ *f* ′ + *inverse m*

have *deriv*: ∃ *D*. ((λ*x*. *ln x* − *?F x*) *has_field_derivative D*) (*at* ξ) ∧ *D* ≥ *0*
if ξ ≥ *x* ξ ≤ *y* **for** ξ
proof −
from *that assms* **have** *inverse* ξ − *inverse m* ≥ *f* ′ ∗ (ξ − *m*)
by (*intro convex_on_imp_above_tangent*[*of* {*0*<..}] *convex_on_inverse*)
(*auto simp*: *m_def interior_open f* ′*_def power2_eq_square intro*!: *derivative_eq_intros*)
hence *1* / ξ − *?f* ξ ≥ *0* **by** (*simp add*: *field_simps f* ′*_def*)
moreover have ((λ*x*. *ln x* − *?F x*) *has_field_derivative 1* / ξ − *?f* ξ) (*at* ξ)
using *that assms m* **by** (*auto intro*!: *derivative_eq_intros simp*: *field_simps*)
ultimately show *?thesis* **by** *blast*
qed
have *ln x* − *?F x* ≤ *ln y* − *?F y*
by (*rule DERIV_nonneg_imp_nondecreasing*[*of x y*, *OF* _ *deriv*]) (*use assms* **in** *auto*)
hence *ln y* − *ln x* ≥ *?F y* − *?F x*
by (*simp add*: *algebra_simps*)
also have *?F y* − *?F x* = *?A*
using *assms* **by** (*simp add*: *f* ′*_def m_def divide_simps*) (*simp add*: *algebra_simps power2_eq_square*)
finally show *?thesis* **.**
qed

lemma *euler_mascheroni_lower*:
euler_mascheroni ≥ *harm* (*Suc n*) − *ln* (*real_of_nat* (*n* + *2*)) + *1*/*real_of_nat* (*2* ∗ (*n* + *2*))
and *euler_mascheroni_upper*:
euler_mascheroni ≤ *harm* (*Suc n*) − *ln* (*real_of_nat* (*n* + *2*)) + *1*/*real_of_nat* (*2* ∗ (*n* + *1*))
proof −
define *D* :: _ ⇒ *real*
where *D n* = *inverse* (*of_nat* (*n*+*1*)) + *ln* (*of_nat* (*n*+*1*)) − *ln* (*of_nat* (*n*+*2*)) **for** *n*
let *?g* = λ*n*. *ln* (*of_nat* (*n*+*2*)) − *ln* (*of_nat* (*n*+*1*)) − *inverse* (*of_nat* (*n*+*1*)) :: *real*
define *inv* **where** [*abs_def*]: *inv n* = *inverse* (*real_of_nat n*) **for** *n*
fix *n* :: *nat*
note *summable* = *sums_summable*[*OF euler_mascheroni_sum_real*, *folded D_def*]
have *sums*: (λ*k*. (*inv* (*Suc* (*k* + (*n*+*1*))) − *inv* (*Suc* (*Suc k* + (*n*+*1*))))/*2*) *sums* ((*inv* (*Suc* (*0* + (*n*+*1*))) − *0*)/*2*)
unfolding *inv_def*
by (*intro sums_divide telescope_sums*′ *LIMSEQ_ignore_initial_segment LIMSEQ_inverse_real_of_nat*)
have *sums*′: (λ*k*. (*inv* (*Suc* (*k* + *n*)) − *inv* (*Suc* (*Suc k* + *n*)))/*2*) *sums* ((*inv* (*Suc* (*0* + *n*)) − *0*)/*2*)

unfolding *inv_def*
by (*intro sums_divide telescope_sums' LIMSEQ_ignore_initial_segment LIMSEQ_inverse_real_of_nat*)
from *euler_mascheroni_sum_real* **have** *euler_mascheroni* = (∑*k*. *D k*)
by (*simp add*: *sums_iff D_def*)
also have ... = (∑*k*. *D* (*k* + *Suc n*)) + (∑*k*≤*n*. *D k*)
by (*subst suminf_split_initial_segment*[*OF summable*, *of Suc n*],
subst lessThan_Suc_atMost) *simp*
finally have *sum*: (∑*k*≤*n*. *D k*) − *euler_mascheroni* = −(∑*k*. *D* (*k* + *Suc n*))
by *simp*

note *sum*
also have ... ≤ −(∑*k*. (*inv* (*k* + *Suc n* + *1*) − *inv* (*k* + *Suc n* + *2*)) / *2*)
proof (*intro le_imp_neg_le suminf_le allI summable_ignore_initial_segment*[*OF summable*])
fix *k'* :: *nat*
define *k* **where** *k* = *k'* + *Suc n*
hence *k*: *k* > *0* **by** (*simp add*: *k_def*)
have *real_of_nat* (*k*+*1*) > *0* **by** (*simp add*: *k_def*)
with *ln_inverse_approx_le*[*OF this zero_less_one*]
have *ln* (*of_nat k* + *2*) − *ln* (*of_nat k* + *1*) ≤ (*inv* (*k*+*1*) + *inv* (*k*+*2*))/*2*
by (*simp add*: *inv_def add_ac*)
hence (*inv* (*k*+*1*) − *inv* (*k*+*2*))/*2* ≤ *inv* (*k*+*1*) + *ln* (*of_nat* (*k*+*1*)) − *ln* (*of_nat* (*k*+*2*))
by (*simp add*: *field_simps*)
also have ... = *D k* **unfolding** *D_def inv_def* **..**
finally show *D* (*k'* + *Suc n*) ≥ (*inv* (*k'* + *Suc n* + *1*) − *inv* (*k'* + *Suc n* + *2*)) / *2*
by (*simp add*: *k_def*)
from *sums_summable*[*OF sums*]
show *summable* (λ*k*. (*inv* (*k* + *Suc n* + *1*) − *inv* (*k* + *Suc n* + *2*))/*2*) **by** *simp*
qed
also from *sums* **have** ... = −*inv* (*n*+*2*) / *2* **by** (*simp add*: *sums_iff*)
finally have *euler_mascheroni* ≥ (∑*k*≤*n*. *D k*) + *1* / (*of_nat* (*2* ∗ (*n*+*2*)))
by (*simp add*: *inv_def field_simps*)
also have (∑*k*≤*n*. *D k*) = *harm* (*Suc n*) − (∑*k*≤*n*. *ln* (*real_of_nat* (*Suc k*+*1*)) − *ln* (*of_nat* (*k*+*1*)))
unfolding *harm_altdef D_def* **by** (*subst lessThan_Suc_atMost*) (*simp add*: *sum.distrib sum_subtractf*)
also have (∑*k*≤*n*. *ln* (*real_of_nat* (*Suc k*+*1*)) − *ln* (*of_nat* (*k*+*1*))) = *ln* (*of_nat* (*n*+*2*))
by (*subst atLeast0AtMost* [*symmetric*], *subst sum_Suc_diff*) *simp_all*
finally show *euler_mascheroni* ≥ *harm* (*Suc n*) − *ln* (*real_of_nat* (*n* + *2*)) + *1*/*real_of_nat* (*2* ∗ (*n* + *2*))
by *simp*

note *sum*
also have −(∑*k*. *D* (*k* + *Suc n*)) ≥ −(∑*k*. (*inv* (*Suc* (*k* + *n*)) − *inv* (*Suc*

(*Suc k* + *n*)))/*2*)
proof (*intro le_imp_neg_le suminf_le allI summable_ignore_initial_segment*[*OF summable*])
fix *k'* :: *nat*
define *k* **where** *k* = *k'* + *Suc n*
hence *k*: *k* > *0* **by** (*simp add*: *k_def*)
have *real_of_nat* (*k*+*1*) > *0* **by** (*simp add*: *k_def*)
from *ln_inverse_approx_ge*[*of of_nat k* + *1 of_nat k* + *2*]
have *2* / (*2* ∗ *real_of_nat k* + *3*) ≤ *ln* (*of_nat* (*k*+*2*)) − *ln* (*real_of_nat* (*k*+*1*))
by (*simp add*: *add_ac*)
hence *D k* ≤ *1* / *real_of_nat* (*k*+*1*) − *2* / (*2* ∗ *real_of_nat k* + *3*)
by (*simp add*: *D_def inverse_eq_divide inv_def*)
also have . . . = *inv* ((*k*+*1*)∗(*2*∗*k*+*3*)) **unfolding** *inv_def* **by** (*simp add*: *field_simps*)
also have . . . ≤ *inv* (*2*∗*k*∗(*k*+*1*)) **unfolding** *inv_def* **using** *k*
by (*intro le_imp_inverse_le*)
(*simp add*: *algebra_simps*, *simp del*: *of_nat_add*)
also have . . . = (*inv k* − *inv* (*k*+*1*))/*2* **unfolding** *inv_def* **using** *k*
by (*simp add*: *divide_simps del*: *of_nat_mult*) (*simp add*: *algebra_simps*)
finally show *D k* ≤ (*inv* (*Suc* (*k'* + *n*)) − *inv* (*Suc* (*Suc k'* + *n*)))/*2* **unfolding** *k_def* **by** *simp*
next
from *sums_summable*[*OF sums'*]
show *summable* (λ*k*. (*inv* (*Suc* (*k* + *n*)) − *inv* (*Suc* (*Suc k* + *n*)))/*2*) **by** *simp*
qed
also from *sums'* **have** (∑ *k*. (*inv* (*Suc* (*k* + *n*)) − *inv* (*Suc* (*Suc k* + *n*)))/*2*) = *inv* (*n*+*1*)/*2*
by (*simp add*: *sums_iff*)
finally have *euler_mascheroni* ≤ (∑ *k*≤*n*. *D k*) + *1* / *of_nat* (*2* ∗ (*n*+*1*))
by (*simp add*: *inv_def field_simps*)
also have (∑ *k*≤*n*. *D k*) = *harm* (*Suc n*) − (∑ *k*≤*n*. *ln* (*real_of_nat* (*Suc k*+*1*)) − *ln* (*of_nat* (*k*+*1*)))
unfolding *harm_altdef D_def* **by** (*subst lessThan_Suc_atMost*) (*simp add*: *sum.distrib sum_subtractf*)
also have (∑ *k*≤*n*. *ln* (*real_of_nat* (*Suc k*+*1*)) − *ln* (*of_nat* (*k*+*1*))) = *ln* (*of_nat* (*n*+*2*))
by (*subst atLeast0AtMost* [*symmetric*], *subst sum_Suc_diff*) *simp_all*
finally show *euler_mascheroni* ≤ *harm* (*Suc n*) − *ln* (*real_of_nat* (*n* + *2*)) + *1*/*real_of_nat* (*2* ∗ (*n* + *1*))
by *simp*
qed

lemma *euler_mascheroni_pos*: *euler_mascheroni* > (*0*::*real*)
using *euler_mascheroni_lower*[*of 0*] *ln_2_less_1* **by** (*simp add*: *harm_def*)

context
begin

private lemma *ln_approx_aux*:
  fixes *n* :: *nat* **and** *x* :: *real*
  defines *y* ≡ (*x*−*1*)/(*x*+*1*)
  assumes *x*: *x* > *0 x* ≠ *1*
  shows *inverse* (*2*∗*y*ˆ(*2*∗*n*+*1*)) ∗ (*ln x* − (∑*k*<*n*. *2*∗*y*ˆ(*2*∗*k*+*1*) / *of_nat* (*2*∗*k*+*1*)))
∈
      {*0*..(*1* / (*1* − *y*ˆ*2*) / *of_nat* (*2*∗*n*+*1*))}
proof −
  from *x* **have** *norm_y*: *norm y* < *1* **unfolding** *y_def* **by** *simp*
  from *power_strict_mono*[*OF this*, *of 2*] **have** *norm_y'*: *norm y*ˆ*2* < *1* **by** *simp*

  let *?f* = λ*k*. *2* ∗ *y* ˆ (*2*∗*k*+*1*) / *of_nat* (*2*∗*k*+*1*)
  note *sums* = *ln_series_quadratic*[*OF x*(*1*)]
  define *c* **where** *c* = *inverse* (*2*∗*y*ˆ(*2*∗*n*+*1*))
  let *?d* = *c* ∗ (*ln x* − (∑*k*<*n*. *?f k*))
  have ⋀*k*. y^2 ˆ*k* / *of_nat* (*2*∗(*k*+*n*)+*1*) ≤ y^2 ˆ *k* / *of_nat* (*2*∗*n*+*1*)
    by (*intro divide_left_mono mult_right_mono mult_pos_pos zero_le_power*[*of y*ˆ*2*])
simp_all
  moreover {
    have (λ*k*. *?f* (*k* + *n*)) *sums* (*ln x* − (∑*k*<*n*. *?f k*))
      using *sums_split_initial_segment*[*OF sums*] **by** (*simp add*: *y_def*)
    hence (λ*k*. *c* ∗ *?f* (*k* + *n*)) *sums ?d* **by** (*rule sums_mult*)
    also have (λ*k*. *c* ∗ (*2*∗*y*ˆ(*2*∗(*k*+*n*)+*1*) / *of_nat* (*2*∗(*k*+*n*)+*1*))) =
               (λ*k*. (*c* ∗ (*2*∗*y*ˆ(*2*∗*n*+*1*))) ∗ ((*y*ˆ*2*)ˆ*k* / *of_nat* (*2*∗(*k*+*n*)+*1*)))
      by (*simp only*: *ring_distribs power_add power_mult*) (*simp add*: *mult_ac*)
    also from *x* **have** *c* ∗ (*2*∗*y*ˆ(*2*∗*n*+*1*)) = *1* **by** (*simp add*: *c_def y_def*)
    finally have (λ*k*. (*y*ˆ*2*)ˆ*k* / *of_nat* (*2*∗(*k*+*n*)+*1*)) *sums ?d* **by** *simp*
  } **note** *sums'* = *this*
  moreover from *norm_y'* **have** (λ*k*. (*y*ˆ*2*)ˆ*k* / *of_nat* (*2*∗*n*+*1*)) *sums* (*1* / (*1*
− *y*ˆ*2*) / *of_nat* (*2*∗*n*+*1*))
    by (*intro sums_divide geometric_sums*) (*simp_all add*: *norm_power*)
  ultimately have *?d* ≤ (*1* / (*1* − *y*ˆ*2*) / *of_nat* (*2*∗*n*+*1*)) **by** (*rule sums_le*)
  moreover have *c* ∗ (*ln x* − (∑*k*<*n*. *2* ∗ *y* ˆ (*2* ∗ *k* + *1*) / *real_of_nat* (*2* ∗ *k*
+ *1*))) ≥ *0*
    by (*intro sums_le*[*OF _ sums_zero sums'*]) *simp_all*
  ultimately show *?thesis* **unfolding** *c_def* **by** *simp*
qed

lemma
  fixes *n* :: *nat* **and** *x* :: *real*
  defines *y* ≡ (*x*−*1*)/(*x*+*1*)
  defines *approx* ≡ (∑*k*<*n*. *2*∗*y*ˆ(*2*∗*k*+*1*) / *of_nat* (*2*∗*k*+*1*))
  defines *d* ≡ *y*ˆ(*2*∗*n*+*1*) / (*1* − *y*ˆ*2*) / *of_nat* (*2*∗*n*+*1*)
  assumes *x*: *x* > *1*
  shows *ln_approx_bounds*: *ln x* ∈ {*approx*..*approx* + *2*∗*d*}
  and *ln_approx_abs*: *abs* (*ln x* − (*approx* + *d*)) ≤ *d*
proof −
  define *c* **where** *c* = *2*∗*y*ˆ(*2*∗*n*+*1*)
  from *x* **have** *c_pos*: *c* > *0* **unfolding** *c_def y_def*

by (*intro mult_pos_pos zero_less_power*) *simp_all*
have *A*: *inverse c* ∗ (*ln x* − (∑ *k*<*n*. *2*∗*y*ˆ(*2*∗*k*+*1*) / *of_nat* (*2*∗*k*+*1*))) ∈
{*0*.. (*1* / (*1* − *y*ˆ*2*) / *of_nat* (*2*∗*n*+*1*))} **using** *assms* **unfolding** *y_def c_def*
by (*intro ln_approx_aux*) *simp_all*
hence *inverse c* ∗ (*ln x* − (∑ *k*<*n*. *2*∗*y*ˆ(*2*∗*k*+*1*)/*of_nat* (*2*∗*k*+*1*))) ≤ (*1* / (*1*−*y*ˆ*2*) / *of_nat* (*2*∗*n*+*1*))
by *simp*
hence (*ln x* − (∑ *k*<*n*. *2*∗*y*ˆ(*2*∗*k*+*1*) / *of_nat* (*2*∗*k*+*1*))) / *c* ≤ (*1* / (*1* − *y*ˆ*2*) / *of_nat* (*2*∗*n*+*1*))
by (*auto simp add*: *field_split_simps*)
with *c_pos* **have** *ln x* ≤ *c* / (*1* − *y*ˆ*2*) / *of_nat* (*2*∗*n*+*1*) + *approx*
by (*subst* (*asm*) *pos_divide_le_eq*) (*simp_all add*: *mult_ac approx_def*)
moreover {
from *A c_pos* **have** *0* ≤ *c* ∗ (*inverse c* ∗ (*ln x* − (∑ *k*<*n*. *2*∗*y*ˆ(*2*∗*k*+*1*) / *of_nat* (*2*∗*k*+*1*))))
by (*intro mult_nonneg_nonneg*[*of c*]) *simp_all*
also have . . . = (*c* ∗ *inverse c*) ∗ (*ln x* − (∑ *k*<*n*. *2*∗*y*ˆ(*2*∗*k*+*1*) / *of_nat* (*2*∗*k*+*1*)))
by (*simp add*: *mult_ac*)
also from *c_pos* **have** *c* ∗ *inverse c* = *1* **by** *simp*
finally have *ln x* ≥ *approx* **by** (*simp add*: *approx_def*)
}
ultimately show *ln x* ∈ {*approx*..*approx* + *2*∗*d*} **by** (*simp add*: *c_def d_def*)
thus *abs* (*ln x* − (*approx* + *d*)) ≤ *d* **by** *auto*
qed

end

lemma *euler_mascheroni_bounds*:
fixes *n* :: *nat* **assumes** *n* ≥ *1* **defines** *t* ≡ *harm n* − *ln* (*of_nat* (*Suc n*)) :: *real*
shows *euler_mascheroni* ∈ {*t* + *inverse* (*of_nat* (*2*∗(*n*+*1*)))..*t* + *inverse* (*of_nat* (*2*∗*n*))}
using *assms euler_mascheroni_upper*[*of n*−*1*] *euler_mascheroni_lower*[*of n*−*1*]
unfolding *t_def* **by** (*cases n*) (*simp_all add*: *harm_Suc t_def inverse_eq_divide*)

lemma *euler_mascheroni_bounds'*:
fixes *n* :: *nat* **assumes** *n* ≥ *1 ln* (*real_of_nat* (*Suc n*)) ∈ {*l*<..<*u*}
shows *euler_mascheroni* ∈
{*harm n* − *u* + *inverse* (*of_nat* (*2*∗(*n*+*1*)))<..<*harm n* − *l* + *inverse* (*of_nat* (*2*∗*n*))}
using *euler_mascheroni_bounds*[*OF assms*(*1*)] *assms*(*2*) **by** *auto*

Approximation of *ln* (*2*::'*a*). The lower bound is accurate to about 0.03; the upper bound is accurate to about 0.0015.

lemma *ln2_ge_two_thirds*: *2*/*3* ≤ *ln* (*2*::*real*)
and *ln2_le_25_over_36*: *ln* (*2*::*real*) ≤ *25*/*36*
using *ln_approx_bounds*[*of 2 1*, *simplified*, *simplified eval_nat_numeral*, *simplified*] **by** *simp_all*

Approximation of the Euler-Mascheroni constant. The lower bound is accurate to about 0.0015; the upper bound is accurate to about 0.015.

lemma *euler_mascheroni_gt_19_over_33*: (*euler_mascheroni* :: *real*) > *19/33* (**is** *?th1*)
  and *euler_mascheroni_less_13_over_22*: (*euler_mascheroni* :: *real*) < *13/22* (**is** *?th2*)
proof −
  have *ln* (*real* (*Suc 7*)) = *3* ∗ *ln 2* **by** (*simp add*: *ln_powr* [*symmetric*])
  also from *ln_approx_bounds*[*of 2 3*] **have** ... ∈ {*3*∗*307*/*443*<..<*3*∗*4615*/*6658*}
    by (*simp add*: *eval_nat_numeral*)
  finally have *ln* (*real* (*Suc 7*)) ∈ ... **.**
  from *euler_mascheroni_bounds'*[*OF _ this*] **have** *?th1* ∧ *?th2* **by** (*simp_all add*: *harm_expand*)
  thus *?th1 ?th2* **by** *blast*+
qed

end

6.23 The Gamma Function

theory *Gamma_Function*
  imports
  Equivalence_Lebesgue_Henstock_Integration
  Summation_Tests
  Harmonic_Numbers
  HOL−Library.Nonpos_Ints
  HOL−Library.Periodic_Fun
begin

Several equivalent definitions of the Gamma function and its most important properties. Also contains the definition and some properties of the log-Gamma function and the Digamma function and the other Polygamma functions.

Based on the Gamma function, we also prove the Weierstraß product form of the sin function and, based on this, the solution of the Basel problem (the sum over all *1* / *real* (n^2).

lemma *pochhammer_eq_0_imp_nonpos_Int*:
  pochhammer (*x*::*'a*::*field_char_0*) *n* = *0* ⟹ *x* ∈ $\mathbb{Z}_{\leq 0}$
  by (*auto simp*: *pochhammer_eq_0_iff*)

lemma *closed_nonpos_Ints* [*simp*]: *closed* ($\mathbb{Z}_{\leq 0}$ :: *'a* :: *real_normed_algebra_1 set*)
proof −
  have $\mathbb{Z}_{\leq 0}$ = (*of_int* ‘ {*n*. *n* ≤ *0*} :: *'a set*)
    by (*auto elim*!: *nonpos_Ints_cases intro*!: *nonpos_Ints_of_int*)
  also have *closed* ... **by** (*rule closed_of_int_image*)
  finally show *?thesis* **.**
qed

lemma *plus_one_in_nonpos_Ints_imp*: $z + 1 \in \mathbb{Z}_{\leq 0} \Longrightarrow z \in \mathbb{Z}_{\leq 0}$
  using *nonpos_Ints_diff_Nats*[*of z+1 1*] **by** *simp_all*

lemma *of_int_in_nonpos_Ints_iff*:
  (*of_int n* :: *'a* :: *ring_char_0*) $\in \mathbb{Z}_{\leq 0} \longleftrightarrow n \leq 0$
  by (*auto simp*: *nonpos_Ints_def*)

lemma *one_plus_of_int_in_nonpos_Ints_iff*:
  (*1* + *of_int n* :: *'a* :: *ring_char_0*) $\in \mathbb{Z}_{\leq 0} \longleftrightarrow n \leq -1$
proof −
  have *1* + *of_int n* = (*of_int* (*n* + *1*) :: *'a*) **by** *simp*
  also have ... $\in \mathbb{Z}_{\leq 0} \longleftrightarrow n + 1 \leq 0$ **by** (*subst of_int_in_nonpos_Ints_iff*) *simp_all*
  also have ... $\longleftrightarrow n \leq -1$ **by** *presburger*
  finally show *?thesis* **.**
qed

lemma *one_minus_of_nat_in_nonpos_Ints_iff*:
  (*1* − *of_nat n* :: *'a* :: *ring_char_0*) $\in \mathbb{Z}_{\leq 0} \longleftrightarrow n > 0$
proof −
  have (*1* − *of_nat n* :: *'a*) = *of_int* (*1* − *int n*) **by** *simp*
  also have ... $\in \mathbb{Z}_{\leq 0} \longleftrightarrow n > 0$ **by** (*subst of_int_in_nonpos_Ints_iff*) *presburger*
  finally show *?thesis* **.**
qed

lemma *fraction_not_in_ints*:
  assumes ¬(*n dvd m*) *n* ≠ *0*
  shows *of_int m* / *of_int n* ∉ (ℤ :: *'a* :: {*division_ring*,*ring_char_0*} *set*)
proof
  assume *of_int m* / (*of_int n* :: *'a*) ∈ ℤ
  then obtain *k* **where** *of_int m* / *of_int n* = (*of_int k* :: *'a*) **by** (*elim Ints_cases*)
  with *assms* **have** *of_int m* = (*of_int* (*k* ∗ *n*) :: *'a*) **by** (*auto simp add*: *field_split_simps*)
  hence *m* = *k* ∗ *n* **by** (*subst* (*asm*) *of_int_eq_iff*)
  hence *n dvd m* **by** *simp*
  with *assms*(*1*) **show** *False* **by** *contradiction*
qed

lemma *fraction_not_in_nats*:
  assumes ¬*n dvd m n* ≠ *0*
  shows *of_int m* / *of_int n* ∉ (ℕ :: *'a* :: {*division_ring*,*ring_char_0*} *set*)
proof
  assume *of_int m* / *of_int n* ∈ (ℕ :: *'a set*)
  also note *Nats_subset_Ints*
  finally have *of_int m* / *of_int n* ∈ (ℤ :: *'a set*) **.**
  moreover have *of_int m* / *of_int n* ∉ (ℤ :: *'a set*)
    using *assms* **by** (*intro fraction_not_in_ints*)
  ultimately show *False* **by** *contradiction*
qed

lemma *not_in_Ints_imp_not_in_nonpos_Ints*: $z \notin \mathbb{Z} \Longrightarrow z \notin \mathbb{Z}_{\leq 0}$
by (*auto simp*: *Ints_def nonpos_Ints_def*)

lemma *double_in_nonpos_Ints_imp*:
assumes $2 * (z :: 'a :: field\_char\_0) \in \mathbb{Z}_{\leq 0}$
shows $z \in \mathbb{Z}_{\leq 0} \vee z + 1/2 \in \mathbb{Z}_{\leq 0}$
proof−
from *assms* **obtain** *k* **where** *k*: $2 * z = - of\_nat\ k$ **by** (*elim nonpos_Ints_cases'*)
thus *?thesis* **by** (*cases even k*) (*auto elim*!: *evenE oddE simp*: *field_simps*)
qed

lemma *sin_series*: (λ*n*. ((−*1*) ^*n* / *fact* (*2*∗*n*+*1*)) ∗$_R$ *z*^(*2*∗*n*+*1*)) *sums sin z*
proof −
from *sin_converges*[*of z*] **have** (λ*n*. *sin_coeff n* ∗$_R$ *z*^*n*) *sums sin z* **.**
also have (λ*n*. *sin_coeff n* ∗$_R$ *z*^*n*) *sums sin z* ⟷
(λ*n*. ((−*1*) ^*n* / *fact* (*2*∗*n*+*1*)) ∗$_R$ *z*^(*2*∗*n*+*1*)) *sums sin z*
by (*subst sums_mono_reindex*[*of* λ*n*. *2*∗*n*+*1*, *symmetric*])
(*auto simp*: *sin_coeff_def strict_mono_def ac_simps elim*!: *oddE*)
finally show *?thesis* **.**
qed

lemma *cos_series*: (λ*n*. ((−*1*) ^*n* / *fact* (*2*∗*n*)) ∗$_R$ *z*^(*2*∗*n*)) *sums cos z*
proof −
from *cos_converges*[*of z*] **have** (λ*n*. *cos_coeff n* ∗$_R$ *z*^*n*) *sums cos z* **.**
also have (λ*n*. *cos_coeff n* ∗$_R$ *z*^*n*) *sums cos z* ⟷
(λ*n*. ((−*1*) ^*n* / *fact* (*2*∗*n*)) ∗$_R$ *z*^(*2*∗*n*)) *sums cos z*
by (*subst sums_mono_reindex*[*of* λ*n*. *2*∗*n*, *symmetric*])
(*auto simp*: *cos_coeff_def strict_mono_def ac_simps elim*!: *evenE*)
finally show *?thesis* **.**
qed

lemma *sin_z_over_z_series*:
fixes *z* :: *'a* :: {*real_normed_field*,*banach*}
assumes *z* ≠ *0*
shows (λ*n*. (−*1*) ^*n* / *fact* (*2*∗*n*+*1*) ∗ *z*^(*2*∗*n*)) *sums* (*sin z* / *z*)
proof −
from *sin_series*[*of z*] **have** (λ*n*. *z* ∗ ((−*1*) ^*n* / *fact* (*2*∗*n*+*1*)) ∗ *z*^(*2*∗*n*)) *sums sin z*
by (*simp add*: *field_simps scaleR_conv_of_real*)
from *sums_mult*[*OF this*, *of inverse z*] **and** *assms* **show** *?thesis*
by (*simp add*: *field_simps*)
qed

lemma *sin_z_over_z_series'*:
fixes *z* :: *'a* :: {*real_normed_field*,*banach*}
assumes *z* ≠ *0*
shows (λ*n*. *sin_coeff* (*n*+*1*) ∗$_R$ *z*^*n*) *sums* (*sin z* / *z*)
proof −

from *sums_split_initial_segment*[*OF sin_converges*[*of z*], *of 1*]
have ($\lambda n.\ z * (sin\_coeff\ (n+1) *_R z\ \hat{}\ n)$) *sums sin z* **by** *simp*
from *sums_mult*[*OF this*, *of inverse z*] *assms* **show** *?thesis* **by** (*simp add*: *field_simps*)
qed

lemma *has_field_derivative_sin_z_over_z*:
fixes *A* :: *'a* :: {*real_normed_field*,*banach*} *set*
shows ((λz. *if z = 0 then 1 else sin z / z*) *has_field_derivative 0*) (*at 0 within A*)
(**is** (*?f has_field_derivative ?f'*) _)
proof (*rule has_field_derivative_at_within*)
have ((λz::*'a*. $\sum n$. *of_real* (*sin_coeff* (*n+1*)) * *z*ˆ*n*)
has_field_derivative ($\sum n$. *diffs* (λn. *of_real* (*sin_coeff* (*n+1*))) *n* * *0*ˆ*n*)) (*at 0*)
proof (*rule termdiffs_strong*)
from *summable_ignore_initial_segment*[*OF sums_summable*[*OF sin_converges*[*of 1*::*'a*]], *of 1*]
show *summable* (λn. *of_real* (*sin_coeff* (*n+1*)) * (*1*::*'a*)ˆ*n*) **by** (*simp add*: *of_real_def*)
qed *simp*
also have (λz::*'a*. $\sum n$. *of_real* (*sin_coeff* (*n+1*)) * *z*ˆ*n*) = *?f*
proof
fix *z*
show ($\sum n$. *of_real* (*sin_coeff* (*n+1*)) * *z*ˆ*n*) = *?f z*
by (*cases z = 0*) (*insert sin_z_over_z_series'*[*of z*],
simp_all add: *scaleR_conv_of_real sums_iff sin_coeff_def*)
qed
also have ($\sum n$. *diffs* (λn. *of_real* (*sin_coeff* (*n + 1*))) *n* * (*0*::*'a*) ˆ *n*) =
diffs (λn. *of_real* (*sin_coeff* (*Suc n*))) *0* **by** *simp*
also have ... = *0* **by** (*simp add*: *sin_coeff_def diffs_def*)
finally show ((λz::*'a*. *if z = 0 then 1 else sin z / z*) *has_field_derivative 0*) (*at 0*) **.**
qed

lemma *round_Re_minimises_norm*:
norm ((*z*::*complex*) − *of_int m*) ≥ *norm* (*z* − *of_int* (*round* (*Re z*)))
proof −
let *?n* = *round* (*Re z*)
have *norm* (*z* − *of_int ?n*) = *sqrt* ((*Re z* − *of_int ?n*)^2 + (*Im z*)^2)
by (*simp add*: *cmod_def*)
also have |*Re z* − *of_int ?n*| ≤ |*Re z* − *of_int m*| **by** (*rule round_diff_minimal*)
hence *sqrt* ((*Re z* − *of_int ?n*)^2 + (*Im z*)^2) ≤ *sqrt* ((*Re z* − *of_int m*)^2 + (*Im z*)^2)
by (*intro real_sqrt_le_mono add_mono*) (*simp_all add*: *abs_le_square_iff*)
also have ... = *norm* (*z* − *of_int m*) **by** (*simp add*: *cmod_def*)
finally show *?thesis* **.**
qed

lemma *Re_pos_in_ball*:

```
  assumes Re z > 0 t ∈ ball z (Re z/2)
  shows   Re t > 0
proof -
  have Re (z - t) ≤ norm (z - t) by (rule complex_Re_le_cmod)
  also from assms have ... < Re z / 2 by (simp add: dist_complex_def)
  finally show Re t > 0 using assms by simp
qed

lemma no_nonpos_Int_in_ball_complex:
  assumes Re z > 0 t ∈ ball z (Re z/2)
  shows   t ∉ ℤ≤0
  using Re_pos_in_ball[OF assms] by (force elim!: nonpos_Ints_cases)

lemma no_nonpos_Int_in_ball:
  assumes t ∈ ball z (dist z (round (Re z)))
  shows   t ∉ ℤ≤0
proof
  assume t ∈ ℤ≤0
  then obtain n where t = of_int n by (auto elim!: nonpos_Ints_cases)
  have dist z (of_int n) ≤ dist z t + dist t (of_int n) by (rule dist_triangle)
  also from assms have dist z t < dist z (round (Re z)) by simp
  also have ... ≤ dist z (of_int n)
    using round_Re_minimises_norm[of z] by (simp add: dist_complex_def)
  finally have dist t (of_int n) > 0 by simp
  with ‹t = of_int n› show False by simp
qed

lemma no_nonpos_Int_in_ball':
  assumes (z :: 'a :: {euclidean_space,real_normed_algebra_1}) ∉ ℤ≤0
  obtains d where d > 0 ⋀t. t ∈ ball z d ⟹ t ∉ ℤ≤0
proof (rule that)
  from assms show setdist {z} ℤ≤0 > 0 by (subst setdist_gt_0_compact_closed)
auto
next
  fix t assume t ∈ ball z (setdist {z} ℤ≤0)
  thus t ∉ ℤ≤0 using setdist_le_dist[of z {z} t ℤ≤0] by force
qed

lemma no_nonpos_Real_in_ball:
  assumes z: z ∉ ℝ≤0 and t: t ∈ ball z (if Im z = 0 then Re z / 2 else abs (Im
z) / 2)
  shows   t ∉ ℝ≤0
using z
proof (cases Im z = 0)
  assume A: Im z = 0
  with z have Re z > 0 by (force simp add: complex_nonpos_Reals_iff)
  with t A Re_pos_in_ball[of z t] show ?thesis by (force simp add: complex_nonpos_Reals_iff)
next
  assume A: Im z ≠ 0
```

have *abs* (*Im z*) − *abs* (*Im t*) ≤ *abs* (*Im z* − *Im t*) **by** *linarith*
also have ... = *abs* (*Im* (*z* − *t*)) **by** *simp*
also have ... ≤ *norm* (*z* − *t*) **by** (*rule abs_Im_le_cmod*)
also from *A t* **have** ... ≤ *abs* (*Im z*) / *2* **by** (*simp add*: *dist_complex_def*)
finally have *abs* (*Im t*) > *0* **using** *A* **by** *simp*
thus *?thesis* **by** (*force simp add*: *complex_nonpos_Reals_iff*)
qed

6.23.1 The Euler form and the logarithmic Gamma function

We define the Gamma function by first defining its multiplicative inverse *rGamma*. This is more convenient because *rGamma* is entire, which makes proofs of its properties more convenient because one does not have to watch out for discontinuities. (e.g. *rGamma* fulfils *rGamma z* = *z* ∗ *rGamma* (*z* + *1*) everywhere, whereas the Γ function does not fulfil the analogous equation on the non-positive integers)

We define the Γ function (resp. its reciprocale) in the Euler form. This form has the advantage that it is a relatively simple limit that converges everywhere. The limit at the poles is 0 (due to division by 0). The functional equation *Gamma* (*z* + *1*) = *z* ∗ *Gamma z* follows immediately from the definition.

definition *Gamma_series* :: (*'a* :: {*banach*,*real_normed_field*}) ⇒ *nat* ⇒ *'a* **where**
Gamma_series z n = *fact n* ∗ *exp* (*z* ∗ *of_real* (*ln* (*of_nat n*))) / *pochhammer z* (*n*+*1*)

definition *Gamma_series'* :: (*'a* :: {*banach*,*real_normed_field*}) ⇒ *nat* ⇒ *'a* **where**
Gamma_series' z n = *fact* (*n* − *1*) ∗ *exp* (*z* ∗ *of_real* (*ln* (*of_nat n*))) / *pochhammer z n*

definition *rGamma_series* :: (*'a* :: {*banach*,*real_normed_field*}) ⇒ *nat* ⇒ *'a* **where**
rGamma_series z n = *pochhammer z* (*n*+*1*) / (*fact n* ∗ *exp* (*z* ∗ *of_real* (*ln* (*of_nat n*))))

lemma *Gamma_series_altdef*: *Gamma_series z n* = *inverse* (*rGamma_series z n*)
and *rGamma_series_altdef*: *rGamma_series z n* = *inverse* (*Gamma_series z n*)
unfolding *Gamma_series_def rGamma_series_def* **by** *simp_all*

lemma *rGamma_series_minus_of_nat*:
eventually (λ*n*. *rGamma_series* (− *of_nat k*) *n* = *0*) *sequentially*
using *eventually_ge_at_top*[*of k*]
by *eventually_elim* (*auto simp*: *rGamma_series_def pochhammer_of_nat_eq_0_iff*)

lemma *Gamma_series_minus_of_nat*:
eventually (λ*n*. *Gamma_series* (− *of_nat k*) *n* = *0*) *sequentially*
using *eventually_ge_at_top*[*of k*]
by *eventually_elim* (*auto simp*: *Gamma_series_def pochhammer_of_nat_eq_0_iff*)

lemma *Gamma_series'_minus_of_nat*:
eventually (λ*n*. *Gamma_series'* (− *of_nat k*) *n* = *0*) *sequentially*
using *eventually_gt_at_top*[*of k*]
by *eventually_elim* (*auto simp*: *Gamma_series'_def pochhammer_of_nat_eq_0_iff*)

lemma *rGamma_series_nonpos_Ints_LIMSEQ*: $z \in \mathbb{Z}_{\leq 0} \Longrightarrow$ *rGamma_series z* ⟶ *0*
by (*elim nonpos_Ints_cases'*, *hypsubst*, *subst tendsto_cong*, *rule rGamma_series_minus_of_nat*, *simp*)

lemma *Gamma_series_nonpos_Ints_LIMSEQ*: $z \in \mathbb{Z}_{\leq 0} \Longrightarrow$ *Gamma_series z* ⟶ *0*
by (*elim nonpos_Ints_cases'*, *hypsubst*, *subst tendsto_cong*, *rule Gamma_series_minus_of_nat*, *simp*)

lemma *Gamma_series'_nonpos_Ints_LIMSEQ*: $z \in \mathbb{Z}_{\leq 0} \Longrightarrow$ *Gamma_series' z* ⟶ *0*
by (*elim nonpos_Ints_cases'*, *hypsubst*, *subst tendsto_cong*, *rule Gamma_series'_minus_of_nat*, *simp*)

lemma *Gamma_series_Gamma_series'*:
assumes *z*: $z \notin \mathbb{Z}_{\leq 0}$
shows (λ*n*. *Gamma_series' z n* / *Gamma_series z n*) ⟶ *1*
proof (*rule Lim_transform_eventually*)
from *eventually_gt_at_top*[*of 0*::*nat*]
show *eventually* (λ*n*. *z* / *of_nat n* + *1* = *Gamma_series' z n* / *Gamma_series z n*) *sequentially*
proof *eventually_elim*
fix *n* :: *nat* **assume** *n*: *n* > *0*
from *n z* **have** *Gamma_series' z n* / *Gamma_series z n* = (*z* + *of_nat n*) / *of_nat n*
by (*cases n*, *simp*)
(*auto simp add*: *Gamma_series_def Gamma_series'_def pochhammer_rec'*
dest: *pochhammer_eq_0_imp_nonpos_Int plus_of_nat_eq_0_imp*)
also from *n* **have** ... = *z* / *of_nat n* + *1* **by** (*simp add*: *field_split_simps*)
finally show *z* / *of_nat n* + *1* = *Gamma_series' z n* / *Gamma_series z n* **..**
qed
have (λ*x*. *z* / *of_nat x*) ⟶ *0*
by (*rule tendsto_norm_zero_cancel*)
(*insert tendsto_mult*[*OF tendsto_const*[*of norm z*] *lim_inverse_n*],
simp add: *norm_divide inverse_eq_divide*)
from *tendsto_add*[*OF this tendsto_const*[*of 1*]] **show** (λ*n*. *z* / *of_nat n* + *1*) ⟶ *1* **by** *simp*
qed

We now show that the series that defines the Γ function in the Euler form converges and that the function defined by it is continuous on the complex halfspace with positive real part.

We do this by showing that the logarithm of the Euler series is continuous and converges locally uniformly, which means that the log-Gamma function defined by its limit is also continuous.

This will later allow us to lift holomorphicity and continuity from the log-Gamma function to the inverse of the Gamma function, and from that to the Gamma function itself.

definition *ln_Gamma_series* :: (*'a* :: {*banach*,*real_normed_field*,*ln*}) ⇒ *nat* ⇒ *'a*
where
ln_Gamma_series z n = *z* ∗ *ln* (*of_nat n*) − *ln z* − (∑ *k*=*1*..*n*. *ln* (*z* / *of_nat k* + *1*))

definition *ln_Gamma_series'* :: (*'a* :: {*banach*,*real_normed_field*,*ln*}) ⇒ *nat* ⇒ *'a*
where
ln_Gamma_series' z n =
− *euler_mascheroni*∗*z* − *ln z* + (∑ *k*=*1*..*n*. *z* / *of_nat n* − *ln* (*z* / *of_nat k* + *1*))

definition *ln_Gamma* :: (*'a* :: {*banach*,*real_normed_field*,*ln*}) ⇒ *'a* **where**
ln_Gamma z = *lim* (*ln_Gamma_series z*)

We now show that the log-Gamma series converges locally uniformly for all complex numbers except the non-positive integers. We do this by proving that the series is locally Cauchy.

context
begin

private lemma *ln_Gamma_series_complex_converges_aux*:
fixes *z* :: *complex* **and** *k* :: *nat*
assumes *z*: *z* ≠ *0* **and** *k*: *of_nat k* ≥ *2*∗*norm z k* ≥ *2*
shows *norm* (*z* ∗ *ln* (*1* − *1*/*of_nat k*) + *ln* (*z*/*of_nat k* + *1*)) ≤ *2*∗(*norm z* + *norm z*ˆ*2*) / *of_nat k*ˆ*2*
proof −
let *?k* = *of_nat k* :: *complex* **and** *?z* = *norm z*
have *z* ∗*ln* (*1* − *1*/*?k*) + *ln* (*z*/*?k*+*1*) = *z*∗(*ln* (*1* − *1*/*?k* :: *complex*) + *1*/*?k*) + (*ln* (*1*+*z*/*?k*) − *z*/*?k*)
by (*simp add*: *algebra_simps*)
also have *norm* ... ≤ *?z* ∗ *norm* (*ln* (*1*−*1*/*?k*) + *1*/*?k* :: *complex*) + *norm* (*ln* (*1*+*z*/*?k*) − *z*/*?k*)
by (*subst norm_mult* [*symmetric*], *rule norm_triangle_ineq*)
also have *norm* (*Ln* (*1* + −*1*/*?k*) − (−*1*/*?k*)) ≤ (*norm* (−*1*/*?k*))^2 / (*1* − *norm*(−*1*/*?k*))
using *k* **by** (*intro Ln_approx_linear*) (*simp add*: *norm_divide*)
hence *?z* ∗ *norm* (*ln* (*1*−*1*/*?k*) + *1*/*?k*) ≤ *?z* ∗ ((*norm* (*1*/*?k*))ˆ*2* / (*1* − *norm* (*1*/*?k*)))
by (*intro mult_left_mono*) *simp_all*
also have ... ≤ (*?z* ∗ (*of_nat k* / (*of_nat k* − *1*))) / *of_nat k*ˆ*2* **using** *k*
by (*simp add*: *field_simps power2_eq_square norm_divide*)
also have ... ≤ (*?z* ∗ *2*) / *of_nat k*ˆ*2* **using** *k*
by (*intro divide_right_mono mult_left_mono*) (*simp_all add*: *field_simps*)

also have *norm* (*ln* (*1*+*z*/*?k*) − *z*/*?k*) ≤ *norm* (*z*/*?k*)ˆ*2* / (*1* − *norm* (*z*/*?k*))
using *k*
by (*intro Ln_approx_linear*) (*simp add*: *norm_divide*)
hence *norm* (*ln* (*1*+*z*/*?k*) − *z*/*?k*) ≤ *?z*ˆ*2* / *of_nat k*ˆ*2* / (*1* − *?z* / *of_nat k*)
by (*simp add*: *field_simps norm_divide*)
also have ... ≤ (*?z*ˆ*2* ∗ (*of_nat k* / (*of_nat k* − *?z*))) / *of_nat k*ˆ*2* **using** *k*
by (*simp add*: *field_simps power2_eq_square*)
also have ... ≤ (*?z*ˆ*2* ∗ *2*) / *of_nat k*ˆ*2* **using** *k*
by (*intro divide_right_mono mult_left_mono*) (*simp_all add*: *field_simps*)
also note *add_divide_distrib* [*symmetric*]
finally show *?thesis* **by** (*simp only*: *distrib_left mult.commute*)
qed

lemma *ln_Gamma_series_complex_converges*:
assumes *z*: *z* ∉ ℤ$_{\leq 0}$
assumes *d*: *d* > *0* ⋀*n*. *n* ∈ ℤ$_{\leq 0}$ ⟹ *norm* (*z* − *of_int n*) > *d*
shows *uniformly_convergent_on* (*ball z d*) (λ*n z*. *ln_Gamma_series z n* :: *complex*)
proof (*intro Cauchy_uniformly_convergent uniformly_Cauchy_onI'*)
fix *e* :: *real* **assume** *e*: *e* > *0*
define *e''* **where** *e''* = (*SUP t*∈*ball z d*. *norm t* + *norm t*ˆ*2*)
define *e'* **where** *e'* = *e* / (*2*∗*e''*)
have *bounded* ((λ*t*. *norm t* + *norm t*ˆ*2*) ' *cball z d*)
by (*intro compact_imp_bounded compact_continuous_image*) (*auto intro*!: *continuous_intros*)
hence *bounded* ((λ*t*. *norm t* + *norm t*ˆ*2*) ' *ball z d*) **by** (*rule bounded_subset*) *auto*
hence *bdd*: *bdd_above* ((λ*t*. *norm t* + *norm t*ˆ*2*) ' *ball z d*) **by** (*rule bounded_imp_bdd_above*)

with *z d*(*1*) *d*(*2*)[*of* −*1*] **have** *e''_pos*: *e''* > *0* **unfolding** *e''_def*
by (*subst less_cSUP_iff*) (*auto intro*!: *add_pos_nonneg bexI*[*of _ z*])
have *e''*: *norm t* + *norm t*ˆ*2* ≤ *e''* **if** *t* ∈ *ball z d* **for** *t* **unfolding** *e''_def* **using** *that*
by (*rule cSUP_upper*[*OF _ bdd*])
from *e z e''_pos* **have** *e'*: *e'* > *0* **unfolding** *e'_def*
by (*intro divide_pos_pos mult_pos_pos add_pos_pos*) (*simp_all add*: *field_simps*)

have *summable* (λ*k*. *inverse* ((*real_of_nat k*)ˆ*2*))
by (*rule inverse_power_summable*) *simp*
from *summable_partial_sum_bound*[*OF this e'*] **guess** *M* **. note** *M* = *this*

define *N* **where** *N* = *max 2* (*max* (*nat* ⌈*2* ∗ (*norm z* + *d*)⌉) *M*)
{
from *d* **have** ⌈*2* ∗ (*cmod z* + *d*)⌉ ≥ ⌈*0*::*real*⌉
by (*intro ceiling_mono mult_nonneg_nonneg add_nonneg_nonneg*) *simp_all*
hence *2* ∗ (*norm z* + *d*) ≤ *of_nat* (*nat* ⌈*2* ∗ (*norm z* + *d*)⌉) **unfolding** *N_def*
by (*simp_all*)
also have ... ≤ *of_nat N* **unfolding** *N_def*
by (*subst of_nat_le_iff*) (*rule max.coboundedI2*, *rule max.cobounded1*)
finally have *of_nat N* ≥ *2* ∗ (*norm z* + *d*) **.**

moreover have *N* ≥ *2 N* ≥ *M* **unfolding** *N_def* **by** *simp_all*
moreover have (∑ *k=m..n. 1*/(*of_nat k*)^2) < *e′* **if** *m* ≥ *N* **for** *m n*
using *M*[*OF order.trans*[*OF* ‹*N* ≥ *M*› *that*]] **unfolding** *real_norm_def*
by (*subst* (*asm*) *abs_of_nonneg*) (*auto intro*: *sum_nonneg simp*: *field_split_simps*)
moreover note *calculation*
} **note** *N* = *this*

show ∃ *M*. ∀ *t*∈*ball z d*. ∀ *m*≥*M*. ∀ *n*>*m*. *dist* (*ln_Gamma_series t m*) (*ln_Gamma_series t n*) < *e*
unfolding *dist_complex_def*
proof (*intro exI*[*of _ N*] *ballI allI impI*)
fix *t m n* **assume** *t*: *t* ∈ *ball z d* **and** *mn*: *m* ≥ *N n* > *m*
from *d*(*2*)[*of 0*] *t* **have** *0* < *dist z 0* − *dist z t* **by** (*simp add*: *field_simps dist_complex_def*)
also have *dist z 0* − *dist z t* ≤ *dist 0 t* **using** *dist_triangle*[*of 0 z t*]
by (*simp add*: *dist_commute*)
finally have *t_nz*: *t* ≠ *0* **by** *auto*

have *norm t* ≤ *norm z* + *norm* (*t* − *z*) **by** (*rule norm_triangle_sub*)
also from *t* **have** *norm* (*t* − *z*) < *d* **by** (*simp add*: *dist_complex_def norm_minus_commute*)
also have *2* ∗ (*norm z* + *d*) ≤ *of_nat N* **by** (*rule N*)
also have *N* ≤ *m* **by** (*rule mn*)
finally have *norm_t*: *2* ∗ *norm t* < *of_nat m* **by** *simp*

have *ln_Gamma_series t m* − *ln_Gamma_series t n* =
(−(*t* ∗ *Ln* (*of_nat n*)) − (−(*t* ∗ *Ln* (*of_nat m*)))) +
((∑ *k=1..n. Ln* (*t* / *of_nat k* + *1*)) − (∑ *k=1..m. Ln* (*t* / *of_nat k* + *1*)))
by (*simp add*: *ln_Gamma_series_def algebra_simps*)
also have (∑ *k=1..n. Ln* (*t* / *of_nat k* + *1*)) − (∑ *k=1..m. Ln* (*t* / *of_nat k* + *1*)) =
(∑ *k*∈{*1..n*}−{*1..m*}. *Ln* (*t* / *of_nat k* + *1*)) **using** *mn*
by (*simp add*: *sum_diff*)
also from *mn* **have** {*1..n*}−{*1..m*} = {*Suc m..n*} **by** *fastforce*
also have −(*t* ∗ *Ln* (*of_nat n*)) − (−(*t* ∗ *Ln* (*of_nat m*))) =
(∑ *k* = *Suc m..n. t* ∗ *Ln* (*of_nat* (*k* − *1*)) − *t* ∗ *Ln* (*of_nat k*)) **using** *mn*
by (*subst sum_telescope″* [*symmetric*]) *simp_all*
also have ... = (∑ *k* = *Suc m..n. t* ∗ *Ln* (*of_nat* (*k* − *1*) / *of_nat k*)) **using** *mn N*
by (*intro sum_cong_Suc*)
(*simp_all del*: *of_nat_Suc add*: *field_simps Ln_of_nat Ln_of_nat_over_of_nat*)
also have *of_nat* (*k* − *1*) / *of_nat k* = *1* − *1* / (*of_nat k* :: *complex*) **if** *k* ∈ {*Suc m..n*} **for** *k*
using *that of_nat_eq_0_iff* [*of Suc i* **for** *i*] **by** (*cases k*) (*simp_all add*: *field_split_simps*)
hence (∑ *k* = *Suc m..n. t* ∗ *Ln* (*of_nat* (*k* − *1*) / *of_nat k*)) =
(∑ *k* = *Suc m..n. t* ∗ *Ln* (*1* − *1* / *of_nat k*)) **using** *mn N*
by (*intro sum.cong*) *simp_all*
also note *sum.distrib* [*symmetric*]

```
    also have norm (∑k=Suc m..n. t * Ln (1 − 1/of_nat k) + Ln (t/of_nat k
+ 1)) ≤
       (∑k=Suc m..n. 2 * (norm t + (norm t)^2) / (real_of_nat k)^2) using t_nz
N(2) mn norm_t
    by (intro order.trans[OF norm_sum sum_mono[OF ln_Gamma_series_complex_converges_aux]])
simp_all
    also have ... ≤ 2 * (norm t + norm t^2) * (∑k=Suc m..n. 1 / (of_nat k)^2)
      by (simp add: sum_distrib_left)
    also have ... < 2 * (norm t + norm t^2) * e' using mn z t_nz
      by (intro mult_strict_left_mono N mult_pos_pos add_pos_pos) simp_all
    also from e''_pos have ... = e * ((cmod t + (cmod t)^2) / e'')
      by (simp add: e'_def field_simps power2_eq_square)
    also from e''[OF t] e''_pos e
      have . . . ≤ e * 1 by (intro mult_left_mono) (simp_all add: field_simps)
    finally show norm (ln_Gamma_series t m − ln_Gamma_series t n) < e by
simp
  qed
qed

end

lemma ln_Gamma_series_complex_converges':
  assumes z: (z :: complex) ∉ ℤ≤0
  shows ∃d>0. uniformly_convergent_on (ball z d) (λn z. ln_Gamma_series z n)
proof −
  define d' where d' = Re z
  define d where d = (if d' > 0 then d' / 2 else norm (z − of_int (round d')) /
2)
  have of_int (round d') ∈ ℤ≤0 if d' ≤ 0 using that
    by (intro nonpos_Ints_of_int) (simp_all add: round_def)
  with assms have d_pos: d > 0 unfolding d_def by (force simp: not_less)

  have d < cmod (z − of_int n) if n ∈ ℤ≤0 for n
  proof (cases Re z > 0)
    case True
    from nonpos_Ints_nonpos[OF that] have n: n ≤ 0 by simp
    from True have d = Re z/2 by (simp add: d_def d'_def)
    also from n True have . . . < Re (z − of_int n) by simp
    also have . . . ≤ norm (z − of_int n) by (rule complex_Re_le_cmod)
    finally show ?thesis .
  next
    case False
    with assms nonpos_Ints_of_int[of round (Re z)]
      have z ≠ of_int (round d') by (auto simp: not_less)
    with False have d < norm (z − of_int (round d')) by (simp add: d_def d'_def)
    also have . . . ≤ norm (z − of_int n) unfolding d'_def by (rule round_Re_minimises_norm)
    finally show ?thesis .
  qed
  hence conv: uniformly_convergent_on (ball z d) (λn z. ln_Gamma_series z n)
```

by (*intro ln_Gamma_series_complex_converges d_pos z*) *simp_all*
from *d_pos conv* **show** *?thesis* **by** *blast*
qed

lemma *ln_Gamma_series_complex_converges''*: (*z* :: *complex*) $\notin \mathbb{Z}_{\leq 0} \Longrightarrow$ *convergent* (*ln_Gamma_series z*)
by (*drule ln_Gamma_series_complex_converges'*) (*auto intro*: *uniformly_convergent_imp_convergent*)

theorem *ln_Gamma_complex_LIMSEQ*: (*z* :: *complex*) $\notin \mathbb{Z}_{\leq 0} \Longrightarrow$ *ln_Gamma_series z* $\longlongrightarrow$ *ln_Gamma z*
using *ln_Gamma_series_complex_converges''* **by** (*simp add*: *convergent_LIMSEQ_iff ln_Gamma_def*)

lemma *exp_ln_Gamma_series_complex*:
assumes *n* > *0 z* $\notin \mathbb{Z}_{\leq 0}$
shows *exp* (*ln_Gamma_series z n* :: *complex*) = *Gamma_series z n*
proof −
from *assms* **obtain** *m* **where** *m*: *n* = *Suc m* **by** (*cases n*) *blast*
from *assms* **have** *z* ≠ *0* **by** (*intro notI*) *auto*
with *assms* **have** *exp* (*ln_Gamma_series z n*) =
(*of_nat n*) *powr z* / (*z* ∗ (∏ *k*=*1*..*n*. *exp* (*Ln* (*z* / *of_nat k* + *1*))))
unfolding *ln_Gamma_series_def powr_def* **by** (*simp add*: *exp_diff exp_sum*)
also from *assms* **have** (∏ *k*=*1*..*n*. *exp* (*Ln* (*z* / *of_nat k* + *1*))) = (∏ *k*=*1*..*n*. *z* / *of_nat k* + *1*)
by (*intro prod.cong*[*OF refl*], *subst exp_Ln*) (*auto simp*: *field_simps plus_of_nat_eq_0_imp*)
also have ... = (∏ *k*=*1*..*n*. *z* + *k*) / *fact n*
by (*simp add*: *fact_prod*)
(*subst prod_dividef* [*symmetric*], *simp_all add*: *field_simps*)
also from *m* **have** *z* ∗ ... = (∏ *k*=*0*..*n*. *z* + *k*) / *fact n*
by (*simp add*: *prod.atLeast0_atMost_Suc_shift prod.atLeast_Suc_atMost_Suc_shift del*: *prod.cl_ivl_Suc*)
also have (∏ *k*=*0*..*n*. *z* + *k*) = *pochhammer z* (*Suc n*)
unfolding *pochhammer_prod*
by (*simp add*: *prod.atLeast0_atMost_Suc atLeastLessThanSuc_atLeastAtMost*)
also have *of_nat n powr z* / (*pochhammer z* (*Suc n*) / *fact n*) = *Gamma_series z n*
unfolding *Gamma_series_def* **using** *assms* **by** (*simp add*: *field_split_simps powr_def*)
finally show *?thesis* **.**
qed

lemma *ln_Gamma_series'_aux*:
assumes (*z*::*complex*) $\notin \mathbb{Z}_{\leq 0}$
shows (λ*k*. *z* / *of_nat* (*Suc k*) − *ln* (*1* + *z* / *of_nat* (*Suc k*))) *sums*
(*ln_Gamma z* + *euler_mascheroni* ∗ *z* + *ln z*) (**is** *?f sums ?s*)
unfolding *sums_def*
proof (*rule Lim_transform*)
show (λ*n*. *ln_Gamma_series z n* + *of_real* (*harm n* − *ln* (*of_nat n*)) ∗ *z* + *ln z*)

```
⟶ ?s
    (is ?g ⟶ _)
  by (intro tendsto_intros ln_Gamma_complex_LIMSEQ euler_mascheroni_LIMSEQ_of_real
assms)

  have A: eventually (λn. (∑k<n. ?f k) − ?g n = 0) sequentially
    using eventually_gt_at_top[of 0::nat]
  proof eventually_elim
    fix n :: nat assume n: n > 0
    have (∑k<n. ?f k) = (∑k=1..n. z / of_nat k − ln (1 + z / of_nat k))
     by (subst atLeast0LessThan [symmetric], subst sum.shift_bounds_Suc_ivl [symmetric],
          subst atLeastLessThanSuc_atLeastAtMost) simp_all
    also have ... = z * of_real (harm n) − (∑k=1..n. ln (1 + z / of_nat k))
      by (simp add: harm_def sum_subtractf sum_distrib_left divide_inverse)
    also from n have ... − ?g n = 0
      by (simp add: ln_Gamma_series_def sum_subtractf algebra_simps)
    finally show (∑k<n. ?f k) − ?g n = 0 .
  qed
   show (λn. (∑k<n. ?f k) − ?g n) ⟶ 0 by (subst tendsto_cong[OF A])
simp_all
qed


lemma uniformly_summable_deriv_ln_Gamma:
  assumes z: (z :: 'a :: {real_normed_field,banach}) ≠ 0 and d: d > 0 d ≤ norm
z/2
  shows uniformly_convergent_on (ball z d)
            (λk z. ∑i<k. inverse (of_nat (Suc i)) − inverse (z + of_nat (Suc i)))
           (is uniformly_convergent_on _ (λk z. ∑i<k. ?f i z))
proof (rule Weierstrass_m_test'_ev)
  {
    fix t assume t: t ∈ ball z d
    have norm z = norm (t + (z − t)) by simp
   have norm (t + (z − t)) ≤ norm t + norm (z − t) by (rule norm_triangle_ineq)
    also from t d have norm (z − t) < norm z / 2 by (simp add: dist_norm)
    finally have A: norm t > norm z / 2 using z by (simp add: field_simps)

    have norm t = norm (z + (t − z)) by simp
    also have ... ≤ norm z + norm (t − z) by (rule norm_triangle_ineq)
     also from t d have norm (t − z) ≤ norm z / 2 by (simp add: dist_norm
norm_minus_commute)
    also from z have ... < norm z by simp
    finally have B: norm t < 2 * norm z by simp
    note A B
  } note ball = this

  show eventually (λn. ∀t∈ball z d. norm (?f n t) ≤ 4 * norm z * inverse (of_nat
(Suc n)^2)) sequentially
    using eventually_gt_at_top apply eventually_elim
```

proof *safe*
fix *t* :: *'a* **assume** *t*: *t* ∈ *ball z d*
from *z ball*[*OF t*] **have** *t_nz*: *t* ≠ *0* **by** *auto*
fix *n* :: *nat* **assume** *n*: *n* > *nat* ⌈*4* ∗ *norm z*⌉
from *ball*[*OF t*] *t_nz* **have** *4* ∗ *norm z* > *2* ∗ *norm t* **by** *simp*
also from *n* **have** ... < *of_nat n* **by** *linarith*
finally have *n*: *of_nat n* > *2* ∗ *norm t* **.**
hence *of_nat n* > *norm t* **by** *simp*
hence *t'*: *t* ≠ −*of_nat* (*Suc n*) **by** (*intro notI*) (*simp del*: *of_nat_Suc*)

with *t_nz* **have** *?f n t* = *1* / (*of_nat* (*Suc n*) ∗ (*1* + *of_nat* (*Suc n*)/*t*))
by (*simp add*: *field_split_simps eq_neg_iff_add_eq_0 del*: *of_nat_Suc*)
also have *norm* ... = *inverse* (*of_nat* (*Suc n*)) ∗ *inverse* (*norm* (*of_nat* (*Suc n*)/*t* + *1*))
by (*simp add*: *norm_divide norm_mult field_split_simps del*: *of_nat_Suc*)
also {
from *z t_nz ball*[*OF t*] **have** *of_nat* (*Suc n*) / (*4* ∗ *norm z*) ≤ *of_nat* (*Suc n*) / (*2* ∗ *norm t*)
by (*intro divide_left_mono mult_pos_pos*) *simp_all*
also have ... < *norm* (*of_nat* (*Suc n*) / *t*) − *norm* (*1* :: *'a*)
using *t_nz n* **by** (*simp add*: *field_simps norm_divide del*: *of_nat_Suc*)
also have ... ≤ *norm* (*of_nat* (*Suc n*)/*t* + *1*) **by** (*rule norm_diff_ineq*)
finally have *inverse* (*norm* (*of_nat* (*Suc n*)/*t* + *1*)) ≤ *4* ∗ *norm z* / *of_nat* (*Suc n*)
using *z* **by** (*simp add*: *field_split_simps norm_divide mult_ac del*: *of_nat_Suc*)
}
also have *inverse* (*real_of_nat* (*Suc n*)) ∗ (*4* ∗ *norm z* / *real_of_nat* (*Suc n*)) =
4 ∗ *norm z* ∗ *inverse* (*of_nat* (*Suc n*) ^*2*)
by (*simp add*: *field_split_simps power2_eq_square del*: *of_nat_Suc*)
finally show *norm* (*?f n t*) ≤ *4* ∗ *norm z* ∗ *inverse* (*of_nat* (*Suc n*) ^*2*)
by (*simp del*: *of_nat_Suc*)
qed
next
show *summable* (λ*n*. *4* ∗ *norm z* ∗ *inverse* ((*of_nat* (*Suc n*)) ^*2*))
by (*subst summable_Suc_iff*) (*simp add*: *summable_mult inverse_power_summable*)
qed

6.23.2 The Polygamma functions

lemma *summable_deriv_ln_Gamma*:
z ≠ (*0* :: *'a* :: {*real_normed_field*,*banach*}) ⟹
summable (λ*n*. *inverse* (*of_nat* (*Suc n*)) − *inverse* (*z* + *of_nat* (*Suc n*)))
unfolding *summable_iff_convergent*
by (*rule uniformly_convergent_imp_convergent*,
rule uniformly_summable_deriv_ln_Gamma[*of z norm z*/*2*]) *simp_all*

definition *Polygamma* :: *nat* ⇒ (*'a* :: {*real_normed_field*,*banach*}) ⇒ *'a* **where**
Polygamma n z = (*if n* = *0 then*
(∑*k*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat k*)) − *euler_mascheroni*

else
  (−*1*) ˆ*Suc n* ∗ *fact n* ∗ (∑*k*. *inverse* ((*z* + *of_nat k*) ˆ*Suc n*)))

abbreviation *Digamma* :: ('*a* :: {*real_normed_field*,*banach*}) ⇒ '*a* **where**
Digamma ≡ *Polygamma 0*

lemma *Digamma_def*:
Digamma z = (∑*k*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat k*)) − *euler_mascheroni*
by (*simp add*: *Polygamma_def*)

lemma *summable_Digamma*:
assumes (*z* :: '*a* :: {*real_normed_field*,*banach*}) ≠ *0*
shows *summable* (λ*n*. *inverse* (*of_nat* (*Suc n*)) − *inverse* (*z* + *of_nat n*))
proof −
have *sums*: (λ*n*. *inverse* (*z* + *of_nat* (*Suc n*)) − *inverse* (*z* + *of_nat n*)) *sums*
(*0* − *inverse* (*z* + *of_nat 0*))
by (*intro telescope_sums filterlim_compose*[*OF tendsto_inverse_0*]
tendsto_add_filterlim_at_infinity[*OF tendsto_const*] *tendsto_of_nat*)
from *summable_add*[*OF summable_deriv_ln_Gamma*[*OF assms*] *sums_summable*[*OF sums*]]
show *summable* (λ*n*. *inverse* (*of_nat* (*Suc n*)) − *inverse* (*z* + *of_nat n*)) **by** *simp*
qed

lemma *summable_offset*:
assumes *summable* (λ*n*. *f* (*n* + *k*) :: '*a* :: *real_normed_vector*)
shows *summable f*
proof −
from *assms* **have** *convergent* (λ*m*. ∑*n*<*m*. *f* (*n* + *k*))
using *summable_iff_convergent* **by** *blast*
hence *convergent* (λ*m*. (∑*n*<*k*. *f n*) + (∑*n*<*m*. *f* (*n* + *k*)))
by (*intro convergent_add convergent_const*)
also have (λ*m*. (∑*n*<*k*. *f n*) + (∑*n*<*m*. *f* (*n* + *k*))) = (λ*m*. ∑*n*<*m*+*k*. *f n*)
proof
fix *m* :: *nat*
have {..<*m*+*k*} = {..<*k*} ∪ {*k*..<*m*+*k*} **by** *auto*
also have (∑*n*∈. . . . *f n*) = (∑*n*<*k*. *f n*) + (∑*n*=*k*..<*m*+*k*. *f n*)
by (*rule sum.union_disjoint*) *auto*
also have (∑*n*=*k*..<*m*+*k*. *f n*) = (∑*n*=*0*..<*m*+*k*−*k*. *f* (*n* + *k*))
using *sum.shift_bounds_nat_ivl* [*of f 0 k m*] **by** *simp*
finally show (∑*n*<*k*. *f n*) + (∑*n*<*m*. *f* (*n* + *k*)) = (∑*n*<*m*+*k*. *f n*) **by** (*simp add*: *atLeast0LessThan*)
qed
finally have (λ*a*. *sum f* {..<*a*}) ⟶ *lim* (λ*m*. *sum f* {..<*m* + *k*})
by (*auto simp*: *convergent_LIMSEQ_iff dest*: *LIMSEQ_offset*)
thus *?thesis* **by** (*auto simp*: *summable_iff_convergent convergent_def*)
qed

lemma *Polygamma_converges*:
fixes *z* :: *'a* :: {*real_normed_field*,*banach*}
assumes *z*: $z \neq 0$ **and** *n*: $n \geq 2$
shows *uniformly_convergent_on* (*ball z d*) (*λk z*. ∑ *i*<*k*. *inverse* ((*z* + *of_nat i*) ^*n*))
proof (*rule Weierstrass_m_test'_ev*)
define *e* **where** *e* = (*1* + *d* / *norm z*)
define *m* **where** *m* = *nat* ⌈*norm z* ∗ *e*⌉
{
fix *t* **assume** *t*: *t* ∈ *ball z d*
have *norm t* = *norm* (*z* + (*t* − *z*)) **by** *simp*
also have ... ≤ *norm z* + *norm* (*t* − *z*) **by** (*rule norm_triangle_ineq*)
also from *t* **have** *norm* (*t* − *z*) < *d* **by** (*simp add*: *dist_norm norm_minus_commute*)
finally have *norm t* < *norm z* ∗ *e* **using** *z* **by** (*simp add*: *divide_simps e_def*)
} **note** *ball* = *this*

show *eventually* (*λk*. ∀ *t*∈*ball z d*. *norm* (*inverse* ((*t* + *of_nat k*) ^*n*)) ≤ *inverse* (*of_nat* (*k* − *m*) ^*n*)) *sequentially*
using *eventually_gt_at_top*[*of m*] **apply** *eventually_elim*
proof (*intro ballI*)
fix *k* :: *nat* **and** *t* :: *'a* **assume** *k*: *k* > *m* **and** *t*: *t* ∈ *ball z d*
from *k* **have** *real_of_nat* (*k* − *m*) = *of_nat k* − *of_nat m* **by** (*simp add*: *of_nat_diff*)
also have ... ≤ *norm* (*of_nat k* :: *'a*) − *norm z* ∗ *e*
unfolding *m_def* **by** (*subst norm_of_nat*) *linarith*
also from *ball*[*OF t*] **have** ... ≤ *norm* (*of_nat k* :: *'a*) − *norm t* **by** *simp*
also have ... ≤ *norm* (*of_nat k* + *t*) **by** (*rule norm_diff_ineq*)
finally have *inverse* ((*norm* (*t* + *of_nat k*)) ^*n*) ≤ *inverse* (*real_of_nat* (*k* − *m*) ^*n*) **using** *k n*
by (*intro le_imp_inverse_le power_mono*) (*simp_all add*: *add_ac del*: *of_nat_Suc*)
thus *norm* (*inverse* ((*t* + *of_nat k*) ^*n*)) ≤ *inverse* (*of_nat* (*k* − *m*) ^*n*)
by (*simp add*: *norm_inverse norm_power power_inverse*)
qed

have *summable* (*λk*. *inverse* ((*real_of_nat k*) ^*n*))
using *inverse_power_summable*[*of n*] *n* **by** *simp*
hence *summable* (*λk*. *inverse* ((*real_of_nat* (*k* + *m* − *m*)) ^*n*)) **by** *simp*
thus *summable* (*λk*. *inverse* ((*real_of_nat* (*k* − *m*)) ^*n*)) **by** (*rule summable_offset*)
qed

lemma *Polygamma_converges'*:
fixes *z* :: *'a* :: {*real_normed_field*,*banach*}
assumes *z*: $z \neq 0$ **and** *n*: $n \geq 2$
shows *summable* (*λk*. *inverse* ((*z* + *of_nat k*) ^*n*))
using *uniformly_convergent_imp_convergent*[*OF Polygamma_converges*[*OF assms*, *of 1*], *of z*]
by (*simp add*: *summable_iff_convergent*)

theorem *Digamma_LIMSEQ*:
fixes *z* :: *'a* :: {*banach*,*real_normed_field*}
assumes *z*: *z* ≠ *0*
shows (λ*m*. *of_real* (*ln* (*real m*)) − (∑ *n*<*m*. *inverse* (*z* + *of_nat n*))) ⟶ *Digamma z*
proof −
have (λ*n*. *of_real* (*ln* (*real n* / (*real* (*Suc n*))))) ⟶ (*of_real* (*ln 1*) :: *'a*)
by (*intro tendsto_intros LIMSEQ_n_over_Suc_n*) *simp_all*
hence (λ*n*. *of_real* (*ln* (*real n* / (*real n* + *1*)))) ⟶ (*0* :: *'a*) **by** (*simp add*: *add_ac*)
hence *lim*: (λ*n*. *of_real* (*ln* (*real n*)) − *of_real* (*ln* (*real n* + *1*))) ⟶ (*0*::*'a*)
proof (*rule Lim_transform_eventually*)
show *eventually* (λ*n*. *of_real* (*ln* (*real n* / (*real n* + *1*))) =
of_real (*ln* (*real n*)) − (*of_real* (*ln* (*real n* + *1*)) :: *'a*)) *at_top*
using *eventually_gt_at_top*[*of 0*::*nat*] **by** *eventually_elim* (*simp add*: *ln_div*)
qed

from *summable_Digamma*[*OF z*]
have (λ*n*. *inverse* (*of_nat* (*n*+*1*)) − *inverse* (*z* + *of_nat n*))
sums (*Digamma z* + *euler_mascheroni*)
by (*simp add*: *Digamma_def summable_sums*)
from *sums_diff*[*OF this euler_mascheroni_sum*]
have (λ*n*. *of_real* (*ln* (*real* (*Suc n*) + *1*)) − *of_real* (*ln* (*real n* + *1*)) − *inverse* (*z* + *of_nat n*))
sums Digamma z **by** (*simp add*: *add_ac*)
hence (λ*m*. (∑ *n*<*m*. *of_real* (*ln* (*real* (*Suc n*) + *1*)) − *of_real* (*ln* (*real n* + *1*))) −
(∑ *n*<*m*. *inverse* (*z* + *of_nat n*))) ⟶ *Digamma z*
by (*simp add*: *sums_def sum_subtractf*)
also have (λ*m*. (∑ *n*<*m*. *of_real* (*ln* (*real* (*Suc n*) + *1*)) − *of_real* (*ln* (*real n* + *1*)))) =
(λ*m*. *of_real* (*ln* (*m* + *1*)) :: *'a*)
by (*subst sum_lessThan_telescope*) *simp_all*
finally show *?thesis* **by** (*rule Lim_transform*) (*insert lim*, *simp*)
qed

theorem *Polygamma_LIMSEQ*:
fixes *z* :: *'a* :: {*banach*,*real_normed_field*}
assumes *z* ≠ *0* **and** *n* > *0*
shows (λ*k*. *inverse* ((*z* + *of_nat k*)^*Suc n*)) *sums* ((−*1*) ^ *Suc n* ∗ *Polygamma n z* / *fact n*)
using *Polygamma_converges'*[*OF assms*(*1*), *of Suc n*] *assms*(*2*)
by (*simp add*: *sums_iff Polygamma_def*)

theorem *has_field_derivative_ln_Gamma_complex* [*derivative_intros*]:
fixes *z* :: *complex*
assumes *z*: *z* ∉ $\mathbb{R}_{\leq 0}$
shows (*ln_Gamma has_field_derivative Digamma z*) (*at z*)
proof −

have *not_nonpos_Int* [*simp*]: $t \notin \mathbb{Z}_{\leq 0}$ **if** *Re t* > *0* **for** *t*
using *that* **by** (*auto elim*!: *nonpos_Ints_cases*′)
from *z* **have** *z*′: $z \notin \mathbb{Z}_{\leq 0}$ **and** *z*′′: *z* ≠ *0* **using** *nonpos_Ints_subset_nonpos_Reals nonpos_Reals_zero_I*
by *blast*+
let *?f*′ = λ*z k*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat* (*Suc k*))
let *?f* = λ*z k*. *z* / *of_nat* (*Suc k*) − *ln* (*1* + *z* / *of_nat* (*Suc k*)) **and** *?F*′ = λ*z*. ∑*n*. *?f*′ *z n*
define *d* **where** *d* = *min* (*norm z*/*2*) (*if Im z* = *0 then Re z* / *2 else abs* (*Im z*) / *2*)
from *z* **have** *d*: *d* > *0 norm z*/*2* ≥ *d* **by** (*auto simp add*: *complex_nonpos_Reals_iff d_def*)
have *ball*: *Im t* = *0* ⟶ *Re t* > *0* **if** *dist z t* < *d* **for** *t*
using *no_nonpos_Real_in_ball*[*OF z*, *of t*] *that* **unfolding** *d_def* **by** (*force simp add*: *complex_nonpos_Reals_iff*)
have *sums*: (λ*n*. *inverse* (*z* + *of_nat* (*Suc n*)) − *inverse* (*z* + *of_nat n*)) *sums* (*0* − *inverse* (*z* + *of_nat 0*))
by (*intro telescope_sums filterlim_compose*[*OF tendsto_inverse_0*] *tendsto_add_filterlim_at_infinity*[*OF tendsto_const*] *tendsto_of_nat*)

have ((λ*z*. ∑*n*. *?f z n*) *has_field_derivative ?F*′ *z*) (*at z*)
using *d z ln_Gamma_series*′*_aux*[*OF z*′]
apply (*intro has_field_derivative_series*′(*2*)[*of ball z d* _ _ *z*] *uniformly_summable_deriv_ln_Gamma*)
apply (*auto intro*!: *derivative_eq_intros add_pos_pos mult_pos_pos dest*!: *ball simp*: *field_simps sums_iff nonpos_Reals_divide_of_nat_iff simp del*: *of_nat_Suc*)
apply (*auto simp add*: *complex_nonpos_Reals_iff*)
done
with *z* **have** ((λ*z*. (∑*k*. *?f z k*) − *euler_mascheroni* ∗ *z* − *Ln z*) *has_field_derivative ?F*′ *z* − *euler_mascheroni* − *inverse z*) (*at z*)
by (*force intro*!: *derivative_eq_intros simp*: *Digamma_def*)
also have *?F*′ *z* − *euler_mascheroni* − *inverse z* = (*?F*′ *z* + −*inverse z*) − *euler_mascheroni* **by** *simp*
also from *sums* **have** −*inverse z* = (∑*n*. *inverse* (*z* + *of_nat* (*Suc n*)) − *inverse* (*z* + *of_nat n*))
by (*simp add*: *sums_iff*)
also from *sums summable_deriv_ln_Gamma*[*OF z*′′]
have *?F*′ *z* + ... = (∑*n*. *inverse* (*of_nat* (*Suc n*)) − *inverse* (*z* + *of_nat n*))
by (*subst suminf_add*) (*simp_all add*: *add_ac sums_iff*)
also have ... − *euler_mascheroni* = *Digamma z* **by** (*simp add*: *Digamma_def*)
finally have ((λ*z*. (∑*k*. *?f z k*) − *euler_mascheroni* ∗ *z* − *Ln z*) *has_field_derivative Digamma z*) (*at z*) .
moreover from *eventually_nhds_ball*[*OF d*(*1*), *of z*]
have *eventually* (λ*z*. *ln_Gamma z* = (∑*k*. *?f z k*) − *euler_mascheroni* ∗ *z* − *Ln z*) (*nhds z*)
proof *eventually_elim*
fix *t* **assume** *t* ∈ *ball z d*
hence $t \notin \mathbb{Z}_{\leq 0}$ **by** (*auto dest*!: *ball elim*!: *nonpos_Ints_cases*)
from *ln_Gamma_series*′*_aux*[*OF this*]

show *ln_Gamma t* = ($\sum$*k. ?f t k*) − *euler_mascheroni* ∗ *t* − *Ln t* **by** (*simp add*: *sums_iff*)
qed
ultimately show *?thesis* **by** (*subst DERIV_cong_ev*[*OF refl _ refl*])
qed

declare *has_field_derivative_ln_Gamma_complex*[*THEN DERIV_chain2*, *derivative_intros*]

lemma *Digamma_1* [*simp*]: *Digamma* (*1* :: *'a* :: {*real_normed_field*,*banach*}) = − *euler_mascheroni*
by (*simp add*: *Digamma_def*)

lemma *Digamma_plus1*:
assumes *z* ≠ *0*
shows *Digamma* (*z*+*1*) = *Digamma z* + *1*/*z*
proof −
have *sums*: (λ*k. inverse* (*z* + *of_nat k*) − *inverse* (*z* + *of_nat* (*Suc k*)))
sums (*inverse* (*z* + *of_nat 0*) − *0*)
by (*intro telescope_sums'*[*OF filterlim_compose*[*OF tendsto_inverse_0*]]
tendsto_add_filterlim_at_infinity[*OF tendsto_const*] *tendsto_of_nat*)
have *Digamma* (*z*+*1*) = ($\sum$*k. inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat* (*Suc k*))) −
euler_mascheroni (**is** _ = *suminf ?f* − _) **by** (*simp add*: *Digamma_def add_ac*)
also have *suminf ?f* = ($\sum$*k. inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat k*)) +
($\sum$*k. inverse* (*z* + *of_nat k*) − *inverse* (*z* + *of_nat* (*Suc k*)))
using *summable_Digamma*[*OF assms*] *sums* **by** (*subst suminf_add*) (*simp_all add*: *add_ac sums_iff*)
also have ($\sum$*k. inverse* (*z* + *of_nat k*) − *inverse* (*z* + *of_nat* (*Suc k*))) = *1*/*z*
using *sums* **by** (*simp add*: *sums_iff inverse_eq_divide*)
finally show *?thesis* **by** (*simp add*: *Digamma_def*[*of z*])
qed

theorem *Polygamma_plus1*:
assumes *z* ≠ *0*
shows *Polygamma n* (*z* + *1*) = *Polygamma n z* + (−*1*) ^*n* ∗ *fact n* / (*z* ^ *Suc n*)
proof (*cases n* = *0*)
assume *n*: *n* ≠ *0*
let *?f* = λ*k. inverse* ((*z* + *of_nat k*) ^ *Suc n*)
have *Polygamma n* (*z* + *1*) = (−*1*) ^ *Suc n* ∗ *fact n* ∗ ($\sum$*k. ?f* (*k*+*1*))
using *n* **by** (*simp add*: *Polygamma_def add_ac*)
also have ($\sum$*k. ?f* (*k*+*1*)) + ($\sum$*k*<*1*. *?f k*) = ($\sum$*k. ?f k*)
using *Polygamma_converges'*[*OF assms*, *of Suc n*] *n*
by (*subst suminf_split_initial_segment* [*symmetric*]) *simp_all*
hence ($\sum$*k. ?f* (*k*+*1*)) = ($\sum$*k. ?f k*) − *inverse* (*z* ^ *Suc n*) **by** (*simp add*: *algebra_simps*)

also have (*−1*) ˆ *Suc n* ∗ *fact n* ∗ ((∑*k*. *?f k*) − *inverse* (*z* ˆ *Suc n*)) =
Polygamma n z + (*−1*)ˆ*n* ∗ *fact n* / (*z* ˆ *Suc n*) **using** *n*
by (*simp add*: *inverse_eq_divide algebra_simps Polygamma_def*)
finally show *?thesis* **.**
qed (*insert assms*, *simp add*: *Digamma_plus1 inverse_eq_divide*)

theorem *Digamma_of_nat*:
Digamma (*of_nat* (*Suc n*) :: *'a* :: {*real_normed_field*,*banach*}) = *harm n* − *euler_mascheroni*
proof (*induction n*)
case (*Suc n*)
have *Digamma* (*of_nat* (*Suc* (*Suc n*)) :: *'a*) = *Digamma* (*of_nat* (*Suc n*) + *1*)
by *simp*
also have ... = *Digamma* (*of_nat* (*Suc n*)) + *inverse* (*of_nat* (*Suc n*))
by (*subst Digamma_plus1*) (*simp_all add*: *inverse_eq_divide del*: *of_nat_Suc*)
also have *Digamma* (*of_nat* (*Suc n*) :: *'a*) = *harm n* − *euler_mascheroni* **by**
(*rule Suc*)
also have ... + *inverse* (*of_nat* (*Suc n*)) = *harm* (*Suc n*) − *euler_mascheroni*
by (*simp add*: *harm_Suc*)
finally show *?case* **.**
qed (*simp add*: *harm_def*)

lemma *Digamma_numeral*: *Digamma* (*numeral n*) = *harm* (*pred_numeral n*) − *euler_mascheroni*
by (*subst of_nat_numeral*[*symmetric*], *subst numeral_eq_Suc*, *subst Digamma_of_nat*)
(*rule refl*)

lemma *Polygamma_of_real*: *x* ≠ *0* ⟹ *Polygamma n* (*of_real x*) = *of_real* (*Polygamma n x*)
unfolding *Polygamma_def* **using** *summable_Digamma*[*of x*] *Polygamma_converges'*[*of x Suc n*]
by (*simp_all add*: *suminf_of_real*)

lemma *Polygamma_Real*: *z* ∈ ℝ ⟹ *z* ≠ *0* ⟹ *Polygamma n z* ∈ ℝ
by (*elim Reals_cases*, *hypsubst*, *subst Polygamma_of_real*) *simp_all*

lemma *Digamma_half_integer*:
Digamma (*of_nat n* + *1/2* :: *'a* :: {*real_normed_field*,*banach*}) =
(∑*k*<*n*. *2* / (*of_nat* (*2*∗*k*+*1*))) − *euler_mascheroni* − *of_real* (*2* ∗ *ln 2*)
proof (*induction n*)
case *0*
have *Digamma* (*1/2* :: *'a*) = *of_real* (*Digamma* (*1/2*)) **by** (*simp add*: *Polygamma_of_real* [*symmetric*])
also have *Digamma* (*1/2*::*real*) =
(∑*k*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*of_nat k* + *1/2*)) − *euler_mascheroni*
by (*simp add*: *Digamma_def add_ac*)
also have (∑*k*. *inverse* (*of_nat* (*Suc k*) :: *real*) − *inverse* (*of_nat k* + *1/2*)) =
(∑*k*. *inverse* (*1/2*) ∗ (*inverse* (*2* ∗ *of_nat* (*Suc k*)) − *inverse* (*2* ∗

of_nat k + 1)))
  by (*simp_all add*: *add_ac inverse_mult_distrib*[*symmetric*] *ring_distribs del*: *inverse_divide*)
 also have ... = − *2* ∗ *ln 2* **using** *sums_minus*[*OF alternating_harmonic_series_sums'*]
  by (*subst suminf_mult*) (*simp_all add*: *algebra_simps sums_iff*)
 finally show *?case* **by** *simp*
next
 case (*Suc n*)
 have *nz*: *2* ∗ *of_nat n* + (*1*:: *'a*) ≠ *0*
  using *of_nat_neq_0*[*of 2∗n*] **by** (*simp only*: *of_nat_Suc*) (*simp add*: *add_ac*)
 hence *nz'*: *of_nat n* + (*1/2*::*'a*) ≠ *0* **by** (*simp add*: *field_simps*)
 have *Digamma* (*of_nat* (*Suc n*) + *1/2* :: *'a*) = *Digamma* (*of_nat n* + *1/2* + *1*)
by *simp*
 also from *nz'* **have** ... = *Digamma* (*of_nat n* + *1/2*) + *1* / (*of_nat n* + *1/2*)
  by (*rule Digamma_plus1*)
 also from *nz nz'* **have** *1* / (*of_nat n* + *1/2* :: *'a*) = *2* / (*2* ∗ *of_nat n* + *1*)
  by (*subst divide_eq_eq*) *simp_all*
 also note *Suc*
 finally show *?case* **by** (*simp add*: *add_ac*)
qed

lemma *Digamma_one_half*: *Digamma* (*1/2*) = − *euler_mascheroni* − *of_real* (*2* ∗ *ln 2*)
 using *Digamma_half_integer*[*of 0*] **by** *simp*

lemma *Digamma_real_three_halves_pos*: *Digamma* (*3/2* :: *real*) > *0*
proof −
 have −*Digamma* (*3/2* :: *real*) = −*Digamma* (*of_nat 1* + *1/2*) **by** *simp*
 also have ... = *2* ∗ *ln 2* + *euler_mascheroni* − *2* **by** (*subst Digamma_half_integer*) *simp*
 also note *euler_mascheroni_less_13_over_22*
 also note *ln2_le_25_over_36*
 finally show *?thesis* **by** *simp*
qed

theorem *has_field_derivative_Polygamma* [*derivative_intros*]:
 fixes *z* :: *'a* :: {*real_normed_field*,*euclidean_space*}
 assumes *z*: $z \notin \mathbb{Z}_{\leq 0}$
 shows (*Polygamma n has_field_derivative Polygamma* (*Suc n*) *z*) (*at z within A*)
proof (*rule has_field_derivative_at_within*, *cases n* = *0*)
 assume *n*: *n* = *0*
 let *?f* = λ*k z*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat k*)
 let *?F* = λ*z*. ∑*k*. *?f k z* **and** *?f'* = λ*k z*. *inverse* ((*z* + *of_nat k*)^2)
 from *no_nonpos_Int_in_ball'*[*OF z*] **guess** *d* **. note** *d* = *this*
 from *z* **have** *summable*: *summable* (λ*k*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat k*))
  by (*intro summable_Digamma*) *force*
 from *z* **have** *conv*: *uniformly_convergent_on* (*ball z d*) (λ*k z*. ∑*i*<*k*. *inverse* ((*z*

+ *of_nat i*)^2))
  by (*intro Polygamma_converges*) *auto*
 with *d* **have** *summable* (λ*k*. *inverse* ((*z* + *of_nat k*)^2)) **unfolding** *summable_iff_convergent*
  by (*auto dest*!: *uniformly_convergent_imp_convergent simp*: *summable_iff_convergent*
)

 have (*?F has_field_derivative* (∑*k*. *?f' k z*)) (*at z*)
 proof (*rule has_field_derivative_series'*[*of ball z d _ _ z*])
  fix *k* :: *nat* **and** *t* :: *'a* **assume** *t*: *t* ∈ *ball z d*
  from *t d*(*2*)[*of t*] **show** ((λ*z*. *?f k z*) *has_field_derivative ?f' k t*) (*at t within ball z d*)
  by (*auto intro*!: *derivative_eq_intros simp*: *power2_eq_square simp del*: *of_nat_Suc dest*!: *plus_of_nat_eq_0_imp elim*!: *nonpos_Ints_cases*)
 qed (*insert d*(*1*) *summable conv*, (*assumption*|*simp*)+)
 with *z* **show** (*Polygamma n has_field_derivative Polygamma* (*Suc n*) *z*) (*at z*)
  unfolding *Digamma_def* [*abs_def*] *Polygamma_def* [*abs_def*] **using** *n*
  by (*force simp*: *power2_eq_square intro*!: *derivative_eq_intros*)
next
 assume *n*: *n* ≠ *0*
 from *z* **have** *z'*: *z* ≠ *0* **by** *auto*
 from *no_nonpos_Int_in_ball'*[*OF z*] **guess** *d* **. note** *d* = *this*
 define *n'* **where** *n'* = *Suc n*
 from *n* **have** *n'*: *n'* ≥ *2* **by** (*simp add*: *n'_def*)
 have ((λ*z*. ∑*k*. *inverse* ((*z* + *of_nat k*) ^ *n'*)) *has_field_derivative*
   (∑*k*. − *of_nat n'* ∗ *inverse* ((*z* + *of_nat k*) ^ (*n'*+*1*)))) (*at z*)
 proof (*rule has_field_derivative_series'*[*of ball z d _ _ z*])
  fix *k* :: *nat* **and** *t* :: *'a* **assume** *t*: *t* ∈ *ball z d*
  with *d* **have** *t'*: $t \notin \mathbb{Z}_{\leq 0}$ *t* ≠ *0* **by** *auto*
  show ((λ*a*. *inverse* ((*a* + *of_nat k*) ^ *n'*)) *has_field_derivative*
   − *of_nat n'* ∗ *inverse* ((*t* + *of_nat k*) ^ (*n'*+*1*))) (*at t within ball z d*)
using *t'*
  by (*fastforce intro*!: *derivative_eq_intros simp*: *divide_simps power_diff dest*: *plus_of_nat_eq_0_imp*)
 next
  have *uniformly_convergent_on* (*ball z d*)
   (λ*k z*. (− *of_nat n'* :: *'a*) ∗ (∑*i*<*k*. *inverse* ((*z* + *of_nat i*) ^ (*n'*+*1*))))
  using *z' n* **by** (*intro uniformly_convergent_mult Polygamma_converges*) (*simp_all add*: *n'_def*)
  thus *uniformly_convergent_on* (*ball z d*)
   (λ*k z*. ∑*i*<*k*. − *of_nat n'* ∗ *inverse* ((*z* + *of_nat i* :: *'a*) ^ (*n'*+*1*)))
   by (*subst* (*asm*) *sum_distrib_left*) *simp*
 qed (*insert Polygamma_converges'*[*OF z' n'*] *d*, *simp_all*)
 also have (∑*k*. − *of_nat n'* ∗ *inverse* ((*z* + *of_nat k*) ^ (*n'* + *1*))) =
   (− *of_nat n'*) ∗ (∑*k*. *inverse* ((*z* + *of_nat k*) ^ (*n'* + *1*)))
  using *Polygamma_converges'*[*OF z'*, *of n'*+*1*] *n'* **by** (*subst suminf_mult*) *simp_all*
 finally have ((λ*z*. ∑*k*. *inverse* ((*z* + *of_nat k*) ^ *n'*)) *has_field_derivative*
   − *of_nat n'* ∗ (∑*k*. *inverse* ((*z* + *of_nat k*) ^ (*n'* + *1*)))) (*at z*) **.**
 from *DERIV_cmult*[*OF this*, *of* (−*1*)^*Suc n* ∗ *fact n* :: *'a*]
  show (*Polygamma n has_field_derivative Polygamma* (*Suc n*) *z*) (*at z*)

unfolding *n'_def Polygamma_def*[*abs_def*] **using** *n* **by** (*simp add*: *algebra_simps*)
qed

declare *has_field_derivative_Polygamma*[*THEN DERIV_chain2*, *derivative_intros*]

lemma *isCont_Polygamma* [*continuous_intros*]:
fixes *f* :: _ ⇒ *'a* :: {*real_normed_field*,*euclidean_space*}
shows *isCont f z* ⟹ *f z* ∉ $\mathbb{Z}_{\leq 0}$ ⟹ *isCont* (λ*x*. *Polygamma n* (*f x*)) *z*
by (*rule isCont_o2*[*OF _ DERIV_isCont*[*OF has_field_derivative_Polygamma*]])

lemma *continuous_on_Polygamma*:
A ∩ $\mathbb{Z}_{\leq 0}$ = {} ⟹ *continuous_on A* (*Polygamma n* :: _ ⇒ *'a* :: {*real_normed_field*,*euclidean_space*})
by (*intro continuous_at_imp_continuous_on isCont_Polygamma*[*OF continuous_ident*] *ballI*) *blast*

lemma *isCont_ln_Gamma_complex* [*continuous_intros*]:
fixes *f* :: *'a*::*t2_space* ⇒ *complex*
shows *isCont f z* ⟹ *f z* ∉ $\mathbb{R}_{\leq 0}$ ⟹ *isCont* (λ*z*. *ln_Gamma* (*f z*)) *z*
by (*rule isCont_o2*[*OF _ DERIV_isCont*[*OF has_field_derivative_ln_Gamma_complex*]])

lemma *continuous_on_ln_Gamma_complex* [*continuous_intros*]:
fixes *A* :: *complex set*
shows *A* ∩ $\mathbb{R}_{\leq 0}$ = {} ⟹ *continuous_on A ln_Gamma*
by (*intro continuous_at_imp_continuous_on ballI isCont_ln_Gamma_complex*[*OF continuous_ident*])
fastforce

lemma *deriv_Polygamma*:
assumes *z* ∉ $\mathbb{Z}_{\leq 0}$
shows *deriv* (*Polygamma m*) *z* =
Polygamma (*Suc m*) (*z* :: *'a* :: {*real_normed_field*,*euclidean_space*})
by (*intro DERIV_imp_deriv has_field_derivative_Polygamma assms*)
thm *has_field_derivative_Polygamma*

lemma *higher_deriv_Polygamma*:
assumes *z* ∉ $\mathbb{Z}_{\leq 0}$
shows (*deriv* ^^ *n*) (*Polygamma m*) *z* =
Polygamma (*m* + *n*) (*z* :: *'a* :: {*real_normed_field*,*euclidean_space*})
proof −
have *eventually* (λ*u*. (*deriv* ^^ *n*) (*Polygamma m*) *u* = *Polygamma* (*m* + *n*) *u*) (*nhds z*)
proof (*induction n*)
case (*Suc n*)
from *Suc.IH* **have** *eventually* (λ*z*. *eventually* (λ*u*. (*deriv* ^^ *n*) (*Polygamma m*) *u* = *Polygamma* (*m* + *n*) *u*) (*nhds z*)) (*nhds z*)
by (*simp add*: *eventually_eventually*)
hence *eventually* (λ*z*. *deriv* ((*deriv* ^^ *n*) (*Polygamma m*)) *z* =
deriv (*Polygamma* (*m* + *n*)) *z*) (*nhds z*)
by *eventually_elim* (*intro deriv_cong_ev refl*)

moreover have *eventually* ($\lambda z.\ z \in UNIV - \mathbb{Z}_{\le 0}$) (*nhds z*) **using** *assms*
by (*intro eventually_nhds_in_open open_Diff open_UNIV*) *auto*
ultimately show *?case* **by** *eventually_elim* (*simp_all add*: *deriv_Polygamma*)
qed *simp_all*
thus *?thesis* **by** (*rule eventually_nhds_x_imp_x*)
qed

lemma *deriv_ln_Gamma_complex*:
assumes $z \notin \mathbb{R}_{\le 0}$
shows *deriv ln_Gamma z = Digamma* (*z :: complex*)
by (*intro DERIV_imp_deriv has_field_derivative_ln_Gamma_complex assms*)

We define a type class that captures all the fundamental properties of the inverse of the Gamma function and defines the Gamma function upon that. This allows us to instantiate the type class both for the reals and for the complex numbers with a minimal amount of proof duplication.

class *Gamma = real_normed_field + complete_space +*
fixes *rGamma* :: $'a \Rightarrow 'a$
assumes *rGamma_eq_zero_iff_aux*: *rGamma z = 0* $\longleftrightarrow$ ($\exists\, n.\ z = -$ *of_nat n*)
assumes *differentiable_rGamma_aux1*:
($\bigwedge n.\ z \neq -$ *of_nat n*) $\Longrightarrow$
let d = (*THE d.* ($\lambda n.\ \sum k{<}n.$ *inverse* (*of_nat* (*Suc k*)) $-$ *inverse* ($z +$ *of_nat k*))
$\longrightarrow$ *d*) $-$ *scaleR euler_mascheroni 1*
in filterlim ($\lambda y.$ (*rGamma y* $-$ *rGamma z* $+$ *rGamma z* $*$ *d* $*$ ($y - z$)) $/_R$
norm ($y - z$)) (*nhds 0*) (*at z*)
assumes *differentiable_rGamma_aux2*:
let z $= -$ *of_nat n*
in filterlim ($\lambda y.$ (*rGamma y* $-$ *rGamma z* $-$ (-1)ˆ*n* $*$ (*prod of_nat* {*1..n*}) $*$ ($y - z$)) $/_R$
norm ($y - z$)) (*nhds 0*) (*at z*)
assumes *rGamma_series_aux*: ($\bigwedge n.\ z \neq -$ *of_nat n*) $\Longrightarrow$
let fact' = ($\lambda n.$ *prod of_nat* {*1..n*});
exp = ($\lambda x.$ *THE e.* ($\lambda n.\ \sum k{<}n.$ *x*ˆ*k* $/_R$ *fact k*) $\longrightarrow$ *e*);
pochhammer' = ($\lambda a\ n.$ ($\prod n = 0..n.\ a +$ *of_nat n*))
in filterlim ($\lambda n.$ *pochhammer' z n* / (*fact' n* $*$ *exp* ($z *$ (*ln* (*of_nat n*) $*_R$ *1*))))
(*nhds* (*rGamma z*)) *sequentially*
begin
subclass *banach* **..**
end

definition *Gamma z = inverse* (*rGamma z*)

6.23.3 Basic properties

lemma *Gamma_nonpos_Int*: $z \in \mathbb{Z}_{\le 0} \Longrightarrow$ *Gamma z = 0*
and *rGamma_nonpos_Int*: $z \in \mathbb{Z}_{\le 0} \Longrightarrow$ *rGamma z = 0*

using *rGamma_eq_zero_iff_aux*[*of z*] **unfolding** *Gamma_def* **by** (*auto elim*!: *nonpos_Ints_cases'*)

lemma *Gamma_nonzero*: $z \notin \mathbb{Z}_{\leq 0} \Longrightarrow Gamma\ z \neq 0$
and *rGamma_nonzero*: $z \notin \mathbb{Z}_{\leq 0} \Longrightarrow rGamma\ z \neq 0$
using *rGamma_eq_zero_iff_aux*[*of z*] **unfolding** *Gamma_def* **by** (*auto elim*!: *nonpos_Ints_cases'*)

lemma *Gamma_eq_zero_iff*: $Gamma\ z = 0 \longleftrightarrow z \in \mathbb{Z}_{\leq 0}$
and *rGamma_eq_zero_iff*: $rGamma\ z = 0 \longleftrightarrow z \in \mathbb{Z}_{\leq 0}$
using *rGamma_eq_zero_iff_aux*[*of z*] **unfolding** *Gamma_def* **by** (*auto elim*!: *nonpos_Ints_cases'*)

lemma *rGamma_inverse_Gamma*: *rGamma z* = *inverse* (*Gamma z*)
unfolding *Gamma_def* **by** *simp*

lemma *rGamma_series_LIMSEQ* [*tendsto_intros*]:
rGamma_series z ⟶ *rGamma z*
proof (*cases* $z \in \mathbb{Z}_{\leq 0}$)
case *False*
hence *z* ≠ − *of_nat n* **for** *n* **by** *auto*
from *rGamma_series_aux*[*OF this*] **show** *?thesis*
by (*simp add*: *rGamma_series_def*[*abs_def*] *fact_prod pochhammer_Suc_prod exp_def of_real_def*[*symmetric*] *suminf_def sums_def*[*abs_def*] *atLeast0AtMost*)
qed (*insert rGamma_eq_zero_iff*[*of z*], *simp_all add*: *rGamma_series_nonpos_Ints_LIMSEQ*)

theorem *Gamma_series_LIMSEQ* [*tendsto_intros*]:
Gamma_series z ⟶ *Gamma z*
proof (*cases* $z \in \mathbb{Z}_{\leq 0}$)
case *False*
hence (λ*n*. *inverse* (*rGamma_series z n*)) ⟶ *inverse* (*rGamma z*)
by (*intro tendsto_intros*) (*simp_all add*: *rGamma_eq_zero_iff*)
also have (λ*n*. *inverse* (*rGamma_series z n*)) = *Gamma_series z*
by (*simp add*: *rGamma_series_def Gamma_series_def*[*abs_def*])
finally show *?thesis* **by** (*simp add*: *Gamma_def*)
qed (*insert Gamma_eq_zero_iff*[*of z*], *simp_all add*: *Gamma_series_nonpos_Ints_LIMSEQ*)

lemma *Gamma_altdef*: *Gamma z* = *lim* (*Gamma_series z*)
using *Gamma_series_LIMSEQ*[*of z*] **by** (*simp add*: *limI*)

lemma *rGamma_1* [*simp*]: *rGamma 1* = *1*
proof −
have *A*: *eventually* (λ*n*. *rGamma_series 1 n* = *of_nat* (*Suc n*) / *of_nat n*) *sequentially*
using *eventually_gt_at_top*[*of 0*::*nat*]
by (*force elim*!: *eventually_mono simp*: *rGamma_series_def exp_of_real pochhammer_fact field_split_simps pochhammer_rec' dest*!: *pochhammer_eq_0_imp_nonpos_Int*)

have *rGamma_series 1* ⟶ *1* **by** (*subst tendsto_cong*[*OF A*]) (*rule LIMSEQ_Suc_n_over_n*)
moreover have *rGamma_series 1* ⟶ *rGamma 1* **by** (*rule tendsto_intros*)
ultimately show *?thesis* **by** (*intro LIMSEQ_unique*)
qed

lemma *rGamma_plus1*: *z* ∗ *rGamma* (*z* + *1*) = *rGamma z*
proof −
let *?f* = λ*n*. (*z* + *1*) ∗ *inverse* (*of_nat n*) + *1*
have *eventually* (λ*n*. *?f n* ∗ *rGamma_series z n* = *z* ∗ *rGamma_series* (*z* + *1*) *n*) *sequentially*
using *eventually_gt_at_top*[*of 0*::*nat*]
proof *eventually_elim*
fix *n* :: *nat* **assume** *n*: *n* > *0*
hence *z* ∗ *rGamma_series* (*z* + *1*) *n* = *inverse* (*of_nat n*) ∗ *pochhammer z* (*Suc* (*Suc n*)) / (*fact n* ∗ *exp* (*z* ∗ *of_real* (*ln* (*of_nat n*))))
by (*subst pochhammer_rec*) (*simp add*: *rGamma_series_def field_simps exp_add exp_of_real*)
also from *n* **have** . . . = *?f n* ∗ *rGamma_series z n*
by (*subst pochhammer_rec′*) (*simp_all add*: *field_split_simps rGamma_series_def*)
finally show *?f n* ∗ *rGamma_series z n* = *z* ∗ *rGamma_series* (*z* + *1*) *n* **..**
qed
moreover have (λ*n*. *?f n* ∗ *rGamma_series z n*) ⟶ ((*z*+*1*) ∗ *0* + *1*) ∗ *rGamma z*
by (*intro tendsto_intros lim_inverse_n*)
hence (λ*n*. *?f n* ∗ *rGamma_series z n*) ⟶ *rGamma z* **by** *simp*
ultimately have (λ*n*. *z* ∗ *rGamma_series* (*z* + *1*) *n*) ⟶ *rGamma z*
by (*blast intro*: *Lim_transform_eventually*)
moreover have (λ*n*. *z* ∗ *rGamma_series* (*z* + *1*) *n*) ⟶ *z* ∗ *rGamma* (*z* + *1*)
by (*intro tendsto_intros*)
ultimately show *z* ∗ *rGamma* (*z* + *1*) = *rGamma z* **using** *LIMSEQ_unique* **by** *blast*
qed

lemma *pochhammer_rGamma*: *rGamma z* = *pochhammer z n* ∗ *rGamma* (*z* + *of_nat n*)
proof (*induction n arbitrary*: *z*)
case (*Suc n z*)
have *rGamma z* = *pochhammer z n* ∗ *rGamma* (*z* + *of_nat n*) **by** (*rule Suc.IH*)
also note *rGamma_plus1* [*symmetric*]
finally show *?case* **by** (*simp add*: *add_ac pochhammer_rec′*)
qed *simp_all*

theorem *Gamma_plus1*: *z* ∉ $\mathbb{Z}_{\leq 0}$ ⟹ *Gamma* (*z* + *1*) = *z* ∗ *Gamma z*
using *rGamma_plus1*[*of z*] **by** (*simp add*: *rGamma_inverse_Gamma field_simps Gamma_eq_zero_iff*)

theorem *pochhammer_Gamma*: $z \notin \mathbb{Z}_{\leq 0} \Longrightarrow$ *pochhammer z n* = *Gamma* (*z* + *of_nat n*) / *Gamma z*
using *pochhammer_rGamma*[*of z*]
by (*simp add*: *rGamma_inverse_Gamma Gamma_eq_zero_iff field_simps*)

lemma *Gamma_0* [*simp*]: *Gamma 0* = *0*
and *rGamma_0* [*simp*]: *rGamma 0* = *0*
and *Gamma_neg_1* [*simp*]: *Gamma* (− *1*) = *0*
and *rGamma_neg_1* [*simp*]: *rGamma* (− *1*) = *0*
and *Gamma_neg_numeral* [*simp*]: *Gamma* (− *numeral n*) = *0*
and *rGamma_neg_numeral* [*simp*]: *rGamma* (− *numeral n*) = *0*
and *Gamma_neg_of_nat* [*simp*]: *Gamma* (− *of_nat m*) = *0*
and *rGamma_neg_of_nat* [*simp*]: *rGamma* (− *of_nat m*) = *0*
by (*simp_all add*: *rGamma_eq_zero_iff Gamma_eq_zero_iff*)

lemma *Gamma_1* [*simp*]: *Gamma 1* = *1* **unfolding** *Gamma_def* **by** *simp*

theorem *Gamma_fact*: *Gamma* (*1* + *of_nat n*) = *fact n*
by (*simp add*: *pochhammer_fact pochhammer_Gamma of_nat_in_nonpos_Ints_iff flip*: *of_nat_Suc*)

lemma *Gamma_numeral*: *Gamma* (*numeral n*) = *fact* (*pred_numeral n*)
by (*subst of_nat_numeral*[*symmetric*], *subst numeral_eq_Suc*,
subst of_nat_Suc, *subst Gamma_fact*) (*rule refl*)

lemma *Gamma_of_int*: *Gamma* (*of_int n*) = (*if n* > *0 then fact* (*nat* (*n* − *1*)) *else 0*)
proof (*cases n* > *0*)
case *True*
hence *Gamma* (*of_int n*) = *Gamma* (*of_nat* (*Suc* (*nat* (*n* − *1*)))) **by** (*subst of_nat_Suc*) *simp_all*
with *True* **show** *?thesis* **by** (*subst* (*asm*) *of_nat_Suc*, *subst* (*asm*) *Gamma_fact*) *simp*
qed (*simp_all add*: *Gamma_eq_zero_iff nonpos_Ints_of_int*)

lemma *rGamma_of_int*: *rGamma* (*of_int n*) = (*if n* > *0 then inverse* (*fact* (*nat* (*n* − *1*))) *else 0*)
by (*simp add*: *Gamma_of_int rGamma_inverse_Gamma*)

lemma *Gamma_seriesI*:
assumes (λ*n*. *g n* / *Gamma_series z n*) ⟶ *1*
shows *g* ⟶ *Gamma z*
proof (*rule Lim_transform_eventually*)
have *1*/*2* > (*0*::*real*) **by** *simp*
from *tendstoD*[*OF assms*, *OF this*]
show *eventually* (λ*n*. *g n* / *Gamma_series z n* ∗ *Gamma_series z n* = *g n*) *sequentially*
by (*force elim*!: *eventually_mono simp*: *dist_real_def*)

from *assms* **have** ($\lambda n.$ *g n* / *Gamma_series z n* ∗ *Gamma_series z n*) ⟶ *1* ∗ *Gamma z*
  by (*intro tendsto_intros*)
thus ($\lambda n.$ *g n* / *Gamma_series z n* ∗ *Gamma_series z n*) ⟶ *Gamma z* **by** *simp*
qed

lemma *Gamma_seriesI'*:
  assumes *f* ⟶ *rGamma z*
  assumes ($\lambda n.$ *g n* ∗ *f n*) ⟶ *1*
  assumes $z \notin \mathbb{Z}_{\leq 0}$
  shows *g* ⟶ *Gamma z*
proof (*rule Lim_transform_eventually*)
  have *1/2* > (*0::real*) **by** *simp*
  from *tendstoD*[*OF assms*(*2*), *OF this*] **show** *eventually* ($\lambda n.$ *g n* ∗ *f n* / *f n* = *g n*) *sequentially*
    by (*force elim!*: *eventually_mono simp*: *dist_real_def*)
  from *assms* **have** ($\lambda n.$ *g n* ∗ *f n* / *f n*) ⟶ *1* / *rGamma z*
    by (*intro tendsto_divide assms*) (*simp_all add*: *rGamma_eq_zero_iff*)
  thus ($\lambda n.$ *g n* ∗ *f n* / *f n*) ⟶ *Gamma z* **by** (*simp add*: *Gamma_def divide_inverse*)
qed

lemma *Gamma_series'_LIMSEQ*: *Gamma_series' z* ⟶ *Gamma z*
  by (*cases* $z \in \mathbb{Z}_{\leq 0}$) (*simp_all add*: *Gamma_nonpos_Int Gamma_seriesI*[*OF Gamma_series_Gamma_serie*
    Gamma_series'_nonpos_Ints_LIMSEQ[*of z*])

6.23.4 Differentiability

lemma *has_field_derivative_rGamma_no_nonpos_int*:
  assumes $z \notin \mathbb{Z}_{\leq 0}$
  shows (*rGamma has_field_derivative* −*rGamma z* ∗ *Digamma z*) (*at z within A*)
proof (*rule has_field_derivative_at_within*)
  from *assms* **have** *z* ≠ − *of_nat n* **for** *n* **by** *auto*
  from *differentiable_rGamma_aux1*[*OF this*]
    show (*rGamma has_field_derivative* −*rGamma z* ∗ *Digamma z*) (*at z*)
      unfolding *Digamma_def suminf_def sums_def*[*abs_def*]
        has_field_derivative_def has_derivative_def netlimit_at
    by (*simp add*: *Let_def bounded_linear_mult_right mult_ac of_real_def* [*symmetric*])
qed

lemma *has_field_derivative_rGamma_nonpos_int*:
  (*rGamma has_field_derivative* (−*1*) ^*n* ∗ *fact n*) (*at* (− *of_nat n*) *within A*)
  apply (*rule has_field_derivative_at_within*)
  using *differentiable_rGamma_aux2*[*of n*]
  unfolding *Let_def has_field_derivative_def has_derivative_def netlimit_at*
  by (*simp only*: *bounded_linear_mult_right mult_ac of_real_def* [*symmetric*] *fact_prod*) *simp*

lemma *has_field_derivative_rGamma* [*derivative_intros*]:
(*rGamma has_field_derivative* (*if z* ∈ $\mathbb{Z}_{\leq 0}$ *then* (−*1*) ^(*nat* ⌊*norm z*⌋) ∗ *fact* (*nat* ⌊*norm z*⌋)
else −*rGamma z* ∗ *Digamma z*)) (*at z within A*)
using *has_field_derivative_rGamma_no_nonpos_int*[*of z A*]
has_field_derivative_rGamma_nonpos_int[*of nat* ⌊*norm z*⌋ *A*]
by (*auto elim*!: *nonpos_Ints_cases'*)

declare *has_field_derivative_rGamma_no_nonpos_int* [*THEN DERIV_chain2*, *derivative_intros*]
declare *has_field_derivative_rGamma* [*THEN DERIV_chain2*, *derivative_intros*]
declare *has_field_derivative_rGamma_nonpos_int* [*derivative_intros*]
declare *has_field_derivative_rGamma_no_nonpos_int* [*derivative_intros*]
declare *has_field_derivative_rGamma* [*derivative_intros*]

theorem *has_field_derivative_Gamma* [*derivative_intros*]:
z ∉ $\mathbb{Z}_{\leq 0}$ ⟹ (*Gamma has_field_derivative Gamma z* ∗ *Digamma z*) (*at z within A*)
unfolding *Gamma_def* [*abs_def*]
by (*fastforce intro*!: *derivative_eq_intros simp*: *rGamma_eq_zero_iff*)

declare *has_field_derivative_Gamma*[*THEN DERIV_chain2*, *derivative_intros*]

hide_fact *rGamma_eq_zero_iff_aux differentiable_rGamma_aux1 differentiable_rGamma_aux2 differentiable_rGamma_aux2 rGamma_series_aux Gamma_class.rGamma_eq_zero_iff_aux*

lemma *continuous_on_rGamma* [*continuous_intros*]: *continuous_on A rGamma*
by (*rule DERIV_continuous_on has_field_derivative_rGamma*)+

lemma *continuous_on_Gamma* [*continuous_intros*]: *A* ∩ $\mathbb{Z}_{\leq 0}$ = {} ⟹ *continuous_on A Gamma*
by (*rule DERIV_continuous_on has_field_derivative_Gamma*)+ *blast*

lemma *isCont_rGamma* [*continuous_intros*]:
isCont f z ⟹ *isCont* (λ*x*. *rGamma* (*f x*)) *z*
by (*rule isCont_o2*[*OF _ DERIV_isCont*[*OF has_field_derivative_rGamma*]])

lemma *isCont_Gamma* [*continuous_intros*]:
isCont f z ⟹ *f z* ∉ $\mathbb{Z}_{\leq 0}$ ⟹ *isCont* (λ*x*. *Gamma* (*f x*)) *z*
by (*rule isCont_o2*[*OF _ DERIV_isCont*[*OF has_field_derivative_Gamma*]])

6.23.5 The complex Gamma function

instantiation *complex* :: *Gamma*
begin

definition *rGamma_complex* :: *complex* ⇒ *complex* **where**

rGamma_complex z = *lim* (*rGamma_series z*)

lemma *rGamma_series_complex_converges*:
convergent (*rGamma_series* (*z* :: *complex*)) (**is** *?thesis1*)
and *rGamma_complex_altdef*:
rGamma z = (*if z* ∈ $\mathbb{Z}_{\leq 0}$ *then 0 else exp* (−*ln_Gamma z*)) (**is** *?thesis2*)
proof −
have *?thesis1* ∧ *?thesis2*
proof (*cases z* ∈ $\mathbb{Z}_{\leq 0}$)
case *False*
have *rGamma_series z* ⟶ *exp* (− *ln_Gamma z*)
proof (*rule Lim_transform_eventually*)
from *ln_Gamma_series_complex_converges'*[*OF False*] **guess** *d* **by** (*elim exE conjE*)
from *this*(*1*) *uniformly_convergent_imp_convergent*[*OF this*(*2*), *of z*]
have *ln_Gamma_series z* ⟶ *lim* (*ln_Gamma_series z*) **by** (*simp add*: *convergent_LIMSEQ_iff*)
thus (λ*n*. *exp* (−*ln_Gamma_series z n*)) ⟶ *exp* (− *ln_Gamma z*)
unfolding *convergent_def ln_Gamma_def* **by** (*intro tendsto_exp tendsto_minus*)
from *eventually_gt_at_top*[*of 0*::*nat*] *exp_ln_Gamma_series_complex False*
show *eventually* (λ*n*. *exp* (−*ln_Gamma_series z n*) = *rGamma_series z n*) *sequentially*
by (*force elim*!: *eventually_mono simp*: *exp_minus Gamma_series_def rGamma_series_def*)
qed
with *False* **show** *?thesis*
by (*auto simp*: *convergent_def rGamma_complex_def intro*!: *limI*)
next
case *True*
then obtain *k* **where** *z* = − *of_nat k* **by** (*erule nonpos_Ints_cases'*)
also have *rGamma_series* ... ⟶ *0*
by (*subst tendsto_cong*[*OF rGamma_series_minus_of_nat*]) (*simp_all add*: *convergent_const*)
finally show *?thesis* **using** *True*
by (*auto simp*: *rGamma_complex_def convergent_def intro*!: *limI*)
qed
thus *?thesis1 ?thesis2* **by** *blast+*
qed

context
begin

private lemma *rGamma_complex_plus1*: *z* ∗ *rGamma* (*z* + *1*) = *rGamma* (*z* :: *complex*)
proof −
let *?f* = λ*n*. (*z* + *1*) ∗ *inverse* (*of_nat n*) + *1*
have *eventually* (λ*n*. *?f n* ∗ *rGamma_series z n* = *z* ∗ *rGamma_series* (*z* + *1*) *n*) *sequentially*
using *eventually_gt_at_top*[*of 0*::*nat*]

```
  proof eventually_elim
    fix n :: nat assume n: n > 0
    hence z * rGamma_series (z + 1) n = inverse (of_nat n) *
              pochhammer z (Suc (Suc n)) / (fact n * exp (z * of_real (ln (of_nat
n))))
      by (subst pochhammer_rec) (simp add: rGamma_series_def field_simps exp_add
exp_of_real)
    also from n have ... = ?f n * rGamma_series z n
     by (subst pochhammer_rec') (simp_all add: field_split_simps rGamma_series_def
add_ac)
    finally show ?f n * rGamma_series z n = z * rGamma_series (z + 1) n ..
  qed
   moreover have (λn. ?f n * rGamma_series z n) ⟶ ((z+1) * 0 + 1) *
rGamma z
    using rGamma_series_complex_converges
    by (intro tendsto_intros lim_inverse_n)
       (simp_all add: convergent_LIMSEQ_iff rGamma_complex_def)
  hence (λn. ?f n * rGamma_series z n) ⟶ rGamma z by simp
  ultimately have (λn. z * rGamma_series (z + 1) n) ⟶ rGamma z
    by (blast intro: Lim_transform_eventually)
  moreover have (λn. z * rGamma_series (z + 1) n) ⟶ z * rGamma (z +
1)
    using rGamma_series_complex_converges
   by (auto intro!: tendsto_mult simp: rGamma_complex_def convergent_LIMSEQ_iff)
  ultimately show z * rGamma (z + 1) = rGamma z using LIMSEQ_unique
by blast
qed

private lemma has_field_derivative_rGamma_complex_no_nonpos_Int:
  assumes (z :: complex) ∉ ℤ≤0
  shows   (rGamma has_field_derivative − rGamma z * Digamma z) (at z)
proof −
  have diff: (rGamma has_field_derivative − rGamma z * Digamma z) (at z) if
Re z > 0 for z
  proof (subst DERIV_cong_ev[OF refl _ refl])
    from that have eventually (λt. t ∈ ball z (Re z/2)) (nhds z)
      by (intro eventually_nhds_in_nhd) simp_all
    thus eventually (λt. rGamma t = exp (− ln_Gamma t)) (nhds z)
      using no_nonpos_Int_in_ball_complex[OF that]
      by (auto elim!: eventually_mono simp: rGamma_complex_altdef)
  next
    have z ∉ ℝ≤0 using that by (simp add: complex_nonpos_Reals_iff)
    with that show ((λt. exp (− ln_Gamma t)) has_field_derivative (−rGamma z
* Digamma z)) (at z)
     by (force elim!: nonpos_Ints_cases intro!: derivative_eq_intros simp: rGamma_complex_altdef)
  qed

  from assms show (rGamma has_field_derivative − rGamma z * Digamma z) (at
z)
```

proof (*induction nat* ⌊*1* − *Re z*⌋ *arbitrary*: *z*)
case (*Suc n z*)
from *Suc.prems* **have** *z*: $z \neq 0$ **by** *auto*
from *Suc.hyps* **have** *n* = *nat* ⌊− *Re z*⌋ **by** *linarith*
hence *A*: *n* = *nat* ⌊*1* − *Re* (*z* + *1*)⌋ **by** *simp*
from *Suc.prems* **have** *B*: $z + 1 \notin \mathbb{Z}_{\leq 0}$ **by** (*force dest*: *plus_one_in_nonpos_Ints_imp*)

have ((λ*z*. *z* ∗ (*rGamma* ∘ (λ*z*. *z* + *1*)) *z*) *has_field_derivative*
−*rGamma* (*z* + *1*) ∗ (*Digamma* (*z* + *1*) ∗ *z* − *1*)) (*at z*)
by (*rule derivative_eq_intros DERIV_chain Suc refl A B*)+ (*simp add*: *algebra_simps*)
also have (λ*z*. *z* ∗ (*rGamma* ∘ (λ*z*. *z* + *1* :: *complex*)) *z*) = *rGamma*
by (*simp add*: *rGamma_complex_plus1*)
also from *z* **have** *Digamma* (*z* + *1*) ∗ *z* − *1* = *z* ∗ *Digamma z*
by (*subst Digamma_plus1*) (*simp_all add*: *field_simps*)
also have −*rGamma* (*z* + *1*) ∗ (*z* ∗ *Digamma z*) = −*rGamma z* ∗ *Digamma z*
by (*simp add*: *rGamma_complex_plus1*[*of z*, *symmetric*])
finally show *?case* **.**
qed (*intro diff*, *simp*)
qed

private lemma *rGamma_complex_1*: *rGamma* (*1* :: *complex*) = *1*
proof −
have *A*: *eventually* (λ*n*. *rGamma_series 1 n* = *of_nat* (*Suc n*) / *of_nat n*) *sequentially*
using *eventually_gt_at_top*[*of 0*::*nat*]
by (*force elim*!: *eventually_mono simp*: *rGamma_series_def exp_of_real pochhammer_fact*
field_split_simps pochhammer_rec′ dest!: *pochhammer_eq_0_imp_nonpos_Int*)
have *rGamma_series 1* ⟶ *1* **by** (*subst tendsto_cong*[*OF A*]) (*rule LIMSEQ_Suc_n_over_n*)
thus *rGamma 1* = (*1* :: *complex*) **unfolding** *rGamma_complex_def* **by** (*rule limI*)
qed

private lemma *has_field_derivative_rGamma_complex_nonpos_Int*:
(*rGamma has_field_derivative* (−*1*)ˆ*n* ∗ *fact n*) (*at* (− *of_nat n* :: *complex*))
proof (*induction n*)
case *0*
have *A*: $(0::complex) + 1 \notin \mathbb{Z}_{\leq 0}$ **by** *simp*
have ((λ*z*. *z* ∗ (*rGamma* ∘ (λ*z*. *z* + *1* :: *complex*)) *z*) *has_field_derivative 1*) (*at 0*)
by (*rule derivative_eq_intros DERIV_chain refl*
has_field_derivative_rGamma_complex_no_nonpos_Int A)+ (*simp add*: *rGamma_complex_1*)
thus *?case* **by** (*simp add*: *rGamma_complex_plus1*)
next
case (*Suc n*)

```
  hence A: (rGamma has_field_derivative (−1) ^n ∗ fact n)
               (at (− of_nat (Suc n) + 1 :: complex)) by simp
   have ((λz. z ∗ (rGamma ∘ (λz. z + 1 :: complex)) z) has_field_derivative
            (− 1) ^ Suc n ∗ fact (Suc n)) (at (− of_nat (Suc n)))
     by (rule derivative_eq_intros refl A DERIV_chain)+
        (simp add: algebra_simps rGamma_complex_altdef)
  thus ?case by (simp add: rGamma_complex_plus1)
qed

instance proof
  fix z :: complex show (rGamma z = 0) ⟷ (∃ n. z = − of_nat n)
    by (auto simp: rGamma_complex_altdef elim!: nonpos_Ints_cases')
next
  fix z :: complex assume ⋀n. z ≠ − of_nat n
  hence z ∉ ℤ≤0 by (auto elim!: nonpos_Ints_cases')
  from has_field_derivative_rGamma_complex_no_nonpos_Int[OF this]
    show let d = (THE d. (λn. ∑k<n. inverse (of_nat (Suc k)) − inverse (z +
of_nat k))
                          ⟶ d) − euler_mascheroni ∗R 1 in (λy. (rGamma y −
rGamma z +
              rGamma z ∗ d ∗ (y − z)) /R cmod (y − z)) −z→ 0
     by (simp add: has_field_derivative_def has_derivative_def Digamma_def sums_def
[abs_def]
                    of_real_def[symmetric] suminf_def)
next
  fix n :: nat
  from has_field_derivative_rGamma_complex_nonpos_Int[of n]
  show let z = − of_nat n in (λy. (rGamma y − rGamma z − (− 1) ^ n ∗ prod
of_nat {1..n} ∗
                  (y − z)) /R cmod (y − z)) −z→ 0
    by (simp add: has_field_derivative_def has_derivative_def fact_prod Let_def)
next
  fix z :: complex
   from rGamma_series_complex_converges[of z] have rGamma_series z ⟶
rGamma z
    by (simp add: convergent_LIMSEQ_iff rGamma_complex_def)
  thus let fact' = λn. prod of_nat {1..n};
             exp = λx. THE e. (λn. ∑k<n. x ^ k /R fact k) ⟶ e;
             pochhammer' = λa n. ∏n = 0..n. a + of_nat n
          in (λn. pochhammer' z n / (fact' n ∗ exp (z ∗ ln (real_of_nat n) ∗R 1)))
⟶ rGamma z
     by (simp add: fact_prod pochhammer_Suc_prod rGamma_series_def [abs_def]
exp_def
               of_real_def [symmetric] suminf_def sums_def [abs_def] atLeast0AtMost)
qed

end
end
```

lemma *Gamma_complex_altdef*:
Gamma z = (*if z* ∈ $\mathbb{Z}_{\leq 0}$ *then 0 else exp* (*ln_Gamma* (*z* :: *complex*)))
unfolding *Gamma_def rGamma_complex_altdef* **by** (*simp add*: *exp_minus*)

lemma *cnj_rGamma*: *cnj* (*rGamma z*) = *rGamma* (*cnj z*)
proof −
have *rGamma_series* (*cnj z*) = (λ*n*. *cnj* (*rGamma_series z n*))
by (*intro ext*) (*simp_all add*: *rGamma_series_def exp_cnj*)
also have ... ⟶ *cnj* (*rGamma z*) **by** (*intro tendsto_cnj tendsto_intros*)
finally show *?thesis* **unfolding** *rGamma_complex_def* **by** (*intro sym*[*OF limI*])
qed

lemma *cnj_Gamma*: *cnj* (*Gamma z*) = *Gamma* (*cnj z*)
unfolding *Gamma_def* **by** (*simp add*: *cnj_rGamma*)

lemma *Gamma_complex_real*:
z ∈ ℝ ⟹ *Gamma z* ∈ (ℝ :: *complex set*) **and** *rGamma_complex_real*: *z* ∈ ℝ ⟹ *rGamma z* ∈ ℝ
by (*simp_all add*: *Reals_cnj_iff cnj_Gamma cnj_rGamma*)

lemma *field_differentiable_rGamma*: *rGamma field_differentiable* (*at z within A*)
using *has_field_derivative_rGamma*[*of z*] **unfolding** *field_differentiable_def* **by** *blast*

lemma *holomorphic_rGamma* [*holomorphic_intros*]: *rGamma holomorphic_on A*
unfolding *holomorphic_on_def* **by** (*auto intro*!: *field_differentiable_rGamma*)

lemma *holomorphic_rGamma'* [*holomorphic_intros*]:
assumes *f holomorphic_on A*
shows (λ*x*. *rGamma* (*f x*)) *holomorphic_on A*
proof −
have *rGamma* ∘ *f holomorphic_on A* **using** *assms*
by (*intro holomorphic_on_compose assms holomorphic_rGamma*)
thus *?thesis* **by** (*simp only*: *o_def*)
qed

lemma *analytic_rGamma*: *rGamma analytic_on A*
unfolding *analytic_on_def* **by** (*auto intro*!: *exI*[*of _ 1*] *holomorphic_rGamma*)

lemma *field_differentiable_Gamma*: *z* ∉ $\mathbb{Z}_{\leq 0}$ ⟹ *Gamma field_differentiable* (*at z within A*)
using *has_field_derivative_Gamma*[*of z*] **unfolding** *field_differentiable_def* **by** *auto*

lemma *holomorphic_Gamma* [*holomorphic_intros*]: *A* ∩ $\mathbb{Z}_{\leq 0}$ = {} ⟹ *Gamma holomorphic_on A*
unfolding *holomorphic_on_def* **by** (*auto intro*!: *field_differentiable_Gamma*)

lemma *holomorphic_Gamma'* [*holomorphic_intros*]:
  assumes *f holomorphic_on A* **and** $\bigwedge x.\ x \in A \Longrightarrow f\ x \notin \mathbb{Z}_{\leq 0}$
  shows (*λx. Gamma* (*f x*)) *holomorphic_on A*
proof −
  have *Gamma* ∘ *f holomorphic_on A* **using** *assms*
    by (*intro holomorphic_on_compose assms holomorphic_Gamma*) *auto*
  thus *?thesis* **by** (*simp only*: *o_def*)
qed

lemma *analytic_Gamma*: $A \cap \mathbb{Z}_{\leq 0} = \{\} \Longrightarrow$ *Gamma analytic_on A*
  by (*rule analytic_on_subset*[*of _ UNIV* − $\mathbb{Z}_{\leq 0}$], *subst analytic_on_open*)
    (*auto intro*!: *holomorphic_Gamma*)

lemma *field_differentiable_ln_Gamma_complex*:
  $z \notin \mathbb{R}_{\leq 0} \Longrightarrow$ *ln_Gamma field_differentiable* (*at* (*z*::*complex*) *within A*)
  by (*rule field_differentiable_within_subset*[*of _ _ UNIV*])
    (*force simp*: *field_differentiable_def intro*!: *derivative_intros*)+

lemma *holomorphic_ln_Gamma* [*holomorphic_intros*]: $A \cap \mathbb{R}_{\leq 0} = \{\} \Longrightarrow$ *ln_Gamma holomorphic_on A*
 unfolding *holomorphic_on_def* **by** (*auto intro*!: *field_differentiable_ln_Gamma_complex*)

lemma *analytic_ln_Gamma*: $A \cap \mathbb{R}_{\leq 0} = \{\} \Longrightarrow$ *ln_Gamma analytic_on A*
  by (*rule analytic_on_subset*[*of _ UNIV* − $\mathbb{R}_{\leq 0}$], *subst analytic_on_open*)
    (*auto intro*!: *holomorphic_ln_Gamma*)

lemma *has_field_derivative_rGamma_complex'* [*derivative_intros*]:
  (*rGamma has_field_derivative* (*if* $z \in \mathbb{Z}_{\leq 0}$ *then* (−*1*) ˆ(*nat* ⌊−*Re z*⌋) ∗ *fact* (*nat* ⌊−*Re z*⌋) *else*
      −*rGamma z* ∗ *Digamma z*)) (*at z within A*)
  using *has_field_derivative_rGamma*[*of z*] **by** (*auto elim*!: *nonpos_Ints_cases'*)

declare *has_field_derivative_rGamma_complex'*[*THEN DERIV_chain2*, *derivative_intros*]

lemma *field_differentiable_Polygamma*:
  fixes *z* :: *complex*
  shows
  $z \notin \mathbb{Z}_{\leq 0} \Longrightarrow$ *Polygamma n field_differentiable* (*at z within A*)
  using *has_field_derivative_Polygamma*[*of z n*] **unfolding** *field_differentiable_def*
by *auto*

lemma *holomorphic_on_Polygamma* [*holomorphic_intros*]: $A \cap \mathbb{Z}_{\leq 0} = \{\} \Longrightarrow$ *Polygamma n holomorphic_on A*
  unfolding *holomorphic_on_def* **by** (*auto intro*!: *field_differentiable_Polygamma*)

lemma *analytic_on_Polygamma*: $A \cap \mathbb{Z}_{\leq 0} = \{\} \Longrightarrow$ *Polygamma n analytic_on A*

by (*rule analytic_on_subset*[*of* _ *UNIV* − $\mathbb{Z}_{\leq 0}$], *subst analytic_on_open*)
(*auto intro*!: *holomorphic_on_Polygamma*)

6.23.6 The real Gamma function

lemma *rGamma_series_real*:
eventually (λ*n*. *rGamma_series x n* = *Re* (*rGamma_series* (*of_real x*) *n*)) *sequentially*
using *eventually_gt_at_top*[*of 0* :: *nat*]
proof *eventually_elim*
fix *n* :: *nat* **assume** *n*: *n* > *0*
have *Re* (*rGamma_series* (*of_real x*) *n*) =
Re (*of_real* (*pochhammer x* (*Suc n*)) / (*fact n* ∗ *exp* (*of_real* (*x* ∗ *ln* (*real_of_nat n*)))))
using *n* **by** (*simp add*: *rGamma_series_def powr_def pochhammer_of_real*)
also from *n* **have** ... = *Re* (*of_real* ((*pochhammer x* (*Suc n*)) /
(*fact n* ∗ (*exp* (*x* ∗ *ln* (*real_of_nat n*))))))
by (*subst exp_of_real*) *simp*
also from *n* **have** ... = *rGamma_series x n*
by (*subst Re_complex_of_real*) (*simp add*: *rGamma_series_def powr_def*)
finally show *rGamma_series x n* = *Re* (*rGamma_series* (*of_real x*) *n*) **..**
qed

instantiation *real* :: *Gamma*
begin

definition *rGamma_real x* = *Re* (*rGamma* (*of_real x* :: *complex*))

instance proof
fix *x* :: *real*
have *rGamma x* = *Re* (*rGamma* (*of_real x*)) **by** (*simp add*: *rGamma_real_def*)
also have *of_real* ... = *rGamma* (*of_real x* :: *complex*)
by (*intro of_real_Re rGamma_complex_real*) *simp_all*
also have ... = *0* ⟷ *x* ∈ $\mathbb{Z}_{\leq 0}$ **by** (*simp add*: *rGamma_eq_zero_iff of_real_in_nonpos_Ints_iff*)
also have ... ⟷ (∃ *n*. *x* = − *of_nat n*) **by** (*auto elim*!: *nonpos_Ints_cases'*)
finally show (*rGamma x*) = *0* ⟷ (∃ *n*. *x* = − *real_of_nat n*) **by** *simp*
next
fix *x* :: *real* **assume** ⋀*n*. *x* ≠ − *of_nat n*
hence *x*: *complex_of_real x* ∉ $\mathbb{Z}_{\leq 0}$
by (*subst of_real_in_nonpos_Ints_iff*) (*auto elim*!: *nonpos_Ints_cases'*)
then have *x* ≠ *0* **by** *auto*
with *x* **have** (*rGamma has_field_derivative* − *rGamma x* ∗ *Digamma x*) (*at x*)
by (*fastforce intro*!: *derivative_eq_intros has_vector_derivative_real_field*
simp: *Polygamma_of_real rGamma_real_def* [*abs_def*])
thus *let d* = (*THE d*. (λ*n*. ∑*k*<*n*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*x* + *of_nat k*))
⟶ *d*) − *euler_mascheroni* $*_R$ *1 in* (λ*y*. (*rGamma y* − *rGamma x* +
rGamma x ∗ *d* ∗ (*y* − *x*)) $/_R$ *norm* (*y* − *x*)) −*x*→ *0*
by (*simp add*: *has_field_derivative_def has_derivative_def Digamma_def sums_def*

[*abs_def*]
of_real_def[*symmetric*] *suminf_def*)
next
fix *n* :: *nat*
have (*rGamma has_field_derivative* (−*1*) ^*n* ∗ *fact n*) (*at* (− *of_nat n* :: *real*))
by (*fastforce intro*!: *derivative_eq_intros has_vector_derivative_real_field*
simp: *Polygamma_of_real rGamma_real_def* [*abs_def*])
thus *let x* = − *of_nat n in* (λ*y*. (*rGamma y* − *rGamma x* − (− *1*) ^ *n* ∗ *prod of_nat* {*1*..*n*} ∗
(*y* − *x*)) /$_R$ *norm* (*y* − *x*)) −*x*::*real*→ *0*
by (*simp add*: *has_field_derivative_def has_derivative_def fact_prod Let_def*)
next
fix *x* :: *real*
have *rGamma_series x* ⟶ *rGamma x*
proof (*rule Lim_transform_eventually*)
show (λ*n*. *Re* (*rGamma_series* (*of_real x*) *n*)) ⟶ *rGamma x* **unfolding** *rGamma_real_def*
by (*intro tendsto_intros*)
qed (*insert rGamma_series_real*, *simp add*: *eq_commute*)
thus *let fact'* = λ*n*. *prod of_nat* {*1*..*n*};
exp = λ*x*. *THE e*. (λ*n*. ∑*k*<*n*. *x* ^ *k* /$_R$ *fact k*) ⟶ *e*;
pochhammer' = λ*a n*. ∏*n* = *0*..*n*. *a* + *of_nat n*
in (λ*n*. *pochhammer' x n* / (*fact' n* ∗ *exp* (*x* ∗ *ln* (*real_of_nat n*) ∗$_R$ *1*))) ⟶ *rGamma x*
by (*simp add*: *fact_prod pochhammer_Suc_prod rGamma_series_def* [*abs_def*] *exp_def*
of_real_def [*symmetric*] *suminf_def sums_def* [*abs_def*] *atLeast0AtMost*)
qed

end

lemma *rGamma_complex_of_real*: *rGamma* (*complex_of_real x*) = *complex_of_real* (*rGamma x*)
unfolding *rGamma_real_def* **using** *rGamma_complex_real* **by** *simp*

lemma *Gamma_complex_of_real*: *Gamma* (*complex_of_real x*) = *complex_of_real* (*Gamma x*)
unfolding *Gamma_def* **by** (*simp add*: *rGamma_complex_of_real*)

lemma *rGamma_real_altdef*: *rGamma x* = *lim* (*rGamma_series* (*x* :: *real*))
by (*rule sym*, *rule limI*, *rule tendsto_intros*)

lemma *Gamma_real_altdef1*: *Gamma x* = *lim* (*Gamma_series* (*x* :: *real*))
by (*rule sym*, *rule limI*, *rule tendsto_intros*)

lemma *Gamma_real_altdef2*: *Gamma x* = *Re* (*Gamma* (*of_real x*))
using *rGamma_complex_real*[*OF Reals_of_real*[*of x*]]
by (*elim Reals_cases*)

(*simp only*: *Gamma_def rGamma_real_def of_real_inverse*[*symmetric*] *Re_complex_of_real*)

lemma *ln_Gamma_series_complex_of_real*:
x > *0* ⟹ *n* > *0* ⟹ *ln_Gamma_series* (*complex_of_real x*) *n* = *of_real* (*ln_Gamma_series x n*)
proof −
assume *xn*: *x* > *0 n* > *0*
have *Ln* (*complex_of_real x* / *of_nat k* + *1*) = *of_real* (*ln* (*x* / *of_nat k* + *1*)) **if** *k* ≥ *1* **for** *k*
using *that xn* **by** (*subst Ln_of_real* [*symmetric*]) (*auto intro*!: *add_nonneg_pos simp*: *field_simps*)
with *xn* **show** *?thesis* **by** (*simp add*: *ln_Gamma_series_def Ln_of_real*)
qed

lemma *ln_Gamma_real_converges*:
assumes (*x*::*real*) > *0*
shows *convergent* (*ln_Gamma_series x*)
proof −
have (λ*n*. *ln_Gamma_series* (*complex_of_real x*) *n*) ⟶ *ln_Gamma* (*of_real x*) **using** *assms*
by (*intro ln_Gamma_complex_LIMSEQ*) (*auto simp*: *of_real_in_nonpos_Ints_iff*)
moreover from *eventually_gt_at_top*[*of 0*::*nat*]
have *eventually* (λ*n*. *complex_of_real* (*ln_Gamma_series x n*) =
ln_Gamma_series (*complex_of_real x*) *n*) *sequentially*
by *eventually_elim* (*simp add*: *ln_Gamma_series_complex_of_real assms*)
ultimately have (λ*n*. *complex_of_real* (*ln_Gamma_series x n*)) ⟶ *ln_Gamma* (*of_real x*)
by (*subst tendsto_cong*) *assumption*+
from *tendsto_Re*[*OF this*] **show** *?thesis* **by** (*auto simp*: *convergent_def*)
qed

lemma *ln_Gamma_real_LIMSEQ*: (*x*::*real*) > *0* ⟹ *ln_Gamma_series x* ⟶ *ln_Gamma x*
using *ln_Gamma_real_converges*[*of x*] **unfolding** *ln_Gamma_def* **by** (*simp add*: *convergent_LIMSEQ_iff*)

lemma *ln_Gamma_complex_of_real*: *x* > *0* ⟹ *ln_Gamma* (*complex_of_real x*) = *of_real* (*ln_Gamma x*)
proof (*unfold ln_Gamma_def*, *rule limI*, *rule Lim_transform_eventually*)
assume *x*: *x* > *0*
show *eventually* (λ*n*. *of_real* (*ln_Gamma_series x n*) =
ln_Gamma_series (*complex_of_real x*) *n*) *sequentially*
using *eventually_gt_at_top*[*of 0*::*nat*]
by *eventually_elim* (*simp add*: *ln_Gamma_series_complex_of_real x*)
qed (*intro tendsto_of_real*, *insert ln_Gamma_real_LIMSEQ*[*of x*], *simp add*: *ln_Gamma_def*)

lemma *Gamma_real_pos_exp*: *x* > (*0* :: *real*) ⟹ *Gamma x* = *exp* (*ln_Gamma x*)
by (*auto simp*: *Gamma_real_altdef2 Gamma_complex_altdef of_real_in_nonpos_Ints_iff*
ln_Gamma_complex_of_real exp_of_real)

lemma *ln_Gamma_real_pos*: $x > 0 \Longrightarrow$ *ln_Gamma* $x =$ *ln* (*Gamma* x :: *real*)
  unfolding *Gamma_real_pos_exp* **by** *simp*

lemma *ln_Gamma_complex_conv_fact*: $n > 0 \Longrightarrow$ *ln_Gamma* (*of_nat* n :: *complex*) $=$ *ln* (*fact* ($n - 1$))
  using *ln_Gamma_complex_of_real*[*of real n*] *Gamma_fact*[*of* $n - 1$, **where** $'a =$ *real*]
  by (*simp add*: *ln_Gamma_real_pos of_nat_diff Ln_of_real* [*symmetric*])

lemma *ln_Gamma_real_conv_fact*: $n > 0 \Longrightarrow$ *ln_Gamma* (*real* n) $=$ *ln* (*fact* ($n - 1$))
  using *Gamma_fact*[*of* $n - 1$, **where** $'a =$ *real*]
  by (*simp add*: *ln_Gamma_real_pos of_nat_diff Ln_of_real* [*symmetric*])

lemma *Gamma_real_pos* [*simp*, *intro*]: $x > (0$::*real*$) \Longrightarrow$ *Gamma* $x > 0$
  by (*simp add*: *Gamma_real_pos_exp*)

lemma *Gamma_real_nonneg* [*simp*, *intro*]: $x > (0$::*real*$) \Longrightarrow$ *Gamma* $x \geq 0$
  by (*simp add*: *Gamma_real_pos_exp*)

lemma *has_field_derivative_ln_Gamma_real* [*derivative_intros*]:
  assumes x: $x > (0$::*real*$)$
  shows (*ln_Gamma has_field_derivative Digamma* x) (*at* x)
proof (*subst DERIV_cong_ev*[*OF refl _ refl*])
  from *assms* **show** ((*Re* ∘ *ln_Gamma* ∘ *complex_of_real*) *has_field_derivative Digamma* x) (*at* x)
    by (*auto intro*!: *derivative_eq_intros has_vector_derivative_real_field*
      simp: *Polygamma_of_real o_def*)
  from *eventually_nhds_in_nhd*[*of* x {*0*<..}] *assms*
    show *eventually* (λ*y*. *ln_Gamma* $y =$ (*Re* ∘ *ln_Gamma* ∘ *of_real*) y) (*nhds* x)
   by (*auto elim*!: *eventually_mono simp*: *ln_Gamma_complex_of_real interior_open*)
qed

lemma *field_differentiable_ln_Gamma_real*:
  $x > 0 \Longrightarrow$ *ln_Gamma field_differentiable* (*at* (x::*real*) *within* A)
  by (*rule field_differentiable_within_subset*[*of _ _ UNIV*])
    (*auto simp*: *field_differentiable_def intro*!: *derivative_intros*)+

declare *has_field_derivative_ln_Gamma_real*[*THEN DERIV_chain2*, *derivative_intros*]

lemma *deriv_ln_Gamma_real*:
  assumes $z > 0$
  shows *deriv ln_Gamma* $z =$ *Digamma* (z :: *real*)
  by (*intro DERIV_imp_deriv has_field_derivative_ln_Gamma_real assms*)

lemma *has_field_derivative_rGamma_real'* [*derivative_intros*]:
  (*rGamma has_field_derivative* (*if* $x \in \mathbb{Z}_{\leq 0}$ *then* (-1) ^(*nat* $\lfloor -x \rfloor$) ∗ *fact* (*nat*

$\lfloor -x \rfloor$) *else*
    $-$*rGamma x* $*$ *Digamma x*)) (*at x within A*)
  using *has_field_derivative_rGamma*[*of x*] **by** (*force elim*!: *nonpos_Ints_cases'*)

declare *has_field_derivative_rGamma_real'*[*THEN DERIV_chain2*, *derivative_intros*]

lemma *Polygamma_real_odd_pos*:
  assumes (*x*::*real*) $\notin \mathbb{Z}_{\leq 0}$ *odd n*
  shows *Polygamma n x* $>$ *0*
proof $-$
  from *assms* **have** *x* $\neq$ *0* **by** *auto*
  with *assms* **show** *?thesis*
    unfolding *Polygamma_def* **using** *Polygamma_converges'*[*of x Suc n*]
    by (*auto simp*: *zero_less_power_eq simp del*: *power_Suc*
          dest: *plus_of_nat_eq_0_imp intro*!: *mult_pos_pos suminf_pos*)
qed

lemma *Polygamma_real_even_neg*:
  assumes (*x*::*real*) $>$ *0 n* $>$ *0 even n*
  shows *Polygamma n x* $<$ *0*
  using *assms* **unfolding** *Polygamma_def* **using** *Polygamma_converges'*[*of x Suc n*]
  by (*auto intro*!: *mult_pos_pos suminf_pos*)

lemma *Polygamma_real_strict_mono*:
  assumes *x* $>$ *0 x* $<$ (*y*::*real*) *even n*
  shows *Polygamma n x* $<$ *Polygamma n y*
proof $-$
  have $\exists\xi$. *x* $<$ ξ $\wedge$ ξ $<$ *y* $\wedge$ *Polygamma n y* $-$ *Polygamma n x* $=$ (*y* $-$ *x*) $*$ *Polygamma* (*Suc n*) ξ
    using *assms* **by** (*intro MVT2 derivative_intros impI allI*) (*auto elim*!: *nonpos_Ints_cases*)
  then guess ξ **by** (*elim exE conjE*) **note** ξ = *this*
  note ξ(*3*)
  also from ξ(*1*,*2*) *assms* **have** (*y* $-$ *x*) $*$ *Polygamma* (*Suc n*) ξ $>$ *0*
   by (*intro mult_pos_pos Polygamma_real_odd_pos*) (*auto elim*!: *nonpos_Ints_cases*)
  finally show *?thesis* **by** *simp*
qed

lemma *Polygamma_real_strict_antimono*:
  assumes *x* $>$ *0 x* $<$ (*y*::*real*) *odd n*
  shows *Polygamma n x* $>$ *Polygamma n y*
proof $-$
  have $\exists\xi$. *x* $<$ ξ $\wedge$ ξ $<$ *y* $\wedge$ *Polygamma n y* $-$ *Polygamma n x* $=$ (*y* $-$ *x*) $*$ *Polygamma* (*Suc n*) ξ
    using *assms* **by** (*intro MVT2 derivative_intros impI allI*) (*auto elim*!: *nonpos_Ints_cases*)
  then guess ξ **by** (*elim exE conjE*) **note** ξ = *this*
  note ξ(*3*)

also from *ξ(1,2) assms* **have** *(y − x) ∗ Polygamma (Suc n) ξ < 0*
by *(intro mult_pos_neg Polygamma_real_even_neg) simp_all*
finally show *?thesis* **by** *simp*
qed

lemma *Polygamma_real_mono*:
assumes *x > 0 x ≤ (y::real) even n*
shows *Polygamma n x ≤ Polygamma n y*
using *Polygamma_real_strict_mono[OF assms(1) _ assms(3), of y] assms(2)*
by *(cases x = y) simp_all*

lemma *Digamma_real_strict_mono*: *(0::real) < x ⟹ x < y ⟹ Digamma x < Digamma y*
by *(rule Polygamma_real_strict_mono) simp_all*

lemma *Digamma_real_mono*: *(0::real) < x ⟹ x ≤ y ⟹ Digamma x ≤ Digamma y*
by *(rule Polygamma_real_mono) simp_all*

lemma *Digamma_real_ge_three_halves_pos*:
assumes *x ≥ 3/2*
shows *Digamma (x :: real) > 0*
proof −
have *0 < Digamma (3/2 :: real)* **by** *(fact Digamma_real_three_halves_pos)*
also from *assms* **have** *. . . ≤ Digamma x* **by** *(intro Polygamma_real_mono) simp_all*
finally show *?thesis* **.**
qed

lemma *ln_Gamma_real_strict_mono*:
assumes *x ≥ 3/2 x < y*
shows *ln_Gamma (x :: real) < ln_Gamma y*
proof −
have *∃ξ. x < ξ ∧ ξ < y ∧ ln_Gamma y − ln_Gamma x = (y − x) ∗ Digamma ξ*
using *assms* **by** *(intro MVT2 derivative_intros impI allI) (auto elim!: nonpos_Ints_cases)*
then guess *ξ* **by** *(elim exE conjE)* **note** *ξ = this*
note *ξ(3)*
also from *ξ(1,2) assms* **have** *(y − x) ∗ Digamma ξ > 0*
by *(intro mult_pos_pos Digamma_real_ge_three_halves_pos) simp_all*
finally show *?thesis* **by** *simp*
qed

lemma *Gamma_real_strict_mono*:
assumes *x ≥ 3/2 x < y*
shows *Gamma (x :: real) < Gamma y*
proof −
from *Gamma_real_pos_exp[of x] assms* **have** *Gamma x = exp (ln_Gamma x)* **by**

simp
also have ... < *exp* (*ln_Gamma y*) **by** (*intro exp_less_mono ln_Gamma_real_strict_mono assms*)
also from *Gamma_real_pos_exp*[*of y*] *assms* **have** ... = *Gamma y* **by** *simp*
finally show *?thesis* **.**
qed

theorem *log_convex_Gamma_real*: *convex_on* {*0*<..} (*ln* ∘ *Gamma* :: *real* ⇒ *real*)
by (*rule convex_on_realI*[*of _ _ Digamma*])
(*auto intro*!: *derivative_eq_intros Polygamma_real_mono Gamma_real_pos simp*: *o_def Gamma_eq_zero_iff elim*!: *nonpos_Ints_cases'*)

6.23.7 The uniqueness of the real Gamma function

The following is a proof of the Bohr–Mollerup theorem, which states that any log-convex function G on the positive reals that fulfils $G(1) = 1$ and satisfies the functional equation $G(x + 1) = x\, G(x)$ must be equal to the Gamma function. In principle, if G is a holomorphic complex function, one could then extend this from the positive reals to the entire complex plane (minus the non-positive integers, where the Gamma function is not defined).

context
fixes *G* :: *real* ⇒ *real*
assumes *G_1*: *G 1* = *1*
assumes *G_plus1*: *x* > *0* ⟹ *G* (*x* + *1*) = *x* ∗ *G x*
assumes *G_pos*: *x* > *0* ⟹ *G x* > *0*
assumes *log_convex_G*: *convex_on* {*0*<..} (*ln* ∘ *G*)
begin

private lemma *G_fact*: *G* (*of_nat n* + *1*) = *fact n*
using *G_plus1*[*of real n* + *1* **for** *n*]
by (*induction n*) (*simp_all add*: *G_1 G_plus1*)

private definition *S* :: *real* ⇒ *real* ⇒ *real* **where**
S x y = (*ln* (*G y*) − *ln* (*G x*)) / (*y* − *x*)

private lemma *S_eq*:
n ≥ *2* ⟹ *S* (*of_nat n*) (*of_nat n* + *x*) = (*ln* (*G* (*real n* + *x*)) − *ln* (*fact* (*n* − *1*))) / *x*
by (*subst G_fact* [*symmetric*]) (*simp add*: *S_def add_ac of_nat_diff*)

private lemma *G_lower*:
assumes *x*: *x* > *0* **and** *n*: *n* ≥ *1*
shows *Gamma_series x n* ≤ *G x*
proof −
have (*ln* ∘ *G*) (*real* (*Suc n*)) ≤ ((*ln* ∘ *G*) (*real* (*Suc n*) + *x*) −
(*ln* ∘ *G*) (*real* (*Suc n*) − *1*)) / (*real* (*Suc n*) + *x* − (*real* (*Suc n*) − *1*)) ∗
(*real* (*Suc n*) − (*real* (*Suc n*) − *1*)) + (*ln* ∘ *G*) (*real* (*Suc n*) − *1*)
using *x n* **by** (*intro convex_onD_Icc' convex_on_subset*[*OF log_convex_G*]) *auto*

```
  hence S (of_nat n) (of_nat (Suc n)) ≤ S (of_nat (Suc n)) (of_nat (Suc n) + x)
    unfolding S_def using x by (simp add: field_simps)
  also have S (of_nat n) (of_nat (Suc n)) = ln (fact n) − ln (fact (n−1))
    unfolding S_def using n
    by (subst (1 2) G_fact [symmetric]) (simp_all add: add_ac of_nat_diff)
  also have ... = ln (fact n / fact (n−1)) by (subst ln_div) simp_all
  also from n have fact n / fact (n − 1) = n by (cases n) simp_all
  finally have x ∗ ln (real n) + ln (fact n) ≤ ln (G (real (Suc n) + x))
    using x n by (subst (asm) S_eq) (simp_all add: field_simps)
  also have x ∗ ln (real n) + ln (fact n) = ln (exp (x ∗ ln (real n)) ∗ fact n)
    using x by (simp add: ln_mult)
  finally have exp (x ∗ ln (real n)) ∗ fact n ≤ G (real (Suc n) + x) using x
    by (subst (asm) ln_le_cancel_iff) (simp_all add: G_pos)
  also have G (real (Suc n) + x) = pochhammer x (Suc n) ∗ G x
    using G_plus1[of real (Suc n) + x for n] G_plus1[of x] x
    by (induction n) (simp_all add: pochhammer_Suc add_ac)
  finally show Gamma_series x n ≤ G x
    using x by (simp add: field_simps pochhammer_pos Gamma_series_def)
qed

private lemma G_upper:
  assumes x: x > 0 x ≤ 1 and n: n ≥ 2
  shows  G x ≤ Gamma_series x n ∗ (1 + x / real n)
proof −
  have (ln ◦ G) (real n + x) ≤ ((ln ◦ G) (real n + 1) −
          (ln ◦ G) (real n)) / (real n + 1 − (real n)) ∗
           ((real n + x) − real n) + (ln ◦ G) (real n)
    using x n by (intro convex_onD_Icc' convex_on_subset[OF log_convex_G]) auto
  hence S (of_nat n) (of_nat n + x) ≤ S (of_nat n) (of_nat n + 1)
    unfolding S_def using x by (simp add: field_simps)
  also from n have S (of_nat n) (of_nat n + 1) = ln (fact n) − ln (fact (n−1))
    by (subst (1 2) G_fact [symmetric]) (simp add: S_def add_ac of_nat_diff)
  also have ... = ln (fact n / (fact (n−1))) using n by (subst ln_div) simp_all
  also from n have fact n / fact (n − 1) = n by (cases n) simp_all
  finally have ln (G (real n + x)) ≤ x ∗ ln (real n) + ln (fact (n − 1))
    using x n by (subst (asm) S_eq) (simp_all add: field_simps)
  also have ... = ln (exp (x ∗ ln (real n)) ∗ fact (n − 1)) using x
    by (simp add: ln_mult)
  finally have G (real n + x) ≤ exp (x ∗ ln (real n)) ∗ fact (n − 1) using x
    by (subst (asm) ln_le_cancel_iff) (simp_all add: G_pos)
  also have G (real n + x) = pochhammer x n ∗ G x
    using G_plus1[of real n + x for n] x
    by (induction n) (simp_all add: pochhammer_Suc add_ac)
  finally have G x ≤ exp (x ∗ ln (real n)) ∗ fact (n− 1) / pochhammer x n
    using x by (simp add: field_simps pochhammer_pos)
  also from n have fact (n − 1) = fact n / n by (cases n) simp_all
  also have exp (x ∗ ln (real n)) ∗ ... / pochhammer x n =
              Gamma_series x n ∗ (1 + x / real n) using n x
    by (simp add: Gamma_series_def divide_simps pochhammer_Suc)
```

```
  finally show ?thesis .
qed

private lemma G_eq_Gamma_aux:
  assumes x: x > 0 x ≤ 1
  shows   G x = Gamma x
proof (rule antisym)
  show G x ≥ Gamma x
  proof (rule tendsto_upperbound)
   from G_lower[of x] show eventually (λn. Gamma_series x n ≤ G x) sequentially
     using x by (auto intro: eventually_mono[OF eventually_ge_at_top[of 1::nat]])
  qed (simp_all add: Gamma_series_LIMSEQ)
next
  show G x ≤ Gamma x
  proof (rule tendsto_lowerbound)
    have (λn. Gamma_series x n * (1 + x / real n)) ⟶ Gamma x * (1 + 0)
      by (rule tendsto_intros real_tendsto_divide_at_top
               Gamma_series_LIMSEQ filterlim_real_sequentially)+
    thus (λn. Gamma_series x n * (1 + x / real n)) ⟶ Gamma x by simp
  next
    from G_upper[of x] show eventually (λn. Gamma_series x n * (1 + x / real
n) ≥ G x) sequentially
      using x by (auto intro: eventually_mono[OF eventually_ge_at_top[of 2::nat]])
  qed simp_all
qed

theorem Gamma_pos_real_unique:
  assumes x: x > 0
  shows   G x = Gamma x
proof -
  have G_eq: G (real n + x) = Gamma (real n + x) if x ∈ {0<..1} for n x using
that
  proof (induction n)
    case (Suc n)
    from Suc have x + real n > 0 by simp
    hence x + real n ∉ ℤ≤0 by auto
    with Suc show ?case using G_plus1[of real n + x] Gamma_plus1[of real n +
x]
      by (auto simp: add_ac)
  qed (simp_all add: G_eq_Gamma_aux)

  show ?thesis
  proof (cases frac x = 0)
    case True
    hence x = of_int (floor x) by (simp add: frac_def)
    with x have x_eq: x = of_nat (nat (floor x) − 1) + 1 by simp
    show ?thesis by (subst (1 2) x_eq, rule G_eq) simp_all
  next
    case False
```

```
    from assms have x_eq: x = of_nat (nat (floor x)) + frac x
      by (simp add: frac_def)
    have frac_le_1: frac x ≤ 1 unfolding frac_def by linarith
    show ?thesis
      by (subst (1 2) x_eq, rule G_eq, insert False frac_le_1) simp_all
  qed
qed

end
```

6.23.8 The Beta function

```
definition Beta where Beta a b = Gamma a * Gamma b / Gamma (a + b)

lemma Beta_altdef: Beta a b = Gamma a * Gamma b * rGamma (a + b)
  by (simp add: inverse_eq_divide Beta_def Gamma_def)

lemma Beta_commute: Beta a b = Beta b a
  unfolding Beta_def by (simp add: ac_simps)

lemma has_field_derivative_Beta1 [derivative_intros]:
  assumes x ∉ ℤ≤0 x + y ∉ ℤ≤0
  shows ((λx. Beta x y) has_field_derivative (Beta x y * (Digamma x − Digamma
(x + y))))
             (at x within A) unfolding Beta_altdef
  by (rule DERIV_cong, (rule derivative_intros assms)+) (simp add: algebra_simps)

lemma Beta_pole1: x ∈ ℤ≤0 ⟹ Beta x y = 0
  by (auto simp add: Beta_def elim!: nonpos_Ints_cases')

lemma Beta_pole2: y ∈ ℤ≤0 ⟹ Beta x y = 0
  by (auto simp add: Beta_def elim!: nonpos_Ints_cases')

lemma Beta_zero: x + y ∈ ℤ≤0 ⟹ Beta x y = 0
  by (auto simp add: Beta_def elim!: nonpos_Ints_cases')

lemma has_field_derivative_Beta2 [derivative_intros]:
  assumes y ∉ ℤ≤0 x + y ∉ ℤ≤0
  shows ((λy. Beta x y) has_field_derivative (Beta x y * (Digamma y − Digamma
(x + y))))
             (at y within A)
  using has_field_derivative_Beta1[of y x A] assms by (simp add: Beta_commute
add_ac)

theorem Beta_plus1_plus1:
  assumes x ∉ ℤ≤0 y ∉ ℤ≤0
  shows Beta (x + 1) y + Beta x (y + 1) = Beta x y
proof −
  have Beta (x + 1) y + Beta x (y + 1) =
```

(*Gamma* (*x* + *1*) ∗ *Gamma y* + *Gamma x* ∗ *Gamma* (*y* + *1*)) ∗ *rGamma* ((*x* + *y*) + *1*)
by (*simp add*: *Beta_altdef add_divide_distrib algebra_simps*)
also have . . . = (*Gamma x* ∗ *Gamma y*) ∗ ((*x* + *y*) ∗ *rGamma* ((*x* + *y*) + *1*))
by (*subst assms*[*THEN Gamma_plus1*])+ (*simp add*: *algebra_simps*)
also from *assms* **have** . . . = *Beta x y* **unfolding** *Beta_altdef* **by** (*subst rGamma_plus1*) *simp*
finally show *?thesis* **.**
qed

theorem *Beta_plus1_left*:
assumes $x \notin \mathbb{Z}_{\le 0}$
shows (*x* + *y*) ∗ *Beta* (*x* + *1*) *y* = *x* ∗ *Beta x y*
proof −
have (*x* + *y*) ∗ *Beta* (*x* + *1*) *y* = *Gamma* (*x* + *1*) ∗ *Gamma y* ∗ ((*x* + *y*) ∗ *rGamma* ((*x* + *y*) + *1*))
unfolding *Beta_altdef* **by** (*simp only*: *ac_simps*)
also have . . . = *x* ∗ *Beta x y* **unfolding** *Beta_altdef*
by (*subst assms*[*THEN Gamma_plus1*] *rGamma_plus1*)+ (*simp only*: *ac_simps*)
finally show *?thesis* **.**
qed

theorem *Beta_plus1_right*:
assumes $y \notin \mathbb{Z}_{\le 0}$
shows (*x* + *y*) ∗ *Beta x* (*y* + *1*) = *y* ∗ *Beta x y*
using *Beta_plus1_left*[*of y x*] *assms* **by** (*simp_all add*: *Beta_commute add.commute*)

lemma *Gamma_Gamma_Beta*:
assumes $x + y \notin \mathbb{Z}_{\le 0}$
shows *Gamma x* ∗ *Gamma y* = *Beta x y* ∗ *Gamma* (*x* + *y*)
unfolding *Beta_altdef* **using** *assms Gamma_eq_zero_iff*[*of x+y*]
by (*simp add*: *rGamma_inverse_Gamma*)

6.23.9 Legendre duplication theorem

context
begin

private lemma *Gamma_legendre_duplication_aux*:
fixes *z* :: *'a* :: *Gamma*
assumes $z \notin \mathbb{Z}_{\le 0}$ $z + 1/2 \notin \mathbb{Z}_{\le 0}$
shows *Gamma z* ∗ *Gamma* (*z* + *1/2*) = *exp* ((*1* − *2*∗*z*) ∗ *of_real* (*ln 2*)) ∗ *Gamma* (*1/2*) ∗ *Gamma* (*2*∗*z*)
proof −
let *?powr* = λ*b a*. *exp* (*a* ∗ *of_real* (*ln* (*of_nat b*)))
let *?h* = λ*n*. (*fact* (*n*−*1*))^2 / *fact* (*2*∗*n*−*1*) ∗ *of_nat* (*2*ˆ(*2*∗*n*)) ∗
exp (*1/2* ∗ *of_real* (*ln* (*real_of_nat n*)))
{
fix *z* :: *'a* **assume** *z*: $z \notin \mathbb{Z}_{\le 0}$ $z + 1/2 \notin \mathbb{Z}_{\le 0}$

```
  let ?g = λn. ?powr 2 (2*z) * Gamma_series' z n * Gamma_series' (z + 1/2)
n /
                   Gamma_series' (2*z) (2*n)
  have eventually (λn. ?g n = ?h n) sequentially using eventually_gt_at_top
  proof eventually_elim
    fix n :: nat assume n: n > 0
    let ?f = fact (n − 1) :: 'a and ?f' = fact (2*n − 1) :: 'a
    have A: exp t * exp t = exp (2*t :: 'a) for t by (subst exp_add [symmetric])
simp
    have A: Gamma_series' z n * Gamma_series' (z + 1/2) n = ?f^2 * ?powr
n (2*z + 1/2) /
             (pochhammer z n * pochhammer (z + 1/2) n)
      by (simp add: Gamma_series'_def exp_add ring_distribs power2_eq_square A
mult_ac)
    have B: Gamma_series' (2*z) (2*n) =
                   ?f' * ?powr 2 (2*z) * ?powr n (2*z) /
                    (of_nat (2^(2*n)) * pochhammer z n * pochhammer (z+1/2)
n) using n
      by (simp add: Gamma_series'_def ln_mult exp_add ring_distribs pochham-
mer_double)
    from z have pochhammer z n ≠ 0 by (auto dest: pochhammer_eq_0_imp_nonpos_Int)
      moreover from z have pochhammer (z + 1/2) n ≠ 0 by (auto dest:
pochhammer_eq_0_imp_nonpos_Int)
    ultimately have ?powr 2 (2*z) * (Gamma_series' z n * Gamma_series' (z
+ 1/2) n) / Gamma_series' (2*z) (2*n) =
      ?f^2 / ?f' * of_nat (2^(2*n)) * (?powr n ((4*z + 1)/2) * ?powr n (−2*z))
      using n unfolding A B by (simp add: field_split_simps exp_minus)
    also have ?powr n ((4*z + 1)/2) * ?powr n (−2*z) = ?powr n (1/2)
      by (simp add: algebra_simps exp_add[symmetric] add_divide_distrib)
    finally show ?g n = ?h n by (simp only: mult_ac)
  qed

  moreover from z double_in_nonpos_Ints_imp[of z] have 2 * z ∉ ℤ≤0 by auto
   hence ?g ⟶ ?powr 2 (2*z) * Gamma z * Gamma (z+1/2) / Gamma
(2*z)
    using LIMSEQ_subseq_LIMSEQ[OF Gamma_series'_LIMSEQ, of (*)2 2*z]
    by (intro tendsto_intros Gamma_series'_LIMSEQ)
      (simp_all add: o_def strict_mono_def Gamma_eq_zero_iff)
  ultimately have ?h ⟶ ?powr 2 (2*z) * Gamma z * Gamma (z+1/2) /
Gamma (2*z)
    by (blast intro: Lim_transform_eventually)
} note lim = this

from assms double_in_nonpos_Ints_imp[of z] have z': 2 * z ∉ ℤ≤0 by auto
from fraction_not_in_ints[of 2 1] have (1/2 :: 'a) ∉ ℤ≤0
  by (intro not_in_Ints_imp_not_in_nonpos_Ints) simp_all
with lim[of 1/2 :: 'a] have ?h ⟶ 2 * Gamma (1/2 :: 'a) by (simp add:
exp_of_real)
from LIMSEQ_unique[OF this lim[OF assms]] z' show ?thesis
```

by (*simp add*: *field_split_simps Gamma_eq_zero_iff ring_distribs exp_diff exp_of_real*)
qed

The following lemma is somewhat annoying. With a little bit of complex analysis (Cauchy's integral theorem, to be exact), this would be completely trivial. However, we want to avoid depending on the complex analysis session at this point, so we prove it the hard way.

private lemma *Gamma_reflection_aux*:
defines $h \equiv \lambda z$::*complex*. *if* $z \in \mathbb{Z}$ *then 0 else*
(*of_real pi* ∗ *cot* (*of_real pi*∗*z*) + *Digamma z* − *Digamma* (*1* − *z*))
defines $a \equiv$ *complex_of_real pi*
obtains h' **where** *continuous_on UNIV* h' $\bigwedge z$. (*h has_field_derivative* (h' *z*)) (*at z*)
proof −
define *f* **where** *f n* = *a* ∗ *of_real* (*cos_coeff* (*n*+*1*) − *sin_coeff* (*n*+*2*)) **for** *n*
define *F* **where** *F z* = (*if z* = *0 then 0 else* (*cos* (*a*∗*z*) − *sin* (*a*∗*z*)/(*a*∗*z*)) / *z*) **for** *z*
define *g* **where** *g n* = *complex_of_real* (*sin_coeff* (*n*+*1*)) **for** *n*
define *G* **where** *G z* = (*if z* = *0 then 1 else sin* (*a*∗*z*)/(*a*∗*z*)) **for** *z*
have *a_nz*: *a* ≠ *0* **unfolding** *a_def* **by** *simp*

have (λ*n*. *f n* ∗ (*a*∗*z*) ˆ*n*) *sums* (*F z*) ∧ (λ*n*. *g n* ∗ (*a*∗*z*) ˆ*n*) *sums* (*G z*)
if *abs* (*Re z*) < *1* **for** *z*
proof (*cases z* = *0*; *rule conjI*)
assume *z* ≠ *0*
note *z* = *this that*

from *z* **have** *sin_nz*: *sin* (*a*∗*z*) ≠ *0* **unfolding** *a_def* **by** (*auto simp*: *sin_eq_0*)
have (λ*n*. *of_real* (*sin_coeff n*) ∗ (*a*∗*z*) ˆ*n*) *sums* (*sin* (*a*∗*z*)) **using** *sin_converges*[*of a*∗*z*]
by (*simp add*: *scaleR_conv_of_real*)
from *sums_split_initial_segment*[*OF this*, *of 1*]
have (λ*n*. (*a*∗*z*) ∗ *of_real* (*sin_coeff* (*n*+*1*)) ∗ (*a*∗*z*) ˆ*n*) *sums* (*sin* (*a*∗*z*)) **by** (*simp add*: *mult_ac*)
from *sums_mult*[*OF this*, *of inverse* (*a*∗*z*)] *z a_nz*
have *A*: (λ*n*. *g n* ∗ (*a*∗*z*) ˆ*n*) *sums* (*sin* (*a*∗*z*)/(*a*∗*z*))
by (*simp add*: *field_simps g_def*)
with *z* **show** (λ*n*. *g n* ∗ (*a*∗*z*) ˆ*n*) *sums* (*G z*) **by** (*simp add*: *G_def*)
from *A z a_nz sin_nz* **have** *g_nz*: (∑*n*. *g n* ∗ (*a*∗*z*) ˆ*n*) ≠ *0* **by** (*simp add*: *sums_iff g_def*)

have [*simp*]: *sin_coeff* (*Suc 0*) = *1* **by** (*simp add*: *sin_coeff_def*)
from *sums_split_initial_segment*[*OF sums_diff*[*OF cos_converges*[*of a*∗*z*] *A*], *of 1*]
have (λ*n*. *z* ∗ *f n* ∗ (*a*∗*z*) ˆ*n*) *sums* (*cos* (*a*∗*z*) − *sin* (*a*∗*z*) / (*a*∗*z*))
by (*simp add*: *mult_ac scaleR_conv_of_real ring_distribs f_def g_def*)
from *sums_mult*[*OF this*, *of inverse z*] *z assms*
show (λ*n*. *f n* ∗ (*a*∗*z*) ˆ*n*) *sums* (*F z*) **by** (*simp add*: *divide_simps mult_ac f_def F_def*)

```
  next
    assume z: z = 0
    have (λn. f n * (a * z) ^ n) sums f 0 using powser_sums_zero[of f] z by simp
    with z show (λn. f n * (a * z) ^ n) sums (F z)
      by (simp add: f_def F_def sin_coeff_def cos_coeff_def)
    have (λn. g n * (a * z) ^ n) sums g 0 using powser_sums_zero[of g] z by simp
    with z show (λn. g n * (a * z) ^ n) sums (G z)
      by (simp add: g_def G_def sin_coeff_def cos_coeff_def)
  qed
  note sums = conjunct1[OF this] conjunct2[OF this]

  define h2 where [abs_def]:
    h2 z = (∑n. f n * (a*z)^n) / (∑n. g n * (a*z)^n) + Digamma (1 + z) −
Digamma (1 − z) for z
  define POWSER where [abs_def]: POWSER f z = (∑n. f n * (z^n :: complex))
for f z
  define POWSER' where [abs_def]: POWSER' f z = (∑n. diffs f n * (z^n)) for
f and z :: complex
  define h2' where [abs_def]:
    h2' z = a * (POWSER g (a*z) * POWSER' f (a*z) − POWSER f (a*z) *
POWSER' g (a*z)) /
      (POWSER g (a*z))^2 + Polygamma 1 (1 + z) + Polygamma 1 (1 − z) for
z

  have h_eq: h t = h2 t if abs (Re t) < 1 for t
  proof −
    from that have t: t ∈ ℤ ⟷ t = 0 by (auto elim!: Ints_cases)
    hence h t = a*cot (a*t) − 1/t + Digamma (1 + t) − Digamma (1 − t)
     unfolding h_def using Digamma_plus1[of t] by (force simp: field_simps a_def)
    also have a*cot (a*t) − 1/t = (F t) / (G t)
      using t by (auto simp add: divide_simps sin_eq_0 cot_def a_def F_def G_def)
    also have ... = (∑n. f n * (a*t)^n) / (∑n. g n * (a*t)^n)
      using sums[of t] that by (simp add: sums_iff)
    finally show h t = h2 t by (simp only: h2_def)
  qed

  let ?A = {z. abs (Re z) < 1}
  have open ({z. Re z < 1} ∩ {z. Re z > −1})
    using open_halfspace_Re_gt open_halfspace_Re_lt by auto
  also have ({z. Re z < 1} ∩ {z. Re z > −1}) = {z. abs (Re z) < 1} by auto
  finally have open_A: open ?A .
  hence [simp]: interior ?A = ?A by (simp add: interior_open)

  have summable_f: summable (λn. f n * z^n) for z
    by (rule powser_inside, rule sums_summable, rule sums[of i * of_real (norm z
+ 1) / a])
       (simp_all add: norm_mult a_def del: of_real_add)
  have summable_g: summable (λn. g n * z^n) for z
    by (rule powser_inside, rule sums_summable, rule sums[of i * of_real (norm z
```

+ *1*) / *a*])
(*simp_all add*: *norm_mult a_def del*: *of_real_add*)
have *summable_fg'*: *summable* (λ*n*. *diffs f n* ∗ *z*ˆ*n*) *summable* (λ*n*. *diffs g n* ∗ *z*ˆ*n*) **for** *z*
by (*intro termdiff_converges_all summable_f summable_g*)+
have (*POWSER f has_field_derivative* (*POWSER' f z*)) (*at z*)
(*POWSER g has_field_derivative* (*POWSER' g z*)) (*at z*) **for** *z*
unfolding *POWSER_def POWSER'_def*
by (*intro termdiffs_strong_converges_everywhere summable_f summable_g*)+
note *derivs* = *this*[*THEN DERIV_chain2*[*OF _ DERIV_cmult*[*OF DERIV_ident*]], *unfolded POWSER_def*]
have *isCont* (*POWSER f*) *z isCont* (*POWSER g*) *z isCont* (*POWSER' f*) *z isCont* (*POWSER' g*) *z*
for *z* **unfolding** *POWSER_def POWSER'_def*
by (*intro isCont_powser_converges_everywhere summable_f summable_g summable_fg'*)+
note *cont* = *this*[*THEN isCont_o2*[*rotated*], *unfolded POWSER_def POWSER'_def*]

{
fix *z* :: *complex* **assume** *z*: *abs* (*Re z*) < *1*
define *d* **where** *d* = i ∗ *of_real* (*norm z* + *1*)
have *d*: *abs* (*Re d*) < *1 norm z* < *norm d* **by** (*simp_all add*: *d_def norm_mult del*: *of_real_add*)
have *eventually* (λ*z*. *h z* = *h2 z*) (*nhds z*)
using *eventually_nhds_in_nhd*[*of z ?A*] **using** *h_eq z*
by (*auto elim*!: *eventually_mono*)

moreover from *sums*(*2*)[*OF z*] *z* **have** *nz*: ($\sum$ *n*. *g n* ∗ (*a* ∗ *z*) ˆ *n*) ≠ *0*
unfolding *G_def* **by** (*auto simp*: *sums_iff sin_eq_0 a_def*)
have *A*: *z* ∈ ℤ ⟷ *z* = *0* **using** *z* **by** (*auto elim*!: *Ints_cases*)
have *no_int*: *1* + *z* ∈ ℤ ⟷ *z* = *0* **using** *z Ints_diff*[*of 1*+*z 1*] *A*
by (*auto elim*!: *nonpos_Ints_cases*)
have *no_int'*: *1* − *z* ∈ ℤ ⟷ *z* = *0* **using** *z Ints_diff*[*of 1 1*−*z*] *A*
by (*auto elim*!: *nonpos_Ints_cases*)
from *no_int no_int'* **have** *no_int*: *1* − *z* ∉ $\mathbb{Z}_{\leq 0}$ *1* + *z* ∉ $\mathbb{Z}_{\leq 0}$ **by** *auto*
have (*h2 has_field_derivative h2' z*) (*at z*) **unfolding** *h2_def*
by (*rule DERIV_cong*, (*rule derivative_intros refl derivs*[*unfolded POWSER_def*] *nz no_int*)+)
(*auto simp*: *h2'_def POWSER_def field_simps power2_eq_square*)
ultimately have *deriv*: (*h has_field_derivative h2' z*) (*at z*)
by (*subst DERIV_cong_ev*[*OF refl _ refl*])

from *sums*(*2*)[*OF z*] *z* **have** ($\sum$ *n*. *g n* ∗ (*a* ∗ *z*) ˆ *n*) ≠ *0*
unfolding *G_def* **by** (*auto simp*: *sums_iff a_def sin_eq_0*)
hence *isCont h2' z* **using** *no_int* **unfolding** *h2'_def*[*abs_def*] *POWSER_def POWSER'_def*
by (*intro continuous_intros cont*
continuous_on_compose2[*OF _ continuous_on_Polygamma*[*of* {*z*. *Re z* > *0*}]]) *auto*
note *deriv* **and** *this*

```
} note A = this

interpret h: periodic_fun_simple' h
proof
  fix z :: complex
  show h (z + 1) = h z
  proof (cases z ∈ ℤ)
    assume z: z ∉ ℤ
    hence A: z + 1 ∉ ℤ z ≠ 0 using Ints_diff[of z+1 1] by auto
    hence Digamma (z + 1) − Digamma (−z) = Digamma z − Digamma (−z
+ 1)
      by (subst (1 2) Digamma_plus1) simp_all
    with A z show h (z + 1) = h z
      by (simp add: h_def sin_plus_pi cos_plus_pi ring_distribs cot_def)
  qed (simp add: h_def)
qed

have h2'_eq: h2' (z − 1) = h2' z if z: Re z > 0 Re z < 1 for z
proof −
  have ((λz. h (z − 1)) has_field_derivative h2' (z − 1)) (at z)
    by (rule DERIV_cong, rule DERIV_chain'[OF _ A(1)])
      (insert z, auto intro!: derivative_eq_intros)
  hence (h has_field_derivative h2' (z − 1)) (at z) by (subst (asm) h.minus_1)
   moreover from z have (h has_field_derivative h2' z) (at z) by (intro A)
simp_all
  ultimately show h2' (z − 1) = h2' z by (rule DERIV_unique)
qed

define h2'' where h2'' z = h2' (z − of_int ⌊Re z⌋) for z
have deriv: (h has_field_derivative h2'' z) (at z) for z
proof −
  fix z :: complex
  have B: |Re z − real_of_int ⌊Re z⌋| < 1 by linarith
  have ((λt. h (t − of_int ⌊Re z⌋)) has_field_derivative h2'' z) (at z)
    unfolding h2''_def by (rule DERIV_cong, rule DERIV_chain'[OF _ A(1)])
                      (insert B, auto intro!: derivative_intros)
  thus (h has_field_derivative h2'' z) (at z) by (simp add: h.minus_of_int)
qed

have cont: continuous_on UNIV h2''
proof (intro continuous_at_imp_continuous_on ballI)
  fix z :: complex
  define r where r = ⌊Re z⌋
  define A where A = {t. of_int r − 1 < Re t ∧ Re t < of_int r + 1}
  have continuous_on A (λt. h2' (t − of_int r)) unfolding A_def
    by (intro continuous_at_imp_continuous_on isCont_o2[OF _ A(2)] ballI con-
tinuous_intros)
      (simp_all add: abs_real_def)
  moreover have h2'' t = h2' (t − of_int r) if t: t ∈ A for t
```

proof (*cases Re t ≥ of_int r*)
case *True*
from *t* **have** *of_int r − 1 < Re t Re t < of_int r + 1* **by** (*simp_all add*: *A_def*)
with *True* **have** ⌊*Re t*⌋ = ⌊*Re z*⌋ **unfolding** *r_def* **by** *linarith*
thus *?thesis* **by** (*auto simp*: *r_def h2′′_def*)
next
case *False*
from *t* **have** *t*: *of_int r − 1 < Re t Re t < of_int r + 1* **by** (*simp_all add*: *A_def*)
with *False* **have** *t′*: ⌊*Re t*⌋ = ⌊*Re z*⌋ − *1* **unfolding** *r_def* **by** *linarith*
moreover from *t False* **have** *h2′* (*t − of_int r + 1 − 1*) = *h2′* (*t − of_int r + 1*)
by (*intro h2′_eq*) *simp_all*
ultimately show *?thesis* **by** (*auto simp*: *r_def h2′′_def algebra_simps t′*)
qed
ultimately have *continuous_on A h2′′* **by** (*subst continuous_on_cong*[*OF refl*])
moreover {
have *open* ({*t*. *of_int r − 1 < Re t*} ∩ {*t*. *of_int r + 1 > Re t*})
by (*intro open_Int open_halfspace_Re_gt open_halfspace_Re_lt*)
also have {*t*. *of_int r − 1 < Re t*} ∩ {*t*. *of_int r + 1 > Re t*} = *A*
unfolding *A_def* **by** *blast*
finally have *open A* **.**
}
ultimately have *C*: *isCont h2′′ t* **if** *t ∈ A* **for** *t* **using** *that*
by (*subst* (*asm*) *continuous_on_eq_continuous_at*) *auto*
have *of_int r − 1 < Re z Re z < of_int r + 1* **unfolding** *r_def* **by** *linarith+*
thus *isCont h2′′ z* **by** (*intro C*) (*simp_all add*: *A_def*)
qed

from *that*[*OF cont deriv*] **show** *?thesis* **.**
qed

lemma *Gamma_reflection_complex*:
fixes *z* :: *complex*
shows *Gamma z ∗ Gamma* (*1 − z*) = *of_real pi* / *sin* (*of_real pi ∗ z*)
proof −
let *?g* = *λz*::*complex*. *Gamma z ∗ Gamma* (*1 − z*) ∗ *sin* (*of_real pi ∗ z*)
define *g* **where** [*abs_def*]: *g z* = (*if z ∈ ℤ then of_real pi else ?g z*) **for** *z* :: *complex*
let *?h* = *λz*::*complex*. (*of_real pi ∗ cot* (*of_real pi∗z*) + *Digamma z − Digamma* (*1 − z*))
define *h* **where** [*abs_def*]: *h z* = (*if z ∈ ℤ then 0 else ?h z*) **for** *z* :: *complex*

— *g* is periodic with period 1.
interpret *g*: *periodic_fun_simple′ g*
proof
fix *z* :: *complex*
show *g* (*z* + *1*) = *g z*

```
    proof (cases z ∈ ℤ)
      case False
      hence z * g z = z * Beta z (- z + 1) * sin (of_real pi * z) by (simp add:
g_def Beta_def)
      also have z * Beta z (- z + 1) = (z + 1 + -z) * Beta (z + 1) (- z + 1)
        using False Ints_diff[of 1 1 - z] nonpos_Ints_subset_Ints
        by (subst Beta_plus1_left [symmetric]) auto
      also have ... * sin (of_real pi * z) = z * (Beta (z + 1) (-z) * sin (of_real
pi * (z + 1)))
        using False Ints_diff[of z+1 1] Ints_minus[of -z] nonpos_Ints_subset_Ints
        by (subst Beta_plus1_right) (auto simp: ring_distribs sin_plus_pi)
      also from False have Beta (z + 1) (-z) * sin (of_real pi * (z + 1)) = g (z
+ 1)
        using Ints_diff[of z+1 1] by (auto simp: g_def Beta_def)
      finally show g (z + 1) = g z using False by (subst (asm) mult_left_cancel)
auto
    qed (simp add: g_def)
  qed

  — g is entire.
  have g_g': (g has_field_derivative (h z * g z)) (at z) for z :: complex
  proof (cases z ∈ ℤ)
    let ?h' = λz. Beta z (1 - z) * ((Digamma z - Digamma (1 - z)) * sin (z *
of_real pi) +
                    of_real pi * cos (z * of_real pi))
    case False
    from False have eventually (λt. t ∈ UNIV - ℤ) (nhds z)
      by (intro eventually_nhds_in_open) (auto simp: open_Diff)
    hence eventually (λt. g t = ?g t) (nhds z) by eventually_elim (simp add: g_def)
    moreover {
      from False Ints_diff[of 1 1-z] have 1 - z ∉ ℤ by auto
      hence (?g has_field_derivative ?h' z) (at z) using nonpos_Ints_subset_Ints
        by (auto intro!: derivative_eq_intros simp: algebra_simps Beta_def)
      also from False have sin (of_real pi * z) ≠ 0 by (subst sin_eq_0) auto
      hence ?h' z = h z * g z
          using False unfolding g_def h_def cot_def by (simp add: field_simps
Beta_def)
      finally have (?g has_field_derivative (h z * g z)) (at z) .
    }
    ultimately show ?thesis by (subst DERIV_cong_ev[OF refl _ refl])
  next
    case True
    then obtain n where z: z = of_int n by (auto elim!: Ints_cases)
    let ?t = (λz::complex. if z = 0 then 1 else sin z / z) ∘ (λz. of_real pi * z)
    have deriv_0: (g has_field_derivative 0) (at 0)
    proof (subst DERIV_cong_ev[OF refl _ refl])
      show eventually (λz. g z = of_real pi * Gamma (1 + z) * Gamma (1 - z)
* ?t z) (nhds 0)
        using eventually_nhds_ball[OF zero_less_one, of 0::complex]
```

proof *eventually_elim*
fix *z* :: *complex* **assume** *z*: $z \in ball\ 0\ 1$
show $g\ z = of\_real\ pi * Gamma\ (1 + z) * Gamma\ (1 - z) * ?t\ z$
proof ($cases\ z = 0$)
assume *z'*: $z \neq 0$
with *z* **have** *z''*: $z \notin \mathbb{Z}_{\leq 0}$ $z \notin \mathbb{Z}$ **by** (*auto elim*!: *Ints_cases*)
from *Gamma_plus1*[*OF this*(*1*)] **have** $Gamma\ z = Gamma\ (z + 1) / z$ **by** *simp*
with *z'' z'* **show** *?thesis* **by** (*simp add*: *g_def ac_simps*)
qed (*simp add*: *g_def*)
qed
have (*?t has_field_derivative* ($0 * of\_real\ pi$)) (*at 0*)
using *has_field_derivative_sin_z_over_z*[*of UNIV* :: *complex set*]
by (*intro DERIV_chain*) *simp_all*
thus (($\lambda z.\ of\_real\ pi * Gamma\ (1 + z) * Gamma\ (1 - z) * ?t\ z$) *has_field_derivative* *0*) (*at 0*)
by (*auto intro*!: *derivative_eq_intros simp*: *o_def*)
qed

have (($g \circ (\lambda x.\ x - of\_int\ n)$) *has_field_derivative* $0 * 1$) (*at* (*of_int n*))
using *deriv_0* **by** (*intro DERIV_chain*) (*auto intro*!: *derivative_eq_intros*)
also have $g \circ (\lambda x.\ x - of\_int\ n) = g$ **by** (*intro ext*) (*simp add*: *g.minus_of_int*)
finally show (*g has_field_derivative* ($h\ z * g\ z$)) (*at z*) **by** (*simp add*: *z h_def*)
qed

have *g_eq*: $g\ (z/2) * g\ ((z+1)/2) = Gamma\ (1/2)\hat{\ }2 * g\ z$ **if** $Re\ z > -1$ $Re\ z < 2$ **for** *z*
proof ($cases\ z \in \mathbb{Z}$)
case *True*
with *that* **have** $z = 0 \lor z = 1$ **by** (*force elim*!: *Ints_cases*)
moreover have $g\ 0 * g\ (1/2) = Gamma\ (1/2)\hat{\ }2 * g\ 0$
using *fraction_not_in_ints*[**where** *'a* = *complex*, *of 2 1*] **by** (*simp add*: *g_def power2_eq_square*)
moreover have $g\ (1/2) * g\ 1 = Gamma\ (1/2)\hat{\ }2 * g\ 1$
using *fraction_not_in_ints*[**where** *'a* = *complex*, *of 2 1*]
by (*simp add*: *g_def power2_eq_square Beta_def algebra_simps*)
ultimately show *?thesis* **by** *force*
next
case *False*
hence *z*: $z/2 \notin \mathbb{Z}$ $(z+1)/2 \notin \mathbb{Z}$ **using** *Ints_diff*[*of z+1 1*] **by** (*auto elim*!: *Ints_cases*)
hence *z'*: $z/2 \notin \mathbb{Z}_{\leq 0}$ $(z+1)/2 \notin \mathbb{Z}_{\leq 0}$ **by** (*auto elim*!: *nonpos_Ints_cases*)
from *z* **have** $1-z/2 \notin \mathbb{Z}$ $1-((z+1)/2) \notin \mathbb{Z}$
using *Ints_diff*[*of 1 1−z/2*] *Ints_diff*[*of 1 1−((z+1)/2)*] **by** *auto*
hence *z''*: $1-z/2 \notin \mathbb{Z}_{\leq 0}$ $1-((z+1)/2) \notin \mathbb{Z}_{\leq 0}$ **by** (*auto elim*!: *nonpos_Ints_cases*)
from *z* **have** $g\ (z/2) * g\ ((z+1)/2) =$
$(Gamma\ (z/2) * Gamma\ ((z+1)/2)) * (Gamma\ (1-z/2) * Gamma\ (1-((z+1)/2))) *$
$(sin\ (of\_real\ pi * z/2) * sin\ (of\_real\ pi * (z+1)/2))$

by (*simp add*: *g_def*)
also from *z′ Gamma_legendre_duplication_aux*[*of z/2*]
have *Gamma* (*z/2*) ∗ *Gamma* ((*z+1*)/*2*) = *exp* ((*1*−*z*) ∗ *of_real* (*ln 2*)) ∗ *Gamma* (*1/2*) ∗ *Gamma z*
by (*simp add*: *add_divide_distrib*)
also from *z′′ Gamma_legendre_duplication_aux*[*of 1*−(*z+1*)/*2*]
have *Gamma* (*1*−*z/2*) ∗ *Gamma* (*1*−(*z+1*)/*2*) =
Gamma (*1*−*z*) ∗ *Gamma* (*1/2*) ∗ *exp* (*z* ∗ *of_real* (*ln 2*))
by (*simp add*: *add_divide_distrib ac_simps*)
finally have *g* (*z/2*) ∗ *g* ((*z+1*)/*2*) = *Gamma* (*1/2*) ˆ*2* ∗ (*Gamma z* ∗ *Gamma* (*1*−*z*) ∗
(*2* ∗ (*sin* (*of_real pi*∗*z/2*) ∗ *sin* (*of_real pi*∗(*z+1*)/*2*))))
by (*simp add*: *add_ac power2_eq_square exp_add ring_distribs exp_diff exp_of_real*)
also have *sin* (*of_real pi*∗(*z+1*)/*2*) = *cos* (*of_real pi*∗*z/2*)
using *cos_sin_eq*[*of* − *of_real pi* ∗ *z/2*, *symmetric*]
by (*simp add*: *ring_distribs add_divide_distrib ac_simps*)
also have *2* ∗ (*sin* (*of_real pi*∗*z/2*) ∗ *cos* (*of_real pi*∗*z/2*)) = *sin* (*of_real pi* ∗ *z*)
by (*subst sin_times_cos*) (*simp add*: *field_simps*)
also have *Gamma z* ∗ *Gamma* (*1* − *z*) ∗ *sin* (*complex_of_real pi* ∗ *z*) = *g z*
using ‹*z* ∉ ℤ› **by** (*simp add*: *g_def*)
finally show *?thesis* **.**
qed
have *g_eq*: *g* (*z/2*) ∗ *g* ((*z+1*)/*2*) = *Gamma* (*1/2*) ˆ*2* ∗ *g z* **for** *z*
proof −
define *r* **where** *r* = ⌊*Re z* / *2*⌋
have *Gamma* (*1/2*) ˆ*2* ∗ *g z* = *Gamma* (*1/2*) ˆ*2* ∗ *g* (*z* − *of_int* (*2*∗*r*)) **by** (*simp only*: *g.minus_of_int*)
also have *of_int* (*2*∗*r*) = *2* ∗ *of_int r* **by** *simp*
also have *Re z* − *2* ∗ *of_int r* > −*1 Re z* − *2* ∗ *of_int r* < *2* **unfolding** *r_def* **by** *linarith+*
hence *Gamma* (*1/2*) ˆ*2* ∗ *g* (*z* − *2* ∗ *of_int r*) =
g ((*z* − *2* ∗ *of_int r*)/*2*) ∗ *g* ((*z* − *2* ∗ *of_int r* + *1*)/*2*)
unfolding *r_def* **by** (*intro g_eq*[*symmetric*]) *simp_all*
also have (*z* − *2* ∗ *of_int r*) / *2* = *z/2* − *of_int r* **by** *simp*
also have *g* ... = *g* (*z/2*) **by** (*rule g.minus_of_int*)
also have (*z* − *2* ∗ *of_int r* + *1*) / *2* = (*z* + *1*)/*2* − *of_int r* **by** *simp*
also have *g* ... = *g* ((*z+1*)/*2*) **by** (*rule g.minus_of_int*)
finally show *?thesis* **..**
qed

have *g_nz* [*simp*]: *g z* ≠ *0* **for** *z* :: *complex*
unfolding *g_def* **using** *Ints_diff*[*of 1 1* − *z*]
by (*auto simp*: *Gamma_eq_zero_iff sin_eq_0 dest*!: *nonpos_Ints_Int*)

have *h_eq*: *h z* = (*h* (*z/2*) + *h* ((*z+1*)/*2*)) / *2* **for** *z*
proof −
have ((λ*t*. *g* (*t/2*) ∗ *g* ((*t+1*)/*2*)) *has_field_derivative*
(*g* (*z/2*) ∗ *g* ((*z+1*)/*2*)) ∗ ((*h* (*z/2*) + *h* ((*z+1*)/*2*)) / *2*)) (*at*

z)
  by (*auto intro*!: *derivative_eq_intros g_g*′[*THEN DERIV_chain2*] *simp*: *field_simps*)
  hence ((*λt*. *Gamma* (*1*/*2*)ˆ*2* ∗ *g t*) *has_field_derivative*
      Gamma (*1*/*2*)ˆ*2* ∗ *g z* ∗ ((*h* (*z*/*2*) + *h* ((*z*+*1*)/*2*)) / *2*)) (*at z*)
   by (*subst* (*1 2*) *g_eq*[*symmetric*]) *simp*
  from *DERIV_cmult*[*OF this*, *of inverse* ((*Gamma* (*1*/*2*))ˆ*2*)]
   have (*g has_field_derivative* (*g z* ∗ ((*h* (*z*/*2*) + *h* ((*z*+*1*)/*2*))/*2*))) (*at z*)
   using *fraction_not_in_ints*[**where** ′*a* = *complex*, *of 2 1*]
  by (*simp add*: *divide_simps Gamma_eq_zero_iff not_in_Ints_imp_not_in_nonpos_Ints*)
  moreover have (*g has_field_derivative* (*g z* ∗ *h z*)) (*at z*)
   using *g_g*′[*of z*] **by** (*simp add*: *ac_simps*)
  ultimately have *g z* ∗ *h z* = *g z* ∗ ((*h* (*z*/*2*) + *h* ((*z*+*1*)/*2*))/*2*)
   by (*intro DERIV_unique*)
  thus *h z* = (*h* (*z*/*2*) + *h* ((*z*+*1*)/*2*)) / *2* **by** *simp*
 qed

 obtain *h*′ **where** *h*′*_cont*: *continuous_on UNIV h*′ **and**
      h_h′: ⋀*z*. (*h has_field_derivative h*′ *z*) (*at z*)
  unfolding *h_def* **by** (*erule Gamma_reflection_aux*)

 have *h*′*_eq*: *h*′ *z* = (*h*′ (*z*/*2*) + *h*′ ((*z*+*1*)/*2*)) / *4* **for** *z*
 proof −
  have ((*λt*. (*h* (*t*/*2*) + *h* ((*t*+*1*)/*2*)) / *2*) *has_field_derivative*
      ((*h*′ (*z*/*2*) + *h*′ ((*z*+*1*)/*2*)) / *4*)) (*at z*)
   by (*fastforce intro*!: *derivative_eq_intros h_h*′[*THEN DERIV_chain2*])
  hence (*h has_field_derivative* ((*h*′ (*z*/*2*) + *h*′ ((*z*+*1*)/*2*))/*4*)) (*at z*)
   by (*subst* (*asm*) *h_eq*[*symmetric*])
  from *h_h*′ **and** *this* **show** *h*′ *z* = (*h*′ (*z*/*2*) + *h*′ ((*z*+*1*)/*2*)) / *4* **by** (*rule DERIV_unique*)
 qed

 have *h*′*_zero*: *h*′ *z* = *0* **for** *z*
 proof −
  define *m* **where** *m* = *max 1* |*Re z*|
  define *B* **where** *B* = {*t*. *abs* (*Re t*) ≤ *m* ∧ *abs* (*Im t*) ≤ *abs* (*Im z*)}
  have *closed* ({*t*. *Re t* ≥ −*m*} ∩ {*t*. *Re t* ≤ *m*} ∩
      {*t*. *Im t* ≥ −|*Im z*|} ∩ {*t*. *Im t* ≤ |*Im z*|})
   (**is** *closed ?B*) **by** (*intro closed_Int closed_halfspace_Re_ge closed_halfspace_Re_le*
      closed_halfspace_Im_ge closed_halfspace_Im_le)
  also have *?B* = *B* **unfolding** *B_def* **by** *fastforce*
  finally have *closed B* **.**
  moreover have *bounded B* **unfolding** *bounded_iff*
  proof (*intro ballI exI*)
   fix *t* **assume** *t*: *t* ∈ *B*
   have *norm t* ≤ |*Re t*| + |*Im t*| **by** (*rule cmod_le*)
   also from *t* **have** |*Re t*| ≤ *m* **unfolding** *B_def* **by** *blast*
   also from *t* **have** |*Im t*| ≤ |*Im z*| **unfolding** *B_def* **by** *blast*
   finally show *norm t* ≤ *m* + |*Im z*| **by** − *simp*
  qed

ultimately have *compact*: *compact B* **by** (*subst compact_eq_bounded_closed*) *blast*

define *M* **where** *M* = (*SUP z*∈*B*. *norm* (*h′ z*))
have *compact* (*h′ ‘ B*)
by (*intro compact_continuous_image continuous_on_subset*[*OF h′_cont*] *compact*) *blast*+
hence *bdd*: *bdd_above* ((λ*z*. *norm* (*h′ z*)) ‘ *B*)
using *bdd_above_norm*[*of h′ ‘ B*] **by** (*simp add*: *image_comp o_def compact_imp_bounded*)
have *norm* (*h′ z*) ≤ *M* **unfolding** *M_def* **by** (*intro cSUP_upper bdd*) (*simp_all add*: *B_def m_def*)
also have *M* ≤ *M*/*2*
proof (*subst M_def*, *subst cSUP_le_iff*)
have *z* ∈ *B* **unfolding** *B_def m_def* **by** *simp*
thus *B* ≠ {} **by** *auto*
next
show ∀ *z*∈*B*. *norm* (*h′ z*) ≤ *M*/*2*
proof
fix *t* :: *complex* **assume** *t*: *t* ∈ *B*
from *h′_eq*[*of t*] *t* **have** *h′ t* = (*h′* (*t*/*2*) + *h′* ((*t*+*1*)/*2*)) / *4* **by** (*simp*)
also have *norm* ... = *norm* (*h′* (*t*/*2*) + *h′* ((*t*+*1*)/*2*)) / *4* **by** *simp*
also have *norm* (*h′* (*t*/*2*) + *h′* ((*t*+*1*)/*2*)) ≤ *norm* (*h′* (*t*/*2*)) + *norm* (*h′* ((*t*+*1*)/*2*))
by (*rule norm_triangle_ineq*)
also from *t* **have** *abs* (*Re* ((*t* + *1*)/*2*)) ≤ *m* **unfolding** *m_def B_def* **by** *auto*
with *t* **have** *t*/*2* ∈ *B* (*t*+*1*)/*2* ∈ *B* **unfolding** *B_def* **by** *auto*
hence *norm* (*h′* (*t*/*2*)) + *norm* (*h′* ((*t*+*1*)/*2*)) ≤ *M* + *M* **unfolding** *M_def*
by (*intro add_mono cSUP_upper bdd*) (*auto simp*: *B_def*)
also have (*M* + *M*) / *4* = *M* / *2* **by** *simp*
finally show *norm* (*h′ t*) ≤ *M*/*2* **by** − *simp_all*
qed
qed (*insert bdd*, *auto*)
hence *M* ≤ *0* **by** *simp*
finally show *h′ z* = *0* **by** *simp*
qed
have *h_h′_2*: (*h has_field_derivative 0*) (*at z*) **for** *z*
using *h_h′*[*of z*] *h′_zero*[*of z*] **by** *simp*

have *g_real*: *g z* ∈ ℝ **if** *z* ∈ ℝ **for** *z*
unfolding *g_def* **using** *that* **by** (*auto intro*!: *Reals_mult Gamma_complex_real*)
have *h_real*: *h z* ∈ ℝ **if** *z* ∈ ℝ **for** *z*
unfolding *h_def* **using** *that* **by** (*auto intro*!: *Reals_mult Reals_add Reals_diff Polygamma_Real*)
have *g_nz*: *g z* ≠ *0* **for** *z* **unfolding** *g_def* **using** *Ints_diff*[*of 1 1*−*z*]
by (*auto simp*: *Gamma_eq_zero_iff sin_eq_0*)

from *h′_zero h_h′_2* **have** ∃ *c*. ∀ *z*∈*UNIV*. *h z* = *c*

by (*intro has_field_derivative_zero_constant*) (*simp_all add*: *dist_0_norm*)
then obtain *c* **where** *c*: ⋀*z*. *h z* = *c* **by** *auto*
have ∃ *u*. *u* ∈ *closed_segment 0 1* ∧ *Re* (*g 1*) − *Re* (*g 0*) = *Re* (*h u* ∗ *g u* ∗ (*1* − *0*))
by (*intro complex_mvt_line g_g′*)
then obtain *u* **where** *u*: *u* ∈ *closed_segment 0 1 Re* (*g 1*) − *Re* (*g 0*) = *Re* (*h u* ∗ *g u*)
by *auto*
from *u*(*1*) **have** *u′*: *u* ∈ ℝ **unfolding** *closed_segment_def*
by (*auto simp*: *scaleR_conv_of_real*)
from *u′ g_real*[*of u*] *g_nz*[*of u*] **have** *Re* (*g u*) ≠ *0* **by** (*auto elim*!: *Reals_cases*)
with *u*(*2*) *c*[*of u*] *g_real*[*of u*] *g_nz*[*of u*] *u′*
have *Re c* = *0* **by** (*simp add*: *complex_is_Real_iff g.of_1*)
with *h_real*[*of 0*] *c*[*of 0*] **have** *c* = *0* **by** (*auto elim*!: *Reals_cases*)
with *c* **have** *A*: *h z* ∗ *g z* = *0* **for** *z* **by** *simp*
hence (*g has_field_derivative 0*) (*at z*) **for** *z* **using** *g_g′*[*of z*] **by** *simp*
hence ∃ *c′*. ∀ *z*∈*UNIV*. *g z* = *c′* **by** (*intro has_field_derivative_zero_constant*) *simp_all*
then obtain *c′* **where** *c*: ⋀*z*. *g z* = *c′* **by** (*force*)
from *this*[*of 0*] **have** *c′* = *pi* **unfolding** *g_def* **by** *simp*
with *c* **have** *g z* = *pi* **by** *simp*

show *?thesis*
proof (*cases z* ∈ ℤ)
case *False*
with ‹*g z* = *pi*› **show** *?thesis* **by** (*auto simp*: *g_def divide_simps*)
next
case *True*
then obtain *n* **where** *n*: *z* = *of_int n* **by** (*elim Ints_cases*)
with *sin_eq_0*[*of of_real pi* ∗ *z*] **have** *sin* (*of_real pi* ∗ *z*) = *0* **by** *force*
moreover have *of_int* (*1* − *n*) ∈ $\mathbb{Z}_{\leq 0}$ **if** *n* > *0* **using** *that* **by** (*intro nonpos_Ints_of_int*) *simp*
ultimately show *?thesis* **using** *n*
by (*cases n* ≤ *0*) (*auto simp*: *Gamma_eq_zero_iff nonpos_Ints_of_int*)
qed
qed

lemma *rGamma_reflection_complex*:
rGamma z ∗ *rGamma* (*1* − *z* :: *complex*) = *sin* (*of_real pi* ∗ *z*) / *of_real pi*
using *Gamma_reflection_complex*[*of z*]
by (*simp add*: *Gamma_def field_split_simps split*: *if_split_asm*)

lemma *rGamma_reflection_complex′*:
rGamma z ∗ *rGamma* (− *z* :: *complex*) = −*z* ∗ *sin* (*of_real pi* ∗ *z*) / *of_real pi*
proof −
have *rGamma z* ∗ *rGamma* (−*z*) = −*z* ∗ (*rGamma z* ∗ *rGamma* (*1* − *z*))
using *rGamma_plus1*[*of* −*z*, *symmetric*] **by** *simp*
also have *rGamma z* ∗ *rGamma* (*1* − *z*) = *sin* (*of_real pi* ∗ *z*) / *of_real pi*
by (*rule rGamma_reflection_complex*)

finally show *?thesis* **by** *simp*
qed

lemma *Gamma_reflection_complex'*:
Gamma z ∗ Gamma (− z :: complex) = − of_real pi / (z ∗ sin (of_real pi ∗ z))
using *rGamma_reflection_complex'[of z]* **by** *(force simp add: Gamma_def field_split_simps)*

lemma *Gamma_one_half_real*: *Gamma (1/2 :: real) = sqrt pi*
proof −
from *Gamma_reflection_complex[of 1/2] fraction_not_in_ints*[**where** *'a = complex, of 2 1*]
have *Gamma (1/2 :: complex)^2 = of_real pi* **by** *(simp add: power2_eq_square)*
hence *of_real pi = Gamma (complex_of_real (1/2))^2* **by** *simp*
also have ... = *of_real ((Gamma (1/2))^2)* **by** *(subst Gamma_complex_of_real) simp_all*
finally have *Gamma (1/2)^2 = pi* **by** *(subst (asm) of_real_eq_iff) simp_all*
moreover have *Gamma (1/2 :: real) ≥ 0* **using** *Gamma_real_pos[of 1/2]* **by** *simp*
ultimately show *?thesis* **by** *(rule real_sqrt_unique [symmetric])*
qed

lemma *Gamma_one_half_complex*: *Gamma (1/2 :: complex) = of_real (sqrt pi)*
proof −
have *Gamma (1/2 :: complex) = Gamma (of_real (1/2))* **by** *simp*
also have ... = *of_real (sqrt pi)* **by** *(simp only: Gamma_complex_of_real Gamma_one_half_real)*
finally show *?thesis* **.**
qed

theorem *Gamma_legendre_duplication*:
fixes *z :: complex*
assumes $z \notin \mathbb{Z}_{\leq 0}$ $z + 1/2 \notin \mathbb{Z}_{\leq 0}$
shows *Gamma z ∗ Gamma (z + 1/2) =*
exp ((1 − 2∗z) ∗ of_real (ln 2)) ∗ of_real (sqrt pi) ∗ Gamma (2∗z)
using *Gamma_legendre_duplication_aux[OF assms]* **by** *(simp add: Gamma_one_half_complex)*

end

6.23.10 Limits and residues

The inverse of the Gamma function has simple zeros:

lemma *rGamma_zeros*:
(λz. rGamma z / (z + of_nat n)) − (− of_nat n) → ((−1)^n ∗ fact n :: 'a :: Gamma)
proof *(subst tendsto_cong)*
let *?f = λz. pochhammer z n ∗ rGamma (z + of_nat (Suc n)) :: 'a*
from *eventually_at_ball'[OF zero_less_one, of − of_nat n :: 'a UNIV]*
show *eventually (λz. rGamma z / (z + of_nat n) = ?f z) (at (− of_nat n))*

by (*subst pochhammer_rGamma*[*of* _ *Suc n*])
(*auto elim*!: *eventually_mono simp*: *field_split_simps pochhammer_rec′ eq_neg_iff_add_eq_0*)
have *isCont ?f* (− *of_nat n*) **by** (*intro continuous_intros*)
thus *?f* − (− *of_nat n*) → (− *1*) ^ *n* ∗ *fact n* **unfolding** *isCont_def*
by (*simp add*: *pochhammer_same*)
qed

The simple zeros of the inverse of the Gamma function correspond to simple poles of the Gamma function, and their residues can easily be computed from the limit we have just proven:

lemma *Gamma_poles*: *filterlim Gamma at_infinity* (*at* (− *of_nat n* :: ′*a* :: *Gamma*))
proof −
from *eventually_at_ball*′[*OF zero_less_one*, *of* − *of_nat n* :: ′*a UNIV*]
have *eventually* (λ*z*. *rGamma z* ≠ (*0* :: ′*a*)) (*at* (− *of_nat n*))
by (*auto elim*!: *eventually_mono nonpos_Ints_cases*′
simp: *rGamma_eq_zero_iff dist_of_nat dist_minus*)
with *isCont_rGamma*[*of* − *of_nat n* :: ′*a*, *OF continuous_ident*]
have *filterlim* (λ*z*. *inverse* (*rGamma z*) :: ′*a*) *at_infinity* (*at* (− *of_nat n*))
unfolding *isCont_def* **by** (*intro filterlim_compose*[*OF filterlim_inverse_at_infinity*])
(*simp_all add*: *filterlim_at*)
moreover have (λ*z*. *inverse* (*rGamma z*) :: ′*a*) = *Gamma*
by (*intro ext*) (*simp add*: *rGamma_inverse_Gamma*)
ultimately show *?thesis* **by** (*simp only*:)
qed

lemma *Gamma_residues*:
(λ*z*. *Gamma z* ∗ (*z* + *of_nat n*)) − (− *of_nat n*) → ((−*1*)^*n* / *fact n* :: ′*a* :: *Gamma*)
proof (*subst tendsto_cong*)
let *?c* = (− *1*) ^ *n* / *fact n* :: ′*a*
from *eventually_at_ball*′[*OF zero_less_one*, *of* − *of_nat n* :: ′*a UNIV*]
show *eventually* (λ*z*. *Gamma z* ∗ (*z* + *of_nat n*) = *inverse* (*rGamma z* / (*z* + *of_nat n*)))
(*at* (− *of_nat n*))
by (*auto elim*!: *eventually_mono simp*: *field_split_simps rGamma_inverse_Gamma*)
have (λ*z*. *inverse* (*rGamma z* / (*z* + *of_nat n*))) − (− *of_nat n*) →
inverse ((− *1*) ^ *n* ∗ *fact n* :: ′*a*)
by (*intro tendsto_intros rGamma_zeros*) *simp_all*
also have *inverse* ((− *1*) ^ *n* ∗ *fact n*) = *?c*
by (*simp_all add*: *field_simps flip*: *power_mult_distrib*)
finally show (λ*z*. *inverse* (*rGamma z* / (*z* + *of_nat n*))) − (− *of_nat n*) → *?c* **.**
qed

6.23.11 Alternative definitions

Variant of the Euler form

definition *Gamma_series_euler*′ **where**
Gamma_series_euler′ *z n* =

inverse z ∗ (∏*k*=*1*..*n*. *exp* (*z* ∗ *of_real* (*ln* (*1* + *inverse* (*of_nat k*)))) / (*1* + *z* / *of_nat k*))

context
begin
private lemma *Gamma_euler'_aux1*:
fixes *z* :: *'a* :: {*real_normed_field*,*banach*}
assumes *n*: *n* > *0*
shows *exp* (*z* ∗ *of_real* (*ln* (*of_nat n* + *1*))) = (∏*k*=*1*..*n*. *exp* (*z* ∗ *of_real* (*ln* (*1* + *1* / *of_nat k*))))
proof −
have (∏*k*=*1*..*n*. *exp* (*z* ∗ *of_real* (*ln* (*1* + *1* / *of_nat k*)))) =
exp (*z* ∗ *of_real* (∑*k* = *1*..*n*. *ln* (*1* + *1* / *real_of_nat k*)))
by (*subst exp_sum* [*symmetric*]) (*simp_all add*: *sum_distrib_left*)
also have (∑*k*=*1*..*n*. *ln* (*1* + *1* / *of_nat k*) :: *real*) = *ln* (∏*k*=*1*..*n*. *1* + *1* / *real_of_nat k*)
by (*subst ln_prod* [*symmetric*]) (*auto intro*!: *add_pos_nonneg*)
also have (∏*k*=*1*..*n*. *1* + *1* / *of_nat k* :: *real*) = (∏*k*=*1*..*n*. (*of_nat k* + *1*) / *of_nat k*)
by (*intro prod.cong*) (*simp_all add*: *field_split_simps*)
also have (∏*k*=*1*..*n*. (*of_nat k* + *1*) / *of_nat k* :: *real*) = *of_nat n* + *1*
by (*induction n*) (*simp_all add*: *prod.nat_ivl_Suc' field_split_simps*)
finally show *?thesis* ..
qed

theorem *Gamma_series_euler'*:
assumes *z*: (*z* :: *'a* :: *Gamma*) ∉ $\mathbb{Z}_{\le 0}$
shows (λ*n*. *Gamma_series_euler' z n*) ⟶ *Gamma z*
proof (*rule Gamma_seriesI*, *rule Lim_transform_eventually*)
let *?f* = λ*n*. *fact n* ∗ *exp* (*z* ∗ *of_real* (*ln* (*of_nat n* + *1*))) / *pochhammer z* (*n* + *1*)
let *?r* = λ*n*. *?f n* / *Gamma_series z n*
let *?r'* = λ*n*. *exp* (*z* ∗ *of_real* (*ln* (*of_nat* (*Suc n*) / *of_nat n*)))
from *z* **have** *z'*: *z* ≠ *0* **by** *auto*

have *eventually* (λ*n*. *?r' n* = *?r n*) *sequentially*
using *z* **by** (*auto simp*: *field_split_simps Gamma_series_def ring_distribs exp_diff ln_div*
intro: *eventually_mono eventually_gt_at_top*[*of 0*::*nat*] *dest*: *pochhammer_eq_0_imp_nonpos_Int*)
moreover have *?r'* ⟶ *exp* (*z* ∗ *of_real* (*ln 1*))
by (*intro tendsto_intros LIMSEQ_Suc_n_over_n*) *simp_all*
ultimately show *?r* ⟶ *1* **by** (*force intro*: *Lim_transform_eventually*)

from *eventually_gt_at_top*[*of 0*::*nat*]
show *eventually* (λ*n*. *?r n* = *Gamma_series_euler' z n* / *Gamma_series z n*) *sequentially*
proof *eventually_elim*
fix *n* :: *nat* **assume** *n*: *n* > *0*

from *n z′* **have** *Gamma_series_euler′ z n* =
exp (*z* ∗ *of_real* (*ln* (*of_nat n* + *1*))) / (*z* ∗ (∏ *k*=*1*..*n*. (*1* + *z* / *of_nat k*)))
by (*subst Gamma_euler′_aux1*)
(*simp_all add*: *Gamma_series_euler′_def prod.distrib*
prod_inversef [*symmetric*] *divide_inverse*)
also have (∏ *k*=*1*..*n*. (*1* + *z* / *of_nat k*)) = *pochhammer* (*z* + *1*) *n* / *fact n*
proof (*cases n*)
case (*Suc n′*)
then show *?thesis*
unfolding *pochhammer_prod fact_prod*
by (*simp add*: *atLeastLessThanSuc_atLeastAtMost field_simps prod_dividef*
prod.atLeast_Suc_atMost_Suc_shift del: *prod.cl_ivl_Suc*)
qed *auto*
also have *z* ∗ ... = *pochhammer z* (*Suc n*) / *fact n* **by** (*simp add*: *pochhammer_rec*)
finally show *?r n* = *Gamma_series_euler′ z n* / *Gamma_series z n* **by** *simp*
qed
qed

end

Weierstrass form

definition *Gamma_series_Weierstrass* :: *′a* :: {*banach*,*real_normed_field*} ⇒ *nat* ⇒ *′a* **where**
Gamma_series_Weierstrass z n =
exp (−*euler_mascheroni* ∗ *z*) / *z* ∗ (∏ *k*=*1*..*n*. *exp* (*z* / *of_nat k*) / (*1* + *z* / *of_nat k*))

definition
rGamma_series_Weierstrass :: *′a* :: {*banach*,*real_normed_field*} ⇒ *nat* ⇒ *′a* **where**
rGamma_series_Weierstrass z n =
exp (*euler_mascheroni* ∗ *z*) ∗ *z* ∗ (∏ *k*=*1*..*n*. (*1* + *z* / *of_nat k*) ∗ *exp* (−*z* / *of_nat k*))

lemma *Gamma_series_Weierstrass_nonpos_Ints*:
eventually (λ*k*. *Gamma_series_Weierstrass* (− *of_nat n*) *k* = *0*) *sequentially*
using *eventually_ge_at_top*[*of n*] **by** *eventually_elim* (*auto simp*: *Gamma_series_Weierstrass_def*)

lemma *rGamma_series_Weierstrass_nonpos_Ints*:
eventually (λ*k*. *rGamma_series_Weierstrass* (− *of_nat n*) *k* = *0*) *sequentially*
using *eventually_ge_at_top*[*of n*] **by** *eventually_elim* (*auto simp*: *rGamma_series_Weierstrass_def*)

theorem *Gamma_Weierstrass_complex*: *Gamma_series_Weierstrass z* ⟶ *Gamma* (*z* :: *complex*)
proof (*cases z* ∈ $\mathbb{Z}_{\leq 0}$)
case *True*
then obtain *n* **where** *z* = − *of_nat n* **by** (*elim nonpos_Ints_cases′*)
also from *True* **have** *Gamma_series_Weierstrass* ... ⟶ *Gamma z*

by (*simp add*: *tendsto_cong*[*OF Gamma_series_Weierstrass_nonpos_Ints*] *Gamma_nonpos_Int*)
finally show *?thesis* **.**
next
case *False*
hence *z*: $z \neq 0$ **by** *auto*
let *?f* = ($\lambda x.\ \prod x = Suc\ 0..x.\ exp\ (z\ /\ of\_nat\ x)\ /\ (1 + z\ /\ of\_nat\ x)$)
have *A*: *exp* (*ln* (*1* + *z* / *of_nat n*)) = (*1* + *z* / *of_nat n*) **if** $n \geq 1$ **for** *n* :: *nat*
using *False that* **by** (*subst exp_Ln*) (*auto simp*: *field_simps dest*!: *plus_of_nat_eq_0_imp*)
have ($\lambda n.\ \sum k=1..n.\ z\ /\ of\_nat\ k - ln\ (1 + z\ /\ of\_nat\ k)$) $\longlongrightarrow$ *ln_Gamma* *z* + *euler_mascheroni* * *z* + *ln z*
using *ln_Gamma_series'_aux*[*OF False*]
by (*simp only*: *atLeastLessThanSuc_atLeastAtMost* [*symmetric*] *One_nat_def* *sum.shift_bounds_Suc_ivl sums_def atLeast0LessThan*)
from *tendsto_exp*[*OF this*] *False z* **have** *?f* $\longrightarrow$ *z* * *exp* (*euler_mascheroni* * *z*) * *Gamma z*
by (*simp add*: *exp_add exp_sum exp_diff mult_ac Gamma_complex_altdef A*)
from *tendsto_mult*[*OF tendsto_const*[*of exp* (−*euler_mascheroni* * *z*) / *z*] *this*] *z*
show *Gamma_series_Weierstrass z* $\longrightarrow$ *Gamma z*
by (*simp add*: *exp_minus field_split_simps Gamma_series_Weierstrass_def* [*abs_def*])
qed

lemma *tendsto_complex_of_real_iff*: ((λ*x*. *complex_of_real* (*f x*)) $\longrightarrow$ *of_real c*) *F* = (*f* $\longrightarrow$ *c*) *F*
by (*rule tendsto_of_real_iff*)

lemma *Gamma_Weierstrass_real*: *Gamma_series_Weierstrass x* $\longrightarrow$ *Gamma* (*x* :: *real*)
using *Gamma_Weierstrass_complex*[*of of_real x*] **unfolding** *Gamma_series_Weierstrass_def*[*abs_def*]
by (*subst tendsto_complex_of_real_iff* [*symmetric*])
(*simp_all add*: *exp_of_real*[*symmetric*] *Gamma_complex_of_real*)

lemma *rGamma_Weierstrass_complex*: *rGamma_series_Weierstrass z* $\longrightarrow$ *rGamma* (*z* :: *complex*)
proof (*cases* $z \in \mathbb{Z}_{\leq 0}$)
case *True*
then obtain *n* **where** *z* = − *of_nat n* **by** (*elim nonpos_Ints_cases'*)
also from *True* **have** *rGamma_series_Weierstrass* ... $\longrightarrow$ *rGamma z*
by (*simp add*: *tendsto_cong*[*OF rGamma_series_Weierstrass_nonpos_Ints*] *rGamma_nonpos_Int*)
finally show *?thesis* **.**
next
case *False*
have *rGamma_series_Weierstrass z* = (λ*n*. *inverse* (*Gamma_series_Weierstrass* *z n*))
by (*simp add*: *rGamma_series_Weierstrass_def*[*abs_def*] *Gamma_series_Weierstrass_def* *exp_minus divide_inverse prod_inversef*[*symmetric*] *mult_ac*)
also from *False* **have** ... $\longrightarrow$ *inverse* (*Gamma z*)
by (*intro tendsto_intros Gamma_Weierstrass_complex*) (*simp add*: *Gamma_eq_zero_iff*)
finally show *?thesis* **by** (*simp add*: *Gamma_def*)
qed

Binomial coefficient form

lemma *Gamma_gbinomial*:
(λ*n*. ((*z* + *of_nat n*) *gchoose n*) ∗ *exp* (−*z* ∗ *of_real* (*ln* (*of_nat n*)))) ⟶ *rGamma* (*z*+*1*)
proof (*cases z* = *0*)
case *False*
show *?thesis*
proof (*rule Lim_transform_eventually*)
let *?powr* = λ*a b*. *exp* (*b* ∗ *of_real* (*ln* (*of_nat a*)))
show *eventually* (λ*n*. *rGamma_series z n* / *z* =
((*z* + *of_nat n*) *gchoose n*) ∗ *?powr n* (−*z*)) *sequentially*
proof (*intro always_eventually allI*)
fix *n* :: *nat*
from *False* **have** ((*z* + *of_nat n*) *gchoose n*) = *pochhammer z* (*Suc n*) / *z* / *fact n*
by (*simp add*: *gbinomial_pochhammer′ pochhammer_rec*)
also have *pochhammer z* (*Suc n*) / *z* / *fact n* ∗ *?powr n* (−*z*) = *rGamma_series z n* / *z*
by (*simp add*: *rGamma_series_def field_split_simps exp_minus*)
finally show *rGamma_series z n* / *z* = ((*z* + *of_nat n*) *gchoose n*) ∗ *?powr n* (−*z*) **..**
qed

from *False* **have** (λ*n*. *rGamma_series z n* / *z*) ⟶ *rGamma z* / *z* **by** (*intro tendsto_intros*)
also from *False* **have** *rGamma z* / *z* = *rGamma* (*z* + *1*) **using** *rGamma_plus1*[*of z*]
by (*simp add*: *field_simps*)
finally show (λ*n*. *rGamma_series z n* / *z*) ⟶ *rGamma* (*z*+*1*) **.**
qed
qed (*simp_all add*: *binomial_gbinomial* [*symmetric*])

lemma *gbinomial_minus′*: (*a* + *of_nat b*) *gchoose b* = (− *1*) ˆ *b* ∗ (− (*a* + *1*) *gchoose b*)
by (*subst gbinomial_minus*) (*simp add*: *power_mult_distrib* [*symmetric*])

lemma *gbinomial_asymptotic*:
fixes *z* :: ′*a* :: *Gamma*
shows (λ*n*. (*z gchoose n*) / ((−*1*)ˆ*n* / *exp* ((*z*+*1*) ∗ *of_real* (*ln* (*real n*))))) ⟶
inverse (*Gamma* (− *z*))
unfolding *rGamma_inverse_Gamma* [*symmetric*] **using** *Gamma_gbinomial*[*of* −*z*−*1*]
by (*subst* (*asm*) *gbinomial_minus′*)
(*simp add*: *add_ac mult_ac divide_inverse power_inverse* [*symmetric*])

lemma *fact_binomial_limit*:
(λ*n*. *of_nat* ((*k* + *n*) *choose n*) / *of_nat* (*n* ˆ *k*) :: ′*a* :: *Gamma*) ⟶ *1* / *fact k*

proof (*rule Lim_transform_eventually*)
have (λn. *of_nat* ((k + n) *choose* n) / *of_real* (*exp* (*of_nat* k ∗ *ln* (*real_of_nat* n))))
⟶ *1* / *Gamma* (*of_nat* (*Suc* k) :: ′a) (**is** *?f* ⟶ _)
using *Gamma_gbinomial*[*of of_nat* k :: ′a]
by (*simp add*: *binomial_gbinomial Gamma_def field_split_simps exp_of_real* [*symmetric*] *exp_minus*)
also have *Gamma* (*of_nat* (*Suc* k)) = *fact* k **by** (*simp add*: *Gamma_fact*)
finally show *?f* ⟶ *1* / *fact* k **.**

show *eventually* (λn. *?f* n = *of_nat* ((k + n) *choose* n) / *of_nat* (n ˆ k)) *sequentially*
using *eventually_gt_at_top*[*of 0*::*nat*]
proof *eventually_elim*
fix n :: *nat* **assume** n: n > *0*
from n **have** *exp* (*real_of_nat* k ∗ *ln* (*real_of_nat* n)) = *real_of_nat* (nˆk)
by (*simp add*: *exp_of_nat_mult*)
thus *?f* n = *of_nat* ((k + n) *choose* n) / *of_nat* (n ˆ k) **by** *simp*
qed
qed

lemma *binomial_asymptotic′*:
(λn. *of_nat* ((k + n) *choose* n) / (*of_nat* (n ˆ k) / *fact* k) :: ′a :: *Gamma*) ⟶ *1*
using *tendsto_mult*[*OF fact_binomial_limit*[*of* k] *tendsto_const*[*of fact* k :: ′a]] **by** *simp*

lemma *gbinomial_Beta*:
assumes z + *1* ∉ $\mathbb{Z}_{\le 0}$
shows ((z::′a::*Gamma*) *gchoose* n) = *inverse* ((z + *1*) ∗ *Beta* (z − *of_nat* n + *1*) (*of_nat* n + *1*))
using *assms*
proof (*induction* n *arbitrary*: z)
case *0*
hence z + *2* ∉ $\mathbb{Z}_{\le 0}$
using *plus_one_in_nonpos_Ints_imp*[*of* z+*1*] **by** (*auto simp*: *add.commute*)
with *0* **show** *?case*
by (*auto simp*: *Beta_def Gamma_eq_zero_iff Gamma_plus1* [*symmetric*] *add.commute*)
next
case (*Suc* n z)
show *?case*
proof (*cases* z ∈ $\mathbb{Z}_{\le 0}$)
case *True*
with *Suc.prems* **have** z = *0*
by (*auto elim*!: *nonpos_Ints_cases simp*: *algebra_simps one_plus_of_int_in_nonpos_Ints_iff*)
show *?thesis*
proof (*cases* n = *0*)
case *True*
with ‹z = *0*› **show** *?thesis*

by (*simp add*: *Beta_def Gamma_eq_zero_iff Gamma_plus1* [*symmetric*])
next
case *False*
with ‹*z* = *0*› **show** *?thesis*
by (*simp_all add*: *Beta_pole1 one_minus_of_nat_in_nonpos_Ints_iff*)
qed
next
case *False*
have (*z gchoose* (*Suc n*)) = ((*z* − *1* + *1*) *gchoose* (*Suc n*)) **by** *simp*
also have ... = (*z* − *1 gchoose n*) ∗ ((*z* − *1*) + *1*) / *of_nat* (*Suc n*)
by (*subst gbinomial_factors*) (*simp add*: *field_simps*)
also from *False* **have** ... = *inverse* (*of_nat* (*Suc n*) ∗ *Beta* (*z* − *of_nat n*) (*of_nat* (*Suc n*)))
(**is** _ = *inverse ?x*) **by** (*subst Suc.IH*) (*simp_all add*: *field_simps Beta_pole1*)
also have *of_nat* (*Suc n*) ∉ ($\mathbb{Z}_{\leq 0}$ :: *'a set*) **by** (*subst of_nat_in_nonpos_Ints_iff*) *simp_all*
hence *?x* = (*z* + *1*) ∗ *Beta* (*z* − *of_nat* (*Suc n*) + *1*) (*of_nat* (*Suc n*) + *1*)
by (*subst Beta_plus1_right* [*symmetric*]) *simp_all*
finally show *?thesis* **.**
qed
qed

theorem *gbinomial_Gamma*:
assumes *z* + *1* ∉ $\mathbb{Z}_{\leq 0}$
shows (*z gchoose n*) = *Gamma* (*z* + *1*) / (*fact n* ∗ *Gamma* (*z* − *of_nat n* + *1*))
proof −
have (*z gchoose n*) = *Gamma* (*z* + *2*) / (*z* + *1*) / (*fact n* ∗ *Gamma* (*z* − *of_nat n* + *1*))
by (*subst gbinomial_Beta*[*OF assms*]) (*simp_all add*: *Beta_def Gamma_fact* [*symmetric*] *add_ac*)
also from *assms* **have** *Gamma* (*z* + *2*) / (*z* + *1*) = *Gamma* (*z* + *1*)
using *Gamma_plus1*[*of z+1*] **by** (*auto simp add*: *field_split_simps*)
finally show *?thesis* **.**
qed

Integral form

lemma *integrable_on_powr_from_0'*:
assumes *a*: *a* > (−*1*::*real*) **and** *c*: *c* ≥ *0*
shows (λ*x*. *x powr a*) *integrable_on* {*0*<..*c*}
proof −
from *c* **have** ∗: {*0*<..*c*} − {*0*..*c*} = {} {*0*..*c*} − {*0*<..*c*} = {*0*} **by** *auto*
show *?thesis*
by (*rule integrable_spike_set* [*OF integrable_on_powr_from_0*[*OF a c*]]) (*simp_all add*: ∗)
qed

lemma *absolutely_integrable_Gamma_integral*:

```
  assumes Re z > 0 a > 0
  shows   (λt. complex_of_real t powr (z − 1) / of_real (exp (a ∗ t)))
            absolutely_integrable_on {0<..} (is ?f absolutely_integrable_on _)
proof −
  have ((λx. (Re z − 1) ∗ (ln x / x)) ⟶ (Re z − 1) ∗ 0) at_top
    by (intro tendsto_intros ln_x_over_x_tendsto_0)
  hence ((λx. ((Re z − 1) ∗ ln x) / x) ⟶ 0) at_top by simp
  from order_tendstoD(2)[OF this, of a/2] and ‹a > 0›
    have eventually (λx. (Re z − 1) ∗ ln x / x < a/2) at_top by simp
  from eventually_conj[OF this eventually_gt_at_top[of 0]]
    obtain x0 where ∀x≥x0. (Re z − 1) ∗ ln x / x < a/2 ∧ x > 0
      by (auto simp: eventually_at_top_linorder)
  hence x0 > 0 by simp
  have x powr (Re z − 1) / exp (a ∗ x) < exp (−(a/2) ∗ x) if x ≥ x0 for x
  proof −
    from that and ‹∀x≥x0. _› have x: (Re z − 1) ∗ ln x / x < a / 2 x > 0 by
auto
    have x powr (Re z − 1) = exp ((Re z − 1) ∗ ln x)
      using ‹x > 0› by (simp add: powr_def)
    also from x have (Re z − 1) ∗ ln x < (a ∗ x) / 2 by (simp add: field_simps)
    finally show ?thesis by (simp add: field_simps exp_add [symmetric])
  qed
  note x0 = ‹x0 > 0› this

  have ?f absolutely_integrable_on ({0<..x0} ∪ {x0..})
  proof (rule set_integrable_Un)
    show ?f absolutely_integrable_on {0<..x0}
      unfolding set_integrable_def
    proof (rule Bochner_Integration.integrable_bound [OF _ _ AE_I2])
      show integrable lebesgue (λx. indicat_real {0<..x0} x ∗R x powr (Re z − 1))

        using x0(1) assms
          by (intro nonnegative_absolutely_integrable_1 [unfolded set_integrable_def]
integrable_on_powr_from_0') auto
      show (λx. indicat_real {0<..x0} x ∗R (x powr (z − 1) / exp (a ∗ x))) ∈
borel_measurable lebesgue
        by (intro measurable_completion)
          (auto intro!: borel_measurable_continuous_on_indicator continuous_intros)
      fix x :: real
      have x powr (Re z − 1) / exp (a ∗ x) ≤ x powr (Re z − 1) / 1 if x ≥ 0
        using that assms by (intro divide_left_mono) auto
      thus norm (indicator {0<..x0} x ∗R ?f x) ≤
              norm (indicator {0<..x0} x ∗R x powr (Re z − 1))
        by (simp_all add: norm_divide norm_powr_real_powr indicator_def)
    qed
  next
    show ?f absolutely_integrable_on {x0..}
      unfolding set_integrable_def
    proof (rule Bochner_Integration.integrable_bound [OF _ _ AE_I2])
```

show *integrable lebesgue* (λx. *indicat_real* {*x0*..} *x* $*_R$ *exp* (− (*a* / *2*) ∗ *x*))
using *assms*
by (*intro nonnegative_absolutely_integrable_1* [*unfolded set_integrable_def*] *integrable_on_exp_minus_to_infinity*) *auto*
show (λx. *indicat_real* {*x0*..} *x* $*_R$ (*x powr* (*z* − *1*) / *exp* (*a* ∗ *x*))) ∈ *borel_measurable lebesgue* **using** *x0*(*1*)
by (*intro measurable_completion*)
(*auto intro*!: *borel_measurable_continuous_on_indicator continuous_intros*)
fix *x* :: *real*
show *norm* (*indicator* {*x0*..} *x* $*_R$ *?f x*) ≤
norm (*indicator* {*x0*..} *x* $*_R$ *exp* (−(*a*/*2*) ∗ *x*)) **using** *x0*
by (*auto simp*: *norm_divide norm_powr_real_powr indicator_def less_imp_le*)
qed
qed *auto*
also have {*0*<..*x0*} ∪ {*x0*..} = {*0*<..} **using** *x0*(*1*) **by** *auto*
finally show *?thesis* **.**
qed

lemma *integrable_Gamma_integral_bound*:
fixes *a c* :: *real*
assumes *a*: *a* > −*1* **and** *c*: *c* ≥ *0*
defines *f* ≡ λx. *if x* ∈ {*0*..*c*} *then x powr a else exp* (−*x*/*2*)
shows *f integrable_on* {*0*..}
proof −
have *f integrable_on* {*0*..*c*}
by (*rule integrable_spike_finite*[*of* {}, *OF* _ _ *integrable_on_powr_from_0*[*of a c*]])
(*insert a c*, *simp_all add*: *f_def*)
moreover have *A*: (λx. *exp* (−*x*/*2*)) *integrable_on* {*c*..}
using *integrable_on_exp_minus_to_infinity*[*of 1*/*2*] **by** *simp*
have *f integrable_on* {*c*..}
by (*rule integrable_spike_finite*[*of* {*c*}, *OF* _ _ *A*]) (*simp_all add*: *f_def*)
ultimately show *f integrable_on* {*0*..}
by (*rule integrable_Un*′) (*insert c*, *auto simp*: *max_def*)
qed

theorem *Gamma_integral_complex*:
assumes *z*: *Re z* > *0*
shows ((λt. *of_real t powr* (*z* − *1*) / *of_real* (*exp t*)) *has_integral Gamma z*) {*0*..}
proof −
have *A*: ((λt. (*of_real t*) *powr* (*z* − *1*) ∗ *of_real* ((*1* − *t*) ˆ *n*))
has_integral (*fact n* / *pochhammer z* (*n*+*1*))) {*0*..*1*}
if *Re z* > *0* **for** *n z* **using** *that*
proof (*induction n arbitrary*: *z*)
case *0*
have ((λt. *complex_of_real t powr* (*z* − *1*)) *has_integral*
(*of_real 1 powr z* / *z* − *of_real 0 powr z* / *z*)) {*0*..*1*} **using** *0*
by (*intro fundamental_theorem_of_calculus_interior*)
(*auto intro*!: *continuous_intros derivative_eq_intros has_vector_derivative_real_field*)

```
    thus ?case by simp
  next
    case (Suc n)
    let ?f = λt. complex_of_real t powr z / z
    let ?f' = λt. complex_of_real t powr (z − 1)
    let ?g = λt. (1 − complex_of_real t) ^ Suc n
    let ?g' = λt. − ((1 − complex_of_real t) ^ n) ∗ of_nat (Suc n)
    have ((λt. ?f' t ∗ ?g t) has_integral
            (of_nat (Suc n)) ∗ fact n / pochhammer z (n+2)) {0..1}
      (is (_ has_integral ?I) _)
    proof (rule integration_by_parts_interior[where f' = ?f' and g = ?g])
      from Suc.prems show continuous_on {0..1} ?f continuous_on {0..1} ?g
        by (auto intro!: continuous_intros)
    next
      fix t :: real assume t: t ∈ {0<..<1}
      show (?f has_vector_derivative ?f' t) (at t) using t Suc.prems
        by (auto intro!: derivative_eq_intros has_vector_derivative_real_field)
      show (?g has_vector_derivative ?g' t) (at t)
        by (rule has_vector_derivative_real_field derivative_eq_intros refl)+ simp_all
    next
      from Suc.prems have [simp]: z ≠ 0 by auto
      from Suc.prems have A: Re (z + of_nat n) > 0 for n by simp
      have [simp]: z + of_nat n ≠ 0 z + 1 + of_nat n ≠ 0 for n
          using A[of n] A[of Suc n] by (auto simp add: add.assoc simp del:
plus_complex.sel)
      have ((λx. of_real x powr z ∗ of_real ((1 − x) ^ n) ∗ (− of_nat (Suc n) / z))
has_integral
              fact n / pochhammer (z+1) (n+1) ∗ (− of_nat (Suc n) / z)) {0..1}
        (is (?A has_integral ?B) _)
        using Suc.IH[of z+1] Suc.prems by (intro has_integral_mult_left) (simp_all
add: add_ac pochhammer_rec)
      also have ?A = (λt. ?f t ∗ ?g' t) by (intro ext) (simp_all add: field_simps)
      also have ?B = − (of_nat (Suc n) ∗ fact n / pochhammer z (n+2))
        by (simp add: field_split_simps pochhammer_rec
              prod.shift_bounds_cl_Suc_ivl del: of_nat_Suc)
       finally show ((λt. ?f t ∗ ?g' t) has_integral (?f 1 ∗ ?g 1 − ?f 0 ∗ ?g 0 −
?I)) {0..1}
        by simp
    qed (simp_all add: bounded_bilinear_mult)
    thus ?case by simp
  qed

  have B: ((λt. if t ∈ {0..of_nat n} then
             of_real t powr (z − 1) ∗ (1 − of_real t / of_nat n) ^ n else 0)
           has_integral (of_nat n powr z ∗ fact n / pochhammer z (n+1))) {0..} for
n
  proof (cases n > 0)
    case [simp]: True
    hence [simp]: n ≠ 0 by auto
```

with *has_integral_affinity01*[*OF A*[*OF z*, *of n*], *of inverse* (*of_nat n*) *0*]
have ((λ*x*. (*of_nat n* − *of_real x*) ˆ *n* ∗ (*of_real x* / *of_nat n*) *powr* (*z* − *1*) / *of_nat n* ˆ *n*)
has_integral fact n ∗ *of_nat n* / *pochhammer z* (*n*+*1*)) ((λ*x*. *real n* ∗ *x*)‘{*0*..*1*})
(**is** (*?f has_integral ?I*) *?ivl*) **by** (*simp add*: *field_simps scaleR_conv_of_real*)
also from *True* **have** ((λ*x*. *real n*∗*x*)‘{*0*..*1*}) = {*0*..*real n*}
by (*subst image_mult_atLeastAtMost*) *simp_all*
also have *?f* = (λ*x*. (*of_real x* / *of_nat n*) *powr* (*z* − *1*) ∗ (*1* − *of_real x* / *of_nat n*) ˆ *n*)
using *True* **by** (*intro ext*) (*simp add*: *field_simps*)
finally have ((λ*x*. (*of_real x* / *of_nat n*) *powr* (*z* − *1*) ∗ (*1* − *of_real x* / *of_nat n*) ˆ *n*)
has_integral ?I) {*0*..*real n*} (**is** *?P*) **.**
also have *?P* ⟷ ((λ*x*. *exp* ((*z* − *1*) ∗ *of_real* (*ln* (*x* / *of_nat n*))) ∗ (*1* − *of_real x* / *of_nat n*) ˆ *n*)
has_integral ?I) {*0*..*real n*}
by (*intro has_integral_spike_finite_eq*[*of* {*0*}]) (*auto simp*: *powr_def Ln_of_real* [*symmetric*])
also have ... ⟷ ((λ*x*. *exp* ((*z* − *1*) ∗ *of_real* (*ln x* − *ln* (*of_nat n*))) ∗ (*1* − *of_real x* / *of_nat n*) ˆ *n*)
has_integral ?I) {*0*..*real n*}
by (*intro has_integral_spike_finite_eq*[*of* {*0*}]) (*simp_all add*: *ln_div*)
finally have ... **.**
note *B* = *has_integral_mult_right*[*OF this*, *of exp* ((*z* − *1*) ∗ *ln* (*of_nat n*))]
have ((λ*x*. *exp* ((*z* − *1*) ∗ *of_real* (*ln x*)) ∗ (*1* − *of_real x* / *of_nat n*) ˆ *n*)
has_integral (*?I* ∗ *exp* ((*z* − *1*) ∗ *ln* (*of_nat n*)))) {*0*..*real n*} (**is** *?P*)
by (*insert B*, *subst* (*asm*) *mult.assoc* [*symmetric*], *subst* (*asm*) *exp_add* [*symmetric*])
(*simp add*: *algebra_simps*)
also have *?P* ⟷ ((λ*x*. *of_real x powr* (*z* − *1*) ∗ (*1* − *of_real x* / *of_nat n*) ˆ *n*)
has_integral (*?I* ∗ *exp* ((*z* − *1*) ∗ *ln* (*of_nat n*)))) {*0*..*real n*}
by (*intro has_integral_spike_finite_eq*[*of* {*0*}]) (*simp_all add*: *powr_def Ln_of_real*)
also have *fact n* ∗ *of_nat n* / *pochhammer z* (*n*+*1*) ∗ *exp* ((*z* − *1*) ∗ *Ln* (*of_nat n*)) =
(*of_nat n powr z* ∗ *fact n* / *pochhammer z* (*n*+*1*))
by (*auto simp add*: *powr_def algebra_simps exp_diff exp_of_real*)
finally show *?thesis* **by** (*subst has_integral_restrict*) *simp_all*
next
case *False*
thus *?thesis* **by** (*subst has_integral_restrict*) (*simp_all add*: *has_integral_refl*)
qed

have *eventually* (λ*n*. *Gamma_series z n* =
of_nat n powr z ∗ *fact n* / *pochhammer z* (*n*+*1*)) *sequentially*
using *eventually_gt_at_top*[*of 0*::*nat*]
by *eventually_elim* (*simp add*: *powr_def algebra_simps Gamma_series_def*)
from *this* **and** *Gamma_series_LIMSEQ*[*of z*]

```
  have C: (λk. of_nat k powr z * fact k / pochhammer z (k+1)) ⟶ Gamma
z
    by (blast intro: Lim_transform_eventually)
  {
    fix x :: real assume x: x ≥ 0
    have lim_exp: (λk. (1 − x / real k) ^ k) ⟶ exp (−x)
      using tendsto_exp_limit_sequentially[of −x] by simp
    have (λk. of_real x powr (z − 1) * of_real ((1 − x / of_nat k) ^ k))
            ⟶ of_real x powr (z − 1) * of_real (exp (−x)) (is ?P)
      by (intro tendsto_intros lim_exp)
    also from eventually_gt_at_top[of nat ⌈x⌉]
      have eventually (λk. of_nat k > x) sequentially by eventually_elim linarith
    hence ?P ⟷ (λk. if x ≤ of_nat k then
                  of_real x powr (z − 1) * of_real ((1 − x / of_nat k) ^ k) else 0)
                    ⟶ of_real x powr (z − 1) * of_real (exp (−x))
      by (intro tendsto_cong) (auto elim!: eventually_mono)
    finally have ... .
  }
  hence D: ∀x∈{0..}. (λk. if x ∈ {0..real k} then
              of_real x powr (z − 1) * (1 − of_real x / of_nat k) ^ k else 0)
            ⟶ of_real x powr (z − 1) / of_real (exp x)
    by (simp add: exp_minus field_simps cong: if_cong)

  have ((λx. (Re z − 1) * (ln x / x)) ⟶ (Re z − 1) * 0) at_top
    by (intro tendsto_intros ln_x_over_x_tendsto_0)
  hence ((λx. ((Re z − 1) * ln x) / x) ⟶ 0) at_top by simp
  from order_tendstoD(2)[OF this, of 1/2]
    have eventually (λx. (Re z − 1) * ln x / x < 1/2) at_top by simp
  from eventually_conj[OF this eventually_gt_at_top[of 0]]
    obtain x0 where ∀x≥x0. (Re z − 1) * ln x / x < 1/2 ∧ x > 0
    by (auto simp: eventually_at_top_linorder)
  hence x0: x0 > 0 ⋀x. x ≥ x0 ⟹ (Re z − 1) * ln x < x / 2 by auto

  define h where h = (λx. if x ∈ {0..x0} then x powr (Re z − 1) else exp (−x/2))
  have le_h: x powr (Re z − 1) * exp (−x) ≤ h x if x: x ≥ 0 for x
  proof (cases x > x0)
    case True
    from True x0(1) have x powr (Re z − 1) * exp (−x) = exp ((Re z − 1) * ln
x − x)
      by (simp add: powr_def exp_diff exp_minus field_simps exp_add)
    also from x0(2)[of x] True have ... < exp (−x/2)
      by (simp add: field_simps)
    finally show ?thesis using True by (auto simp add: h_def)
  next
    case False
    from x have x powr (Re z − 1) * exp (− x) ≤ x powr (Re z − 1) * 1
      by (intro mult_left_mono) simp_all
    with False show ?thesis by (auto simp add: h_def)
  qed
```

```
  have E: ∀x∈{0..}. cmod (if x ∈ {0..real k} then of_real x powr (z − 1) *
                 (1 − complex_of_real x / of_nat k) ^ k else 0) ≤ h x
    (is ∀x∈_. ?f x ≤ _) for k
  proof safe
    fix x :: real assume x: x ≥ 0
    {
      fix x :: real and n :: nat assume x: x ≤ of_nat n
      have (1 − complex_of_real x / of_nat n) = complex_of_real ((1 − x / of_nat
n)) by simp
      also have norm ... = |(1 − x / real n)| by (subst norm_of_real) (rule refl)
      also from x have ... = (1 − x / real n) by (intro abs_of_nonneg) (simp_all
add: field_split_simps)
      finally have cmod (1 − complex_of_real x / of_nat n) = 1 − x / real n .
    } note D = this
    from D[of x k] x
      have ?f x ≤ (if of_nat k ≥ x ∧ k > 0 then x powr (Re z − 1) * (1 − x /
real k) ^ k else 0)
     by (auto simp: norm_mult norm_powr_real_powr norm_power intro!: mult_nonneg_nonneg)
    also have ... ≤ x powr (Re z − 1) * exp (−x)
      by (auto intro!: mult_left_mono exp_ge_one_minus_x_over_n_power_n)
    also from x have ... ≤ h x by (rule le_h)
    finally show ?f x ≤ h x .
  qed

  have F: h integrable_on {0..} unfolding h_def
    by (rule integrable_Gamma_integral_bound) (insert assms x0(1), simp_all)
  show ?thesis
    by (rule has_integral_dominated_convergence[OF B F E D C])
qed

lemma Gamma_integral_real:
  assumes x: x > (0 :: real)
  shows   ((λt. t powr (x − 1) / exp t) has_integral Gamma x) {0..}
proof −
  have A: ((λt. complex_of_real t powr (complex_of_real x − 1) /
          complex_of_real (exp t)) has_integral complex_of_real (Gamma x)) {0..}
   using Gamma_integral_complex[of x] assms by (simp_all add: Gamma_complex_of_real
powr_of_real)
  have ((λt. complex_of_real (t powr (x − 1) / exp t)) has_integral of_real (Gamma
x)) {0..}
    by (rule has_integral_eq[OF _ A]) (simp_all add: powr_of_real [symmetric])
  from has_integral_linear[OF this bounded_linear_Re] show ?thesis by (simp add:
o_def)
qed

lemma absolutely_integrable_Gamma_integral':
  assumes Re z > 0
  shows  (λt. complex_of_real t powr (z − 1) / of_real (exp t)) absolutely_integrable_on
```

{0<..}
using *absolutely_integrable_Gamma_integral* [*OF assms zero_less_one*] **by** *simp*

lemma *Gamma_integral_complex'*:
assumes *z*: *Re z > 0*
shows ((λ*t*. *of_real t powr* (*z* − *1*) / *of_real* (*exp t*)) *has_integral Gamma z*) {*0*<..}
proof −
have ((λ*t*. *of_real t powr* (*z* − *1*) / *of_real* (*exp t*)) *has_integral Gamma z*) {*0*..}
by (*rule Gamma_integral_complex*) *fact+*
hence ((λ*t*. *if t* ∈ {*0*<..} *then of_real t powr* (*z* − *1*) / *of_real* (*exp t*) *else 0*)
has_integral Gamma z) {*0*..}
by (*rule has_integral_spike* [*of* {*0*}, *rotated 2*]) *auto*
also have *?this* = *?thesis*
by (*subst has_integral_restrict*) *auto*
finally show *?thesis* **.**
qed

lemma *Gamma_conv_nn_integral_real*:
assumes *s* > (*0*::*real*)
shows *Gamma s* = *nn_integral lborel* (λ*t*. *ennreal* (*indicator* {*0*..} *t* ∗ *t powr* (*s* − *1*) / *exp t*))
using *nn_integral_has_integral_lebesgue*[*OF* _ *Gamma_integral_real*[*OF assms*]] **by** *simp*

lemma *integrable_Beta*:
assumes *a* > *0* *b* > (*0*::*real*)
shows *set_integrable lborel* {*0*..*1*} (λ*t*. *t powr* (*a* − *1*) ∗ (*1* − *t*) *powr* (*b* − *1*))
proof −
define *C* **where** *C* = *max 1* ((*1*/*2*) *powr* (*b* − *1*))
define *D* **where** *D* = *max 1* ((*1*/*2*) *powr* (*a* − *1*))
have *C*: (*1* − *x*) *powr* (*b* − *1*) ≤ *C* **if** *x* ∈ {*0*..*1*/*2*} **for** *x*
proof (*cases b* < *1*)
case *False*
with *that* **have** (*1* − *x*) *powr* (*b* − *1*) ≤ (*1 powr* (*b* − *1*)) **by** (*intro powr_mono2*) *auto*
thus *?thesis* **by** (*auto simp*: *C_def*)
qed (*insert that*, *auto simp*: *max.coboundedI1 max.coboundedI2 powr_mono2' powr_mono2 C_def*)
have *D*: *x powr* (*a* − *1*) ≤ *D* **if** *x* ∈ {*1*/*2*..*1*} **for** *x*
proof (*cases a* < *1*)
case *False*
with *that* **have** *x powr* (*a* − *1*) ≤ (*1 powr* (*a* − *1*)) **by** (*intro powr_mono2*) *auto*
thus *?thesis* **by** (*auto simp*: *D_def*)
next
case *True*
qed (*insert that*, *auto simp*: *max.coboundedI1 max.coboundedI2 powr_mono2' powr_mono2 D_def*)

have [*simp*]: $C \geq 0$ $D \geq 0$ **by** (*simp_all add*: *C_def D_def*)

have *I1*: *set_integrable lborel* {*0..1/2*} (λ*t*. *t powr* (*a* − *1*) ∗ (*1* − *t*) *powr* (*b* − *1*))
  unfolding *set_integrable_def*
 proof (*rule Bochner_Integration.integrable_bound*[*OF* _ _ *AE_I2*])
  have (λ*t*. *t powr* (*a* − *1*)) *integrable_on* {*0..1/2*}
   by (*rule integrable_on_powr_from_0*) (*use assms* **in** *auto*)
  hence (λ*t*. *t powr* (*a* − *1*)) *absolutely_integrable_on* {*0..1/2*}
   by (*subst absolutely_integrable_on_iff_nonneg*) *auto*
  from *integrable_mult_right*[*OF this* [*unfolded set_integrable_def*], *of C*]
  show *integrable lborel* (λ*x*. *indicat_real* {*0..1/2*} *x* $*_R$ (*C* ∗ *x powr* (*a* − *1*)))
   by (*subst* (*asm*) *integrable_completion*) (*auto simp*: *mult_ac*)
 next
  fix *x* :: *real*
  have *x powr* (*a* − *1*) ∗ (*1* − *x*) *powr* (*b* − *1*) ≤ *x powr* (*a* − *1*) ∗ *C* **if** *x* ∈ {*0..1/2*}
   using *that* **by** (*intro mult_left_mono powr_mono2 C*) *auto*
  thus *norm* (*indicator* {*0..1/2*} *x* $*_R$ (*x powr* (*a* − *1*) ∗ (*1* − *x*) *powr* (*b* − *1*))) ≤
     norm (*indicator* {*0..1/2*} *x* $*_R$ (*C* ∗ *x powr* (*a* − *1*)))
   by (*auto simp*: *indicator_def abs_mult mult_ac*)
 qed (*auto intro*!: *AE_I2 simp*: *indicator_def*)

have *I2*: *set_integrable lborel* {*1/2..1*} (λ*t*. *t powr* (*a* − *1*) ∗ (*1* − *t*) *powr* (*b* − *1*))
  unfolding *set_integrable_def*
 proof (*rule Bochner_Integration.integrable_bound*[*OF* _ _ *AE_I2*])
  have (λ*t*. *t powr* (*b* − *1*)) *integrable_on* {*0..1/2*}
   by (*rule integrable_on_powr_from_0*) (*use assms* **in** *auto*)
  hence (λ*t*. *t powr* (*b* − *1*)) *integrable_on* (*cbox 0* (*1/2*)) **by** *simp*
  from *integrable_affinity*[*OF this*, *of* −*1 1*]
   have (λ*t*. (*1* − *t*) *powr* (*b* − *1*)) *integrable_on* {*1/2..1*} **by** *simp*
  hence (λ*t*. (*1* − *t*) *powr* (*b* − *1*)) *absolutely_integrable_on* {*1/2..1*}
   by (*subst absolutely_integrable_on_iff_nonneg*) *auto*
  from *integrable_mult_right*[*OF this* [*unfolded set_integrable_def*], *of D*]
  show *integrable lborel* (λ*x*. *indicat_real* {*1/2..1*} *x* $*_R$ (*D* ∗ (*1* − *x*) *powr* (*b* − *1*)))
   by (*subst* (*asm*) *integrable_completion*) (*auto simp*: *mult_ac*)
 next
  fix *x* :: *real*
  have *x powr* (*a* − *1*) ∗ (*1* − *x*) *powr* (*b* − *1*) ≤ *D* ∗ (*1* − *x*) *powr* (*b* − *1*) **if** *x* ∈ {*1/2..1*}
   using *that* **by** (*intro mult_right_mono powr_mono2 D*) *auto*
  thus *norm* (*indicator* {*1/2..1*} *x* $*_R$ (*x powr* (*a* − *1*) ∗ (*1* − *x*) *powr* (*b* − *1*))) ≤
     norm (*indicator* {*1/2..1*} *x* $*_R$ (*D* ∗ (*1* − *x*) *powr* (*b* − *1*)))
   by (*auto simp*: *indicator_def abs_mult mult_ac*)
 qed (*auto intro*!: *AE_I2 simp*: *indicator_def*)

have *set_integrable lborel* ({*0..1/2*} ∪ {*1/2..1*}) (λ*t. t powr* (*a* − *1*) ∗ (*1* − *t*) *powr* (*b* − *1*))
  by (*intro set_integrable_Un I1 I2*) *auto*
also have {*0..1/2*} ∪ {*1/2..1*} = {*0..*(*1*::*real*)} **by** *auto*
finally show *?thesis* **.**
qed

lemma *integrable_Beta′*:
  assumes *a* > *0 b* > (*0*::*real*)
  shows (λ*t. t powr* (*a* − *1*) ∗ (*1* − *t*) *powr* (*b* − *1*)) *integrable_on* {*0..1*}
  using *integrable_Beta*[*OF assms*] **by** (*rule set_borel_integral_eq_integral*)

theorem *has_integral_Beta_real*:
  assumes *a*: *a* > *0* **and** *b*: *b* > (*0* :: *real*)
  shows ((λ*t. t powr* (*a* − *1*) ∗ (*1* − *t*) *powr* (*b* − *1*)) *has_integral Beta a b*) {*0..1*}
proof −
  define *B* **where** *B* = *integral* {*0..1*} (λ*x. x powr* (*a* − *1*) ∗ (*1* − *x*) *powr* (*b* − *1*))
  have [*simp*]: *B* ≥ *0* **unfolding** *B_def* **using** *a b*
    by (*intro integral_nonneg integrable_Beta′*) *auto*
  from *a b* **have** *ennreal* (*Gamma a* ∗ *Gamma b*) =
    (∫ + *t. ennreal* (*indicator* {*0..*} *t* ∗ *t powr* (*a* − *1*) / *exp t*) ∂*lborel*) ∗
    (∫ + *t. ennreal* (*indicator* {*0..*} *t* ∗ *t powr* (*b* − *1*) / *exp t*) ∂*lborel*)
    by (*subst ennreal_mult′*) (*simp_all add*: *Gamma_conv_nn_integral_real*)
  also have ... = (∫ +*t.* ∫ +*u. ennreal* (*indicator* {*0..*} *t* ∗ *t powr* (*a* − *1*) / *exp t*) ∗
                    ennreal (*indicator* {*0..*} *u* ∗ *u powr* (*b* − *1*) / *exp u*) ∂*lborel* ∂*lborel*)
    by (*simp add*: *nn_integral_cmult nn_integral_multc*)
  also have ... = (∫ +*t.* ∫ +*u. ennreal* (*indicator* ({*0..*}×{*0..*}) (*t*,*u*) ∗ *t powr* (*a* − *1*) ∗ *u powr* (*b* − *1*)
                    / *exp* (*t* + *u*)) ∂*lborel* ∂*lborel*)
    by (*intro nn_integral_cong*)
       (*auto simp*: *indicator_def divide_ennreal ennreal_mult′* [*symmetric*] *exp_add*)
  also have ... = (∫ +*t.* ∫ +*u. ennreal* (*indicator* ({*0..*}×{*t..*}) (*t*,*u*) ∗ *t powr* (*a* − *1*) ∗
                    (*u* − *t*) *powr* (*b* − *1*) / *exp u*) ∂*lborel* ∂*lborel*)
  proof (*rule nn_integral_cong*, *goal_cases*)
    case (*1 t*)
    have (∫ +*u. ennreal* (*indicator* ({*0..*}×{*0..*}) (*t*,*u*) ∗ *t powr* (*a* − *1*) ∗
                    u powr (*b* − *1*) / *exp* (*t* + *u*)) ∂*distr lborel borel* ((+) (−*t*))) =
           (∫ +*u. ennreal* (*indicator* ({*0..*}×{*t..*}) (*t*,*u*) ∗ *t powr* (*a* − *1*) ∗
                    (*u* − *t*) *powr* (*b* − *1*) / *exp u*) ∂*lborel*)
      by (*subst nn_integral_distr*) (*auto intro*!: *nn_integral_cong simp*: *indicator_def*)
    thus *?case* **by** (*subst* (*asm*) *lborel_distr_plus*)
  qed

also have ... = (∫ $^+$*u*. ∫ $^+$*t*. *ennreal* (*indicator* ({*0*..}×{*t*..}) (*t*,*u*) ∗ *t powr* (*a* − *1*) ∗
(*u* − *t*) *powr* (*b* − *1*) / *exp u*) ∂*lborel* ∂*lborel*)
by (*subst lborel_pair*.*Fubini′*)
(*auto simp*: *case_prod_unfold indicator_def cong*: *measurable_cong_sets*)
also have ... = (∫ $^+$*u*. ∫ $^+$*t*. *ennreal* (*indicator* {*0*..*u*} *t* ∗ *t powr* (*a* − *1*) ∗ (*u* − *t*) *powr* (*b* − *1*)) ∗
ennreal (*indicator* {*0*..} *u* / *exp u*) ∂*lborel* ∂*lborel*)
by (*intro nn_integral_cong*) (*auto simp*: *indicator_def ennreal_mult′* [*symmetric*])
also have ... = (∫ $^+$*u*. (∫ $^+$*t*. *ennreal* (*indicator* {*0*..*u*} *t* ∗ *t powr* (*a* − *1*) ∗ (*u* − *t*) *powr* (*b* − *1*))
∂*lborel*) ∗ *ennreal* (*indicator* {*0*..} *u* / *exp u*) ∂*lborel*)
by (*subst nn_integral_multc* [*symmetric*]) *auto*
also have ... = (∫ $^+$*u*. (∫ $^+$*t*. *ennreal* (*indicator* {*0*..*u*} *t* ∗ *t powr* (*a* − *1*) ∗ (*u* − *t*) *powr* (*b* − *1*))
∂*lborel*) ∗ *ennreal* (*indicator* {*0*<..} *u* / *exp u*) ∂*lborel*)
by (*intro nn_integral_cong_AE eventually_mono*[*OF AE_lborel_singleton*[*of 0*]])
(*auto simp*: *indicator_def*)
also have ... = (∫ $^+$*u*. *ennreal B* ∗ *ennreal* (*indicator* {*0*..} *u* / *exp u* ∗ *u powr* (*a* + *b* − *1*)) ∂*lborel*)
proof (*intro nn_integral_cong*, *goal_cases*)
case (*1 u*)
show *?case*
proof (*cases u* > *0*)
case *True*
have (∫ $^+$*t*. *ennreal* (*indicator* {*0*..*u*} *t* ∗ *t powr* (*a* − *1*) ∗ (*u* − *t*) *powr* (*b* − *1*)) ∂*lborel*) =
(∫ $^+$*t*. *ennreal* (*indicator* {*0*..*1*} *t* ∗ (*u* ∗ *t*) *powr* (*a* − *1*) ∗ (*u* − *u* ∗ *t*) *powr* (*b* − *1*))
∂*distr lborel borel* ((∗) (*1* / *u*))) (**is** _ = *nn_integral* _ *?f*)
using *True*
by (*subst nn_integral_distr*) (*auto simp*: *indicator_def field_simps intro*!: *nn_integral_cong*)
also have *distr lborel borel* ((∗) (*1* / *u*)) = *density lborel* (λ_. *u*)
using ‹*u* > *0*› **by** (*subst lborel_distr_mult*) *auto*
also have *nn_integral* ... *?f* = (∫ $^+$*x*. *ennreal* (*indicator* {*0*..*1*} *x* ∗ (*u* ∗ (*u* ∗ *x*) *powr* (*a* − *1*) ∗
(*u* ∗ (*1* − *x*)) *powr* (*b* − *1*))) ∂*lborel*) **using** ‹*u* > *0*›
by (*subst nn_integral_density*) (*auto simp*: *ennreal_mult′* [*symmetric*] *algebra_simps*)
also have ... = (∫ $^+$*x*. *ennreal* (*u powr* (*a* + *b* − *1*)) ∗
ennreal (*indicator* {*0*..*1*} *x* ∗ *x powr* (*a* − *1*) ∗
(*1* − *x*) *powr* (*b* − *1*)) ∂*lborel*) **using** ‹*u* > *0*› *a b*
by (*intro nn_integral_cong*)
(*auto simp*: *indicator_def powr_mult powr_add powr_diff mult_ac ennreal_mult′* [*symmetric*])
also have ... = *ennreal* (*u powr* (*a* + *b* − *1*)) ∗
(∫ $^+$*x*. *ennreal* (*indicator* {*0*..*1*} *x* ∗ *x powr* (*a* − *1*) ∗

```
                                    (1 − x) powr (b − 1)) ∂lborel)
        by (subst nn_integral_cmult) auto
      also have ((λx. x powr (a − 1) ∗ (1 − x) powr (b − 1)) has_integral
                     integral {0..1} (λx. x powr (a − 1) ∗ (1 − x) powr (b − 1)))
{0..1}
        using a b by (intro integrable_integral integrable_Beta')
      from nn_integral_has_integral_lebesgue[OF _ this] a b
        have (∫+x. ennreal (indicator {0..1} x ∗ x powr (a − 1) ∗
                        (1 − x) powr (b − 1)) ∂lborel) = B by (simp add: mult_ac
B_def)
      finally show ?thesis using ‹u > 0› by (simp add: ennreal_mult' [symmetric]
mult_ac)
    qed auto
  qed
  also have ... = ennreal B ∗ ennreal (Gamma (a + b))
   using a b by (subst nn_integral_cmult) (auto simp: Gamma_conv_nn_integral_real)
  also have ... = ennreal (B ∗ Gamma (a + b))
     by (subst (1 2) mult.commute, intro ennreal_mult' [symmetric]) (use a b in
auto)
  finally have B = Beta a b using a b Gamma_real_pos[of a + b]
   by (subst (asm) ennreal_inj) (auto simp: field_simps Beta_def Gamma_eq_zero_iff)
   moreover have (λt. t powr (a − 1) ∗ (1 − t) powr (b − 1)) integrable_on
{0..1}
    by (intro integrable_Beta' a b)
  ultimately show ?thesis by (simp add: has_integral_iff B_def)
qed
```

6.23.12 The Weierstraß product formula for the sine

```
theorem sin_product_formula_complex:
  fixes z :: complex
  shows (λn. of_real pi ∗ z ∗ (∏k=1..n. 1 − zˆ2 / of_nat kˆ2)) ⟶ sin (of_real
pi ∗ z)
proof −
  let ?f = rGamma_series_Weierstrass
  have (λn. (− of_real pi ∗ inverse z) ∗ (?f z n ∗ ?f (− z) n))
           ⟶ (− of_real pi ∗ inverse z) ∗ (rGamma z ∗ rGamma (− z))
    by (intro tendsto_intros rGamma_Weierstrass_complex)
  also have (λn. (− of_real pi ∗ inverse z) ∗ (?f z n ∗ ?f (−z) n)) =
                  (λn. of_real pi ∗ z ∗ (∏k=1..n. 1 − zˆ2 / of_nat k ˆ 2))
  proof
    fix n :: nat
    have (− of_real pi ∗ inverse z) ∗ (?f z n ∗ ?f (−z) n) =
            of_real pi ∗ z ∗ (∏k=1..n. (of_nat k − z) ∗ (of_nat k + z) / of_nat k
ˆ 2)
      by (simp add: rGamma_series_Weierstrass_def mult_ac exp_minus
                  divide_simps prod.distrib[symmetric] power2_eq_square)
    also have (∏k=1..n. (of_nat k − z) ∗ (of_nat k + z) / of_nat k ˆ 2) =
                (∏k=1..n. 1 − zˆ2 / of_nat k ˆ 2)
```

by (*intro prod.cong*) (*simp_all add*: *power2_eq_square field_simps*)
finally show (− *of_real pi* ∗ *inverse z*) ∗ (*?f z n* ∗ *?f* (−*z*) *n*) = *of_real pi* ∗ *z* ∗ . . .
by (*simp add*: *field_split_simps*)
qed
also have (− *of_real pi* ∗ *inverse z*) ∗ (*rGamma z* ∗ *rGamma* (− *z*)) = *sin* (*of_real pi* ∗ *z*)
by (*subst rGamma_reflection_complex′*) (*simp add*: *field_split_simps*)
finally show *?thesis* **.**
qed

lemma *sin_product_formula_real*:
(λ*n*. *pi* ∗ (*x*::*real*) ∗ (∏ *k*=*1*..*n*. *1* − *x*ˆ*2* / *of_nat k*ˆ*2*)) ⟶ *sin* (*pi* ∗ *x*)
proof −
from *sin_product_formula_complex*[*of of_real x*]
have (λ*n*. *of_real pi* ∗ *of_real x* ∗ (∏ *k*=*1*..*n*. *1* − (*of_real x*)ˆ*2* / (*of_nat k*)ˆ*2*)) ⟶ *sin* (*of_real pi* ∗ *of_real x* :: *complex*) (**is** *?f* ⟶ *?y*) **.**
also have *?f* = (λ*n*. *of_real* (*pi* ∗ *x* ∗ (∏ *k*=*1*..*n*. *1* − *x*ˆ*2* / (*of_nat k*ˆ*2*)))) **by** *simp*
also have *?y* = *of_real* (*sin* (*pi* ∗ *x*)) **by** (*simp only*: *sin_of_real* [*symmetric*] *of_real_mult*)
finally show *?thesis* **by** (*subst* (*asm*) *tendsto_of_real_iff*)
qed

lemma *sin_product_formula_real′*:
assumes *x* ≠ (*0*::*real*)
shows (λ*n*. (∏ *k*=*1*..*n*. *1* − *x*ˆ*2* / *of_nat k*ˆ*2*)) ⟶ *sin* (*pi* ∗ *x*) / (*pi* ∗ *x*)
using *tendsto_divide*[*OF sin_product_formula_real*[*of x*] *tendsto_const*[*of pi* ∗ *x*]] *assms*
by *simp*

theorem *wallis*: (λ*n*. ∏ *k*=*1*..*n*. (*4*∗*real k*ˆ*2*) / (*4*∗*real k*ˆ*2* − *1*)) ⟶ *pi* / *2*
proof −
from *tendsto_inverse*[*OF tendsto_mult*[*OF*
sin_product_formula_real[*of 1*/*2*] *tendsto_const*[*of 2*/*pi*]]]
have (λ*n*. (∏ *k*=*1*..*n*. *inverse* (*1* − (*1*/*2*)^2 / (*real k*)^2))) ⟶ *pi*/*2*
by (*simp add*: *prod_inversef* [*symmetric*])
also have (λ*n*. (∏ *k*=*1*..*n*. *inverse* (*1* − (*1*/*2*)^2 / (*real k*)^2))) =
(λ*n*. (∏ *k*=*1*..*n*. (*4*∗*real k*ˆ*2*)/(*4*∗*real k*ˆ*2* − *1*)))
by (*intro ext prod.cong refl*) (*simp add*: *field_split_simps*)
finally show *?thesis* **.**
qed

6.23.13 The Solution to the Basel problem

theorem *inverse_squares_sums*: (λ*n*. *1* / (*n* + *1*)^2) *sums* (*pi*^{2} / *6*)
proof −
define *P* **where** *P x n* = (∏ *k*=*1*..*n*. *1* − *x*ˆ*2* / *of_nat k*ˆ*2*) **for** *x* :: *real* **and** *n*
define *K* **where** *K* = (∑ *n*. *inverse* (*real_of_nat* (*Suc n*))ˆ*2*)

```
  define f where [abs_def]: f x = (∑n. P x n / of_nat (Suc n)^2) for x
  define g where [abs_def]: g x = (1 − sin (pi ∗ x) / (pi ∗ x)) for x

  have sums: (λn. P x n / of_nat (Suc n)^2) sums (if x = 0 then K else g x / x^2) for x
  proof (cases x = 0)
    assume x: x = 0
    have summable (λn. inverse ((real_of_nat (Suc n))²))
      using inverse_power_summable[of 2] by (subst summable_Suc_iff) simp
    thus ?thesis by (simp add: x g_def P_def K_def inverse_eq_divide power_divide summable_sums)
  next
    assume x: x ≠ 0
    have (λn. P x n − P x (Suc n)) sums (P x 0 − sin (pi ∗ x) / (pi ∗ x))
      unfolding P_def using x by (intro telescope_sums' sin_product_formula_real')
    also have (λn. P x n − P x (Suc n)) = (λn. (x^2 / of_nat (Suc n)^2) ∗ P x n)
      unfolding P_def by (simp add: prod.nat_ivl_Suc' algebra_simps)
    also have P x 0 = 1 by (simp add: P_def)
    finally have (λn. x² / (of_nat (Suc n))² ∗ P x n) sums (1 − sin (pi ∗ x) / (pi ∗ x)) .
    from sums_divide[OF this, of x^2] x show ?thesis unfolding g_def by simp
  qed

  have continuous_on (ball 0 1) f
  proof (rule uniform_limit_theorem; (intro always_eventually allI)?)
    show uniform_limit (ball 0 1) (λn x. ∑k<n. P x k / of_nat (Suc k)^2) f sequentially
    proof (unfold f_def, rule Weierstrass_m_test)
      fix n :: nat and x :: real assume x: x ∈ ball 0 1
      {
        fix k :: nat assume k: k ≥ 1
        from x have x^2 < 1 by (auto simp: abs_square_less_1)
        also from k have ... ≤ of_nat k^2 by simp
        finally have (1 − x^2 / of_nat k^2) ∈ {0..1} using k
          by (simp_all add: field_simps del: of_nat_Suc)
      }
      hence (∏k=1..n. abs (1 − x^2 / of_nat k^2)) ≤ (∏k=1..n. 1) by (intro prod_mono) simp
      thus norm (P x n / (of_nat (Suc n)^2)) ≤ 1 / of_nat (Suc n)^2
        unfolding P_def by (simp add: field_simps abs_prod del: of_nat_Suc)
    qed (subst summable_Suc_iff, insert inverse_power_summable[of 2], simp add: inverse_eq_divide)
  qed (auto simp: P_def intro!: continuous_intros)
  hence isCont f 0 by (subst (asm) continuous_on_eq_continuous_at) simp_all
  hence (f −0→ f 0) by (simp add: isCont_def)
  also have f 0 = K unfolding f_def P_def K_def by (simp add: inverse_eq_divide power_divide)
  finally have f −0→ K .
```

moreover have $f\ -\ 0 \to pi\hat{}2\ /\ 6$
proof (*rule Lim_transform_eventually*)
define f' **where** [*abs_def*]: $f'\ x = (\sum n.\ -\ sin\_coeff\ (n+3) * pi\ \hat{}\ (n+2) * x\hat{}n)$ **for** x
have *eventually* $(\lambda x.\ x \neq (0::real))$ (*at 0*)
by (*auto simp add*: *eventually_at intro*!: *exI*[*of _ 1*])
thus *eventually* $(\lambda x.\ f'\ x = f\ x)$ (*at 0*)
proof *eventually_elim*
fix x :: *real* **assume** x: $x \neq 0$
have *sin_coeff 1* = (*1* :: *real*) *sin_coeff 2* = (*0*::*real*) **by** (*simp_all add*: *sin_coeff_def*)
with *sums_split_initial_segment*[*OF sums_minus*[*OF sin_converges*], *of 3 pi*∗*x*]
have $(\lambda n.\ -\ (sin\_coeff\ (n+3) * (pi*x)\ \hat{}(n+3)))$ *sums* $(pi * x - sin\ (pi*x))$
by (*simp add*: *eval_nat_numeral*)
from *sums_divide*[*OF this*, *of x*ˆ*3* ∗ *pi*] x
have $(\lambda n.\ -\ (sin\_coeff\ (n+3) * pi\hat{}(n+2) * x\hat{}n))$ *sums* $((1\ -\ sin\ (pi*x)\ /\ (pi*x))\ /\ x\hat{}2)$
by (*simp add*: *field_split_simps eval_nat_numeral*)
with x **have** $(\lambda n.\ -\ (sin\_coeff\ (n+3) * pi\hat{}(n+2) * x\hat{}n))$ *sums* $(g\ x\ /\ x\hat{}2)$
by (*simp add*: *g_def*)
hence $f'\ x = g\ x\ /\ x\hat{}2$ **by** (*simp add*: *sums_iff f'_def*)
also have ... = $f\ x$ **using** *sums*[*of x*] x **by** (*simp add*: *sums_iff g_def f_def*)
finally show $f'\ x = f\ x$ **.**
qed

have *isCont* f' *0* **unfolding** *f'_def*
proof (*intro isCont_powser_converges_everywhere*)
fix x :: *real* **show** *summable* $(\lambda n.\ -sin\_coeff\ (n+3) * pi\hat{}(n+2) * x\hat{}n)$
proof (*cases* $x = 0$)
assume x: $x \neq 0$
from *summable_divide*[*OF sums_summable*[*OF sums_split_initial_segment*[*OF sin_converges*[*of pi*∗*x*]], *of 3*], *of* −*pi*∗*x*ˆ*3*] x
show *?thesis* **by** (*simp add*: *field_split_simps eval_nat_numeral*)
qed (*simp only*: *summable_0_powser*)
qed
hence $f'\ -\ 0 \to f'\ 0$ **by** (*simp add*: *isCont_def*)
also have $f'\ 0 = pi * pi\ /\ fact\ 3$ **unfolding** *f'_def*
by (*subst powser_zero*) (*simp add*: *sin_coeff_def*)
finally show $f'\ -\ 0 \to pi\hat{}2\ /\ 6$ **by** (*simp add*: *eval_nat_numeral*)
qed

ultimately have $K = pi\hat{}2\ /\ 6$ **by** (*rule LIM_unique*)
moreover from *inverse_power_summable*[*of 2*]
have *summable* $(\lambda n.\ (inverse\ (real\_of\_nat\ (Suc\ n)))^2)$
by (*subst summable_Suc_iff*) (*simp add*: *power_inverse*)
ultimately show *?thesis* **unfolding** *K_def*
by (*auto simp add*: *sums_iff power_divide inverse_eq_divide*)
qed

end

theory *Interval_Integral*
imports *Equivalence_Lebesgue_Henstock_Integration*
begin

definition *einterval a b* = {*x*. *a* < *ereal x* ∧ *ereal x* < *b*}

lemma *einterval_eq*[*simp*]:
shows *einterval_eq_Icc*: *einterval* (*ereal a*) (*ereal b*) = {*a* <..< *b*}
and *einterval_eq_Ici*: *einterval* (*ereal a*) ∞ = {*a* <..}
and *einterval_eq_Iic*: *einterval* (− ∞) (*ereal b*) = {..< *b*}
and *einterval_eq_UNIV*: *einterval* (− ∞) ∞ = *UNIV*
by (*auto simp*: *einterval_def*)

lemma *einterval_same*: *einterval a a* = {}
by (*auto simp*: *einterval_def*)

lemma *einterval_iff*: *x* ∈ *einterval a b* ⟷ *a* < *ereal x* ∧ *ereal x* < *b*
by (*simp add*: *einterval_def*)

lemma *einterval_nonempty*: *a* < *b* ⟹ ∃ *c*. *c* ∈ *einterval a b*
by (*cases a b rule*: *ereal2_cases*, *auto simp*: *einterval_def intro*!: *dense gt_ex lt_ex*)

lemma *open_einterval*[*simp*]: *open* (*einterval a b*)
by (*cases a b rule*: *ereal2_cases*)
(*auto simp*: *einterval_def intro*!: *open_Collect_conj open_Collect_less continuous_intros*)

lemma *borel_einterval*[*measurable*]: *einterval a b* ∈ *sets borel*
unfolding *einterval_def* **by** *measurable*

6.23.14 Approximating a (possibly infinite) interval

lemma *filterlim_sup1*: (*LIM x F*. *f x* :> *G1*) ⟹ (*LIM x F*. *f x* :> (*sup G1 G2*))
unfolding *filterlim_def* **by** (*auto intro*: *le_supI1*)

lemma *ereal_incseq_approx*:
fixes *a b* :: *ereal*
assumes *a* < *b*
obtains *X* :: *nat* ⇒ *real* **where** *incseq X* ⋀*i*. *a* < *X i* ⋀*i*. *X i* < *b X* ⟶ *b*
proof (*cases b*)
case *PInf*
with ‹*a* < *b*› **have** *a* = −∞ ∨ (∃ *r*. *a* = *ereal r*)
by (*cases a*) *auto*
moreover have (λ*x*. *ereal* (*real* (*Suc x*))) ⟶ ∞
by (*simp add*: *Lim_PInfty filterlim_sequentially_Suc*) (*metis le_SucI of_nat_Suc*

of_nat_mono order_trans real_arch_simple)
 moreover have $\bigwedge r.\ (\lambda x.\ ereal\ (r + real\ (Suc\ x))) \longrightarrow \infty$
 by (*simp add*: *filterlim_sequentially_Suc Lim_PInfty*) (*metis add.commute diff_le_eq nat_ceiling_le_eq*)
 ultimately show *thesis*
 by (*intro that*[*of* $\lambda i.$ *real_of_ereal a* + *Suc i*])
 (*auto simp*: *incseq_def PInf*)
next
 case (*real b'*)
 define *d* **where** $d = b' - (if\ a = -\infty\ then\ b' - 1\ else\ real\_of\_ereal\ a)$
 with ‹$a < b$› **have** *a'*: $0 < d$
 by (*cases a*) (*auto simp*: *real*)
 moreover
 have $\bigwedge i\ r.\ r < b' \Longrightarrow (b' - r) * 1 < (b' - r) * real\ (Suc\ (Suc\ i))$
 by (*intro mult_strict_left_mono*) *auto*
 with ‹$a < b$› *a'* **have** $\bigwedge i.\ a < ereal\ (b' - d\ /\ real\ (Suc\ (Suc\ i)))$
 by (*cases a*) (*auto simp*: *real d_def field_simps*)
 moreover
 have $(\lambda i.\ b' - d\ /\ real\ i) \longrightarrow b'$
 by (*force intro*: *tendsto_eq_intros tendsto_divide_0*[*OF tendsto_const*] *filterlim_sup1*
 simp: *at_infinity_eq_at_top_bot filterlim_real_sequentially*)
 then have $(\lambda i.\ b' - d\ /\ Suc\ (Suc\ i)) \longrightarrow b'$
 by (*blast intro*: *dest*: *filterlim_sequentially_Suc* [*THEN iffD2*])
 ultimately show *thesis*
 by (*intro that*[*of* $\lambda i.\ b' - d\ /\ Suc\ (Suc\ i)$])
 (*auto simp*: *real incseq_def intro*!: *divide_left_mono*)
qed (*insert* ‹$a < b$›, *auto*)

lemma *ereal_decseq_approx*:
 fixes *a b* :: *ereal*
 assumes $a < b$
 obtains *X* :: *nat* ⇒ *real* **where**
 decseq X $\bigwedge i.\ a < X\ i$ $\bigwedge i.\ X\ i < b$ $X \longrightarrow a$
proof −
 have $-b < -a$ **using** ‹$a < b$› **by** *simp*
 from *ereal_incseq_approx*[*OF this*] **guess** *X* **.**
 then show *thesis*
 apply (*intro that*[*of* $\lambda i.\ -\ X\ i$])
 apply (*auto simp*: *decseq_def incseq_def simp flip*: *uminus_ereal.simps*)
 apply (*metis ereal_minus_less_minus ereal_uminus_uminus ereal_Lim_uminus*)+
 done
qed

proposition *einterval_Icc_approximation*:
 fixes *a b* :: *ereal*
 assumes $a < b$
 obtains *u l* :: *nat* ⇒ *real* **where**
 einterval a b = $(\bigcup i.\ \{l\ i\ ..\ u\ i\})$

```
  incseq u decseq l ⋀i. l i < u i ⋀i. a < l i ⋀i. u i < b
  l ⟶ a u ⟶ b
proof –
  from dense[OF ‹a < b›] obtain c where a < c c < b by safe
  from ereal_incseq_approx[OF ‹c < b›] guess u . note u = this
  from ereal_decseq_approx[OF ‹a < c›] guess l . note l = this
  { fix i from less_trans[OF ‹l i < c› ‹c < u i›] have l i < u i by simp }
  have einterval a b = (⋃ i. {l i .. u i})
  proof (auto simp: einterval_iff)
    fix x assume a < ereal x ereal x < b
    have eventually (λi. ereal (l i) < ereal x) sequentially
      using l(4) ‹a < ereal x› by (rule order_tendstoD)
    moreover
    have eventually (λi. ereal x < ereal (u i)) sequentially
      using u(4) ‹ereal x< b› by (rule order_tendstoD)
    ultimately have eventually (λi. l i < x ∧ x < u i) sequentially
      by eventually_elim auto
    then show ∃ i. l i ≤ x ∧ x ≤ u i
      by (auto intro: less_imp_le simp: eventually_sequentially)
  next
    fix x i assume l i ≤ x x ≤ u i
    with ‹a < ereal (l i)› ‹ereal (u i) < b›
    show a < ereal x ereal x < b
      by (auto simp flip: ereal_less_eq(3))
  qed
  show thesis
    by (intro that) fact+
qed


definition interval_lebesgue_integral :: real measure ⇒ ereal ⇒ ereal ⇒ (real ⇒
'a) ⇒ 'a::{banach, second_countable_topology} where
  interval_lebesgue_integral M a b f =
    (if a ≤ b then (LINT x:einterval a b|M. f x) else – (LINT x:einterval b a|M.
f x))

syntax
  _ascii_interval_lebesgue_integral :: pttrn ⇒ real ⇒ real ⇒ real measure ⇒ real ⇒
real
  ((5LINT _=_.._|_. _) [0,60,60,61,100] 60)

translations
  LINT x=a..b|M. f == CONST interval_lebesgue_integral M a b (λx. f)

definition interval_lebesgue_integrable :: real measure ⇒ ereal ⇒ ereal ⇒ (real ⇒
'a::{banach, second_countable_topology}) ⇒ bool where
  interval_lebesgue_integrable M a b f =
    (if a ≤ b then set_integrable M (einterval a b) f else set_integrable M (einterval
b a) f)
```

syntax
_ascii_interval_lebesgue_borel_integral :: *pttrn* ⇒ *real* ⇒ *real* ⇒ *real* ⇒ *real*
((*4LBINT* _=_.._. _) [*0*,*60*,*60*,*61*] *60*)

translations
LBINT x=*a*..*b*. *f* == *CONST interval_lebesgue_integral CONST lborel a b* (λ*x*. *f*)

6.23.15 Basic properties of integration over an interval

lemma *interval_lebesgue_integral_cong*:
a ≤ *b* ⟹ (⋀*x*. *x* ∈ *einterval a b* ⟹ *f x* = *g x*) ⟹ *einterval a b* ∈ *sets M* ⟹
interval_lebesgue_integral M a b f = *interval_lebesgue_integral M a b g*
by (*auto intro*: *set_lebesgue_integral_cong simp*: *interval_lebesgue_integral_def*)

lemma *interval_lebesgue_integral_cong_AE*:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹
a ≤ *b* ⟹ *AE x* ∈ *einterval a b in M*. *f x* = *g x* ⟹ *einterval a b* ∈ *sets M* ⟹
interval_lebesgue_integral M a b f = *interval_lebesgue_integral M a b g*
by (*auto intro*: *set_lebesgue_integral_cong_AE simp*: *interval_lebesgue_integral_def*)

lemma *interval_integrable_mirror*:
shows *interval_lebesgue_integrable lborel a b* (λ*x*. *f* (−*x*)) ⟷
interval_lebesgue_integrable lborel (−*b*) (−*a*) *f*
proof −
have ∗: *indicator* (*einterval a b*) (− *x*) = (*indicator* (*einterval* (−*b*) (−*a*)) *x* :: *real*)
for *a b* :: *ereal* **and** *x* :: *real*
by (*cases a b rule*: *ereal2_cases*) (*auto simp*: *einterval_def split*: *split_indicator*)
show *?thesis*
unfolding *interval_lebesgue_integrable_def*
using *lborel_integrable_real_affine_iff* [*symmetric*, *of* −*1* λ*x*. *indicator* (*einterval* _ _) *x* $*_R$ *f x 0*]
by (*simp add*: ∗ *set_integrable_def*)
qed

lemma *interval_lebesgue_integral_add* [*intro*, *simp*]:
fixes *M a b f*
assumes *interval_lebesgue_integrable M a b f interval_lebesgue_integrable M a b g*
shows *interval_lebesgue_integrable M a b* (λ*x*. *f x* + *g x*) **and**
interval_lebesgue_integral M a b (λ*x*. *f x* + *g x*) =
interval_lebesgue_integral M a b f + *interval_lebesgue_integral M a b g*
using *assms* **by** (*auto simp*: *interval_lebesgue_integral_def interval_lebesgue_integrable_def field_simps*)

lemma *interval_lebesgue_integral_diff* [*intro*, *simp*]:
fixes *M a b f*

assumes *interval_lebesgue_integrable M a b f*
interval_lebesgue_integrable M a b g
shows *interval_lebesgue_integrable M a b* (*λx. f x − g x*) **and**
interval_lebesgue_integral M a b (*λx. f x − g x*) =
interval_lebesgue_integral M a b f − interval_lebesgue_integral M a b g
using *assms* **by** (*auto simp*: *interval_lebesgue_integral_def interval_lebesgue_integrable_def field_simps*)

lemma *interval_lebesgue_integrable_mult_right* [*intro*, *simp*]:
fixes *M a b c* **and** *f* :: *real* ⇒ *'a*::{*banach*, *real_normed_field*, *second_countable_topology*}
shows (*c* ≠ *0* ⟹ *interval_lebesgue_integrable M a b f*) ⟹
interval_lebesgue_integrable M a b (*λx. c ∗ f x*)
by (*simp add*: *interval_lebesgue_integrable_def*)

lemma *interval_lebesgue_integrable_mult_left* [*intro*, *simp*]:
fixes *M a b c* **and** *f* :: *real* ⇒ *'a*::{*banach*, *real_normed_field*, *second_countable_topology*}
shows (*c* ≠ *0* ⟹ *interval_lebesgue_integrable M a b f*) ⟹
interval_lebesgue_integrable M a b (*λx. f x ∗ c*)
by (*simp add*: *interval_lebesgue_integrable_def*)

lemma *interval_lebesgue_integrable_divide* [*intro*, *simp*]:
fixes *M a b c* **and** *f* :: *real* ⇒ *'a*::{*banach*, *real_normed_field*, *field*, *second_countable_topology*}
shows (*c* ≠ *0* ⟹ *interval_lebesgue_integrable M a b f*) ⟹
interval_lebesgue_integrable M a b (*λx. f x / c*)
by (*simp add*: *interval_lebesgue_integrable_def*)

lemma *interval_lebesgue_integral_mult_right* [*simp*]:
fixes *M a b c* **and** *f* :: *real* ⇒ *'a*::{*banach*, *real_normed_field*, *second_countable_topology*}
shows *interval_lebesgue_integral M a b* (*λx. c ∗ f x*) =
c ∗ interval_lebesgue_integral M a b f
by (*simp add*: *interval_lebesgue_integral_def*)

lemma *interval_lebesgue_integral_mult_left* [*simp*]:
fixes *M a b c* **and** *f* :: *real* ⇒ *'a*::{*banach*, *real_normed_field*, *second_countable_topology*}
shows *interval_lebesgue_integral M a b* (*λx. f x ∗ c*) =
interval_lebesgue_integral M a b f ∗ c
by (*simp add*: *interval_lebesgue_integral_def*)

lemma *interval_lebesgue_integral_divide* [*simp*]:
fixes *M a b c* **and** *f* :: *real* ⇒ *'a*::{*banach*, *real_normed_field*, *field*, *second_countable_topology*}
shows *interval_lebesgue_integral M a b* (*λx. f x / c*) =
interval_lebesgue_integral M a b f / c
by (*simp add*: *interval_lebesgue_integral_def*)

lemma *interval_lebesgue_integral_uminus*:
interval_lebesgue_integral M a b (*λx. − f x*) = − *interval_lebesgue_integral M a b f*
by (*auto simp*: *interval_lebesgue_integral_def interval_lebesgue_integrable_def set_lebesgue_integral_def*)

lemma *interval_lebesgue_integral_of_real*:
interval_lebesgue_integral M a b (λ*x*. *complex_of_real* (*f x*)) =
of_real (*interval_lebesgue_integral M a b f*)
unfolding *interval_lebesgue_integral_def*
by (*auto simp*: *interval_lebesgue_integral_def set_integral_complex_of_real*)

lemma *interval_lebesgue_integral_le_eq*:
fixes *a b f*
assumes $a \le b$
shows *interval_lebesgue_integral M a b f* = (*LINT x* : *einterval a b* | *M*. *f x*)
using *assms* **by** (*auto simp*: *interval_lebesgue_integral_def*)

lemma *interval_lebesgue_integral_gt_eq*:
fixes *a b f*
assumes $a > b$
shows *interval_lebesgue_integral M a b f* = −(*LINT x* : *einterval b a* | *M*. *f x*)
using *assms* **by** (*auto simp*: *interval_lebesgue_integral_def less_imp_le einterval_def*)

lemma *interval_lebesgue_integral_gt_eq′*:
fixes *a b f*
assumes $a > b$
shows *interval_lebesgue_integral M a b f* = − *interval_lebesgue_integral M b a f*
using *assms* **by** (*auto simp*: *interval_lebesgue_integral_def less_imp_le einterval_def*)

lemma *interval_integral_endpoints_same* [*simp*]: (*LBINT x*=*a*..*a*. *f x*) = *0*
by (*simp add*: *interval_lebesgue_integral_def set_lebesgue_integral_def einterval_same*)

lemma *interval_integral_endpoints_reverse*: (*LBINT x*=*a*..*b*. *f x*) = −(*LBINT x*=*b*..*a*. *f x*)
by (*cases a b rule*: *linorder_cases*) (*auto simp*: *interval_lebesgue_integral_def set_lebesgue_integral_def einterval_same*)

lemma *interval_integrable_endpoints_reverse*:
interval_lebesgue_integrable lborel a b f ⟷
interval_lebesgue_integrable lborel b a f
by (*cases a b rule*: *linorder_cases*) (*auto simp*: *interval_lebesgue_integrable_def einterval_same*)

lemma *interval_integral_reflect*:
(*LBINT x*=*a*..*b*. *f x*) = (*LBINT x*=−*b*..−*a*. *f* (−*x*))
proof (*induct a b rule*: *linorder_wlog*)
case (*sym a b*) **then show** *?case*
by (*auto simp*: *interval_lebesgue_integral_def interval_integrable_endpoints_reverse*
split: *if_split_asm*)
next
case (*le a b*)
have *LBINT x*:{*x*. − *x* ∈ *einterval a b*}. *f* (− *x*) = *LBINT x*:*einterval* (− *b*) (− *a*). *f* (− *x*)
unfolding *interval_lebesgue_integrable_def set_lebesgue_integral_def*

apply (*rule Bochner_Integration.integral_cong* [*OF refl*])
by (*auto simp*: *einterval_iff ereal_uminus_le_reorder ereal_uminus_less_reorder not_less*
simp flip: *uminus_ereal.simps*
split: *split_indicator*)
then show *?case*
unfolding *interval_lebesgue_integral_def*
by (*subst set_integral_reflect*) (*simp add*: *le*)
qed

lemma *interval_lebesgue_integral_0_infty*:
interval_lebesgue_integrable M 0 ∞ *f* $\longleftrightarrow$ *set_integrable M* {*0*<..} *f*
interval_lebesgue_integral M 0 ∞ *f* = (*LINT x*:{*0*<..}|*M*. *f x*)
unfolding *zero_ereal_def*
by (*auto simp*: *interval_lebesgue_integral_le_eq interval_lebesgue_integrable_def*)

lemma *interval_integral_to_infinity_eq*: (*LINT x*=*ereal a*..∞ | *M*. *f x*) = (*LINT x* : {*a*<..} | *M*. *f x*)
unfolding *interval_lebesgue_integral_def* **by** *auto*

proposition *interval_integrable_to_infinity_eq*: (*interval_lebesgue_integrable M a* ∞ *f*) =
(*set_integrable M* {*a*<..} *f*)
unfolding *interval_lebesgue_integrable_def* **by** *auto*

6.23.16 Basic properties of integration over an interval wrt lebesgue measure

lemma *interval_integral_zero* [*simp*]:
fixes *a b* :: *ereal*
shows *LBINT x*=*a*..*b*. *0* = *0*
unfolding *interval_lebesgue_integral_def set_lebesgue_integral_def einterval_eq*
by *simp*

lemma *interval_integral_const* [*intro*, *simp*]:
fixes *a b c* :: *real*
shows *interval_lebesgue_integrable lborel a b* (λx. *c*) **and** *LBINT x*=*a*..*b*. *c* = *c* $*$ (*b* − *a*)
unfolding *interval_lebesgue_integral_def interval_lebesgue_integrable_def einterval_eq*
by (*auto simp*: *less_imp_le field_simps measure_def set_integrable_def set_lebesgue_integral_def*)

lemma *interval_integral_cong_AE*:
assumes [*measurable*]: *f* $\in$ *borel_measurable borel g* $\in$ *borel_measurable borel*
assumes *AE x* $\in$ *einterval* (*min a b*) (*max a b*) *in lborel*. *f x* = *g x*
shows *interval_lebesgue_integral lborel a b f* = *interval_lebesgue_integral lborel a b g*
using *assms*
proof (*induct a b rule*: *linorder_wlog*)
case (*sym a b*) **then show** *?case*

by (*simp add*: *min.commute max.commute interval_integral_endpoints_reverse*[*of a b*])
next
case (*le a b*) **then show** *?case*
by (*auto simp*: *interval_lebesgue_integral_def max_def min_def*
intro!: *set_lebesgue_integral_cong_AE*)
qed

lemma *interval_integral_cong*:
assumes $\bigwedge x.\ x \in$ *einterval* (*min a b*) (*max a b*) $\Longrightarrow f\ x = g\ x$
shows *interval_lebesgue_integral lborel a b f* = *interval_lebesgue_integral lborel a b g*
using *assms*
proof (*induct a b rule*: *linorder_wlog*)
case (*sym a b*) **then show** *?case*
by (*simp add*: *min.commute max.commute interval_integral_endpoints_reverse*[*of a b*])
next
case (*le a b*) **then show** *?case*
by (*auto simp*: *interval_lebesgue_integral_def max_def min_def*
intro!: *set_lebesgue_integral_cong*)
qed

lemma *interval_lebesgue_integrable_cong_AE*:
$f \in$ *borel_measurable lborel* $\Longrightarrow g \in$ *borel_measurable lborel* $\Longrightarrow$
AE x $\in$ *einterval* (*min a b*) (*max a b*) *in lborel*. $f\ x = g\ x \Longrightarrow$
interval_lebesgue_integrable lborel a b f = *interval_lebesgue_integrable lborel a b g*
apply (*simp add*: *interval_lebesgue_integrable_def*)
apply (*intro conjI impI set_integrable_cong_AE*)
apply (*auto simp*: *min_def max_def*)
done

lemma *interval_integrable_abs_iff*:
fixes f :: *real* $\Rightarrow$ *real*
shows $f \in$ *borel_measurable lborel* $\Longrightarrow$
interval_lebesgue_integrable lborel a b ($\lambda x.\ |f\ x|$) = *interval_lebesgue_integrable lborel a b f*
unfolding *interval_lebesgue_integrable_def*
by (*subst* (*1 2*) *set_integrable_abs_iff'*) *simp_all*

lemma *interval_integral_Icc*:
fixes *a b* :: *real*
shows $a \le b \Longrightarrow$ (*LBINT x=a..b. f x*) = (*LBINT x* : {*a..b*}. *f x*)
by (*auto intro*!: *set_integral_discrete_difference*[**where** *X*={*a*, *b*}]
simp add: *interval_lebesgue_integral_def*)

lemma *interval_integral_Icc'*:
$a \le b \Longrightarrow$ (*LBINT x=a..b. f x*) = (*LBINT x* : {*x*. $a \le$ *ereal x* $\wedge$ *ereal x* $\le b$}. *f x*)

by (*auto intro*!: *set_integral_discrete_difference*[**where** X={*real_of_ereal a*, *real_of_ereal b*}]
simp add: *interval_lebesgue_integral_def einterval_iff*)

lemma *interval_integral_Ioc*:
$a \leq b \Longrightarrow$ (*LBINT x*=*a*..*b*. *f x*) = (*LBINT x* : {*a*<..*b*}. *f x*)
by (*auto intro*!: *set_integral_discrete_difference*[**where** X={*a*, *b*}]
simp add: *interval_lebesgue_integral_def einterval_iff*)

lemma *interval_integral_Ioc'*:
$a \leq b \Longrightarrow$ (*LBINT x*=*a*..*b*. *f x*) = (*LBINT x* : {*x*. $a <$ *ereal x* $\wedge$ *ereal x* $\leq b$}. *f x*)
by (*auto intro*!: *set_integral_discrete_difference*[**where** X={*real_of_ereal a*, *real_of_ereal b*}]
simp add: *interval_lebesgue_integral_def einterval_iff*)

lemma *interval_integral_Ico*:
$a \leq b \Longrightarrow$ (*LBINT x*=*a*..*b*. *f x*) = (*LBINT x* : {*a*..<*b*}. *f x*)
by (*auto intro*!: *set_integral_discrete_difference*[**where** X={*a*, *b*}]
simp add: *interval_lebesgue_integral_def einterval_iff*)

lemma *interval_integral_Ioi*:
$|a| < \infty \Longrightarrow$ (*LBINT x*=*a*..∞. *f x*) = (*LBINT x* : {*real_of_ereal a* <..}. *f x*)
by (*auto simp*: *interval_lebesgue_integral_def einterval_iff*)

lemma *interval_integral_Ioo*:
$a \leq b \Longrightarrow |a| < \infty$ ==> $|b| < \infty \Longrightarrow$ (*LBINT x*=*a*..*b*. *f x*) = (*LBINT x* : {*real_of_ereal a* <..< *real_of_ereal b*}. *f x*)
by (*auto simp*: *interval_lebesgue_integral_def einterval_iff*)

lemma *interval_integral_discrete_difference*:
fixes *f* :: *real* ⇒ *'b*::{*banach*, *second_countable_topology*} **and** *a b* :: *ereal*
assumes *countable X*
and *eq*: $\bigwedge x.\ a \leq b \Longrightarrow a < x \Longrightarrow x < b \Longrightarrow x \notin X \Longrightarrow f\ x = g\ x$
and *anti_eq*: $\bigwedge x.\ b \leq a \Longrightarrow b < x \Longrightarrow x < a \Longrightarrow x \notin X \Longrightarrow f\ x = g\ x$
assumes $\bigwedge x.\ x \in X \Longrightarrow$ *emeasure M* $\{x\} = 0$ $\bigwedge x.\ x \in X \Longrightarrow \{x\} \in$ *sets M*
shows *interval_lebesgue_integral M a b f* = *interval_lebesgue_integral M a b g*
unfolding *interval_lebesgue_integral_def set_lebesgue_integral_def*
apply (*intro if_cong refl arg_cong*[**where** *f*=λ*x*. − *x*] *integral_discrete_difference*[*of X*] *assms*)
apply (*auto simp*: *eq anti_eq einterval_iff split*: *split_indicator*)
done

lemma *interval_integral_sum*:
fixes *a b c* :: *ereal*
assumes *integrable*: *interval_lebesgue_integrable lborel* (*min a* (*min b c*)) (*max a* (*max b c*)) *f*
shows (*LBINT x*=*a*..*b*. *f x*) + (*LBINT x*=*b*..*c*. *f x*) = (*LBINT x*=*a*..*c*. *f x*)

proof −
  let *?I* = λ*a b*. *LBINT x*=*a*..*b*. *f x*
  { fix *a b c* :: *ereal* **assume** *interval_lebesgue_integrable lborel a c f a* ≤ *b b* ≤ *c*
    then have *ord*: *a* ≤ *b b* ≤ *c a* ≤ *c* **and** *f'*: *set_integrable lborel* (*einterval a c*) *f*
      by (*auto simp*: *interval_lebesgue_integrable_def*)
    then have *f*: *set_borel_measurable borel* (*einterval a c*) *f*
      unfolding *set_integrable_def set_borel_measurable_def*
      by (*drule_tac borel_measurable_integrable*) *simp*
    have (*LBINT x*:*einterval a c*. *f x*) = (*LBINT x*:*einterval a b* ∪ *einterval b c*. *f x*)
    proof (*rule set_integral_cong_set*)
      show *AE x in lborel*. (*x* ∈ *einterval a b* ∪ *einterval b c*) = (*x* ∈ *einterval a c*)
        using *AE_lborel_singleton*[*of real_of_ereal b*] *ord*
        by (*cases a b c rule*: *ereal3_cases*) (*auto simp*: *einterval_iff*)
      show *set_borel_measurable lborel* (*einterval a c*) *f set_borel_measurable lborel* (*einterval a b* ∪ *einterval b c*) *f*
        unfolding *set_borel_measurable_def*
        using *ord* **by** (*auto simp*: *einterval_iff intro*!: *set_borel_measurable_subset*[*OF f*, *unfolded set_borel_measurable_def*])
    qed
    also have ... = (*LBINT x*:*einterval a b*. *f x*) + (*LBINT x*:*einterval b c*. *f x*)
      using *ord*
      by (*intro set_integral_Un_AE*) (*auto intro*!: *set_integrable_subset*[*OF f'*] *simp*: *einterval_iff not_less*)
    finally have *?I a b* + *?I b c* = *?I a c*
      using *ord* **by** (*simp add*: *interval_lebesgue_integral_def*)
  } note *1* = *this*
  { fix *a b c* :: *ereal* **assume** *interval_lebesgue_integrable lborel a c f a* ≤ *b b* ≤ *c*
    from *1*[*OF this*] **have** *?I b c* + *?I a b* = *?I a c*
      by (*metis add.commute*)
  } note *2* = *this*
  have *3*: ⋀*a b*. *b* ≤ *a* ⟹ (*LBINT x*=*a*..*b*. *f x*) = − (*LBINT x*=*b*..*a*. *f x*)
    by (*rule interval_integral_endpoints_reverse*)
  show *?thesis*
    using *integrable*
      by (*cases a b b c a c rule*: *linorder_le_cases*[*case_product linorder_le_cases linorder_cases*])
      (*simp_all add*: *min_absorb1 min_absorb2 max_absorb1 max_absorb2 field_simps 1 2 3*)
qed

lemma *interval_integrable_isCont*:
  fixes *a b* **and** *f* :: *real* ⇒ *'a*::{*banach*, *second_countable_topology*}
  shows (⋀*x*. *min a b* ≤ *x* ⟹ *x* ≤ *max a b* ⟹ *isCont f x*) ⟹
    interval_lebesgue_integrable lborel a b f
proof (*induct a b rule*: *linorder_wlog*)
  case (*le a b*) **then show** *?case*
    unfolding *interval_lebesgue_integrable_def set_integrable_def*

by (*auto simp*: *interval_lebesgue_integrable_def*
intro!: *set_integrable_subset*[*unfolded set_integrable_def*, *OF borel_integrable_compact*[*of* {*a* .. *b*}]]
continuous_at_imp_continuous_on)
qed (*auto intro*: *interval_integrable_endpoints_reverse*[*THEN iffD1*])

lemma *interval_integrable_continuous_on*:
fixes *a b* :: *real* **and** *f*
assumes $a \leq b$ **and** *continuous_on* {*a*..*b*} *f*
shows *interval_lebesgue_integrable lborel a b f*
using *assms* **unfolding** *interval_lebesgue_integrable_def* **apply** *simp*
by (*rule set_integrable_subset*, *rule borel_integrable_atLeastAtMost'* [*of a b*], *auto*)

lemma *interval_integral_eq_integral*:
fixes *f* :: *real* ⇒ *'a*::*euclidean_space*
shows $a \leq b$ ⟹ *set_integrable lborel* {*a*..*b*} *f* ⟹ *LBINT x*=*a*..*b*. *f x* = *integral* {*a*..*b*} *f*
by (*subst interval_integral_Icc*, *simp*) (*rule set_borel_integral_eq_integral*)

lemma *interval_integral_eq_integral'*:
fixes *f* :: *real* ⇒ *'a*::*euclidean_space*
shows $a \leq b$ ⟹ *set_integrable lborel* (*einterval a b*) *f* ⟹ *LBINT x*=*a*..*b*. *f x* = *integral* (*einterval a b*) *f*
by (*subst interval_lebesgue_integral_le_eq*, *simp*) (*rule set_borel_integral_eq_integral*)

6.23.17 General limit approximation arguments

proposition *interval_integral_Icc_approx_nonneg*:
fixes *a b* :: *ereal*
assumes $a < b$
fixes *u l* :: *nat* ⇒ *real*
assumes *approx*: *einterval a b* = (⋃*i*. {*l i* .. *u i*})
incseq u decseq l ⋀*i*. *l i* < *u i* ⋀*i*. *a* < *l i* ⋀*i*. *u i* < *b*
l ⟶ *a u* ⟶ *b*
fixes *f* :: *real* ⇒ *real*
assumes *f_integrable*: ⋀*i*. *set_integrable lborel* {*l i*..*u i*} *f*
assumes *f_nonneg*: *AE x in lborel*. *a* < *ereal x* ⟶ *ereal x* < *b* ⟶ *0* ≤ *f x*
assumes *f_measurable*: *set_borel_measurable lborel* (*einterval a b*) *f*
assumes *lbint_lim*: (λ*i*. *LBINT x*=*l i*.. *u i*. *f x*) ⟶ *C*
shows
set_integrable lborel (*einterval a b*) *f*
(*LBINT x*=*a*..*b*. *f x*) = *C*
proof −
have *1* [*unfolded set_integrable_def*]: ⋀*i*. *set_integrable lborel* {*l i*..*u i*} *f* **by** (*rule f_integrable*)
have *2*: *AE x in lborel*. *mono* (λ*n*. *indicator* {*l n*..*u n*} *x* $*_R$ *f x*)
proof −
from *f_nonneg* **have** *AE x in lborel*. ∀*i*. *l i* ≤ *x* ⟶ *x* ≤ *u i* ⟶ *0* ≤ *f x*
by *eventually_elim*

(*metis approx*(*5*) *approx*(*6*) *dual_order.strict_trans1 ereal_less_eq*(*3*) *le_less_trans*)
then show *?thesis*
apply *eventually_elim*
apply (*auto simp*: *mono_def split*: *split_indicator*)
apply (*metis approx*(*3*) *decseqD order_trans*)
apply (*metis approx*(*2*) *incseqD order_trans*)
done
qed
have *3*: *AE x in lborel*. (λ*i*. *indicator* {*l i*..*u i*} *x* $*_R$ *f x*) $\longrightarrow$ *indicator* (*einterval a b*) *x* $*_R$ *f x*
proof −
{ **fix** *x i* **assume** *l i* ≤ *x x* ≤ *u i*
then have *eventually* (λ*i*. *l i* ≤ *x* ∧ *x* ≤ *u i*) *sequentially*
apply (*auto simp*: *eventually_sequentially intro*!: *exI*[*of _ i*])
apply (*metis approx*(*3*) *decseqD order_trans*)
apply (*metis approx*(*2*) *incseqD order_trans*)
done
then have *eventually* (λ*i*. *f x* ∗ *indicator* {*l i*..*u i*} *x* = *f x*) *sequentially*
by *eventually_elim auto* }
then show *?thesis*
unfolding *approx*(*1*) **by** (*auto intro*!: *AE_I2 tendsto_eventually split*: *split_indicator*)
qed
have *4*: (λ*i*. ∫ *x*. *indicator* {*l i*..*u i*} *x* $*_R$ *f x* ∂*lborel*) $\longrightarrow$ *C*
using *lbint_lim* **by** (*simp add*: *interval_integral_Icc* [*unfolded set_lebesgue_integral_def*] *approx less_imp_le*)
have *5*: (λ*x*. *indicat_real* (*einterval a b*) *x* $*_R$ *f x*) ∈ *borel_measurable lborel*
using *f_measurable set_borel_measurable_def* **by** *blast*
have (*LBINT x*=*a*..*b*. *f x*) = *lebesgue_integral lborel* (λ*x*. *indicator* (*einterval a b*) *x* $*_R$ *f x*)
using *assms* **by** (*simp add*: *interval_lebesgue_integral_def set_lebesgue_integral_def less_imp_le*)
also have . . . = *C*
by (*rule integral_monotone_convergence* [*OF 1 2 3 4 5*])
finally show (*LBINT x*=*a*..*b*. *f x*) = *C* **.**
show *set_integrable lborel* (*einterval a b*) *f*
unfolding *set_integrable_def*
by (*rule integrable_monotone_convergence*[*OF 1 2 3 4 5*])
qed

proposition *interval_integral_Icc_approx_integrable*:
fixes *u l* :: *nat* ⇒ *real* **and** *a b* :: *ereal*
fixes *f* :: *real* ⇒ ′*a*::{*banach*, *second_countable_topology*}
assumes *a* < *b*
assumes *approx*: *einterval a b* = (⋃ *i*. {*l i* .. *u i*})
incseq u decseq l ⋀*i*. *l i* < *u i* ⋀*i*. *a* < *l i* ⋀*i*. *u i* < *b*
l $\longrightarrow$ *a u* $\longrightarrow$ *b*
assumes *f_integrable*: *set_integrable lborel* (*einterval a b*) *f*
shows (λ*i*. *LBINT x*=*l i*.. *u i*. *f x*) $\longrightarrow$ (*LBINT x*=*a*..*b*. *f x*)
proof −

have (λ*i*. *LBINT x*:{*l i*.. *u i*}. *f x*) ⟶ (*LBINT x*:*einterval a b*. *f x*)
  unfolding *set_lebesgue_integral_def*
proof (*rule integral_dominated_convergence*)
  show *integrable lborel* (λ*x*. *norm* (*indicator* (*einterval a b*) *x* $*_R$ *f x*))
    using *f_integrable integrable_norm set_integrable_def* **by** *blast*
  show (λ*x*. *indicat_real* (*einterval a b*) *x* $*_R$ *f x*) ∈ *borel_measurable lborel*
    using *f_integrable* **by** (*simp add*: *set_integrable_def*)
  then show ⋀*i*. (λ*x*. *indicat_real* {*l i*..*u i*} *x* $*_R$ *f x*) ∈ *borel_measurable lborel*
    by (*rule set_borel_measurable_subset* [*unfolded set_borel_measurable_def*]) (*auto simp*: *approx*)
  show ⋀*i*. *AE x in lborel*. *norm* (*indicator* {*l i*..*u i*} *x* $*_R$ *f x*) ≤ *norm* (*indicator* (*einterval a b*) *x* $*_R$ *f x*)
    by (*intro AE_I2*) (*auto simp*: *approx split*: *split_indicator*)
    show *AE x in lborel*. (λ*i*. *indicator* {*l i*..*u i*} *x* $*_R$ *f x*) ⟶ *indicator* (*einterval a b*) *x* $*_R$ *f x*
  proof (*intro AE_I2 tendsto_intros tendsto_eventually*)
    fix *x*
    { **fix** *i* **assume** *l i* ≤ *x x* ≤ *u i*
      with ‹*incseq u*›[*THEN incseqD*, *of i*] ‹*decseq l*›[*THEN decseqD*, *of i*]
      have *eventually* (λ*i*. *l i* ≤ *x* ∧ *x* ≤ *u i*) *sequentially*
        by (*auto simp*: *eventually_sequentially decseq_def incseq_def intro*: *order_trans*) }
    then show *eventually* (λ*xa*. *indicator* {*l xa*..*u xa*} *x* = (*indicator* (*einterval a b*) *x*::*real*)) *sequentially*
    using *approx order_tendstoD*(*2*)[*OF* ‹*l* ⟶ *a*›, *of x*] *order_tendstoD*(*1*)[*OF* ‹*u* ⟶ *b*›, *of x*]
      by (*auto split*: *split_indicator*)
  qed
qed
with ‹*a* < *b*› ‹⋀*i*. *l i* < *u i*› **show** *?thesis*
  by (*simp add*: *interval_lebesgue_integral_le_eq*[*symmetric*] *interval_integral_Icc less_imp_le*)
qed

6.23.18 A slightly stronger Fundamental Theorem of Calculus

Three versions: first, for finite intervals, and then two versions for arbitrary intervals.

lemma *interval_integral_FTC_finite*:
  fixes *f F* :: *real* ⇒ ′*a*::*euclidean_space* **and** *a b* :: *real*
  assumes *f*: *continuous_on* {*min a b*..*max a b*} *f*
  assumes *F*: ⋀*x*. *min a b* ≤ *x* ⟹ *x* ≤ *max a b* ⟹ (*F has_vector_derivative* (*f x*)) (*at x within* {*min a b*..*max a b*})
  shows (*LBINT x*=*a*..*b*. *f x*) = *F b* − *F a*
proof (*cases a* ≤ *b*)
  case *True*

have (*LBINT x=a..b. f x*) = (*LBINT x. indicat_real* {*a..b*} *x* $*_R$ *f x*)
by (*simp add*: *True interval_integral_Icc set_lebesgue_integral_def*)
also have ... = *F b* − *F a*
proof (*rule integral_FTC_atLeastAtMost* [*OF True*])
show *continuous_on* {*a..b*} *f*
using *True f* **by** *linarith*
show $\bigwedge$*x*. ⟦*a* ≤ *x*; *x* ≤ *b*⟧ ⟹ (*F has_vector_derivative f x*) (*at x within* {*a..b*})
by (*metis F True max.commute max_absorb1 min_def*)
qed
finally show *?thesis* **.**
next
case *False*
then have *b* ≤ *a*
by *simp*
have − *interval_lebesgue_integral lborel* (*ereal b*) (*ereal a*) *f* = − (*LBINT x. indicat_real* {*b..a*} *x* $*_R$ *f x*)
by (*simp add*: ‹*b* ≤ *a*› *interval_integral_Icc set_lebesgue_integral_def*)
also have ... = *F b* − *F a*
proof (*subst integral_FTC_atLeastAtMost* [*OF* ‹*b* ≤ *a*›])
show *continuous_on* {*b..a*} *f*
using *False f* **by** *linarith*
show $\bigwedge$*x*. ⟦*b* ≤ *x*; *x* ≤ *a*⟧
⟹ (*F has_vector_derivative f x*) (*at x within* {*b..a*})
by (*metis F False max_def min_def*)
qed *auto*
finally show *?thesis*
by (*metis interval_integral_endpoints_reverse*)
qed

lemma *interval_integral_FTC_nonneg*:
fixes *f F* :: *real* ⇒ *real* **and** *a b* :: *ereal*
assumes *a* < *b*
assumes *F*: $\bigwedge$*x*. *a* < *ereal x* ⟹ *ereal x* < *b* ⟹ *DERIV F x* :> *f x*
assumes *f*: $\bigwedge$*x*. *a* < *ereal x* ⟹ *ereal x* < *b* ⟹ *isCont f x*
assumes *f_nonneg*: *AE x in lborel*. *a* < *ereal x* ⟶ *ereal x* < *b* ⟶ *0* ≤ *f x*
assumes *A*: ((*F* ∘ *real_of_ereal*) ⟶ *A*) (*at_right a*)
assumes *B*: ((*F* ∘ *real_of_ereal*) ⟶ *B*) (*at_left b*)
shows
set_integrable lborel (*einterval a b*) *f*
(*LBINT x=a..b. f x*) = *B* − *A*
proof −
obtain *u l* **where** *approx*:
einterval a b = ($\bigcup$*i*. {*l i* .. *u i*})
incseq u decseq l $\bigwedge$*i*. *l i* < *u i* $\bigwedge$*i*. *a* < *l i* $\bigwedge$*i*. *u i* < *b*
l ⟶ *a u* ⟶ *b*
by (*blast intro*: *einterval_Icc_approximation*[*OF* ‹*a* < *b*›])
have [*simp*]: $\bigwedge$*x i*. *l i* ≤ *x* ⟹ *a* < *ereal x*
by (*rule order_less_le_trans*, *rule approx*, *force*)

have [*simp*]: ⋀*x i*. *x* ≤ *u i* ⟹ *ereal x* < *b*
  by (*rule order_le_less_trans*, *subst ereal_less_eq*(*3*), *assumption*, *rule approx*)
have *FTCi*: ⋀*i*. (*LBINT x*=*l i*..*u i*. *f x*) = *F* (*u i*) − *F* (*l i*)
  using *assms approx* **apply** (*intro interval_integral_FTC_finite*)
  apply (*auto simp*: *less_imp_le min_def max_def*
    has_field_derivative_iff_has_vector_derivative[*symmetric*])
  apply (*rule continuous_at_imp_continuous_on*, *auto intro*!: *f*)
  by (*rule DERIV_subset* [*OF F*], *auto*)
have *1*: ⋀*i*. *set_integrable lborel* {*l i*..*u i*} *f*
proof −
  fix *i* **show** *set_integrable lborel* {*l i* .. *u i*} *f*
    using ‹*a* < *l i*› ‹*u i* < *b*› **unfolding** *set_integrable_def*
      by (*intro borel_integrable_compact f continuous_at_imp_continuous_on compact_Icc ballI*)
        (*auto simp flip*: *ereal_less_eq*)
qed
have *2*: *set_borel_measurable lborel* (*einterval a b*) *f*
  unfolding *set_borel_measurable_def*
  by (*auto simp del*: *real_scaleR_def intro*!: *borel_measurable_continuous_on_indicator*
    simp: *continuous_on_eq_continuous_at einterval_iff f*)
have *3*: (λ*i*. *LBINT x*=*l i*..*u i*. *f x*) ⟶ *B* − *A*
  apply (*subst FTCi*)
  apply (*intro tendsto_intros*)
  using *B approx* **unfolding** *tendsto_at_iff_sequentially comp_def*
  using *tendsto_at_iff_sequentially*[**where** ′*a*=*real*]
  apply (*elim allE*[*of* _ λ*i*. *ereal* (*u i*)], *auto*)
  using *A approx* **unfolding** *tendsto_at_iff_sequentially comp_def*
  by (*elim allE*[*of* _ λ*i*. *ereal* (*l i*)], *auto*)
show (*LBINT x*=*a*..*b*. *f x*) = *B* − *A*
  by (*rule interval_integral_Icc_approx_nonneg* [*OF* ‹*a* < *b*› *approx 1 f_nonneg 2 3*])
show *set_integrable lborel* (*einterval a b*) *f*
  by (*rule interval_integral_Icc_approx_nonneg* [*OF* ‹*a* < *b*› *approx 1 f_nonneg 2 3*])
qed

theorem *interval_integral_FTC_integrable*:
  fixes *f F* :: *real* ⇒ ′*a*::*euclidean_space* **and** *a b* :: *ereal*
  assumes *a* < *b*
  assumes *F*: ⋀*x*. *a* < *ereal x* ⟹ *ereal x* < *b* ⟹ (*F has_vector_derivative f x*) (*at x*)
  assumes *f*: ⋀*x*. *a* < *ereal x* ⟹ *ereal x* < *b* ⟹ *isCont f x*
  assumes *f_integrable*: *set_integrable lborel* (*einterval a b*) *f*
  assumes *A*: ((*F* ∘ *real_of_ereal*) ⟶ *A*) (*at_right a*)
  assumes *B*: ((*F* ∘ *real_of_ereal*) ⟶ *B*) (*at_left b*)
  shows (*LBINT x*=*a*..*b*. *f x*) = *B* − *A*
proof −
  obtain *u l* **where** *approx*:
    einterval a b = (⋃*i*. {*l i* .. *u i*})

```
    incseq u decseq l ⋀i. l i < u i ⋀i. a < l i ⋀i. u i < b
    l ⟶ a u ⟶ b
    by (blast intro: einterval_Icc_approximation[OF ‹a < b›])
  have [simp]: ⋀x i. l i ≤ x ⟹ a < ereal x
    by (rule order_less_le_trans, rule approx, force)
  have [simp]: ⋀x i. x ≤ u i ⟹ ereal x < b
    by (rule order_le_less_trans, subst ereal_less_eq(3), assumption, rule approx)
  have FTCi: ⋀i. (LBINT x=l i..u i. f x) = F (u i) − F (l i)
    using assms approx
    by (auto simp: less_imp_le min_def max_def
          intro!: f continuous_at_imp_continuous_on interval_integral_FTC_finite
          intro: has_vector_derivative_at_within)
  have (λi. LBINT x=l i..u i. f x) ⟶ B − A
    unfolding FTCi
  proof (intro tendsto_intros)
    show (λx. F (l x)) ⟶ A
      using A approx unfolding tendsto_at_iff_sequentially comp_def
      by (elim allE[of _ λi. ereal (l i)], auto)
    show (λx. F (u x)) ⟶ B
      using B approx unfolding tendsto_at_iff_sequentially comp_def
      by (elim allE[of _ λi. ereal (u i)], auto)
  qed
  moreover have (λi. LBINT x=l i..u i. f x) ⟶ (LBINT x=a..b. f x)
   by (rule interval_integral_Icc_approx_integrable [OF ‹a < b› approx f_integrable])
  ultimately show ?thesis
    by (elim LIMSEQ_unique)
qed


theorem interval_integral_FTC2:
  fixes a b c :: real and f :: real ⇒ 'a::euclidean_space
  assumes a ≤ c c ≤ b
  and contf: continuous_on {a..b} f
  fixes x :: real
  assumes a ≤ x and x ≤ b
  shows ((λu. LBINT y=c..u. f y) has_vector_derivative (f x)) (at x within {a..b})
proof −
  let ?F = (λu. LBINT y=a..u. f y)
  have intf: set_integrable lborel {a..b} f
    by (rule borel_integrable_atLeastAtMost', rule contf)
  have ((λu. integral {a..u} f) has_vector_derivative f x) (at x within {a..b})
    using ‹a ≤ x› ‹x ≤ b›
    by (auto intro: integral_has_vector_derivative continuous_on_subset [OF contf])
  then have ((λu. integral {a..u} f) has_vector_derivative (f x)) (at x within {a..b})
    by simp
  then have (?F has_vector_derivative (f x)) (at x within {a..b})
    by (rule has_vector_derivative_weaken)
      (auto intro!: assms interval_integral_eq_integral[symmetric] set_integrable_subset
```

```
[OF intf])
  then have ((λx. (LBINT y=c..a. f y) + ?F x) has_vector_derivative (f x)) (at x
within {a..b})
    by (auto intro!: derivative_eq_intros)
  then show ?thesis
  proof (rule has_vector_derivative_weaken)
    fix u assume u ∈ {a .. b}
    then show (LBINT y=c..a. f y) + (LBINT y=a..u. f y) = (LBINT y=c..u. f
y)
      using assms
      apply (intro interval_integral_sum)
      apply (auto simp: interval_lebesgue_integrable_def simp del: real_scaleR_def)
      by (rule set_integrable_subset [OF intf], auto simp: min_def max_def)
  qed (insert assms, auto)
qed

proposition einterval_antiderivative:
  fixes a b :: ereal and f :: real ⇒ 'a::euclidean_space
  assumes a < b and contf: ⋀x :: real. a < x ⟹ x < b ⟹ isCont f x
  shows ∃F. ∀x :: real. a < x ⟶ x < b ⟶ (F has_vector_derivative f x) (at x)
proof –
  from einterval_nonempty [OF ‹a < b›] obtain c :: real where [simp]: a < c c
< b
    by (auto simp: einterval_def)
  let ?F = (λu. LBINT y=c..u. f y)
  show ?thesis
  proof (rule exI, clarsimp)
    fix x :: real
    assume [simp]: a < x x < b
    have 1: a < min c x by simp
    from einterval_nonempty [OF 1] obtain d :: real where [simp]: a < d d < c
d < x
      by (auto simp: einterval_def)
    have 2: max c x < b by simp
    from einterval_nonempty [OF 2] obtain e :: real where [simp]: c < e x < e
e < b
      by (auto simp: einterval_def)
    have (?F has_vector_derivative f x) (at x within {d<..<e})
    proof (rule has_vector_derivative_within_subset [of _ _ _ {d..e}])
      have continuous_on {d..e} f
      proof (intro continuous_at_imp_continuous_on ballI contf; clarsimp)
        show ⋀x. ⟦d ≤ x; x ≤ e⟧ ⟹ a < ereal x
          using ‹a < ereal d› ereal_less_ereal_Ex by auto
        show ⋀x. ⟦d ≤ x; x ≤ e⟧ ⟹ ereal x < b
          using ‹ereal e < b› ereal_less_eq(3) le_less_trans by blast
      qed
      then show (?F has_vector_derivative f x) (at x within {d..e})
        by (intro interval_integral_FTC2) (use ‹d < c› ‹c < e› ‹d < x› ‹x < e› in
‹linarith+›)
```

qed *auto*
then show (*?F has_vector_derivative f x*) (*at x*)
by (*force simp*: *has_vector_derivative_within_open* [*of _* {*d*<..<*e*}])
qed
qed

6.23.19 The substitution theorem

Once again, three versions: first, for finite intervals, and then two versions for arbitrary intervals.

theorem *interval_integral_substitution_finite*:
fixes *a b* :: *real* **and** *f* :: *real* ⇒ *'a::euclidean_space*
assumes $a \leq b$
and *derivg*: ⋀*x*. $a \leq x \Longrightarrow x \leq b \Longrightarrow$ (*g has_real_derivative* (*g' x*)) (*at x within* {*a..b*})
and *contf* : *continuous_on* (*g* ‘ {*a..b*}) *f*
and *contg'*: *continuous_on* {*a..b*} *g'*
shows *LBINT x=a..b. g' x* $*_R$ *f* (*g x*) = *LBINT y=g a..g b. f y*
proof−
have *v_derivg*: ⋀*x*. $a \leq x \Longrightarrow x \leq b \Longrightarrow$ (*g has_vector_derivative* (*g' x*)) (*at x within* {*a..b*})
using *derivg* **unfolding** *has_field_derivative_iff_has_vector_derivative* **.**
then have *contg* [*simp*]: *continuous_on* {*a..b*} *g*
by (*rule continuous_on_vector_derivative*) *auto*
have *1*: ∃*x*∈{*a..b*}. *u* = *g x* **if** *min* (*g a*) (*g b*) ≤ *u u* ≤ *max* (*g a*) (*g b*) **for** *u*
by (*cases g a* ≤ *g b*) (*use that assms IVT'* [*of g a u b*] *IVT2'* [*of g b u a*] **in** ‹*auto simp*: *min_def max_def*›)
obtain *c d* **where** *g_im*: *g* ‘ {*a..b*} = {*c..d*} **and** *c* ≤ *d*
by (*metis continuous_image_closed_interval contg* ‹*a* ≤ *b*›)
obtain *F* **where** *derivF*:
⋀*x*. ⟦*a* ≤ *x*; *x* ≤ *b*⟧ ⟹ (*F has_vector_derivative* (*f* (*g x*))) (*at* (*g x*) *within* (*g* ‘ {*a..b*}))
using *continuous_on_subset* [*OF contf*] *g_im*
by (*metis antiderivative_continuous atLeastAtMost_iff image_subset_iff set_eq_subset*)
have *contfg*: *continuous_on* {*a..b*} (λ*x*. *f* (*g x*))
by (*blast intro*: *continuous_on_compose2 contf contg*)
have *LBINT x. indicat_real* {*a..b*} *x* $*_R$ *g' x* $*_R$ *f* (*g x*) = *F* (*g b*) − *F* (*g a*)
apply (*rule integral_FTC_atLeastAtMost*
[*OF* ‹*a* ≤ *b*› *vector_diff_chain_within*[*OF v_derivg derivF*, *unfolded comp_def*]])
apply (*auto intro*!: *continuous_on_scaleR contg' contfg*)
done
then have *LBINT x=a..b. g' x* $*_R$ *f* (*g x*) = *F* (*g b*) − *F* (*g a*)
by (*simp add*: *assms interval_integral_Icc set_lebesgue_integral_def*)
moreover have *LBINT y=*(*g a*)..(*g b*). *f y* = *F* (*g b*) − *F* (*g a*)
proof (*rule interval_integral_FTC_finite*)
show *continuous_on* {*min* (*g a*) (*g b*)..*max* (*g a*) (*g b*)} *f*
by (*rule continuous_on_subset* [*OF contf*]) (*auto simp*: *image_def 1*)
show (*F has_vector_derivative f y*) (*at y within* {*min* (*g a*) (*g b*)..*max* (*g a*) (*g*

b)})
  if y: min (g a) (g b) ≤ y y ≤ max (g a) (g b) **for** y
  proof −
   obtain x **where** a ≤ x x ≤ b y = g x
    using 1 y **by** force
   then show ?thesis
    by (auto simp: image_def intro!: 1 has_vector_derivative_within_subset [OF derivF])
  qed
 qed
 ultimately show ?thesis **by** simp
qed

theorem interval_integral_substitution_integrable:
 fixes f :: real ⇒ 'a::euclidean_space **and** a b u v :: ereal
 assumes a < b
 and deriv_g: ⋀x. a < ereal x ⟹ ereal x < b ⟹ DERIV g x :> g' x
 and contf: ⋀x. a < ereal x ⟹ ereal x < b ⟹ isCont f (g x)
 and contg': ⋀x. a < ereal x ⟹ ereal x < b ⟹ isCont g' x
 and g'_nonneg: ⋀x. a ≤ ereal x ⟹ ereal x ≤ b ⟹ 0 ≤ g' x
 and A: ((ereal ∘ g ∘ real_of_ereal) ⟶ A) (at_right a)
 and B: ((ereal ∘ g ∘ real_of_ereal) ⟶ B) (at_left b)
 and integrable: set_integrable lborel (einterval a b) (λx. g' x *R f (g x))
 and integrable2: set_integrable lborel (einterval A B) (λx. f x)
 shows (LBINT x=A..B. f x) = (LBINT x=a..b. g' x *R f (g x))
proof −
 obtain u l **where** approx [simp]:
  einterval a b = (⋃i. {l i .. u i})
  incseq u decseq l ⋀i. l i < u i ⋀i. a < l i ⋀i. u i < b
  l ⟶ a u ⟶ b
  by (blast intro: einterval_Icc_approximation[OF ‹a < b›])
 note less_imp_le [simp]
 have [simp]: ⋀x i. l i ≤ x ⟹ a < ereal x
  by (rule order_less_le_trans, rule approx, force)
 have [simp]: ⋀x i. x ≤ u i ⟹ ereal x < b
  by (rule order_le_less_trans, subst ereal_less_eq(3), assumption, rule approx)
 then have lessb[simp]: ⋀i. l i < b
  using approx(4) less_eq_real_def **by** blast
 have [simp]: ⋀i. a < u i
  by (rule order_less_trans, rule approx, auto, rule approx)
 have lle[simp]: ⋀i j. i ≤ j ⟹ l j ≤ l i **by** (rule decseqD, rule approx)
 have [simp]: ⋀i j. i ≤ j ⟹ u i ≤ u j **by** (rule incseqD, rule approx)
 have g_nondec [simp]: g x ≤ g y **if** a < x x ≤ y y < b **for** x y
 proof (rule DERIV_nonneg_imp_nondecreasing [OF ‹x ≤ y›], intro exI conjI)
  show ⋀u. x ≤ u ⟹ u ≤ y ⟹ (g has_real_derivative g' u) (at u)
   by (meson deriv_g ereal_less_eq(3) le_less_trans less_le_trans that)
  show ⋀u. x ≤ u ⟹ u ≤ y ⟹ 0 ≤ g' u

by (*meson assms*(*5*) *dual_order.trans le_ereal_le less_imp_le order_refl that*)
qed
have $A \le B$ **and** *un*: *einterval A B* = ($\bigcup$*i*. {*g*(*l i*)<..<*g*(*u i*)})
proof −
have *A2*: (λ*i*. *g* (*l i*)) ⟶ *A*
using *A* **apply** (*auto simp*: *einterval_def tendsto_at_iff_sequentially comp_def*)
by (*drule_tac x* = λ*i*. *ereal* (*l i*) **in** *spec*, *auto*)
hence *A3*: $\bigwedge$*i*. *g* (*l i*) $\ge$ *A*
by (*intro decseq_ge*, *auto simp*: *decseq_def*)
have *B2*: (λ*i*. *g* (*u i*)) ⟶ *B*
using *B* **apply** (*auto simp*: *einterval_def tendsto_at_iff_sequentially comp_def*)
by (*drule_tac x* = λ*i*. *ereal* (*u i*) **in** *spec*, *auto*)
hence *B3*: $\bigwedge$*i*. *g* (*u i*) $\le$ *B*
by (*intro incseq_le*, *auto simp*: *incseq_def*)
have *ereal* (*g* (*l 0*)) $\le$ *ereal* (*g* (*u 0*))
by *auto*
then show $A \le B$
by (*meson A3 B3 order.trans*)
{ **fix** *x* :: *real*
assume $A < x$ **and** $x < B$
then have *eventually* (λ*i*. *ereal* (*g* (*l i*)) < *x* ∧ *x* < *ereal* (*g* (*u i*))) *sequentially*
by (*fast intro*: *eventually_conj order_tendstoD A2 B2*)
hence ∃ *i*. *g* (*l i*) < *x* ∧ *x* < *g* (*u i*)
by (*simp add*: *eventually_sequentially*, *auto*)
} **note** *AB* = *this*
show *einterval A B* = ($\bigcup$*i*. {*g*(*l i*)<..<*g*(*u i*)})
proof
show *einterval A B* ⊆ ($\bigcup$*i*. {*g*(*l i*)<..<*g*(*u i*)})
by (*auto simp*: *einterval_def AB*)
show ($\bigcup$*i*. {*g*(*l i*)<..<*g*(*u i*)}) ⊆ *einterval A B*
proof (*clarsimp simp add*: *einterval_def*, *intro conjI*)
show $\bigwedge$*x i*. ⟦*g* (*l i*) < *x*; *x* < *g* (*u i*)⟧ ⟹ *A* < *ereal x*
using *A3 le_ereal_less* **by** *blast*
show $\bigwedge$*x i*. ⟦*g* (*l i*) < *x*; *x* < *g* (*u i*)⟧ ⟹ *ereal x* < *B*
using *B3 ereal_le_less* **by** *blast*
qed
qed
qed

have *eq1*: (*LBINT x*=*l i*.. *u i*. *g′ x* $*_R$ *f* (*g x*)) = (*LBINT y*=*g* (*l i*)..*g* (*u i*). *f y*) **for** *i*
apply (*rule interval_integral_substitution_finite* [*OF* _ *DERIV_subset* [*OF deriv_g*]])
unfolding *has_field_derivative_iff_has_vector_derivative*[*symmetric*]
apply (*auto intro*!: *continuous_at_imp_continuous_on contf contg′*)
done
have (λ*i*. *LBINT x*=*l i*..*u i*. *g′ x* $*_R$ *f* (*g x*)) ⟶ (*LBINT x*=*a*..*b*. *g′ x* $*_R$ *f* (*g x*))
apply (*rule interval_integral_Icc_approx_integrable* [*OF* ‹*a* < *b*› *approx*])

by (*rule assms*)
hence *2*: (λ*i*. (*LBINT y*=*g* (*l i*)..*g* (*u i*). *f y*)) ⟶ (*LBINT x*=*a*..*b*. *g′ x* $*_R$ *f* (*g x*))
by (*simp add*: *eq1*)
have *incseq*: *incseq* (λ*i*. {*g* (*l i*)<..<*g* (*u i*)})
apply (*auto simp*: *incseq_def*)
using *lessb lle approx*(*5*) *g_nondec le_less_trans* **apply** *blast*
by (*force intro*: *less_le_trans*)
have (λ*i*. *set_lebesgue_integral lborel* {*g* (*l i*)<..<*g* (*u i*)} *f*)
⟶ *set_lebesgue_integral lborel* (*einterval A B*) *f*
unfolding *un* **by** (*rule set_integral_cont_up*) (*use incseq integrable2 un* **in** *auto*)
then have (λ*i*. (*LBINT y*=*g* (*l i*)..*g* (*u i*). *f y*)) ⟶ (*LBINT x* = *A*..*B*. *f x*)
by (*simp add*: *interval_lebesgue_integral_le_eq* ‹*A* ≤ *B*›)
thus *?thesis* **by** (*intro LIMSEQ_unique* [*OF* _ *2*])
qed

theorem *interval_integral_substitution_nonneg*:
fixes *f g g′*:: *real* ⇒ *real* **and** *a b u v* :: *ereal*
assumes *a* < *b*
and *deriv_g*: ⋀*x*. *a* < *ereal x* ⟹ *ereal x* < *b* ⟹ *DERIV g x* :> *g′ x*
and *contf*: ⋀*x*. *a* < *ereal x* ⟹ *ereal x* < *b* ⟹ *isCont f* (*g x*)
and *contg′*: ⋀*x*. *a* < *ereal x* ⟹ *ereal x* < *b* ⟹ *isCont g′ x*
and *f_nonneg*: ⋀*x*. *a* < *ereal x* ⟹ *ereal x* < *b* ⟹ *0* ≤ *f* (*g x*)
and *g′_nonneg*: ⋀*x*. *a* ≤ *ereal x* ⟹ *ereal x* ≤ *b* ⟹ *0* ≤ *g′ x*
and *A*: ((*ereal* ∘ *g* ∘ *real_of_ereal*) ⟶ *A*) (*at_right a*)
and *B*: ((*ereal* ∘ *g* ∘ *real_of_ereal*) ⟶ *B*) (*at_left b*)
and *integrable_fg*: *set_integrable lborel* (*einterval a b*) (λ*x*. *f* (*g x*) ∗ *g′ x*)
shows
set_integrable lborel (*einterval A B*) *f*
(*LBINT x*=*A*..*B*. *f x*) = (*LBINT x*=*a*..*b*. (*f* (*g x*) ∗ *g′ x*))
proof −
from *einterval_Icc_approximation*[*OF* ‹*a* < *b*›] **guess** *u l* **. note** *approx* [*simp*] = *this*
note *less_imp_le* [*simp*]
have *aless*[*simp*]: ⋀*x i*. *l i* ≤ *x* ⟹ *a* < *ereal x*
by (*rule order_less_le_trans*, *rule approx*, *force*)
have *lessb*[*simp*]: ⋀*x i*. *x* ≤ *u i* ⟹ *ereal x* < *b*
by (*rule order_le_less_trans*, *subst ereal_less_eq*(*3*), *assumption*, *rule approx*)
have *llb*[*simp*]: ⋀*i*. *l i* < *b*
using *lessb approx*(*4*) *less_eq_real_def* **by** *blast*
have *alu*[*simp*]: ⋀*i*. *a* < *u i*
by (*rule order_less_trans*, *rule approx*, *auto*, *rule approx*)
have [*simp*]: ⋀*i j*. *i* ≤ *j* ⟹ *l j* ≤ *l i* **by** (*rule decseqD*, *rule approx*)
have *uleu*[*simp*]: ⋀*i j*. *i* ≤ *j* ⟹ *u i* ≤ *u j* **by** (*rule incseqD*, *rule approx*)
have *g_nondec* [*simp*]: *g x* ≤ *g y* **if** *a* < *x x* ≤ *y y* < *b* **for** *x y*
proof (*rule DERIV_nonneg_imp_nondecreasing* [*OF* ‹*x* ≤ *y*›], *intro exI conjI*)

show $\bigwedge u.\ x \leq u \Longrightarrow u \leq y \Longrightarrow (g\ has\_real\_derivative\ g'\ u)\ (at\ u)$
by (*meson deriv_g ereal_less_eq*(*3*) *le_less_trans less_le_trans that*)
show $\bigwedge u.\ x \leq u \Longrightarrow u \leq y \Longrightarrow 0 \leq g'\ u$
by (*meson g'_nonneg less_ereal.simps*(*1*) *less_trans not_less that*)
qed
have $A \leq B$ **and** *un*: *einterval A B* = ($\bigcup i.$ {$g(l\ i)$<..<$g(u\ i)$})
proof −
have *A2*: $(\lambda i.\ g\ (l\ i)) \longrightarrow A$
using *A* **apply** (*auto simp*: *einterval_def tendsto_at_iff_sequentially comp_def*)
by (*drule_tac* $x = \lambda i.\ ereal\ (l\ i)$ **in** *spec*, *auto*)
hence *A3*: $\bigwedge i.\ g\ (l\ i) \geq A$
by (*intro decseq_ge*, *auto simp*: *decseq_def*)
have *B2*: $(\lambda i.\ g\ (u\ i)) \longrightarrow B$
using *B* **apply** (*auto simp*: *einterval_def tendsto_at_iff_sequentially comp_def*)
by (*drule_tac* $x = \lambda i.\ ereal\ (u\ i)$ **in** *spec*, *auto*)
hence *B3*: $\bigwedge i.\ g\ (u\ i) \leq B$
by (*intro incseq_le*, *auto simp*: *incseq_def*)
have *ereal* $(g\ (l\ 0)) \leq$ *ereal* $(g\ (u\ 0))$
by *auto*
then show $A \leq B$
by (*meson A3 B3 order.trans*)
{ fix *x* :: *real*
assume $A < x$ **and** $x < B$
then have *eventually* $(\lambda i.\ ereal\ (g\ (l\ i)) < x \wedge x < ereal\ (g\ (u\ i)))$ *sequentially*
by (*fast intro*: *eventually_conj order_tendstoD A2 B2*)
hence $\exists i.\ g\ (l\ i) < x \wedge x < g\ (u\ i)$
by (*simp add*: *eventually_sequentially*, *auto*)
} note *AB* = *this*
show *einterval A B* = ($\bigcup i.$ {$g(l\ i)$<..<$g(u\ i)$})
proof
show *einterval A B* $\subseteq$ ($\bigcup i.$ {$g\ (l\ i)$<..<$g\ (u\ i)$})
by (*auto simp*: *einterval_def AB*)
show ($\bigcup i.$ {$g\ (l\ i)$<..<$g\ (u\ i)$}) $\subseteq$ *einterval A B*
apply (*clarsimp simp*: *einterval_def*, *intro conjI*)
using *A3 le_ereal_less* **apply** *blast*
using *B3 ereal_le_less* **by** *blast*
qed
qed

have *eq1*: (*LBINT x=l i.. u i.* $(f\ (g\ x) * g'\ x)$) = (*LBINT y=g* $(l\ i)$..*g* $(u\ i)$. *f y*) **for** *i*
proof −
have (*LBINT x=l i.. u i.* $g'\ x *_R f\ (g\ x)$) = (*LBINT y=g* $(l\ i)$..*g* $(u\ i)$. *f y*)
apply (*rule interval_integral_substitution_finite* [*OF* _ *DERIV_subset* [*OF deriv_g*]])
unfolding *has_field_derivative_iff_has_vector_derivative*[*symmetric*]
apply (*auto intro*!: *continuous_at_imp_continuous_on contf contg'*)
done
then show *?thesis*

```
      by (simp add: ac_simps)
  qed
  have incseq: incseq (λi. {g (l i)<..<g (u i)})
    apply (clarsimp simp add: incseq_def, intro conjI)
    apply (meson llb antimono_def approx(3) approx(5) g_nondec le_less_trans)
    using alu uleu approx(6) g_nondec less_le_trans by blast
  have img: ∃ c ≥ l i. c ≤ u i ∧ x = g c if g (l i) ≤ x x ≤ g (u i) for x i
  proof −
    have continuous_on {l i..u i} g
      by (force intro!: DERIV_isCont deriv_g continuous_at_imp_continuous_on)
    with that show ?thesis
      using IVT' [of g] approx(4) dual_order.strict_implies_order by blast
  qed
  have continuous_on {g (l i)..g (u i)} f for i
    apply (intro continuous_intros continuous_at_imp_continuous_on)
    using contf img by force
  then have int_f: ⋀i. set_integrable lborel {g (l i)<..<g (u i)} f
   by (rule set_integrable_subset [OF borel_integrable_atLeastAtMost']) (auto intro:
less_imp_le)
  have integrable: set_integrable lborel (⋃i. {g (l i)<..<g (u i)}) f
  proof (intro pos_integrable_to_top incseq int_f)
    let ?l = (LBINT x=a..b. f (g x) ∗ g' x)
    have (λi. LBINT x=l i..u i. f (g x) ∗ g' x) ⟶ ?l
      by (intro assms interval_integral_Icc_approx_integrable [OF ‹a < b› approx])
    hence (λi. (LBINT y=g (l i)..g (u i). f y)) ⟶ ?l
      by (simp add: eq1)
    then show (λi. set_lebesgue_integral lborel {g (l i)<..<g (u i)} f) ⟶ ?l
      unfolding interval_lebesgue_integral_def by auto
    have ⋀x i. g (l i) ≤ x ⟹ x ≤ g (u i) ⟹ 0 ≤ f x
      using aless f_nonneg img lessb by blast
    then show ⋀x i. x ∈ {g (l i)<..<g (u i)} ⟹ 0 ≤ f x
      using less_eq_real_def by auto
  qed (auto simp: greaterThanLessThan_borel)
  thus set_integrable lborel (einterval A B) f
    by (simp add: un)

  have (LBINT x=A..B. f x) = (LBINT x=a..b. g' x ∗R f (g x))
  proof (rule interval_integral_substitution_integrable)
    show set_integrable lborel (einterval a b) (λx. g' x ∗R f (g x))
      using integrable_fg by (simp add: ac_simps)
  qed fact+
  then show (LBINT x=A..B. f x) = (LBINT x=a..b. (f (g x) ∗ g' x))
    by (simp add: ac_simps)
qed


syntax _complex_lebesgue_borel_integral :: pttrn ⇒ real ⇒ complex
  ((2CLBINT _. _) [0,60] 60)
```

translations *CLBINT x. f == CONST complex_lebesgue_integral CONST lborel* (*λx. f*)

syntax *_complex_set_lebesgue_borel_integral* :: *pttrn* ⇒ *real set* ⇒ *real* ⇒ *complex*
((*3CLBINT _:_. _*) [*0*,*60*,*61*] *60*)

translations
CLBINT x:*A*. *f == CONST complex_set_lebesgue_integral CONST lborel A* (*λx. f*)

abbreviation *complex_interval_lebesgue_integral* ::
real measure ⇒ *ereal* ⇒ *ereal* ⇒ (*real* ⇒ *complex*) ⇒ *complex* **where**
complex_interval_lebesgue_integral M a b f ≡ *interval_lebesgue_integral M a b f*

abbreviation *complex_interval_lebesgue_integrable* ::
real measure ⇒ *ereal* ⇒ *ereal* ⇒ (*real* ⇒ *complex*) ⇒ *bool* **where**
complex_interval_lebesgue_integrable M a b f ≡ *interval_lebesgue_integrable M a b f*

syntax
_ascii_complex_interval_lebesgue_borel_integral :: *pttrn* ⇒ *ereal* ⇒ *ereal* ⇒ *real* ⇒ *complex*
((*4CLBINT _=_.._. _*) [*0*,*60*,*60*,*61*] *60*)

translations
CLBINT x=*a*..*b*. *f == CONST complex_interval_lebesgue_integral CONST lborel a b* (*λx. f*)

proposition *interval_integral_norm*:
fixes *f* :: *real* ⇒ *'a* :: {*banach*, *second_countable_topology*}
shows *interval_lebesgue_integrable lborel a b f* ⟹ *a* ≤ *b* ⟹
norm (*LBINT t*=*a*..*b*. *f t*) ≤ *LBINT t*=*a*..*b*. *norm* (*f t*)
using *integral_norm_bound*[*of lborel λx. indicator* (*einterval a b*) *x* $*_R$ *f x*]
by (*auto simp*: *interval_lebesgue_integral_def interval_lebesgue_integrable_def set_lebesgue_integral_def*)

proposition *interval_integral_norm2*:
interval_lebesgue_integrable lborel a b f ⟹
norm (*LBINT t*=*a*..*b*. *f t*) ≤ |*LBINT t*=*a*..*b*. *norm* (*f t*)|
proof (*induct a b rule*: *linorder_wlog*)
case (*sym a b*) **then show** *?case*
by (*simp add*: *interval_integral_endpoints_reverse*[*of a b*] *interval_integrable_endpoints_reverse*[*of a b*])
next
case (*le a b*)
then have |*LBINT t*=*a*..*b*. *norm* (*f t*)| = *LBINT t*=*a*..*b*. *norm* (*f t*)
using *integrable_norm*[*of lborel λx. indicator* (*einterval a b*) *x* $*_R$ *f x*]
by (*auto simp*: *interval_lebesgue_integral_def interval_lebesgue_integrable_def set_lebesgue_integral_def intro*!: *integral_nonneg_AE abs_of_nonneg*)
then show *?case*

  using *le* **by** (*simp add*: *interval_integral_norm*)
qed

lemma *integral_cos*: $t \neq 0 \Longrightarrow$ *LBINT x=a..b. cos* $(t * x) =$ *sin* $(t * b) / t -$ *sin* $(t * a) / t$
 apply (*intro interval_integral_FTC_finite continuous_intros*)
 by (*auto intro*!: *derivative_eq_intros simp*: *has_field_derivative_iff_has_vector_derivative*[*symmetric*])

end

6.24 Integration by Substition for the Lebesgue Integral

theory *Lebesgue_Integral_Substitution*
imports *Interval_Integral*
begin

lemma *nn_integral_substitution_aux*:
 fixes *f* :: *real* ⇒ *ennreal*
 assumes *Mf*: *f* ∈ *borel_measurable borel*
 assumes *nonnegf*: $\bigwedge x.\ f\ x \geq 0$
 assumes *derivg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow$ (*g has_real_derivative* g' *x*) (*at x*)
 assumes *contg'*: *continuous_on* $\{a..b\}$ g'
 assumes *derivg_nonneg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g'\ x \geq 0$
 assumes $a < b$
 shows ($\int^+ x.\ f\ x *$ *indicator* $\{g\ a..g\ b\}$ *x* ∂*lborel*) =
   ($\int^+ x.\ f\ (g\ x) * g'\ x *$ *indicator* $\{a..b\}$ *x* ∂*lborel*)
proof−
 from ‹$a < b$› **have** [*simp*]: $a \leq b$ **by** *simp*
 from *derivg* **have** *contg*: *continuous_on* $\{a..b\}$ *g* **by** (*rule has_real_derivative_imp_continuous_on*)
 from *this* **and** *contg'* **have** *Mg*: *set_borel_measurable borel* $\{a..b\}$ *g* **and**
   Mg': *set_borel_measurable borel* $\{a..b\}$ g'
  by (*simp_all only*: *set_measurable_continuous_on_ivl*)
 from *derivg* **have** *derivg'*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow$ (*g has_vector_derivative* g' *x*) (*at x*)
  by (*simp only*: *has_field_derivative_iff_has_vector_derivative*)

 have *real_ind*[*simp*]: $\bigwedge A\ x.$ *enn2real* (*indicator A x*) = *indicator A x*
  by (*auto split*: *split_indicator*)
 have *ennreal_ind*[*simp*]: $\bigwedge A\ x.$ *ennreal* (*indicator A x*) = *indicator A x*
  by (*auto split*: *split_indicator*)
 have [*simp*]: $\bigwedge x\ A.$ *indicator A* (*g x*) = *indicator* (*g* −‘ *A*) *x*
  by (*auto split*: *split_indicator*)

 from *derivg derivg_nonneg* **have** *monog*: $\bigwedge x\ y.\ a \leq x \Longrightarrow x \leq y \Longrightarrow y \leq b \Longrightarrow g\ x \leq g\ y$

by (*rule deriv_nonneg_imp_mono*) *simp_all*
with *monog* **have** [*simp*]: $g\ a \leq g\ b$ **by** (*auto intro*: *mono_onD*)

show *?thesis*
proof (*induction rule*: *borel_measurable_induct*[*OF Mf*, *case_names cong set mult add sup*])
case (*cong f1 f2*)
from *cong.hyps*(*3*) **have** *f1* = *f2* **by** *auto*
with *cong* **show** *?case* **by** *simp*
next
case (*set A*)
from *set.hyps* **show** *?case*
proof (*induction rule*: *borel_set_induct*)
case *empty*
thus *?case* **by** *simp*
next
case (*interval c d*)
{
fix *u v* :: *real* **assume** *asm*: $\{u..v\} \subseteq \{g\ a..g\ b\}$ $u \leq v$

obtain u' v' **where** $u'v'$: $\{a..b\} \cap g-`\{u..v\} = \{u'..v'\}$ $u' \leq v'$ $g\ u' = u$ $g\ v' = v$
using *asm* **by** (*rule_tac continuous_interval_vimage_Int*[*OF contg monog*, *of u v*]) *simp_all*
hence $\{u'..v'\} \subseteq \{a..b\}$ $\{u'..v'\} \subseteq g -` \{u..v\}$ **by** *blast+*
with $u'v'(2)$ **have** $u' \in g -` \{u..v\}$ $v' \in g -` \{u..v\}$ **by** *auto*
from $u'v'(1)$ **have** [*simp*]: $\{a..b\} \cap g -` \{u..v\} \in$ *sets borel* **by** *simp*

have *A*: *continuous_on* $\{min\ u'\ v'..max\ u'\ v'\}$ g'
by (*simp only*: $u'v'$ *max_absorb2 min_absorb1*)
(*intro continuous_on_subset*[*OF contg'*], *insert* $u'v'$, *auto*)
have $\bigwedge x.\ x \in \{u'..v'\} \Longrightarrow$ (*g has_real_derivative* $g'\ x$) (*at x within* $\{u'..v'\}$)
using *asm* **by** (*intro has_field_derivative_subset*[*OF derivg*] *subsetD*[*OF* ‹$\{u'..v'\} \subseteq \{a..b\}$›]) *auto*
hence *B*: $\bigwedge x.\ min\ u'\ v' \leq x \Longrightarrow x \leq max\ u'\ v' \Longrightarrow$
(*g has_vector_derivative* $g'\ x$) (*at x within* $\{min\ u'\ v'..max\ u'\ v'\}$)
by (*simp only*: $u'v'$ *max_absorb2 min_absorb1*)
(*auto simp*: *has_field_derivative_iff_has_vector_derivative*)
have *integrable lborel* ($\lambda x.$ *indicator* ($\{a..b\} \cap g -` \{u..v\}$) $x *_R g'\ x$)
using *set_integrable_subset borel_integrable_atLeastAtMost'*[*OF contg'*]
by (*metis* ‹$\{u'..v'\} \subseteq \{a..b\}$› *eucl_ivals*(*5*) *set_integrable_def sets_lborel* $u'v'(1)$)
hence ($\int^+ x.$ *ennreal* ($g'\ x$) $*$ *indicator* ($\{a..b\} \cap g-` \{u..v\}$) $x\ \partial lborel$) =
LBINT x:$\{a..b\} \cap g-`\{u..v\}$. $g'\ x$
unfolding *set_lebesgue_integral_def*
by (*subst nn_integral_eq_integral*[*symmetric*])
(*auto intro*!: *derivg_nonneg nn_integral_cong split*: *split_indicator*)
also from *interval_integral_FTC_finite*[*OF A B*]
have *LBINT* x:$\{a..b\} \cap g-`\{u..v\}$. $g'\ x = v - u$

by (*simp add*: *u'v' interval_integral_Icc* ‹*u* ≤ *v*›)
finally have (∫⁺ *x*. *ennreal* (*g' x*) ∗ *indicator* ({*a..b*} ∩ *g* −‘ {*u..v*}) *x* ∂*lborel*) =
ennreal (*v* − *u*) **.**
} **note** *A* = *this*

have (∫⁺*x*. *indicator* {*c..d*} (*g x*) ∗ *ennreal* (*g' x*) ∗ *indicator* {*a..b*} *x* ∂*lborel*) =
(∫⁺ *x*. *ennreal* (*g' x*) ∗ *indicator* ({*a..b*} ∩ *g* −‘ {*c..d*}) *x* ∂*lborel*)
by (*intro nn_integral_cong*) (*simp split*: *split_indicator*)
also have {*a..b*} ∩ *g*−‘{*c..d*} = {*a..b*} ∩ *g*−‘{*max* (*g a*) *c..min* (*g b*) *d*}
using ‹*a* ≤ *b*› ‹*c* ≤ *d*›
by (*auto intro*!: *monog intro*: *order.trans*)
also have (∫⁺ *x*. *ennreal* (*g' x*) ∗ *indicator* ... *x* ∂*lborel*) =
(*if max* (*g a*) *c* ≤ *min* (*g b*) *d then min* (*g b*) *d* − *max* (*g a*) *c else 0*)
using ‹*c* ≤ *d*› **by** (*simp add*: *A*)
also have ... = (∫⁺ *x*. *indicator* ({*g a..g b*} ∩ {*c..d*}) *x* ∂*lborel*)
by (*subst nn_integral_indicator*) (*auto intro*!: *measurable_sets Mg simp*:)
also have ... = (∫⁺ *x*. *indicator* {*c..d*} *x* ∗ *indicator* {*g a..g b*} *x* ∂*lborel*)
by (*intro nn_integral_cong*) (*auto split*: *split_indicator*)
finally show *?case* **..**

next

case (*compl A*)
note ‹*A* ∈ *sets borel*›[*measurable*]
from *emeasure_mono*[*of A* ∩ {*g a..g b*} {*g a..g b*} *lborel*]
have [*simp*]: *emeasure lborel* (*A* ∩ {*g a..g b*}) ≠ *top* **by** (*auto simp*: *top_unique*)
have [*simp*]: *g* −‘ *A* ∩ {*a..b*} ∈ *sets borel*
by (*rule set_borel_measurable_sets*[*OF Mg*]) *auto*
have [*simp*]: *g* −‘ (−*A*) ∩ {*a..b*} ∈ *sets borel*
by (*rule set_borel_measurable_sets*[*OF Mg*]) *auto*

have (∫⁺*x*. *indicator* (−*A*) *x* ∗ *indicator* {*g a..g b*} *x* ∂*lborel*) =
(∫⁺*x*. *indicator* (−*A* ∩ {*g a..g b*}) *x* ∂*lborel*)
by (*rule nn_integral_cong*) (*simp split*: *split_indicator*)
also from *compl* **have** ... = *emeasure lborel* ({*g a..g b*} − *A*) **using** *derivg_nonneg*
by (*simp add*: *vimage_Compl diff_eq Int_commute*[*of* −*A*])
also have {*g a..g b*} − *A* = {*g a..g b*} − *A* ∩ {*g a..g b*} **by** *blast*
also have *emeasure lborel* ... = *g b* − *g a* − *emeasure lborel* (*A* ∩ {*g a..g b*})
using ‹*A* ∈ *sets borel*› **by** (*subst emeasure_Diff*) (*auto simp*:)
also have *emeasure lborel* (*A* ∩ {*g a..g b*}) =
∫⁺*x*. *indicator A x* ∗ *indicator* {*g a..g b*} *x* ∂*lborel*
using ‹*A* ∈ *sets borel*›
by (*subst nn_integral_indicator*[*symmetric*], *simp*, *intro nn_integral_cong*)
(*simp split*: *split_indicator*)
also have ... = ∫⁺ *x*. *indicator* (*g*−‘*A* ∩ {*a..b*}) *x* ∗ *ennreal* (*g' x* ∗ *indicator* {*a..b*} *x*) ∂*lborel* (**is** _ = *?I*)

by (*subst compl.IH*, *intro nn_integral_cong*) (*simp split*: *split_indicator*)
also have $g\ b - g\ a = LBINT\ x{:}\{a..b\}.\ g'\ x$ **using** *derivg′*
unfolding *set_lebesgue_integral_def*
by (*intro integral_FTC_atLeastAtMost*[*symmetric*])
(*auto intro*: *continuous_on_subset*[*OF contg′*] *has_field_derivative_subset*[*OF derivg*]
has_vector_derivative_at_within)
also have *ennreal* ... $= \int^+ x.\ g'\ x * indicator\ \{a..b\}\ x\ \partial lborel$
using *borel_integrable_atLeastAtMost′*[*OF contg′*] **unfolding** *set_lebesgue_integral_def*
by (*subst nn_integral_eq_integral*)
(*simp_all add*: *mult.commute derivg_nonneg set_integrable_def split*: *split_indicator*)
also have *Mg′′*: $(\lambda x.\ indicator\ (g -`\ A \cap \{a..b\})\ x * ennreal\ (g'\ x * indicator\ \{a..b\}\ x))$
$\in$ *borel_measurable borel* **using** *Mg′*
by (*intro borel_measurable_times_ennreal borel_measurable_indicator*)
(*simp_all add*: *mult.commute set_borel_measurable_def*)
have *le*: $(\int^+ x.\ indicator\ (g-`A \cap \{a..b\})\ x * ennreal\ (g'\ x * indicator\ \{a..b\}\ x)\ \partial lborel) \le$
$(\int^+ x.\ ennreal\ (g'\ x) * indicator\ \{a..b\}\ x\ \partial lborel)$
by (*intro nn_integral_mono*) (*simp split*: *split_indicator add*: *derivg_nonneg*)
note *integrable* = *borel_integrable_atLeastAtMost′*[*OF contg′*]
with *le* **have** *notinf*: $(\int^+ x.\ indicator\ (g-`A \cap \{a..b\})\ x * ennreal\ (g'\ x * indicator\ \{a..b\}\ x)\ \partial lborel) \neq top$
by (*auto simp*: *real_integrable_def nn_integral_set_ennreal mult.commute top_unique set_integrable_def*)
have $(\int^+ x.\ g'\ x * indicator\ \{a..b\}\ x\ \partial lborel) - ?I =$
$\int^+ x.\ ennreal\ (g'\ x * indicator\ \{a..b\}\ x) -$
$indicator\ (g -`\ A \cap \{a..b\})\ x * ennreal\ (g'\ x * indicator\ \{a..b\}\ x)\ \partial lborel$
apply (*intro nn_integral_diff*[*symmetric*])
apply (*insert Mg′*, *simp add*: *mult.commute set_borel_measurable_def*) []
apply (*insert Mg′′*, *simp*) []
apply (*simp split*: *split_indicator add*: *derivg_nonneg*)
apply (*rule notinf*)
apply (*simp split*: *split_indicator add*: *derivg_nonneg*)
done
also have ... $= \int^+ x.\ indicator\ (-A)\ (g\ x) * ennreal\ (g'\ x) * indicator\ \{a..b\}\ x\ \partial lborel$
by (*intro nn_integral_cong*) (*simp split*: *split_indicator*)
finally show *?case* **.**

next
case (*union f*)
then have [*simp*]: $\bigwedge i.\ \{a..b\} \cap g -`\ f\ i \in sets\ borel$
by (*subst Int_commute*, *intro set_borel_measurable_sets*[*OF Mg*]) *auto*
have $g -`\ (\bigcup i.\ f\ i) \cap \{a..b\} = (\bigcup i.\ \{a..b\} \cap g -`\ f\ i)$ **by** *auto*
hence $g -`\ (\bigcup i.\ f\ i) \cap \{a..b\} \in sets\ borel$ **by** (*auto simp del*: *UN_simps*)

have ($\int^+ x.$ *indicator* ($\bigcup i.$ *f i*) *x* ∗ *indicator* {*g a..g b*} *x* ∂*lborel*) =
$\int^+ x.$ *indicator* ($\bigcup i.$ {*g a..g b*} ∩ *f i*) *x* ∂*lborel*
by (*intro nn_integral_cong*) (*simp split*: *split_indicator*)
also from *union* **have** ... = *emeasure lborel* ($\bigcup i.$ {*g a..g b*} ∩ *f i*) **by** *simp*
also from *union* **have** ... = ($\sum i.$ *emeasure lborel* ({*g a..g b*} ∩ *f i*))
by (*intro suminf_emeasure*[*symmetric*]) (*auto simp*: *disjoint_family_on_def*)
also from *union* **have** ... = ($\sum i.$ $\int^+ x.$ *indicator* ({*g a..g b*} ∩ *f i*) *x* ∂*lborel*)
by *simp*
also have ($\lambda i.$ $\int^+ x.$ *indicator* ({*g a..g b*} ∩ *f i*) *x* ∂*lborel*) =
($\lambda i.$ $\int^+ x.$ *indicator* (*f i*) *x* ∗ *indicator* {*g a..g b*} *x* ∂*lborel*)
by (*intro ext nn_integral_cong*) (*simp split*: *split_indicator*)
also from *union.IH* **have** ($\sum i.$ $\int^+ x.$ *indicator* (*f i*) *x* ∗ *indicator* {*g a..g b*}
x ∂*lborel*) =
($\sum i.$ $\int^+$ *x*. *indicator* (*f i*) (*g x*) ∗ *ennreal* (*g′ x*) ∗ *indicator* {*a..b*} *x*
∂*lborel*) **by** *simp*
also have ($\lambda i.$ $\int^+$ *x*. *indicator* (*f i*) (*g x*) ∗ *ennreal* (*g′ x*) ∗ *indicator* {*a..b*}
x ∂*lborel*) =
($\lambda i.$ $\int^+$ *x*. *ennreal* (*g′ x* ∗ *indicator* {*a..b*} *x*) ∗ *indicator*
({*a..b*} ∩ *g* −‘ *f i*) *x* ∂*lborel*)
by (*intro ext nn_integral_cong*) (*simp split*: *split_indicator*)
also have ($\sum i.$ ... *i*) = $\int^+$ *x*. ($\sum i.$ *ennreal* (*g′ x* ∗ *indicator* {*a..b*} *x*) ∗
indicator ({*a..b*} ∩ *g* −‘ *f i*) *x*) ∂*lborel*
using *Mg′*
apply (*intro nn_integral_suminf*[*symmetric*])
apply (*rule borel_measurable_times_ennreal*, *simp add*: *mult.commute set_borel_measurable_def*)
apply (*rule borel_measurable_indicator*, *subst sets_lborel*)
apply (*simp_all split*: *split_indicator add*: *derivg_nonneg*)
done
also have ($\lambda x\ i.$ *ennreal* (*g′ x* ∗ *indicator* {*a..b*} *x*) ∗ *indicator* ({*a..b*} ∩ *g*
−‘ *f i*) *x*) =
($\lambda x\ i.$ *ennreal* (*g′ x* ∗ *indicator* {*a..b*} *x*) ∗ *indicator* (*g* −‘ *f i*) *x*)
by (*intro ext*) (*simp split*: *split_indicator*)
also have ($\int^+$ *x*. ($\sum i.$ *ennreal* (*g′ x* ∗ *indicator* {*a..b*} *x*) ∗ *indicator* (*g* −‘
f i) *x*) ∂*lborel*) =
$\int^+$ *x*. *ennreal* (*g′ x* ∗ *indicator* {*a..b*} *x*) ∗ ($\sum i.$ *indicator* (*g* −‘
f i) *x*) ∂*lborel*
by (*intro nn_integral_cong*) (*auto split*: *split_indicator simp*: *derivg_nonneg*)
also from *union* **have** ($\lambda x.$ $\sum i.$ *indicator* (*g* −‘ *f i*) *x* :: *ennreal*) = ($\lambda x.$
indicator ($\bigcup i.$ *g* −‘ *f i*) *x*)
by (*intro ext suminf_indicator*) (*auto simp*: *disjoint_family_on_def*)
also have ($\int^+ x.$ *ennreal* (*g′ x* ∗ *indicator* {*a..b*} *x*) ∗ ... *x* ∂*lborel*) =
($\int^+ x.$ *indicator* ($\bigcup i.$ *f i*) (*g x*) ∗ *ennreal* (*g′ x*) ∗ *indicator* {*a..b*}
x ∂*lborel*)
by (*intro nn_integral_cong*) (*simp split*: *split_indicator*)
finally show *?case* .
qed

next
case (*mult f c*)

note *Mf*[*measurable*] = ‹*f* ∈ *borel_measurable borel*›
let *?I* = *indicator* {*a*..*b*}
have (λ*x*. *f* (*g x* ∗ *?I x*) ∗ *ennreal* (*g′ x* ∗ *?I x*)) ∈ *borel_measurable borel* **using** *Mg Mg′*
by (*intro borel_measurable_times_ennreal measurable_compose*[*OF* _ *Mf*])
(*simp_all add*: *mult.commute set_borel_measurable_def*)
also have (λ*x*. *f* (*g x* ∗ *?I x*) ∗ *ennreal* (*g′ x* ∗ *?I x*)) = (λ*x*. *f* (*g x*) ∗ *ennreal* (*g′ x*) ∗ *?I x*)
by (*intro ext*) (*simp split*: *split_indicator*)
finally have *Mf′*: (λ*x*. *f* (*g x*) ∗ *ennreal* (*g′ x*) ∗ *?I x*) ∈ *borel_measurable borel* **.**
with *mult* **show** *?case*
by (*subst* (*1 2 3*) *mult_ac*, *subst* (*1 2*) *nn_integral_cmult*) (*simp_all add*: *mult_ac*)

next
case (*add f2 f1*)
let *?I* = *indicator* {*a*..*b*}
{
fix *f* :: *real* ⇒ *ennreal* **assume** *Mf*: *f* ∈ *borel_measurable borel*
have (λ*x*. *f* (*g x* ∗ *?I x*) ∗ *ennreal* (*g′ x* ∗ *?I x*)) ∈ *borel_measurable borel* **using** *Mg Mg′*
by (*intro borel_measurable_times_ennreal measurable_compose*[*OF* _ *Mf*])
(*simp_all add*: *mult.commute set_borel_measurable_def*)
also have (λ*x*. *f* (*g x* ∗ *?I x*) ∗ *ennreal* (*g′ x* ∗ *?I x*)) = (λ*x*. *f* (*g x*) ∗ *ennreal* (*g′ x*) ∗ *?I x*)
by (*intro ext*) (*simp split*: *split_indicator*)
finally have (λ*x*. *f* (*g x*) ∗ *ennreal* (*g′ x*) ∗ *?I x*) ∈ *borel_measurable borel* **.**
} **note** *Mf′* = *this*[*OF* ‹*f1* ∈ *borel_measurable borel*›] *this*[*OF* ‹*f2* ∈ *borel_measurable borel*›]

have (∫⁺ *x*. (*f1 x* + *f2 x*) ∗ *indicator* {*g a*..*g b*} *x* ∂*lborel*) =
(∫⁺ *x*. *f1 x* ∗ *indicator* {*g a*..*g b*} *x* + *f2 x* ∗ *indicator* {*g a*..*g b*} *x* ∂*lborel*)
by (*intro nn_integral_cong*) (*simp split*: *split_indicator*)
also from *add* **have** ... = (∫⁺ *x*. *f1* (*g x*) ∗ *ennreal* (*g′ x*) ∗ *indicator* {*a*..*b*} *x* ∂*lborel*) +
(∫⁺ *x*. *f2* (*g x*) ∗ *ennreal* (*g′ x*) ∗ *indicator* {*a*..*b*} *x* ∂*lborel*)
by (*simp_all add*: *nn_integral_add*)
also from *add* **have** ... = (∫⁺ *x*. *f1* (*g x*) ∗ *ennreal* (*g′ x*) ∗ *indicator* {*a*..*b*} *x* +
f2 (*g x*) ∗ *ennreal* (*g′ x*) ∗ *indicator* {*a*..*b*} *x* ∂*lborel*)
by (*intro nn_integral_add*[*symmetric*])
(*auto simp add*: *Mf′ derivg_nonneg split*: *split_indicator*)
also have ... = ∫⁺ *x*. (*f1* (*g x*) + *f2* (*g x*)) ∗ *ennreal* (*g′ x*) ∗ *indicator* {*a*..*b*} *x* ∂*lborel*
by (*intro nn_integral_cong*) (*simp split*: *split_indicator add*: *distrib_right*)
finally show *?case* **.**

next
  case (*sup F*)
  {
    fix *i*
    let *?I* = *indicator* {*a..b*}
    have (λ*x*. *F i* (*g x* ∗ *?I x*) ∗ *ennreal* (*g′ x* ∗ *?I x*)) ∈ *borel_measurable borel*
using *Mg Mg′*
      by (*rule_tac borel_measurable_times_ennreal*, *rule_tac measurable_compose*[*OF _ sup.hyps*(*1*)])
         (*simp_all add*: *mult.commute set_borel_measurable_def*)
    also have (λ*x*. *F i* (*g x* ∗ *?I x*) ∗ *ennreal* (*g′ x* ∗ *?I x*)) = (λ*x*. *F i* (*g x*) ∗ *ennreal* (*g′ x*) ∗ *?I x*)
      by (*intro ext*) (*simp split*: *split_indicator*)
    finally have ... ∈ *borel_measurable borel* **.**
  } **note** *Mf′* = *this*

    have (∫⁺*x*. (*SUP i*. *F i x*) ∗ *indicator* {*g a..g b*} *x* ∂*lborel*) =
              ∫⁺*x*. (*SUP i*. *F i x*∗ *indicator* {*g a..g b*} *x*) ∂*lborel*
      by (*intro nn_integral_cong*) (*simp split*: *split_indicator*)
    also from *sup* **have** ... = (*SUP i*. ∫⁺*x*. *F i x*∗ *indicator* {*g a..g b*} *x* ∂*lborel*)
      by (*intro nn_integral_monotone_convergence_SUP*)
         (*auto simp*: *incseq_def le_fun_def split*: *split_indicator*)
    also from *sup* **have** ... = (*SUP i*. ∫⁺*x*. *F i* (*g x*) ∗ *ennreal* (*g′ x*) ∗ *indicator* {*a..b*} *x* ∂*lborel*)
      by *simp*
    also from *sup* **have** ... = ∫⁺*x*. (*SUP i*. *F i* (*g x*) ∗ *ennreal* (*g′ x*) ∗ *indicator* {*a..b*} *x*) ∂*lborel*
      by (*intro nn_integral_monotone_convergence_SUP*[*symmetric*])
         (*auto simp*: *incseq_def le_fun_def derivg_nonneg Mf′ split*: *split_indicator*
              intro!: *mult_right_mono*)
    also from *sup* **have** ... = ∫⁺*x*. (*SUP i*. *F i* (*g x*)) ∗ *ennreal* (*g′ x*) ∗ *indicator* {*a..b*} *x* ∂*lborel*
      by (*subst mult.assoc*, *subst mult.commute*, *subst SUP_mult_left_ennreal*)
         (*auto split*: *split_indicator simp*: *derivg_nonneg mult_ac*)
    finally show *?case* **by** (*simp add*: *image_comp*)
  qed
qed

theorem *nn_integral_substitution*:
  fixes *f* :: *real* ⇒ *real*
  assumes *Mf*[*measurable*]: *set_borel_measurable borel* {*g a..g b*} *f*
  assumes *derivg*: ⋀*x*. *x* ∈ {*a..b*} ⟹ (*g has_real_derivative g′ x*) (*at x*)
  assumes *contg′*: *continuous_on* {*a..b*} *g′*
  assumes *derivg_nonneg*: ⋀*x*. *x* ∈ {*a..b*} ⟹ *g′ x* ≥ *0*
  assumes *a* ≤ *b*
  shows (∫⁺*x*. *f x* ∗ *indicator* {*g a..g b*} *x* ∂*lborel*) =
            (∫⁺*x*. *f* (*g x*) ∗ *g′ x* ∗ *indicator* {*a..b*} *x* ∂*lborel*)
proof (*cases a* = *b*)

assume $a \neq b$
with ‹$a \leq b$› **have** $a < b$ **by** *auto*
let *?f'* = $\lambda x.\ f\ x * indicator\ \{g\ a..g\ b\}\ x$

from *derivg derivg_nonneg* **have** *monog*: $\bigwedge x\ y.\ a \leq x \Longrightarrow x \leq y \Longrightarrow y \leq b \Longrightarrow g\ x \leq g\ y$
by (*rule deriv_nonneg_imp_mono*) *simp_all*
have *bounds*: $\bigwedge x.\ x \geq a \Longrightarrow x \leq b \Longrightarrow g\ x \geq g\ a\ \bigwedge x.\ x \geq a \Longrightarrow x \leq b \Longrightarrow g\ x \leq g\ b$
by (*auto intro*: *monog*)

from *derivg_nonneg* **have** *nonneg*:
$\bigwedge f\ x.\ x \geq a \Longrightarrow x \leq b \Longrightarrow g'\ x \neq 0 \Longrightarrow f\ x * ennreal\ (g'\ x) \geq 0 \Longrightarrow f\ x \geq 0$
by (*force simp*: *field_simps*)
have *nonneg'*: $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow \neg\ 0 \leq f\ (g\ x) \Longrightarrow 0 \leq f\ (g\ x) * g'\ x \Longrightarrow g'\ x = 0$
by (*metis atLeastAtMost_iff derivg_nonneg eq_iff mult_eq_0_iff mult_le_0_iff*)

have $(\int^+ x.\ f\ x * indicator\ \{g\ a..g\ b\}\ x\ \partial lborel) =$
$(\int^+ x.\ ennreal\ (?f'\ x) * indicator\ \{g\ a..g\ b\}\ x\ \partial lborel)$
by (*intro nn_integral_cong*)
(*auto split*: *split_indicator split_max simp*: *zero_ennreal.rep_eq ennreal_neg*)
also have ... = $\int^+ x.\ ?f'\ (g\ x) * ennreal\ (g'\ x) * indicator\ \{a..b\}\ x\ \partial lborel$ **using** *Mf*
by (*subst nn_integral_substitution_aux*[*OF _ _ derivg contg' derivg_nonneg* ‹$a < b$›])
(*auto simp add*: *mult.commute set_borel_measurable_def*)
also have ... = $\int^+ x.\ f\ (g\ x) * ennreal\ (g'\ x) * indicator\ \{a..b\}\ x\ \partial lborel$
by (*intro nn_integral_cong*) (*auto split*: *split_indicator simp*: *max_def dest*: *bounds*)
also have ... = $\int^+ x.\ ennreal\ (f\ (g\ x) * g'\ x * indicator\ \{a..b\}\ x)\ \partial lborel$
by (*intro nn_integral_cong*) (*auto simp*: *mult.commute derivg_nonneg ennreal_mult' split*: *split_indicator*)
finally show *?thesis* **.**
qed *auto*

theorem *integral_substitution*:
assumes *integrable*: *set_integrable lborel* $\{g\ a..g\ b\}$ *f*
assumes *derivg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow$ (*g has_real_derivative g' x*) (*at x*)
assumes *contg'*: *continuous_on* $\{a..b\}$ *g'*
assumes *derivg_nonneg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g'\ x \geq 0$
assumes $a \leq b$
shows *set_integrable lborel* $\{a..b\}$ $(\lambda x.\ f\ (g\ x) * g'\ x)$
and $(LBINT\ x.\ f\ x * indicator\ \{g\ a..g\ b\}\ x) = (LBINT\ x.\ f\ (g\ x) * g'\ x * indicator\ \{a..b\}\ x)$
proof−
from *derivg* **have** *contg*: *continuous_on* $\{a..b\}$ *g* **by** (*rule has_real_derivative_imp_continuous_on*)
with *contg'* **have** *Mg*: *set_borel_measurable borel* $\{a..b\}$ *g*
and *Mg'*: *set_borel_measurable borel* $\{a..b\}$ *g'*

by (*simp_all only*: *set_measurable_continuous_on_ivl*)
from *derivg derivg_nonneg* **have** *monog*: $\bigwedge x\ y.\ a \leq x \Longrightarrow x \leq y \Longrightarrow y \leq b \Longrightarrow g\ x \leq g\ y$
by (*rule deriv_nonneg_imp_mono*) *simp_all*

have ($\lambda x.$ *ennreal* (*f x*) $*$ *indicator* {*g a..g b*} *x*) =
($\lambda x.$ *ennreal* (*f x* $*$ *indicator* {*g a..g b*} *x*))
by (*intro ext*) (*simp split*: *split_indicator*)
with *integrable* **have** *M1*: ($\lambda x.$ *f x* $*$ *indicator* {*g a..g b*} *x*) $\in$ *borel_measurable borel*
by (*force simp*: *mult.commute set_integrable_def*)
from *integrable* **have** *M2*: ($\lambda x.$ $-$*f x* $*$ *indicator* {*g a..g b*} *x*) $\in$ *borel_measurable borel*
by (*force simp*: *mult.commute set_integrable_def*)

have *LBINT x*. (*f x* :: *real*) $*$ *indicator* {*g a..g b*} *x* =
enn2real ($\int^+$ *x*. *ennreal* (*f x*) $*$ *indicator* {*g a..g b*} *x* ∂*lborel*) $-$
enn2real ($\int^+$ *x*. *ennreal* ($-$ (*f x*)) $*$ *indicator* {*g a..g b*} *x* ∂*lborel*) **using** *integrable*
unfolding *set_integrable_def*
by (*subst real_lebesgue_integral_def*) (*simp_all add*: *nn_integral_set_ennreal mult.commute*)
also have $*$: ($\int^+$*x*. *ennreal* (*f x*) $*$ *indicator* {*g a..g b*} *x* ∂*lborel*) =
($\int^+$*x*. *ennreal* (*f x* $*$ *indicator* {*g a..g b*} *x*) ∂*lborel*)
by (*intro nn_integral_cong*) (*simp split*: *split_indicator*)
also from *M1* $*$ **have** *A*: ($\int^+$ *x*. *ennreal* (*f x* $*$ *indicator* {*g a..g b*} *x*) ∂*lborel*) =
($\int^+$ *x*. *ennreal* (*f* (*g x*) $*$ *g′ x* $*$ *indicator* {*a..b*} *x*) ∂*lborel*)
by (*subst nn_integral_substitution*[*OF _ derivg contg′ derivg_nonneg* ‹*a* ≤ *b*›])
(*auto simp*: *nn_integral_set_ennreal mult.commute set_borel_measurable_def*)
also have $**$: ($\int^+$ *x*. *ennreal* ($-$ (*f x*)) $*$ *indicator* {*g a..g b*} *x* ∂*lborel*) =
($\int^+$ *x*. *ennreal* ($-$ (*f x*) $*$ *indicator* {*g a..g b*} *x*) ∂*lborel*)
by (*intro nn_integral_cong*) (*simp split*: *split_indicator*)
also from *M2* $**$ **have** *B*: ($\int^+$ *x*. *ennreal* ($-$ (*f x*) $*$ *indicator* {*g a..g b*} *x*) ∂*lborel*) =
($\int^+$ *x*. *ennreal* ($-$ (*f* (*g x*)) $*$ *g′ x* $*$ *indicator* {*a..b*} *x*) ∂*lborel*)
by (*subst nn_integral_substitution*[*OF _ derivg contg′ derivg_nonneg* ‹*a* ≤ *b*›])
(*auto simp*: *nn_integral_set_ennreal mult.commute set_borel_measurable_def*)

also {
from *integrable* **have** *Mf*: *set_borel_measurable borel* {*g a..g b*} *f*
unfolding *set_borel_measurable_def set_integrable_def* **by** *simp*
from *measurable_compose Mg Mf Mg′ borel_measurable_times*
have ($\lambda x.$ *f* (*g x* $*$ *indicator* {*a..b*} *x*) $*$ *indicator* {*g a..g b*} (*g x* $*$ *indicator* {*a..b*} *x*) $*$
(*g′ x* $*$ *indicator* {*a..b*} *x*)) $\in$ *borel_measurable borel* (**is** *?f* $\in$ _)
by (*simp add*: *mult.commute set_borel_measurable_def*)
also have *?f* = ($\lambda x.$ *f* (*g x*) $*$ *g′ x* $*$ *indicator* {*a..b*} *x*)
using *monog* **by** (*intro ext*) (*auto split*: *split_indicator*)
finally show *set_integrable lborel* {*a..b*} ($\lambda x.$ *f* (*g x*) $*$ *g′ x*)

```
    using A B integrable unfolding real_integrable_def set_integrable_def
    by (simp_all add: nn_integral_set_ennreal mult.commute)
  } note integrable' = this

  have enn2real (∫+ x. ennreal (f (g x) * g' x * indicator {a..b} x) ∂lborel) −
                  enn2real (∫+ x. ennreal (−f (g x) * g' x * indicator {a..b} x)
∂lborel) =
              (LBINT x. f (g x) * g' x * indicator {a..b} x)
    using integrable' unfolding set_integrable_def
    by (subst real_lebesgue_integral_def) (simp_all add: field_simps)
  finally show (LBINT x. f x * indicator {g a..g b} x) =
                  (LBINT x. f (g x) * g' x * indicator {a..b} x) .
qed

theorem interval_integral_substitution:
  assumes integrable: set_integrable lborel {g a..g b} f
  assumes derivg: ⋀x. x ∈ {a..b} ⟹ (g has_real_derivative g' x) (at x)
  assumes contg': continuous_on {a..b} g'
  assumes derivg_nonneg: ⋀x. x ∈ {a..b} ⟹ g' x ≥ 0
  assumes a ≤ b
  shows set_integrable lborel {a..b} (λx. f (g x) * g' x)
    and (LBINT x=g a..g b. f x) = (LBINT x=a..b. f (g x) * g' x)
  apply (rule integral_substitution[OF assms], simp, simp)
  apply (subst (1 2) interval_integral_Icc, fact)
  apply (rule deriv_nonneg_imp_mono[OF derivg derivg_nonneg], simp, simp, fact)
  using integral_substitution(2)[OF assms]
  apply (simp add: mult.commute set_lebesgue_integral_def)
  done

lemma set_borel_integrable_singleton[simp]: set_integrable lborel {x} (f :: real ⇒
real)
  unfolding set_integrable_def
  by (subst integrable_discrete_difference[where X={x} and g=λ_. 0]) auto

end
```

6.25 The Volume of an n-Dimensional Ball

```
theory Ball_Volume
  imports Gamma_Function Lebesgue_Integral_Substitution
begin
```

We define the volume of the unit ball in terms of the Gamma function. Note that the dimension need not be an integer; we also allow fractional dimensions, although we do not use this case or prove anything about it for now.

```
definition unit_ball_vol :: real ⇒ real where
  unit_ball_vol n = pi powr (n / 2) / Gamma (n / 2 + 1)
```

```
lemma unit_ball_vol_pos [simp]: n ≥ 0 ⟹ unit_ball_vol n > 0
  by (force simp: unit_ball_vol_def intro: divide_nonneg_pos)

lemma unit_ball_vol_nonneg [simp]: n ≥ 0 ⟹ unit_ball_vol n ≥ 0
  by (simp add: dual_order.strict_implies_order)
```

We first need the value of the following integral, which is at the core of computing the measure of an $n + 1$-dimensional ball in terms of the measure of an n-dimensional one.

```
lemma emeasure_cball_aux_integral:
  (∫+x. indicator {−1..1} x ∗ sqrt (1 − x²) ^ n ∂lborel) =
    ennreal (Beta (1 / 2) (real n / 2 + 1))
proof −
  have ((λt. t powr (−1 / 2) ∗ (1 − t) powr (real n / 2)) has_integral
        Beta (1 / 2) (real n / 2 + 1)) {0..1}
    using has_integral_Beta_real[of 1/2 n / 2 + 1] by simp
  from nn_integral_has_integral_lebesgue[OF _ this] have
     ennreal (Beta (1 / 2) (real n / 2 + 1)) =
       nn_integral lborel (λt. ennreal (t powr (−1 / 2) ∗ (1 − t) powr (real n / 2)
∗
                                   indicator {0^2..1^2} t))
    by (simp add: mult_ac ennreal_mult′ ennreal_indicator)
  also have ... = (∫+ x. ennreal (x² powr − (1 / 2) ∗ (1 − x²) powr (real n /
2) ∗ (2 ∗ x) ∗
                              indicator {0..1} x) ∂lborel)
    by (subst nn_integral_substitution[where g = λx. x ^ 2 and g′ = λx. 2 ∗ x])
     (auto intro!: derivative_eq_intros continuous_intros simp: set_borel_measurable_def)
  also have ... = (∫+ x. 2 ∗ ennreal ((1 − x²) powr (real n / 2) ∗ indicator
{0..1} x) ∂lborel)
    by (intro nn_integral_cong_AE AE_I[of _ _ {0}])
         (auto simp: indicator_def powr_minus powr_half_sqrt field_split_simps en-
nreal_mult′)
  also have ... = (∫+ x. ennreal ((1 − x²) powr (real n / 2) ∗ indicator {0..1}
x) ∂lborel) +
                      (∫+ x. ennreal ((1 − x²) powr (real n / 2) ∗ indicator {0..1} x)
∂lborel)
    (is _ = ?I + _) by (simp add: mult_2 nn_integral_add)
  also have ?I = (∫+ x. ennreal ((1 − x²) powr (real n / 2) ∗ indicator {−1..0}
x) ∂lborel)
    by (subst nn_integral_real_affine[of _ −1 0])
      (auto simp: indicator_def intro!: nn_integral_cong)
  hence ?I + ?I = ... + ?I by simp
  also have ... = (∫+ x. ennreal ((1 − x²) powr (real n / 2) ∗
                      (indicator {−1..0} x + indicator{0..1} x)) ∂lborel)
    by (subst nn_integral_add [symmetric]) (auto simp: algebra_simps)
  also have ... = (∫+ x. ennreal ((1 − x²) powr (real n / 2) ∗ indicator {−1..1}
x) ∂lborel)
    by (intro nn_integral_cong_AE AE_I[of _ _ {0}]) (auto simp: indicator_def)
```

also have ... $= (\int^+ x.$ *ennreal* (*indicator* $\{-1..1\}$ $x *$ *sqrt* $(1 - x^2)$ ^ n) ∂*lborel*)
  by (*intro nn_integral_cong_AE AE_I*[*of* _ _ {*1*, −*1*}])
    (*auto simp*: *powr_half_sqrt* [*symmetric*] *indicator_def abs_square_le_1*
    abs_square_eq_1 powr_def exp_of_nat_mult [*symmetric*] *emeasure_lborel_countable*)
 finally show *?thesis* **..**
qed

lemma *real_sqrt_le_iff'*: $x \geq 0 \Longrightarrow y \geq 0 \Longrightarrow$ *sqrt* $x \leq y \longleftrightarrow x \leq y$ ^ *2*
 using *real_le_lsqrt sqrt_le_D* **by** *blast*

lemma *power2_le_iff_abs_le*: $y \geq 0 \Longrightarrow$ (*x*::*real*) ^ *2* $\leq y$ ^ *2* $\longleftrightarrow$ *abs* $x \leq y$
 by (*subst real_sqrt_le_iff'* [*symmetric*]) *auto*

Isabelle's type system makes it very difficult to do an induction over the dimension of a Euclidean space type, because the type would change in the inductive step. To avoid this problem, we instead formulate the problem in a more concrete way by unfolding the definition of the Euclidean norm.

lemma *emeasure_cball_aux*:
 assumes *finite A* $r > 0$
 shows *emeasure* (Pi_M *A* (λ_. *lborel*))
   ({*f*. *sqrt* ($\sum$ *i*∈*A*. (*f i*)^2) ≤ *r*} ∩ *space* (Pi_M *A* (λ_. *lborel*))) =
   ennreal (*unit_ball_vol* (*real* (*card A*)) ∗ *r* ^ *card A*)
 using *assms*
proof (*induction arbitrary*: *r*)
 case (*empty r*)
 thus *?case*
  by (*simp add*: *unit_ball_vol_def space_PiM*)
next
 case (*insert i A r*)
 interpret *product_sigma_finite* λ_. *lborel*
  by *standard*
 have *emeasure* (Pi_M (*insert i A*) (λ_. *lborel*))
   ({*f*. *sqrt* ($\sum$ *i*∈*insert i A*. (*f i*)^2) ≤ *r*} ∩ *space* (Pi_M (*insert i A*) (λ_. *lborel*))) =
  nn_integral (Pi_M (*insert i A*) (λ_. *lborel*))
   (*indicator* ({*f*. *sqrt* ($\sum$ *i*∈*insert i A*. (*f i*)^2) ≤ *r*} ∩
   space (Pi_M (*insert i A*) (λ_. *lborel*))))
  by (*subst nn_integral_indicator*) *auto*
 also have ... = ($\int^+$ *y*. $\int^+$ *x*. *indicator* ({*f*. *sqrt* ((*f i*)^2 + ($\sum$ *i*∈*A*. (*f i*)^2)) ≤ *r*} ∩
     space (Pi_M (*insert i A*) (λ_. *lborel*))) (*x*(*i* := *y*))
   ∂Pi_M *A* (λ_. *lborel*) ∂*lborel*)
  using *insert.prems insert.hyps* **by** (*subst product_nn_integral_insert_rev*) *auto*
 also have ... = ($\int^+$ (*y*::*real*). $\int^+$ *x*. *indicator* {−*r*..*r*} *y* ∗ *indicator* ({*f*. *sqrt* (($\sum$ *i*∈*A*. (*f i*)^2)) ≤
    sqrt (*r* ^ *2* − *y* ^ *2*)} ∩ *space* (Pi_M *A* (λ_. *lborel*))) *x* ∂Pi_M *A* (λ_. *lborel*) ∂*lborel*)
 proof (*intro nn_integral_cong*, *goal_cases*)

```
    case (1 y f)
    have *: y ∈ {−r..r} if y ˆ 2 + c ≤ r ˆ 2 c ≥ 0 for c
    proof −
      have y ˆ 2 ≤ y ˆ 2 + c using that by simp
      also have . . . ≤ r ˆ 2 by fact
      finally show ?thesis
        using ‹r > 0› by (simp add: power2_le_iff_abs_le abs_if split: if_splits)
    qed
    have (∑x∈A. (if x = i then y else f x)²) = (∑x∈A. (f x)²)
      using insert.hyps by (intro sum.cong) auto
    thus ?case using 1 ‹r > 0›
      by (auto simp: sum_nonneg real_sqrt_le_iff' indicator_def PiE_def space_PiM
dest!: *)
  qed
  also have . . . = (∫+ (y::real). indicator {−r..r} y * (∫+ x. indicator ({f. sqrt
((∑i∈A. (f i)²))
                                   ≤ sqrt (r ˆ 2 − y ˆ 2)} ∩ space (Pi_M A (λ_. lborel))) x
                   ∂Pi_M A (λ_. lborel)) ∂lborel) by (subst nn_integral_cmult) auto
   also have . . . = (∫+ (y::real). indicator {−r..r} y * emeasure (PiM A (λ_.
lborel))
       ({f. sqrt ((∑i∈A. (f i)²)) ≤ sqrt (r ˆ 2 − y ˆ 2)} ∩ space (Pi_M A (λ_.
lborel))) ∂lborel)
    using ‹finite A› by (intro nn_integral_cong, subst nn_integral_indicator) auto
  also have . . . = (∫+ (y::real). indicator {−r..r} y * ennreal (unit_ball_vol (real
(card A)) *
                                  (sqrt (r ˆ 2 − y ˆ 2)) ˆ card A) ∂lborel)
  proof (intro nn_integral_cong_AE, goal_cases)
    case 1
    have AE y in lborel. y ∉ {−r,r}
      by (intro AE_not_in countable_imp_null_set_lborel) auto
    thus ?case
    proof eventually_elim
      case (elim y)
      show ?case
      proof (cases y ∈ {−r<..<r})
        case True
        hence y² < r² by (subst real_sqrt_less_iff [symmetric]) auto
        thus ?thesis by (subst insert.IH) (auto)
      qed (insert elim, auto)
    qed
  qed
  also have . . . = ennreal (unit_ball_vol (real (card A))) *
                   (∫+ (y::real). indicator {−r..r} y * (sqrt (r ˆ 2 − y ˆ 2)) ˆ card
A ∂lborel)
    by (subst nn_integral_cmult [symmetric])
     (auto simp: mult_ac ennreal_mult' [symmetric] indicator_def intro!: nn_integral_cong)
  also have (∫+ (y::real). indicator {−r..r} y * (sqrt (r ˆ 2 − y ˆ 2)) ˆ card A
∂lborel) =
               (∫+ (y::real). r ˆ card A * indicator {−1..1} y * (sqrt (1 − y ˆ 2))
```

^ *card A*
∂(*distr lborel borel* ((∗) (*1*/*r*)))) **using** ‹*r* > *0*›
by (*subst nn_integral_distr*)
(*auto simp*: *indicator_def field_simps real_sqrt_divide intro*!: *nn_integral_cong*)
also have ... = (∫ + *x*. *ennreal* (*r* ^ *Suc* (*card A*)) ∗
(*indicator* {− *1*..*1*} *x* ∗ *sqrt* (*1* − x^2) ^ *card A*) ∂*lborel*) **using** ‹*r* > *0*›
by (*subst lborel_distr_mult*) (*auto simp*: *nn_integral_density ennreal_mult′* [*symmetric*] *mult_ac*)
also have ... = *ennreal* (*r* ^ *Suc* (*card A*)) ∗ (∫ + *x*. *indicator* {− *1*..*1*} *x* ∗
sqrt (*1* − x^2) ^ *card A* ∂*lborel*)
by (*subst nn_integral_cmult*) *auto*
also note *emeasure_cball_aux_integral*
also have *ennreal* (*unit_ball_vol* (*real* (*card A*))) ∗ (*ennreal* (*r* ^ *Suc* (*card A*)) ∗
ennreal (*Beta* (*1*/*2*) (*card A* / *2* + *1*))) =
ennreal (*unit_ball_vol* (*card A*) ∗ *Beta* (*1*/*2*) (*card A* / *2* + *1*) ∗ *r* ^ *Suc* (*card A*))
using ‹*r* > *0*› **by** (*simp add*: *ennreal_mult′* [*symmetric*] *mult_ac*)
also have *unit_ball_vol* (*card A*) ∗ *Beta* (*1*/*2*) (*card A* / *2* + *1*) = *unit_ball_vol* (*Suc* (*card A*))
by (*auto simp*: *unit_ball_vol_def Beta_def Gamma_eq_zero_iff field_simps*
Gamma_one_half_real powr_half_sqrt [*symmetric*] *powr_add* [*symmetric*])
also have *Suc* (*card A*) = *card* (*insert i A*) **using** *insert.hyps* **by** *simp*
finally show *?case* **.**
qed

We now get the main theorem very easily by just applying the above lemma.

context
fixes *c* :: ′*a* :: *euclidean_space* **and** *r* :: *real*
assumes *r*: *r* ≥ *0*
begin

theorem *emeasure_cball*:
emeasure lborel (*cball c r*) = *ennreal* (*unit_ball_vol* (*DIM*(′*a*)) ∗ *r* ^ *DIM*(′*a*))
proof (*cases r* = *0*)
case *False*
with *r* **have** *r*: *r* > *0* **by** *simp*
have (*lborel* :: ′*a measure*) =
distr (Pi_M *Basis* (λ_. *lborel*)) *borel* (λ*f*. ∑ *b*∈*Basis*. *f b* $*_R$ *b*)
by (*rule lborel_eq*)
also have *emeasure* ... (*cball 0 r*) =
emeasure (Pi_M *Basis* (λ_. *lborel*))
({*y*. *dist 0* (∑ *b*∈*Basis*. *y b* $*_R$ *b* :: ′*a*) ≤ *r*} ∩ *space* (Pi_M *Basis* (λ_. *lborel*)))
by (*subst emeasure_distr*) (*auto simp*: *cball_def*)
also have {*f*. *dist 0* (∑ *b*∈*Basis*. *f b* $*_R$ *b* :: ′*a*) ≤ *r*} = {*f*. *sqrt* (∑ *i*∈*Basis*. (*f i*)^2) ≤ *r*}
by (*subst euclidean_dist_l2*) (*auto simp*: *L2_set_def*)
also have *emeasure* (Pi_M *Basis* (λ_. *lborel*)) (... ∩ *space* (Pi_M *Basis* (λ_. *lborel*))) =

```
            ennreal (unit_ball_vol (real DIM('a)) * r ^ DIM('a))
    using r by (subst emeasure_cball_aux) simp_all
  also have emeasure lborel (cball 0 r :: 'a set) =
            emeasure (distr lborel borel (λx. c + x)) (cball c r)
   by (subst emeasure_distr) (auto simp: cball_def dist_norm norm_minus_commute)
  also have distr lborel borel (λx. c + x) = lborel
    using lborel_affine[of 1 c] by (simp add: density_1)
  finally show ?thesis .
qed auto

corollary content_cball:
  content (cball c r) = unit_ball_vol (DIM('a)) * r ^ DIM('a)
  by (simp add: measure_def emeasure_cball r)

corollary emeasure_ball:
  emeasure lborel (ball c r) = ennreal (unit_ball_vol (DIM('a)) * r ^ DIM('a))
proof -
  from negligible_sphere[of c r] have sphere c r ∈ null_sets lborel
   by (auto simp: null_sets_completion_iff negligible_iff_null_sets negligible_convex_frontier)
  hence emeasure lborel (ball c r ∪ sphere c r :: 'a set) = emeasure lborel (ball c
r :: 'a set)
    by (intro emeasure_Un_null_set) auto
  also have ball c r ∪ sphere c r = (cball c r :: 'a set) by auto
   also have emeasure lborel ... = ennreal (unit_ball_vol (real DIM('a)) * r ^
DIM('a))
    by (rule emeasure_cball)
  finally show ?thesis ..
qed

corollary content_ball:
  content (ball c r) = unit_ball_vol (DIM('a)) * r ^ DIM('a)
  by (simp add: measure_def r emeasure_ball)

end
```

Lastly, we now prove some nicer explicit formulas for the volume of the unit balls in the cases of even and odd integer dimensions.

```
lemma unit_ball_vol_even:
  unit_ball_vol (real (2 * n)) = pi ^ n / fact n
  by (simp add: unit_ball_vol_def add_ac powr_realpow Gamma_fact)

lemma unit_ball_vol_odd':
      unit_ball_vol (real (2 * n + 1)) = pi ^ n / pochhammer (1 / 2) (Suc n)
  and unit_ball_vol_odd:
      unit_ball_vol (real (2 * n + 1)) =
        (2 ^ (2 * Suc n) * fact (Suc n)) / fact (2 * Suc n) * pi ^ n
proof -
  have unit_ball_vol (real (2 * n + 1)) =
         pi powr (real n + 1 / 2) / Gamma (1 / 2 + real (Suc n))
```

by (*simp add*: *unit_ball_vol_def field_simps*)
also have *pochhammer* (*1* / *2*) (*Suc n*) = *Gamma* (*1* / *2* + *real* (*Suc n*)) / *Gamma* (*1* / *2*)
by (*intro pochhammer_Gamma*) *auto*
hence *Gamma* (*1* / *2* + *real* (*Suc n*)) = *sqrt pi* ∗ *pochhammer* (*1* / *2*) (*Suc n*)
by (*simp add*: *Gamma_one_half_real*)
also have *pi powr* (*real n* + *1* / *2*) / ... = *pi* ˆ *n* / *pochhammer* (*1* / *2*) (*Suc n*)
by (*simp add*: *powr_add powr_half_sqrt powr_realpow*)
finally show *unit_ball_vol* (*real* (*2* ∗ *n* + *1*)) = ... **.**
also have *pochhammer* (*1* / *2* :: *real*) (*Suc n*) =
fact (*2* ∗ *Suc n*) / (*2* ˆ (*2* ∗ *Suc n*) ∗ *fact* (*Suc n*))
using *fact_double*[*of Suc n*, **where** *?'a* = *real*] **by** (*simp add*: *divide_simps mult_ac*)
also have *pi* ˆ*n* / ... = (*2* ˆ (*2* ∗ *Suc n*) ∗ *fact* (*Suc n*)) / *fact* (*2* ∗ *Suc n*) ∗ *pi* ˆ *n*
by *simp*
finally show *unit_ball_vol* (*real* (*2* ∗ *n* + *1*)) = ... **.**
qed

lemma *unit_ball_vol_numeral*:
unit_ball_vol (*numeral* (*Num.Bit0 n*)) = *pi* ˆ *numeral n* / *fact* (*numeral n*) (**is** *?th1*)
unit_ball_vol (*numeral* (*Num.Bit1 n*)) = *2* ˆ (*2* ∗ *Suc* (*numeral n*)) ∗ *fact* (*Suc* (*numeral n*)) /
fact (*2* ∗ *Suc* (*numeral n*)) ∗ *pi* ˆ *numeral n* (**is** *?th2*)
proof −
have *numeral* (*Num.Bit0 n*) = (*2* ∗ *numeral n* :: *nat*)
by (*simp only*: *numeral_Bit0 mult_2 ring_distribs*)
also have *unit_ball_vol* ... = *pi* ˆ *numeral n* / *fact* (*numeral n*)
by (*rule unit_ball_vol_even*)
finally show *?th1* **by** *simp*
next
have *numeral* (*Num.Bit1 n*) = (*2* ∗ *numeral n* + *1* :: *nat*)
by (*simp only*: *numeral_Bit1 mult_2*)
also have *unit_ball_vol* ... = *2* ˆ (*2* ∗ *Suc* (*numeral n*)) ∗ *fact* (*Suc* (*numeral n*)) /
fact (*2* ∗ *Suc* (*numeral n*)) ∗ *pi* ˆ *numeral n*
by (*rule unit_ball_vol_odd*)
finally show *?th2* **by** *simp*
qed

lemmas *eval_unit_ball_vol* = *unit_ball_vol_numeral fact_numeral*

Just for fun, we compute the volume of unit balls for a few dimensions.

lemma *unit_ball_vol_0* [*simp*]: *unit_ball_vol 0* = *1*
using *unit_ball_vol_even*[*of 0*] **by** *simp*

lemma *unit_ball_vol_1* [*simp*]: *unit_ball_vol 1* = *2*

using *unit_ball_vol_odd*[*of 0*] **by** *simp*

corollary
unit_ball_vol_2: *unit_ball_vol 2 = pi*
and *unit_ball_vol_3*: *unit_ball_vol 3 = 4 / 3 ∗ pi*
and *unit_ball_vol_4*: *unit_ball_vol 4 = pi*^{2} */ 2*
and *unit_ball_vol_5*: *unit_ball_vol 5 = 8 / 15 ∗ pi*^{2}
by (*simp_all add*: *eval_unit_ball_vol*)

corollary *circle_area*:
r ≥ 0 ⟹ content (*ball c r* :: (*real ^ 2*) *set*) = *r ^ 2 ∗ pi*
by (*simp add*: *content_ball unit_ball_vol_2*)

corollary *sphere_volume*:
r ≥ 0 ⟹ content (*ball c r* :: (*real ^ 3*) *set*) = *4 / 3 ∗ r ^ 3 ∗ pi*
by (*simp add*: *content_ball unit_ball_vol_3*)

Useful equivalent forms

corollary *content_ball_eq_0_iff* [*simp*]: *content* (*ball c r*) = *0* ⟷ *r ≤ 0*
proof −
have *r > 0 ⟹ content* (*ball c r*) > *0*
by (*simp add*: *content_ball unit_ball_vol_def*)
then show *?thesis*
by (*fastforce simp*: *ball_empty*)
qed

corollary *content_ball_gt_0_iff* [*simp*]: *0 < content* (*ball z r*) ⟷ *0 < r*
by (*auto simp*: *zero_less_measure_iff*)

corollary *content_cball_eq_0_iff* [*simp*]: *content* (*cball c r*) = *0* ⟷ *r ≤ 0*
proof (*cases r = 0*)
case *False*
moreover have *r > 0 ⟹ content* (*cball c r*) > *0*
by (*simp add*: *content_cball unit_ball_vol_def*)
ultimately show *?thesis*
by *fastforce*
qed *auto*

corollary *content_cball_gt_0_iff* [*simp*]: *0 < content* (*cball z r*) ⟷ *0 < r*
by (*auto simp*: *zero_less_measure_iff*)

end

6.26 Integral Test for Summability

theory *Integral_Test*
imports *Henstock_Kurzweil_Integration*
begin

The integral test for summability. We show here that for a decreasing non-negative function, the infinite sum over that function evaluated at the natural numbers converges iff the corresponding integral converges.

As a useful side result, we also provide some results on the difference between the integral and the partial sum. (This is useful e.g. for the definition of the Euler-Mascheroni constant)

locale *antimono_fun_sum_integral_diff* =
fixes *f* :: *real* ⇒ *real*
assumes *dec*: $\bigwedge x\ y.\ x \geq 0 \Longrightarrow x \leq y \Longrightarrow f\ x \geq f\ y$
assumes *nonneg*: $\bigwedge x.\ x \geq 0 \Longrightarrow f\ x \geq 0$
assumes *cont*: *continuous_on* {*0*..} *f*
begin

definition *sum_integral_diff_series n* = (∑ *k*≤*n*. *f* (*of_nat k*)) − (*integral* {*0*..*of_nat n*} *f*)

lemma *sum_integral_diff_series_nonneg*:
sum_integral_diff_series n ≥ *0*
proof −
note *int* = *integrable_continuous_real*[*OF continuous_on_subset*[*OF cont*]]
let *?int* = λ*a b*. *integral* {*of_nat a*..*of_nat b*} *f*
have −*sum_integral_diff_series n* = *?int 0 n* − (∑ *k*≤*n*. *f* (*of_nat k*))
by (*simp add*: *sum_integral_diff_series_def*)
also have *?int 0 n* = (∑ *k*<*n*. *?int k* (*Suc k*))
proof (*induction n*)
case (*Suc n*)
have *?int 0* (*Suc n*) = *?int 0 n* + *?int n* (*Suc n*)
by (*intro integral_combine*[*symmetric*] *int*) *simp_all*
with *Suc* **show** *?case* **by** *simp*
qed *simp_all*
also have ... ≤ (∑ *k*<*n*. *integral* {*of_nat k*..*of_nat* (*Suc k*)} (λ_::*real*. *f* (*of_nat k*)))
by (*intro sum_mono integral_le int*) (*auto intro*: *dec*)
also have ... = (∑ *k*<*n*. *f* (*of_nat k*)) **by** *simp*
also have ... − (∑ *k*≤*n*. *f* (*of_nat k*)) = −(∑ *k*∈{..*n*} − {..<*n*}. *f* (*of_nat k*))
by (*subst sum_diff*) *auto*
also have ... ≤ *0* **by** (*auto intro*!: *sum_nonneg nonneg*)
finally show *sum_integral_diff_series n* ≥ *0* **by** *simp*
qed

lemma *sum_integral_diff_series_antimono*:
assumes *m* ≤ *n*
shows *sum_integral_diff_series m* ≥ *sum_integral_diff_series n*
proof −
let *?int* = λ*a b*. *integral* {*of_nat a*..*of_nat b*} *f*
note *int* = *integrable_continuous_real*[*OF continuous_on_subset*[*OF cont*]]
have *d_mono*: *sum_integral_diff_series* (*Suc n*) ≤ *sum_integral_diff_series n* **for** *n*
proof −

```
    fix n :: nat
    have sum_integral_diff_series (Suc n) − sum_integral_diff_series n =
          f (of_nat (Suc n)) + (?int 0 n − ?int 0 (Suc n))
      unfolding sum_integral_diff_series_def by (simp add: algebra_simps)
    also have ?int 0 n − ?int 0 (Suc n) = −?int n (Suc n)
      by (subst integral_combine [symmetric, of of_nat 0 of_nat n of_nat (Suc n)])
        (auto intro!: int simp: algebra_simps)
     also have ?int n (Suc n) ≥ integral {of_nat n..of_nat (Suc n)} (λ_::real. f
(of_nat (Suc n)))
      by (intro integral_le int) (auto intro: dec)
   hence f (of_nat (Suc n)) + −?int n (Suc n) ≤ 0 by (simp add: algebra_simps)
    finally show sum_integral_diff_series (Suc n) ≤ sum_integral_diff_series n by
simp
  qed
  with assms show ?thesis
    by (induction rule: inc_induct) (auto intro: order.trans[OF _ d_mono])
qed

lemma sum_integral_diff_series_Bseq: Bseq sum_integral_diff_series
proof −
  from sum_integral_diff_series_nonneg and sum_integral_diff_series_antimono
    have norm (sum_integral_diff_series n) ≤ sum_integral_diff_series 0 for n by
simp
  thus Bseq sum_integral_diff_series by (rule BseqI')
qed

lemma sum_integral_diff_series_monoseq: monoseq sum_integral_diff_series
  using sum_integral_diff_series_antimono unfolding monoseq_def by blast

lemma sum_integral_diff_series_convergent: convergent sum_integral_diff_series
  using sum_integral_diff_series_Bseq sum_integral_diff_series_monoseq
  by (blast intro!: Bseq_monoseq_convergent)

theorem integral_test:
  summable (λn. f (of_nat n)) ⟷ convergent (λn. integral {0..of_nat n} f)
proof −
  have summable (λn. f (of_nat n)) ⟷ convergent (λn. ∑k≤n. f (of_nat k))
    by (simp add: summable_iff_convergent')
  also have ... ⟷ convergent (λn. integral {0..of_nat n} f)
  proof
    assume convergent (λn. ∑k≤n. f (of_nat k))
    from convergent_diff[OF this sum_integral_diff_series_convergent]
      show convergent (λn. integral {0..of_nat n} f)
        unfolding sum_integral_diff_series_def by simp
  next
    assume convergent (λn. integral {0..of_nat n} f)
    from convergent_add[OF this sum_integral_diff_series_convergent]
    show convergent (λn. ∑k≤n. f (of_nat k)) unfolding sum_integral_diff_series_def
by simp
```

qed
finally show *?thesis* **by** *simp*
qed

end

end

6.27 Continuity of the indefinite integral; improper integral theorem

theory *Improper_Integral*
imports *Equivalence_Lebesgue_Henstock_Integration*
begin

6.27.1 Equiintegrability

The definition here only really makes sense for an elementary set. We just use compact intervals in applications below.

definition *equiintegrable_on* (**infixr** *equiintegrable′_on 46*)
where *F equiintegrable_on I* ≡
(∀ *f* ∈ *F*. *f integrable_on I*) ∧
(∀ *e* > *0*. ∃ γ. *gauge* γ ∧
(∀ *f* 𝒟. *f* ∈ *F* ∧ 𝒟 *tagged_division_of I* ∧ γ *fine* 𝒟
⟶ *norm* ((∑ (*x*,*K*) ∈ 𝒟. *content K* $*_R$ *f x*) − *integral I f*)
< *e*))

lemma *equiintegrable_on_integrable*:
⟦*F equiintegrable_on I*; *f* ∈ *F*⟧ ⟹ *f integrable_on I*
using *equiintegrable_on_def* **by** *metis*

lemma *equiintegrable_on_sing* [*simp*]:
{*f*} *equiintegrable_on cbox a b* ⟷ *f integrable_on cbox a b*
by (*simp add*: *equiintegrable_on_def has_integral_integral has_integral integrable_on_def*)

lemma *equiintegrable_on_subset*: ⟦*F equiintegrable_on I*; *G* ⊆ *F*⟧ ⟹ *G equiintegrable_on I*
unfolding *equiintegrable_on_def Ball_def*
by (*erule conj_forward imp_forward all_forward ex_forward* | *blast*)+

lemma *equiintegrable_on_Un*:
assumes *F equiintegrable_on I G equiintegrable_on I*
shows (*F* ∪ *G*) *equiintegrable_on I*
unfolding *equiintegrable_on_def*
proof (*intro conjI impI allI*)
show ∀ *f*∈*F* ∪ *G*. *f integrable_on I*
using *assms* **unfolding** *equiintegrable_on_def* **by** *blast*
show ∃ γ. *gauge* γ ∧

(∀f 𝒟. f ∈ F ∪ G ∧
  𝒟 tagged_division_of I ∧ γ fine 𝒟 ⟶
  norm ((∑(x,K) ∈ 𝒟. content K *R f x) − integral I f) < ε)
 if ε > 0 **for** ε
proof −
 obtain γ1 **where** gauge γ1
  and γ1: ⋀f 𝒟. f ∈ F ∧ 𝒟 tagged_division_of I ∧ γ1 fine 𝒟
   ⟹ norm ((∑(x,K) ∈ 𝒟. content K *R f x) − integral I f) < ε
  using assms ‹ε > 0› **unfolding** equiintegrable_on_def **by** auto
 obtain γ2 **where** gauge γ2
  and γ2: ⋀f 𝒟. f ∈ G ∧ 𝒟 tagged_division_of I ∧ γ2 fine 𝒟
   ⟹ norm ((∑(x,K) ∈ 𝒟. content K *R f x) − integral I f) < ε
  using assms ‹ε > 0› **unfolding** equiintegrable_on_def **by** auto
 have gauge (λx. γ1 x ∩ γ2 x)
  using ‹gauge γ1› ‹gauge γ2› **by** blast
 moreover have ∀f 𝒟. f ∈ F ∪ G ∧ 𝒟 tagged_division_of I ∧ (λx. γ1 x ∩ γ2
x) fine 𝒟 ⟶
  norm ((∑(x,K) ∈ 𝒟. content K *R f x) − integral I f) < ε
  using γ1 γ2 **by** (auto simp: fine_Int)
 ultimately show ?thesis
  by (intro exI conjI) assumption+
qed
qed

lemma equiintegrable_on_insert:
 assumes f integrable_on cbox a b F equiintegrable_on cbox a b
 shows (insert f F) equiintegrable_on cbox a b
 by (metis assms equiintegrable_on_Un equiintegrable_on_sing insert_is_Un)

lemma equiintegrable_cmul:
 assumes F: F equiintegrable_on I
 shows (⋃c ∈ {−k..k}. ⋃f ∈ F. {(λx. c *R f x)}) equiintegrable_on I
 unfolding equiintegrable_on_def
 proof (intro conjI impI allI ballI)
 show f integrable_on I
  if f ∈ (⋃c∈{− k..k}. ⋃f∈F. {λx. c *R f x})
  for f :: 'a ⇒ 'b
  using that assms equiintegrable_on_integrable integrable_cmul **by** blast
 show ∃γ. gauge γ ∧ (∀f 𝒟. f ∈ (⋃c∈{− k..k}. ⋃f∈F. {λx. c *R f x}) ∧ 𝒟
tagged_division_of I
  ∧ γ fine 𝒟 ⟶ norm ((∑(x, K)∈𝒟. content K *R f x) − integral I f) <
ε)
  if ε > 0 **for** ε
 proof −
  obtain γ **where** gauge γ
   and γ: ⋀f 𝒟. ⟦f ∈ F; 𝒟 tagged_division_of I; γ fine 𝒟⟧
    ⟹ norm ((∑(x,K) ∈ 𝒟. content K *R f x) − integral I f) < ε

/ (|*k*| + *1*)
using *assms* ‹*ε* > *0*› **unfolding** *equiintegrable_on_def*
by (*metis add.commute add.right_neutral add_strict_mono divide_pos_pos norm_eq_zero real_norm_def zero_less_norm_iff zero_less_one*)
moreover have *norm* (($\sum$(*x*, *K*)∈𝒟. *content K* $*_R$ *c* $*_R$ (*f x*)) − *integral I* (λ*x*. *c* $*_R$ *f x*)) < *ε*
if *c*: *c* ∈ {− *k*..*k*}
and *f* ∈ *F* 𝒟 *tagged_division_of I* γ *fine* 𝒟
for 𝒟 *c f*
proof −
have *norm* (($\sum$ *x*∈𝒟. *case x of* (*x*, *K*) ⇒ *content K* $*_R$ *c* $*_R$ *f x*) − *integral I* (λ*x*. *c* $*_R$ *f x*))
= |*c*| ∗ *norm* (($\sum$ *x*∈𝒟. *case x of* (*x*, *K*) ⇒ *content K* $*_R$ *f x*) − *integral I f*)
by (*simp add*: *algebra_simps scale_sum_right case_prod_unfold flip*: *norm_scaleR*)
also have . . . ≤ (|*k*| + *1*) ∗ *norm* (($\sum$ *x*∈𝒟. *case x of* (*x*, *K*) ⇒ *content K* $*_R$ *f x*) − *integral I f*)
using *c* **by** (*auto simp*: *mult_right_mono*)
also have . . . < (|*k*| + *1*) ∗ (*ε* / (|*k*| + *1*))
by (*rule mult_strict_left_mono*) (*use* γ *less_eq_real_def that* **in** *auto*)
also have . . . = *ε*
by *auto*
finally show *?thesis* **.**
qed
ultimately show *?thesis*
by (*rule_tac x*=γ **in** *exI*) *auto*
qed
qed

lemma *equiintegrable_add*:
assumes *F*: *F equiintegrable_on I* **and** *G*: *G equiintegrable_on I*
shows ($\bigcup$*f* ∈ *F*. $\bigcup$*g* ∈ *G*. {(λ*x*. *f x* + *g x*)}) *equiintegrable_on I*
unfolding *equiintegrable_on_def*
proof (*intro conjI impI allI ballI*)
show *f integrable_on I*
if *f* ∈ ($\bigcup$*f*∈*F*. $\bigcup$*g*∈*G*. {λ*x*. *f x* + *g x*}) **for** *f*
using *that equiintegrable_on_integrable assms* **by** (*auto intro*: *integrable_add*)
show ∃γ. *gauge* γ ∧ (∀*f* 𝒟. *f* ∈ ($\bigcup$*f*∈*F*. $\bigcup$*g*∈*G*. {λ*x*. *f x* + *g x*}) ∧ 𝒟 *tagged_division_of I*
∧ γ *fine* 𝒟 ⟶ *norm* (($\sum$(*x*, *K*)∈𝒟. *content K* $*_R$ *f x*) − *integral I f*) < *ε*)
if *ε* > *0* **for** *ε*
proof −
obtain γ*1* **where** *gauge* γ*1*
and γ*1*: ⋀*f* 𝒟. ⟦*f* ∈ *F*; 𝒟 *tagged_division_of I*; γ*1 fine* 𝒟⟧
⟹ *norm* (($\sum$(*x*,*K*) ∈ 𝒟. *content K* $*_R$ *f x*) − *integral I f*) < *ε*/*2*
using *assms* ‹*ε* > *0*› **unfolding** *equiintegrable_on_def* **by** (*meson half_gt_zero_iff*)
obtain γ*2* **where** *gauge* γ*2*

```
      and γ2: ⋀g 𝒟. ⟦g ∈ G; 𝒟 tagged_division_of I; γ2 fine 𝒟⟧
              ⟹ norm ((∑(x,K) ∈ 𝒟. content K *R g x) − integral I g) < ε/2
    using assms ‹ε > 0› unfolding equiintegrable_on_def by (meson half_gt_zero_iff)
    have gauge (λx. γ1 x ∩ γ2 x)
      using ‹gauge γ1› ‹gauge γ2› by blast
    moreover have norm ((∑(x,K) ∈ 𝒟. content K *R h x) − integral I h) < ε
      if h: h ∈ (⋃f∈F. ⋃g∈G. {λx. f x + g x})
        and 𝒟: 𝒟 tagged_division_of I and fine: (λx. γ1 x ∩ γ2 x) fine 𝒟
      for h 𝒟
    proof −
      obtain f g where f ∈ F g ∈ G and heq: h = (λx. f x + g x)
        using h by blast
      then have int: f integrable_on I g integrable_on I
        using F G equiintegrable_on_def by blast+
      have norm ((∑(x,K) ∈ 𝒟. content K *R h x) − integral I h)
         = norm ((∑(x,K) ∈ 𝒟. content K *R f x + content K *R g x) − (integral
I f + integral I g))
        by (simp add: heq algebra_simps integral_add int)
      also have ... = norm (((∑(x,K) ∈ 𝒟. content K *R f x) − integral I f +
(∑(x,K) ∈ 𝒟. content K *R g x) − integral I g))
        by (simp add: sum.distrib algebra_simps case_prod_unfold)
      also have ... ≤ norm ((∑(x,K) ∈ 𝒟. content K *R f x) − integral I f) +
norm ((∑(x,K) ∈ 𝒟. content K *R g x) − integral I g)
        by (metis (mono_tags) add_diff_eq norm_triangle_ineq)
      also have ... < ε/2 + ε/2
        using γ1 [OF ‹f ∈ F› 𝒟] γ2 [OF ‹g ∈ G› 𝒟] fine by (simp add: fine_Int)
      finally show ?thesis by simp
    qed
    ultimately show ?thesis
      by meson
  qed
qed

lemma equiintegrable_minus:
  assumes F equiintegrable_on I
  shows (⋃f ∈ F. {(λx. − f x)}) equiintegrable_on I
   by (force intro: equiintegrable_on_subset [OF equiintegrable_cmul [OF assms, of
1]])

lemma equiintegrable_diff:
  assumes F: F equiintegrable_on I and G: G equiintegrable_on I
  shows (⋃f ∈ F. ⋃g ∈ G. {(λx. f x − g x)}) equiintegrable_on I
  by (rule equiintegrable_on_subset [OF equiintegrable_add [OF F equiintegrable_minus
[OF G]]]) auto


lemma equiintegrable_sum:
  fixes F :: ('a::euclidean_space ⇒ 'b::euclidean_space) set
  assumes F equiintegrable_on cbox a b
```

shows ($\bigcup I \in$ *Collect finite*. $\bigcup c \in \{c.\ (\forall\, i \in I.\ c\ i \geq 0) \wedge sum\ c\ I = 1\}$.
$\bigcup f \in I \to F.\ \{(\lambda x.\ sum\ (\lambda i{::}'j.\ c\ i *_R f\ i\ x)\ I)\}$) *equiintegrable_on cbox a b*
(**is** *?G equiintegrable_on _*)
unfolding *equiintegrable_on_def*
proof (*intro conjI impI allI ballI*)
show *f integrable_on cbox a b* **if** $f \in$ *?G* **for** *f*
using *that assms* **by** (*auto simp*: *equiintegrable_on_def intro*!: *integrable_sum integrable_cmul*)
show $\exists\gamma$. *gauge* γ
$\wedge$ ($\forall\, g\ \mathcal{D}.\ g \in$ *?G* $\wedge\ \mathcal{D}$ *tagged_division_of cbox a b* $\wedge\ \gamma$ *fine* $\mathcal{D}$
$\longrightarrow$ *norm* $((\sum (x,K) \in \mathcal{D}.\ content\ K *_R g\ x) - integral\ (cbox\ a\ b)\ g) < \varepsilon)$
if $\varepsilon > 0$ **for** ε
proof −
obtain γ **where** *gauge* γ
and γ: $\bigwedge f\ \mathcal{D}.\ [\![f \in F;\ \mathcal{D}$ *tagged_division_of cbox a b*; γ *fine* $\mathcal{D}]\!]$
$\Longrightarrow$ *norm* $((\sum (x,K) \in \mathcal{D}.\ content\ K *_R f\ x) - integral\ (cbox\ a\ b)\ f) < \varepsilon\ /\ 2$
using *assms* ‹$\varepsilon > 0$› **unfolding** *equiintegrable_on_def* **by** (*meson half_gt_zero_iff*)
moreover have *norm* $((\sum (x,K) \in \mathcal{D}.\ content\ K *_R g\ x) - integral\ (cbox\ a\ b)\ g) < \varepsilon$
if *g*: $g \in$ *?G*
and $\mathcal{D}$: $\mathcal{D}$ *tagged_division_of cbox a b*
and *fine*: γ *fine* $\mathcal{D}$
for $\mathcal{D}$ *g*
proof −
obtain *I c f* **where** *finite I* **and** *0*: $\bigwedge i{::}'j.\ i \in I \Longrightarrow 0 \leq c\ i$
and *1*: *sum c I = 1* **and** *f*: $f \in I \to F$ **and** *geq*: $g = (\lambda x.\ \sum i{\in}I.\ c\ i *_R f\ i\ x)$
using *g* **by** *auto*
have *fi_int*: *f i integrable_on cbox a b* **if** $i \in I$ **for** *i*
by (*metis Pi_iff assms equiintegrable_on_def f that*)
have $*$: *integral* (*cbox a b*) $(\lambda x.\ c\ i *_R f\ i\ x) = (\sum (x,\ K){\in}\mathcal{D}.\ integral\ K\ (\lambda x.\ c\ i *_R f\ i\ x))$
if $i \in I$ **for** *i*
proof −
have *f i integrable_on cbox a b*
by (*metis Pi_iff assms equiintegrable_on_def f that*)
then show *?thesis*
by (*intro* $\mathcal{D}$ *integrable_cmul integral_combine_tagged_division_topdown*)
qed
have *finite* $\mathcal{D}$
using $\mathcal{D}$ **by** *blast*
have *swap*: $(\sum (x,K){\in}\mathcal{D}.\ content\ K *_R (\sum i{\in}I.\ c\ i *_R f\ i\ x))$
$= (\sum i{\in}I.\ c\ i *_R (\sum (x,K){\in}\mathcal{D}.\ content\ K *_R f\ i\ x))$
by (*simp add*: *scale_sum_right case_prod_unfold algebra_simps*) (*rule sum.swap*)
have *norm* $((\sum (x,\ K){\in}\mathcal{D}.\ content\ K *_R g\ x) - integral\ (cbox\ a\ b)\ g)$
$=$ *norm* $((\sum i{\in}I.\ c\ i *_R ((\sum (x,K){\in}\mathcal{D}.\ content\ K *_R f\ i\ x) - integral$

(*cbox a b*) (*f i*))))
    unfolding *geq swap*
      by (*simp add*: *scaleR_right.sum algebra_simps integral_sum fi_int integrable_cmul* ‹*finite I*› *sum_subtractf flip*: *sum_diff*)
   also have ... ≤ (∑*i*∈*I*. *c i* ∗ ε / *2*)
   proof (*rule sum_norm_le*)
    show *norm* (*c i* ∗$_R$ ((∑(*xa*, *K*)∈𝒟. *content K* ∗$_R$ *f i xa*) − *integral* (*cbox a b*) (*f i*))) ≤ *c i* ∗ ε / *2*
     if *i* ∈ *I* **for** *i*
    proof −
      have *norm* ((∑(*x*, *K*)∈𝒟. *content K* ∗$_R$ *f i x*) − *integral* (*cbox a b*) (*f i*)) ≤ ε/*2*
       using γ [*OF* _ 𝒟 *fine*, *of f i*] *funcset_mem* [*OF f*] *that* **by** *auto*
     then show *?thesis*
       using *that* **by** (*auto simp*: *0 mult.assoc intro*: *mult_left_mono*)
    qed
   qed
   also have ... < ε
    using *1* ‹ε > *0*› **by** (*simp add*: *flip*: *sum_divide_distrib sum_distrib_right*)
   finally show *?thesis* .
  qed
  ultimately show *?thesis*
   by (*rule_tac x*=γ **in** *exI*) *auto*
 qed
qed

corollary *equiintegrable_sum_real*:
 fixes *F* :: (*real* ⇒ '*b*::*euclidean_space*) *set*
 assumes *F equiintegrable_on* {*a*..*b*}
 shows (⋃*I* ∈ *Collect finite*. ⋃*c* ∈ {*c*. (∀*i* ∈ *I*. *c i* ≥ *0*) ∧ *sum c I* = *1*}.
     ⋃*f* ∈ *I* → *F*. {(λ*x*. *sum* (λ*i*. *c i* ∗$_R$ *f i x*) *I*)})
    equiintegrable_on {*a*..*b*}
 using *equiintegrable_sum* [*of F a b*] *assms* **by** *auto*

Basic combining theorems for the interval of integration.

lemma *equiintegrable_on_null* [*simp*]:
  content(*cbox a b*) = *0* ⟹ *F equiintegrable_on cbox a b*
 unfolding *equiintegrable_on_def*
 by (*metis diff_zero gauge_trivial integrable_on_null integral_null norm_zero sum_content_null*)

Main limit theorem for an equiintegrable sequence.

theorem *equiintegrable_limit*:
 fixes *g* :: '*a* :: *euclidean_space* ⇒ '*b* :: *banach*
 assumes *feq*: *range f equiintegrable_on cbox a b*
   and *to_g*: ⋀*x*. *x* ∈ *cbox a b* ⟹ (λ*n*. *f n x*) ⟶ *g x*
  shows *g integrable_on cbox a b* ∧ (λ*n*. *integral* (*cbox a b*) (*f n*)) ⟶ *integral* (*cbox a b*) *g*
proof −
 have *Cauchy* (λ*n*. *integral*(*cbox a b*) (*f n*))

proof (*clarsimp simp add*: *Cauchy_def*)
fix *e*::*real*
assume *0* < *e*
then have *e3*: *0* < *e*/*3*
by *simp*
then obtain γ **where** *gauge* γ
and γ: $\bigwedge n$ $\mathcal{D}$. ⟦$\mathcal{D}$ *tagged_division_of cbox a b*; γ *fine* $\mathcal{D}$⟧
$\Longrightarrow$ *norm*(($\sum$(*x*,*K*) $\in$ $\mathcal{D}$. *content K* $*_R$ *f n x*) − *integral* (*cbox a b*) (*f n*)) < *e*/*3*
using *feq* **unfolding** *equiintegrable_on_def*
by (*meson image_eqI iso_tuple_UNIV_I*)
obtain $\mathcal{D}$ **where** $\mathcal{D}$: $\mathcal{D}$ *tagged_division_of* (*cbox a b*) **and** γ *fine* $\mathcal{D}$ *finite* $\mathcal{D}$
by (*meson* ‹*gauge* γ› *fine_division_exists tagged_division_of_finite*)
with γ **have** δT: $\bigwedge n$. *dist* (($\sum$(*x*,*K*)$\in\mathcal{D}$. *content K* $*_R$ *f n x*)) (*integral* (*cbox a b*) (*f n*)) < *e*/*3*
by (*force simp*: *dist_norm*)
have (λn. $\sum$(*x*,*K*)$\in\mathcal{D}$. *content K* $*_R$ *f n x*) $\longrightarrow$ ($\sum$(*x*,*K*)$\in\mathcal{D}$. *content K* $*_R$ *g x*)
using $\mathcal{D}$ *to_g* **by** (*auto intro*!: *tendsto_sum tendsto_scaleR*)
then have *Cauchy* (λn. $\sum$(*x*,*K*)$\in\mathcal{D}$. *content K* $*_R$ *f n x*)
by (*meson convergent_eq_Cauchy*)
with *e3* **obtain** *M* **where**
M: $\bigwedge m$ *n*. ⟦*m*≥*M*; *n*≥*M*⟧ $\Longrightarrow$ *dist* ($\sum$(*x*,*K*)$\in\mathcal{D}$. *content K* $*_R$ *f m x*) ($\sum$(*x*,*K*)$\in\mathcal{D}$. *content K* $*_R$ *f n x*)
< *e*/*3*
unfolding *Cauchy_def* **by** *blast*
have $\bigwedge m$ *n*. ⟦*m*≥*M*; *n*≥*M*;
dist ($\sum$(*x*,*K*)$\in\mathcal{D}$. *content K* $*_R$ *f m x*) ($\sum$(*x*,*K*)$\in\mathcal{D}$. *content K* $*_R$ *f n x*) < *e*/*3*⟧
$\Longrightarrow$ *dist* (*integral* (*cbox a b*) (*f m*)) (*integral* (*cbox a b*) (*f n*)) < *e*
by (*metis* δT *dist_commute dist_triangle_third* [*OF* _ _ δT])
then show $\exists M$. $\forall m\geq M$. $\forall n\geq M$. *dist* (*integral* (*cbox a b*) (*f m*)) (*integral* (*cbox a b*) (*f n*)) < *e*
using *M* **by** *auto*
qed
then obtain *L* **where** *L*: (λn. *integral* (*cbox a b*) (*f n*)) $\longrightarrow$ *L*
by (*meson convergent_eq_Cauchy*)
have (*g has_integral L*) (*cbox a b*)
proof (*clarsimp simp*: *has_integral*)
fix *e*::*real* **assume** *0* < *e*
then have *e2*: *0* < *e*/*2*
by *simp*
then obtain γ **where** *gauge* γ
and γ: $\bigwedge n$ $\mathcal{D}$. ⟦$\mathcal{D}$ *tagged_division_of cbox a b*; γ *fine* $\mathcal{D}$⟧
$\Longrightarrow$ *norm*(($\sum$(*x*,*K*)$\in\mathcal{D}$. *content K* $*_R$ *f n x*) − *integral* (*cbox a b*) (*f n*)) < *e*/*2*
using *feq* **unfolding** *equiintegrable_on_def*
by (*meson image_eqI iso_tuple_UNIV_I*)
moreover

have *norm* ((∑(*x*,*K*)∈𝒟. *content K* $*_R$ *g x*) − *L*) < *e*
if 𝒟 *tagged_division_of cbox a b* γ *fine* 𝒟 **for** 𝒟
proof −
have *norm* ((∑(*x*,*K*)∈𝒟. *content K* $*_R$ *g x*) − *L*) ≤ *e*/*2*
proof (*rule Lim_norm_ubound*)
show (λ*n*. (∑(*x*,*K*)∈𝒟. *content K* $*_R$ *f n x*) − *integral* (*cbox a b*) (*f n*))
⟶ (∑(*x*,*K*)∈𝒟. *content K* $*_R$ *g x*) − *L*
using *to_g that L*
by (*intro tendsto_diff tendsto_sum*) (*auto simp*: *tag_in_interval tendsto_scaleR*)
show $\forall_F$ *n in sequentially*.
norm ((∑(*x*,*K*) ∈ 𝒟. *content K* $*_R$ *f n x*) − *integral* (*cbox a b*) (*f n*)) ≤ *e*/*2*
by (*intro eventuallyI less_imp_le* γ *that*)
qed *auto*
with ‹*0* < *e*› **show** *?thesis*
by *linarith*
qed
ultimately
show ∃γ. *gauge* γ ∧
(∀𝒟. 𝒟 *tagged_division_of cbox a b* ∧ γ *fine* 𝒟 ⟶
norm ((∑(*x*,*K*)∈𝒟. *content K* $*_R$ *g x*) − *L*) < *e*)
by *meson*
qed
with *L* **show** *?thesis*
by (*simp add*: ‹(λ*n*. *integral* (*cbox a b*) (*f n*)) ⟶ *L*› *has_integral_integrable_integral*)
qed

lemma *equiintegrable_reflect*:
assumes *F equiintegrable_on cbox a b*
shows (λ*f*. *f* ∘ *uminus*) ‘ *F equiintegrable_on cbox* (−*b*) (−*a*)
proof −
have §: ∃γ. *gauge* γ ∧
(∀*f* 𝒟. *f* ∈ (λ*f*. *f* ∘ *uminus*) ‘ *F* ∧ 𝒟 *tagged_division_of cbox* (− *b*) (− *a*) ∧ γ *fine* 𝒟 ⟶
norm ((∑(*x*,*K*) ∈ 𝒟. *content K* $*_R$ *f x*) − *integral* (*cbox* (− *b*) (− *a*)) *f*) < *e*)
if *gauge* γ **and**
γ: ⋀*f* 𝒟. ⟦*f* ∈ *F*; 𝒟 *tagged_division_of cbox a b*; γ *fine* 𝒟⟧ ⟹
norm ((∑(*x*,*K*) ∈ 𝒟. *content K* $*_R$ *f x*) − *integral* (*cbox a b*) *f*) < *e* **for** *e* γ
proof (*intro exI*, *safe*)
show *gauge* (λ*x*. *uminus* ‘ γ (−*x*))
by (*metis* ‹*gauge* γ› *gauge_reflect*)
show *norm* ((∑(*x*,*K*) ∈ 𝒟. *content K* $*_R$ (*f* ∘ *uminus*) *x*) − *integral* (*cbox* (− *b*) (− *a*)) (*f* ∘ *uminus*)) < *e*
if *f* ∈ *F* **and** *tag*: 𝒟 *tagged_division_of cbox* (− *b*) (− *a*)
and *fine*: (λ*x*. *uminus* ‘ γ (− *x*)) *fine* 𝒟 **for** *f* 𝒟

proof −
have *1*: (λ(*x*,*K*). (− *x*, *uminus* ' *K*)) ' 𝒟 *tagged_partial_division_of cbox a b*
if 𝒟 *tagged_partial_division_of cbox* (− *b*) (− *a*)
proof −
have − *y* ∈ *cbox a b*
if ⋀*x K*. (*x*,*K*) ∈ 𝒟 ⟹ *x* ∈ *K* ∧ *K* ⊆ *cbox* (− *b*) (− *a*) ∧ (∃ *a b*. *K* = *cbox a b*)
(*x*, *Y*) ∈ 𝒟 *y* ∈ *Y* **for** *x Y y*
proof −
have *y* ∈ *uminus* ' *cbox a b*
using *that* **by** *auto*
then show − *y* ∈ *cbox a b*
by *force*
qed
with *that* **show** *?thesis*
by (*fastforce simp*: *tagged_partial_division_of_def interior_negations image_iff*)
qed
have *2*: ∃ *K*. (∃ *x*. (*x*,*K*) ∈ (λ(*x*,*K*). (− *x*, *uminus* ' *K*)) ' 𝒟) ∧ *x* ∈ *K*
if ⋃{*K*. ∃ *x*. (*x*,*K*) ∈ 𝒟} = *cbox* (− *b*) (− *a*) *x* ∈ *cbox a b* **for** *x*
proof −
have *xm*: *x* ∈ *uminus* ' ⋃{*A*. ∃ *a*. (*a*, *A*) ∈ 𝒟}
by (*simp add*: *that*)
then obtain *a X* **where** −*x* ∈ *X* (*a*, *X*) ∈ 𝒟
by *auto*
then show *?thesis*
by (*metis* (*no_types*, *lifting*) *add.inverse_inverse image_iff pair_imageI*)
qed
have *3*: ⋀*x X y*. ⟦𝒟 *tagged_partial_division_of cbox* (− *b*) (− *a*); (*x*, *X*) ∈ 𝒟; *y* ∈ *X*⟧ ⟹ − *y* ∈ *cbox a b*
by (*metis* (*no_types*, *lifting*) *equation_minus_iff imageE subsetD tagged_partial_division_ofD*(*3*) *uminus_interval_vector*)
have *tag'*: (λ(*x*,*K*). (− *x*, *uminus* ' *K*)) ' 𝒟 *tagged_division_of cbox a b*
using *tag* **by** (*auto simp*: *tagged_division_of_def dest*: *1 2 3*)
have *fine'*: γ *fine* (λ(*x*,*K*). (− *x*, *uminus* ' *K*)) ' 𝒟
using *fine* **by** (*fastforce simp*: *fine_def*)
have *inj*: *inj_on* (λ(*x*,*K*). (− *x*, *uminus* ' *K*)) 𝒟
unfolding *inj_on_def* **by** *force*
have *eq*: *content* (*uminus* ' *I*) = *content I*
if *I*: (*x*, *I*) ∈ 𝒟 **and** *fnz*: *f* (− *x*) ≠ *0* **for** *x I*
proof −
obtain *a b* **where** *I* = *cbox a b*
using *tag I that* **by** (*force simp*: *tagged_division_of_def tagged_partial_division_of_def*)
then show *?thesis*
using *content_image_affinity_cbox* [*of* −*1 0*] **by** *auto*
qed
have (∑(*x*,*K*) ∈ (λ(*x*,*K*). (− *x*, *uminus* ' *K*)) ' 𝒟. *content K* $*_R$ *f x*) =
(∑(*x*,*K*) ∈ 𝒟. *content K* $*_R$ *f* (− *x*))
by (*auto simp add*: *eq sum.reindex* [*OF inj*] *intro*!: *sum.cong*)

then show *?thesis*
using γ [*OF* ‹*f* ∈ *F*› *tag'* *fine'*] *integral_reflect*
by (*metis* (*mono_tags*, *lifting*) *Henstock_Kurzweil_Integration.integral_cong comp_apply split_def sum.cong*)
qed
qed
show *?thesis*
using *assms*
apply (*auto simp*: *equiintegrable_on_def*)
subgoal for *f*
by (*metis* (*mono_tags*, *lifting*) *comp_apply integrable_eq integrable_reflect*)
using § **by** *fastforce*
qed

6.27.2 Subinterval restrictions for equiintegrable families

First, some technical lemmas about minimizing a "flat" part of a sum over a division.

lemma *lemma0*:
assumes $i \in Basis$
shows *content* (*cbox u v*) / (*interval_upperbound* (*cbox u v*) · *i* − *interval_lowerbound* (*cbox u v*) · *i*) =
(*if content* (*cbox u v*) = *0 then 0*
else $\prod j \in Basis - \{i\}$. *interval_upperbound* (*cbox u v*) · *j* − *interval_lowerbound* (*cbox u v*) · *j*)
proof (*cases content* (*cbox u v*) = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then show *?thesis*
using *prod.subset_diff* [*of* {*i*} *Basis*] *assms*
by (*force simp*: *content_cbox_if divide_simps split*: *if_split_asm*)
qed

lemma *content_division_lemma1*:
assumes *div*: $\mathcal{D}$ *division_of S* **and** *S*: $S \subseteq$ *cbox a b* **and** *i*: $i \in Basis$
and *mt*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow$ *content K* ≠ *0*
and *disj*: $(\forall K \in \mathcal{D}.\ K \cap \{x.\ x \cdot i = a \cdot i\} \neq \{\}) \vee (\forall K \in \mathcal{D}.\ K \cap \{x.\ x \cdot i = b \cdot i\} \neq \{\})$
shows $(b \cdot i - a \cdot i) * (\sum K{\in}\mathcal{D}.$ *content K* / (*interval_upperbound K* · *i* − *interval_lowerbound K* · *i*))
≤ *content*(*cbox a b*) (**is** *?lhs* ≤ *?rhs*)
proof −
have *finite* $\mathcal{D}$
using *div* **by** *blast*
define *extend* **where**
extend ≡ $\lambda K.$ *cbox* ($\sum j \in Basis.$ *if j* = *i then* (*a* · *i*) $*_R$ *i else* (*interval_lowerbound*

$K \cdot j) *_R j)$
$(\sum j \in Basis.\ if\ j = i\ then\ (b \cdot i) *_R i\ else\ (interval\_upperbound\ K \cdot j) *_R j)$
have *div_subset_cbox*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow K \subseteq cbox\ a\ b$
using *S div* **by** *auto*
have $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow K \neq \{\}$
using *div* **by** *blast*
have *extend_cbox*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow \exists\, a\ b.\ extend\ K = cbox\ a\ b$
using *extend_def* **by** *blast*
have *extend*: $extend\ K \neq \{\}$ $extend\ K \subseteq cbox\ a\ b$ **if** K: $K \in \mathcal{D}$ **for** K
proof −
obtain *u v* **where** K: $K = cbox\ u\ v$ $K \neq \{\}$ $K \subseteq cbox\ a\ b$
using *K cbox_division_memE* [*OF* _ *div*] **by** (*meson div_subset_cbox*)
with *i* **show** $extend\ K \subseteq cbox\ a\ b$
by (*auto simp*: *extend_def subset_box box_ne_empty*)
have $a \cdot i \leq b \cdot i$
using *K* **by** (*metis bot.extremum_uniqueI box_ne_empty*(*1*) *i*)
with *K* **show** $extend\ K \neq \{\}$
by (*simp add*: *extend_def i box_ne_empty*)
qed
have *int_extend_disjoint*:
$interior(extend\ K1) \cap interior(extend\ K2) = \{\}$ **if** K: $K1 \in \mathcal{D}$ $K2 \in \mathcal{D}$ $K1 \neq K2$ **for** *K1 K2*
proof −
obtain *u v* **where** *K1*: $K1 = cbox\ u\ v$ $K1 \neq \{\}$ $K1 \subseteq cbox\ a\ b$
using *K cbox_division_memE* [*OF* _ *div*] **by** (*meson div_subset_cbox*)
obtain *w z* **where** *K2*: $K2 = cbox\ w\ z$ $K2 \neq \{\}$ $K2 \subseteq cbox\ a\ b$
using *K cbox_division_memE* [*OF* _ *div*] **by** (*meson div_subset_cbox*)
have *cboxes*: $cbox\ u\ v \in \mathcal{D}$ $cbox\ w\ z \in \mathcal{D}$ $cbox\ u\ v \neq cbox\ w\ z$
using *K1 K2 that* **by** *auto*
with *div* **have** $interior\ (cbox\ u\ v) \cap interior\ (cbox\ w\ z) = \{\}$
by *blast*
moreover
have $\exists x.\ x \in box\ u\ v \wedge x \in box\ w\ z$
if $x \in interior\ (extend\ K1)$ $x \in interior\ (extend\ K2)$ **for** x
proof −
have $a \cdot i < x \cdot i$ $x \cdot i < b \cdot i$
and *ux*: $\bigwedge k.\ k \in Basis - \{i\} \Longrightarrow u \cdot k < x \cdot k$
and *xv*: $\bigwedge k.\ k \in Basis - \{i\} \Longrightarrow x \cdot k < v \cdot k$
and *wx*: $\bigwedge k.\ k \in Basis - \{i\} \Longrightarrow w \cdot k < x \cdot k$
and *xz*: $\bigwedge k.\ k \in Basis - \{i\} \Longrightarrow x \cdot k < z \cdot k$
using *that K1 K2 i* **by** (*auto simp*: *extend_def box_ne_empty mem_box*)
have $box\ u\ v \neq \{\}$ $box\ w\ z \neq \{\}$
using *cboxes interior_cbox* **by** (*auto simp*: *content_eq_0_interior dest*: *mt*)
then obtain *q s*
where *q*: $\bigwedge k.\ k \in Basis \Longrightarrow w \cdot k < q \cdot k \wedge q \cdot k < z \cdot k$
and *s*: $\bigwedge k.\ k \in Basis \Longrightarrow u \cdot k < s \cdot k \wedge s \cdot k < v \cdot k$
by (*meson all_not_in_conv mem_box*(*1*))
show *?thesis* **using** *disj*

```
proof
  assume ∀K∈𝒟. K ∩ {x. x · i = a · i} ≠ {}
  then have uva: (cbox u v) ∩ {x. x · i = a · i} ≠ {}
      and wza: (cbox w z) ∩ {x. x · i = a · i} ≠ {}
    using cboxes by (auto simp: content_eq_0_interior)
  then obtain r t where r · i = a · i and r: ⋀k. k ∈ Basis ⟹ w · k ≤ r
· k ∧ r · k ≤ z · k
              and t · i = a · i and t: ⋀k. k ∈ Basis ⟹ u · k ≤ t · k ∧ t ·
k ≤ v · k
    by (fastforce simp: mem_box)
  have u: u · i < q · i
   using i K2(1) K2(3) ‹t · i = a · i› q s t [OF i] by (force simp: subset_box)
  have w: w · i < s · i
   using i K1(1) K1(3) ‹r · i = a · i› s r [OF i] by (force simp: subset_box)
  define ξ where ξ ≡ (∑j ∈ Basis. if j = i then min (q · i) (s · i) *R i else
(x · j) *R j)
  have [simp]: ξ · j = (if j = i then min (q · j) (s · j) else x · j) if j ∈ Basis
for j
    unfolding ξ_def
    by (intro sum_if_inner that ‹i ∈ Basis›)
  show ?thesis
  proof (intro exI conjI)
    have min (q · i) (s · i) < v · i
      using i s by fastforce
    with ‹i ∈ Basis› s u ux xv
    show ξ ∈ box u v
      by (force simp: mem_box)
    have min (q · i) (s · i) < z · i
      using i q by force
    with ‹i ∈ Basis› q w wx xz
    show ξ ∈ box w z
      by (force simp: mem_box)
  qed
next
  assume ∀K∈𝒟. K ∩ {x. x · i = b · i} ≠ {}
  then have uva: (cbox u v) ∩ {x. x · i = b · i} ≠ {}
      and wza: (cbox w z) ∩ {x. x · i = b · i} ≠ {}
    using cboxes by (auto simp: content_eq_0_interior)
  then obtain r t where r · i = b · i and r: ⋀k. k ∈ Basis ⟹ w · k ≤ r
· k ∧ r · k ≤ z · k
              and t · i = b · i and t: ⋀k. k ∈ Basis ⟹ u · k ≤ t · k ∧ t ·
k ≤ v · k
    by (fastforce simp: mem_box)
  have z: s · i < z · i
   using K1(1) K1(3) ‹r · i = b · i› r [OF i] i s by (force simp: subset_box)
  have v: q · i < v · i
   using K2(1) K2(3) ‹t · i = b · i› t [OF i] i q by (force simp: subset_box)
  define ξ where ξ ≡ (∑j ∈ Basis. if j = i then max (q · i) (s · i) *R i
else (x · j) *R j)
```

have [*simp*]: $\xi \cdot j = (if\ j = i\ then\ max\ (q \cdot j)\ (s \cdot j)\ else\ x \cdot j)$ **if** $j \in Basis$ **for** j
unfolding *ξ_def*
by (*intro sum_if_inner that* ‹$i \in Basis$›)
show *?thesis*
proof (*intro exI conjI*)
show $\xi \in box\ u\ v$
using ‹$i \in Basis$› *s* **by** (*force simp*: *mem_box ux v xv*)
show $\xi \in box\ w\ z$
using ‹$i \in Basis$› *q* **by** (*force simp*: *mem_box wx xz z*)
qed
qed
qed
ultimately show *?thesis* **by** *auto*
qed
define *interv_diff* **where** $interv\_diff \equiv \lambda K.\ \lambda i::'a.\ interval\_upperbound\ K \cdot i - interval\_lowerbound\ K \cdot i$
have $?lhs = (\sum K\in\mathcal{D}.\ (b \cdot i - a \cdot i) * content\ K\ /\ (interv\_diff\ K\ i))$
by (*simp add*: *sum_distrib_left interv_diff_def*)
also have $\ldots = sum\ (content \circ extend)\ \mathcal{D}$
proof (*rule sum.cong* [*OF refl*])
fix *K* **assume** $K \in \mathcal{D}$
then obtain *u v* **where** *K*: $K = cbox\ u\ v\ cbox\ u\ v \neq \{\}\ K \subseteq cbox\ a\ b$
using *cbox_division_memE* [*OF _ div*] *div_subset_cbox* **by** *metis*
then have *uv*: $u \cdot i < v \cdot i$
using *mt* [*OF* ‹$K \in \mathcal{D}$›] ‹$i \in Basis$› *content_eq_0* **by** *fastforce*
have $insert\ i\ (Basis \cap -\{i\}) = Basis$
using ‹$i \in Basis$› **by** *auto*
then have $(b \cdot i - a \cdot i) * content\ K\ /\ (interv\_diff\ K\ i)$
$= (b \cdot i - a \cdot i) * (\prod i \in insert\ i\ (Basis \cap -\{i\}).\ v \cdot i - u \cdot i)\ /\ (interv\_diff\ (cbox\ u\ v)\ i)$
using *K box_ne_empty*(*1*) *content_cbox* **by** *fastforce*
also have ... $= (\prod x\in Basis.\ if\ x = i\ then\ b \cdot x - a \cdot x$
$else\ (interval\_upperbound\ (cbox\ u\ v) - interval\_lowerbound\ (cbox\ u\ v)) \cdot x)$
using ‹$i \in Basis$› *K uv* **by** (*simp add*: *prod.If_cases interv_diff_def*) (*simp add*: *algebra_simps*)
also have ... $= (\prod k\in Basis.$
$(\sum j\in Basis.\ if\ j = i\ then\ (b \cdot i - a \cdot i) *_R i$
$else\ ((interval\_upperbound\ (cbox\ u\ v) - interval\_lowerbound\ (cbox\ u\ v)) \cdot j) *_R j) \cdot k)$
using ‹$i \in Basis$› **by** (*subst prod.cong* [*OF refl sum_if_inner*]; *simp*)
also have ... $= (\prod k\in Basis.$
$(\sum j\in Basis.\ if\ j = i\ then\ (b \cdot i) *_R i\ else\ (interval\_upperbound\ (cbox\ u\ v) \cdot j) *_R j) \cdot k -$
$(\sum j\in Basis.\ if\ j = i\ then\ (a \cdot i) *_R i\ else\ (interval\_lowerbound\ (cbox\ u\ v) \cdot j) *_R j) \cdot k)$
using ‹$i \in Basis$›
by (*intro prod.cong* [*OF refl*]) (*subst sum_if_inner*; *simp add*: *algebra_simps*)+

also have ... = (*content* ∘ *extend*) *K*
using ‹*i* ∈ *Basis*› *K box_ne_empty* ‹*K* ∈ *𝒟*› *extend*(*1*)
by (*auto simp add*: *extend_def content_cbox_if*)
finally show (*b* · *i* − *a* · *i*) ∗ *content K* / (*interv_diff K i*) = (*content* ∘ *extend*) *K* **.**
qed
also have ... = *sum content* (*extend* ‘ *𝒟*)
proof −
have ⟦*K1* ∈ *𝒟*; *K2* ∈ *𝒟*; *K1* ≠ *K2*; *extend K1* = *extend K2*⟧ ⟹ *content* (*extend K1*) = *0* **for** *K1 K2*
using *int_extend_disjoint* [*of K1 K2*] *extend_def* **by** (*simp add*: *content_eq_0_interior*)
then show *?thesis*
by (*simp add*: *comm_monoid_add_class.sum.reindex_nontrivial* [*OF* ‹*finite 𝒟*›])
qed
also have ... ≤ *?rhs*
proof (*rule subadditive_content_division*)
show *extend* ‘ *𝒟 division_of* ⋃ (*extend* ‘ *𝒟*)
using *int_extend_disjoint* **by** (*auto simp*: *division_of_def* ‹*finite 𝒟*› *extend extend_cbox*)
show ⋃ (*extend* ‘ *𝒟*) ⊆ *cbox a b*
using *extend* **by** *fastforce*
qed
finally show *?thesis* **.**
qed

proposition *sum_content_area_over_thin_division*:
assumes *div*: *𝒟 division_of S* **and** *S*: *S* ⊆ *cbox a b* **and** *i*: *i* ∈ *Basis*
and *a* · *i* ≤ *c c* ≤ *b* · *i*
and *nonmt*: ⋀*K*. *K* ∈ *𝒟* ⟹ *K* ∩ {*x*. *x* · *i* = *c*} ≠ {}
shows (*b* · *i* − *a* · *i*) ∗ (∑*K*∈*𝒟*. *content K* / (*interval_upperbound K* · *i* − *interval_lowerbound K* · *i*))
≤ *2* ∗ *content*(*cbox a b*)
proof (*cases content*(*cbox a b*) = *0*)
case *True*
have (∑*K*∈*𝒟*. *content K* / (*interval_upperbound K* · *i* − *interval_lowerbound K* · *i*)) = *0*
using *S div* **by** (*force intro*!: *sum.neutral content_0_subset* [*OF True*])
then show *?thesis*
by (*auto simp*: *True*)
next
case *False*
then have *content*(*cbox a b*) > *0*
using *zero_less_measure_iff* **by** *blast*
then have *a* · *i* < *b* · *i* **if** *i* ∈ *Basis* **for** *i*
using *content_pos_lt_eq that* **by** *blast*
have *finite 𝒟*
using *div* **by** *blast*
define *Dlec* **where** *Dlec* ≡ {*L* ∈ (λ*L*. *L* ∩ {*x*. *x* · *i* ≤ *c*}) ‘ *𝒟*. *content L* ≠ *0*}

define *Dgec* **where** $Dgec \equiv \{L \in (\lambda L.\ L \cap \{x.\ x \cdot i \geq c\})$ ' $\mathcal{D}.\ content\ L \neq 0\}$
define a' **where** $a' \equiv (\sum j{\in}Basis.\ (if\ j = i\ then\ c\ else\ a \cdot j) *_R j)$
define b' **where** $b' \equiv (\sum j{\in}Basis.\ (if\ j = i\ then\ c\ else\ b \cdot j) *_R j)$
define *interv_diff* **where** $interv\_diff \equiv \lambda K.\ \lambda i{::}'a.\ interval\_upperbound\ K \cdot i - interval\_lowerbound\ K \cdot i$
have *Dlec_cbox*: $\bigwedge K.\ K \in Dlec \Longrightarrow \exists\, a\ b.\ K = cbox\ a\ b$
using *interval_split* [*OF i*] *div* **by** (*fastforce simp*: *Dlec_def division_of_def*)
then have *lec_is_cbox*: $[\![content\ (L \cap \{x.\ x \cdot i \leq c\}) \neq 0;\ L \in \mathcal{D}]\!] \Longrightarrow \exists\, a\ b.\ L \cap \{x.\ x \cdot i \leq c\} = cbox\ a\ b$ **for** *L*
using *Dlec_def* **by** *blast*
have *Dgec_cbox*: $\bigwedge K.\ K \in Dgec \Longrightarrow \exists\, a\ b.\ K = cbox\ a\ b$
using *interval_split* [*OF i*] *div* **by** (*fastforce simp*: *Dgec_def division_of_def*)
then have *gec_is_cbox*: $[\![content\ (L \cap \{x.\ x \cdot i \geq c\}) \neq 0;\ L \in \mathcal{D}]\!] \Longrightarrow \exists\, a\ b.\ L \cap \{x.\ x \cdot i \geq c\} = cbox\ a\ b$ **for** *L*
using *Dgec_def* **by** *blast*

have *zero_left*: $\bigwedge x\ y.\ [\![x \in \mathcal{D};\ y \in \mathcal{D};\ x \neq y;\ x \cap \{x.\ x \cdot i \leq c\} = y \cap \{x.\ x \cdot i \leq c\}]\!]$
$\Longrightarrow content\ (y \cap \{x.\ x \cdot i \leq c\}) = 0$
by (*metis division_split_left_inj* [*OF div*] *lec_is_cbox content_eq_0_interior*)
have *zero_right*: $\bigwedge x\ y.\ [\![x \in \mathcal{D};\ y \in \mathcal{D};\ x \neq y;\ x \cap \{x.\ c \leq x \cdot i\} = y \cap \{x.\ c \leq x \cdot i\}]\!]$
$\Longrightarrow content\ (y \cap \{x.\ c \leq x \cdot i\}) = 0$
by (*metis division_split_right_inj* [*OF div*] *gec_is_cbox content_eq_0_interior*)

have $(b' \cdot i - a \cdot i) * (\sum K{\in}Dlec.\ content\ K\ /\ interv\_diff\ K\ i) \leq content(cbox\ a\ b')$
unfolding *interv_diff_def*
proof (*rule content_division_lemma1*)
show *Dlec division_of* $\bigcup$ *Dlec*
unfolding *division_of_def*
proof (*intro conjI ballI Dlec_cbox*)
show $\bigwedge K1\ K2.\ [\![K1 \in Dlec;\ K2 \in Dlec]\!] \Longrightarrow K1 \neq K2 \longrightarrow interior\ K1 \cap interior\ K2 = \{\}$
by (*clarsimp simp*: *Dlec_def*) (*use div* **in** *auto*)
qed (*use* ‹*finite* $\mathcal{D}$› *Dlec_def* **in** *auto*)
show $\bigcup Dlec \subseteq cbox\ a\ b'$
using *Dlec_def div S* **by** (*auto simp*: *b'_def division_of_def mem_box*)
show $(\forall K{\in}Dlec.\ K \cap \{x.\ x \cdot i = a \cdot i\} \neq \{\}) \vee (\forall K{\in}Dlec.\ K \cap \{x.\ x \cdot i = b' \cdot i\} \neq \{\})$
using *nonmt* **by** (*fastforce simp*: *Dlec_def b'_def i*)
qed (*use i Dlec_def* **in** *auto*)
moreover
have $(\sum K{\in}Dlec.\ content\ K\ /\ (interv\_diff\ K\ i)) = (\sum K{\in}(\lambda K.\ K \cap \{x.\ x \cdot i \leq c\})$ ' $\mathcal{D}.\ content\ K\ /\ interv\_diff\ K\ i)$
unfolding *Dlec_def* **using** ‹*finite* $\mathcal{D}$› **by** (*auto simp*: *sum.mono_neutral_left*)
moreover have ... =
$(\sum K{\in}\mathcal{D}.\ ((\lambda K.\ content\ K\ /\ (interv\_diff\ K\ i)) \circ ((\lambda K.\ K \cap \{x.\ x \cdot i \leq c\}))) K)$

```
    by (simp add: zero_left sum.reindex_nontrivial [OF ‹finite 𝒟›])
  moreover have (b' · i − a · i) = (c − a · i)
    by (simp add: b'_def i)
  ultimately
  have lec: (c − a · i) ∗ (∑K∈𝒟. ((λK. content K / (interv_diff K i)) ∘ ((λK.
K ∩ {x. x · i ≤ c}))) K)
             ≤ content(cbox a b')
    by simp

  have (b · i − a' · i) ∗ (∑K∈Dgec. content K / (interv_diff K i)) ≤ content(cbox
a' b)
    unfolding interv_diff_def
  proof (rule content_division_lemma1)
    show Dgec division_of ⋃Dgec
      unfolding division_of_def
    proof (intro conjI ballI Dgec_cbox)
      show ⋀K1 K2. ⟦K1 ∈ Dgec; K2 ∈ Dgec⟧ ⟹ K1 ≠ K2 ⟶ interior K1 ∩
interior K2 = {}
        by (clarsimp simp: Dgec_def) (use div in auto)
    qed (use ‹finite 𝒟› Dgec_def in auto)
    show ⋃Dgec ⊆ cbox a' b
      using Dgec_def div S by (auto simp: a'_def division_of_def mem_box)
    show (∀K∈Dgec. K ∩ {x. x · i = a' · i} ≠ {}) ∨ (∀K∈Dgec. K ∩ {x. x · i
= b · i} ≠ {})
      using nonmt by (fastforce simp: Dgec_def a'_def i)
  qed (use i Dgec_def in auto)
  moreover
  have (∑K∈Dgec. content K / (interv_diff K i)) = (∑K∈(λK. K ∩ {x. c ≤ x
· i}) ` 𝒟.
       content K / interv_diff K i)
    unfolding Dgec_def using ‹finite 𝒟› by (auto simp: sum.mono_neutral_left)
  moreover have ... =
       (∑K∈𝒟. ((λK. content K / (interv_diff K i)) ∘ ((λK. K ∩ {x. x · i ≥
c}))) K)
    by (simp add: zero_right sum.reindex_nontrivial [OF ‹finite 𝒟›])
  moreover have (b · i − a' · i) = (b · i − c)
    by (simp add: a'_def i)
  ultimately
  have gec: (b · i − c) ∗ (∑K∈𝒟. ((λK. content K / (interv_diff K i)) ∘ ((λK.
K ∩ {x. x · i ≥ c}))) K)
             ≤ content(cbox a' b)
    by simp

  show ?thesis
  proof (cases c = a · i ∨ c = b · i)
    case True
    then show ?thesis
    proof
      assume c: c = a · i
```

moreover
have $(\sum j \in Basis.\ (if\ j = i\ then\ a \cdot i\ else\ a \cdot j) *_R j) = a$
using *euclidean_representation* [*of a*] *sum.cong* [*OF refl*, *of Basis* $\lambda i.\ (a \cdot i) *_R i$] **by** *presburger*
ultimately have $a' = a$
by (*simp add*: *i a'_def cong*: *if_cong*)
then have *content* (*cbox a' b*) $\leq 2 *$ *content* (*cbox a b*) **by** *simp*
moreover
have *eq*: $(\sum K \in \mathcal{D}.\ content\ (K \cap \{x.\ a \cdot i \leq x \cdot i\})\ /\ interv\_diff\ (K \cap \{x.\ a \cdot i \leq x \cdot i\})\ i)$
$= (\sum K \in \mathcal{D}.\ content\ K\ /\ interv\_diff\ K\ i)$
(**is** *sum ?f _ = sum ?g _*)
proof (*rule sum.cong* [*OF refl*])
fix K **assume** $K \in \mathcal{D}$
then have $a \cdot i \leq x \cdot i$ **if** $x \in K$ **for** x
by (*metis S UnionI div division_ofD*(*6*) *i mem_box*(*2*) *subsetCE that*)
then have $K \cap \{x.\ a \cdot i \leq x \cdot i\} = K$
by *blast*
then show *?f K = ?g K*
by *simp*
qed
ultimately show *?thesis*
using *gec c eq interv_diff_def* **by** *auto*
next
assume *c*: $c = b \cdot i$
moreover have $(\sum j \in Basis.\ (if\ j = i\ then\ b \cdot i\ else\ b \cdot j) *_R j) = b$
using *euclidean_representation* [*of b*] *sum.cong* [*OF refl*, *of Basis* $\lambda i.\ (b \cdot i) *_R i$] **by** *presburger*
ultimately have $b' = b$
by (*simp add*: *i b'_def cong*: *if_cong*)
then have *content* (*cbox a b'*) $\leq 2 *$ *content* (*cbox a b*) **by** *simp*
moreover
have *eq*: $(\sum K \in \mathcal{D}.\ content\ (K \cap \{x.\ x \cdot i \leq b \cdot i\})\ /\ interv\_diff\ (K \cap \{x.\ x \cdot i \leq b \cdot i\})\ i)$
$= (\sum K \in \mathcal{D}.\ content\ K\ /\ interv\_diff\ K\ i)$
(**is** *sum ?f _ = sum ?g _*)
proof (*rule sum.cong* [*OF refl*])
fix K **assume** $K \in \mathcal{D}$
then have $x \cdot i \leq b \cdot i$ **if** $x \in K$ **for** x
by (*metis S UnionI div division_ofD*(*6*) *i mem_box*(*2*) *subsetCE that*)
then have $K \cap \{x.\ x \cdot i \leq b \cdot i\} = K$
by *blast*
then show *?f K = ?g K*
by *simp*
qed
ultimately show *?thesis*
using *lec c eq interv_diff_def* **by** *auto*
qed
next

case *False*
have *prod_if*: $(\prod k\in Basis \cap - \{i\}.\ f\ k) = (\prod k\in Basis.\ f\ k)\ /\ f\ i$ **if** $f\ i \neq (0::real)$ **for** *f*
proof –
have $f\ i * prod\ f\ (Basis \cap - \{i\}) = prod\ f\ Basis$
using *that mk_disjoint_insert* [*OF i*]
by (*metis Int_insert_left_if0 finite_Basis finite_insert le_iff_inf order_refl prod.insert subset_Compl_singleton*)
then show *?thesis*
by (*metis nonzero_mult_div_cancel_left that*)
qed
have *abc*: $a \cdot i < c$ $c < b \cdot i$
using *False assms* **by** *auto*
then have $(\sum K\in\mathcal{D}.\ ((\lambda K.\ content\ K\ /\ (interv\_diff\ K\ i)) \circ ((\lambda K.\ K \cap \{x.\ x \cdot i \le c\}))) K)$
$\le content(cbox\ a\ b')\ /\ (c - a \cdot i)$
$(\sum K\in\mathcal{D}.\ ((\lambda K.\ content\ K\ /\ (interv\_diff\ K\ i)) \circ ((\lambda K.\ K \cap \{x.\ x \cdot i \ge c\}))) K)$
$\le content(cbox\ a'\ b)\ /\ (b \cdot i - c)$
using *lec gec* **by** (*simp_all add*: *field_split_simps*)
moreover
have $(\sum K\in\mathcal{D}.\ content\ K\ /\ (interv\_diff\ K\ i))$
$\le (\sum K\in\mathcal{D}.\ ((\lambda K.\ content\ K\ /\ (interv\_diff\ K\ i)) \circ ((\lambda K.\ K \cap \{x.\ x \cdot i \le c\}))) K) +$
$(\sum K\in\mathcal{D}.\ ((\lambda K.\ content\ K\ /\ (interv\_diff\ K\ i)) \circ ((\lambda K.\ K \cap \{x.\ x \cdot i \ge c\}))) K)$
(**is** $?lhs \le ?rhs$)
proof –
have $?lhs \le$
$(\sum K\in\mathcal{D}.\ ((\lambda K.\ content\ K\ /\ (interv\_diff\ K\ i)) \circ ((\lambda K.\ K \cap \{x.\ x \cdot i \le c\}))) K +$
$((\lambda K.\ content\ K\ /\ (interv\_diff\ K\ i)) \circ ((\lambda K.\ K \cap \{x.\ x \cdot i \ge c\}))) K)$
(**is** $sum\ ?f\ \_ \le sum\ ?g\ \_$)
proof (*rule sum_mono*)
fix *K* **assume** $K \in \mathcal{D}$
then obtain *u v* **where** *uv*: $K = cbox\ u\ v$
using *div* **by** *blast*
obtain *u' v'* **where** *uv'*: $cbox\ u\ v \cap \{x.\ x \cdot i \le c\} = cbox\ u\ v'$
$cbox\ u\ v \cap \{x.\ c \le x \cdot i\} = cbox\ u'\ v$
$\bigwedge k.\ k \in Basis \Longrightarrow u' \cdot k = (if\ k = i\ then\ max\ (u \cdot i)\ c\ else\ u \cdot k)$
$\bigwedge k.\ k \in Basis \Longrightarrow v' \cdot k = (if\ k = i\ then\ min\ (v \cdot i)\ c\ else\ v \cdot k)$
using *i* **by** (*auto simp*: *interval_split*)
have $*$: $[\![content\ (cbox\ u\ v') = 0;\ content\ (cbox\ u'\ v) = 0]\!] \Longrightarrow content\ (cbox\ u\ v) = 0$
$content\ (cbox\ u'\ v) \neq 0 \Longrightarrow content\ (cbox\ u\ v) \neq 0$
$content\ (cbox\ u\ v') \neq 0 \Longrightarrow content\ (cbox\ u\ v) \neq 0$

using *i uv uv'* **by** (*auto simp*: *content_eq_0 le_max_iff_disj min_le_iff_disj split*: *if_split_asm intro*: *order_trans*)
have *uniq*: $\bigwedge j$. ⟦*j* ∈ *Basis*; ¬ *u* · *j* ≤ *v* · *j*⟧ ⟹ *j* = *i*
by (*metis* ‹*K* ∈ $\mathcal{D}$› *box_ne_empty*(*1*) *div division_of_def uv*)
show *?f K* ≤ *?g K*
using *i uv uv'* **by** (*auto simp add*: *interv_diff_def lemma0 dest*: *uniq* ∗ *intro*!: *prod_nonneg*)
qed
also have ... = *?rhs*
by (*simp add*: *sum.distrib*)
finally show *?thesis* **.**
qed
moreover have *content* (*cbox a b'*) / (*c* − *a* · *i*) = *content* (*cbox a b*) / (*b* · *i* − *a* · *i*)
using *i abc*
apply (*simp add*: *field_simps a'_def b'_def measure_lborel_cbox_eq inner_diff*)
apply (*auto simp*: *if_distrib if_distrib* [*of* λ*f*. *f x* **for** *x*] *prod.If_cases* [*of Basis* λ*x*. *x* = *i*, *simplified*] *prod_if field_simps*)
done
moreover have *content* (*cbox a' b*) / (*b* · *i* − *c*) = *content* (*cbox a b*) / (*b* · *i* − *a* · *i*)
using *i abc*
apply (*simp add*: *field_simps a'_def b'_def measure_lborel_cbox_eq inner_diff*)
apply (*auto simp*: *if_distrib prod.If_cases* [*of Basis* λ*x*. *x* = *i*, *simplified*] *prod_if field_simps*)
done
ultimately
have ($\sum$ *K*∈$\mathcal{D}$. *content K* / (*interv_diff K i*)) ≤ *2* ∗ *content* (*cbox a b*) / (*b* · *i* − *a* · *i*)
by *linarith*
then show *?thesis*
using *abc interv_diff_def* **by** (*simp add*: *field_split_simps*)
qed
qed

proposition *bounded_equiintegral_over_thin_tagged_partial_division*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *F*: *F equiintegrable_on cbox a b* **and** *f*: *f* ∈ *F* **and** *0* < ε
and *norm_f*: $\bigwedge$*h x*. ⟦*h* ∈ *F*; *x* ∈ *cbox a b*⟧ ⟹ *norm*(*h x*) ≤ *norm*(*f x*)
obtains γ **where** *gauge* γ
$\bigwedge$*c i S h*. ⟦*c* ∈ *cbox a b*; *i* ∈ *Basis*; *S tagged_partial_division_of cbox a b*; γ *fine S*; *h* ∈ *F*; $\bigwedge$*x K*. (*x*,*K*) ∈ *S* ⟹ (*K* ∩ {*x*. *x* · *i* = *c* · *i*} ≠ {})⟧
⟹ ($\sum$(*x*,*K*) ∈ *S*. *norm* (*integral K h*)) < ε
proof (*cases content*(*cbox a b*) = *0*)
case *True*
show *?thesis*
proof

```
    show gauge (λx. ball x 1)
      by (simp add: gauge_trivial)
    show (∑(x,K) ∈ S. norm (integral K h)) < ε
          if S tagged_partial_division_of cbox a b (λx. ball x 1) fine S for S and h::
'a ⇒ 'b
    proof -
      have (∑(x,K) ∈ S. norm (integral K h)) = 0
          using that True content_0_subset
          by (fastforce simp: tagged_partial_division_of_def intro: sum.neutral)
      with ‹0 < ε› show ?thesis
        by simp
    qed
  qed
next
  case False
  then have contab_gt0:  content(cbox a b) > 0
    by (simp add: zero_less_measure_iff)
  then have a_less_b: ⋀i. i ∈ Basis ⟹ a•i < b•i
    by (auto simp: content_pos_lt_eq)
  obtain γ0 where gauge γ0
            and γ0: ⋀S h. ⟦S tagged_partial_division_of cbox a b; γ0 fine S; h ∈ F⟧
                            ⟹ (∑(x,K) ∈ S. norm (content K *R h x − integral K
h)) < ε/2
  proof -
    obtain γ where gauge γ
               and γ: ⋀f D. ⟦f ∈ F; D tagged_division_of cbox a b; γ fine D⟧
                              ⟹ norm ((∑(x,K) ∈ D. content K *R f x) − integral
(cbox a b) f)
                                 < ε/(5 * (Suc DIM('b)))
    proof -
      have e5: ε/(5 * (Suc DIM('b))) > 0
        using ‹ε > 0› by auto
      then show ?thesis
        using F that by (auto simp: equiintegrable_on_def)
    qed
    show ?thesis
    proof
      show gauge γ
        by (rule ‹gauge γ›)
      show (∑(x,K) ∈ S. norm (content K *R h x − integral K h)) < ε/2
          if S tagged_partial_division_of cbox a b γ fine S h ∈ F for S h
      proof -
        have (∑(x,K) ∈ S. norm (content K *R h x − integral K h)) ≤ 2 * real
DIM('b) * (ε/(5 * Suc DIM('b)))
        proof (rule Henstock_lemma_part2 [of h a b])
          show h integrable_on cbox a b
            using that F equiintegrable_on_def by metis
          show gauge γ
            by (rule ‹gauge γ›)
```

qed (*use that* ‹$\varepsilon > 0$› γ **in** *auto*)
also have ... < $\varepsilon/2$
using ‹$\varepsilon > 0$› **by** (*simp add*: *divide_simps*)
finally show *?thesis* **.**
qed
qed
qed
define γ **where** $\gamma \equiv \lambda x.\ \gamma 0\ x \cap$
ball x ((ε/*8* / (*norm*(*f x*) + *1*)) * (*INF m*∈*Basis*. *b* · *m* − *a* · *m*) / *content*(*cbox a b*))
define *interv_diff* **where** *interv_diff* ≡ λ*K*. λ*i*::′*a*. *interval_upperbound K* · *i* − *interval_lowerbound K* · *i*
have *8* * *content* (*cbox a b*) + *norm* (*f x*) * (*8* * *content* (*cbox a b*)) > *0* **for** *x*
by (*metis add.right_neutral add_pos_pos contab_gt0 mult_pos_pos mult_zero_left norm_eq_zero zero_less_norm_iff zero_less_numeral*)
then have *gauge* (λ*x*. *ball x*
(ε * (*INF m*∈*Basis*. *b* · *m* − *a* · *m*) / ((*8* * *norm* (*f x*) + *8*) * *content* (*cbox a b*))))
using ‹*0* < *content* (*cbox a b*)› ‹*0* < ε› *a_less_b*
by (*auto simp add*: *gauge_def field_split_simps add_nonneg_eq_0_iff finite_less_Inf_iff*)
then have *gauge* γ
unfolding γ*_def* **using** ‹*gauge* $\gamma 0$› *gauge_Int* **by** *auto*
moreover
have ($\sum$(*x*,*K*) ∈ *S*. *norm* (*integral K h*)) < ε
if *c* ∈ *cbox a b i* ∈ *Basis* **and** *S*: *S tagged_partial_division_of cbox a b*
and γ *fine S h* ∈ *F* **and** *ne*: ⋀*x K*. (*x*,*K*) ∈ *S* ⟹ *K* ∩ {*x*. *x* · *i* = *c* · *i*} ≠ {} **for** *c i S h*
proof −
have *cbox c b* ⊆ *cbox a b*
by (*meson mem_box*(*2*) *order_refl subset_box*(*1*) *that*(*1*))
have *finite S*
using *S* **unfolding** *tagged_partial_division_of_def* **by** *blast*
have $\gamma 0$ *fine S* **and** *fineS*:
(λ*x*. *ball x* (ε * (*INF m*∈*Basis*. *b* · *m* − *a* · *m*) / ((*8* * *norm* (*f x*) + *8*) * *content* (*cbox a b*)))) *fine S*
using ‹γ *fine S*› **by** (*auto simp*: γ*_def fine_Int*)
then have ($\sum$(*x*,*K*) ∈ *S*. *norm* (*content K* $*_R$ *h x* − *integral K h*)) < $\varepsilon/2$
by (*intro* $\gamma 0$ *that fineS*)
moreover have ($\sum$(*x*,*K*) ∈ *S*. *norm* (*integral K h*) − *norm* (*content K* $*_R$ *h x* − *integral K h*)) ≤ $\varepsilon/2$
proof −
have ($\sum$(*x*,*K*) ∈ *S*. *norm* (*integral K h*) − *norm* (*content K* $*_R$ *h x* − *integral K h*))
≤ ($\sum$(*x*,*K*) ∈ *S*. *norm* (*content K* $*_R$ *h x*))
proof (*clarify intro*!: *sum_mono*)
fix *x K*
assume *xK*: (*x*,*K*) ∈ *S*
have *norm* (*integral K h*) − *norm* (*content K* $*_R$ *h x* − *integral K h*) ≤ *norm* (*integral K h* − (*integral K h* − *content K* $*_R$ *h x*))

by (*metis norm_minus_commute norm_triangle_ineq2*)
also have ... ≤ *norm* (*content K* $*_R$ *h x*)
by *simp*
finally show *norm* (*integral K h*) − *norm* (*content K* $*_R$ *h x* − *integral K h*) ≤ *norm* (*content K* $*_R$ *h x*) **.**
qed
also have ... ≤ (∑(*x*,*K*) ∈ *S*. ε/*4* ∗ (*b* · *i* − *a* · *i*) / *content* (*cbox a b*) ∗ *content K* / *interv_diff K i*)
proof (*clarify intro*!: *sum_mono*)
fix *x K*
assume *xK*: (*x*,*K*) ∈ *S*
then have *x*: *x* ∈ *cbox a b*
using *S* **unfolding** *tagged_partial_division_of_def* **by** (*meson subset_iff*)
show *norm* (*content K* $*_R$ *h x*) ≤ ε/*4* ∗ (*b* · *i* − *a* · *i*) / *content* (*cbox a b*) ∗ *content K* / *interv_diff K i*
proof (*cases content K* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then have *Kgt0*: *content K* > *0*
using *zero_less_measure_iff* **by** *blast*
moreover
obtain *u v* **where** *uv*: *K* = *cbox u v*
using *S* ‹(*x*,*K*) ∈ *S*› **unfolding** *tagged_partial_division_of_def* **by** *blast*
then have *u_less_v*: ⋀*i*. *i* ∈ *Basis* ⟹ *u* · *i* < *v* · *i*
using *content_pos_lt_eq uv Kgt0* **by** *blast*
then have *dist_uv*: *dist u v* > *0*
using *that* **by** *auto*
ultimately have *norm* (*h x*) ≤ (ε ∗ (*b* · *i* − *a* · *i*)) / (*4* ∗ *content* (*cbox a b*) ∗ *interv_diff K i*)
proof −
have *dist x u* < ε ∗ (*INF m*∈*Basis*. *b* · *m* − *a* · *m*) / (*4* ∗ (*norm* (*f x*) + *1*) ∗ *content* (*cbox a b*)) / *2*
dist x v < ε ∗ (*INF m*∈*Basis*. *b* · *m* − *a* · *m*) / (*4* ∗ (*norm* (*f x*) + *1*) ∗ *content* (*cbox a b*)) / *2*
using *fineS u_less_v uv xK*
by (*force simp*: *fine_def mem_box field_simps dest*!: *bspec*)+
moreover have ε ∗ (*INF m*∈*Basis*. *b* · *m* − *a* · *m*) / (*4* ∗ (*norm* (*f x*) + *1*) ∗ *content* (*cbox a b*)) / *2*
≤ ε ∗ (*b* · *i* − *a* · *i*) / (*4* ∗ (*norm* (*f x*) + *1*) ∗ *content* (*cbox a b*)) / *2*
proof (*intro mult_left_mono divide_right_mono*)
show (*INF m*∈*Basis*. *b* · *m* − *a* · *m*) ≤ *b* · *i* − *a* · *i*
using ‹*i* ∈ *Basis*› **by** (*auto intro*!: *cInf_le_finite*)
qed (*use* ‹*0* < ε› **in** *auto*)
ultimately
have *dist x u* < ε ∗ (*b* · *i* − *a* · *i*) / (*4* ∗ (*norm* (*f x*) + *1*) ∗ *content* (*cbox a b*)) / *2*

dist x v < ε ∗ (*b* · *i* − *a* · *i*) / (*4* ∗ (*norm* (*f x*) + *1*) ∗ *content* (*cbox a b*)) / *2*
by *linarith+*
then have *duv*: *dist u v* < ε ∗ (*b* · *i* − *a* · *i*) / (*4* ∗ (*norm* (*f x*) + *1*) ∗ *content* (*cbox a b*))
using *dist_triangle_half_r* **by** *blast*
have *uvi*: |*v* · *i* − *u* · *i*| ≤ *norm* (*v* − *u*)
by (*metis inner_commute inner_diff_right* ‹*i* ∈ *Basis*› *Basis_le_norm*)
have *norm* (*h x*) ≤ *norm* (*f x*)
using *x that* **by** (*auto simp*: *norm_f*)
also have ... < (*norm* (*f x*) + *1*)
by *simp*
also have ... < ε ∗ (*b* · *i* − *a* · *i*) / *dist u v* / (*4* ∗ *content* (*cbox a b*))
proof −
have *0* < *norm* (*f x*) + *1*
by (*simp add*: *add.commute add_pos_nonneg*)
then show *?thesis*
using *duv dist_uv contab_gt0*
by (*simp only*: *mult_ac divide_simps*) *auto*
qed
also have ... = ε ∗ (*b* · *i* − *a* · *i*) / *norm* (*v* − *u*) / (*4* ∗ *content* (*cbox a b*))
by (*simp add*: *dist_norm norm_minus_commute*)
also have ... ≤ ε ∗ (*b* · *i* − *a* · *i*) / |*v* · *i* − *u* · *i*| / (*4* ∗ *content* (*cbox a b*))
proof (*intro mult_right_mono divide_left_mono divide_right_mono uvi*)
show *norm* (*v* − *u*) ∗ |*v* · *i* − *u* · *i*| > *0*
using *u_less_v* [*OF* ‹*i* ∈ *Basis*›]
by (*auto simp*: *less_eq_real_def zero_less_mult_iff that*)
show ε ∗ (*b* · *i* − *a* · *i*) ≥ *0*
using *a_less_b* ‹*0* < ε› ‹*i* ∈ *Basis*› **by** *force*
qed *auto*
also have ... = ε ∗ (*b* · *i* − *a* · *i*) / (*4* ∗ *content* (*cbox a b*) ∗ *interv_diff K i*)
using *uv False that*(*2*) *u_less_v interv_diff_def* **by** *fastforce*
finally show *?thesis* **by** *simp*
qed
with *Kgt0* **have** *norm* (*content K* $*_R$ *h x*) ≤ *content K* ∗ ((ε/*4* ∗ (*b* · *i* − *a* · *i*) / *content* (*cbox a b*)) / *interv_diff K i*)
using *mult_left_mono* **by** *fastforce*
also have ... = ε/*4* ∗ (*b* · *i* − *a* · *i*) / *content* (*cbox a b*) ∗ *content K* / *interv_diff K i*
by (*simp add*: *field_split_simps*)
finally show *?thesis* **.**
qed
qed
also have ... = (∑ *K*∈*snd* ‘ *S*. ε/*4* ∗ (*b* · *i* − *a* · *i*) / *content* (*cbox a b*) ∗ *content K* / *interv_diff K i*)
unfolding *interv_diff_def*

apply (*rule sum.over_tagged_division_lemma* [*OF tagged_partial_division_of_Union_self* [*OF S*]])
apply (*simp add*: *box_eq_empty*(*1*) *content_eq_0*)
done
also have ... = $\varepsilon/2$ ∗ ((*b* · *i* − *a* · *i*) / (*2* ∗ *content* (*cbox a b*)) ∗ (∑ *K*∈*snd* ‘ *S*. *content K* / *interv_diff K i*))
by (*simp add*: *interv_diff_def sum_distrib_left mult.assoc*)
also have ... ≤ ($\varepsilon/2$) ∗ *1*
proof (*rule mult_left_mono*)
have (*b* · *i* − *a* · *i*) ∗ (∑ *K*∈*snd* ‘ *S*. *content K* / *interv_diff K i*) ≤ *2* ∗ *content* (*cbox a b*)
unfolding *interv_diff_def*
proof (*rule sum_content_area_over_thin_division*)
show *snd* ‘ *S division_of* ⋃(*snd* ‘ *S*)
by (*auto intro*: *S tagged_partial_division_of_Union_self division_of_tagged_division*)
show ⋃(*snd* ‘ *S*) ⊆ *cbox a b*
using *S* **unfolding** *tagged_partial_division_of_def* **by** *force*
show *a* · *i* ≤ *c* · *i* *c* · *i* ≤ *b* · *i*
using *mem_box*(*2*) *that* **by** *blast*+
qed (*use that* **in** *auto*)
then show (*b* · *i* − *a* · *i*) / (*2* ∗ *content* (*cbox a b*)) ∗ (∑ *K*∈*snd* ‘ *S*. *content K* / *interv_diff K i*) ≤ *1*
by (*simp add*: *contab_gt0*)
qed (*use* ‹*0* < ε› **in** *auto*)
finally show *?thesis* **by** *simp*
qed
then have (∑(*x*,*K*) ∈ *S*. *norm* (*integral K h*)) − (∑(*x*,*K*) ∈ *S*. *norm* (*content K* $*_R$ *h x* − *integral K h*)) ≤ $\varepsilon/2$
by (*simp add*: *Groups_Big.sum_subtractf* [*symmetric*])
ultimately show (∑(*x*,*K*) ∈ *S*. *norm* (*integral K h*)) < ε
by *linarith*
qed
ultimately show *?thesis* **using** *that* **by** *auto*
qed

proposition *equiintegrable_halfspace_restrictions_le*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
assumes *F*: *F equiintegrable_on cbox a b* **and** *f*: *f* ∈ *F*
and *norm_f*: ⋀*h x*. ⟦*h* ∈ *F*; *x* ∈ *cbox a b*⟧ ⟹ *norm*(*h x*) ≤ *norm*(*f x*)
shows (⋃*i* ∈ *Basis*. ⋃*c*. ⋃*h* ∈ *F*. {(λ*x*. *if x* · *i* ≤ *c then h x else 0*)})
equiintegrable_on cbox a b
proof (*cases content*(*cbox a b*) = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then have *content*(*cbox a b*) > *0*

using *zero_less_measure_iff* **by** *blast*
then have $a \cdot i < b \cdot i$ **if** $i \in$ *Basis* **for** *i*
using *content_pos_lt_eq that* **by** *blast*
have *int_F*: *f integrable_on cbox a b* **if** $f \in F$ **for** *f*
using *F that* **by** (*simp add*: *equiintegrable_on_def*)
let *?CI* = $\lambda K\ h\ x.$ *content K* $*_R$ *h x* − *integral K h*
show *?thesis*
unfolding *equiintegrable_on_def*
proof (*intro conjI*; *clarify*)
show *int_lec*: ⟦$i \in$ *Basis*; $h \in F$⟧ ⟹ ($\lambda x.$ *if* $x \cdot i \leq c$ *then h x else 0*) *integrable_on cbox a b* **for** *i c h*
using *integrable_restrict_Int* [*of* {$x.\ x \cdot i \leq c$} *h*]
by (*simp add*: *inf_commute int_F integrable_split*(*1*))
show ∃γ. *gauge* γ ∧
(∀*f T*. $f \in$ (⋃*i*∈*Basis*. ⋃*c*. ⋃*h*∈*F*. {λ*x*. *if* $x \cdot i \leq c$ *then h x else 0*}) ∧
T tagged_division_of cbox a b ∧ γ *fine T* ⟶
norm ((∑(*x*,*K*) ∈ *T*. *content K* $*_R$ *f x*) − *integral* (*cbox a b*) *f*) < ε)
if ε > *0* **for** ε
proof −
obtain γ*0* **where** *gauge* γ*0* **and** γ*0*:
⋀*c i S h*. ⟦$c \in$ *cbox a b*; $i \in$ *Basis*; *S tagged_partial_division_of cbox a b*;
γ*0 fine S*; $h \in F$; ⋀*x K*. (*x*,*K*) ∈ *S* ⟹ (*K* ∩ {$x.\ x \cdot i = c \cdot i$} ≠ {})⟧
⟹ (∑(*x*,*K*) ∈ *S*. *norm* (*integral K h*)) < ε/*12*
proof (*rule bounded_equiintegral_over_thin_tagged_partial_division* [*OF F f*, *of* ‹ε/*12*›])
show ⋀*h x*. ⟦$h \in F$; $x \in$ *cbox a b*⟧ ⟹ *norm* (*h x*) ≤ *norm* (*f x*)
by (*auto simp*: *norm_f*)
qed (*use* ‹ε > *0*› **in** *auto*)
obtain γ*1* **where** *gauge* γ*1*
and γ*1*: ⋀*h T*. ⟦$h \in F$; *T tagged_division_of cbox a b*; γ*1 fine T*⟧
⟹ *norm* ((∑(*x*,*K*) ∈ *T*. *content K* $*_R$ *h x*) − *integral* (*cbox a b*) *h*)
< ε/(*7* ∗ (*Suc DIM*('*b*)))
proof −
have *e5*: ε/(*7* ∗ (*Suc DIM*('*b*))) > *0*
using ‹ε > *0*› **by** *auto*
then show *?thesis*
using *F that* **by** (*auto simp*: *equiintegrable_on_def*)
qed
have *h_less3*: (∑(*x*,*K*) ∈ *T*. *norm* (*?CI K h x*)) < ε/*3*
if *T tagged_partial_division_of cbox a b* γ*1 fine T* $h \in F$ **for** *T h*
proof −
have (∑(*x*,*K*) ∈ *T*. *norm* (*?CI K h x*)) ≤ *2* ∗ *real DIM*('*b*) ∗ (ε/(*7* ∗ *Suc DIM*('*b*)))
proof (*rule Henstock_lemma_part2* [*of h a b*])
show *h integrable_on cbox a b*

```
      using that F equiintegrable_on_def by metis
    qed (use that ‹ε > 0› ‹gauge γ1› γ1 in auto)
    also have ... < ε/3
      using ‹ε > 0› by (simp add: divide_simps)
    finally show ?thesis .
  qed
  have *: norm ((∑(x,K) ∈ T. content K *R f x) − integral (cbox a b) f) < ε
           if f: f = (λx. if x · i ≤ c then h x else 0)
           and T: T tagged_division_of cbox a b
          and fine: (λx. γ0 x ∩ γ1 x) fine T and i ∈ Basis h ∈ F for f T i c h
  proof (cases a · i ≤ c ∧ c ≤ b · i)
    case True
    have finite T
      using T by blast
    define T' where T' ≡ {(x,K) ∈ T. K ∩ {x. x · i ≤ c} ≠ {}}
    then have T' ⊆ T
      by auto
    then have finite T'
      using ‹finite T› infinite_super by blast
    have T'_tagged: T' tagged_partial_division_of cbox a b
     by (meson T ‹T' ⊆ T› tagged_division_of_def tagged_partial_division_subset)
    have fine': γ0 fine T' γ1 fine T'
      using ‹T' ⊆ T› fine_Int fine_subset fine by blast+
   have int_KK': (∑(x,K) ∈ T. integral K f) = (∑(x,K) ∈ T'. integral K f)
    proof (rule sum.mono_neutral_right [OF ‹finite T› ‹T' ⊆ T›])
      show ∀i ∈ T − T'. (case i of (x, K) ⇒ integral K f) = 0
        using f ‹finite T› ‹T' ⊆ T› integral_restrict_Int [of _ {x. x · i ≤ c} h]
        by (auto simp: T'_def Int_commute)
    qed
    have (∑(x,K) ∈ T. content K *R f x) = (∑(x,K) ∈ T'. content K *R f
x)
    proof (rule sum.mono_neutral_right [OF ‹finite T› ‹T' ⊆ T›])
      show ∀i ∈ T − T'. (case i of (x, K) ⇒ content K *R f x) = 0
        using T f ‹finite T› ‹T' ⊆ T› by (force simp: T'_def)
    qed
   moreover have norm ((∑(x,K) ∈ T'. content K *R f x) − integral (cbox
a b) f) < ε
    proof −
      have *: norm y < ε if norm x < ε/3 norm(x − y) ≤ 2 * ε/3 for x y::'b
      proof −
        have norm y ≤ norm x + norm(x − y)
          by (metis norm_minus_commute norm_triangle_sub)
        also have . . . < ε/3 + 2*ε/3
          using that by linarith
        also have ... = ε
          by simp
        finally show ?thesis .
      qed
      have norm (∑(x,K) ∈ T'. ?CI K h x)
```

$\leq (\sum(x,K) \in T'.\ norm\ (?CI\ K\ h\ x))$
by (*simp add*: *norm_sum split_def*)
also have ... $< \varepsilon/3$
by (*intro h_less3 T'_tagged fine' that*)
finally have $norm\ (\sum(x,K) \in T'.\ ?CI\ K\ h\ x) < \varepsilon/3$ **.**
moreover have $integral\ (cbox\ a\ b)\ f = (\sum(x,K) \in T.\ integral\ K\ f)$
using *int_lec that* **by** (*auto simp*: *integral_combine_tagged_division_topdown*)
moreover have $norm\ (\sum(x,K) \in T'.\ ?CI\ K\ h\ x - ?CI\ K\ f\ x)$
$\leq 2{*}\varepsilon/3$
proof −
define T'' **where** $T'' \equiv \{(x,K) \in T'.\ \neg\ (K \subseteq \{x.\ x \cdot i \leq c\})\}$
then have $T'' \subseteq T'$
by *auto*
then have *finite* T''
using ‹*finite* T'› *infinite_super* **by** *blast*
have T''*_tagged*: T'' *tagged_partial_division_of cbox a b*
using T'*_tagged* ‹$T'' \subseteq T'$› *tagged_partial_division_subset* **by** *blast*
have *fine''*: $\gamma 0$ *fine* T'' $\gamma 1$ *fine* T''
using ‹$T'' \subseteq T'$› *fine'* **by** (*blast intro*: *fine_subset*)+
have $(\sum(x,K) \in T'.\ ?CI\ K\ h\ x - ?CI\ K\ f\ x)$
$= (\sum(x,K) \in T''.\ ?CI\ K\ h\ x - ?CI\ K\ f\ x)$
proof (*clarify intro*!: *sum.mono_neutral_right* [*OF* ‹*finite* T'› ‹$T'' \subseteq T'$›])
fix $x\ K$
assume $(x,K) \in T'\ (x,K) \notin T''$
then have $x \in K\ x \cdot i \leq c\ \{x.\ x \cdot i \leq c\} \cap K = K$
using T''*_def* T'*_tagged tagged_partial_division_of_def* **by** *blast*+
then show $?CI\ K\ h\ x - ?CI\ K\ f\ x = 0$
using *integral_restrict_Int* [*of* _ $\{x.\ x \cdot i \leq c\}$ *h*] **by** (*auto simp*: *f*)
qed
moreover have $norm\ (\sum(x,K) \in T''.\ ?CI\ K\ h\ x - ?CI\ K\ f\ x) \leq 2{*}\varepsilon/3$
proof −
define A **where** $A \equiv \{(x,K) \in T''.\ x \cdot i \leq c\}$
define B **where** $B \equiv \{(x,K) \in T''.\ x \cdot i > c\}$
then have $A \subseteq T''\ B \subseteq T''$ **and** *disj*: $A \cap B = \{\}$ **and** T''*_eq*: T''
$= A \cup B$
by (*auto simp*: *A_def B_def*)
then have *finite A finite B*
using ‹*finite* T''› **by** (*auto intro*: *finite_subset*)
have *A_tagged*: *A tagged_partial_division_of cbox a b*
using T''*_tagged* ‹$A \subseteq T''$› *tagged_partial_division_subset* **by** *blast*
have *fineA*: $\gamma 0$ *fine A* $\gamma 1$ *fine A*
using ‹$A \subseteq T''$› *fine''* **by** (*blast intro*: *fine_subset*)+
have *B_tagged*: *B tagged_partial_division_of cbox a b*
using T''*_tagged* ‹$B \subseteq T''$› *tagged_partial_division_subset* **by** *blast*
have *fineB*: $\gamma 0$ *fine B* $\gamma 1$ *fine B*
using ‹$B \subseteq T''$› *fine''* **by** (*blast intro*: *fine_subset*)+
have $norm\ (\sum(x,K) \in T''.\ ?CI\ K\ h\ x - ?CI\ K\ f\ x)$
$\leq (\sum(x,K) \in T''.\ norm\ (?CI\ K\ h\ x - ?CI\ K\ f\ x))$
by (*simp add*: *norm_sum split_def*)

```
also have ... = (∑(x,K) ∈ A. norm (?CI K h x − ?CI K f x)) +
                (∑(x,K) ∈ B. norm (?CI K h x − ?CI K f x))
  by (simp add: sum.union_disjoint T''_eq disj ‹finite A› ‹finite B›)
also have ... = (∑(x,K) ∈ A. norm (integral K h − integral K f)) +
                (∑(x,K) ∈ B. norm (?CI K h x + integral K f))
  by (auto simp: A_def B_def f norm_minus_commute intro!: sum.cong
arg_cong2 [where f= (+)])
also have ... ≤ (∑(x,K)∈A. norm (integral K h)) +
                 (∑(x,K)∈(λ(x,K). (x,K ∩ {x. x · i ≤ c})) ` A. norm
(integral K h))
              + ((∑(x,K)∈B. norm (?CI K h x)) +
                 (∑(x,K)∈B. norm (integral K h)) +
                 (∑(x,K)∈(λ(x,K). (x,K ∩ {x. c ≤ x · i})) ` B. norm
(integral K h)))
proof (rule add_mono)
  show (∑(x,K)∈A. norm (integral K h − integral K f))
       ≤ (∑(x,K)∈A. norm (integral K h)) +
         (∑(x,K)∈(λ(x,K). (x,K ∩ {x. x · i ≤ c})) ` A.
            norm (integral K h))
  proof (subst sum.reindex_nontrivial [OF ‹finite A›], clarsimp)
    fix x K L
    assume (x,K) ∈ A (x,L) ∈ A
      and int_ne0: integral (L ∩ {x. x · i ≤ c}) h ≠ 0
      and eq: K ∩ {x. x · i ≤ c} = L ∩ {x. x · i ≤ c}
    have False if K ≠ L
    proof −
      obtain u v where uv: L = cbox u v
          using T'_tagged ‹(x, L) ∈ A› ‹A ⊆ T''› ‹T'' ⊆ T'› by (blast
dest: tagged_partial_division_ofD)
      have interior (K ∩ {x. x · i ≤ c}) = {}
      proof (rule tagged_division_split_left_inj [OF _ ‹(x,K) ∈ A› ‹(x,L)
∈ A›])
        show A tagged_division_of ⋃(snd ` A)
          using A_tagged tagged_partial_division_of_Union_self by auto
        show K ∩ {x. x · i ≤ c} = L ∩ {x. x · i ≤ c}
          using eq ‹i ∈ Basis› by auto
      qed (use that in auto)
      then show False
       using interval_split [OF ‹i ∈ Basis›] int_ne0 content_eq_0_interior
eq uv by fastforce
    qed
    then show K = L by blast
  next
    show (∑(x,K) ∈ A. norm (integral K h − integral K f))
         ≤ (∑(x,K) ∈ A. norm (integral K h)) +
           sum ((λ(x,K). norm (integral K h)) ∘ (λ(x,K). (x,K ∩ {x.
x · i ≤ c}))) A
      using integral_restrict_Int [of _ {x. x · i ≤ c} h] f
         by (auto simp: Int_commute A_def [symmetric] sum.distrib
```

[*symmetric*] *intro*!: *sum_mono norm_triangle_ineq4*)
qed
next
show ($\sum$(*x*,*K*)∈*B*. *norm* (*?CI K h x* + *integral K f*))
≤ ($\sum$(*x*,*K*)∈*B*. *norm* (*?CI K h x*)) + ($\sum$(*x*,*K*)∈*B*. *norm* (*integral K h*)) +
($\sum$(*x*,*K*)∈(λ(*x*,*K*). (*x*,*K* ∩ {*x*. *c* ≤ *x* · *i*})) ' *B*. *norm* (*integral K h*))
proof (*subst sum.reindex_nontrivial* [*OF* ‹*finite B*›], *clarsimp*)
fix *x K L*
assume (*x*,*K*) ∈ *B* (*x*,*L*) ∈ *B*
and *int_ne0*: *integral* (*L* ∩ {*x*. *c* ≤ *x* · *i*}) *h* ≠ *0*
and *eq*: *K* ∩ {*x*. *c* ≤ *x* · *i*} = *L* ∩ {*x*. *c* ≤ *x* · *i*}
have *False* **if** *K* ≠ *L*
proof −
obtain *u v* **where** *uv*: *L* = *cbox u v*
using *T'_tagged* ‹(*x*, *L*) ∈ *B*› ‹*B* ⊆ *T''*› ‹*T''* ⊆ *T'*› **by** (*blast dest*: *tagged_partial_division_ofD*)
have *interior* (*K* ∩ {*x*. *c* ≤ *x* · *i*}) = {}
proof (*rule tagged_division_split_right_inj* [*OF* _ ‹(*x*,*K*) ∈ *B*› ‹(*x*,*L*) ∈ *B*›])
show *B tagged_division_of* ⋃(*snd* ' *B*)
using *B_tagged tagged_partial_division_of_Union_self* **by** *auto*
show *K* ∩ {*x*. *c* ≤ *x* · *i*} = *L* ∩ {*x*. *c* ≤ *x* · *i*}
using *eq* ‹*i* ∈ *Basis*› **by** *auto*
qed (*use that* **in** *auto*)
then show *False*
using *interval_split* [*OF* ‹*i* ∈ *Basis*›] *int_ne0*
content_eq_0_interior eq uv **by** *fastforce*
qed
then show *K* = *L* **by** *blast*
next
show ($\sum$(*x*,*K*) ∈ *B*. *norm* (*?CI K h x* + *integral K f*))
≤ ($\sum$(*x*,*K*) ∈ *B*. *norm* (*?CI K h x*)) +
($\sum$(*x*,*K*) ∈ *B*. *norm* (*integral K h*)) + *sum* ((λ(*x*,*K*). *norm* (*integral K h*)) ∘ (λ(*x*,*K*). (*x*,*K* ∩ {*x*. *c* ≤ *x* · *i*}))) *B*
proof (*clarsimp simp*: *B_def* [*symmetric*] *sum.distrib* [*symmetric*] *intro*!: *sum_mono*)
fix *x K*
assume (*x*,*K*) ∈ *B*
have ∗: *i* = *i1* + *i2* ⟹ *norm*(*c* + *i1*) ≤ *norm c* + *norm i* + *norm*(*i2*)
for *i*::'*b* **and** *c i1 i2*
by (*metis add.commute add.left_commute add_diff_cancel_right' dual_order.refl norm_add_rule_thm norm_triangle_ineq4*)
obtain *u v* **where** *uv*: *K* = *cbox u v*
using *T'_tagged* ‹(*x*,*K*) ∈ *B*› ‹*B* ⊆ *T''*› ‹*T''* ⊆ *T'*› **by** (*blast dest*: *tagged_partial_division_ofD*)
have *huv*: *h integrable_on cbox u v*

```
          proof (rule integrable_on_subcbox)
            show cbox u v ⊆ cbox a b
         using B_tagged ‹(x,K) ∈ B› uv by (blast dest: tagged_partial_division_ofD)
            show h integrable_on cbox a b
              by (simp add: int_F ‹h ∈ F›)
          qed
          have integral K h = integral K f + integral (K ∩ {x. c ≤ x · i}) h
            using integral_restrict_Int [of _ {x. x · i ≤ c} h] f uv ‹i ∈ Basis›
            by (simp add: Int_commute integral_split [OF huv ‹i ∈ Basis›])
          then show norm (?CI K h x + integral K f)
                       ≤ norm (?CI K h x) + norm (integral K h) + norm
(integral (K ∩ {x. c ≤ x · i}) h)
            by (rule *)
        qed
      qed
    qed
    also have ... ≤ 2*ε/3
    proof -
      have overlap: K ∩ {x. x · i = c} ≠ {} if (x,K) ∈ T'' for x K
      proof -
        obtain y y' where y: y' ∈ K c < y' · i y ∈ K y · i ≤ c
          using that  T''_def T'_def ‹(x,K) ∈ T''› by fastforce
        obtain u v where uv: K = cbox u v
       using T''_tagged ‹(x,K) ∈ T''› by (blast dest: tagged_partial_division_ofD)
        then have connected K
          by (simp add: is_interval_connected)
        then have (∃z ∈ K. z · i = c)
          using y connected_ivt_component by fastforce
        then show ?thesis
          by fastforce
      qed
      have **: ⟦x < ε/12; y < ε/12; z ≤ ε/2⟧ ⟹ x + y + z ≤ 2 * ε/3
for x y z
        by auto
      show ?thesis
      proof (rule **)
        have cb_ab: (∑j ∈ Basis. if j = i then c *R i else (a · j) *R j) ∈
cbox a b
          using ‹i ∈ Basis› True ‹⋀i. i ∈ Basis ⟹ a · i < b · i›
          by (force simp add: mem_box sum_if_inner [where f = λj. c])
        show (∑(x,K) ∈ A. norm (integral K h)) < ε/12
          using ‹i ∈ Basis› ‹A ⊆ T''› overlap
          by (force simp add: sum_if_inner [where f = λj. c]
              intro!: γ0 [OF cb_ab ‹i ∈ Basis› A_tagged fineA(1) ‹h ∈ F›])
        let ?F = λ(x,K). (x, K ∩ {x. x · i ≤ c})
        have 1: ?F ` A tagged_partial_division_of cbox a b
          unfolding tagged_partial_division_of_def
        proof (intro conjI strip)
          show ⋀x K. (x, K) ∈ ?F ` A ⟹ ∃a b. K = cbox a b
```

using *A_tagged interval_split*(*1*) [*OF* ‹*i* ∈ *Basis*›, *of* _ _ *c*]
by (*force dest*: *tagged_partial_division_ofD*(*4*))
show ⋀*x K*. (*x*, *K*) ∈ *?F* ‘ *A* ⟹ *x* ∈ *K*
using *A_def A_tagged* **by** (*fastforce dest*: *tagged_partial_division_ofD*)
qed (*use A_tagged* **in** ‹*fastforce dest*: *tagged_partial_division_ofD*›)+
have *2*: *γ0 fine* (λ(*x*,*K*). (*x*,*K* ∩ {*x*. *x* · *i* ≤ *c*})) ‘ *A*
using *fineA*(*1*) *fine_def* **by** *fastforce*
show (∑(*x*,*K*) ∈ (λ(*x*,*K*). (*x*,*K* ∩ {*x*. *x* · *i* ≤ *c*})) ‘ *A*. *norm* (*integral K h*)) < ε/*12*
using ‹*i* ∈ *Basis*› ‹*A* ⊆ *T''*› *overlap*
by (*force simp add*: *sum_if_inner* [**where** *f* = λ*j*. *c*]
intro!: *γ0* [*OF cb_ab* ‹*i* ∈ *Basis*› *1 2* ‹*h* ∈ *F*›])
have ∗: ⟦*x* < ε/*3*; *y* < ε/*12*; *z* < ε/*12*⟧ ⟹ *x* + *y* + *z* ≤ ε/*2* **for** *x y z*
by *auto*
show (∑(*x*,*K*) ∈ *B*. *norm* (*?CI K h x*)) +
(∑(*x*,*K*) ∈ *B*. *norm* (*integral K h*)) +
(∑(*x*,*K*) ∈ (λ(*x*,*K*). (*x*,*K* ∩ {*x*. *c* ≤ *x* · *i*})) ‘ *B*. *norm* (*integral K h*))
≤ ε/*2*
proof (*rule* ∗)
show (∑(*x*,*K*) ∈ *B*. *norm* (*?CI K h x*)) < ε/*3*
by (*intro h_less3 B_tagged fineB that*)
show (∑(*x*,*K*) ∈ *B*. *norm* (*integral K h*)) < ε/*12*
using ‹*i* ∈ *Basis*› ‹*B* ⊆ *T''*› *overlap*
by (*force simp add*: *sum_if_inner* [**where** *f* = λ*j*. *c*]
intro!: *γ0* [*OF cb_ab* ‹*i* ∈ *Basis*› *B_tagged fineB*(*1*) ‹*h* ∈ *F*›])
let *?F* = λ(*x*,*K*). (*x*, *K* ∩ {*x*. *c* ≤ *x* · *i*})
have *1*: *?F* ‘ *B tagged_partial_division_of cbox a b*
unfolding *tagged_partial_division_of_def*
proof (*intro conjI strip*)
show ⋀*x K*. (*x*, *K*) ∈ *?F* ‘ *B* ⟹ ∃ *a b*. *K* = *cbox a b*
using *B_tagged interval_split*(*2*) [*OF* ‹*i* ∈ *Basis*›, *of* _ _ *c*]
by (*force dest*: *tagged_partial_division_ofD*(*4*))
show ⋀*x K*. (*x*, *K*) ∈ *?F* ‘ *B* ⟹ *x* ∈ *K*
using *B_def B_tagged* **by** (*fastforce dest*: *tagged_partial_division_ofD*)
qed (*use B_tagged* **in** ‹*fastforce dest*: *tagged_partial_division_ofD*›)+
have *2*: *γ0 fine* (λ(*x*,*K*). (*x*,*K* ∩ {*x*. *c* ≤ *x* · *i*})) ‘ *B*
using *fineB*(*1*) *fine_def* **by** *fastforce*
show (∑(*x*,*K*) ∈ (λ(*x*,*K*). (*x*,*K* ∩ {*x*. *c* ≤ *x* · *i*})) ‘ *B*. *norm* (*integral K h*)) < ε/*12*
using ‹*i* ∈ *Basis*› ‹*A* ⊆ *T''*› *overlap*
by (*force simp add*: *B_def sum_if_inner* [**where** *f* = λ*j*. *c*]
intro!: *γ0* [*OF cb_ab* ‹*i* ∈ *Basis*› *1 2* ‹*h* ∈ *F*›])
qed
qed
qed
finally show *?thesis* **.**
qed

ultimately show *?thesis* **by** *metis*
qed
ultimately show *?thesis*
by (*simp add*: *sum_subtractf* [*symmetric*] *int_KK′* ∗)
qed
ultimately show *?thesis* **by** *metis*
next
case *False*
then consider $c < a \cdot i \mid b \cdot i < c$
by *auto*
then show *?thesis*
proof *cases*
case *1*
then have *f0*: *f x* = *0* **if** $x \in$ *cbox a b* **for** *x*
using *that f* ‹*i* ∈ *Basis*› *mem_box*(*2*) **by** *force*
then have *int_f0*: *integral* (*cbox a b*) *f* = *0*
by (*simp add*: *integral_cong*)
have *f0_tag*: *f x* = *0* **if** (*x*,*K*) ∈ *T* **for** *x K*
using *T f0 that* **by** (*meson tag_in_interval*)
then have (∑(*x*,*K*) ∈ *T*. *content K* $*_R$ *f x*) = *0*
by (*metis* (*mono_tags*, *lifting*) *real_vector.scale_eq_0_iff split_conv sum.neutral surj_pair*)
then show *?thesis*
using ‹*0* < ε› **by** (*simp add*: *int_f0*)
next
case *2*
then have *fh*: *f x* = *h x* **if** *x* ∈ *cbox a b* **for** *x*
using *that f* ‹*i* ∈ *Basis*› *mem_box*(*2*) **by** *force*
then have *int_f*: *integral* (*cbox a b*) *f* = *integral* (*cbox a b*) *h*
using *integral_cong* **by** *blast*
have *fh_tag*: *f x* = *h x* **if** (*x*,*K*) ∈ *T* **for** *x K*
using *T fh that* **by** (*meson tag_in_interval*)
then have *fh*: (∑(*x*,*K*) ∈ *T*. *content K* $*_R$ *f x*) = (∑(*x*,*K*) ∈ *T*. *content K* $*_R$ *h x*)
by (*metis* (*mono_tags*, *lifting*) *split_cong sum.cong*)
show *?thesis*
unfolding *fh int_f*
proof (*rule less_trans* [*OF* γ*1*])
show γ*1 fine T*
by (*meson fine fine_Int*)
show ε / (*7* ∗ *Suc DIM*(*′b*)) < ε
using ‹*0* < ε› **by** (*force simp*: *divide_simps*)+
qed (*use that* **in** *auto*)
qed
qed
have *gauge* (λ*x*. γ*0 x* ∩ γ*1 x*)
by (*simp add*: ‹*gauge* γ*0*› ‹*gauge* γ*1*› *gauge_Int*)
then show *?thesis*
by (*auto intro*: ∗)

qed
qed
qed

corollary *equiintegrable_halfspace_restrictions_ge*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *F*: *F equiintegrable_on cbox a b* **and** *f*: *f* ∈ *F*
and *norm_f*: ⋀*h x*. ⟦*h* ∈ *F*; *x* ∈ *cbox a b*⟧ ⟹ *norm*(*h x*) ≤ *norm*(*f x*)
shows (⋃ *i* ∈ *Basis*. ⋃ *c*. ⋃ *h* ∈ *F*. {(λ*x*. *if x* · *i* ≥ *c then h x else 0*)})
equiintegrable_on cbox a b
proof −
have ∗: (⋃*i*∈*Basis*. ⋃ *c*. ⋃*h*∈(λ*f*. *f* ∘ *uminus*) ‘ *F*. {λ*x*. *if x* · *i* ≤ *c then h x else 0*})
equiintegrable_on cbox (− *b*) (− *a*)
proof (*rule equiintegrable_halfspace_restrictions_le*)
show (λ*f*. *f* ∘ *uminus*) ‘ *F equiintegrable_on cbox* (− *b*) (− *a*)
using *F equiintegrable_reflect* **by** *blast*
show *f* ∘ *uminus* ∈ (λ*f*. *f* ∘ *uminus*) ‘ *F*
using *f* **by** *auto*
show ⋀*h x*. ⟦*h* ∈ (λ*f*. *f* ∘ *uminus*) ‘ *F*; *x* ∈ *cbox* (− *b*) (− *a*)⟧ ⟹ *norm* (*h x*) ≤ *norm* ((*f* ∘ *uminus*) *x*)
using *f* **unfolding** *comp_def image_iff*
by (*metis* (*no_types*, *lifting*) *equation_minus_iff imageE norm_f uminus_interval_vector*)
qed
have *eq*: (λ*f*. *f* ∘ *uminus*) ‘
(⋃*i*∈*Basis*. ⋃ *c*. ⋃*h*∈*F*. {λ*x*. *if x* · *i* ≤ *c then* (*h* ∘ *uminus*) *x else 0*}) =
(⋃*i*∈*Basis*. ⋃ *c*. ⋃*h*∈*F*. {λ*x*. *if c* ≤ *x* · *i then h x else 0*}) (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
using *minus_le_iff* **by** *fastforce*
show *?rhs* ⊆ *?lhs*
apply *clarsimp*
apply (*rule_tac x*=λ*x*. *if c* ≤ (−*x*) · *i then h*(−*x*) *else 0* **in** *image_eqI*)
using *le_minus_iff* **by** *fastforce*+
qed
show *?thesis*
using *equiintegrable_reflect* [*OF* ∗] **by** (*auto simp*: *eq*)
qed

corollary *equiintegrable_halfspace_restrictions_lt*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *F*: *F equiintegrable_on cbox a b* **and** *f*: *f* ∈ *F*
and *norm_f*: ⋀*h x*. ⟦*h* ∈ *F*; *x* ∈ *cbox a b*⟧ ⟹ *norm*(*h x*) ≤ *norm*(*f x*)
shows (⋃ *i* ∈ *Basis*. ⋃ *c*. ⋃ *h* ∈ *F*. {(λ*x*. *if x* · *i* < *c then h x else 0*)}) *equiintegrable_on cbox a b*
(**is** *?G equiintegrable_on cbox a b*)

proof –
have ∗: (⋃ *i*∈*Basis*. ⋃ *c*. ⋃ *h*∈*F*. {λ*x*. *if c* ≤ *x* · *i then h x else 0*}) *equiintegrable_on cbox a b*
using *equiintegrable_halfspace_restrictions_ge* [*OF F f*] *norm_f* **by** *auto*
have (λ*x*. *if x* · *i* < *c then h x else 0*) = (λ*x*. *h x* − (*if c* ≤ *x* · *i then h x else 0*))
if *i* ∈ *Basis h* ∈ *F* **for** *i c h*
using *that* **by** *force*
then show *?thesis*
by (*blast intro*: *equiintegrable_on_subset* [*OF equiintegrable_diff* [*OF F* ∗]])
qed

corollary *equiintegrable_halfspace_restrictions_gt*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
assumes *F*: *F equiintegrable_on cbox a b* **and** *f*: *f* ∈ *F*
and *norm_f*: ⋀*h x*. ⟦*h* ∈ *F*; *x* ∈ *cbox a b*⟧ ⟹ *norm*(*h x*) ≤ *norm*(*f x*)
shows (⋃ *i* ∈ *Basis*. ⋃ *c*. ⋃ *h* ∈ *F*. {(λ*x*. *if x* · *i* > *c then h x else 0*)}) *equiintegrable_on cbox a b*
(**is** *?G equiintegrable_on cbox a b*)
proof –
have ∗: (⋃ *i*∈*Basis*. ⋃ *c*. ⋃ *h*∈*F*. {λ*x*. *if c* ≥ *x* · *i then h x else 0*}) *equiintegrable_on cbox a b*
using *equiintegrable_halfspace_restrictions_le* [*OF F f*] *norm_f* **by** *auto*
have (λ*x*. *if x* · *i* > *c then h x else 0*) = (λ*x*. *h x* − (*if c* ≥ *x* · *i then h x else 0*))
if *i* ∈ *Basis h* ∈ *F* **for** *i c h*
using *that* **by** *force*
then show *?thesis*
by (*blast intro*: *equiintegrable_on_subset* [*OF equiintegrable_diff* [*OF F* ∗]])
qed

proposition *equiintegrable_closed_interval_restrictions*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
assumes *f*: *f integrable_on cbox a b*
shows (⋃ *c d*. {(λ*x*. *if x* ∈ *cbox c d then f x else 0*)}) *equiintegrable_on cbox a b*
proof –
let *?g* = λ*B c d x*. *if* ∀ *i*∈*B*. *c* · *i* ≤ *x* · *i* ∧ *x* · *i* ≤ *d* · *i then f x else 0*
have ∗: *insert f* (⋃ *c d*. {*?g B c d*}) *equiintegrable_on cbox a b* **if** *B* ⊆ *Basis* **for** *B*
proof –
have *finite B*
using *finite_Basis finite_subset* ‹*B* ⊆ *Basis*› **by** *blast*
then show *?thesis* **using** ‹*B* ⊆ *Basis*›
proof (*induction B*)
case *empty*
with *f* **show** *?case* **by** *auto*
next
case (*insert i B*)
then have *i* ∈ *Basis B* ⊆ *Basis*

by *auto*
have ∗: *norm* (*h x*) ≤ *norm* (*f x*)
if *h* ∈ *insert f* (⋃*c d*. {*?g B c d*}) *x* ∈ *cbox a b* **for** *h x*
using *that* **by** *auto*
define *F* **where** *F* ≡ (⋃*i*∈*Basis*.
⋃*ξ*. ⋃*h*∈*insert f* (⋃*i*∈*Basis*. ⋃*ψ*. ⋃*h*∈*insert f* (⋃*c d*. {*?g B c d*}).
{*λx*. *if x* · *i* ≤ *ψ then h x else 0*}).
{*λx*. *if ξ* ≤ *x* · *i then h x else 0*})
show *?case*
proof (*rule equiintegrable_on_subset*)
have *F equiintegrable_on cbox a b*
unfolding *F_def*
proof (*rule equiintegrable_halfspace_restrictions_ge*)
show *insert f* (⋃*i*∈*Basis*. ⋃*ξ*. ⋃*h*∈*insert f* (⋃*c d*. {*?g B c d*}).
{*λx*. *if x* · *i* ≤ *ξ then h x else 0*}) *equiintegrable_on cbox a b*
by (*intro* ∗ *f equiintegrable_on_insert equiintegrable_halfspace_restrictions_le* [*OF insert.IH insertI1*] ‹*B* ⊆ *Basis*›)
show *norm*(*h x*) ≤ *norm*(*f x*)
if *h* ∈ *insert f* (⋃*i*∈*Basis*. ⋃*ξ*. ⋃*h*∈*insert f* (⋃*c d*. {*?g B c d*}). {*λx*. *if x* · *i* ≤ *ξ then h x else 0*})
x ∈ *cbox a b* **for** *h x*
using *that* **by** *auto*
qed *auto*
then show *insert f F*
equiintegrable_on cbox a b
by (*blast intro*: *f equiintegrable_on_insert*)
show *insert f* (⋃*c d*. {*λx*. *if* ∀*j*∈*insert i B*. *c* · *j* ≤ *x* · *j* ∧ *x* · *j* ≤ *d* · *j then f x else 0*})
⊆ *insert f F*
using ‹*i* ∈ *Basis*›
apply *clarify*
apply (*simp add*: *F_def*)
apply (*drule_tac x*=*i* **in** *bspec*, *assumption*)
apply (*drule_tac x*=*c* · *i* **in** *spec*, *clarify*)
apply (*drule_tac x*=*i* **in** *bspec*, *assumption*)
apply (*drule_tac x*=*d* · *i* **in** *spec*)
apply (*clarsimp simp*: *fun_eq_iff*)
apply (*drule_tac x*=*c* **in** *spec*)
apply (*drule_tac x*=*d* **in** *spec*)
apply (*simp split*: *if_split_asm*)
done
qed
qed
qed
show *?thesis*
by (*rule equiintegrable_on_subset* [*OF* ∗ [*OF subset_refl*]]) (*auto simp*: *mem_box*)
qed

6.27.3 Continuity of the indefinite integral

proposition *indefinite_integral_continuous*:
fixes f :: *'a* :: *euclidean_space* ⇒ *'b* :: *euclidean_space*
assumes *int_f*: *f integrable_on cbox a b*
and *c*: $c \in$ *cbox a b* **and** *d*: $d \in$ *cbox a b* $0 < \varepsilon$
obtains δ **where** $0 < \delta$
$\bigwedge c'\ d'.\ [\![c' \in$ *cbox a b*; $d' \in$ *cbox a b*; $norm(c' - c) \le \delta$; $norm(d' - d) \le \delta]\!]$
$\Longrightarrow norm(integral(cbox\ c'\ d')\ f - integral(cbox\ c\ d)\ f) < \varepsilon$
proof –
{ **assume** $\exists\, c'\ d'.\ c' \in cbox\ a\ b \wedge d' \in cbox\ a\ b \wedge norm(c' - c) \le \delta \wedge norm(d' - d) \le \delta \wedge$
$norm(integral(cbox\ c'\ d')\ f - integral(cbox\ c\ d)\ f) \ge \varepsilon$
(**is** $\exists\, c'\ d'.\ ?\Phi\ c'\ d'\ \delta$) **if** $0 < \delta$ **for** δ
then have $\exists\, c'\ d'.\ ?\Phi\ c'\ d'\ (1\ /\ Suc\ n)$ **for** n
by *simp*
then obtain $u\ v$ **where** $\bigwedge n.\ ?\Phi\ (u\ n)\ (v\ n)\ (1\ /\ Suc\ n)$
by *metis*
then have *u*: $u\ n \in cbox\ a\ b$ **and** *norm_u*: $norm(u\ n - c) \le 1\ /\ Suc\ n$
and *v*: $v\ n \in cbox\ a\ b$ **and** *norm_v*: $norm(v\ n - d) \le 1\ /\ Suc\ n$
and ε: $\varepsilon \le norm\ (integral\ (cbox\ (u\ n)\ (v\ n))\ f - integral\ (cbox\ c\ d)\ f)$ **for** n
by *blast+*
then have *False*
proof –
have *uvn*: $cbox\ (u\ n)\ (v\ n) \subseteq cbox\ a\ b$ **for** n
by (*meson u v mem_box*(*2*) *subset_box*(*1*))
define S **where** $S \equiv \bigcup i \in Basis.\ \{x.\ x \cdot i = c \cdot i\} \cup \{x.\ x \cdot i = d \cdot i\}$
have *negligible S*
unfolding *S_def* **by** *force*
then have *int_f'*: $(\lambda x.\ if\ x \in S\ then\ 0\ else\ f\ x)$ *integrable_on cbox a b*
by (*force intro*: *integrable_spike assms*)
have *get_n*: $\exists\, n.\ \forall\, m{\ge}n.\ x \in cbox\ (u\ m)\ (v\ m) \longleftrightarrow x \in cbox\ c\ d$ **if** x: $x \notin S$ **for** x
proof –
define ε **where** $\varepsilon \equiv Min\ ((\lambda i.\ min\ |x \cdot i - c \cdot i|\ |x \cdot i - d \cdot i|)$ ` *Basis*)
have $\varepsilon > 0$
using ‹$x \notin S$› **by** (*auto simp*: *S_def* ε*_def*)
then obtain n **where** $n \ne 0$ **and** n: $1\ /\ (real\ n) < \varepsilon$
by (*metis inverse_eq_divide real_arch_inverse*)
have *emin*: $\varepsilon \le min\ |x \cdot i - c \cdot i|\ |x \cdot i - d \cdot i|$ **if** $i \in Basis$ **for** i
unfolding ε*_def*
by (*meson Min.coboundedI euclidean_space_class.finite_Basis finite_imageI image_iff that*)
have $1\ /\ real\ (Suc\ n) < \varepsilon$
using n ‹$n \ne 0$› ‹$\varepsilon > 0$› **by** (*simp add*: *field_simps*)
have $x \in cbox\ (u\ m)\ (v\ m) \longleftrightarrow x \in cbox\ c\ d$ **if** $m \ge n$ **for** m
proof –
have $*$: $[\![|u - c| \le n;\ |v - d| \le n;\ N < |x - c|;\ N < |x - d|;\ n \le N]\!]$

$\Longrightarrow u \leq x \wedge x \leq v \longleftrightarrow c \leq x \wedge x \leq d$ **for** *N n u v c d* **and** *x::real*
by *linarith*
have $(u\ m \cdot i \leq x \cdot i \wedge x \cdot i \leq v\ m \cdot i) = (c \cdot i \leq x \cdot i \wedge x \cdot i \leq d \cdot i)$
if $i \in Basis$ **for** *i*
proof (*rule* ∗)
show $|u\ m \cdot i - c \cdot i| \leq 1\ /\ Suc\ m$
using *norm_u* [*of m*]
by (*metis* (*full_types*) *order_trans Basis_le_norm inner_commute inner_diff_right that*)
show $|v\ m \cdot i - d \cdot i| \leq 1\ /\ real\ (Suc\ m)$
using *norm_v* [*of m*]
by (*metis* (*full_types*) *order_trans Basis_le_norm inner_commute inner_diff_right that*)
show $1/n < |x \cdot i - c \cdot i|\ 1/n < |x \cdot i - d \cdot i|$
using *n* ‹$n \neq 0$› *emin* [*OF* ‹$i \in Basis$›]
by (*simp_all add*: *inverse_eq_divide*)
show $1\ /\ real\ (Suc\ m) \leq 1\ /\ real\ n$
using ‹$n \neq 0$› ‹$m \geq n$› **by** (*simp add*: *field_split_simps*)
qed
then show *?thesis* **by** (*simp add*: *mem_box*)
qed
then show *?thesis* **by** *blast*
qed
have *1*: *range* ($\lambda n\ x.$ *if* $x \in$ *cbox* (*u n*) (*v n*) *then if* $x \in S$ *then 0 else f x else 0*) *equiintegrable_on cbox a b*
by (*blast intro*: *equiintegrable_on_subset* [*OF equiintegrable_closed_interval_restrictions* [*OF int_f'*]])
have *2*: ($\lambda n.$ *if* $x \in$ *cbox* (*u n*) (*v n*) *then if* $x \in S$ *then 0 else f x else 0*)
$\longrightarrow$ (*if* $x \in$ *cbox c d then if* $x \in S$ *then 0 else f x else 0*) **for** *x*
by (*fastforce simp*: *dest*: *get_n intro*: *tendsto_eventually eventually_sequentiallyI*)
have [*simp*]: *cbox c d* ∩ *cbox a b* = *cbox c d*
using *c d* **by** (*force simp*: *mem_box*)
have [*simp*]: *cbox* (*u n*) (*v n*) ∩ *cbox a b* = *cbox* (*u n*) (*v n*) **for** *n*
using *u v* **by** (*fastforce simp*: *mem_box intro*: *order.trans*)
have $\bigwedge y\ A.\ y \in A - S \Longrightarrow f\ y = (\lambda x.$ *if* $x \in S$ *then 0 else f x*) *y*
by *simp*
then have $\bigwedge A.$ *integral A* ($\lambda x.$ *if* $x \in S$ *then 0 else f* (*x*)) = *integral A* ($\lambda x.$ *f* (*x*))
by (*blast intro*: *integral_spike* [*OF* ‹*negligible S*›])
moreover
obtain *N* **where** *dist* (*integral* (*cbox* (*u N*) (*v N*)) ($\lambda x.$ *if* $x \in S$ *then 0 else f x*))
(*integral* (*cbox c d*) ($\lambda x.$ *if* $x \in S$ *then 0 else f x*)) < ε
using *equiintegrable_limit* [*OF 1 2*] ‹$0 < ε$› **by** (*force simp*: *integral_restrict_Int lim_sequentially*)
ultimately have *dist* (*integral* (*cbox* (*u N*) (*v N*)) *f*) (*integral* (*cbox c d*) *f*) < ε
by *simp*
then show *False*

```
      by (metis dist_norm not_le ε)
    qed
  }
  then show ?thesis
    by (meson not_le that)
qed

corollary indefinite_integral_uniformly_continuous:
  fixes f :: 'a :: euclidean_space ⇒ 'b :: euclidean_space
  assumes f integrable_on cbox a b
  shows uniformly_continuous_on (cbox (Pair a a) (Pair b b)) (λy. integral (cbox
(fst y) (snd y)) f)
proof –
  show ?thesis
  proof (rule compact_uniformly_continuous, clarsimp simp add: continuous_on_iff)
    fix c d and ε::real
    assume c: c ∈ cbox a b and d: d ∈ cbox a b and 0 < ε
    obtain δ where 0 < δ and δ:
           ⋀c' d'. ⟦c' ∈ cbox a b; d' ∈ cbox a b; norm(c' − c) ≤ δ; norm(d' −
d) ≤ δ⟧
                          ⟹ norm(integral(cbox c' d') f −
                                  integral(cbox c d) f) < ε
      using indefinite_integral_continuous ‹0 < ε› assms c d by blast
    show ∃δ > 0. ∀x' ∈ cbox (a, a) (b, b).
                  dist x' (c, d) < δ ⟶
                  dist (integral (cbox (fst x') (snd x')) f)
                       (integral (cbox c d) f)
                  < ε
      using ‹0 < δ›
       by (force simp: dist_norm intro: δ order_trans [OF norm_fst_le] order_trans
[OF norm_snd_le] less_imp_le)
  qed auto
qed


corollary bounded_integrals_over_subintervals:
  fixes f :: 'a :: euclidean_space ⇒ 'b :: euclidean_space
  assumes f integrable_on cbox a b
  shows bounded {integral (cbox c d) f |c d. cbox c d ⊆ cbox a b}
proof –
  have bounded ((λy. integral (cbox (fst y) (snd y)) f) ` cbox (a, a) (b, b))
       (is bounded ?I)
   by (blast intro: bounded_cbox bounded_uniformly_continuous_image indefinite_integral_uniformly_continuous
[OF assms])
  then obtain B where B > 0 and B: ⋀x. x ∈ ?I ⟹ norm x ≤ B
    by (auto simp: bounded_pos)
  have norm x ≤ B if x = integral (cbox c d) f cbox c d ⊆ cbox a b for x c d
  proof (cases cbox c d = {})
    case True
```

with ‹*0* < *B*› *that* **show** *?thesis* **by** *auto*
next
case *False*
then have ∃ *x* ∈ *cbox* (*a*,*a*) (*b*,*b*). *integral* (*cbox c d*) *f* = *integral* (*cbox* (*fst x*) (*snd x*)) *f*
using *that* **by** (*metis cbox_Pair_iff interval_subset_is_interval is_interval_cbox prod.sel*)
then show *?thesis*
using *B that*(*1*) **by** *blast*
qed
then show *?thesis*
by (*blast intro*: *boundedI*)
qed

An existence theorem for "improper" integrals. Hake's theorem implies that if the integrals over subintervals have a limit, the integral exists. We only need to assume that the integrals are bounded, and we get absolute integrability, but we also need a (rather weak) bound assumption on the function.

theorem *absolutely_integrable_improper*:
fixes *f* :: *'M*::*euclidean_space* ⇒ *'N*::*euclidean_space*
assumes *int_f*: ⋀*c d*. *cbox c d* ⊆ *box a b* ⟹ *f integrable_on cbox c d*
and *bo*: *bounded* {*integral* (*cbox c d*) *f* |*c d*. *cbox c d* ⊆ *box a b*}
and *absi*: ⋀*i*. *i* ∈ *Basis*
⟹ ∃ *g*. *g absolutely_integrable_on cbox a b* ∧
((∀ *x* ∈ *cbox a b*. *f x* • *i* ≤ *g x*) ∨ (∀ *x* ∈ *cbox a b*. *f x* • *i* ≥ *g x*))
shows *f absolutely_integrable_on cbox a b*
proof (*cases content*(*cbox a b*) = *0*)
case *True*
then show *?thesis*
by *auto*
next
case *False*
then have *pos*: *content*(*cbox a b*) > *0*
using *zero_less_measure_iff* **by** *blast*
show *?thesis*
unfolding *absolutely_integrable_componentwise_iff* [**where** *f* = *f*]
proof
fix *j*::*'N*
assume *j* ∈ *Basis*
then obtain *g* **where** *absint_g*: *g absolutely_integrable_on cbox a b*
and *g*: (∀ *x* ∈ *cbox a b*. *f x* • *j* ≤ *g x*) ∨ (∀ *x* ∈ *cbox a b*. *f x* • *j* ≥ *g x*)
using *absi* **by** *blast*
have *int_gab*: *g integrable_on cbox a b*
using *absint_g set_lebesgue_integral_eq_integral*(*1*) **by** *blast*
define α **where** α ≡ λ*k*. *a* + (*b* − *a*) $/_R$ *real k*
define β **where** β ≡ λ*k*. *b* − (*b* − *a*) $/_R$ *real k*
define *I* **where** *I* ≡ λ*k*. *cbox* (α *k*) (β *k*)

have *ISuc_box*: $I\ (Suc\ n) \subseteq box\ a\ b$ **for** n
  using *pos* **unfolding** *I_def*
   by (*intro subset_box_imp*) (*auto simp*: *α_def β_def content_pos_lt_eq algebra_simps*)
have *ISucSuc*: $I\ (Suc\ n) \subseteq I\ (Suc\ (Suc\ n))$ **for** n
proof −
  have $\bigwedge i.\ i \in Basis$
     $\Longrightarrow a \cdot i\ /\ Suc\ n + b \cdot i\ /\ (real\ n + 2)\ \le b \cdot i\ /\ Suc\ n + a \cdot i\ /\ (real\ n + 2)$
    using *pos*
    by (*simp add*: *content_pos_lt_eq divide_simps*) (*auto simp*: *algebra_simps*)
  then show *?thesis*
    unfolding *I_def*
    by (*intro subset_box_imp*) (*auto simp*: *algebra_simps inverse_eq_divide α_def β_def*)
qed
have *getN*: $\exists N{::}nat.\ \forall k.\ k \ge N \longrightarrow x \in I\ k$
  if x: $x \in box\ a\ b$ **for** x
proof −
  define Δ **where** $\Delta \equiv (\bigcup i \in Basis.\ \{((x - a) \cdot i)\ /\ ((b - a) \cdot i),\ (b - x) \cdot i\ /\ ((b - a) \cdot i)\})$
  obtain N **where** N: $real\ N > 1\ /\ Inf\ \Delta$
    using *reals_Archimedean2* **by** *blast*
  moreover have Δ: $Inf\ \Delta > 0$
     using *that* **by** (*auto simp*: *Δ_def finite_less_Inf_iff mem_box algebra_simps divide_simps*)
  ultimately have $N > 0$
    using *of_nat_0_less_iff* **by** *fastforce*
  show *?thesis*
  proof (*intro exI impI allI*)
    fix k **assume** $N \le k$
    with ‹$0 < N$› **have** $k > 0$
      by *linarith*
    have *xa_gt*: $(x - a) \cdot i > ((b - a) \cdot i)\ /\ (real\ k)$ **if** $i \in Basis$ **for** i
    proof −
      have $*$: $Inf\ \Delta \le ((x - a) \cdot i)\ /\ ((b - a) \cdot i)$
        unfolding *Δ_def* **using** *that* **by** (*force intro*: *cInf_le_finite*)
      have $1\ /\ Inf\ \Delta \ge ((b - a) \cdot i)\ /\ ((x - a) \cdot i)$
        using *le_imp_inverse_le* [*OF* $*$ Δ]
        by (*simp add*: *field_simps*)
      with N **have** $k > ((b - a) \cdot i)\ /\ ((x - a) \cdot i)$
        using ‹$N \le k$› **by** *linarith*
      with x *that* **show** *?thesis*
        by (*auto simp*: *mem_box algebra_simps field_split_simps*)
    qed
    have *bx_gt*: $(b - x) \cdot i > ((b - a) \cdot i)\ /\ k$ **if** $i \in Basis$ **for** i
    proof −
      have $*$: $Inf\ \Delta \le ((b - x) \cdot i)\ /\ ((b - a) \cdot i)$
        using *that* **unfolding** *Δ_def* **by** (*force intro*: *cInf_le_finite*)

```
          have 1 / Inf Δ ≥ ((b − a) · i) / ((b − x) · i)
            using le_imp_inverse_le [OF ∗ Δ]
            by (simp add: field_simps)
          with N have k > ((b − a) · i) / ((b − x) · i)
            using ‹N ≤ k› by linarith
          with x that show ?thesis
            by (auto simp: mem_box algebra_simps field_split_simps)
        qed
        show x ∈ I k
          using that Δ ‹k > 0› unfolding I_def
           by (auto simp: α_def β_def mem_box algebra_simps divide_inverse dest:
xa_gt bx_gt)
      qed
    qed
    obtain Bf where  Bf: ⋀c d. cbox c d ⊆ box a b ⟹ norm (integral (cbox c
d) f) ≤ Bf
      using bo unfolding bounded_iff by blast
    obtain Bg where Bg:⋀c d. cbox c d ⊆ cbox a b ⟹ |integral (cbox c d) g| ≤
Bg
      using bounded_integrals_over_subintervals [OF int_gab] unfolding bounded_iff
real_norm_def by blast
    show (λx. f x · j) absolutely_integrable_on cbox a b
      using g
    proof     — A lot of duplication in the two proofs
      assume fg [rule_format]: ∀x∈cbox a b. f x · j ≤ g x
      have (λx. (f x · j)) = (λx. g x − (g x − (f x · j)))
        by simp
      moreover have (λx. g x − (g x − (f x · j))) integrable_on cbox a b
      proof (rule Henstock_Kurzweil_Integration.integrable_diff [OF int_gab])
        define φ where φ ≡ λk x. if x ∈ I (Suc k) then g x − f x · j else 0
        have (λx. g x − f x · j) integrable_on box a b
        proof (rule monotone_convergence_increasing [of φ, THEN conjunct1])
          have ∗: I (Suc k) ∩ box a b = I (Suc k) for k
            using box_subset_cbox ISuc_box by fastforce
          show φ k integrable_on box a b for k
          proof −
            have I (Suc k) ⊆ cbox a b
              using ∗ box_subset_cbox by blast
            moreover have (λm. f m · j) integrable_on I (Suc k)
              by (metis ISuc_box I_def int_f integrable_component)
            ultimately have (λm. g m − f m · j) integrable_on I (Suc k)
                by (metis Henstock_Kurzweil_Integration.integrable_diff I_def int_gab
integrable_on_subcbox)
            then show ?thesis
              by (simp add: ∗ φ_def integrable_restrict_Int)
          qed
          show φ k x ≤ φ (Suc k) x if x ∈ box a b for k x
            using ISucSuc box_subset_cbox that by (force simp: φ_def intro!: fg)
          show (λk. φ k x) ⟶ g x − f x · j if x: x ∈ box a b for x
```

```
        proof (rule tendsto_eventually)
          obtain N::nat where N: ⋀k. k ≥ N ⟹ x ∈ I k
            using getN [OF x] by blast
          show ∀F k in sequentially. φ k x = g x − f x · j
          proof
            fix k::nat assume N ≤ k
            have x ∈ I (Suc k)
              by (metis ‹N ≤ k› le_Suc_eq N)
            then show φ k x = g x − f x · j
              by (simp add: φ_def)
          qed
        qed
       have |integral (box a b) (λx. if x ∈ I (Suc k) then g x − f x · j else 0)| ≤
Bg + Bf for k
        proof −
          have ABK_def [simp]: I (Suc k) ∩ box a b = I (Suc k)
            using ISuc_box by (simp add: Int_absorb2)
          have int_fI: f integrable_on I (Suc k)
            using ISuc_box I_def int_f by auto
          moreover
          have |integral (I (Suc k)) (λx. f x · j)| ≤ norm (integral (I (Suc k)) f)
            by (simp add: Basis_le_norm int_fI ‹j ∈ Basis›)
          with ISuc_box ABK_def have |integral (I (Suc k)) (λx. f x · j)| ≤ Bf
        by (metis Bf I_def ‹j ∈ Basis› int_fI integral_component_eq norm_bound_Basis_le)

          ultimately
          have |integral (I (Suc k)) g − integral (I (Suc k)) (λx. f x · j)|  ≤ Bg
+ Bf
            using ∗ box_subset_cbox unfolding I_def
            by (blast intro: Bg add_mono order_trans [OF abs_triangle_ineq4])
          moreover have g integrable_on I (Suc k)
                by (metis ISuc_box I_def int_gab integrable_on_open_interval inte-
grable_on_subcbox)
          moreover have (λx. f x · j) integrable_on I (Suc k)
            using int_fI by (simp add: integrable_component)
         ultimately show ?thesis
            by (simp add: integral_restrict_Int integral_diff)
        qed
        then show bounded (range (λk. integral (box a b) (φ k)))
          by (auto simp add: bounded_iff φ_def)
      qed
      then show (λx. g x − f x · j) integrable_on cbox a b
        by (simp add: integrable_on_open_interval)
    qed
    ultimately have (λx. f x · j) integrable_on cbox a b
      by auto
    then show ?thesis
      using absolutely_integrable_component_ubound [OF _ absint_g] fg by force
  next
```

assume *gf* [*rule_format*]: ∀ *x*∈*cbox a b*. *g x* ≤ *f x* · *j*
have (λ*x*. (*f x* · *j*)) = (λ*x*. ((*f x* · *j*) − *g x*) + *g x*)
by *simp*
moreover have (λ*x*. (*f x* · *j* − *g x*) + *g x*) *integrable_on cbox a b*
proof (*rule Henstock_Kurzweil_Integration.integrable_add* [*OF _ int_gab*])
let *?φ* = λ*k x*. *if x* ∈ *I*(*Suc k*) *then f x* · *j* − *g x else 0*
have (λ*x*. *f x* · *j* − *g x*) *integrable_on box a b*
proof (*rule monotone_convergence_increasing* [*of ?φ*, *THEN conjunct1*])
have ∗: *I* (*Suc k*) ∩ *box a b* = *I* (*Suc k*) **for** *k*
using *box_subset_cbox ISuc_box* **by** *fastforce*
show *?φ k integrable_on box a b* **for** *k*
proof (*simp add*: *integrable_restrict_Int integral_restrict_Int* ∗)
show (λ*x*. *f x* · *j* − *g x*) *integrable_on I* (*Suc k*)
by (*metis ISuc_box Henstock_Kurzweil_Integration.integrable_diff I_def int_f int_gab integrable_component integrable_on_open_interval integrable_on_subcbox*)
qed
show *?φ k x* ≤ *?φ* (*Suc k*) *x* **if** *x* ∈ *box a b* **for** *k x*
using *ISucSuc box_subset_cbox that* **by** (*force simp*: *I_def intro*!: *gf*)
show (λ*k*. *?φ k x*) ⟶ *f x* · *j* − *g x* **if** *x*: *x* ∈ *box a b* **for** *x*
proof (*rule tendsto_eventually*)
obtain *N*::*nat* **where** *N*: ⋀*k*. *k* ≥ *N* ⟹ *x* ∈ *I k*
using *getN* [*OF x*] **by** *blast*
then show $\forall_F$ *k in sequentially*. *?φ k x* = *f x* · *j* − *g x*
by (*metis* (*no_types*, *lifting*) *eventually_at_top_linorderI le_Suc_eq*)
qed
have |*integral* (*box a b*)
(λ*x*. *if x* ∈ *I* (*Suc k*) *then f x* · *j* − *g x else 0*)| ≤ *Bf* + *Bg* **for** *k*
proof −
define *ABK* **where** *ABK* ≡ *cbox* (*a* + (*b* − *a*) $/_R$ (*1* + *real k*)) (*b* − (*b* − *a*) $/_R$ (*1* + *real k*))
have *ABK_eq* [*simp*]: *ABK* ∩ *box a b* = *ABK*
using ∗ *I_def* *α_def* *β_def* *ABK_def* **by** *auto*
have *int_fI*: *f integrable_on ABK*
unfolding *ABK_def*
using *ISuc_box I_def* *α_def* *β_def* *int_f* **by** *force*
then have (λ*x*. *f x* · *j*) *integrable_on ABK*
by (*simp add*: *integrable_component*)
moreover have *g integrable_on ABK*
by (*metis ABK_def ABK_eq IntE box_subset_cbox int_gab integrable_on_subcbox subset_eq*)
moreover
have |*integral ABK* (λ*x*. *f x* · *j*)| ≤ *norm* (*integral ABK f*)
by (*simp add*: *Basis_le_norm int_fI* ‹*j* ∈ *Basis*›)
then have |*integral ABK* (λ*x*. *f x* · *j*)| ≤ *Bf*
by (*metis ABK_eq ABK_def Bf IntE dual_order.trans subset_eq*)
ultimately show *?thesis*
using ∗ *box_subset_cbox*
apply (*simp add*: *integral_restrict_Int integral_diff ABK_def I_def* *α_def* *β_def*)

```
          by (blast intro: Bg add_mono order_trans [OF abs_triangle_ineq4])
        qed
        then show bounded (range (λk. integral (box a b) (?φ k)))
          by (auto simp add: bounded_iff)
      qed
      then show (λx. f x · j − g x) integrable_on cbox a b
        by (simp add: integrable_on_open_interval)
    qed
    ultimately have (λx. f x · j) integrable_on cbox a b
      by auto
    then show ?thesis
      using absint_g absolutely_integrable_absolutely_integrable_lbound gf by blast
  qed
 qed
qed
```

6.27.4 Second mean value theorem and corollaries

```
lemma level_approx:
  fixes f :: real ⇒ real and n::nat
  assumes f: ⋀x. x ∈ S ⟹ 0 ≤ f x ∧ f x ≤ 1 and x ∈ S n ≠ 0
  shows |f x − (∑k = Suc 0..n. if k / n ≤ f x then inverse n else 0)| < inverse n
      (is ?lhs < _)
proof −
  have n ∗ f x ≥ 0
    using assms by auto
  then obtain m::nat where m: floor(n ∗ f x) = int m
    using nonneg_int_cases zero_le_floor by blast
  then have kn: real k / real n ≤ f x ⟷ k ≤ m for k
    using ‹n ≠ 0› by (simp add: field_split_simps) linarith
  then have Suc n / real n ≤ f x ⟷ Suc n ≤ m
    by blast
  have real n ∗ f x ≤ real n
    by (simp add: ‹x ∈ S› f mult_left_le)
  then have m ≤ n
    using m by linarith
  have ?lhs = |f x − (∑k ∈ {Suc 0..n} ∩ {..m}. inverse n)|
    by (subst sum.inter_restrict) (auto simp: kn)
  also have ... < inverse n
    using ‹m ≤ n› ‹n ≠ 0› m
    by (simp add: min_absorb2 field_split_simps) linarith
  finally show ?thesis .
qed


lemma SMVT_lemma2:
  fixes f :: real ⇒ real
  assumes f: f integrable_on {a..b}
    and g: ⋀x y. x ≤ y ⟹ g x ≤ g y
```

shows $(\bigcup y{::}real.\ \{\lambda x.\ if\ g\ x \geq y\ then\ f\ x\ else\ 0\})$ *equiintegrable_on* $\{a..b\}$
proof −
have *ffab*: $\{f\}$ *equiintegrable_on* $\{a..b\}$
by (*metis equiintegrable_on_sing f interval_cbox*)
then have *ff*: $\{f\}$ *equiintegrable_on* (*cbox a b*)
by *simp*
have *ge*: $(\bigcup c.\ \{\lambda x.\ if\ x \geq c\ then\ f\ x\ else\ 0\})$ *equiintegrable_on* $\{a..b\}$
using *equiintegrable_halfspace_restrictions_ge* [*OF ff*] **by** *auto*
have *gt*: $(\bigcup c.\ \{\lambda x.\ if\ x > c\ then\ f\ x\ else\ 0\})$ *equiintegrable_on* $\{a..b\}$
using *equiintegrable_halfspace_restrictions_gt* [*OF ff*] **by** *auto*
have *0*: $\{(\lambda x.\ 0)\}$ *equiintegrable_on* $\{a..b\}$
by (*metis box_real*(*2*) *equiintegrable_on_sing integrable_0*)
have †: $(\lambda x.\ if\ g\ x \geq y\ then\ f\ x\ else\ 0) \in \{(\lambda x.\ 0), f\} \cup (\bigcup z.\ \{\lambda x.\ if\ z < x\ then\ f\ x\ else\ 0\}) \cup (\bigcup z.\ \{\lambda x.\ if\ z \leq x\ then\ f\ x\ else\ 0\})$
for *y*
proof (*cases* $(\forall x.\ g\ x \geq y) \vee (\forall x.\ \neg\ (g\ x \geq y))$)
let $?\mu = Inf\ \{x.\ g\ x \geq y\}$
case *False*
have *lower*: $?\mu \leq x$ **if** $g\ x \geq y$ **for** *x*
proof (*rule cInf_lower*)
show $x \in \{x.\ y \leq g\ x\}$
using *False* **by** (*auto simp*: *that*)
show *bdd_below* $\{x.\ y \leq g\ x\}$
by (*metis False bdd_belowI dual_order.trans g linear mem_Collect_eq*)
qed
have *greatest*: $?\mu \geq z$ **if** $(\bigwedge x.\ g\ x \geq y \Longrightarrow z \leq x)$ **for** *z*
by (*metis False cInf_greatest empty_iff mem_Collect_eq that*)
show *?thesis*
proof (*cases* $g\ ?\mu \geq y$)
case *True*
then obtain ζ **where** ζ: $\bigwedge x.\ g\ x \geq y \longleftrightarrow x \geq \zeta$
by (*metis g lower order.trans*) — in fact y is $Inf\ \{x.\ y \leq g\ x\}$
then show *?thesis*
by (*force simp*: ζ)
next
case *False*
have $(y \leq g\ x) \longleftrightarrow (?\mu < x)$ **for** *x*
proof
show $?\mu < x$ **if** $y \leq g\ x$
using *that False less_eq_real_def lower* **by** *blast*
show $y \leq g\ x$ **if** $?\mu < x$
by (*metis g greatest le_less_trans that less_le_trans linear not_less*)
qed
then obtain ζ **where** ζ: $\bigwedge x.\ g\ x \geq y \longleftrightarrow x > \zeta$ **..**
then show *?thesis*
by (*force simp*: ζ)
qed
qed *auto*
show *?thesis*

using † **by** (*simp add*: *UN_subset_iff equiintegrable_on_subset* [*OF equiintegrable_on_Un* [*OF gt equiintegrable_on_Un* [*OF ge equiintegrable_on_Un* [*OF ffab 0*]]]]])
qed

lemma *SMVT_lemma4*:
fixes *f* :: *real* ⇒ *real*
assumes *f*: *f integrable_on* {*a*..*b*}
and *a* ≤ *b*
and *g*: ⋀*x y*. *x* ≤ *y* ⟹ *g x* ≤ *g y*
and *01*: ⋀*x*. ⟦*a* ≤ *x*; *x* ≤ *b*⟧ ⟹ *0* ≤ *g x* ∧ *g x* ≤ *1*
obtains *c* **where** *a* ≤ *c c* ≤ *b* ((λ*x*. *g x* $*_R$ *f x*) *has_integral integral* {*c*..*b*} *f*) {*a*..*b*}
proof −
have *connected* ((λ*x*. *integral* {*x*..*b*} *f*) ‘ {*a*..*b*})
by (*simp add*: *f indefinite_integral_continuous_1′ connected_continuous_image*)
moreover have *compact* ((λ*x*. *integral* {*x*..*b*} *f*) ‘ {*a*..*b*})
by (*simp add*: *compact_continuous_image f indefinite_integral_continuous_1′*)
ultimately obtain *m M* **where** *int_fab*: (λ*x*. *integral* {*x*..*b*} *f*) ‘ {*a*..*b*} = {*m*..*M*}
using *connected_compact_interval_1* **by** *meson*
have ∃ *c*. *c* ∈ {*a*..*b*} ∧
integral {*c*..*b*} *f* =
integral {*a*..*b*} (λ*x*. (∑ *k* = *1*..*n*. *if g x* ≥ *real k* / *real n then inverse n* $*_R$ *f x else 0*)) **for** *n*
proof (*cases n*=*0*)
case *True*
then show *?thesis*
using ‹*a* ≤ *b*› **by** *auto*
next
case *False*
have (⋃ *c*::*real*. {λ*x*. *if g x* ≥ *c then f x else 0*}) *equiintegrable_on* {*a*..*b*}
using *SMVT_lemma2* [*OF f g*] **.**
then have *int*: (λ*x*. *if g x* ≥ *c then f x else 0*) *integrable_on* {*a*..*b*} **for** *c*
by (*simp add*: *equiintegrable_on_def*)
have *int′*: (λ*x*. *if g x* ≥ *c then u* ∗ *f x else 0*) *integrable_on* {*a*..*b*} **for** *c u*
proof −
have (λ*x*. *if g x* ≥ *c then u* ∗ *f x else 0*) = (λ*x*. *u* ∗ (*if g x* ≥ *c then f x else 0*))
by (*force simp*: *if_distrib*)
then show *?thesis*
using *integrable_on_cmult_left* [*OF int*] **by** *simp*
qed
have ∃ *d*. *d* ∈ {*a*..*b*} ∧ *integral* {*a*..*b*} (λ*x*. *if g x* ≥ *y then f x else 0*) = *integral* {*d*..*b*} *f* **for** *y*
proof −
let *?X* = {*x*. *g x* ≥ *y*}
have ∗: ∃ *a*. *?X* = {*a*..} ∨ *?X* = {*a*<..}

if *1*: *?X* $\neq$ {} **and** *2*: *?X* $\neq$ *UNIV*
proof −
let *?μ* = *Inf*{*x*. *g x* $\geq$ *y*}
have *lower*: *?μ* $\leq$ *x* **if** *g x* $\geq$ *y* **for** *x*
proof (*rule cInf_lower*)
show *x* $\in$ {*x*. *y* $\leq$ *g x*}
using *1 2* **by** (*auto simp*: *that*)
show *bdd_below* {*x*. *y* $\leq$ *g x*}
unfolding *bdd_below_def*
by (*metis 2 UNIV_eq_I dual_order.trans g less_eq_real_def mem_Collect_eq not_le*)
qed
have *greatest*: *?μ* $\geq$ *z* **if** $\bigwedge$*x*. *g x* $\geq$ *y* $\Longrightarrow$ *z* $\leq$ *x* **for** *z*
by (*metis cInf_greatest mem_Collect_eq that 1*)
show *?thesis*
proof (*cases g ?μ* $\geq$ *y*)
case *True*
then obtain ζ **where** ζ: $\bigwedge$*x*. *g x* $\geq$ *y* $\longleftrightarrow$ *x* $\geq$ ζ
by (*metis g lower order.trans*) — in fact y is *Inf* {*x*. *y* $\leq$ *g x*}
then show *?thesis*
by (*force simp*: ζ)
next
case *False*
have (*y* $\leq$ *g x*) = (*?μ* < *x*) **for** *x*
proof
show *?μ* < *x* **if** *y* $\leq$ *g x*
using *that False less_eq_real_def lower* **by** *blast*
show *y* $\leq$ *g x* **if** *?μ* < *x*
by (*metis g greatest le_less_trans that less_le_trans linear not_less*)
qed
then obtain ζ **where** ζ: $\bigwedge$*x*. *g x* $\geq$ *y* $\longleftrightarrow$ *x* > ζ **..**
then show *?thesis*
by (*force simp*: ζ)
qed
qed
then consider *?X* = {} | *?X* = *UNIV* | (*intv*) *d* **where** *?X* = {*d*..} $\vee$ *?X* = {*d*<..}
by *metis*
then have $\exists$ *d*. *d* $\in$ {*a*..*b*} $\wedge$ *integral* {*a*..*b*} (λ*x*. *if x* $\in$ *?X then f x else 0*) = *integral* {*d*..*b*} *f*
proof *cases*
case (*intv d*)
show *?thesis*
proof (*cases d* < *a*)
case *True*
with *intv* **have** *integral* {*a*..*b*} (λ*x*. *if y* $\leq$ *g x then f x else 0*) = *integral* {*a*..*b*} *f*
by (*intro Henstock_Kurzweil_Integration.integral_cong*) *force*
then show *?thesis*

```
          by (rule_tac x=a in exI) (simp add: ‹a ≤ b›)
      next
        case False
        show ?thesis
        proof (cases b < d)
          case True
          have integral {a..b} (λx. if x ∈ {x. y ≤ g x} then f x else 0) = integral
{a..b} (λx. 0)
            by (rule Henstock_Kurzweil_Integration.integral_cong) (use intv True in
fastforce)
          then show ?thesis
            using ‹a ≤ b› by auto
        next
          case False
          with ‹¬ d < a› have eq: {d..} ∩ {a..b} = {d..b} {d<..} ∩ {a..b} =
{d<..b}
            by force+
          moreover have integral {d<..b} f = integral {d..b} f
              by (rule integral_spike_set [OF empty_imp_negligible negligible_subset
[OF negligible_sing [of d]]]) auto
          ultimately
          have integral {a..b} (λx. if x ∈ {x. y ≤ g x} then f x else 0) = integral
{d..b} f
            unfolding integral_restrict_Int using intv by presburger
          moreover have d ∈ {a..b}
            using ‹¬ d < a› ‹a ≤ b› False by auto
          ultimately show ?thesis
            by auto
        qed
      qed
    qed (use ‹a ≤ b› in auto)
    then show ?thesis
      by auto
  qed
  then have ∀k. ∃d. d ∈ {a..b} ∧ integral {a..b} (λx. if real k / real n ≤ g x
then f x else 0) = integral {d..b} f
    by meson
  then obtain d where dab: ⋀k. d k ∈ {a..b}
    and deq: ⋀k::nat. integral {a..b} (λx. if k/n ≤ g x then f x else 0) = integral
{d k..b} f
    by metis
  have (∑k = 1..n. integral {a..b} (λx. if real k / real n ≤ g x then f x else 0))
/R n ∈ {m..M}
    unfolding scaleR_right.sum
  proof (intro conjI allI impI convex [THEN iffD1, rule_format])
    show integral {a..b} (λxa. if real k / real n ≤ g xa then f xa else 0) ∈ {m..M}
for k
      by (metis (no_types, lifting) deq image_eqI int_fab dab)
  qed (use ‹n ≠ 0› in auto)
```

then have $\exists\, c.\ c \in \{a..b\} \wedge$
$integral\ \{c..b\}\ f = inverse\ n *_R (\sum k = 1..n.\ integral\ \{a..b\}\ (\lambda x.\ if\ g\ x \geq real\ k\ /\ real\ n\ then\ f\ x\ else\ 0))$
by (*metis* (*no_types*, *lifting*) *int_fab imageE*)
then show *?thesis*
by (*simp add*: *sum_distrib_left if_distrib integral_sum int′ flip*: *integral_mult_right cong*: *if_cong*)
qed
then obtain *c* **where** *cab*: $\bigwedge n.\ c\ n \in \{a..b\}$
and *c*: $\bigwedge n.\ integral\ \{c\ n..b\}\ f = integral\ \{a..b\}\ (\lambda x.\ (\sum k = 1..n.\ if\ g\ x \geq real\ k\ /\ real\ n\ then\ f\ x\ /_R\ n\ else\ 0))$
by *metis*
obtain *d* **and** $\sigma :: nat \Rightarrow nat$
where $d \in \{a..b\}$ **and** σ: *strict_mono* σ **and** *d*: $(c \circ \sigma) \longrightarrow d$ **and** *non0*: $\bigwedge n.\ \sigma\ n \geq Suc\ 0$
proof −
have *compact*$\{a..b\}$
by *auto*
with *cab* **obtain** *d* **and** *s0*
where $d \in \{a..b\}$ **and** *s0*: *strict_mono s0* **and** *tends*: $(c \circ s0) \longrightarrow d$
unfolding *compact_def*
using *that* **by** *blast*
show *thesis*
proof
show $d \in \{a..b\}$
by *fact*
show *strict_mono* $(s0 \circ Suc)$
using *s0* **by** (*auto simp*: *strict_mono_def*)
show $(c \circ (s0 \circ Suc)) \longrightarrow d$
by (*metis tends LIMSEQ_subseq_LIMSEQ Suc_less_eq comp_assoc strict_mono_def*)
show $\bigwedge n.\ (s0 \circ Suc)\ n \geq Suc\ 0$
by (*metis comp_apply le0 not_less_eq_eq old.nat.exhaust s0 seq_suble*)
qed
qed
define φ **where** $\varphi \equiv \lambda n\ x.\ \sum k = Suc\ 0..\sigma\ n.\ if\ k/(\sigma\ n) \leq g\ x\ then\ f\ x\ /_R\ (\sigma\ n)\ else\ 0$
define ψ **where** $\psi \equiv \lambda n\ x.\ \sum k = Suc\ 0..\sigma\ n.\ if\ k/(\sigma\ n) \leq g\ x\ then\ inverse\ (\sigma\ n)\ else\ 0$
have $**$: $(\lambda x.\ g\ x *_R f\ x)$ *integrable_on cbox a b* $\wedge$
$(\lambda n.\ integral\ (cbox\ a\ b)\ (\varphi\ n)) \longrightarrow integral\ (cbox\ a\ b)\ (\lambda x.\ g\ x *_R f\ x)$
proof (*rule equiintegrable_limit*)
have †: $((\lambda n.\ \lambda x.\ (\sum k = Suc\ 0..n.\ if\ k\ /\ n \leq g\ x\ then\ inverse\ n *_R f\ x\ else\ 0))\ `\ \{Suc\ 0..\})$ *equiintegrable_on* $\{a..b\}$
proof −
have $*$: $(\bigcup c::real.\ \{\lambda x.\ if\ g\ x \geq c\ then\ f\ x\ else\ 0\})$ *equiintegrable_on* $\{a..b\}$
using *SMVT_lemma2* [*OF f g*] **.**
show *?thesis*
apply (*rule equiintegrable_on_subset* [*OF equiintegrable_sum_real* [*OF* $*$]], *clarify*)

```
      apply (rule_tac a={Suc 0..n} in UN_I, force)
      apply (rule_tac a=λk. inverse n in UN_I, auto)
      apply (rule_tac x=λk x. if real k / real n ≤ g x then f x else 0 in bexI)
       apply (force intro: sum.cong)+
      done
  qed
  show range φ equiintegrable_on cbox a b
    unfolding φ_def
    by (auto simp: non0 intro: equiintegrable_on_subset [OF †])
  show (λn. φ n x) ⟶ g x *R f x
    if x: x ∈ cbox a b for x
  proof −
    have eq: φ n x = ψ n x *R f x  for n
      by (auto simp: φ_def ψ_def sum_distrib_right if_distrib intro: sum.cong)
    show ?thesis
      unfolding eq
    proof (rule tendsto_scaleR [OF _ tendsto_const])
      show (λn. ψ n x) ⟶ g x
        unfolding lim_sequentially dist_real_def
      proof (intro allI impI)
        fix e :: real
        assume e > 0
       then obtain N where N ≠ 0 0 < inverse (real N) and N: inverse (real
N) < e
          using real_arch_inverse by metis
        moreover have |ψ n x − g x| < inverse (real N) if n≥N for n
        proof −
          have |g x − ψ n x| < inverse (real (σ n))
            unfolding ψ_def
          proof (rule level_approx [of {a..b} g])
            show σ n ≠ 0
              by (metis Suc_n_not_le_n non0)
          qed (use x 01 non0 in auto)
          also have ... ≤ inverse N
            using seq_suble [OF σ] ‹N ≠ 0› non0 that by (auto intro: order_trans
simp: field_split_simps)
          finally show ?thesis
            by linarith
        qed
        ultimately show ∃N. ∀n≥N. |ψ n x − g x| < e
          using less_trans by blast
      qed
    qed
  qed
 qed
 show thesis
 proof
  show a ≤ d d ≤ b
    using ‹d ∈ {a..b}› atLeastAtMost_iff by blast+
```

show ((*λx. g x* $*_R$ *f x*) *has_integral integral* {*d..b*} *f*) {*a..b*}
unfolding *has_integral_iff*
proof
show (*λx. g x* $*_R$ *f x*) *integrable_on* {*a..b*}
using ∗∗ **by** *simp*
show *integral* {*a..b*} (*λx. g x* $*_R$ *f x*) = *integral* {*d..b*} *f*
proof (*rule tendsto_unique*)
show (*λn. integral* {*c*(*σ n*)*..b*} *f*) ⟶ *integral* {*a..b*} (*λx. g x* $*_R$ *f x*)
using ∗∗ **by** (*simp add*: *c φ_def*)
have *continuous* (*at d within* {*a..b*}) (*λx. integral* {*x..b*} *f*)
using *indefinite_integral_continuous_1'* [*OF f*] ‹*d* ∈ {*a..b*}›
by (*simp add*: *continuous_on_eq_continuous_within*)
then show (*λn. integral* {*c*(*σ n*)*..b*} *f*) ⟶ *integral* {*d..b*} *f*
using *d cab* **unfolding** *o_def*
by (*simp add*: *continuous_within_sequentially o_def*)
qed *auto*
qed
qed
qed

theorem *second_mean_value_theorem_full*:
fixes *f* :: *real* ⇒ *real*
assumes *f*: *f integrable_on* {*a..b*} **and** *a* ≤ *b*
and *g*: ⋀*x y*. ⟦*a* ≤ *x*; *x* ≤ *y*; *y* ≤ *b*⟧ ⟹ *g x* ≤ *g y*
obtains *c* **where** *c* ∈ {*a..b*}
and ((*λx. g x* ∗ *f x*) *has_integral* (*g a* ∗ *integral* {*a..c*} *f* + *g b* ∗ *integral* {*c..b*} *f*)) {*a..b*}
proof −
have *gab*: *g a* ≤ *g b*
using ‹*a* ≤ *b*› *g* **by** *blast*
then consider *g a* < *g b* | *g a* = *g b*
by *linarith*
then show *thesis*
proof *cases*
case *1*
define *h* **where** *h* ≡ *λx. if x* < *a then 0 else if b* < *x then 1*
else (*g x* − *g a*) / (*g b* − *g a*)
obtain *c* **where** *a* ≤ *c c* ≤ *b* **and** *c*: ((*λx. h x* $*_R$ *f x*) *has_integral integral* {*c..b*} *f*) {*a..b*}
proof (*rule SMVT_lemma4* [*OF f* ‹*a* ≤ *b*›, *of h*])
show *h x* ≤ *h y 0* ≤ *h x* ∧ *h x* ≤ *1* **if** *x* ≤ *y* **for** *x y*
using *that gab* **by** (*auto simp*: *divide_simps g h_def*)
qed
show *?thesis*
proof
show *c* ∈ {*a..b*}
using ‹*a* ≤ *c*› ‹*c* ≤ *b*› **by** *auto*
have *I*: ((*λx. g x* ∗ *f x* − *g a* ∗ *f x*) *has_integral* (*g b* − *g a*) ∗ *integral* {*c..b*}

f) {*a..b*}
  proof (*subst has_integral_cong*)
   show *g x* ∗ *f x* − *g a* ∗ *f x* = (*g b* − *g a*) ∗ *h x* ∗$_R$ *f x*
    if *x* ∈ {*a..b*} **for** *x*
    using *1 that* **by** (*simp add*: *h_def field_split_simps*)
   show ((λ*x*. (*g b* − *g a*) ∗ *h x* ∗$_R$ *f x*) *has_integral* (*g b* − *g a*) ∗ *integral* {*c..b*} *f*) {*a..b*}
    using *has_integral_mult_right* [*OF c*, *of g b* − *g a*] .
  qed
  have *II*: ((λ*x*. *g a* ∗ *f x*) *has_integral g a* ∗ *integral* {*a..b*} *f*) {*a..b*}
   using *has_integral_mult_right* [**where** *c* = *g a*, *OF integrable_integral* [*OF f*]] .
  have ((λ*x*. *g x* ∗ *f x*) *has_integral* (*g b* − *g a*) ∗ *integral* {*c..b*} *f* + *g a* ∗ *integral* {*a..b*} *f*) {*a..b*}
   using *has_integral_add* [*OF I II*] **by** *simp*
  then show ((λ*x*. *g x* ∗ *f x*) *has_integral g a* ∗ *integral* {*a..c*} *f* + *g b* ∗ *integral* {*c..b*} *f*) {*a..b*}
   by (*simp add*: *algebra_simps flip*: *integral_combine* [*OF* ‹*a* ≤ *c*› ‹*c* ≤ *b*› *f*])
 qed
next
 case *2*
 show *?thesis*
 proof
  show *a* ∈ {*a..b*}
   by (*simp add*: ‹*a* ≤ *b*›)
  have ((λ*x*. *g x* ∗ *f x*) *has_integral g a* ∗ *integral* {*a..b*} *f*) {*a..b*}
  proof (*rule has_integral_eq*)
   show ((λ*x*. *g a* ∗ *f x*) *has_integral g a* ∗ *integral* {*a..b*} *f*) {*a..b*}
    using *f has_integral_mult_right* **by** *blast*
   show *g a* ∗ *f x* = *g x* ∗ *f x*
    if *x* ∈ {*a..b*} **for** *x*
    by (*metis atLeastAtMost_iff g less_eq_real_def not_le that 2*)
  qed
  then show ((λ*x*. *g x* ∗ *f x*) *has_integral g a* ∗ *integral* {*a..a*} *f* + *g b* ∗ *integral* {*a..b*} *f*) {*a..b*}
   by (*simp add*: *2*)
 qed
qed
qed

corollary *second_mean_value_theorem*:
 fixes *f* :: *real* ⇒ *real*
 assumes *f*: *f integrable_on* {*a..b*} **and** *a* ≤ *b*
 and *g*: ⋀*x y*. ⟦*a* ≤ *x*; *x* ≤ *y*; *y* ≤ *b*⟧ ⟹ *g x* ≤ *g y*
obtains *c* **where** *c* ∈ {*a..b*}
   integral {*a..b*} (λ*x*. *g x* ∗ *f x*) = *g a* ∗ *integral* {*a..c*} *f* + *g b* ∗ *integral* {*c..b*} *f*
 using *second_mean_value_theorem_full* [**where** *g*=*g*, *OF assms*]

by (*metis* (*full_types*) *integral_unique*)

end

6.28 Continuous Extensions of Functions

theory *Continuous_Extension*
imports *Starlike*
begin

6.28.1 Partitions of unity subordinate to locally finite open coverings

A difference from HOL Light: all summations over infinite sets equal zero, so the "support" must be made explicit in the summation below!

proposition *subordinate_partition_of_unity*:
fixes S :: *'a::metric_space set*
assumes $S \subseteq \bigcup \mathcal{C}$ **and** *opC*: $\bigwedge T.\ T \in \mathcal{C} \Longrightarrow$ *open* T
and *fin*: $\bigwedge x.\ x \in S \Longrightarrow \exists V.$ *open* $V \wedge x \in V \wedge$ *finite* $\{U \in \mathcal{C}.\ U \cap V \neq \{\}\}$
obtains F :: [*'a set*, *'a*] ⇒ *real*
where $\bigwedge U.\ U \in \mathcal{C} \Longrightarrow$ *continuous_on* S $(F\ U) \wedge (\forall x \in S.\ 0 \leq F\ U\ x)$
and $\bigwedge x\ U.\ [\![U \in \mathcal{C}; x \in S; x \notin U]\!] \Longrightarrow F\ U\ x = 0$
and $\bigwedge x.\ x \in S \Longrightarrow$ *supp_sum* $(\lambda W.\ F\ W\ x)\ \mathcal{C} = 1$
and $\bigwedge x.\ x \in S \Longrightarrow \exists V.$ *open* $V \wedge x \in V \wedge$ *finite* $\{U \in \mathcal{C}.\ \exists x{\in}V.\ F\ U\ x \neq 0\}$
proof (*cases* $\exists W.\ W \in \mathcal{C} \wedge S \subseteq W$)
case *True*
then obtain W **where** $W \in \mathcal{C}$ $S \subseteq W$ **by** *metis*
then show *?thesis*
by (*rule_tac* $F = \lambda V\ x.$ *if* $V = W$ *then* 1 *else* 0 **in** *that*) (*auto simp*: *supp_sum_def support_on_def*)
next
case *False*
have *nonneg*: $0 \leq$ *supp_sum* $(\lambda V.$ *setdist* $\{x\}$ $(S - V))\ \mathcal{C}$ **for** x
by (*simp add*: *supp_sum_def sum_nonneg*)
have *sd_pos*: $0 <$ *setdist* $\{x\}$ $(S - V)$ **if** $V \in \mathcal{C}$ $x \in S$ $x \in V$ **for** $V\ x$
proof −
have *closedin* (*top_of_set* S) $(S - V)$
by (*simp add*: *Diff_Diff_Int closedin_def opC openin_open_Int* ‹$V \in \mathcal{C}$›)
with *that False setdist_pos_le* [*of* $\{x\}$ $S - V$]
show *?thesis*
using *setdist_gt_0_closedin* **by** *fastforce*
qed
have *ss_pos*: $0 <$ *supp_sum* $(\lambda V.$ *setdist* $\{x\}$ $(S - V))\ \mathcal{C}$ **if** $x \in S$ **for** x
proof −
obtain U **where** $U \in \mathcal{C}$ $x \in U$ **using** ‹$x \in S$› ‹$S \subseteq \bigcup \mathcal{C}$›
by *blast*

```
    obtain V where open V x ∈ V finite {U ∈ C. U ∩ V ≠ {}}
      using ‹x ∈ S› fin by blast
    then have *: finite {A ∈ C. ¬ S ⊆ A ∧ x ∉ closure (S − A)}
      using closure_def that by (blast intro: rev_finite_subset)
    have x ∉ closure (S − U)
      using ‹U ∈ C› ‹x ∈ U› opC open_Int_closure_eq_empty by fastforce
    then show ?thesis
      apply (simp add: setdist_eq_0_sing_1 supp_sum_def support_on_def)
      apply (rule ordered_comm_monoid_add_class.sum_pos2 [OF *, of U])
      using ‹U ∈ C› ‹x ∈ U› False
      apply (auto simp: sd_pos that)
      done
  qed
  define F where
    F ≡ λW x. if x ∈ S then setdist {x} (S − W) / supp_sum (λV. setdist {x}
(S − V)) C else 0
  show ?thesis
  proof (rule_tac F = F in that)
    have continuous_on S (F U) if U ∈ C for U
    proof −
      have *: continuous_on S (λx. supp_sum (λV. setdist {x} (S − V)) C)
      proof (clarsimp simp add: continuous_on_eq_continuous_within)
        fix x assume x ∈ S
         then obtain X where open X and x: x ∈ S ∩ X and finX: finite {U
∈ C. U ∩ X ≠ {}}
          using assms by blast
        then have OSX: openin (top_of_set S) (S ∩ X) by blast
        have sumeq: ⋀x. x ∈ S ∩ X ⟹
                    (∑ V | V ∈ C ∧ V ∩ X ≠ {}. setdist {x} (S − V))
                    = supp_sum (λV. setdist {x} (S − V)) C
          apply (simp add: supp_sum_def)
          apply (rule sum.mono_neutral_right [OF finX])
          apply (auto simp: setdist_eq_0_sing_1 support_on_def subset_iff)
          apply (meson DiffI closure_subset disjoint_iff_not_equal subsetCE)
          done
         show continuous (at x within S) (λx. supp_sum (λV. setdist {x} (S −
V)) C)
          apply (rule continuous_transform_within_openin
                   [where f = λx. (sum (λV. setdist {x} (S − V)) {V ∈ C. V ∩
X ≠ {}})
                     and S =S ∩ X])
        apply (rule continuous_intros continuous_at_setdist continuous_at_imp_continuous_at_within
OSX x)+
          apply (simp add: sumeq)
          done
      qed
      show ?thesis
        apply (simp add: F_def)
        apply (rule continuous_intros *)+
```

using *ss_pos* **apply** *force*
done
qed
moreover have $[\![U \in \mathcal{C}; x \in S]\!] \Longrightarrow 0 \leq F\ U\ x$ **for** *U x*
using *nonneg* [*of x*] **by** (*simp add*: *F_def field_split_simps*)
ultimately show $\bigwedge U.\ U \in \mathcal{C} \Longrightarrow$ *continuous_on S* (*F U*) $\wedge$ ($\forall x{\in}S.\ 0 \leq F$ *U x*)
by *metis*
next
show $\bigwedge x\ U.\ [\![U \in \mathcal{C}; x \in S; x \notin U]\!] \Longrightarrow F\ U\ x = 0$
by (*simp add*: *setdist_eq_0_sing_1 closure_def F_def*)
next
show *supp_sum* ($\lambda W.\ F\ W\ x$) $\mathcal{C} = 1$ **if** $x \in S$ **for** *x*
using *that ss_pos* [*OF that*]
by (*simp add*: *F_def field_split_simps supp_sum_divide_distrib* [*symmetric*])
next
show $\exists V.$ *open* $V \wedge x \in V \wedge$ *finite* $\{U \in \mathcal{C}.\ \exists x{\in}V.\ F\ U\ x \neq 0\}$ **if** $x \in S$ **for** *x*
using *fin* [*OF that*] *that*
by (*fastforce simp*: *setdist_eq_0_sing_1 closure_def F_def elim*!: *rev_finite_subset*)
qed
qed

6.28.2 Urysohn's Lemma for Euclidean Spaces

For Euclidean spaces the proof is easy using distances.

lemma *Urysohn_both_ne*:
assumes *US*: *closedin* (*top_of_set U*) *S*
and *UT*: *closedin* (*top_of_set U*) *T*
and $S \cap T = \{\}$ $S \neq \{\}$ $T \neq \{\}$ $a \neq b$
obtains *f* :: *'a::euclidean_space* ⇒ *'b::real_normed_vector*
where *continuous_on U f*
$\bigwedge x.\ x \in U \Longrightarrow f\ x \in$ *closed_segment a b*
$\bigwedge x.\ x \in U \Longrightarrow (f\ x = a \longleftrightarrow x \in S)$
$\bigwedge x.\ x \in U \Longrightarrow (f\ x = b \longleftrightarrow x \in T)$
proof −
have *S0*: $\bigwedge x.\ x \in U \Longrightarrow$ *setdist* $\{x\}$ $S = 0 \longleftrightarrow x \in S$
using ‹$S \neq \{\}$› *US setdist_eq_0_closedin* **by** *auto*
have *T0*: $\bigwedge x.\ x \in U \Longrightarrow$ *setdist* $\{x\}$ $T = 0 \longleftrightarrow x \in T$
using ‹$T \neq \{\}$› *UT setdist_eq_0_closedin* **by** *auto*
have *sdpos*: $0 <$ *setdist* $\{x\}$ S + *setdist* $\{x\}$ T **if** $x \in U$ **for** *x*
proof −
have ¬ (*setdist* $\{x\}$ $S = 0 \wedge$ *setdist* $\{x\}$ $T = 0$)
using *assms* **by** (*metis IntI empty_iff setdist_eq_0_closedin that*)
then show *?thesis*
by (*metis add.left_neutral add.right_neutral add_pos_pos linorder_neqE_linordered_idom not_le setdist_pos_le*)
qed
define *f* **where** $f \equiv \lambda x.\ a + ($*setdist* $\{x\}$ S / (*setdist* $\{x\}$ S + *setdist* $\{x\}$ T))

```
*R (b − a)
  show ?thesis
  proof (rule_tac f = f in that)
    show continuous_on U f
      using sdpos unfolding f_def
      by (intro continuous_intros | force)+
    show f x ∈ closed_segment a b if x ∈ U for x
        unfolding f_def
      apply (simp add: closed_segment_def)
      apply (rule_tac x=(setdist {x} S / (setdist {x} S + setdist {x} T)) in exI)
      using sdpos that apply (simp add: algebra_simps)
      done
    show ⋀x. x ∈ U ⟹ (f x = a ⟷ x ∈ S)
      using S0 ‹a ≠ b› f_def sdpos by force
    show (f x = b ⟷ x ∈ T) if x ∈ U for x
    proof −
      have f x = b ⟷ (setdist {x} S / (setdist {x} S + setdist {x} T)) = 1
        unfolding f_def
        apply (rule iffI)
       apply (metis ‹a ≠ b› add_diff_cancel_left' eq_iff_diff_eq_0 pth_1 real_vector.scale_right_imp_eq,
force)
        done
      also have ... ⟷ setdist {x} T = 0 ∧ setdist {x} S ≠ 0
        using sdpos that
        by (simp add: field_split_simps) linarith
      also have ... ⟷ x ∈ T
        using ‹S ≠ {}› ‹T ≠ {}› ‹S ∩ T = {}› that
        by (force simp: S0 T0)
      finally show ?thesis .
    qed
  qed
qed

proposition Urysohn_local_strong:
  assumes US: closedin (top_of_set U) S
      and UT: closedin (top_of_set U) T
      and S ∩ T = {} a ≠ b
  obtains f :: 'a::euclidean_space ⇒ 'b::euclidean_space
    where continuous_on U f
          ⋀x. x ∈ U ⟹ f x ∈ closed_segment a b
          ⋀x. x ∈ U ⟹ (f x = a ⟷ x ∈ S)
          ⋀x. x ∈ U ⟹ (f x = b ⟷ x ∈ T)
proof (cases S = {})
  case True show ?thesis
  proof (cases T = {})
    case True show ?thesis
    proof (rule_tac f = λx. midpoint a b in that)
      show continuous_on U (λx. midpoint a b)
        by (intro continuous_intros)
```

show *midpoint a b* ∈ *closed_segment a b*
using *csegment_midpoint_subset* **by** *blast*
show (*midpoint a b* = *a*) = (*x* ∈ *S*) **for** *x*
using ‹*S* = {}› ‹*a* ≠ *b*› **by** *simp*
show (*midpoint a b* = *b*) = (*x* ∈ *T*) **for** *x*
using ‹*T* = {}› ‹*a* ≠ *b*› **by** *simp*
qed
next
case *False*
show *?thesis*
proof (*cases T* = *U*)
case *True* **with** ‹*S* = {}› ‹*a* ≠ *b*› **show** *?thesis*
by (*rule_tac f* = λ*x*. *b* **in** *that*) (*auto*)
next
case *False*
with *UT closedin_subset* **obtain** *c* **where** *c*: *c* ∈ *U c* ∉ *T*
by *fastforce*
obtain *f* **where** *f*: *continuous_on U f*
⋀*x*. *x* ∈ *U* ⟹ *f x* ∈ *closed_segment* (*midpoint a b*) *b*
⋀*x*. *x* ∈ *U* ⟹ (*f x* = *midpoint a b* ⟷ *x* = *c*)
⋀*x*. *x* ∈ *U* ⟹ (*f x* = *b* ⟷ *x* ∈ *T*)
apply (*rule Urysohn_both_ne* [*of U* {*c*} *T midpoint a b b*])
using *c* ‹*T* ≠ {}› *assms* **apply** *simp_all*
done
show *?thesis*
apply (*rule_tac f*=*f* **in** *that*)
using ‹*S* = {}› ‹*T* ≠ {}› *f csegment_midpoint_subset notin_segment_midpoint*
[*OF* ‹*a* ≠ *b*›]
apply *force+*
done
qed
qed
next
case *False*
show *?thesis*
proof (*cases T* = {})
case *True* **show** *?thesis*
proof (*cases S* = *U*)
case *True* **with** ‹*T* = {}› ‹*a* ≠ *b*› **show** *?thesis*
by (*rule_tac f* = λ*x*. *a* **in** *that*) (*auto*)
next
case *False*
with *US closedin_subset* **obtain** *c* **where** *c*: *c* ∈ *U c* ∉ *S*
by *fastforce*
obtain *f* **where** *f*: *continuous_on U f*
⋀*x*. *x* ∈ *U* ⟹ *f x* ∈ *closed_segment a* (*midpoint a b*)
⋀*x*. *x* ∈ *U* ⟹ (*f x* = *midpoint a b* ⟷ *x* = *c*)
⋀*x*. *x* ∈ *U* ⟹ (*f x* = *a* ⟷ *x* ∈ *S*)
apply (*rule Urysohn_both_ne* [*of U S* {*c*} *a midpoint a b*])

```
        using c ‹S ≠ {}› assms apply simp_all
        apply (metis midpoint_eq_endpoint)
        done
      show ?thesis
        apply (rule_tac f=f in that)
        using ‹S ≠ {}› ‹T = {}› f ‹a ≠ b›
        apply simp_all
        apply (metis (no_types) closed_segment_commute csegment_midpoint_subset
midpoint_sym subset_iff)
      apply (metis closed_segment_commute midpoint_sym notin_segment_midpoint)
        done
    qed
  next
    case False
    show ?thesis
      using Urysohn_both_ne [OF US UT ‹S ∩ T = {}› ‹S ≠ {}› ‹T ≠ {}› ‹a ≠
b›] that
      by blast
  qed
qed

lemma Urysohn_local:
  assumes US: closedin (top_of_set U) S
      and UT: closedin (top_of_set U) T
      and S ∩ T = {}
  obtains f :: 'a::euclidean_space ⇒ 'b::euclidean_space
    where continuous_on U f
          ⋀x. x ∈ U ⟹ f x ∈ closed_segment a b
          ⋀x. x ∈ S ⟹ f x = a
          ⋀x. x ∈ T ⟹ f x = b
proof (cases a = b)
  case True then show ?thesis
    by (rule_tac f = λx. b in that) (auto)
next
  case False
  then show ?thesis
    apply (rule Urysohn_local_strong [OF assms])
    apply (erule that, assumption)
    apply (meson US closedin_singleton closedin_trans)
    apply (meson UT closedin_singleton closedin_trans)
    done
qed

lemma Urysohn_strong:
  assumes US: closed S
      and UT: closed T
      and S ∩ T = {} a ≠ b
  obtains f :: 'a::euclidean_space ⇒ 'b::euclidean_space
    where continuous_on UNIV f
```

$\bigwedge x.\ f\ x \in closed\_segment\ a\ b$
$\bigwedge x.\ f\ x = a \longleftrightarrow x \in S$
$\bigwedge x.\ f\ x = b \longleftrightarrow x \in T$
using *assms* **by** (*auto intro*: *Urysohn_local_strong* [*of UNIV S T*])

proposition *Urysohn*:
assumes *US*: *closed S*
and *UT*: *closed T*
and $S \cap T = \{\}$
obtains f :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
where *continuous_on UNIV f*
$\bigwedge x.\ f\ x \in closed\_segment\ a\ b$
$\bigwedge x.\ x \in S \Longrightarrow f\ x = a$
$\bigwedge x.\ x \in T \Longrightarrow f\ x = b$
using *assms* **by** (*auto intro*: *Urysohn_local* [*of UNIV S T a b*])

6.28.3 Dugundji's Extension Theorem and Tietze Variants

See [2].

lemma *convex_supp_sum*:
assumes *convex S* **and** *1*: *supp_sum u I* = *1*
and $\bigwedge i.\ i \in I \Longrightarrow 0 \le u\ i \wedge (u\ i = 0 \vee f\ i \in S)$
shows *supp_sum* $(\lambda i.\ u\ i *_R f\ i)\ I \in S$
proof −
have *fin*: *finite* $\{i \in I.\ u\ i \neq 0\}$
using *1 sum.infinite* **by** (*force simp*: *supp_sum_def support_on_def*)
then have *supp_sum* $(\lambda i.\ u\ i *_R f\ i)\ I = sum\ (\lambda i.\ u\ i *_R f\ i)\ \{i \in I.\ u\ i \neq 0\}$
by (*force intro*: *sum.mono_neutral_left simp*: *supp_sum_def support_on_def*)
also have ... $\in S$
using *1 assms* **by** (*force simp*: *supp_sum_def support_on_def intro*: *convex_sum* [*OF fin* ‹*convex S*›])
finally show *?thesis* **.**
qed

theorem *Dugundji*:
fixes f :: $'a$::{*metric_space*,*second_countable_topology*} $\Rightarrow$ $'b$::*real_inner*
assumes *convex C* $C \neq \{\}$
and *cloin*: *closedin* (*top_of_set U*) *S*
and *contf*: *continuous_on S f* **and** f ‘ $S \subseteq C$
obtains g **where** *continuous_on U g* g ‘ $U \subseteq C$
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
proof (*cases* $S = \{\}$)
case *True* **then show** *thesis*
apply (*rule_tac* $g{=}\lambda x.$ *SOME* $y.\ y \in C$ **in** *that*)
apply (*rule continuous_intros*)
apply (*meson all_not_in_conv* ‹$C \neq \{\}$› *image_subsetI someI_ex*, *simp*)
done
next
case *False*

```
then have sd_pos: ⋀x. ⟦x ∈ U; x ∉ S⟧ ⟹ 0 < setdist {x} S
  using setdist_eq_0_closedin [OF cloin] le_less setdist_pos_le by fastforce
define ℬ where ℬ = {ball x (setdist {x} S / 2) |x. x ∈ U − S}
have [simp]: ⋀T. T ∈ ℬ ⟹ open T
  by (auto simp: ℬ_def)
have USS: U − S ⊆ ⋃ℬ
  by (auto simp: sd_pos ℬ_def)
obtain 𝒞 where USsub: U − S ⊆ ⋃𝒞
    and nbrhd: ⋀U. U ∈ 𝒞 ⟹ open U ∧ (∃T. T ∈ ℬ ∧ U ⊆ T)
    and fin: ⋀x. x ∈ U − S ⟹ ∃V. open V ∧ x ∈ V ∧ finite {U. U ∈ 𝒞 ∧
U ∩ V ≠ {}}
  by (rule paracompact [OF USS]) auto
have ∃v a. v ∈ U ∧ v ∉ S ∧ a ∈ S ∧
            T ⊆ ball v (setdist {v} S / 2) ∧
            dist v a ≤ 2 ∗ setdist {v} S if T ∈ 𝒞 for T
proof −
  obtain v where v: T ⊆ ball v (setdist {v} S / 2) v ∈ U v ∉ S
    using ‹T ∈ 𝒞› nbrhd by (force simp: ℬ_def)
  then obtain a where a ∈ S dist v a < 2 ∗ setdist {v} S
    using setdist_ltE [of {v} S 2 ∗ setdist {v} S]
    using False sd_pos by force
  with v show ?thesis
    apply (rule_tac x=v in exI)
    apply (rule_tac x=a in exI, auto)
    done
qed
then obtain 𝒱 𝒜 where
  VA: ⋀T. T ∈ 𝒞 ⟹ 𝒱 T ∈ U ∧ 𝒱 T ∉ S ∧ 𝒜 T ∈ S ∧
            T ⊆ ball (𝒱 T) (setdist {𝒱 T} S / 2) ∧
            dist (𝒱 T) (𝒜 T) ≤ 2 ∗ setdist {𝒱 T} S
  by metis
have sdle: setdist {𝒱 T} S ≤ 2 ∗ setdist {v} S if T ∈ 𝒞 v ∈ T for T v
  using setdist_Lipschitz [of 𝒱 T S v] VA [OF ‹T ∈ 𝒞›] ‹v ∈ T› by auto
have d6: dist a (𝒜 T) ≤ 6 ∗ dist a v if T ∈ 𝒞 v ∈ T a ∈ S for T v a
proof −
  have dist (𝒱 T) v < setdist {𝒱 T} S / 2
    using that VA mem_ball by blast
  also have ... ≤ setdist {v} S
    using sdle [OF ‹T ∈ 𝒞› ‹v ∈ T›] by simp
  also have vS: setdist {v} S ≤ dist a v
    by (simp add: setdist_le_dist setdist_sym ‹a ∈ S›)
  finally have VTV: dist (𝒱 T) v < dist a v .
  have VTS: setdist {𝒱 T} S ≤ 2 ∗ dist a v
    using sdle that vS by force
  have dist a (𝒜 T) ≤ dist a v + dist v (𝒱 T) + dist (𝒱 T) (𝒜 T)
   by (metis add.commute add_le_cancel_left dist_commute dist_triangle2 dist_triangle_le)
  also have ... ≤ dist a v + dist a v + dist (𝒱 T) (𝒜 T)
    using VTV by (simp add: dist_commute)
  also have ... ≤ 2 ∗ dist a v + 2 ∗ setdist {𝒱 T} S
```

using *VA* [*OF* ‹*T* ∈ 𝒞›] **by** *auto*
finally show *?thesis*
using *VTS* **by** *linarith*
qed
obtain *H* :: [′*a set*, ′*a*] ⇒ *real*
where *Hcont*: ⋀*Z*. *Z* ∈ 𝒞 ⟹ *continuous_on* (*U*−*S*) (*H Z*)
and *Hge0*: ⋀*Z x*. ⟦*Z* ∈ 𝒞; *x* ∈ *U*−*S*⟧ ⟹ *0* ≤ *H Z x*
and *Heq0*: ⋀*x Z*. ⟦*Z* ∈ 𝒞; *x* ∈ *U*−*S*; *x* ∉ *Z*⟧ ⟹ *H Z x* = *0*
and *H1*: ⋀*x*. *x* ∈ *U*−*S* ⟹ *supp_sum* (λ*W*. *H W x*) 𝒞 = *1*
and *Hfin*: ⋀*x*. *x* ∈ *U*−*S* ⟹ ∃ *V*. *open V* ∧ *x* ∈ *V* ∧ *finite* {*U* ∈ 𝒞. ∃*x*∈*V*. *H U x* ≠ *0*}
apply (*rule subordinate_partition_of_unity* [*OF USsub* _ *fin*])
using *nbrhd* **by** *auto*
define *g* **where** *g* ≡ λ*x*. *if x* ∈ *S then f x else supp_sum* (λ*T*. *H T x* $*_R$ *f*(𝒜 *T*)) 𝒞
show *?thesis*
proof (*rule that*)
show *continuous_on U g*
proof (*clarsimp simp*: *continuous_on_eq_continuous_within*)
fix *a* **assume** *a* ∈ *U*
show *continuous* (*at a within U*) *g*
proof (*cases a* ∈ *S*)
case *True* **show** *?thesis*
proof (*clarsimp simp add*: *continuous_within_topological*)
fix *W*
assume *open W g a* ∈ *W*
then obtain *e* **where** *0* < *e* **and** *e*: *ball* (*f a*) *e* ⊆ *W*
using *openE True g_def* **by** *auto*
have *continuous* (*at a within S*) *f*
using *True contf continuous_on_eq_continuous_within* **by** *blast*
then obtain *d* **where** *0* < *d*
and *d*: ⋀*x*. ⟦*x* ∈ *S*; *dist x a* < *d*⟧ ⟹ *dist* (*f x*) (*f a*) < *e*
using *continuous_within_eps_delta* ‹*0* < *e*› **by** *force*
have *g y* ∈ *ball* (*f a*) *e* **if** *y* ∈ *U* **and** *y*: *y* ∈ *ball a* (*d* / *6*) **for** *y*
proof (*cases y* ∈ *S*)
case *True*
then have *dist* (*f a*) (*f y*) < *e*
by (*metis ball_divide_subset_numeral dist_commute in_mono mem_ball y d*)
then show *?thesis*
by (*simp add*: *True g_def*)
next
case *False*
have ∗: *dist* (*f* (𝒜 *T*)) (*f a*) < *e* **if** *T* ∈ 𝒞 *H T y* ≠ *0* **for** *T*
proof −
have *y* ∈ *T*
using *Heq0 that False* ‹*y* ∈ *U*› **by** *blast*
have *dist* (𝒜 *T*) *a* < *d*
using *d6* [*OF* ‹*T* ∈ 𝒞› ‹*y* ∈ *T*› ‹*a* ∈ *S*›] *y*

```
          by (simp add: dist_commute mult.commute)
        then show ?thesis
          using VA [OF ‹T ∈ 𝒞›] by (auto simp: d)
      qed
      have supp_sum (λT. H T y *R f (𝒜 T)) 𝒞 ∈ ball (f a) e
        apply (rule convex_supp_sum [OF convex_ball])
        apply (simp_all add: False H1 Hge0 ‹y ∈ U›)
        by (metis dist_commute *)
      then show ?thesis
        by (simp add: False g_def)
    qed
    then show ∃A. open A ∧ a ∈ A ∧ (∀y∈U. y ∈ A ⟶ g y ∈ W)
      apply (rule_tac x = ball a (d / 6) in exI)
      using e ‹0 < d› by fastforce
  qed
next
  case False
  obtain N where N: open N a ∈ N
           and finN: finite {U ∈ 𝒞. ∃a∈N. H U a ≠ 0}
    using Hfin False ‹a ∈ U› by auto
  have oUS: openin (top_of_set U) (U − S)
    using cloin by (simp add: openin_diff)
  have HcontU: continuous (at a within U) (H T) if T ∈ 𝒞 for T
    using Hcont [OF ‹T ∈ 𝒞›] False  ‹a ∈ U› ‹T ∈ 𝒞›
    apply (simp add: continuous_on_eq_continuous_within continuous_within)
    apply (rule Lim_transform_within_set)
    using oUS
     apply (force simp: eventually_at openin_contains_ball dist_commute dest!:
bspec)+
    done
  show ?thesis
  proof (rule continuous_transform_within_openin [OF _ oUS])
   show continuous (at a within U) (λx. supp_sum (λT. H T x *R f (𝒜 T))
𝒞)
    proof (rule continuous_transform_within_openin)
      show continuous (at a within U)
            (λx. ∑T∈{U ∈ 𝒞. ∃x∈N. H U x ≠ 0}. H T x *R f (𝒜 T))
        by (force intro: continuous_intros HcontU)+
    next
      show openin (top_of_set U) ((U − S) ∩ N)
        using N oUS openin_trans by blast
    next
      show a ∈ (U − S) ∩ N using False ‹a ∈ U› N by blast
    next
      show ⋀x. x ∈ (U − S) ∩ N ⟹
                 (∑T ∈ {U ∈ 𝒞. ∃x∈N. H U x ≠ 0}. H T x *R f (𝒜 T))
                 = supp_sum (λT. H T x *R f (𝒜 T)) 𝒞
        by (auto simp: supp_sum_def support_on_def
                intro: sum.mono_neutral_right [OF finN])
```

qed
next
show $a \in U - S$ **using** *False* ‹$a \in U$› **by** *blast*
next
show $\bigwedge x.\ x \in U - S \Longrightarrow$ *supp_sum* $(\lambda T.\ H\ T\ x *_R f\ (\mathcal{A}\ T))\ \mathcal{C} = g\ x$
by (*simp add*: *g_def*)
qed
qed
qed
show g ‘ $U \subseteq C$
using ‹f ‘ $S \subseteq C$› *VA*
by (*fastforce simp*: *g_def Hge0 intro*!: *convex_supp_sum* [*OF* ‹*convex C*›] *H1*)
show $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
by (*simp add*: *g_def*)
qed
qed

corollary *Tietze*:
fixes *f* :: ′*a*::{*metric_space*,*second_countable_topology*} ⇒ ′*b*::*real_inner*
assumes *continuous_on S f*
and *closedin* (*top_of_set U*) *S*
and $0 \le B$
and $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) \le B$
obtains *g* **where** *continuous_on U g* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
$\bigwedge x.\ x \in U \Longrightarrow norm(g\ x) \le B$
using *assms* **by** (*auto simp*: *image_subset_iff intro*: *Dugundji* [*of cball 0 B U S f*])

corollary *Tietze_closed_interval*:
fixes *f* :: ′*a*::{*metric_space*,*second_countable_topology*} ⇒ ′*b*::*euclidean_space*
assumes *continuous_on S f*
and *closedin* (*top_of_set U*) *S*
and *cbox a b* ≠ {}
and $\bigwedge x.\ x \in S \Longrightarrow f\ x \in cbox\ a\ b$
obtains *g* **where** *continuous_on U g* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
$\bigwedge x.\ x \in U \Longrightarrow g\ x \in cbox\ a\ b$
apply (*rule Dugundji* [*of cbox a b U S f*])
using *assms* **by** *auto*

corollary *Tietze_closed_interval_1*:
fixes *f* :: ′*a*::{*metric_space*,*second_countable_topology*} ⇒ *real*
assumes *continuous_on S f*
and *closedin* (*top_of_set U*) *S*
and $a \le b$
and $\bigwedge x.\ x \in S \Longrightarrow f\ x \in cbox\ a\ b$
obtains *g* **where** *continuous_on U g* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
$\bigwedge x.\ x \in U \Longrightarrow g\ x \in cbox\ a\ b$
apply (*rule Dugundji* [*of cbox a b U S f*])

using *assms* **by** (*auto simp*: *image_subset_iff*)

corollary *Tietze_open_interval*:
fixes *f* :: *'a*::{*metric_space*,*second_countable_topology*} ⇒ *'b*::*euclidean_space*
assumes *continuous_on S f*
and *closedin* (*top_of_set U*) *S*
and *box a b* ≠ {}
and ⋀*x*. *x* ∈ *S* ⟹ *f x* ∈ *box a b*
obtains *g* **where** *continuous_on U g* ⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x*
⋀*x*. *x* ∈ *U* ⟹ *g x* ∈ *box a b*
apply (*rule Dugundji* [*of box a b U S f*])
using *assms* **by** *auto*

corollary *Tietze_open_interval_1*:
fixes *f* :: *'a*::{*metric_space*,*second_countable_topology*} ⇒ *real*
assumes *continuous_on S f*
and *closedin* (*top_of_set U*) *S*
and *a* < *b*
and *no*: ⋀*x*. *x* ∈ *S* ⟹ *f x* ∈ *box a b*
obtains *g* **where** *continuous_on U g* ⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x*
⋀*x*. *x* ∈ *U* ⟹ *g x* ∈ *box a b*
apply (*rule Dugundji* [*of box a b U S f*])
using *assms* **by** (*auto simp*: *image_subset_iff*)

corollary *Tietze_unbounded*:
fixes *f* :: *'a*::{*metric_space*,*second_countable_topology*} ⇒ *'b*::*real_inner*
assumes *continuous_on S f*
and *closedin* (*top_of_set U*) *S*
obtains *g* **where** *continuous_on U g* ⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x*
apply (*rule Dugundji* [*of UNIV U S f*])
using *assms* **by** *auto*

end

6.29 Equivalence Between Classical Borel Measurability and HOL Light's

theory *Equivalence_Measurable_On_Borel*
imports *Equivalence_Lebesgue_Henstock_Integration Improper_Integral Continuous_Extension*
begin

abbreviation *sym_diff* :: *'a set* ⇒ *'a set* ⇒ *'a set* **where**
sym_diff A B ≡ ((*A* − *B*) ∪ (*B*−*A*))

6.29.1 Austin's Lemma

lemma *Austin_Lemma*:
fixes $\mathcal{D}$ :: *'a::euclidean_space set set*
assumes *finite* $\mathcal{D}$ **and** $\mathcal{D}$: $\bigwedge D.\ D \in \mathcal{D} \Longrightarrow \exists\, k\ a\ b.\ D = cbox\ a\ b \wedge (\forall\, i \in Basis.\ b{\cdot}i - a{\cdot}i = k)$
obtains $\mathcal{C}$ **where** $\mathcal{C} \subseteq \mathcal{D}$ *pairwise disjnt* $\mathcal{C}$
measure lebesgue $(\bigcup\mathcal{C}) \geq$ *measure lebesgue* $(\bigcup\mathcal{D})$ / *3* ^ (*DIM*('*a*))
using *assms*
proof (*induction card* $\mathcal{D}$ *arbitrary*: $\mathcal{D}$ *thesis rule*: *less_induct*)
case *less*
show *?case*
proof (*cases* $\mathcal{D} = \{\}$)
case *True*
then show *thesis*
using *less* **by** *auto*
next
case *False*
then have *Max* (*Sigma_Algebra.measure lebesgue* ' $\mathcal{D}$) ∈ *Sigma_Algebra.measure lebesgue* ' $\mathcal{D}$
using *Max_in finite_imageI* ‹*finite* $\mathcal{D}$› **by** *blast*
then obtain *D* **where** $D \in \mathcal{D}$ **and** *measure lebesgue D* = *Max* (*measure lebesgue* ' $\mathcal{D}$)
by *auto*
then have *D*: $\bigwedge C.\ C \in \mathcal{D} \Longrightarrow$ *measure lebesgue C* ≤ *measure lebesgue D*
by (*simp add*: ‹*finite* $\mathcal{D}$›)
let $?\mathcal{E} = \{C.\ C \in \mathcal{D} - \{D\} \wedge disjnt\ C\ D\}$
obtain $\mathcal{D}'$ **where** $\mathcal{D}'$*sub*: $\mathcal{D}' \subseteq ?\mathcal{E}$ **and** $\mathcal{D}'$*dis*: *pairwise disjnt* $\mathcal{D}'$
and $\mathcal{D}'$*m*: *measure lebesgue* $(\bigcup\mathcal{D}') \geq$ *measure lebesgue* $(\bigcup ?\mathcal{E})$ / *3* ^ (*DIM*('*a*))
proof (*rule less.hyps*)
have ∗: $?\mathcal{E} \subset \mathcal{D}$
using ‹$D \in \mathcal{D}$› **by** *auto*
then show *card* $?\mathcal{E}$ < *card* $\mathcal{D}$ *finite* $?\mathcal{E}$
by (*auto simp*: ‹*finite* $\mathcal{D}$› *psubset_card_mono*)
show $\exists\, k\ a\ b.\ D = cbox\ a\ b \wedge (\forall\, i{\in}Basis.\ b \cdot i - a \cdot i = k)$ **if** $D \in ?\mathcal{E}$ **for** *D*
using *less.prems*(*3*) *that* **by** *auto*
qed
then have [*simp*]: $\bigcup\mathcal{D}' - D = \bigcup\mathcal{D}'$
by (*auto simp*: *disjnt_iff*)
show *?thesis*
proof (*rule less.prems*)
show *insert D* $\mathcal{D}' \subseteq \mathcal{D}$
using $\mathcal{D}'$*sub* ‹$D \in \mathcal{D}$› **by** *blast*
show *disjoint* (*insert D* $\mathcal{D}'$)
using $\mathcal{D}'$*dis* $\mathcal{D}'$*sub* **by** (*fastforce simp add*: *pairwise_def disjnt_sym*)
obtain *a3 b3* **where** *m3*: *content* (*cbox a3 b3*) = *3* ^ *DIM*('*a*) ∗ *measure lebesgue D*
and *sub3*: $\bigwedge C.\ [\![C \in \mathcal{D};\ \neg\ disjnt\ C\ D]\!] \Longrightarrow C \subseteq cbox\ a3\ b3$
proof −
obtain *k a b* **where** *ab*: *D* = *cbox a b* **and** *k*: $\bigwedge i.\ i \in Basis \Longrightarrow b{\cdot}i - a{\cdot}i$

= k
using less.prems ‹D ∈ 𝒟› **by** meson
then have eqk: ⋀i. i ∈ Basis ⟹ a · i ≤ b · i ⟷ k ≥ 0
by force
show thesis
proof
let ?a = (a + b) /_R 2 − (3/2) *_R (b − a)
let ?b = (a + b) /_R 2 + (3/2) *_R (b − a)
have eq: (∏i∈Basis. b · i * 3 − a · i * 3) = (∏i∈Basis. b · i − a · i) * 3 ^ DIM('a)
by (simp add: comm_monoid_mult_class.prod.distrib flip: left_diff_distrib inner_diff_left)
show content (cbox ?a ?b) = 3 ^ DIM('a) * measure lebesgue D
by (simp add: content_cbox_if box_eq_empty algebra_simps eq ab k)
show C ⊆ cbox ?a ?b **if** C ∈ 𝒟 **and** CD: ¬ disjnt C D **for** C
proof −
obtain k' a' b' **where** ab': C = cbox a' b' **and** k': ⋀i. i ∈ Basis ⟹ b'·i − a'·i = k'
using less.prems ‹C ∈ 𝒟› **by** meson
then have eqk': ⋀i. i ∈ Basis ⟹ a' · i ≤ b' · i ⟷ k' ≥ 0
by force
show ?thesis
proof (clarsimp simp add: disjoint_interval disjnt_def ab ab' not_less subset_box algebra_simps)
show a · i * 2 ≤ a' · i + b · i ∧ a · i + b' · i ≤ b · i * 2
if * [rule_format]: ∀j∈Basis. a' · j ≤ b' · j **and** i ∈ Basis **for** i
proof −
have a' · i ≤ b' · i ∧ a · i ≤ b · i ∧ a · i ≤ b' · i ∧ a' · i ≤ b · i
using ‹i ∈ Basis› CD **by** (simp_all add: disjoint_interval disjnt_def ab ab' not_less)
then show ?thesis
using D [OF ‹C ∈ 𝒟›] ‹i ∈ Basis›
apply (simp add: ab ab' k k' eqk eqk' content_cbox_cases)
using k k' **by** fastforce
qed
qed
qed
qed
qed
have 𝒟lm: ⋀D. D ∈ 𝒟 ⟹ D ∈ lmeasurable
using less.prems(3) **by** blast
have measure lebesgue (⋃𝒟) ≤ measure lebesgue (cbox a3 b3 ∪ (⋃𝒟 − cbox a3 b3))
proof (rule measure_mono_fmeasurable)
show ⋃𝒟 ∈ sets lebesgue
using 𝒟lm ‹finite 𝒟› **by** blast
show cbox a3 b3 ∪ (⋃𝒟 − cbox a3 b3) ∈ lmeasurable
by (simp add: 𝒟lm fmeasurable.Un fmeasurable.finite_Union less.prems(2) subset_eq)

qed *auto*
also have ... = *content* (*cbox a3 b3*) + *measure lebesgue* ($\bigcup\mathcal{D}$ − *cbox a3 b3*)
by (*simp add*: $\mathcal{D}$*lm fmeasurable.finite_Union less.prems*(*2*) *measure_Un2 subsetI*)
also have ... ≤ (*measure lebesgue D* + *measure lebesgue* ($\bigcup\mathcal{D}'$)) ∗ *3* ˆ *DIM*(′*a*)
proof −
have ($\bigcup\mathcal{D}$ − *cbox a3 b3*) ⊆ $\bigcup$ *?*$\mathcal{E}$
using *sub3* **by** *fastforce*
then have *measure lebesgue* ($\bigcup\mathcal{D}$ − *cbox a3 b3*) ≤ *measure lebesgue* ($\bigcup$ *?*$\mathcal{E}$)
proof (*rule measure_mono_fmeasurable*)
show $\bigcup$ $\mathcal{D}$ − *cbox a3 b3* ∈ *sets lebesgue*
by (*simp add*: $\mathcal{D}$*lm fmeasurableD less.prems*(*2*) *sets.Diff sets.finite_Union subsetI*)
show $\bigcup$ {*C* ∈ $\mathcal{D}$ − {*D*}. *disjnt C D*} ∈ *lmeasurable*
using $\mathcal{D}$*lm less.prems*(*2*) **by** *auto*
qed
then have *measure lebesgue* ($\bigcup\mathcal{D}$ − *cbox a3 b3*) / *3* ˆ *DIM*(′*a*) ≤ *measure lebesgue* ($\bigcup$ $\mathcal{D}'$)
using $\mathcal{D}'$*m* **by** (*simp add*: *field_split_simps*)
then show *?thesis*
by (*simp add*: *m3 field_simps*)
qed
also have ... ≤ *measure lebesgue* ($\bigcup$(*insert D* $\mathcal{D}'$)) ∗ *3* ˆ *DIM*(′*a*)
proof (*simp add*: $\mathcal{D}$*lm* ‹*D* ∈ $\mathcal{D}$›)
show *measure lebesgue D* + *measure lebesgue* ($\bigcup\mathcal{D}'$) ≤ *measure lebesgue* (*D* ∪ $\bigcup$ $\mathcal{D}'$)
proof (*subst measure_Un2*)
show $\bigcup$ $\mathcal{D}'$ ∈ *lmeasurable*
by (*meson* $\mathcal{D}$*lm* ‹*insert D* $\mathcal{D}'$ ⊆ $\mathcal{D}$› *fmeasurable.finite_Union less.prems*(*2*) *finite_subset subset_eq subset_insertI*)
show *measure lebesgue D* + *measure lebesgue* ($\bigcup$ $\mathcal{D}'$) ≤ *measure lebesgue D* + *measure lebesgue* ($\bigcup$ $\mathcal{D}'$ − *D*)
using ‹*insert D* $\mathcal{D}'$ ⊆ $\mathcal{D}$› *infinite_super less.prems*(*2*) **by** *force*
qed (*simp add*: $\mathcal{D}$*lm* ‹*D* ∈ $\mathcal{D}$›)
qed
finally show *measure lebesgue* ($\bigcup\mathcal{D}$) / *3* ˆ *DIM*(′*a*) ≤ *measure lebesgue* ($\bigcup$(*insert D* $\mathcal{D}'$))
by (*simp add*: *field_split_simps*)
qed
qed
qed

6.29.2 A differentiability-like property of the indefinite integral.

proposition *integrable_ccontinuous_explicit*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*

assumes $\bigwedge a\ b::'a.\ f\ integrable\_on\ cbox\ a\ b$
obtains N **where**
$negligible\ N$
$\bigwedge x\ e.\ [\![x \notin N;\ 0 < e]\!] \Longrightarrow$
$\exists\, d{>}0.\ \forall\, h.\ 0 < h \wedge h < d \longrightarrow$
$norm(integral\ (cbox\ x\ (x + h *_R\ One))\ f\ /_R\ h\ \hat{}\ DIM('a) - f$
$x) < e$
proof –
define BOX **where** $BOX \equiv \lambda h.\ \lambda x::'a.\ cbox\ x\ (x + h *_R\ One)$
define $BOX2$ **where** $BOX2 \equiv \lambda h.\ \lambda x::'a.\ cbox\ (x - h *_R\ One)\ (x + h *_R\ One)$
define i **where** $i \equiv \lambda h\ x.\ integral\ (BOX\ h\ x)\ f\ /_R\ h\ \hat{}\ DIM('a)$
define Ψ **where** $\Psi \equiv \lambda x\ r.\ \forall\, d{>}0.\ \exists\, h.\ 0 < h \wedge h < d \wedge r \le norm(i\ h\ x - f$
$x)$
let $?N = \{x.\ \exists\, e{>}0.\ \Psi\ x\ e\}$
have $\exists\, N.\ negligible\ N \wedge (\forall\, x\ e.\ x \notin N \wedge 0 < e \longrightarrow \neg\ \Psi\ x\ e)$
proof (*rule exI* ; *intro conjI allI impI*)
let $?M = \bigcup n.\ \{x.\ \Psi\ x\ (inverse(real\ n + 1))\}$
have $negligible\ (\{x.\ \Psi\ x\ \mu\} \cap cbox\ a\ b)$
if $\mu > 0$ **for** $a\ b\ \mu$
proof (*cases negligible*(*cbox a b*))
case *True*
then show *?thesis*
by (*simp add*: *negligible_Int*)
next
case *False*
then have $box\ a\ b \neq \{\}$
by (*simp add*: *negligible_interval*)
then have ab: $\bigwedge i.\ i \in Basis \Longrightarrow a{\cdot}i < b{\cdot}i$
by (*simp add*: *box_ne_empty*)
show *?thesis*
unfolding *negligible_outer_le*
proof (*intro allI impI*)
fix *e::real*
let $?ee = (e * \mu)\ /\ 2\ /\ 6\ \hat{}\ (DIM('a))$
assume $e > 0$
then have $gt0$: $?ee > 0$
using ‹$\mu > 0$› **by** *auto*
have f': $f\ integrable\_on\ cbox\ (a - One)\ (b + One)$
using *assms* **by** *blast*
obtain γ **where** $gauge\ \gamma$
and γ: $\bigwedge p.\ [\![p\ tagged\_partial\_division\_of\ (cbox\ (a - One)\ (b + One));\ \gamma$
$fine\ p]\!]$
$\Longrightarrow (\sum (x,\ k){\in}p.\ norm\ (content\ k *_R\ f\ x - integral\ k\ f)) < ?ee$
using *Henstock_lemma* [*OF f′ gt0*] *that* **by** *auto*
let $?E = \{x.\ x \in cbox\ a\ b \wedge \Psi\ x\ \mu\}$
have $\exists\, h{>}0.\ BOX\ h\ x \subseteq \gamma\ x\ \wedge$
$BOX\ h\ x \subseteq cbox\ (a - One)\ (b + One) \wedge \mu \le norm\ (i\ h\ x - f\ x)$
if $x \in cbox\ a\ b\ \Psi\ x\ \mu$ **for** x
proof –

obtain d **where** $d > 0$ **and** d: $ball\ x\ d \subseteq \gamma\ x$
using $gaugeD$ [OF ‹$gauge\ \gamma$›, $of\ x$] $openE$ **by** $blast$
then obtain h **where** $0 < h\ h < 1$ **and** $hless$: $h < d$ / $real\ DIM('a)$
and $mule$: $\mu \leq norm\ (i\ h\ x - f\ x)$
using ‹$\Psi\ x\ \mu$› [$unfolded\ \Psi\_def$, $rule\_format$, $of\ min\ 1\ (d\ /\ DIM('a))$]
by $auto$
show $?thesis$
proof ($intro\ exI\ conjI$)
show $0 < h\ \mu \leq norm\ (i\ h\ x - f\ x)$ **by** $fact+$
have $BOX\ h\ x \subseteq ball\ x\ d$
proof ($clarsimp\ simp$: $BOX\_def\ mem\_box\ dist\_norm\ algebra\_simps$)
fix y
assume $\forall i \in Basis.\ x \cdot i \leq y \cdot i \wedge y \cdot i \leq h + x \cdot i$
then have lt: $|(x - y) \cdot i| < d$ / $real\ DIM('a)$ **if** $i \in Basis$ **for** i
using $hless\ that$ **by** ($force\ simp$: $inner\_diff\_left$)
have $norm\ (x - y) \leq (\sum i \in Basis.\ |(x - y) \cdot i|)$
using $norm\_le\_l1$ **by** $blast$
also have $\ldots < d$
using $sum\_bounded\_above\_strict$ [$of\ Basis\ \lambda i.\ |(x - y) \cdot i|\ d\ /\ DIM('a)$, $OF\ lt$]
by $auto$
finally show $norm\ (x - y) < d$ **.**
qed
with d **show** $BOX\ h\ x \subseteq \gamma\ x$
by $blast$
show $BOX\ h\ x \subseteq cbox\ (a - One)\ (b + One)$
using $that$ ‹$h < 1$›
by ($force\ simp$: $BOX\_def\ mem\_box\ algebra\_simps\ intro$: $subset\_box\_imp$)
qed
qed
then obtain η **where** $h0$: $\bigwedge x.\ x \in ?E \Longrightarrow \eta\ x > 0$
and BOX_γ: $\bigwedge x.\ x \in ?E \Longrightarrow BOX\ (\eta\ x)\ x \subseteq \gamma\ x$
and $\bigwedge x.\ x \in ?E \Longrightarrow BOX\ (\eta\ x)\ x \subseteq cbox\ (a - One)\ (b + One) \wedge \mu \leq norm\ (i\ (\eta\ x)\ x - f\ x)$
by $simp\ metis$
then have BOX_cbox: $\bigwedge x.\ x \in ?E \Longrightarrow BOX\ (\eta\ x)\ x \subseteq cbox\ (a - One)\ (b + One)$
and μ_le: $\bigwedge x.\ x \in ?E \Longrightarrow \mu \leq norm\ (i\ (\eta\ x)\ x - f\ x)$
by $blast+$
define γ' **where** $\gamma' \equiv \lambda x.\ if\ x \in cbox\ a\ b \wedge \Psi\ x\ \mu\ then\ ball\ x\ (\eta\ x)\ else\ \gamma\ x$
have $gauge\ \gamma'$
using ‹$gauge\ \gamma$› **by** ($auto\ simp$: $h0\ gauge\_def\ \gamma'\_def$)
obtain $\mathcal{D}$ **where** $countable\ \mathcal{D}$
and $\mathcal{D}$: $\bigcup \mathcal{D} \subseteq cbox\ a\ b$
$\bigwedge K.\ K \in \mathcal{D} \Longrightarrow interior\ K \neq \{\} \wedge (\exists c\ d.\ K = cbox\ c\ d)$
and $Dcovered$: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow \exists x.\ x \in cbox\ a\ b \wedge \Psi\ x\ \mu \wedge x \in K \wedge K \subseteq \gamma'\ x$
and $subUD$: $?E \subseteq \bigcup \mathcal{D}$
by ($rule\ covering\_lemma$ [$of\ ?E\ a\ b\ \gamma'$]) ($simp\_all\ add$: Bex_def ‹$box\ a\ b \neq$

{}› ‹gauge γ'›)
then have $\mathcal{D} \subseteq$ *sets lebesgue*
by *fastforce*
show $\exists T. \{x. \Psi\ x\ \mu\} \cap$ *cbox a b* $\subseteq T \wedge T \in$ *lmeasurable* $\wedge$ *measure lebesgue* $T \leq e$
proof (*intro exI conjI*)
show $\{x. \Psi\ x\ \mu\} \cap$ *cbox a b* $\subseteq \bigcup\mathcal{D}$
apply *auto*
using *subUD* **by** *auto*
have *mUE*: *measure lebesgue* $(\bigcup\ \mathcal{E}) \leq$ *measure lebesgue* (*cbox a b*)
if $\mathcal{E} \subseteq \mathcal{D}$ *finite* $\mathcal{E}$ **for** $\mathcal{E}$
proof (*rule measure_mono_fmeasurable*)
show $\bigcup\ \mathcal{E} \subseteq$ *cbox a b*
using $\mathcal{D}$(*1*) *that*(*1*) **by** *blast*
show $\bigcup\ \mathcal{E} \in$ *sets lebesgue*
by (*metis* $\mathcal{D}$(*2*) *fmeasurable.finite_Union fmeasurableD lmeasurable_cbox subset_eq that*)
qed *auto*
then show $\bigcup\mathcal{D} \in$ *lmeasurable*
by (*metis* $\mathcal{D}$(*2*) ‹*countable* $\mathcal{D}$› *fmeasurable_Union_bound lmeasurable_cbox*)
then have *leab*: *measure lebesgue* $(\bigcup\mathcal{D}) \leq$ *measure lebesgue* (*cbox a b*)
by (*meson* $\mathcal{D}$(*1*) *fmeasurableD lmeasurable_cbox measure_mono_fmeasurable*)
obtain $\mathcal{F}$ **where** $\mathcal{F} \subseteq \mathcal{D}$ *finite* $\mathcal{F}$
and $\mathcal{F}$: *measure lebesgue* $(\bigcup\mathcal{D}) \leq 2 *$ *measure lebesgue* $(\bigcup\mathcal{F})$
proof (*cases measure lebesgue* $(\bigcup\mathcal{D}) = 0$)
case *True*
then show *?thesis*
by (*force intro*: *that* [**where** $\mathcal{F} = \{\}$])
next
case *False*
obtain $\mathcal{F}$ **where** $\mathcal{F} \subseteq \mathcal{D}$ *finite* $\mathcal{F}$
and $\mathcal{F}$: *measure lebesgue* $(\bigcup\mathcal{D})/2 <$ *measure lebesgue* $(\bigcup\mathcal{F})$
proof (*rule measure_countable_Union_approachable* [*of* $\mathcal{D}$ *measure lebesgue* $(\bigcup\mathcal{D})$ / *2 content* (*cbox a b*)])
show *countable* $\mathcal{D}$
by *fact*
show *0* < *measure lebesgue* $(\bigcup\ \mathcal{D})$ / *2*
using *False* **by** (*simp add*: *zero_less_measure_iff*)
show *Dlm*: $D \in$ *lmeasurable* **if** $D \in \mathcal{D}$ **for** D
using $\mathcal{D}$(*2*) *that* **by** *blast*
show *measure lebesgue* $(\bigcup\ \mathcal{F}) \leq$ *content* (*cbox a b*)
if $\mathcal{F} \subseteq \mathcal{D}$ *finite* $\mathcal{F}$ **for** $\mathcal{F}$
proof −
have *measure lebesgue* $(\bigcup\ \mathcal{F}) \leq$ *measure lebesgue* $(\bigcup\mathcal{D})$
proof (*rule measure_mono_fmeasurable*)
show $\bigcup\ \mathcal{F} \subseteq \bigcup\ \mathcal{D}$
by (*simp add*: *Sup_subset_mono* ‹$\mathcal{F} \subseteq \mathcal{D}$›)
show $\bigcup\ \mathcal{F} \in$ *sets lebesgue*
by (*meson Dlm fmeasurableD sets.finite_Union subset_eq that*)

show $\bigcup \mathcal{D} \in$ *lmeasurable*
by *fact*
qed
also have ... $\leq$ *measure lebesgue* (*cbox a b*)
proof (*rule measure_mono_fmeasurable*)
show $\bigcup \mathcal{D} \in$ *sets lebesgue*
by (*simp add*: ‹$\bigcup \mathcal{D} \in$ *lmeasurable*› *fmeasurableD*)
qed (*auto simp*:$\mathcal{D}$(*1*))
finally show *?thesis*
by *simp*
qed
qed *auto*
then show *?thesis*
using *that* **by** *auto*
qed
obtain *tag* **where** *tag_in_E*: $\bigwedge D.\ D \in \mathcal{D} \Longrightarrow tag\ D \in\ ?E$
and *tag_in_self*: $\bigwedge D.\ D \in \mathcal{D} \Longrightarrow tag\ D \in D$
and *tag_sub*: $\bigwedge D.\ D \in \mathcal{D} \Longrightarrow D \subseteq \gamma'\ (tag\ D)$
using *Dcovered* **by** *simp metis*
then have *sub_ball_tag*: $\bigwedge D.\ D \in \mathcal{D} \Longrightarrow D \subseteq ball\ (tag\ D)\ (\eta\ (tag\ D))$
by (*simp add*: γ'*_def*)
define Φ **where** $\Phi \equiv \lambda D.\ BOX\ (\eta(tag\ D))\ (tag\ D)$
define $\Phi 2$ **where** $\Phi 2 \equiv \lambda D.\ BOX2\ (\eta(tag\ D))\ (tag\ D)$
obtain $\mathcal{C}$ **where** $\mathcal{C} \subseteq \Phi 2$ ‘ $\mathcal{F}$ *pairwise disjnt* $\mathcal{C}$
measure lebesgue $(\bigcup \mathcal{C}) \geq$ *measure lebesgue* $(\bigcup(\Phi 2‘\mathcal{F}))$ / *3* ˆ (*DIM*('*a*))
proof (*rule Austin_Lemma*)
show *finite* ($\Phi 2‘\mathcal{F}$)
using ‹*finite* $\mathcal{F}$› **by** *blast*
have $\exists k\ a\ b.\ \Phi 2\ D = cbox\ a\ b \wedge (\forall i{\in}Basis.\ b \cdot i - a \cdot i = k)$ **if** $D \in \mathcal{F}$ **for** D
apply (*rule_tac* $x{=}2 * \eta(tag\ D)$ **in** *exI*)
apply (*rule_tac* $x{=}tag\ D - \eta(tag\ D) *_R$ *One* **in** *exI*)
apply (*rule_tac* $x{=}tag\ D + \eta(tag\ D) *_R$ *One* **in** *exI*)
using *that*
apply (*auto simp*: $\Phi 2$*_def BOX2_def algebra_simps*)
done
then show $\bigwedge D.\ D \in \Phi 2$ ‘ $\mathcal{F} \Longrightarrow \exists k\ a\ b.\ D = cbox\ a\ b \wedge (\forall i{\in}Basis.\ b \cdot i - a \cdot i = k)$
by *blast*
qed *auto*
then obtain $\mathcal{G}$ **where** $\mathcal{G} \subseteq \mathcal{F}$ **and** *disj*: *pairwise disjnt* ($\Phi 2$ ‘ $\mathcal{G}$)
and *measure lebesgue* $(\bigcup(\Phi 2$ ‘ $\mathcal{G})) \geq$ *measure lebesgue* $(\bigcup(\Phi 2‘\mathcal{F}))$ / *3* ˆ (*DIM*('*a*))
unfolding $\Phi 2$*_def subset_image_iff*
by (*meson empty_subsetI equals0D pairwise_imageI*)
moreover
have *measure lebesgue* $(\bigcup(\Phi 2$ ‘ $\mathcal{G})) * 3$ ˆ *DIM*('*a*) $\leq e/2$
proof –
have *finite* $\mathcal{G}$

```
      using ‹finite ℱ› ‹𝒢 ⊆ ℱ› infinite_super by blast
    have BOX2_m: ⋀x. x ∈ tag ` 𝒢 ⟹ BOX2 (η x) x ∈ lmeasurable
      by (auto simp: BOX2_def)
    have BOX_m: ⋀x. x ∈ tag ` 𝒢 ⟹ BOX (η x) x ∈ lmeasurable
      by (auto simp: BOX_def)
    have BOX_sub: BOX (η x) x ⊆ BOX2 (η x) x for x
      by (auto simp: BOX_def BOX2_def subset_box algebra_simps)
    have DISJ2: BOX2 (η (tag X)) (tag X) ∩ BOX2 (η (tag Y)) (tag Y)
= {}
      if X ∈ 𝒢 Y ∈ 𝒢 tag X ≠ tag Y for X Y
    proof -
      obtain i where i: i ∈ Basis tag X · i ≠ tag Y · i
        using ‹tag X ≠ tag Y› by (auto simp: euclidean_eq_iff [of tag X])
      have XY: X ∈ 𝒟 Y ∈ 𝒟
        using ‹ℱ ⊆ 𝒟› ‹𝒢 ⊆ ℱ› that by auto
      then have 0 ≤ η (tag X) 0 ≤ η (tag Y)
        by (meson h0 le_cases not_le tag_in_E)+
      with XY i have BOX2 (η (tag X)) (tag X) ≠ BOX2 (η (tag Y)) (tag
Y)
        unfolding eq_iff
        by (fastforce simp add: BOX2_def subset_box algebra_simps)
      then show ?thesis
        using disj that by (auto simp: pairwise_def disjnt_def Φ2_def)
    qed
  then have BOX2_disj: pairwise (λx y. negligible (BOX2 (η x) x ∩ BOX2
(η y) y)) (tag ` 𝒢)
      by (simp add: pairwise_imageI)
    then have BOX_disj: pairwise (λx y. negligible (BOX (η x) x ∩ BOX
(η y) y)) (tag ` 𝒢)
    proof (rule pairwise_mono)
      show negligible (BOX (η x) x ∩ BOX (η y) y)
        if negligible (BOX2 (η x) x ∩ BOX2 (η y) y) for x y
          by (metis (no_types, hide_lams) that Int_mono negligible_subset
BOX_sub)
    qed auto

    have eq: ⋀box. (λD. box (η (tag D)) (tag D)) ` 𝒢 = (λt. box (η t) t) `
tag ` 𝒢
      by (simp add: image_comp)
    have measure lebesgue (BOX2 (η t) t) * 3 ^ DIM('a)
        = measure lebesgue (BOX (η t) t) * (2*3) ^ DIM('a)
      if t ∈ tag ` 𝒢 for t
    proof -
      have content (cbox (t - η t *R One) (t + η t *R One))
          = content (cbox t (t + η t *R One)) * 2 ^ DIM('a)
       using that by (simp add: algebra_simps content_cbox_if box_eq_empty)
      then show ?thesis
        by (simp add: BOX2_def BOX_def flip: power_mult_distrib)
    qed
```

then have *measure lebesgue* $(\bigcup(\Phi 2$ ‘ $\mathcal{G})) * 3$ ^ $DIM('a)$ = *measure lebesgue* $(\bigcup(\Phi$ ‘ $\mathcal{G})) * 6$ ^ $DIM('a)$
unfolding Φ*_def* $\Phi 2$*_def eq*
by (*simp add*: *measure_negligible_finite_Union_image*
‹*finite* $\mathcal{G}$› *BOX2_m BOX_m BOX2_disj BOX_disj sum_distrib_right*
del: *UN_simps*)
also have ... $\leq e/2$
proof −
have μ * *measure lebesgue* $(\bigcup D{\in}\mathcal{G}.\ \Phi\ D) \leq \mu * (\sum D \in \Phi$‘$\mathcal{G}$. *measure lebesgue D*)
using ‹$\mu > 0$› ‹*finite* $\mathcal{G}$› **by** (*force simp*: *BOX_m* Φ*_def fmeasurableD intro*: *measure_Union_le*)
also have ... = $(\sum D \in \Phi$‘$\mathcal{G}$. *measure lebesgue D* * μ)
by (*metis mult.commute sum_distrib_right*)
also have ... $\leq (\sum (x, K) \in (\lambda D.\ (tag\ D,\ \Phi\ D))$ ‘ $\mathcal{G}$. *norm* (*content K* $*_R$ *f x* − *integral K f*))
proof (*rule sum_le_included*; *clarify?*)
fix *D*
assume $D \in \mathcal{G}$
then have η (*tag D*) > *0*
using ‹$\mathcal{F} \subseteq \mathcal{D}$› ‹$\mathcal{G} \subseteq \mathcal{F}$› *h0 tag_in_E* **by** *auto*
then have *m_*Φ: *measure lebesgue* (Φ *D*) > *0*
by (*simp add*: Φ*_def BOX_def algebra_simps*)
have $\mu \leq$ *norm* (*i* (η(*tag D*)) (*tag D*) − *f*(*tag D*))
using μ*_le* ‹$D \in \mathcal{G}$› ‹$\mathcal{F} \subseteq \mathcal{D}$› ‹$\mathcal{G} \subseteq \mathcal{F}$› *tag_in_E* **by** *auto*
also have ... = *norm* ((*content* (Φ *D*) $*_R$ *f*(*tag D*) − *integral* (Φ *D*) *f*) $/_R$ *measure lebesgue* (Φ *D*))
using *m_*Φ
unfolding *i_def* Φ*_def BOX_def*
by (*simp add*: *algebra_simps content_cbox_plus norm_minus_commute*)
finally have *measure lebesgue* (Φ *D*) * $\mu \leq$ *norm* (*content* (Φ *D*) $*_R$ *f*(*tag D*) − *integral* (Φ *D*) *f*)
using *m_*Φ **by** *simp* (*simp add*: *field_simps*)
then show $\exists\, y{\in}(\lambda D.\ (tag\ D,\ \Phi\ D))$ ‘ $\mathcal{G}$.
snd y = Φ *D* $\wedge$ *measure lebesgue* (Φ *D*) * $\mu \leq$ (*case y of* (*x*, *k*) $\Rightarrow$ *norm* (*content k* $*_R$ *f x* − *integral k f*))
using ‹$D \in \mathcal{G}$› **by** *auto*
qed (*use* ‹*finite* $\mathcal{G}$› **in** *auto*)
also have ... < *?ee*
proof (*rule* γ)
show (λD. (*tag D*, Φ *D*)) ‘ $\mathcal{G}$ *tagged_partial_division_of cbox* (*a* − *One*) (*b* + *One*)
unfolding *tagged_partial_division_of_def*
proof (*intro conjI allI impI* ; *clarify ?*)
show *tag D* $\in \Phi$ *D*
if $D \in \mathcal{G}$ **for** *D*
using *that* ‹$\mathcal{F} \subseteq \mathcal{D}$› ‹$\mathcal{G} \subseteq \mathcal{F}$› *h0 tag_in_E*
by (*auto simp*: Φ*_def BOX_def mem_box algebra_simps eucl_less_le_not_le in_mono*)

show $y \in cbox\ (a - One)\ (b + One)$ **if** $D \in \mathcal{G}\ y \in \Phi\ D$ **for** $D\ y$
using *that BOX_cbox Φ_def* ‹$\mathcal{F} \subseteq \mathcal{D}$› ‹$\mathcal{G} \subseteq \mathcal{F}$› *tag_in_E* **by** *blast*
show $tag\ D = tag\ E \wedge \Phi\ D = \Phi\ E$
if $D \in \mathcal{G}\ E \in \mathcal{G}$ **and** ne: $interior\ (\Phi\ D) \cap interior\ (\Phi\ E) \neq \{\}$
for $D\ E$
proof –
have $BOX2\ (\eta\ (tag\ D))\ (tag\ D) \cap BOX2\ (\eta\ (tag\ E))\ (tag\ E) = \{\} \vee tag\ E = tag\ D$
using *DISJ2* ‹$D \in \mathcal{G}$› ‹$E \in \mathcal{G}$› **by** *force*
then have $BOX\ (\eta\ (tag\ D))\ (tag\ D) \cap BOX\ (\eta\ (tag\ E))\ (tag\ E) = \{\} \vee tag\ E = tag\ D$
using *BOX_sub* **by** *blast*
then show $tag\ D = tag\ E \wedge \Phi\ D = \Phi\ E$
by (*metis Φ_def interior_Int interior_empty ne*)
qed
qed (*use* ‹*finite* $\mathcal{G}$› *Φ_def BOX_def* **in** *auto*)
show γ *fine* $(\lambda D.\ (tag\ D, \Phi\ D))$ ‘ $\mathcal{G}$
unfolding *fine_def Φ_def* **using** *BOX_γ* ‹$\mathcal{F} \subseteq \mathcal{D}$› ‹$\mathcal{G} \subseteq \mathcal{F}$› *tag_in_E*
by *blast*
qed
finally show *?thesis*
using ‹$\mu > 0$› **by** (*auto simp*: *field_split_simps*)
qed
finally show *?thesis* **.**
qed
moreover
have *measure lebesgue* $(\bigcup \mathcal{F}) \leq$ *measure lebesgue* $(\bigcup (\Phi 2 ‘\mathcal{F}))$
proof (*rule measure_mono_fmeasurable*)
have $D \subseteq ball\ (tag\ D)\ (\eta(tag\ D))$ **if** $D \in \mathcal{F}$ **for** D
using ‹$\mathcal{F} \subseteq \mathcal{D}$› *sub_ball_tag that* **by** *blast*
moreover have $ball\ (tag\ D)\ (\eta(tag\ D)) \subseteq BOX2\ (\eta\ (tag\ D))\ (tag\ D)$ **if**
$D \in \mathcal{F}$ **for** D
proof (*clarsimp simp*: *Φ2_def BOX2_def mem_box algebra_simps dist_norm*)
fix x **and** $i{::}'a$
assume $norm\ (tag\ D - x) < \eta\ (tag\ D)$ **and** $i \in Basis$
then have $|tag\ D \cdot i - x \cdot i| \leq \eta\ (tag\ D)$
by (*metis eucl_less_le_not_le inner_commute inner_diff_right norm_bound_Basis_le*)
then show $tag\ D \cdot i \leq x \cdot i + \eta\ (tag\ D) \wedge x \cdot i \leq \eta\ (tag\ D) + tag\ D$
$\cdot\ i$
by (*simp add*: *abs_diff_le_iff*)
qed
ultimately show $\bigcup \mathcal{F} \subseteq \bigcup (\Phi 2 ‘\mathcal{F})$
by (*force simp*: *Φ2_def*)
show $\bigcup \mathcal{F} \in$ *sets lebesgue*
using ‹*finite* $\mathcal{F}$› ‹$\mathcal{D} \subseteq$ *sets lebesgue*› ‹$\mathcal{F} \subseteq \mathcal{D}$› **by** *blast*
show $\bigcup (\Phi 2 ‘\mathcal{F}) \in$ *lmeasurable*
unfolding *Φ2_def BOX2_def* **using** ‹*finite* $\mathcal{F}$› **by** *blast*
qed
ultimately

have *measure lebesgue* ($\bigcup\mathcal{F}$) ≤ *e/2*
by (*auto simp*: *field_split_simps*)
then show *measure lebesgue* ($\bigcup\mathcal{D}$) ≤ *e*
using $\mathcal{F}$ **by** *linarith*
qed
qed
qed
then have ⋀*j*. *negligible* {*x*. Ψ *x* (*inverse*(*real j* + *1*))}
using *negligible_on_intervals*
by (*metis* (*full_types*) *inverse_positive_iff_positive le_add_same_cancel1 linorder_not_le nat_le_real_less not_add_less1 of_nat_0*)
then have *negligible ?M*
by *auto*
moreover have *?N* ⊆ *?M*
proof (*clarsimp simp*: *dist_norm*)
fix *y e*
assume *0* < *e*
and *ye* [*rule_format*]: Ψ *y e*
then obtain *k* **where** *k*: *0* < *k inverse* (*real k* + *1*) < *e*
by (*metis One_nat_def add.commute less_add_same_cancel2 less_imp_inverse_less less_trans neq0_conv of_nat_1 of_nat_Suc reals_Archimedean zero_less_one*)
with *ye* **show** ∃ *n*. Ψ *y* (*inverse* (*real n* + *1*))
apply (*rule_tac x*=*k* **in** *exI*)
unfolding Ψ*_def*
by (*force intro*: *less_le_trans*)
qed
ultimately show *negligible ?N*
by (*blast intro*: *negligible_subset*)
show ¬ Ψ *x e* **if** *x* ∉ *?N* ∧ *0* < *e* **for** *x e*
using *that* **by** *blast*
qed
with *that* **show** *?thesis*
unfolding *i_def BOX_def* Ψ*_def* **by** (*fastforce simp add*: *not_le*)
qed

6.29.3 HOL Light measurability

definition *measurable_on* :: (*'a*::*euclidean_space* ⇒ *'b*::*real_normed_vector*) ⇒ *'a set* ⇒ *bool*
(**infixr** *measurable'_on 46*)
where *f measurable_on S* ≡
∃ *N g*. *negligible N* ∧
(∀ *n*. *continuous_on UNIV* (*g n*)) ∧
(∀ *x*. *x* ∉ *N* ⟶ (λ*n*. *g n x*) ⟶ (*if x* ∈ *S then f x else 0*))

lemma *measurable_on_UNIV*:
(λ*x*. *if x* ∈ *S then f x else 0*) *measurable_on UNIV* ⟷ *f measurable_on S*
by (*auto simp*: *measurable_on_def*)

lemma *measurable_on_spike_set*:
  assumes *f*: *f measurable_on S* **and** *neg*: *negligible* $((S - T) \cup (T - S))$
  shows *f measurable_on T*
proof –
  obtain *N* **and** *F*
    where *N*: *negligible N*
      and *conF*: $\bigwedge n$. *continuous_on UNIV* (*F n*)
      and *tendsF*: $\bigwedge x.\ x \notin N \Longrightarrow (\lambda n.\ F\ n\ x) \longrightarrow$ (*if* $x \in S$ *then f x else 0*)
    using *f* **by** (*auto simp*: *measurable_on_def*)
  show *?thesis*
    unfolding *measurable_on_def*
  proof (*intro exI conjI allI impI*)
    show *continuous_on UNIV* ($\lambda x.\ F\ n\ x$) **for** *n*
      by (*intro conF continuous_intros*)
    show *negligible* $(N \cup (S - T) \cup (T - S))$
      by (*metis* (*full_types*) *N neg negligible_Un_eq*)
    show $(\lambda n.\ F\ n\ x) \longrightarrow$ (*if* $x \in T$ *then f x else 0*)
      if $x \notin (N \cup (S - T) \cup (T - S))$ **for** *x*
      using *that tendsF* [*of x*] **by** *auto*
  qed
qed

Various common equivalent forms of function measurability.

lemma *measurable_on_0* [*simp*]: ($\lambda x.\ 0$) *measurable_on S*
  unfolding *measurable_on_def*
proof (*intro exI conjI allI impI*)
  show $(\lambda n.\ 0) \longrightarrow$ (*if* $x \in S$ *then* 0::'*b else 0*) **for** *x*
    by *force*
qed *auto*

lemma *measurable_on_scaleR_const*:
  assumes *f*: *f measurable_on S*
  shows ($\lambda x.\ c *_R f\ x$) *measurable_on S*
proof –
  obtain *NF* **and** *F*
    where *NF*: *negligible NF*
      and *conF*: $\bigwedge n$. *continuous_on UNIV* (*F n*)
      and *tendsF*: $\bigwedge x.\ x \notin NF \Longrightarrow (\lambda n.\ F\ n\ x) \longrightarrow$ (*if* $x \in S$ *then f x else 0*)
    using *f* **by** (*auto simp*: *measurable_on_def*)
  show *?thesis*
    unfolding *measurable_on_def*
  proof (*intro exI conjI allI impI*)
    show *continuous_on UNIV* ($\lambda x.\ c *_R F\ n\ x$) **for** *n*
      by (*intro conF continuous_intros*)
    show $(\lambda n.\ c *_R F\ n\ x) \longrightarrow$ (*if* $x \in S$ *then* $c *_R f\ x$ *else 0*)
      if $x \notin NF$ **for** *x*
      using *tendsto_scaleR* [*OF tendsto_const tendsF*, *of x*] *that* **by** *auto*
  qed (*auto simp*: *NF*)
qed

lemma *measurable_on_cmul*:
  fixes *c* :: *real*
  assumes *f measurable_on S*
  shows (*λx. c ∗ f x*) *measurable_on S*
  using *measurable_on_scaleR_const* [*OF assms*] **by** *simp*

lemma *measurable_on_cdivide*:
  fixes *c* :: *real*
  assumes *f measurable_on S*
  shows (*λx. f x / c*) *measurable_on S*
proof (*cases c=0*)
  case *False*
  then show *?thesis*
    using *measurable_on_cmul* [*of f S 1/c*]
    by (*simp add*: *assms*)
qed *auto*

lemma *measurable_on_minus*:
   f measurable_on S ⟹ (*λx.* −(*f x*)) *measurable_on S*
  using *measurable_on_scaleR_const* [*of f S −1*] **by** *auto*

lemma *continuous_imp_measurable_on*:
   continuous_on UNIV f ⟹ *f measurable_on UNIV*
  unfolding *measurable_on_def*
  apply (*rule_tac x*={} **in** *exI*)
  apply (*rule_tac x*=*λn. f* **in** *exI*, *auto*)
  done

proposition *integrable_subintervals_imp_measurable*:
  fixes *f* :: *'a::euclidean_space ⇒ 'b::euclidean_space*
  assumes ⋀*a b. f integrable_on cbox a b*
  shows *f measurable_on UNIV*
proof −
  define *BOX* **where** *BOX* ≡ *λh. λx::'a. cbox x* (*x* + *h* $*_R$ *One*)
  define *i* **where** *i* ≡ *λh x. integral* (*BOX h x*) *f* $/_R$ *h* ˆ *DIM*(*'a*)
  obtain *N* **where** *negligible N*
    and *k*: ⋀*x e.* ⟦*x* ∉ *N*; *0* < *e*⟧
          ⟹ ∃ *d*>*0*. ∀ *h*. *0* < *h* ∧ *h* < *d* ⟶
                norm (*integral* (*cbox x* (*x* + *h* $*_R$ *One*)) *f* $/_R$ *h* ˆ *DIM*(*'a*) − *f x*)
< *e*
    using *integrable_ccontinuous_explicit assms* **by** *blast*
  show *?thesis*
    unfolding *measurable_on_def*
  proof (*intro exI conjI allI impI*)
    show *continuous_on UNIV* ((*λn x. i* (*inverse*(*Suc n*)) *x*) *n*) **for** *n*

```
  proof (clarsimp simp: continuous_on_iff)
    show ∃d>0. ∀x'. dist x' x < d ⟶ dist (i (inverse (1 + real n)) x') (i
(inverse (1 + real n)) x) < e
      if 0 < e
      for x e
    proof −
      let ?e = e / (1 + real n) ^ DIM('a)
      have ?e > 0
        using ‹e > 0› by auto
      moreover have x ∈ cbox (x − 2 *R One) (x + 2 *R One)
        by (simp add: mem_box inner_diff_left inner_left_distrib)
      moreover have x + One /R real (Suc n) ∈ cbox (x − 2 *R One) (x + 2
*R One)
        by (auto simp: mem_box inner_diff_left inner_left_distrib field_simps)
      ultimately obtain δ where δ > 0
        and δ: ⋀c' d'. ⟦c' ∈ cbox (x − 2 *R One) (x + 2 *R One);
                         d' ∈ cbox (x − 2 *R One) (x + 2 *R One);
                         norm(c' − x) ≤ δ; norm(d' − (x + One /R Suc n)) ≤ δ⟧
                        ⟹ norm(integral(cbox c' d') f − integral(cbox x (x + One
/R Suc n)) f) < ?e
        by (blast intro: indefinite_integral_continuous [of f _ _ x] assms)
      show ?thesis
      proof (intro exI impI conjI allI)
        show min δ 1 > 0
          using ‹δ > 0› by auto
        show dist (i (inverse (1 + real n)) y) (i (inverse (1 + real n)) x) < e
          if dist y x < min δ 1 for y
        proof −
          have no: norm (y − x) < 1
            using that by (auto simp: dist_norm)
          have le1: inverse (1 + real n) ≤ 1
            by (auto simp: field_split_simps)
          have norm (integral (cbox y (y + One /R real (Suc n))) f
              − integral (cbox x (x + One /R real (Suc n))) f)
              < e / (1 + real n) ^ DIM('a)
          proof (rule δ)
            show y ∈ cbox (x − 2 *R One) (x + 2 *R One)
              using no by (auto simp: mem_box algebra_simps dest: Basis_le_norm
[of _ y−x])
             show y + One /R real (Suc n) ∈ cbox (x − 2 *R One) (x + 2 *R
One)
            proof (simp add: dist_norm mem_box algebra_simps, intro ballI conjI)
              fix i::'a
              assume i ∈ Basis
              then have 1: |y · i − x · i| < 1
                by (metis inner_commute inner_diff_right no norm_bound_Basis_lt)
              moreover have ... < (2 + inverse (1 + real n)) 1 ≤ 2 − inverse
(1 + real n)
                by (auto simp: field_simps)
```

ultimately show *x* · *i* ≤ *y* · *i* + (*2* + *inverse* (*1* + *real n*))
y · *i* + *inverse* (*1* + *real n*) ≤ *x* · *i* + *2*
by *linarith+*
qed
show *norm* (*y* − *x*) ≤ δ *norm* (*y* + *One* $/_R$ *real* (*Suc n*) − (*x* + *One* $/_R$ *real* (*Suc n*))) ≤ δ
using *that* **by** (*auto simp*: *dist_norm*)
qed
then show *?thesis*
using *that* **by** (*simp add*: *dist_norm i_def BOX_def flip*: *scaleR_diff_right*) (*simp add*: *field_simps*)
qed
qed
qed
qed
show *negligible N*
by (*simp add*: ‹*negligible N*›)
show (λ*n*. *i* (*inverse* (*Suc n*)) *x*) ⟶ (*if x* ∈ *UNIV then f x else 0*)
if *x* ∉ *N* **for** *x*
unfolding *lim_sequentially*
proof *clarsimp*
show ∃ *no*. ∀ *n*≥*no*. *dist* (*i* (*inverse* (*1* + *real n*)) *x*) (*f x*) < *e*
if *0* < *e* **for** *e*
proof −
obtain *d* **where** *d* > *0*
and *d*: ⋀*h*. ⟦*0* < *h*; *h* < *d*⟧ ⟹
norm (*integral* (*cbox x* (*x* + *h* $*_R$ *One*)) *f* $/_R$ *h* ˆ *DIM*(′*a*) − *f x*) < *e*
using *k* [*of x e*] ‹*x* ∉ *N*› ‹*0* < *e*› **by** *blast*
then obtain *M* **where** *M*: *M* ≠ *0 0* < *inverse* (*real M*) *inverse* (*real M*) < *d*
using *real_arch_invD* **by** *auto*
show *?thesis*
proof (*intro exI allI impI*)
show *dist* (*i* (*inverse* (*1* + *real n*)) *x*) (*f x*) < *e*
if *M* ≤ *n* **for** *n*
proof −
have ∗: *0* < *inverse* (*1* + *real n*) *inverse* (*1* + *real n*) ≤ *inverse M*
using *that* ‹*M* ≠ *0*› **by** *auto*
show *?thesis*
using *that M*
apply (*simp add*: *i_def BOX_def dist_norm*)
apply (*blast intro*: *le_less_trans* ∗ *d*)
done
qed
qed
qed
qed
qed
qed

6.29.4 Composing continuous and measurable functions; a few variants

lemma *measurable_on_compose_continuous*:
  assumes *f*: *f measurable_on UNIV* **and** *g*: *continuous_on UNIV g*
  shows $(g \circ f)$ *measurable_on UNIV*
proof −
  obtain *N* **and** *F*
    where *negligible N*
      and *conF*: $\bigwedge n$. *continuous_on UNIV* $(F\ n)$
      and *tendsF*: $\bigwedge x.\ x \notin N \Longrightarrow (\lambda n.\ F\ n\ x) \longrightarrow f\ x$
    using *f* **by** (*auto simp*: *measurable_on_def*)
  show *?thesis*
    unfolding *measurable_on_def*
  proof (*intro exI conjI allI impI*)
    show *negligible N*
      by *fact*
    show *continuous_on UNIV* $(g \circ (F\ n))$ **for** *n*
      using *conF continuous_on_compose continuous_on_subset g* **by** *blast*
    show $(\lambda n.\ (g \circ F\ n)\ x) \longrightarrow$ (*if* $x \in$ *UNIV then* $(g \circ f)\ x$ *else 0*)
      if $x \notin N$ **for** $x :: 'a$
     using *that g tendsF* **by** (*auto simp*: *continuous_on_def intro*: *tendsto_compose*)
  qed
qed

lemma *measurable_on_compose_continuous_0*:
  assumes *f*: *f measurable_on S* **and** *g*: *continuous_on UNIV g* **and** $g\ 0 = 0$
  shows $(g \circ f)$ *measurable_on S*
proof −
  have f': $(\lambda x.$ *if* $x \in S$ *then f x else 0*) *measurable_on UNIV*
    using *f measurable_on_UNIV* **by** *blast*
  show *?thesis*
    using *measurable_on_compose_continuous* [*OF f' g*]
    by (*simp add*: *measurable_on_UNIV o_def if_distrib* ‹$g\ 0 = 0$› *cong*: *if_cong*)
qed

lemma *measurable_on_compose_continuous_box*:
  assumes *fm*: *f measurable_on UNIV* **and** *fab*: $\bigwedge x.\ f\ x \in$ *box a b*
    and *contg*: *continuous_on* (*box a b*) *g*
  shows $(g \circ f)$ *measurable_on UNIV*
proof −
   have $\exists \gamma.\ (\forall n.$ *continuous_on UNIV* $(\gamma\ n)) \wedge (\forall x.\ x \notin N \longrightarrow (\lambda n.\ \gamma\ n\ x)$
$\longrightarrow g\ (f\ x))$
    if *negligible N*
      and *conth* [*rule_format*]: $\forall n.$ *continuous_on UNIV* $(\lambda x.\ h\ n\ x)$
      and *tends* [*rule_format*]: $\forall x.\ x \notin N \longrightarrow (\lambda n.\ h\ n\ x) \longrightarrow f\ x$
    for *N* **and** $h :: nat \Rightarrow 'a \Rightarrow 'b$
  proof −
   define ϑ **where** $\vartheta \equiv \lambda n\ x.\ (\sum i{\in}Basis.\ (max\ (a{\cdot}i + (b{\cdot}i - a{\cdot}i)\ /\ real\ (n{+}2))$

$(min\ ((h\ n\ x){\cdot}i)$
$(b{\cdot}i - (b{\cdot}i - a{\cdot}i)\ /\ real\ (n{+}2)))) *_R\ i)$

have *aibi*: $\bigwedge i.\ i \in Basis \Longrightarrow a \cdot i < b \cdot i$
using *box_ne_empty*(*2*) *fab* **by** *auto*
then have ∗: $\bigwedge i\ n.\ i \in Basis \Longrightarrow a \cdot i + real\ n * (a \cdot i) < b \cdot i + real\ n * (b \cdot i)$
by (*meson add_mono_thms_linordered_field*(*3*) *less_eq_real_def mult_left_mono of_nat_0_le_iff*)
show *?thesis*
proof (*intro exI conjI allI impI*)
show *continuous_on UNIV* $(g \circ (\vartheta\ n))$ **for** *n* :: *nat*
unfolding *ϑ_def*
apply (*intro continuous_on_compose2* [*OF contg*] *continuous_intros conth*)
apply (*auto simp*: *aibi* ∗ *mem_box less_max_iff_disj min_less_iff_disj field_split_simps*)
done
show $(\lambda n.\ (g \circ \vartheta\ n)\ x) \longrightarrow g\ (f\ x)$
if $x \notin N$ **for** *x*
unfolding *o_def*
proof (*rule isCont_tendsto_compose* [**where** *g*=*g*])
show *isCont g* (*f x*)
using *contg fab continuous_on_eq_continuous_at* **by** *blast*
have $(\lambda n.\ \vartheta\ n\ x) \longrightarrow (\sum i{\in}Basis.\ max\ (a \cdot i)\ (min\ (f\ x \cdot i)\ (b \cdot i)) *_R i)$
unfolding *ϑ_def*
proof (*intro tendsto_intros* ‹$x \notin N$› *tends*)
fix *i*::*'b*
assume $i \in Basis$
have *a*: $(\lambda n.\ a \cdot i + (b \cdot i - a \cdot i)\ /\ real\ n) \longrightarrow a{\cdot}i + 0$
by (*intro tendsto_add lim_const_over_n tendsto_const*)
show $(\lambda n.\ a \cdot i + (b \cdot i - a \cdot i)\ /\ real\ (n + 2)) \longrightarrow a \cdot i$
using *LIMSEQ_ignore_initial_segment* [**where** *k*=*2*, *OF a*] **by** *simp*
have *b*: $(\lambda n.\ b{\cdot}i - (b \cdot i - a \cdot i)\ /\ (real\ n)) \longrightarrow b{\cdot}i - 0$
by (*intro tendsto_diff lim_const_over_n tendsto_const*)
show $(\lambda n.\ b \cdot i - (b \cdot i - a \cdot i)\ /\ real\ (n + 2)) \longrightarrow b \cdot i$
using *LIMSEQ_ignore_initial_segment* [**where** *k*=*2*, *OF b*] **by** *simp*
qed
also have $(\sum i{\in}Basis.\ max\ (a \cdot i)\ (min\ (f\ x \cdot i)\ (b \cdot i)) *_R i) = (\sum i{\in}Basis.\ (f\ x \cdot i) *_R i)$
apply (*rule sum.cong*)
using *fab*
apply *auto*
apply (*intro order_antisym*)
apply (*auto simp*: *mem_box*)
using *less_imp_le* **apply** *blast*
by (*metis* (*full_types*) *linear max_less_iff_conj min.bounded_iff not_le*)
also have $\ldots = f\ x$
using *euclidean_representation* **by** *blast*
finally show $(\lambda n.\ \vartheta\ n\ x) \longrightarrow f\ x$ **.**
qed

```
    qed
  qed
  then show ?thesis
    using fm by (auto simp: measurable_on_def)
qed

lemma measurable_on_Pair:
  assumes f: f measurable_on S and g: g measurable_on S
  shows (λx. (f x, g x)) measurable_on S
proof –
  obtain NF and F
    where NF: negligible NF
      and conF: ⋀n. continuous_on UNIV (F n)
      and tendsF: ⋀x. x ∉ NF ⟹ (λn. F n x) ⟶ (if x ∈ S then f x else 0)
    using f by (auto simp: measurable_on_def)
  obtain NG and G
    where NG: negligible NG
      and conG: ⋀n. continuous_on UNIV (G n)
      and tendsG: ⋀x. x ∉ NG ⟹ (λn. G n x) ⟶ (if x ∈ S then g x else 0)
    using g by (auto simp: measurable_on_def)
  show ?thesis
    unfolding measurable_on_def
  proof (intro exI conjI allI impI)
    show negligible (NF ∪ NG)
      by (simp add: NF NG)
    show continuous_on UNIV (λx. (F n x, G n x)) for n
      using conF conG continuous_on_Pair by blast
    show (λn. (F n x, G n x)) ⟶ (if x ∈ S then (f x, g x) else 0)
      if x ∉ NF ∪ NG for x
      using tendsto_Pair [OF tendsF tendsG, of x x] that unfolding zero_prod_def
      by (simp add: split: if_split_asm)
  qed
qed

lemma measurable_on_combine:
  assumes f: f measurable_on S and g: g measurable_on S
    and h: continuous_on UNIV (λx. h (fst x) (snd x)) and h 0 0 = 0
  shows (λx. h (f x) (g x)) measurable_on S
proof –
  have *: (λx. h (f x) (g x)) = (λx. h (fst x) (snd x)) ∘ (λx. (f x, g x))
    by auto
  show ?thesis
      unfolding * by (auto simp: measurable_on_compose_continuous_0 measur-
able_on_Pair assms)
qed

lemma measurable_on_add:
  assumes f: f measurable_on S and g: g measurable_on S
  shows (λx. f x + g x) measurable_on S
```

by (*intro continuous_intros measurable_on_combine* [*OF assms*]) *auto*

lemma *measurable_on_diff*:
assumes *f*: *f measurable_on S* **and** *g*: *g measurable_on S*
shows (*λx. f x − g x*) *measurable_on S*
by (*intro continuous_intros measurable_on_combine* [*OF assms*]) *auto*

lemma *measurable_on_scaleR*:
assumes *f*: *f measurable_on S* **and** *g*: *g measurable_on S*
shows (*λx. f x* $*_R$ *g x*) *measurable_on S*
by (*intro continuous_intros measurable_on_combine* [*OF assms*]) *auto*

lemma *measurable_on_sum*:
assumes *finite I* ⋀*i. i ∈ I* ⟹ *f i measurable_on S*
shows (*λx. sum* (*λi. f i x*) *I*) *measurable_on S*
using *assms* **by** (*induction I*) (*auto simp*: *measurable_on_add*)

lemma *measurable_on_spike*:
assumes *f*: *f measurable_on T* **and** *negligible S* **and** *gf*: ⋀*x. x ∈ T − S* ⟹ *g x* = *f x*
shows *g measurable_on T*
proof −
obtain *NF* **and** *F*
where *NF*: *negligible NF*
and *conF*: ⋀*n. continuous_on UNIV* (*F n*)
and *tendsF*: ⋀*x. x ∉ NF* ⟹ (*λn. F n x*) ⟶ (*if x ∈ T then f x else 0*)
using *f* **by** (*auto simp*: *measurable_on_def*)
show *?thesis*
unfolding *measurable_on_def*
proof (*intro exI conjI allI impI*)
show *negligible* (*NF ∪ S*)
by (*simp add*: *NF* ‹*negligible S*›)
show ⋀*x. x ∉ NF ∪ S* ⟹ (*λn. F n x*) ⟶ (*if x ∈ T then g x else 0*)
by (*metis* (*full_types*) *Diff_iff Un_iff gf tendsF*)
qed (*auto simp*: *conF*)
qed

proposition *indicator_measurable_on*:
assumes *S ∈ sets lebesgue*
shows *indicat_real S measurable_on UNIV*
proof −
{ **fix** *n*::*nat*
let *?ε* = (*1*::*real*) / (*2* ∗ *2*^*n*)
have *ε*: *?ε* > *0*
by *auto*
obtain *T* **where** *closed T T ⊆ S S−T ∈ lmeasurable* **and** *ST*: *emeasure lebesgue* (*S − T*) < *?ε*
by (*meson ε assms sets_lebesgue_inner_closed*)
obtain *U* **where** *open U S ⊆ U* (*U − S*) *∈ lmeasurable* **and** *US*: *emeasure*

lebesgue (*U* − *S*) < *?ε*
  by (*meson ε assms sets_lebesgue_outer_open*)
 have *eq*: −*T* ∩ *U* = (*S*−*T*) ∪ (*U* − *S*)
  using ‹*T* ⊆ *S*› ‹*S* ⊆ *U*› **by** *auto*
 have *emeasure lebesgue* ((*S*−*T*) ∪ (*U* − *S*)) ≤ *emeasure lebesgue* (*S* − *T*) + *emeasure lebesgue* (*U* − *S*)
  using ‹*S* − *T* ∈ *lmeasurable*› ‹*U* − *S* ∈ *lmeasurable*› *emeasure_subadditive* **by** *blast*
 also have ... < *?ε* + *?ε*
  using *ST US add_mono_ennreal* **by** *metis*
 finally have *le*: *emeasure lebesgue* (−*T* ∩ *U*) < *ennreal* (*1* / *2*ˆ*n*)
  by (*simp add*: *eq*)
 have *1*: *continuous_on* (*T* ∪ −*U*) (*indicat_real S*)
  unfolding *indicator_def*
 proof (*rule continuous_on_cases* [*OF* ‹*closed T*›])
  show *closed* (− *U*)
   using ‹*open U*› **by** *blast*
  show *continuous_on T* (λ*x*. *1*::*real*) *continuous_on* (− *U*) (λ*x*. *0*::*real*)
   by (*auto simp*: *continuous_on*)
  show ∀ *x*. *x* ∈ *T* ∧ *x* ∉ *S* ∨ *x* ∈ − *U* ∧ *x* ∈ *S* ⟶ (*1*::*real*) = *0*
   using ‹*T* ⊆ *S*› ‹*S* ⊆ *U*› **by** *auto*
 qed
 have *2*: *closedin* (*top_of_set UNIV*) (*T* ∪ −*U*)
  using ‹*closed T*› ‹*open U*› **by** *auto*
 obtain *g* **where** *continuous_on UNIV g* ⋀*x*. *x* ∈ *T* ∪ −*U* ⟹ *g x* = *indicat_real S x* ⋀*x*. *norm*(*g x*) ≤ *1*
  by (*rule Tietze* [*OF 1 2*, *of 1*]) *auto*
 with *le* **have** ∃ *g E*. *continuous_on UNIV g* ∧ (∀ *x* ∈ −*E*. *g x* = *indicat_real S x*) ∧
   (∀ *x*. *norm*(*g x*) ≤ *1*) ∧ *E* ∈ *sets lebesgue* ∧ *emeasure lebesgue E* < *ennreal* (*1* / *2*ˆ*n*)
  apply (*rule_tac x*=*g* **in** *exI*)
  apply (*rule_tac x*=−*T* ∩ *U* **in** *exI*)
  using ‹*S* − *T* ∈ *lmeasurable*› ‹*U* − *S* ∈ *lmeasurable*› *eq* **by** *auto*
}
then obtain *g E* **where** *cont*: ⋀*n*. *continuous_on UNIV* (*g n*)
 and *geq*: ⋀*n x*. *x* ∈ − *E n* ⟹ *g n x* = *indicat_real S x*
 and *ng1*: ⋀*n x*. *norm*(*g n x*) ≤ *1*
 and *Eset*: ⋀*n*. *E n* ∈ *sets lebesgue*
 and *Em*: ⋀*n*. *emeasure lebesgue* (*E n*) < *ennreal* (*1* / *2*ˆ*n*)
 by *metis*
have *null*: *limsup E* ∈ *null_sets lebesgue*
proof (*rule borel_cantelli_limsup1* [*OF Eset*])
 show *emeasure lebesgue* (*E n*) < ∞ **for** *n*
  by (*metis Em infinity_ennreal_def order.asym top.not_eq_extremum*)
 show *summable* (λ*n*. *measure lebesgue* (*E n*))
 proof (*rule summable_comparison_test'* [*OF summable_geometric*, *of 1/2 0*])
  show *norm* (*measure lebesgue* (*E n*)) ≤ (*1*/*2*) ˆ *n* **for** *n*
   using *Em* [*of n*] **by** (*simp add*: *measure_def enn2real_leI power_one_over*)

qed *auto*
qed
have *tends*: $(\lambda n.\ g\ n\ x) \longrightarrow$ *indicat_real S x* **if** $x \notin$ *limsup E* **for** *x*
proof −
have $\forall_F$ *n in sequentially*. $x \in -$ *E n*
using *that* **by** (*simp add*: *mem_limsup_iff not_frequently*)
then show *?thesis*
unfolding *tendsto_iff dist_real_def*
by (*simp add*: *eventually_mono geq*)
qed
show *?thesis*
unfolding *measurable_on_def*
proof (*intro exI conjI allI impI*)
show *negligible* (*limsup E*)
using *negligible_iff_null_sets null* **by** *blast*
show *continuous_on UNIV* (*g n*) **for** *n*
using *cont* **by** *blast*
qed (*use tends* **in** *auto*)
qed

lemma *measurable_on_restrict*:
assumes *f*: *f measurable_on UNIV* **and** *S*: $S \in$ *sets lebesgue*
shows ($\lambda x.$ *if* $x \in S$ *then f x else 0*) *measurable_on UNIV*
proof −
have *indicat_real S measurable_on UNIV*
by (*simp add*: *S indicator_measurable_on*)
then show *?thesis*
using *measurable_on_scaleR* [*OF _ f*, *of indicat_real S*]
by (*simp add*: *indicator_scaleR_eq_if*)
qed

lemma *measurable_on_const_UNIV*: ($\lambda x.\ k$) *measurable_on UNIV*
by (*simp add*: *continuous_imp_measurable_on*)

lemma *measurable_on_const* [*simp*]: $S \in$ *sets lebesgue* $\Longrightarrow$ ($\lambda x.\ k$) *measurable_on S*
using *measurable_on_UNIV measurable_on_const_UNIV measurable_on_restrict* **by** *blast*

lemma *simple_function_indicator_representation_real*:
fixes *f* ::*'a* $\Rightarrow$ *real*
assumes *f*: *simple_function M f* **and** *x*: $x \in$ *space M* **and** *nn*: $\bigwedge x.\ f\ x \geq 0$
shows $f\ x = (\sum y \in f$ ` *space M*. $y *$ *indicator* ($f$ −` $\{y\} \cap$ *space M*) *x*)
proof −
have *f'*: *simple_function M* (*ennreal* $\circ$ *f*)
by (*simp add*: *f*)
have $*$: $f\ x =$
enn2real
($\sum y \in$ *ennreal* ` *f* ` *space M*.

$y * indicator ((ennreal \circ f) -` \{y\} \cap space\ M)\ x)$
using *arg_cong* [*OF simple_function_indicator_representation* [*OF f' x*], *of enn2real*, *simplified nn o_def*] *nn*
unfolding *o_def image_comp*
by (*metis enn2real_ennreal*)
have *enn2real* ($\sum y \in ennreal$ ` $f$ ` *space M*. *if ennreal* ($f\ x$) = $y \wedge x \in$ *space M then y else 0*)
= *sum* (*enn2real* $\circ$ (λy. *if ennreal* ($f\ x$) = $y \wedge x \in$ *space M then y else 0*))
(*ennreal* ` $f$ ` *space M*)
by (*rule enn2real_sum*) *auto*
also have ... = *sum* (*enn2real* $\circ$ (λy. *if ennreal* ($f\ x$) = $y \wedge x \in$ *space M then y else 0*) $\circ$ *ennreal*)
(f ` *space M*)
by (*rule sum.reindex*) (*use nn* **in** ‹*auto simp*: *inj_on_def intro*: *sum.cong*›)
also have ... = ($\sum y \in f$ ` *space M*. *if f x* = $y \wedge x \in$ *space M then y else 0*)
using *nn*
by (*auto simp*: *inj_on_def intro*: *sum.cong*)
finally show *?thesis*
by (*subst* $*$) (*simp add*: *enn2real_sum indicator_def if_distrib cong*: *if_cong*)
qed

lemma *simple_function_induct_real*
[*consumes 1*, *case_names cong set mult add*, *induct set*: *simple_function*]:
fixes $u :: 'a \Rightarrow real$
assumes *u*: *simple_function M u*
assumes *cong*: $\bigwedge f\ g$. *simple_function M f* $\Longrightarrow$ *simple_function M g* $\Longrightarrow$ (*AE x in M*. $f\ x = g\ x$) $\Longrightarrow P\ f \Longrightarrow P\ g$
assumes *set*: $\bigwedge A$. $A \in$ *sets M* $\Longrightarrow P$ (*indicator A*)
assumes *mult*: $\bigwedge u\ c.\ P\ u \Longrightarrow P\ (\lambda x.\ c * u\ x)$
assumes *add*: $\bigwedge u\ v.\ P\ u \Longrightarrow P\ v \Longrightarrow P\ (\lambda x.\ u\ x + v\ x)$
and *nn*: $\bigwedge x.\ u\ x \geq 0$
shows $P\ u$
proof (*rule cong*)
from *AE_space* **show** *AE x in M*. ($\sum y \in u$ ` *space M*. $y *$ *indicator* ($u$ -` $\{y\} \cap$ *space M*) x) = $u\ x$
proof *eventually_elim*
fix x **assume** x: $x \in$ *space M*
from *simple_function_indicator_representation_real*[*OF u x*] *nn*
show ($\sum y \in u$ ` *space M*. $y *$ *indicator* ($u$ -` $\{y\} \cap$ *space M*) x) = $u\ x$
by *metis*
qed
next
from *u* **have** *finite* (u ` *space M*)
unfolding *simple_function_def* **by** *auto*
then show P (λx. $\sum y \in u$ ` *space M*. $y *$ *indicator* ($u$ -` $\{y\} \cap$ *space M*) x)
proof *induct*
case *empty*
then show *?case*
using *set*[*of* {}] **by** (*simp add*: *indicator_def*[*abs_def*])

next
case (*insert a F*)
have *eq*: $\sum$ {*y*. *u x* = *y* ∧ (*y* = *a* ∨ *y* ∈ *F*) ∧ *x* ∈ *space M*}
= (*if u x* = *a* ∧ *x* ∈ *space M then a else 0*) + $\sum$ {*y*. *u x* = *y* ∧ *y* ∈ *F* ∧ *x* ∈ *space M*} **for** *x*
proof (*cases x* ∈ *space M*)
case *True*
have ∗: {*y*. *u x* = *y* ∧ (*y* = *a* ∨ *y* ∈ *F*)} = {*y*. *u x* = *a* ∧ *y* = *a*} ∪ {*y*. *u x* = *y* ∧ *y* ∈ *F*}
by *auto*
show *?thesis*
using *insert* **by** (*simp add*: ∗ *True*)
qed *auto*
have *a*: *P* (λ*x*. *a* ∗ *indicator* (*u* −' {*a*} ∩ *space M*) *x*)
proof (*intro mult set*)
show *u* −' {*a*} ∩ *space M* ∈ *sets M*
using *u* **by** *auto*
qed
show *?case*
using *nn insert a*
by (*simp add*: *eq indicator_times_eq_if* [**where** *f* = λ*x*. *a*] *add*)
qed
next
show *simple_function M* (λ*x*. ($\sum$ *y*∈*u* ' *space M*. *y* ∗ *indicator* (*u* −' {*y*} ∩ *space M*) *x*))
apply (*subst simple_function_cong*)
apply (*rule simple_function_indicator_representation_real*[*symmetric*])
apply (*auto intro*: *u nn*)
done
qed *fact*

proposition *simple_function_measurable_on_UNIV*:
fixes *f* :: '*a*::*euclidean_space* ⇒ *real*
assumes *f*: *simple_function lebesgue f* **and** *nn*: ⋀*x*. *f x* ≥ *0*
shows *f measurable_on UNIV*
using *f*
proof (*induction f*)
case (*cong f g*)
then obtain *N* **where** *negligible N* {*x*. *g x* ≠ *f x*} ⊆ *N*
by (*auto simp*: *eventually_ae_filter_negligible eq_commute*)
then show *?case*
by (*blast intro*: *measurable_on_spike cong*)
next
case (*set S*)
then show *?case*
by (*simp add*: *indicator_measurable_on*)
next
case (*mult u c*)
then show *?case*

    by (simp add: measurable_on_cmul)
  case (add u v)
  then show ?case
    by (simp add: measurable_on_add)
qed (auto simp: nn)

lemma *simple_function_lebesgue_if*:
  fixes f :: 'a::euclidean_space ⇒ real
  assumes f: simple_function lebesgue f **and** S: $S \in$ sets lebesgue
  shows simple_function lebesgue (λx. if $x \in S$ then $f\ x$ else 0)
proof −
  have ffin: finite (range f) **and** fsets: ∀ x. f −‘ {$f\ x$} ∈ sets lebesgue
    using f **by** (auto simp: simple_function_def)
  have finite (f ‘ S)
    by (meson finite_subset subset_image_iff ffin top_greatest)
  moreover have finite ((λx. 0::real) ‘ T) **for** T :: 'a set
    by (auto simp: image_def)
  moreover have if_sets: (λx. if $x \in S$ then $f\ x$ else 0) −‘ {$f\ a$} ∈ sets lebesgue
for a
  proof −
    have ∗: (λx. if $x \in S$ then $f\ x$ else 0) −‘ {$f\ a$}
          = (if $f\ a = 0$ then −S ∪ f −‘ {$f\ a$} else (f −‘ {$f\ a$}) ∩ S)
      by (auto simp: split: if_split_asm)
    show ?thesis
      unfolding ∗ **by** (metis Compl_in_sets_lebesgue S sets.Int sets.Un fsets)
  qed
  moreover have (λx. if $x \in S$ then $f\ x$ else 0) −‘ {0} ∈ sets lebesgue
  proof (cases 0 ∈ range f)
    case True
    then show ?thesis
      by (metis (no_types, lifting) if_sets rangeE)
  next
    case False
    then have (λx. if $x \in S$ then $f\ x$ else 0) −‘ {0} = −S
      by auto
    then show ?thesis
      by (simp add: Compl_in_sets_lebesgue S)
  qed
  ultimately show ?thesis
    by (auto simp: simple_function_def)
qed

corollary *simple_function_measurable_on*:
  fixes f :: 'a::euclidean_space ⇒ real
  assumes f: simple_function lebesgue f **and** nn: ⋀x. $f\ x \geq 0$ **and** S: $S \in$ sets lebesgue
  shows f measurable_on S
  by (simp add: measurable_on_UNIV [symmetric, of f] S f simple_function_lebesgue_if nn simple_function_measurable_on_UNIV)

lemma
fixes *f* :: *'a::euclidean_space* ⇒ *'b::ordered_euclidean_space*
assumes *f*: *f measurable_on S* **and** *g*: *g measurable_on S*
shows *measurable_on_sup*: (λ*x*. *sup* (*f x*) (*g x*)) *measurable_on S*
and *measurable_on_inf*: (λ*x*. *inf* (*f x*) (*g x*)) *measurable_on S*
proof −
obtain *NF* **and** *F*
where *NF*: *negligible NF*
and *conF*: ⋀*n*. *continuous_on UNIV* (*F n*)
and *tendsF*: ⋀*x*. *x* ∉ *NF* ⟹ (λ*n*. *F n x*) ⟶ (*if x* ∈ *S then f x else 0*)
using *f* **by** (*auto simp*: *measurable_on_def*)
obtain *NG* **and** *G*
where *NG*: *negligible NG*
and *conG*: ⋀*n*. *continuous_on UNIV* (*G n*)
and *tendsG*: ⋀*x*. *x* ∉ *NG* ⟹ (λ*n*. *G n x*) ⟶ (*if x* ∈ *S then g x else 0*)
using *g* **by** (*auto simp*: *measurable_on_def*)
show (λ*x*. *sup* (*f x*) (*g x*)) *measurable_on S*
unfolding *measurable_on_def*
proof (*intro exI conjI allI impI*)
show *continuous_on UNIV* (λ*x*. *sup* (*F n x*) (*G n x*)) **for** *n*
unfolding *sup_max eucl_sup* **by** (*intro conF conG continuous_intros*)
show (λ*n*. *sup* (*F n x*) (*G n x*)) ⟶ (*if x* ∈ *S then sup* (*f x*) (*g x*) *else 0*)
if *x* ∉ *NF* ∪ *NG* **for** *x*
using *tendsto_sup* [*OF tendsF tendsG*, *of x x*] *that* **by** *auto*
qed (*simp add*: *NF NG*)
show (λ*x*. *inf* (*f x*) (*g x*)) *measurable_on S*
unfolding *measurable_on_def*
proof (*intro exI conjI allI impI*)
show *continuous_on UNIV* (λ*x*. *inf* (*F n x*) (*G n x*)) **for** *n*
unfolding *inf_min eucl_inf* **by** (*intro conF conG continuous_intros*)
show (λ*n*. *inf* (*F n x*) (*G n x*)) ⟶ (*if x* ∈ *S then inf* (*f x*) (*g x*) *else 0*)
if *x* ∉ *NF* ∪ *NG* **for** *x*
using *tendsto_inf* [*OF tendsF tendsG*, *of x x*] *that* **by** *auto*
qed (*simp add*: *NF NG*)
qed

proposition *measurable_on_componentwise_UNIV*:
f measurable_on UNIV ⟷ (∀ *i*∈*Basis*. (λ*x*. (*f x* · *i*) $*_R$ *i*) *measurable_on UNIV*)
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
show *?rhs*
proof
fix *i*::*'b*
assume *i* ∈ *Basis*
have *cont*: *continuous_on UNIV* (λ*x*. (*x* · *i*) $*_R$ *i*)
by (*intro continuous_intros*)
show (λ*x*. (*f x* · *i*) $*_R$ *i*) *measurable_on UNIV*

using *measurable_on_compose_continuous* [*OF L cont*]
by (*simp add*: *o_def*)
qed
next
assume *?rhs*
then have $\exists N\ g.\ negligible\ N \wedge$
$(\forall n.\ continuous\_on\ UNIV\ (g\ n)) \wedge$
$(\forall x.\ x \notin N \longrightarrow (\lambda n.\ g\ n\ x) \longrightarrow (f\ x \cdot i) *_R\ i)$
if $i \in Basis$ **for** i
by (*simp add*: *measurable_on_def that*)
then obtain $N\ g$ **where** N: $\bigwedge i.\ i \in Basis \Longrightarrow negligible\ (N\ i)$
and *cont*: $\bigwedge i\ n.\ i \in Basis \Longrightarrow continuous\_on\ UNIV\ (g\ i\ n)$
and *tends*: $\bigwedge i\ x.\ [\![i \in Basis;\ x \notin N\ i]\!] \Longrightarrow (\lambda n.\ g\ i\ n\ x) \longrightarrow (f\ x \cdot i) *_R$
i
by *metis*
show *?lhs*
unfolding *measurable_on_def*
proof (*intro exI conjI allI impI*)
show *negligible* $(\bigcup i \in Basis.\ N\ i)$
using *N eucl.finite_Basis* **by** *blast*
show *continuous_on UNIV* $(\lambda x.\ (\sum i{\in}Basis.\ g\ i\ n\ x))$ **for** n
by (*intro continuous_intros cont*)
next
fix x
assume $x \notin (\bigcup i \in Basis.\ N\ i)$
then have $\bigwedge i.\ i \in Basis \Longrightarrow x \notin N\ i$
by *auto*
then have $(\lambda n.\ (\sum i{\in}Basis.\ g\ i\ n\ x)) \longrightarrow (\sum i{\in}Basis.\ (f\ x \cdot i) *_R\ i)$
by (*intro tends tendsto_intros*)
then show $(\lambda n.\ (\sum i{\in}Basis.\ g\ i\ n\ x)) \longrightarrow (if\ x \in UNIV\ then\ f\ x\ else\ 0)$
by (*simp add*: *euclidean_representation*)
qed
qed

corollary *measurable_on_componentwise*:
f *measurable_on* $S \longleftrightarrow (\forall i{\in}Basis.\ (\lambda x.\ (f\ x \cdot i) *_R\ i)$ *measurable_on* $S)$
apply (*subst measurable_on_UNIV* [*symmetric*])
apply (*subst measurable_on_componentwise_UNIV*)
apply (*simp add*: *measurable_on_UNIV if_distrib* [*of* $\lambda x.$ *inner* x _] *if_distrib* [*of*
$\lambda x.$ *scaleR* x _] *cong*: *if_cong*)
done

lemma *borel_measurable_implies_simple_function_sequence_real*:
fixes u :: $'a \Rightarrow real$
assumes u[*measurable*]: $u \in borel\_measurable\ M$ **and** *nn*: $\bigwedge x.\ u\ x \geq 0$
shows $\exists f.\ incseq\ f \wedge (\forall i.\ simple\_function\ M\ (f\ i)) \wedge (\forall x.\ bdd\_above\ (range\ (\lambda i.$
$f\ i\ x))) \wedge$

$(\forall i\ x.\ 0 \le f\ i\ x) \wedge u = (SUP\ i.\ f\ i)$

proof –
define *f* **where** [*abs_def*]:
f i x = *real_of_int* (*floor* ((*min i* (*u x*)) ∗ *2^i*)) / *2^i* **for** *i x*

have [*simp*]: *0* ≤ *f i x* **for** *i x*
by (*auto simp*: *f_def intro*!: *divide_nonneg_nonneg mult_nonneg_nonneg nn*)

have ∗: *2^n* ∗ *real_of_int x* = *real_of_int* (*2^n* ∗ *x*) **for** *n x*
by *simp*

have *real_of_int* ⌊*real i* ∗ *2* ^ *i*⌋ = *real_of_int* ⌊*i* ∗ *2* ^ *i*⌋ **for** *i*
by (*intro arg_cong*[**where** *f*=*real_of_int*]) *simp*
then have [*simp*]: *real_of_int* ⌊*real i* ∗ *2* ^ *i*⌋ = *i* ∗ *2* ^ *i* **for** *i*
unfolding *floor_of_nat* **by** *simp*

have *bdd*: *bdd_above* (*range* (λ*i*. *f i x*)) **for** *x*
by (*rule bdd_aboveI* [**where** *M* = *u x*]) (*auto simp*: *f_def field_simps min_def*)

have *incseq f*
proof (*intro monoI le_funI*)
fix *m n* :: *nat* **and** *x* **assume** *m* ≤ *n*
moreover
{ **fix** *d* :: *nat*
have ⌊*2^d*::*real*⌋ ∗ ⌊*2^m* ∗ (*min* (*of_nat m*) (*u x*))⌋ ≤ ⌊*2^d* ∗ (*2^m* ∗ (*min* (*of_nat m*) (*u x*)))⌋
by (*rule le_mult_floor*) (*auto simp*: *nn*)
also have ... ≤ ⌊*2^d* ∗ (*2^m* ∗ (*min* (*of_nat d* + *of_nat m*) (*u x*)))⌋
by (*intro floor_mono mult_mono min.mono*)
(*auto simp*: *nn min_less_iff_disj of_nat_less_top*)
finally have *f m x* ≤ *f*(*m* + *d*) *x*
unfolding *f_def*
by (*auto simp*: *field_simps power_add* ∗ *simp del*: *of_int_mult*) }
ultimately show *f m x* ≤ *f n x*
by (*auto simp*: *le_iff_add*)
qed
then have *inc_f*: *incseq* (λ*i*. *f i x*) **for** *x*
by (*auto simp*: *incseq_def le_fun_def*)
moreover
have *simple_function M* (*f i*) **for** *i*
proof (*rule simple_function_borel_measurable*)
have ⌊(*min* (*of_nat i*) (*u x*)) ∗ *2* ^ *i*⌋ ≤ ⌊*int i* ∗ *2* ^ *i*⌋ **for** *x*
by (*auto split*: *split_min intro*!: *floor_mono*)
then have *f i* ‘ *space M* ⊆ (λ*n*. *real_of_int n* / *2^i*) ‘ {*0* .. *of_nat i* ∗ *2^i*}
unfolding *floor_of_int* **by** (*auto simp*: *f_def nn intro*!: *imageI*)
then show *finite* (*f i* ‘ *space M*)
by (*rule finite_subset*) *auto*
show *f i* ∈ *borel_measurable M*
unfolding *f_def enn2real_def* **by** *measurable*

```
  qed
  moreover
  { fix x
    have (SUP i. (f i x)) = u x
    proof –
      obtain n where u x ≤ of_nat n using real_arch_simple by auto
      then have min_eq_r: ∀F i in sequentially. min (real i) (u x) = u x
        by (auto simp: eventually_sequentially intro!: exI[of _ n] split: split_min)
      have (λi. real_of_int ⌊min (real i) (u x) ∗ 2^i⌋ / 2^i) ⟶ u x
      proof (rule tendsto_sandwich)
        show (λn. u x − (1/2)^n) ⟶ u x
          by (auto intro!: tendsto_eq_intros LIMSEQ_power_zero)
        show ∀F n in sequentially. real_of_int ⌊min (real n) (u x) ∗ 2 ^ n⌋ / 2 ^ n
≤ u x
          using min_eq_r by eventually_elim (auto simp: field_simps)
        have ∗: u x ∗ (2 ^ n ∗ 2 ^ n) ≤ 2^n + 2^n ∗ real_of_int ⌊u x ∗ 2 ^ n⌋ for n
          using real_of_int_floor_ge_diff_one[of u x ∗ 2^n, THEN mult_left_mono, of
2^n]
          by (auto simp: field_simps)
        show ∀F n in sequentially. u x − (1/2)^n ≤ real_of_int ⌊min (real n) (u
x) ∗ 2 ^ n⌋ / 2 ^ n
          using min_eq_r by eventually_elim (insert ∗, auto simp: field_simps)
      qed auto
      then have (λi. (f i x)) ⟶ u x
        by (simp add: f_def)
      from LIMSEQ_unique LIMSEQ_incseq_SUP [OF bdd inc_f] this
      show ?thesis
        by blast
    qed }
  ultimately show ?thesis
    by (intro exI [of _ λi x. f i x]) (auto simp: ‹incseq f› bdd image_comp)
qed


lemma homeomorphic_open_interval_UNIV:
  fixes a b:: real
  assumes a < b
  shows {a<..<b} homeomorphic (UNIV::real set)
proof –
  have {a<..<b} = ball ((b+a) / 2) ((b−a) / 2)
    using assms
    by (auto simp: dist_real_def abs_if field_split_simps split: if_split_asm)
  then show ?thesis
    by (simp add: homeomorphic_ball_UNIV assms)
qed

proposition homeomorphic_box_UNIV:
  fixes a b:: 'a::euclidean_space
  assumes box a b ≠ {}
```

shows *box a b homeomorphic* (*UNIV*::'*a set*)
proof −
have {*a* · *i* <..<*b* · *i*} *homeomorphic* (*UNIV*::*real set*) **if** *i* ∈ *Basis* **for** *i*
using *assms box_ne_empty that* **by** (*blast intro*: *homeomorphic_open_interval_UNIV*)
then have ∃*f g*. (∀ *x*. *a* · *i* < *x* ∧ *x* < *b* · *i* ⟶ *g* (*f x*) = *x*) ∧
(∀ *y*. *a* · *i* < *g y* ∧ *g y* < *b* · *i* ∧ *f*(*g y*) = *y*) ∧
continuous_on {*a* · *i*<..<*b* · *i*} *f* ∧
continuous_on (*UNIV*::*real set*) *g*
if *i* ∈ *Basis* **for** *i*
using *that* **by** (*auto simp*: *homeomorphic_minimal mem_box Ball_def*)
then obtain *f g* **where** *gf*: ⋀*i x*. ⟦*i* ∈ *Basis*; *a* · *i* < *x*; *x* < *b* · *i*⟧ ⟹ *g i* (*f i x*) = *x*
and *fg*: ⋀*i y*. *i* ∈ *Basis* ⟹ *a* · *i* < *g i y* ∧ *g i y* < *b* · *i* ∧ *f i* (*g i y*) = *y*
and *contf*: ⋀*i*. *i* ∈ *Basis* ⟹ *continuous_on* {*a* · *i*<..<*b* · *i*} (*f i*)
and *contg*: ⋀*i*. *i* ∈ *Basis* ⟹ *continuous_on* (*UNIV*::*real set*) (*g i*)
by *metis*
define *F* **where** *F* ≡ λ*x*. ∑*i*∈*Basis*. (*f i* (*x* · *i*)) $*_R$ *i*
define *G* **where** *G* ≡ λ*x*. ∑*i*∈*Basis*. (*g i* (*x* · *i*)) $*_R$ *i*
show *?thesis*
unfolding *homeomorphic_minimal*
proof (*intro exI conjI ballI*)
show *G y* ∈ *box a b* **for** *y*
using *fg* **by** (*simp add*: *G_def mem_box*)
show *G* (*F x*) = *x* **if** *x* ∈ *box a b* **for** *x*
using *that* **by** (*simp add*: *F_def G_def gf mem_box euclidean_representation*)
show *F* (*G y*) = *y* **for** *y*
by (*simp add*: *F_def G_def fg mem_box euclidean_representation*)
show *continuous_on* (*box a b*) *F*
unfolding *F_def*
proof (*intro continuous_intros continuous_on_compose2* [*OF contf continuous_on_inner*])
show (λ*x*. *x* · *i*) ‘ *box a b* ⊆ {*a* · *i*<..<*b* · *i*} **if** *i* ∈ *Basis* **for** *i*
using *that* **by** (*auto simp*: *mem_box*)
qed
show *continuous_on UNIV G*
unfolding *G_def*
by (*intro continuous_intros continuous_on_compose2* [*OF contg continuous_on_inner*]) *auto*
qed *auto*
qed

lemma *diff_null_sets_lebesgue*: ⟦*N* ∈ *null_sets* (*lebesgue_on S*); *X*−*N* ∈ *sets* (*lebesgue_on S*); *N* ⊆ *X*⟧
⟹ *X* ∈ *sets* (*lebesgue_on S*)
by (*metis Int_Diff_Un inf.commute inf.orderE null_setsD2 sets.Un*)

lemma *borel_measurable_diff_null*:
  fixes f :: *'a::euclidean_space ⇒ 'b::euclidean_space*
  assumes N: $N \in$ *null_sets* (*lebesgue_on* S) **and** S: $S \in$ *sets lebesgue*
  shows $f \in$ *borel_measurable* (*lebesgue_on* $(S-N)$) $\longleftrightarrow f \in$ *borel_measurable* (*lebesgue_on* S)
  unfolding *in_borel_measurable space_lebesgue_on sets_restrict_UNIV*
proof (*intro ball_cong iffI*)
  show f −' $T \cap S \in$ *sets* (*lebesgue_on* S)
    if f −' $T \cap (S-N) \in$ *sets* (*lebesgue_on* $(S-N)$) **for** T
  proof −
    have $N \cap S = N$
      by (*metis N S inf.orderE null_sets_restrict_space*)
    moreover have $N \cap S \in$ *sets lebesgue*
      by (*metis N S inf.orderE null_setsD2 null_sets_restrict_space*)
    moreover have f −' $T \cap S \cap (f$ −' $T \cap N) \in$ *sets lebesgue*
      by (*metis N S completion.complete inf.absorb2 inf_le2 inf_mono null_sets_restrict_space*)
    ultimately show *?thesis*
      by (*metis Diff_Int_distrib Int_Diff_Un S inf_le2 sets.Diff sets.Un sets_restrict_space_iff space_lebesgue_on space_restrict_space that*)
  qed
  show f −' $T \cap (S-N) \in$ *sets* (*lebesgue_on* $(S-N)$)
    if f −' $T \cap S \in$ *sets* (*lebesgue_on* S) **for** T
  proof −
    have $(S - N) \cap f$ −' $T = (S - N) \cap (f$ −' $T \cap S)$
      by *blast*
    then have $(S - N) \cap f$ −' $T \in$ *sets.restricted_space lebesgue* $(S - N)$
      by (*metis S image_iff sets.Int_space_eq2 sets_restrict_space_iff that*)
    then show *?thesis*
      by (*simp add*: *inf.commute sets_restrict_space*)
  qed
qed *auto*

lemma *lebesgue_measurable_diff_null*:
  fixes f :: *'a::euclidean_space ⇒ 'b::euclidean_space*
  assumes $N \in$ *null_sets lebesgue*
  shows $f \in$ *borel_measurable* (*lebesgue_on* $(-N)$) $\longleftrightarrow f \in$ *borel_measurable lebesgue*
  by (*simp add*: *Compl_eq_Diff_UNIV assms borel_measurable_diff_null lebesgue_on_UNIV_eq*)

proposition *measurable_on_imp_borel_measurable_lebesgue_UNIV*:
  fixes f :: *'a::euclidean_space ⇒ 'b::euclidean_space*
  assumes f *measurable_on UNIV*
  shows $f \in$ *borel_measurable lebesgue*
proof −
  obtain N **and** F
    where *NF*: *negligible* N
      and *conF*: $\bigwedge n.$ *continuous_on UNIV* $(F\ n)$
      and *tendsF*: $\bigwedge x.\ x \notin N \Longrightarrow (\lambda n.\ F\ n\ x) \longlongrightarrow f\ x$

```
    using assms by (auto simp: measurable_on_def)
  obtain N where N ∈ null_sets lebesgue f ∈ borel_measurable (lebesgue_on (−N))
  proof
    show f ∈ borel_measurable (lebesgue_on (− N))
    proof (rule borel_measurable_LIMSEQ_metric)
      show F i ∈ borel_measurable (lebesgue_on (− N)) for i
      by (meson Compl_in_sets_lebesgue NF conF continuous_imp_measurable_on_sets_lebesgue
continuous_on_subset negligible_imp_sets subset_UNIV)
      show (λi. F i x) ⟶ f x if x ∈ space (lebesgue_on (− N)) for x
        using that
        by (simp add: tendsF)
    qed
    show N ∈ null_sets lebesgue
      using NF negligible_iff_null_sets by blast
  qed
  then show ?thesis
    using lebesgue_measurable_diff_null by blast
qed

corollary measurable_on_imp_borel_measurable_lebesgue:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes f measurable_on S and S: S ∈ sets lebesgue
  shows f ∈ borel_measurable (lebesgue_on S)
proof −
  have (λx. if x ∈ S then f x else 0) measurable_on UNIV
    using assms(1) measurable_on_UNIV by blast
  then show ?thesis
   by (simp add: borel_measurable_if_D measurable_on_imp_borel_measurable_lebesgue_UNIV)
qed


proposition measurable_on_limit:
  fixes f :: nat ⇒ 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes f: ⋀n. f n measurable_on S and N: negligible N
    and lim: ⋀x. x ∈ S − N ⟹ (λn. f n x) ⟶ g x
  shows g measurable_on S
proof −
  have box (0::'b) One homeomorphic (UNIV::'b set)
    by (simp add: homeomorphic_box_UNIV)
  then obtain h h':: 'b⇒'b where hh': ⋀x. x ∈ box 0 One ⟹ h (h' x) = x
                and h'im:  h' ` box 0 One = UNIV
                and conth: continuous_on UNIV h
                and conth': continuous_on (box 0 One) h'
                and h'h:   ⋀y. h' (h y) = y
                and rangeh: range h = box 0 One
    by (auto simp: homeomorphic_def homeomorphism_def)
  have norm y ≤ DIM('b) if y: y ∈ box 0 One for y::'b
  proof −
    have y01: 0 < y · i y · i < 1 if i ∈ Basis for i
```

```
      using that y by (auto simp: mem_box)
    have norm y ≤ (∑i∈Basis. |y · i|)
      using norm_le_l1 by blast
    also have ... ≤ (∑i::'b∈Basis. 1)
    proof (rule sum_mono)
      show |y · i| ≤ 1 if i ∈ Basis for i
        using y01 that by fastforce
    qed
    also have ... ≤ DIM('b)
      by auto
    finally show ?thesis .
  qed
  then have norm_le: norm(h y) ≤ DIM('b) for y
    by (metis UNIV_I image_eqI rangeh)
  have (h' ∘ (h ∘ (λx. if x ∈ S then g x else 0))) measurable_on UNIV
  proof (rule measurable_on_compose_continuous_box)
    let ?χ =  h ∘ (λx. if x ∈ S then g x else 0)
    let ?f = λn. h ∘ (λx. if x ∈ S then f n x else 0)
    show ?χ measurable_on UNIV
    proof (rule integrable_subintervals_imp_measurable)
      show ?χ integrable_on cbox a b for a b
      proof (rule integrable_spike_set)
        show ?χ integrable_on (cbox a b − N)
        proof (rule dominated_convergence_integrable)
          show const: (λx. DIM('b)) integrable_on cbox a b − N
              by (simp add: N has_integral_iff integrable_const integrable_negligible
integrable_setdiff negligible_diff)
          show norm ((h ∘ (λx. if x ∈ S then g x else 0)) x) ≤ DIM('b) if x ∈ cbox
a b − N for x
            using that norm_le  by (simp add: o_def)
          show (λk. ?f k x) ⟶ ?χ x if x ∈ cbox a b − N for x
            using that lim [of x] conth
            by (auto simp: continuous_on_def intro: tendsto_compose)
          show (?f n) absolutely_integrable_on cbox a b − N for n
          proof (rule measurable_bounded_by_integrable_imp_absolutely_integrable)
            show ?f n ∈ borel_measurable (lebesgue_on (cbox a b − N))
            proof (rule measurable_on_imp_borel_measurable_lebesgue [OF measur-
able_on_spike_set])
              show ?f n measurable_on cbox a b
                unfolding measurable_on_UNIV [symmetric, of _ cbox a b]
              proof (rule measurable_on_restrict)
                have f': (λx. if x ∈ S then f n x else 0) measurable_on UNIV
                  by (simp add: f measurable_on_UNIV)
                show ?f n measurable_on UNIV
                  using measurable_on_compose_continuous [OF f' conth] by auto
              qed auto
              show negligible (sym_diff (cbox a b) (cbox a b − N))
                by (auto intro: negligible_subset [OF N])
              show cbox a b − N ∈ sets lebesgue
```

```
            by (simp add: N negligible_imp_sets sets.Diff)
          qed
          show cbox a b − N ∈ sets lebesgue
            by (simp add: N negligible_imp_sets sets.Diff)
          show norm (?f n x) ≤ DIM('b)
            if x ∈ cbox a b − N for x
            using that local.norm_le by simp
        qed (auto simp: const)
      qed
      show negligible {x ∈ cbox a b − N − cbox a b. ?χ x ≠ 0}
        by (auto simp: empty_imp_negligible)
      have {x ∈ cbox a b − (cbox a b − N). ?χ x ≠ 0} ⊆ N
        by auto
      then show negligible {x ∈ cbox a b − (cbox a b − N). ?χ x ≠ 0}
        using N negligible_subset by blast
    qed
  qed
  show ?χ x ∈ box 0 One for x
    using rangeh by auto
  show continuous_on (box 0 One) h'
    by (rule conth')
 qed
 then show ?thesis
  by (simp add: o_def h'h measurable_on_UNIV)
qed
```

```
lemma measurable_on_if_simple_function_limit:
 fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
 shows ⟦⋀n. g n measurable_on UNIV; ⋀n. finite (range (g n)); ⋀x. (λn. g n
x) ⟶ f x⟧
  ⟹ f measurable_on UNIV
 by (force intro: measurable_on_limit [where N={}])
```

```
lemma lebesgue_measurable_imp_measurable_on_nnreal_UNIV:
 fixes u :: 'a::euclidean_space ⇒ real
 assumes u: u ∈ borel_measurable lebesgue and nn: ⋀x. u x ≥ 0
 shows u measurable_on UNIV
proof −
 obtain f where incseq f and f: ∀ i. simple_function lebesgue (f i)
  and bdd: ⋀x. bdd_above (range (λi. f i x))
  and nnf: ⋀i x. 0 ≤ f i x and ∗: u = (SUP i. f i)
  using borel_measurable_implies_simple_function_sequence_real nn u by metis
 show ?thesis
  unfolding ∗
 proof (rule measurable_on_if_simple_function_limit [of concl: Sup (range f)])
  show (f i) measurable_on UNIV for i
   by (simp add: f nnf simple_function_measurable_on_UNIV)
```

```
    show finite (range (f i)) for i
      by (metis f simple_function_def space_borel space_completion space_lborel)
    show (λi. f i x) ⟶ Sup (range f) x for x
    proof −
      have incseq (λi. f i x)
        using ‹incseq f› apply (auto simp: incseq_def)
        by (simp add: le_funD)
      then show ?thesis
        by (metis SUP_apply bdd LIMSEQ_incseq_SUP)
    qed
  qed
qed

lemma lebesgue_measurable_imp_measurable_on_nnreal:
  fixes u :: 'a::euclidean_space ⇒ real
  assumes u ∈ borel_measurable lebesgue ⋀x. u x ≥ 0S ∈ sets lebesgue
  shows u measurable_on S
  unfolding measurable_on_UNIV [symmetric, of u]
  using assms
  by (auto intro: lebesgue_measurable_imp_measurable_on_nnreal_UNIV)

lemma lebesgue_measurable_imp_measurable_on_real:
  fixes u :: 'a::euclidean_space ⇒ real
  assumes u: u ∈ borel_measurable lebesgue and S: S ∈ sets lebesgue
  shows u measurable_on S
proof −
  let ?f = λx. |u x| + u x
  let ?g = λx. |u x| − u x
  have ?f measurable_on S ?g measurable_on S
    using S u by (auto intro: lebesgue_measurable_imp_measurable_on_nnreal)
  then have (λx. (?f x − ?g x) / 2) measurable_on S
    using measurable_on_cdivide measurable_on_diff by blast
  then show ?thesis
    by auto
qed


proposition lebesgue_measurable_imp_measurable_on:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes f: f ∈ borel_measurable lebesgue and S: S ∈ sets lebesgue
  shows f measurable_on S
  unfolding measurable_on_componentwise [of f]
proof
  fix i::'b
  assume i ∈ Basis
  have (λx. (f x · i)) ∈ borel_measurable lebesgue
    using ‹i ∈ Basis› borel_measurable_euclidean_space f by blast
  then have (λx. (f x · i)) measurable_on S
    using S lebesgue_measurable_imp_measurable_on_real by blast
```

then show ($\lambda x.$ ($f\ x \cdot i$) $*_R$ i) *measurable_on S*
by (*intro measurable_on_scaleR measurable_on_const S*)
qed

proposition *measurable_on_iff_borel_measurable*:
fixes f :: *'a::euclidean_space* $\Rightarrow$ *'b::euclidean_space*
assumes $S \in$ *sets lebesgue*
shows *f measurable_on S* $\longleftrightarrow$ $f \in$ *borel_measurable* (*lebesgue_on S*) (**is** *?lhs* = *?rhs*)
proof
show $f \in$ *borel_measurable* (*lebesgue_on S*)
if *f measurable_on S*
using *that* **by** (*simp add*: *assms measurable_on_imp_borel_measurable_lebesgue*)
next
assume $f \in$ *borel_measurable* (*lebesgue_on S*)
then have ($\lambda a.$ *if* $a \in S$ *then f a else 0*) *measurable_on UNIV*
by (*simp add*: *assms borel_measurable_if lebesgue_measurable_imp_measurable_on*)
then show *f measurable_on S*
using *measurable_on_UNIV* **by** *blast*
qed

6.29.5 Measurability on generalisations of the binary product

lemma *measurable_on_bilinear*:
fixes h :: *'a::euclidean_space* $\Rightarrow$ *'b::euclidean_space* $\Rightarrow$ *'c::euclidean_space*
assumes *h*: *bilinear h* **and** *f*: *f measurable_on S* **and** *g*: *g measurable_on S*
shows ($\lambda x.$ *h* (*f x*) (*g x*)) *measurable_on S*
proof (*rule measurable_on_combine* [**where** $h = h$])
show *continuous_on UNIV* ($\lambda x.$ *h* (*fst x*) (*snd x*))
by (*simp add*: *bilinear_continuous_on_compose* [*OF continuous_on_fst continuous_on_snd h*])
show *h 0 0* = *0*
by (*simp add*: *bilinear_lzero h*)
qed (*auto intro*: *assms*)

lemma *borel_measurable_bilinear*:
fixes h :: *'a::euclidean_space* $\Rightarrow$ *'b::euclidean_space* $\Rightarrow$ *'c::euclidean_space*
assumes *bilinear h* $f \in$ *borel_measurable* (*lebesgue_on S*) $g \in$ *borel_measurable* (*lebesgue_on S*)
and *S*: $S \in$ *sets lebesgue*
shows ($\lambda x.$ *h* (*f x*) (*g x*)) $\in$ *borel_measurable* (*lebesgue_on S*)
using *assms measurable_on_bilinear* [*of h f S g*]
by (*simp flip*: *measurable_on_iff_borel_measurable*)

lemma *absolutely_integrable_bounded_measurable_product*:
fixes h :: *'a::euclidean_space* $\Rightarrow$ *'b::euclidean_space* $\Rightarrow$ *'c::euclidean_space*
assumes *bilinear h* **and** *f*: $f \in$ *borel_measurable* (*lebesgue_on S*) $S \in$ *sets lebesgue*
and *bou*: *bounded* (*f ‘ S*) **and** *g*: *g absolutely_integrable_on S*

shows (λx. h (f x) (g x)) absolutely_integrable_on S
proof –
obtain B **where** B > 0 **and** B: ⋀x y. norm (h x y) ≤ B ∗ norm x ∗ norm y
using bilinear_bounded_pos ‹bilinear h› **by** blast
obtain C **where** C > 0 **and** C: ⋀x. x ∈ S ⟹ norm (f x) ≤ C
using bounded_pos **by** (metis bou imageI)
show ?thesis
proof (rule measurable_bounded_by_integrable_imp_absolutely_integrable [OF _ ‹S ∈ sets lebesgue›])
show norm (h (f x) (g x)) ≤ B ∗ C ∗ norm(g x) **if** x ∈ S **for** x
by (meson less_le mult_left_mono mult_right_mono norm_ge_zero order_trans that ‹B > 0› B C)
show (λx. h (f x) (g x)) ∈ borel_measurable (lebesgue_on S)
using ‹bilinear h› f g
by (blast intro: borel_measurable_bilinear dest: absolutely_integrable_measurable)
show (λx. B ∗ C ∗ norm(g x)) integrable_on S
using ‹0 < B› ‹0 < C› absolutely_integrable_on_def g **by** auto
qed
qed

lemma absolutely_integrable_bounded_measurable_product_real:
fixes f :: real ⇒ real
assumes f ∈ borel_measurable (lebesgue_on S) S ∈ sets lebesgue
and bounded (f ‘ S) **and** g absolutely_integrable_on S
shows (λx. f x ∗ g x) absolutely_integrable_on S
using absolutely_integrable_bounded_measurable_product bilinear_times assms **by** blast

lemma borel_measurable_AE:
fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
assumes f ∈ borel_measurable lebesgue **and** ae: AE x in lebesgue. f x = g x
shows g ∈ borel_measurable lebesgue
proof –
obtain N **where** N: N ∈ null_sets lebesgue ⋀x. x ∉ N ⟹ f x = g x
using ae **unfolding** completion.AE_iff_null_sets **by** auto
have f measurable_on UNIV
by (simp add: assms lebesgue_measurable_imp_measurable_on)
then have g measurable_on UNIV
by (metis Diff_iff N measurable_on_spike negligible_iff_null_sets)
then show ?thesis
using measurable_on_imp_borel_measurable_lebesgue_UNIV **by** blast
qed

lemma has_bochner_integral_combine:
fixes f :: real ⇒ 'a::euclidean_space
assumes a ≤ c c ≤ b
and ac: has_bochner_integral (lebesgue_on {a..c}) f i
and cb: has_bochner_integral (lebesgue_on {c..b}) f j

shows *has_bochner_integral* (*lebesgue_on* {*a..b*}) *f* (*i* + *j*)
proof −
have *i*: *has_bochner_integral lebesgue* (λ*x*. *indicator* {*a..c*} *x* $*_R$ *f x*) *i*
and *j*: *has_bochner_integral lebesgue* (λ*x*. *indicator* {*c..b*} *x* $*_R$ *f x*) *j*
using *assms* **by** (*auto simp*: *has_bochner_integral_restrict_space*)
have *AE*: *AE x in lebesgue*. *indicat_real* {*a..c*} *x* $*_R$ *f x* + *indicat_real* {*c..b*} *x* $*_R$ *f x* = *indicat_real* {*a..b*} *x* $*_R$ *f x*
proof (*rule AE_I'*)
have *eq*: *indicat_real* {*a..c*} *x* $*_R$ *f x* + *indicat_real* {*c..b*} *x* $*_R$ *f x* = *indicat_real* {*a..b*} *x* $*_R$ *f x* **if** *x* ≠ *c* **for** *x*
using *assms that* **by** (*auto simp*: *indicator_def*)
then show {*x* ∈ *space lebesgue*. *indicat_real* {*a..c*} *x* $*_R$ *f x* + *indicat_real* {*c..b*} *x* $*_R$ *f x* ≠ *indicat_real* {*a..b*} *x* $*_R$ *f x*} ⊆ {*c*}
by *auto*
qed *auto*
have *has_bochner_integral lebesgue* (λ*x*. *indicator* {*a..b*} *x* $*_R$ *f x*) (*i* + *j*)
proof (*rule has_bochner_integralI_AE* [*OF has_bochner_integral_add* [*OF i j*] _ *AE*])
have *eq*: *indicat_real* {*a..c*} *x* $*_R$ *f x* + *indicat_real* {*c..b*} *x* $*_R$ *f x* = *indicat_real* {*a..b*} *x* $*_R$ *f x* **if** *x* ≠ *c* **for** *x*
using *assms that* **by** (*auto simp*: *indicator_def*)
show (λ*x*. *indicat_real* {*a..b*} *x* $*_R$ *f x*) ∈ *borel_measurable lebesgue*
proof (*rule borel_measurable_AE* [*OF borel_measurable_add AE*])
show (λ*x*. *indicator* {*a..c*} *x* $*_R$ *f x*) ∈ *borel_measurable lebesgue*
(λ*x*. *indicator* {*c..b*} *x* $*_R$ *f x*) ∈ *borel_measurable lebesgue*
using *i j* **by** *auto*
qed
qed
then show *?thesis*
by (*simp add*: *has_bochner_integral_restrict_space*)
qed

lemma *integrable_combine*:
fixes *f* :: *real* ⇒ *'a::euclidean_space*
assumes *integrable* (*lebesgue_on* {*a..c*}) *f integrable* (*lebesgue_on* {*c..b*}) *f*
and *a* ≤ *c c* ≤ *b*
shows *integrable* (*lebesgue_on* {*a..b*}) *f*
using *assms has_bochner_integral_combine has_bochner_integral_iff* **by** *blast*

lemma *integral_combine*:
fixes *f* :: *real* ⇒ *'a::euclidean_space*
assumes *f*: *integrable* (*lebesgue_on* {*a..b*}) *f* **and** *a* ≤ *c c* ≤ *b*
shows $integral^L$ (*lebesgue_on* {*a..b*}) *f* = $integral^L$ (*lebesgue_on* {*a..c*}) *f* + $integral^L$ (*lebesgue_on* {*c..b*}) *f*
proof −
have *i*: *has_bochner_integral* (*lebesgue_on* {*a..c*}) *f* ($integral^L$ (*lebesgue_on* {*a..c*}) *f*)
using *integrable_subinterval* ‹*c* ≤ *b*› *f has_bochner_integral_iff* **by** *fastforce*
have *j*: *has_bochner_integral* (*lebesgue_on* {*c..b*}) *f* ($integral^L$ (*lebesgue_on* {*c..b*})

```
f)
    using integrable_subinterval ‹a ≤ c› f has_bochner_integral_iff by fastforce
  show ?thesis
    by (meson ‹a ≤ c› ‹c ≤ b› has_bochner_integral_combine has_bochner_integral_iff
i j)
qed

lemma has_bochner_integral_null [intro]:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes N ∈ null_sets lebesgue
  shows has_bochner_integral (lebesgue_on N) f 0
  unfolding has_bochner_integral_iff — strange that the proof's so long
proof
  show integrable (lebesgue_on N) f
  proof (subst integrable_restrict_space)
    show N ∩ space lebesgue ∈ sets lebesgue
      using assms by force
    show integrable lebesgue (λx. indicat_real N x *R f x)
    proof (rule integrable_cong_AE_imp)
      show integrable lebesgue (λx. 0)
        by simp
      show *: AE x in lebesgue. 0 = indicat_real N x *R f x
        using assms
        by (simp add: indicator_def completion.null_sets_iff_AE eventually_mono)
      show (λx. indicat_real N x *R f x) ∈ borel_measurable lebesgue
        by (auto intro: borel_measurable_AE [OF _ *])
    qed
  qed
  show integral^L (lebesgue_on N) f = 0
  proof (rule integral_eq_zero_AE)
    show AE x in lebesgue_on N. f x = 0
     by (rule AE_I' [where N=N]) (auto simp: assms null_setsD2 null_sets_restrict_space)
  qed
qed

lemma has_bochner_integral_null_eq[simp]:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes N ∈ null_sets lebesgue
  shows has_bochner_integral (lebesgue_on N) f i ⟷ i = 0
  using assms has_bochner_integral_eq by blast

end
```

6.30 Embedding Measure Spaces with a Function

```
theory Embed_Measure
imports Binary_Product_Measure
begin
```

Given a measure space on some carrier set Ω and a function *f*, we can define a push-forward measure on the carrier set $f(\Omega)$ whose σ-algebra is the one generated by mapping f over the original sigma algebra.

This is useful e. g. when f is injective, i. e. it is some kind of "tagging" function. For instance, suppose we have some algebraaic datatype of values with various constructors, including a constructor *RealVal* for real numbers. Then *embed_measure* allows us to lift a measure on real numbers to the appropriate subset of that algebraic datatype.

definition *embed_measure* :: *'a measure* ⇒ (*'a* ⇒ *'b*) ⇒ *'b measure* **where**
  embed_measure M f = *measure_of* (*f* ' *space M*) {*f* ' *A* |*A*. *A* ∈ *sets M*}
    (λ*A*. *emeasure M* (*f* −' *A* ∩ *space M*))

lemma *space_embed_measure*: *space* (*embed_measure M f*) = *f* ' *space M*
  unfolding *embed_measure_def*
  by (*subst space_measure_of*) (*auto dest*: *sets.sets_into_space*)

lemma *sets_embed_measure'*:
  assumes *inj*: *inj_on f* (*space M*)
  shows *sets* (*embed_measure M f*) = {*f* ' *A* |*A*. *A* ∈ *sets M*}
  unfolding *embed_measure_def*
proof (*intro sigma_algebra.sets_measure_of_eq sigma_algebra_iff2*[*THEN iffD2*] *conjI allI ballI impI*)
  fix *s* **assume** *s* ∈ {*f* ' *A* |*A*. *A* ∈ *sets M*}
  then obtain *s'* **where** *s'_props*: *s* = *f* ' *s'* *s'* ∈ *sets M* **by** *auto*
  hence *f* ' *space M* − *s* = *f* ' (*space M* − *s'*) **using** *inj*
    by (*auto dest*: *inj_onD sets.sets_into_space*)
  also have ... ∈ {*f* ' *A* |*A*. *A* ∈ *sets M*} **using** *s'_props* **by** *auto*
  finally show *f* ' *space M* − *s* ∈ {*f* ' *A* |*A*. *A* ∈ *sets M*} **.**
next
  fix *A* :: *nat* ⇒ _ **assume** *range A* ⊆ {*f* ' *A* |*A*. *A* ∈ *sets M*}
  then obtain *A'* **where** *A'*: ⋀*i*. *A i* = *f* ' *A' i* ⋀*i*. *A' i* ∈ *sets M*
    by (*auto simp*: *subset_eq choice_iff*)
  then have (⋃*x*. *f* ' *A' x*) = *f* ' (⋃*x*. *A' x*) **by** *blast*
  with *A'* **show** (⋃*i*. *A i*) ∈ {*f* ' *A* |*A*. *A* ∈ *sets M*}
    by *simp blast*
qed (*auto dest*: *sets.sets_into_space*)

lemma *the_inv_into_vimage*:
  inj_on f X ⟹ *A* ⊆ *X* ⟹ *the_inv_into X f* −' *A* ∩ (*f*'*X*) = *f* ' *A*
  by (*auto simp*: *the_inv_into_f_f*)

lemma *sets_embed_eq_vimage_algebra*:
  assumes *inj_on f* (*space M*)
  shows *sets* (*embed_measure M f*) = *sets* (*vimage_algebra* (*f*'*space M*) (*the_inv_into* (*space M*) *f*) *M*)
  by (*auto simp*: *sets_embed_measure'*[*OF assms*] *Pi_iff the_inv_into_f_f assms sets_vimage_algebra2 Setcompr_eq_image*
    dest: *sets.sets_into_space*

```
    intro!: image_cong the_inv_into_vimage[symmetric])

lemma sets_embed_measure:
  assumes inj: inj f
  shows sets (embed_measure M f) = {f ` A |A. A ∈ sets M}
  using assms by (subst sets_embed_measure') (auto intro!: inj_onI dest: injD)

lemma in_sets_embed_measure: A ∈ sets M ⟹ f ` A ∈ sets (embed_measure M f)
  unfolding embed_measure_def
  by (intro in_measure_of) (auto dest: sets.sets_into_space)

lemma measurable_embed_measure1:
  assumes g: (λx. g (f x)) ∈ measurable M N
  shows g ∈ measurable (embed_measure M f) N
  unfolding measurable_def
proof safe
  fix A assume A ∈ sets N
  with g have (λx. g (f x)) −` A ∩ space M ∈ sets M
    by (rule measurable_sets)
  then have f ` ((λx. g (f x)) −` A ∩ space M) ∈ sets (embed_measure M f)
    by (rule in_sets_embed_measure)
  also have f ` ((λx. g (f x)) −` A ∩ space M) = g −` A ∩ space (embed_measure
M f)
    by (auto simp: space_embed_measure)
  finally show g −` A ∩ space (embed_measure M f) ∈ sets (embed_measure M f)
.
qed (insert measurable_space[OF assms], auto simp: space_embed_measure)

lemma measurable_embed_measure2':
  assumes inj_on f (space M)
  shows f ∈ measurable M (embed_measure M f)
proof−
  {
    fix A assume A: A ∈ sets M
    also from A have A = A ∩ space M by auto
    also have ... = f −` f ` A ∩ space M using A assms
      by (auto dest: inj_onD sets.sets_into_space)
    finally have f −` f ` A ∩ space M ∈ sets M .
  }
  thus ?thesis using assms unfolding embed_measure_def
    by (intro measurable_measure_of) (auto dest: sets.sets_into_space)
qed

lemma measurable_embed_measure2:
  assumes [simp]: inj f shows f ∈ measurable M (embed_measure M f)
  by (auto simp: inj_vimage_image_eq embed_measure_def
        intro!: measurable_measure_of dest: sets.sets_into_space)

lemma embed_measure_eq_distr':
```

assumes *inj_on f* (*space M*)
shows *embed_measure M f* = *distr M* (*embed_measure M f*) *f*
proof−
have *distr M* (*embed_measure M f*) *f* =
measure_of (*f* ‘ *space M*) {*f* ‘ *A* |*A*. *A* ∈ *sets M*}
(λ*A*. *emeasure M* (*f* −‘ *A* ∩ *space M*)) **unfolding** *distr_def*
by (*simp add*: *space_embed_measure sets_embed_measure′*[*OF assms*])
also have ... = *embed_measure M f* **unfolding** *embed_measure_def* **..**
finally show *?thesis* **..**
qed

lemma *embed_measure_eq_distr*:
inj f ⟹ *embed_measure M f* = *distr M* (*embed_measure M f*) *f*
by (*rule embed_measure_eq_distr′*) (*auto intro*!: *inj_onI dest*: *injD*)

lemma *nn_integral_embed_measure′*:
inj_on f (*space M*) ⟹ *g* ∈ *borel_measurable* (*embed_measure M f*) ⟹
nn_integral (*embed_measure M f*) *g* = *nn_integral M* (λ*x*. *g* (*f x*))
apply (*subst embed_measure_eq_distr′*, *simp*)
apply (*subst nn_integral_distr*)
apply (*simp_all add*: *measurable_embed_measure2′*)
done

lemma *nn_integral_embed_measure*:
inj f ⟹ *g* ∈ *borel_measurable* (*embed_measure M f*) ⟹
nn_integral (*embed_measure M f*) *g* = *nn_integral M* (λ*x*. *g* (*f x*))
by(*erule nn_integral_embed_measure′*[*OF subset_inj_on*]) *simp*

lemma *emeasure_embed_measure′*:
assumes *inj_on f* (*space M*) *A* ∈ *sets* (*embed_measure M f*)
shows *emeasure* (*embed_measure M f*) *A* = *emeasure M* (*f* −‘ *A* ∩ *space M*)
by (*subst embed_measure_eq_distr′*[*OF assms*(*1*)])
(*simp add*: *emeasure_distr*[*OF measurable_embed_measure2′*[*OF assms*(*1*)] *assms*(*2*)])

lemma *emeasure_embed_measure*:
assumes *inj f A* ∈ *sets* (*embed_measure M f*)
shows *emeasure* (*embed_measure M f*) *A* = *emeasure M* (*f* −‘ *A* ∩ *space M*)
using *assms* **by** (*intro emeasure_embed_measure′*) (*auto intro*!: *inj_onI dest*: *injD*)

lemma *embed_measure_comp*:
assumes [*simp*]: *inj f inj g*
shows *embed_measure* (*embed_measure M f*) *g* = *embed_measure M* (*g* ∘ *f*)
proof−
have [*simp*]: *inj* (λ*x*. *g* (*f x*)) **by** (*subst o_def*[*symmetric*]) (*auto intro*: *inj_compose*)
note *measurable_embed_measure2*[*measurable*]
have *embed_measure* (*embed_measure M f*) *g* =
distr M (*embed_measure* (*embed_measure M f*) *g*) (*g* ∘ *f*)
by (*subst* (*1 2*) *embed_measure_eq_distr*)
(*simp_all add*: *distr_distr sets_embed_measure cong*: *distr_cong*)

also have ... = *embed_measure M* (*g* ∘ *f*)
  by (*subst* (*3*) *embed_measure_eq_distr*, *simp add*: *o_def*, *rule distr_cong*)
   (*auto simp*: *sets_embed_measure o_def image_image*[*symmetric*]
     intro: *inj_compose cong*: *distr_cong*)
finally show *?thesis* **.**
qed

lemma *sigma_finite_embed_measure*:
 assumes *sigma_finite_measure M* **and** *inj*: *inj f*
 shows *sigma_finite_measure* (*embed_measure M f*)
proof −
 from *assms*(*1*) **interpret** *sigma_finite_measure M* **.**
 from *sigma_finite_countable* **obtain** *A* **where**
  A_props: *countable A A* ⊆ *sets M* ⋃*A* = *space M* ⋀*X*. *X*∈*A* ⟹ *emeasure M X* ≠ ∞ **by** *blast*
 from *A_props* **have** *countable* ((‘) *f*‘*A*) **by** *auto*
 moreover
 from *inj* **and** *A_props* **have** (‘) *f*‘*A* ⊆ *sets* (*embed_measure M f*)
  by (*auto simp*: *sets_embed_measure*)
 moreover
 from *A_props* **and** *inj* **have** ⋃((‘) *f*‘*A*) = *space* (*embed_measure M f*)
  by (*auto simp*: *space_embed_measure intro*!: *imageI*)
 moreover
 from *A_props* **and** *inj* **have** ∀ *a*∈(‘) *f* ‘ *A*. *emeasure* (*embed_measure M f*) *a* ≠ ∞
  by (*intro ballI*, *subst emeasure_embed_measure*)
   (*auto simp*: *inj_vimage_image_eq intro*: *in_sets_embed_measure*)
 ultimately show *?thesis* **by** − (*standard*, *blast*)
qed

lemma *embed_measure_count_space'*:
  inj_on f A ⟹ *embed_measure* (*count_space A*) *f* = *count_space* (*f*‘*A*)
 apply (*subst distr_bij_count_space*[*of f A f*‘*A*, *symmetric*])
 apply (*simp add*: *inj_on_def bij_betw_def*)
 apply (*subst embed_measure_eq_distr'*)
 apply *simp*
 apply(*auto 4 3 intro*!: *measure_eqI imageI simp add*: *sets_embed_measure' subset_image_iff*)
 apply (*subst* (*1 2*) *emeasure_distr*)
 apply (*auto simp*: *space_embed_measure sets_embed_measure'*)
 done

lemma *embed_measure_count_space*:
  inj f ⟹ *embed_measure* (*count_space A*) *f* = *count_space* (*f*‘*A*)
 by(*rule embed_measure_count_space'*)(*erule subset_inj_on*, *simp*)

lemma *sets_embed_measure_alt*:
  inj f ⟹ *sets* (*embed_measure M f*) = ((‘) *f*) ‘ *sets M*
 by (*auto simp*: *sets_embed_measure*)

lemma *emeasure_embed_measure_image'*:
assumes *inj_on f* (*space M*) *X* ∈ *sets M*
shows *emeasure* (*embed_measure M f*) (*f'X*) = *emeasure M X*
proof−
from *assms* **have** *emeasure* (*embed_measure M f*) (*f'X*) = *emeasure M* (*f* −' *f* ' *X* ∩ *space M*)
by (*subst emeasure_embed_measure'*) (*auto simp*: *sets_embed_measure'*)
also from *assms* **have** *f* −' *f* ' *X* ∩ *space M* = *X* **by** (*auto dest*: *inj_onD sets.sets_into_space*)
finally show *?thesis* **.**
qed

lemma *emeasure_embed_measure_image*:
inj f ⟹ *X* ∈ *sets M* ⟹ *emeasure* (*embed_measure M f*) (*f'X*) = *emeasure M X*
by (*simp_all add*: *emeasure_embed_measure in_sets_embed_measure inj_vimage_image_eq*)

lemma *embed_measure_eq_iff*:
assumes *inj f*
shows *embed_measure A f* = *embed_measure B f* ⟷ *A* = *B* (**is** *?M* = *?N* ⟷ _)
proof
from *assms* **have** *I*: *inj* ((') *f*) **by** (*auto intro*: *injI dest*: *injD*)
assume *asm*: *?M* = *?N*
hence *sets* (*embed_measure A f*) = *sets* (*embed_measure B f*) **by** *simp*
with *assms* **have** *sets A* = *sets B* **by** (*simp only*: *I inj_image_eq_iff sets_embed_measure_alt*)
moreover {
fix *X* **assume** *X* ∈ *sets A*
from *asm* **have** *emeasure ?M* (*f'X*) = *emeasure ?N* (*f'X*) **by** *simp*
with ⟨*X* ∈ *sets A*⟩ **and** ⟨*sets A* = *sets B*⟩ **and** *assms*
have *emeasure A X* = *emeasure B X* **by** (*simp add*: *emeasure_embed_measure_image*)
}
ultimately show *A* = *B* **by** (*rule measure_eqI*)
qed *simp*

lemma *the_inv_into_in_Pi*: *inj_on f A* ⟹ *the_inv_into A f* ∈ *f* ' *A* → *A*
by (*auto simp*: *the_inv_into_f_f*)

lemma *map_prod_image*: *map_prod f g* ' (*A* × *B*) = (*f'A*) × (*g'B*)
using *map_prod_surj_on*[*OF refl refl*] **.**

lemma *map_prod_vimage*: *map_prod f g* −' (*A* × *B*) = (*f*−'*A*) × (*g*−'*B*)
by *auto*

lemma *embed_measure_prod*:
assumes *f*: *inj f* **and** *g*: *inj g* **and** [*simp*]: *sigma_finite_measure M sigma_finite_measure N*
shows *embed_measure M f* $\bigotimes_M$ *embed_measure N g* = *embed_measure* (*M* $\bigotimes_M$

N) (λ(*x*, *y*). (*f x*, *g y*))
  (**is** *?L* = _)
 unfolding *map_prod_def*[*symmetric*]
proof (*rule pair_measure_eqI*)
 have *fg*[*simp*]: ⋀*A*. *inj_on* (*map_prod f g*) *A* ⋀*A*. *inj_on f A* ⋀*A*. *inj_on g A*
  using *f g* **by** (*auto simp*: *inj_on_def*)

 note *complete_lattice_class.Sup_insert*[*simp del*] *ccSup_insert*[*simp del*]
  ccSUP_insert[*simp del*]
 show *sets*: *sets ?L* = *sets* (*embed_measure* (*M* ⨂_{*M*} *N*) (*map_prod f g*))
  unfolding *map_prod_def*[*symmetric*]
  apply (*simp add*: *sets_pair_eq_sets_fst_snd sets_embed_eq_vimage_algebra*
   cong: *vimage_algebra_cong*)
  apply (*subst sets_vimage_Sup_eq*[**where** *Y*=*space* (*M* ⨂_{*M*} *N*)])
  apply (*simp_all add*: *space_pair_measure*[*symmetric*])
  apply (*auto simp add*: *the_inv_into_f_f*
      simp del: *map_prod_simp*
      del: *prod_fun_imageE*) []
  apply *auto* []
  apply (*subst* (*1 2 3 4*) *vimage_algebra_vimage_algebra_eq*)
  apply (*simp_all add*: *the_inv_into_in_Pi Pi_iff*[*of snd*] *Pi_iff*[*of fst*] *space_pair_measure*)
  apply (*simp_all add*: *Pi_iff*[*of snd*] *Pi_iff*[*of fst*] *the_inv_into_in_Pi vimage_algebra_vimage_algebra_eq*
   space_pair_measure[*symmetric*] *map_prod_image*[*symmetric*])
  apply (*intro arg_cong*[**where** *f*=*sets*] *arg_cong*[**where** *f*=*Sup*] *arg_cong2*[**where**
f=*insert*] *vimage_algebra_cong*)
  apply (*auto simp*: *map_prod_image the_inv_into_f_f*
      simp del: *map_prod_simp del*: *prod_fun_imageE*)
  apply (*simp_all add*: *the_inv_into_f_f space_pair_measure*)
  done

 note *measurable_embed_measure2*[*measurable*]
 fix *A B* **assume** *AB*: *A* ∈ *sets* (*embed_measure M f*) *B* ∈ *sets* (*embed_measure*
N g)
 moreover have *f* −' *A* × *g* −' *B* ∩ *space* (*M* ⨂_{*M*} *N*) = (*f* −' *A* ∩ *space M*)
× (*g* −' *B* ∩ *space N*)
  by (*auto simp*: *space_pair_measure*)
 ultimately show *emeasure* (*embed_measure M f*) *A* ∗ *emeasure* (*embed_measure*
N g) *B* =
     emeasure (*embed_measure* (*M* ⨂_{*M*} *N*) (*map_prod f g*)) (*A* × *B*)
  by (*simp add*: *map_prod_vimage sets*[*symmetric*] *emeasure_embed_measure*
     sigma_finite_measure.emeasure_pair_measure_Times)
qed (*insert assms*, *simp_all add*: *sigma_finite_embed_measure*)

lemma *mono_embed_measure*:
 space M = *space M'* ⟹ *sets M* ⊆ *sets M'* ⟹ *sets* (*embed_measure M f*) ⊆
sets (*embed_measure M' f*)
 unfolding *embed_measure_def*
 apply (*subst* (*1 2*) *sets_measure_of*)
 apply (*blast dest*: *sets.sets_into_space*)

apply (*blast dest*: *sets.sets_into_space*)
apply *simp*
apply (*intro sigma_sets_mono′*)
apply *safe*
apply (*simp add*: *subset_eq*)
apply *metis*
done

lemma *density_embed_measure*:
assumes *inj*: *inj f* **and** *Mg*[*measurable*]: $g \in$ *borel_measurable* (*embed_measure M f*)
shows *density* (*embed_measure M f*) g = *embed_measure* (*density M* ($g \circ f$)) *f* (**is** *?M1* = *?M2*)
proof (*rule measure_eqI*)
fix *X* **assume** *X*: $X \in$ *sets ?M1*
from *inj* **have** *Mf*[*measurable*]: $f \in$ *measurable M* (*embed_measure M f*)
by (*rule measurable_embed_measure2*)
from *Mg* **and** *X* **have** *emeasure ?M1 X* = $\int^+ x.\ g\ x * \mathit{indicator}\ X\ x\ \partial$*embed_measure M f*
by (*subst emeasure_density*) *simp_all*
also from *X* **have** ... = $\int^+ x.\ g\ (f\ x) * \mathit{indicator}\ X\ (f\ x)\ \partial M$
by (*subst embed_measure_eq_distr*[*OF inj*], *subst nn_integral_distr*) *auto*
also have ... = $\int^+ x.\ g\ (f\ x) * \mathit{indicator}\ (f\ -`\ X \cap \mathit{space}\ M)\ x\ \partial M$
by (*intro nn_integral_cong*) (*auto split*: *split_indicator*)
also from *X* **have** ... = *emeasure* (*density M* ($g \circ f$)) (f −‘ $X \cap$ *space M*)
by (*subst emeasure_density*) (*simp_all add*: *measurable_comp*[*OF Mf Mg*] *measurable_sets*[*OF Mf*])
also from *X* **and** *inj* **have** ... = *emeasure ?M2 X*
by (*subst emeasure_embed_measure*) (*simp_all add*: *sets_embed_measure*)
finally show *emeasure ?M1 X* = *emeasure ?M2 X* **.**
qed (*simp_all add*: *sets_embed_measure inj*)

lemma *density_embed_measure′*:
assumes *inj*: *inj f* **and** *inv*: $\bigwedge x.\ f'\ (f\ x) = x$ **and** *Mg*[*measurable*]: $g \in$ *borel_measurable M*
shows *density* (*embed_measure M f*) ($g \circ f'$) = *embed_measure* (*density M g*) *f*
proof−
have *density* (*embed_measure M f*) ($g \circ f'$) = *embed_measure* (*density M* ($g \circ f' \circ f$)) *f*
by (*rule density_embed_measure*[*OF inj*])
(*rule measurable_comp*, *rule measurable_embed_measure1*, *subst measurable_cong*,
rule inv, *rule measurable_ident_sets*, *simp*, *rule Mg*)
also have *density M* ($g \circ f' \circ f$) = *density M g*
by (*intro density_cong*) (*subst measurable_cong*, *simp add*: *o_def inv*, *simp_all add*: *Mg inv*)
finally show *?thesis* **.**
qed

lemma *inj_on_image_subset_iff*:
  assumes *inj_on f C A ⊆ C B ⊆ C*
  shows *f ‘ A ⊆ f ‘ B ⟷ A ⊆ B*
proof (*intro iffI subsetI*)
  fix *x* **assume** *A*: *f ‘ A ⊆ f ‘ B* **and** *B*: *x ∈ A*
  from *B* **have** *f x ∈ f ‘ A* **by** *blast*
  with *A* **have** *f x ∈ f ‘ B* **by** *blast*
  then obtain *y* **where** *f x = f y* **and** *y ∈ B* **by** *blast*
  with *assms* **and** *B* **have** *x = y* **by** (*auto dest*: *inj_onD*)
  with ‹*y ∈ B*› **show** *x ∈ B* **by** *simp*
qed *auto*

lemma *AE_embed_measure′*:
  assumes *inj*: *inj_on f* (*space M*)
  shows (*AE x in embed_measure M f. P x*) ⟷ (*AE x in M. P* (*f x*))
proof
  let *?M = embed_measure M f*
  assume *AE x in ?M. P x*
  then obtain *A* **where** *A_props*: *A ∈ sets ?M emeasure ?M A = 0* {*x∈space ?M. ¬P x*} ⊆ *A*
    by (*force elim*: *AE_E*)
  then obtain *A′* **where** *A′_props*: *A = f ‘ A′ A′ ∈ sets M* **by** (*auto simp*: *sets_embed_measure′ inj*)
  moreover have *B*: {*x∈space ?M. ¬P x*} = *f ‘* {*x∈space M. ¬P* (*f x*)}
    by (*auto simp*: *inj space_embed_measure*)
  from *A_props*(*3*) **have** {*x∈space M. ¬P* (*f x*)} ⊆ *A′*
  by (*subst* (*asm*) *B*, *subst* (*asm*) *A′_props*, *subst* (*asm*) *inj_on_image_subset_iff*[*OF inj*])
      (*insert A′_props*, *auto dest*: *sets.sets_into_space*)
  moreover from *A_props A′_props* **have** *emeasure M A′ = 0*
    by (*simp add*: *emeasure_embed_measure_image′ inj*)
  ultimately show *AE x in M. P* (*f x*) **by** (*intro AE_I*)
next
  let *?M = embed_measure M f*
  assume *AE x in M. P* (*f x*)
  then obtain *A* **where** *A_props*: *A ∈ sets M emeasure M A = 0* {*x∈space M. ¬P* (*f x*)} ⊆ *A*
    by (*force elim*: *AE_E*)
  hence *f‘A ∈ sets ?M emeasure ?M* (*f‘A*) = *0* {*x∈space ?M. ¬P x*} ⊆ *f‘A*
  by (*auto simp*: *space_embed_measure emeasure_embed_measure_image′ sets_embed_measure′ inj*)
  thus *AE x in ?M. P x* **by** (*intro AE_I*)
qed

lemma *AE_embed_measure*:
  assumes *inj*: *inj f*
  shows (*AE x in embed_measure M f. P x*) ⟷ (*AE x in M. P* (*f x*))
  using *assms* **by** (*intro AE_embed_measure′*) (*auto intro*!: *inj_onI dest*: *injD*)

lemma *nn_integral_monotone_convergence_SUP_countable*:
fixes *f* :: *'a* ⇒ *'b* ⇒ *ennreal*
assumes *nonempty*: *Y* ≠ {}
and *chain*: *Complete_Partial_Order.chain* (≤) (*f* ' *Y*)
and *countable*: *countable B*
shows (∫⁺ *x*. (*SUP i*∈*Y*. *f i x*) ∂*count_space B*) = (*SUP i*∈*Y*. (∫⁺ *x*. *f i x* ∂*count_space B*))
(**is** *?lhs* = *?rhs*)
proof −
let *?f* = (λ*i x*. *f i* (*from_nat_into B x*) ∗ *indicator* (*to_nat_on B* ' *B*) *x*)
have *?lhs* = ∫⁺ *x*. (*SUP i*∈*Y*. *f i* (*from_nat_into B* (*to_nat_on B x*))) ∂*count_space B*
by(*rule nn_integral_cong*)(*simp add*: *countable*)
also have ... = ∫⁺ *x*. (*SUP i*∈*Y*. *f i* (*from_nat_into B x*)) ∂*count_space* (*to_nat_on B* ' *B*)
by(*simp add*: *embed_measure_count_space'*[*symmetric*] *inj_on_to_nat_on countable nn_integral_embed_measure' measurable_embed_measure1*)
also have ... = ∫⁺ *x*. (*SUP i*∈*Y*. *?f i x*) ∂*count_space UNIV*
by(*simp add*: *nn_integral_count_space_indicator ennreal_indicator*[*symmetric*] *SUP_mult_right_ennreal nonempty*)
also have ... = (*SUP i*∈*Y*. ∫⁺ *x*. *?f i x* ∂*count_space UNIV*)
proof(*rule nn_integral_monotone_convergence_SUP_nat*)
show *Complete_Partial_Order.chain* (≤) (*?f* ' *Y*)
by(*rule chain_imageI*[*OF chain*, *unfolded image_image*])(*auto intro*!: *le_funI split*: *split_indicator dest*: *le_funD*)
qed *fact*
also have ... = (*SUP i*∈*Y*. ∫⁺ *x*. *f i* (*from_nat_into B x*) ∂*count_space* (*to_nat_on B* ' *B*))
by(*simp add*: *nn_integral_count_space_indicator*)
also have ... = (*SUP i*∈*Y*. ∫⁺ *x*. *f i* (*from_nat_into B* (*to_nat_on B x*)) ∂*count_space B*)
by(*simp add*: *embed_measure_count_space'*[*symmetric*] *inj_on_to_nat_on countable nn_integral_embed_measure' measurable_embed_measure1*)
also have ... = *?rhs*
by(*intro arg_cong2*[**where** *f* = λ*A f*. *Sup* (*f* ' *A*)] *ext nn_integral_cong_AE*)(*simp_all add*: *AE_count_space countable*)
finally show *?thesis* **.**
qed

end

6.31 Brouwer's Fixed Point Theorem

theory *Brouwer_Fixpoint*
imports *Homeomorphism Derivative*
begin

6.31.1 Retractions

lemma *retract_of_contractible*:
  assumes *contractible T S retract_of T*
    shows *contractible S*
using *assms*
apply (*clarsimp simp add*: *retract_of_def contractible_def retraction_def homotopic_with*)
apply (*rule_tac x=r a* **in** *exI*)
apply (*rule_tac x=r ∘ h* **in** *exI*)
apply (*intro conjI continuous_intros continuous_on_compose*)
apply (*erule continuous_on_subset* | *force*)+
done

lemma *retract_of_path_connected*:
  ⟦*path_connected T*; *S retract_of T*⟧ ⟹ *path_connected S*
  by (*metis path_connected_continuous_image retract_of_def retraction*)

lemma *retract_of_simply_connected*:
  ⟦*simply_connected T*; *S retract_of T*⟧ ⟹ *simply_connected S*
apply (*simp add*: *retract_of_def retraction_def*, *clarify*)
apply (*rule simply_connected_retraction_gen*)
apply (*force elim*!: *continuous_on_subset*)+
done

lemma *retract_of_homotopically_trivial*:
  assumes *ts*: *T retract_of S*
    and *hom*: ⋀*f g*. ⟦*continuous_on U f*; *f* ‘ *U* ⊆ *S*;
                continuous_on U g; *g* ‘ *U* ⊆ *S*⟧
                ⟹ *homotopic_with_canon* (λ*x*. *True*) *U S f g*
    and *continuous_on U f f* ‘ *U* ⊆ *T*
    and *continuous_on U g g* ‘ *U* ⊆ *T*
  shows *homotopic_with_canon* (λ*x*. *True*) *U T f g*
proof −
  obtain *r* **where** *r* ‘ *S* ⊆ *S continuous_on S r* ∀ *x*∈*S*. *r* (*r x*) = *r x T* = *r* ‘ *S*
    using *ts* **by** (*auto simp*: *retract_of_def retraction*)
  then obtain *k* **where** *Retracts S r T k*
    unfolding *Retracts_def*
    by (*metis continuous_on_subset dual_order.trans image_iff image_mono*)
  then show *?thesis*
    apply (*rule Retracts.homotopically_trivial_retraction_gen*)
    using *assms*
    apply (*force simp*: *hom*)+
    done
qed

lemma *retract_of_homotopically_trivial_null*:
  assumes *ts*: *T retract_of S*
    and *hom*: ⋀*f*. ⟦*continuous_on U f*; *f* ‘ *U* ⊆ *S*⟧
              ⟹ ∃ *c*. *homotopic_with_canon* (λ*x*. *True*) *U S f* (λ*x*. *c*)
    and *continuous_on U f f* ‘ *U* ⊆ *T*

obtains *c* **where** *homotopic_with_canon* ($\lambda x.$ *True*) *U T f* ($\lambda x.$ *c*)
proof −
obtain *r* **where** *r* ‘ *S* ⊆ *S continuous_on S r* ∀ *x*∈*S*. *r* (*r x*) = *r x T* = *r* ‘ *S*
using *ts* **by** (*auto simp*: *retract_of_def retraction*)
then obtain *k* **where** *Retracts S r T k*
unfolding *Retracts_def*
by (*metis continuous_on_subset dual_order.trans image_iff image_mono*)
then show *?thesis*
apply (*rule Retracts.homotopically_trivial_retraction_null_gen*)
apply (*rule TrueI refl assms that* | *assumption*)+
done
qed

lemma *retraction_openin_vimage_iff*:
openin (*top_of_set S*) (*S* ∩ *r* −‘ *U*) ⟷ *openin* (*top_of_set T*) *U*
if *retraction*: *retraction S T r* **and** *U* ⊆ *T*
using *retraction* **apply** (*rule retractionE*)
apply (*rule continuous_right_inverse_imp_quotient_map* [**where** *g*=*r*])
using ‹*U* ⊆ *T*› **apply** (*auto elim*: *continuous_on_subset*)
done

lemma *retract_of_locally_compact*:
fixes *S* :: ′*a* :: {*heine_borel*,*real_normed_vector*} *set*
shows ⟦ *locally compact S*; *T retract_of S*⟧ ⟹ *locally compact T*
by (*metis locally_compact_closedin closedin_retract*)

lemma *homotopic_into_retract*:
⟦*f* ‘ *S* ⊆ *T*; *g* ‘ *S* ⊆ *T*; *T retract_of U*; *homotopic_with_canon* ($\lambda x.$ *True*) *S U f g*⟧
⟹ *homotopic_with_canon* ($\lambda x.$ *True*) *S T f g*
apply (*subst* (*asm*) *homotopic_with_def*)
apply (*simp add*: *homotopic_with retract_of_def retraction_def*, *clarify*)
apply (*rule_tac x*=*r* ∘ *h* **in** *exI*)
apply (*rule conjI continuous_intros* | *erule continuous_on_subset* | *force simp*: *image_subset_iff*)+
done

lemma *retract_of_locally_connected*:
assumes *locally connected T S retract_of T*
shows *locally connected S*
using *assms*
by (*auto simp*: *idempotent_imp_retraction intro*!: *retraction_openin_vimage_iff elim*!: *locally_connected_quotient_image retract_ofE*)

lemma *retract_of_locally_path_connected*:
assumes *locally path_connected T S retract_of T*
shows *locally path_connected S*
using *assms*
by (*auto simp*: *idempotent_imp_retraction intro*!: *retraction_openin_vimage_iff elim*!:

locally_path_connected_quotient_image retract_ofE)

A few simple lemmas about deformation retracts

```
lemma deformation_retract_imp_homotopy_eqv:
  fixes S :: 'a::euclidean_space set
  assumes homotopic_with_canon (λx. True) S S id r and r: retraction S T r
  shows S homotopy_eqv T
proof -
  have homotopic_with_canon (λx. True) S S (id ∘ r) id
    by (simp add: assms(1) homotopic_with_symD)
  moreover have homotopic_with_canon (λx. True) T T (r ∘ id) id
    using r unfolding retraction_def
    by (metis eq_id_iff homotopic_with_id2 topspace_euclidean_subtopology)
  ultimately
  show ?thesis
    unfolding homotopy_equivalent_space_def
     by (metis (no_types, lifting) continuous_map_subtopology_eu continuous_on_id'
id_def image_id r retraction_def)
qed

lemma deformation_retract:
  fixes S :: 'a::euclidean_space set
    shows (∃r. homotopic_with_canon (λx. True) S S id r ∧ retraction S T r) ⟷
           T retract_of S ∧ (∃f. homotopic_with_canon (λx. True) S S id f ∧ f ` S
⊆ T)
    (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    by (auto simp: retract_of_def retraction_def)
next
  assume ?rhs
  then show ?lhs
    apply (clarsimp simp add: retract_of_def retraction_def)
    apply (rule_tac x=r in exI, simp)
     apply (rule homotopic_with_trans, assumption)
     apply (rule_tac f = r ∘ f and g=r ∘ id in homotopic_with_eq)
        apply (rule_tac Y=S in homotopic_with_compose_continuous_left)
         apply (auto simp: homotopic_with_sym)
    done
qed

lemma deformation_retract_of_contractible_sing:
  fixes S :: 'a::euclidean_space set
  assumes contractible S a ∈ S
  obtains r where homotopic_with_canon (λx. True) S S id r retraction S {a} r
proof -
  have {a} retract_of S
    by (simp add: ‹a ∈ S›)
```

moreover have *homotopic_with_canon* (λ*x*. *True*) *S S id* (λ*x*. *a*)
  using *assms*
  by (*auto simp*: *contractible_def homotopic_into_contractible image_subset_iff*)
moreover have (λ*x*. *a*) ' *S* ⊆ {*a*}
 by (*simp add*: *image_subsetI*)
ultimately show *?thesis*
 using *that deformation_retract* **by** *metis*
qed

lemma *continuous_on_compact_surface_projection_aux*:
 fixes *S* :: '*a*::*t2_space set*
 assumes *compact S S* ⊆ *T image q T* ⊆ *S*
  and *contp*: *continuous_on T p*
  and ⋀*x*. *x* ∈ *S* ⟹ *q x* = *x*
  and [*simp*]: ⋀*x*. *x* ∈ *T* ⟹ *q*(*p x*) = *q x*
  and ⋀*x*. *x* ∈ *T* ⟹ *p*(*q x*) = *p x*
 shows *continuous_on T q*
proof −
 have ∗: *image p T* = *image p S*
  using *assms* **by** *auto* (*metis imageI subset_iff*)
 have *contp'*: *continuous_on S p*
  by (*rule continuous_on_subset* [*OF contp* ‹*S* ⊆ *T*›])
 have *continuous_on* (*p* ' *T*) *q*
  by (*simp add*: ∗ *assms*(*1*) *assms*(*2*) *assms*(*5*) *continuous_on_inv contp' rev_subsetD*)
 then have *continuous_on T* (*q* ∘ *p*)
  by (*rule continuous_on_compose* [*OF contp*])
 then show *?thesis*
  by (*rule continuous_on_eq* [*of _ q* ∘ *p*]) (*simp add*: *o_def*)
qed

lemma *continuous_on_compact_surface_projection*:
 fixes *S* :: '*a*::*real_normed_vector set*
 assumes *compact S*
  and *S*: *S* ⊆ *V* − {*0*} **and** *cone V*
  and *iff*: ⋀*x k*. *x* ∈ *V* − {*0*} ⟹ *0* < *k* ∧ (*k* $*_R$ *x*) ∈ *S* ⟷ *d x* = *k*
 shows *continuous_on* (*V* − {*0*}) (λ*x*. *d x* $*_R$ *x*)
proof (*rule continuous_on_compact_surface_projection_aux* [*OF* ‹*compact S*› *S*])
 show (λ*x*. *d x* $*_R$ *x*) ' (*V* − {*0*}) ⊆ *S*
  using *iff* **by** *auto*
 show *continuous_on* (*V* − {*0*}) (λ*x*. *inverse*(*norm x*) $*_R$ *x*)
  by (*intro continuous_intros*) *force*
 show ⋀*x*. *x* ∈ *S* ⟹ *d x* $*_R$ *x* = *x*
  by (*metis S zero_less_one local.iff scaleR_one subset_eq*)
 show *d* (*x* $/_R$ *norm x*) $*_R$ (*x* $/_R$ *norm x*) = *d x* $*_R$ *x* **if** *x* ∈ *V* − {*0*} **for** *x*
  using *iff* [*of inverse*(*norm x*) $*_R$ *x norm x* ∗ *d x*, *symmetric*] *iff that* ‹*cone V*›
  by (*simp add*: *field_simps cone_def zero_less_mult_iff*)
 show *d x* $*_R$ *x* $/_R$ *norm* (*d x* $*_R$ *x*) = *x* $/_R$ *norm x* **if** *x* ∈ *V* − {*0*} **for** *x*
 proof −

have *0 < d x*
  using *local.iff that* **by** *blast*
then show *?thesis*
  by *simp*
qed
qed

6.31.2 Kuhn Simplices

lemma *bij_betw_singleton_eq*:
  assumes *f*: *bij_betw f A B* **and** *g*: *bij_betw g A B* **and** *a*: *a ∈ A*
  assumes *eq*: (⋀*x. x ∈ A ⟹ x ≠ a ⟹ f x = g x*)
  shows *f a = g a*
proof –
  have *f ‘ (A − {a}) = g ‘ (A − {a})*
    by (*intro image_cong*) (*simp_all add*: *eq*)
  then have *B − {f a} = B − {g a}*
    using *f g a* **by** (*auto simp*: *bij_betw_def inj_on_image_set_diff set_eq_iff*)
  moreover have *f a ∈ B g a ∈ B*
    using *f g a* **by** (*auto simp*: *bij_betw_def*)
  ultimately show *?thesis*
    by *auto*
qed

lemma *swap_image*:
  Fun.swap i j f ‘ A = (if i ∈ A then (if j ∈ A then f ‘ A else f ‘ ((A − {i}) ∪ {j}))
    else (if j ∈ A then f ‘ ((A − {j}) ∪ {i}) else f ‘ A))
  by (*auto simp*: *swap_def cong*: *image_cong_simp*)

lemmas *swap_apply1 = swap_apply(1)*
lemmas *swap_apply2 = swap_apply(2)*

lemma *pointwise_minimal_pointwise_maximal*:
  fixes *s* :: (*nat ⇒ nat*) *set*
  assumes *finite s*
    and *s ≠ {}*
    and *∀x∈s. ∀y∈s. x ≤ y ∨ y ≤ x*
  shows *∃a∈s. ∀x∈s. a ≤ x*
    and *∃a∈s. ∀x∈s. x ≤ a*
  using *assms*
proof (*induct s rule*: *finite_ne_induct*)
  case (*insert b s*)
  assume *∗*: *∀x∈insert b s. ∀y∈insert b s. x ≤ y ∨ y ≤ x*
  then obtain *u l* **where** *l ∈ s ∀b∈s. l ≤ b u ∈ s ∀b∈s. b ≤ u*
    using *insert* **by** *auto*
  with ∗ **show** *∃a∈insert b s. ∀x∈insert b s. a ≤ x ∃a∈insert b s. ∀x∈insert b s. x ≤ a*
    using ∗[*rule_format, of b u*] ∗[*rule_format, of b l*] **by** (*metis insert_iff or-*

der.trans)+
qed *auto*

lemma *kuhn_labelling_lemma*:
fixes $P\ Q$ :: $'a$::*euclidean_space* ⇒ *bool*
assumes $\forall x.\ P\ x \longrightarrow P\ (f\ x)$
and $\forall x.\ P\ x \longrightarrow (\forall i{\in}Basis.\ Q\ i \longrightarrow 0 \leq x{\cdot}i \wedge x{\cdot}i \leq 1)$
shows $\exists l.\ (\forall x.\forall i{\in}Basis.\ l\ x\ i \leq (1{::}nat)) \wedge$
$(\forall x.\forall i{\in}Basis.\ P\ x \wedge Q\ i \wedge (x{\cdot}i = 0) \longrightarrow (l\ x\ i = 0)) \wedge$
$(\forall x.\forall i{\in}Basis.\ P\ x \wedge Q\ i \wedge (x{\cdot}i = 1) \longrightarrow (l\ x\ i = 1)) \wedge$
$(\forall x.\forall i{\in}Basis.\ P\ x \wedge Q\ i \wedge (l\ x\ i = 0) \longrightarrow x{\cdot}i \leq f\ x{\cdot}i) \wedge$
$(\forall x.\forall i{\in}Basis.\ P\ x \wedge Q\ i \wedge (l\ x\ i = 1) \longrightarrow f\ x{\cdot}i \leq x{\cdot}i)$
proof −
{ fix $x\ i$
let $?R = \lambda y.\ (P\ x \wedge Q\ i \wedge x \cdot i = 0 \longrightarrow y = (0{::}nat)) \wedge$
$(P\ x \wedge Q\ i \wedge x \cdot i = 1 \longrightarrow y = 1) \wedge$
$(P\ x \wedge Q\ i \wedge y = 0 \longrightarrow x \cdot i \leq f\ x \cdot i) \wedge$
$(P\ x \wedge Q\ i \wedge y = 1 \longrightarrow f\ x \cdot i \leq x \cdot i)$
{ assume $P\ x\ Q\ i\ i \in Basis$ **with** *assms* **have** $0 \leq f\ x \cdot i \wedge f\ x \cdot i \leq 1$ **by** *auto* **}**
then have $i \in Basis \Longrightarrow ?R\ 0 \vee ?R\ 1$ **by** *auto* **}**
then show *?thesis*
unfolding *all_conj_distrib*[*symmetric*] *Ball_def*
by (*subst choice_iff*[*symmetric*])+ *blast*
qed

The key "counting" observation, somewhat abstracted

lemma *kuhn_counting_lemma*:
fixes *bnd compo compo′ face S F*
defines $nF\ s == card\ \{f{\in}F.\ face\ f\ s \wedge compo'\ f\}$
assumes [*simp*, *intro*]: *finite F* — faces **and** [*simp*, *intro*]: *finite S* — simplices
and $\bigwedge f.\ f \in F \Longrightarrow bnd\ f \Longrightarrow card\ \{s{\in}S.\ face\ f\ s\} = 1$
and $\bigwedge f.\ f \in F \Longrightarrow \neg\ bnd\ f \Longrightarrow card\ \{s{\in}S.\ face\ f\ s\} = 2$
and $\bigwedge s.\ s \in S \Longrightarrow compo\ s \Longrightarrow nF\ s = 1$
and $\bigwedge s.\ s \in S \Longrightarrow \neg\ compo\ s \Longrightarrow nF\ s = 0 \vee nF\ s = 2$
and $odd\ (card\ \{f{\in}F.\ compo'\ f \wedge bnd\ f\})$
shows $odd\ (card\ \{s{\in}S.\ compo\ s\})$
proof −
have $(\sum s\ |\ s \in S \wedge \neg\ compo\ s.\ nF\ s) + (\sum s\ |\ s \in S \wedge compo\ s.\ nF\ s) = (\sum s{\in}S.\ nF\ s)$
by (*subst sum.union_disjoint*[*symmetric*]) (*auto intro*!: *sum.cong*)
also have $\ldots = (\sum s{\in}S.\ card\ \{f \in \{f{\in}F.\ compo'\ f \wedge bnd\ f\}.\ face\ f\ s\}) +$
$(\sum s{\in}S.\ card\ \{f \in \{f{\in}F.\ compo'\ f \wedge \neg\ bnd\ f\}.\ face\ f\ s\})$
unfolding *sum.distrib*[*symmetric*]
by (*subst card_Un_disjoint*[*symmetric*])
(*auto simp*: *nF_def intro*!: *sum.cong arg_cong*[**where** *f*=*card*])
also have $\ldots = 1 * card\ \{f{\in}F.\ compo'\ f \wedge bnd\ f\} + 2 * card\ \{f{\in}F.\ compo'\ f \wedge \neg\ bnd\ f\}$

using *assms*(*4*,*5*) **by** (*fastforce intro*!: *arg_cong2*[**where** *f*=(+)] *sum_multicount*)
finally have *odd* ((∑ *s* | *s* ∈ *S* ∧ ¬ *compo s*. *nF s*) + *card* {*s*∈*S*. *compo s*})
using *assms*(*6*,*8*) **by** *simp*
moreover have (∑ *s* | *s* ∈ *S* ∧ ¬ *compo s*. *nF s*) =
(∑ *s* | *s* ∈ *S* ∧ ¬ *compo s* ∧ *nF s* = *0*. *nF s*) + (∑ *s* | *s* ∈ *S* ∧ ¬ *compo s* ∧ *nF s* = *2*. *nF s*)
using *assms*(*7*) **by** (*subst sum.union_disjoint*[*symmetric*]) (*fastforce intro*!: *sum.cong*)+
ultimately show *?thesis*
by *auto*
qed

The odd/even result for faces of complete vertices, generalized

lemma *kuhn_complete_lemma*:
assumes [*simp*]: *finite simplices*
and *face*: ⋀*f s*. *face f s* ⟷ (∃ *a*∈*s*. *f* = *s* − {*a*})
and *card_s*[*simp*]: ⋀*s*. *s* ∈ *simplices* ⟹ *card s* = *n* + *2*
and *rl_bd*: ⋀*s*. *s* ∈ *simplices* ⟹ *rl* ‘ *s* ⊆ {..*Suc n*}
and *bnd*: ⋀*f s*. *s* ∈ *simplices* ⟹ *face f s* ⟹ *bnd f* ⟹ *card* {*s*∈*simplices*. *face f s*} = *1*
and *nbnd*: ⋀*f s*. *s* ∈ *simplices* ⟹ *face f s* ⟹ ¬ *bnd f* ⟹ *card* {*s*∈*simplices*. *face f s*} = *2*
and *odd_card*: *odd* (*card* {*f*. (∃ *s*∈*simplices*. *face f s*) ∧ *rl* ‘ *f* = {..*n*} ∧ *bnd f*})
shows *odd* (*card* {*s*∈*simplices*. (*rl* ‘ *s* = {..*Suc n*})})
proof (*rule kuhn_counting_lemma*)
have *finite_s*[*simp*]: ⋀*s*. *s* ∈ *simplices* ⟹ *finite s*
by (*metis add_is_0 zero_neq_numeral card.infinite assms*(*3*))

let *?F* = {*f*. ∃ *s*∈*simplices*. *face f s*}
have *F_eq*: *?F* = (⋃ *s*∈*simplices*. ⋃ *a*∈*s*. {*s* − {*a*}})
by (*auto simp*: *face*)
show *finite ?F*
using ‹*finite simplices*› **unfolding** *F_eq* **by** *auto*

show *card* {*s* ∈ *simplices*. *face f s*} = *1* **if** *f* ∈ *?F bnd f* **for** *f*
using *bnd that* **by** *auto*

show *card* {*s* ∈ *simplices*. *face f s*} = *2* **if** *f* ∈ *?F* ¬ *bnd f* **for** *f*
using *nbnd that* **by** *auto*

show *odd* (*card* {*f* ∈ {*f*. ∃ *s*∈*simplices*. *face f s*}. *rl* ‘ *f* = {..*n*} ∧ *bnd f*})
using *odd_card* **by** *simp*

fix *s* **assume** *s*[*simp*]: *s* ∈ *simplices*
let *?S* = {*f* ∈ {*f*. ∃ *s*∈*simplices*. *face f s*}. *face f s* ∧ *rl* ‘ *f* = {..*n*}}
have *?S* = (λ*a*. *s* − {*a*}) ‘ {*a*∈*s*. *rl* ‘ (*s* − {*a*}) = {..*n*}}
using *s* **by** (*fastforce simp*: *face*)
then have *card_S*: *card ?S* = *card* {*a*∈*s*. *rl* ‘ (*s* − {*a*}) = {..*n*}}

```
  by (auto intro!: card_image inj_onI)

{ assume rl: rl ` s = {..Suc n}
  then have inj_rl: inj_on rl s
    by (intro eq_card_imp_inj_on) auto
  moreover obtain a where rl a = Suc n a ∈ s
    by (metis atMost_iff image_iff le_Suc_eq rl)
  ultimately have n: {..n} = rl ` (s − {a})
    by (auto simp: inj_on_image_set_diff rl)
  have {a∈s. rl ` (s − {a}) = {..n}} = {a}
    using inj_rl ‹a ∈ s› by (auto simp: n inj_on_image_eq_iff[OF inj_rl])
  then show card ?S = 1
    unfolding card_S by simp }

{ assume rl: rl ` s ≠ {..Suc n}
  show card ?S = 0 ∨ card ?S = 2
  proof cases
    assume *: {..n} ⊆ rl ` s
    with rl rl_bd[OF s] have rl_s: rl ` s = {..n}
      by (auto simp: atMost_Suc subset_insert_iff split: if_split_asm)
    then have ¬ inj_on rl s
      by (intro pigeonhole) simp
    then obtain a b where ab: a ∈ s b ∈ s rl a = rl b a ≠ b
      by (auto simp: inj_on_def)
    then have eq: rl ` (s − {a}) = rl ` s
      by auto
    with ab have inj: inj_on rl (s − {a})
      by (intro eq_card_imp_inj_on) (auto simp: rl_s card_Diff_singleton_if)

    { fix x assume x ∈ s x ∉ {a, b}
      then have rl ` s − {rl x} = rl ` ((s − {a}) − {x})
        by (auto simp: eq inj_on_image_set_diff[OF inj])
      also have ... = rl ` (s − {x})
        using ab ‹x ∉ {a, b}› by auto
      also assume ... = rl ` s
      finally have False
        using ‹x∈s› by auto }
    moreover
    { fix x assume x ∈ {a, b} with ab have x ∈ s ∧ rl ` (s − {x}) = rl ` s
        by (simp add: set_eq_iff image_iff Bex_def) metis }
    ultimately have {a∈s. rl ` (s − {a}) = {..n}} = {a, b}
      unfolding rl_s[symmetric] by fastforce
    with ‹a ≠ b› show card ?S = 0 ∨ card ?S = 2
      unfolding card_S by simp
  next
    assume ¬ {..n} ⊆ rl ` s
    then have ⋀x. rl ` (s − {x}) ≠ {..n}
      by auto
    then show card ?S = 0 ∨ card ?S = 2
```

```
        unfolding card_S by simp
    qed }
qed fact

locale kuhn_simplex =
  fixes p n and base upd and s :: (nat ⇒ nat) set
  assumes base: base ∈ {..< n} → {..< p}
  assumes base_out: ⋀i. n ≤ i ⟹ base i = p
  assumes upd: bij_betw upd {..< n} {..< n}
  assumes s_pre: s = (λi j. if j ∈ upd‘{..< i} then Suc (base j) else base j) ‘ {..
n}
begin

definition enum i j = (if j ∈ upd‘{..< i} then Suc (base j) else base j)

lemma s_eq: s = enum ‘ {.. n}
  unfolding s_pre enum_def[abs_def] ..

lemma upd_space: i < n ⟹ upd i < n
  using upd by (auto dest!: bij_betwE)

lemma s_space: s ⊆ {..< n} → {.. p}
proof –
  { fix i assume i ≤ n then have enum i ∈ {..< n} → {.. p}
    proof (induct i)
      case 0 then show ?case
        using base by (auto simp: Pi_iff less_imp_le enum_def)
    next
      case (Suc i) with base show ?case
        by (auto simp: Pi_iff Suc_le_eq less_imp_le enum_def intro: upd_space)
    qed }
  then show ?thesis
    by (auto simp: s_eq)
qed

lemma inj_upd: inj_on upd {..< n}
  using upd by (simp add: bij_betw_def)

lemma inj_enum: inj_on enum {.. n}
proof –
  { fix x y :: nat assume x ≠ y x ≤ n y ≤ n
    with upd have upd ‘ {..< x} ≠ upd ‘ {..< y}
      by (subst inj_on_image_eq_iff[where C={..< n}]) (auto simp: bij_betw_def)
    then have enum x ≠ enum y
      by (auto simp: enum_def fun_eq_iff) }
  then show ?thesis
    by (auto simp: inj_on_def)
qed
```

lemma *enum_0*: *enum 0* = *base*
  by (*simp add*: *enum_def*[*abs_def*])

lemma *base_in_s*: *base* ∈ *s*
  unfolding *s_eq* **by** (*subst enum_0*[*symmetric*]) *auto*

lemma *enum_in*: *i* ≤ *n* ⟹ *enum i* ∈ *s*
  unfolding *s_eq* **by** *auto*

lemma *one_step*:
  assumes *a*: *a* ∈ *s j* < *n*
  assumes ∗: ⋀*a'*. *a'* ∈ *s* ⟹ *a'* ≠ *a* ⟹ *a' j* = *p'*
  shows *a j* ≠ *p'*
proof
  assume *a j* = *p'*
  with ∗ *a* **have** ⋀*a'*. *a'* ∈ *s* ⟹ *a' j* = *p'*
    by *auto*
  then have ⋀*i*. *i* ≤ *n* ⟹ *enum i j* = *p'*
    unfolding *s_eq* **by** *auto*
  from *this*[*of 0*] *this*[*of n*] **have** *j* ∉ *upd* ‘ {..< *n*}
    by (*auto simp*: *enum_def fun_eq_iff split*: *if_split_asm*)
  with *upd* ‹*j* < *n*› **show** *False*
    by (*auto simp*: *bij_betw_def*)
qed

lemma *upd_inj*: *i* < *n* ⟹ *j* < *n* ⟹ *upd i* = *upd j* ⟷ *i* = *j*
  using *upd* **by** (*auto simp*: *bij_betw_def inj_on_eq_iff*)

lemma *upd_surj*: *upd* ‘ {..< *n*} = {..< *n*}
  using *upd* **by** (*auto simp*: *bij_betw_def*)

lemma *in_upd_image*: *A* ⊆ {..< *n*} ⟹ *i* < *n* ⟹ *upd i* ∈ *upd* ‘ *A* ⟷ *i* ∈ *A*
  using *inj_on_image_mem_iff*[*of upd* {..< *n*}] *upd*
  by (*auto simp*: *bij_betw_def*)

lemma *enum_inj*: *i* ≤ *n* ⟹ *j* ≤ *n* ⟹ *enum i* = *enum j* ⟷ *i* = *j*
  using *inj_enum* **by** (*auto simp*: *inj_on_eq_iff*)

lemma *in_enum_image*: *A* ⊆ {.. *n*} ⟹ *i* ≤ *n* ⟹ *enum i* ∈ *enum* ‘ *A* ⟷ *i* ∈ *A*
  using *inj_on_image_mem_iff*[*OF inj_enum*] **by** *auto*

lemma *enum_mono*: *i* ≤ *n* ⟹ *j* ≤ *n* ⟹ *enum i* ≤ *enum j* ⟷ *i* ≤ *j*
  by (*auto simp*: *enum_def le_fun_def in_upd_image Ball_def*[*symmetric*])

lemma *enum_strict_mono*: *i* ≤ *n* ⟹ *j* ≤ *n* ⟹ *enum i* < *enum j* ⟷ *i* < *j*
  using *enum_mono*[*of i j*] *enum_inj*[*of i j*] **by** (*auto simp*: *le_less*)

lemma *chain*: *a* ∈ *s* ⟹ *b* ∈ *s* ⟹ *a* ≤ *b* ∨ *b* ≤ *a*
  by (*auto simp*: *s_eq enum_mono*)

lemma *less*: $a \in s \Longrightarrow b \in s \Longrightarrow a\ i < b\ i \Longrightarrow a < b$
  using *chain*[*of a b*] **by** (*auto simp*: *less_fun_def le_fun_def not_le*[*symmetric*])

lemma *enum_0_bot*: $a \in s \Longrightarrow a = enum\ 0 \longleftrightarrow (\forall a' \in s.\ a \le a')$
  unfolding *s_eq* **by** (*auto simp*: *enum_mono Ball_def*)

lemma *enum_n_top*: $a \in s \Longrightarrow a = enum\ n \longleftrightarrow (\forall a' \in s.\ a' \le a)$
  unfolding *s_eq* **by** (*auto simp*: *enum_mono Ball_def*)

lemma *enum_Suc*: $i < n \Longrightarrow enum\ (Suc\ i) = (enum\ i)(upd\ i := Suc\ (enum\ i\ (upd\ i)))$
  by (*auto simp*: *fun_eq_iff enum_def upd_inj*)

lemma *enum_eq_p*: $i \le n \Longrightarrow n \le j \Longrightarrow enum\ i\ j = p$
  by (*induct i*) (*auto simp*: *enum_Suc enum_0 base_out upd_space not_less*[*symmetric*])

lemma *out_eq_p*: $a \in s \Longrightarrow n \le j \Longrightarrow a\ j = p$
  unfolding *s_eq* **by** (*auto simp*: *enum_eq_p*)

lemma *s_le_p*: $a \in s \Longrightarrow a\ j \le p$
  using *out_eq_p*[*of a j*] *s_space* **by** (*cases* $j < n$) *auto*

lemma *le_Suc_base*: $a \in s \Longrightarrow a\ j \le Suc\ (base\ j)$
  unfolding *s_eq* **by** (*auto simp*: *enum_def*)

lemma *base_le*: $a \in s \Longrightarrow base\ j \le a\ j$
  unfolding *s_eq* **by** (*auto simp*: *enum_def*)

lemma *enum_le_p*: $i \le n \Longrightarrow j < n \Longrightarrow enum\ i\ j \le p$
  using *enum_in*[*of i*] *s_space* **by** *auto*

lemma *enum_less*: $a \in s \Longrightarrow i < n \Longrightarrow enum\ i < a \longleftrightarrow enum\ (Suc\ i) \le a$
  unfolding *s_eq* **by** (*auto simp*: *enum_strict_mono enum_mono*)

lemma *ksimplex_0*:
  $n = 0 \Longrightarrow s = \{(\lambda x.\ p)\}$
  using *s_eq enum_def base_out* **by** *auto*

lemma *replace_0*:
  assumes $j < n$ $a \in s$ **and** *p*: $\forall x \in s - \{a\}.\ x\ j = 0$ **and** $x \in s$
  shows $x \le a$
proof *cases*
  assume $x \ne a$
  have $a\ j \ne 0$
    using *assms* **by** (*intro one_step*[**where** *a*=*a*]) *auto*
  with *less*[*OF* ‹$x \in s$› ‹$a \in s$›, *of j*] *p*[*rule_format*, *of x*] ‹$x \in s$› ‹$x \ne a$›
  show *?thesis*
    by *auto*

qed *simp*

lemma *replace_1*:
  assumes $j < n$ $a \in s$ **and** p: $\forall x \in s - \{a\}.\ x\ j = p$ **and** $x \in s$
  shows $a \leq x$
proof *cases*
  assume $x \neq a$
  have $a\ j \neq p$
    using *assms* **by** (*intro one_step*[**where** $a=a$]) *auto*
  with *enum_le_p*[*of* _ *j*] ‹$j < n$› ‹$a \in s$›
  have $a\ j < p$
    by (*auto simp*: *less_le s_eq*)
  with *less*[*OF* ‹$a \in s$› ‹$x \in s$›, *of j*] *p*[*rule_format*, *of x*] ‹$x \in s$› ‹$x \neq a$›
  show *?thesis*
    by *auto*
qed *simp*

end

locale *kuhn_simplex_pair* = *s*: *kuhn_simplex p n b_s u_s s* + *t*: *kuhn_simplex p n b_t u_t t*
  for *p n b_s u_s s b_t u_t t*
begin

lemma *enum_eq*:
  assumes *l*: $i \leq l$ $l \leq j$ **and** $j + d \leq n$
  assumes *eq*: *s.enum* ‘ $\{i .. j\}$ = *t.enum* ‘ $\{i + d .. j + d\}$
  shows *s.enum* l = *t.enum* $(l + d)$
using *l* **proof** (*induct l rule*: *dec_induct*)
  case *base*
  then have *s*: *s.enum* $i \in$ *t.enum* ‘ $\{i + d .. j + d\}$ **and** *t*: *t.enum* $(i + d) \in$ *s.enum* ‘ $\{i .. j\}$
    using *eq* **by** *auto*
  from *t* ‹$i \leq j$› ‹$j + d \leq n$› **have** *s.enum* $i \leq$ *t.enum* $(i + d)$
    by (*auto simp*: *s.enum_mono*)
  moreover from *s* ‹$i \leq j$› ‹$j + d \leq n$› **have** *t.enum* $(i + d) \leq$ *s.enum* i
    by (*auto simp*: *t.enum_mono*)
  ultimately show *?case*
    by *auto*
next
  case (*step l*)
  moreover from *step.prems* ‹$j + d \leq n$› **have**
      s.enum $l <$ *s.enum* (*Suc* l)
      t.enum $(l + d) <$ *t.enum* (*Suc* $l + d$)
    by (*simp_all add*: *s.enum_strict_mono t.enum_strict_mono*)
  moreover have
      s.enum (*Suc* l) $\in$ *t.enum* ‘ $\{i + d .. j + d\}$
      t.enum (*Suc* $l + d$) $\in$ *s.enum* ‘ $\{i .. j\}$
    using *step* ‹$j + d \leq n$› *eq* **by** (*auto simp*: *s.enum_inj t.enum_inj*)

```
  ultimately have s.enum (Suc l) = t.enum (Suc (l + d))
    using ‹j + d ≤ n›
    by (intro antisym s.enum_less[THEN iffD1] t.enum_less[THEN iffD1])
       (auto intro!: s.enum_in t.enum_in)
  then show ?case by simp
qed

lemma ksimplex_eq_bot:
  assumes a: a ∈ s ⋀a'. a' ∈ s ⟹ a ≤ a'
  assumes b: b ∈ t ⋀b'. b' ∈ t ⟹ b ≤ b'
  assumes eq: s − {a} = t − {b}
  shows s = t
proof cases
  assume n = 0 with s.ksimplex_0 t.ksimplex_0 show ?thesis by simp
next
  assume n ≠ 0
  have s.enum 0 = (s.enum (Suc 0)) (u_s 0 := s.enum (Suc 0) (u_s 0) − 1)
       t.enum 0 = (t.enum (Suc 0)) (u_t 0 := t.enum (Suc 0) (u_t 0) − 1)
    using ‹n ≠ 0› by (simp_all add: s.enum_Suc t.enum_Suc)
  moreover have e0: a = s.enum 0 b = t.enum 0
    using a b by (simp_all add: s.enum_0_bot t.enum_0_bot)
  moreover
  { fix j assume 0 < j j ≤ n
    moreover have s − {a} = s.enum ` {Suc 0 .. n} t − {b} = t.enum ` {Suc 0
.. n}
      unfolding s.s_eq t.s_eq e0 by (auto simp: s.enum_inj t.enum_inj)
    ultimately have s.enum j = t.enum j
      using enum_eq[of 1 j n 0] eq by auto }
  note enum_eq = this
  then have s.enum (Suc 0) = t.enum (Suc 0)
    using ‹n ≠ 0› by auto
  moreover
  { fix j assume Suc j < n
    with enum_eq[of Suc j] enum_eq[of Suc (Suc j)]
    have u_s (Suc j) = u_t (Suc j)
      using s.enum_Suc[of Suc j] t.enum_Suc[of Suc j]
      by (auto simp: fun_eq_iff split: if_split_asm) }
  then have ⋀j. 0 < j ⟹ j < n ⟹ u_s j = u_t j
    by (auto simp: gr0_conv_Suc)
  with ‹n ≠ 0› have u_t 0 = u_s 0
    by (intro bij_betw_singleton_eq[OF t.upd s.upd, of 0]) auto
  ultimately have a = b
    by simp
  with assms show s = t
    by auto
qed

lemma ksimplex_eq_top:
  assumes a: a ∈ s ⋀a'. a' ∈ s ⟹ a' ≤ a
```

assumes *b*: $b \in t \bigwedge b'. \ b' \in t \Longrightarrow b' \leq b$
assumes *eq*: $s - \{a\} = t - \{b\}$
shows $s = t$
proof (*cases n*)
assume $n = 0$ **with** *s.ksimplex_0 t.ksimplex_0* **show** *?thesis* **by** *simp*
next
case (*Suc n'*)
have *s.enum n* = (*s.enum n'*) (*u_s n'* := *Suc* (*s.enum n'* (*u_s n'*)))
t.enum n = (*t.enum n'*) (*u_t n'* := *Suc* (*t.enum n'* (*u_t n'*)))
using *Suc* **by** (*simp_all add*: *s.enum_Suc t.enum_Suc*)
moreover have *en*: *a* = *s.enum n b* = *t.enum n*
using *a b* **by** (*simp_all add*: *s.enum_n_top t.enum_n_top*)
moreover
{ fix *j* **assume** $j < n$
moreover have *s* − {*a*} = *s.enum* ' {*0* .. *n'*} *t* − {*b*} = *t.enum* ' {*0* .. *n'*}
unfolding *s.s_eq t.s_eq en* **by** (*auto simp*: *s.enum_inj t.enum_inj Suc*)
ultimately have *s.enum j* = *t.enum j*
using *enum_eq*[*of 0 j n' 0*] *eq Suc* **by** *auto* **}**
note *enum_eq* = *this*
then have *s.enum n'* = *t.enum n'*
using *Suc* **by** *auto*
moreover
{ fix *j* **assume** $j < n'$
with *enum_eq*[*of j*] *enum_eq*[*of Suc j*]
have *u_s j* = *u_t j*
using *s.enum_Suc*[*of j*] *t.enum_Suc*[*of j*]
by (*auto simp*: *Suc fun_eq_iff split*: *if_split_asm*) **}**
then have $\bigwedge j. \ j < n' \Longrightarrow$ *u_s j* = *u_t j*
by (*auto simp*: *gr0_conv_Suc*)
then have *u_t n'* = *u_s n'*
by (*intro bij_betw_singleton_eq*[*OF t.upd s.upd*, *of n'*]) (*auto simp*: *Suc*)
ultimately have $a = b$
by *simp*
with *assms* **show** $s = t$
by *auto*
qed

end

inductive *ksimplex* **for** *p n* :: *nat* **where**
ksimplex: *kuhn_simplex p n base upd s* ⟹ *ksimplex p n s*

lemma *finite_ksimplexes*: *finite* {*s*. *ksimplex p n s*}
proof (*rule finite_subset*)
{ fix *a s* **assume** *ksimplex p n s* $a \in s$
then obtain *b u* **where** *kuhn_simplex p n b u s* **by** (*auto elim*: *ksimplex.cases*)
then interpret *kuhn_simplex p n b u s* **.**
from *s_space* ‹$a \in s$› *out_eq_p*[*OF* ‹$a \in s$›]
have $a \in (\lambda f\ x.\ if\ n \leq x\ then\ p\ else\ f\ x)$ ' ({..< *n*} $\rightarrow_E$ {.. *p*})

```
    by (auto simp: image_iff subset_eq Pi_iff split: if_split_asm
          intro!: bexI[of _ restrict a {..< n}]) }
  then show {s. ksimplex p n s} ⊆ Pow ((λf x. if n ≤ x then p else f x) ‘ ({..<
n} →E {.. p}))
    by auto
qed (simp add: finite_PiE)

lemma ksimplex_card:
  assumes ksimplex p n s shows card s = Suc n
using assms proof cases
  case (ksimplex u b)
  then interpret kuhn_simplex p n u b s .
  show ?thesis
    by (simp add: card_image s_eq inj_enum)
qed

lemma simplex_top_face:
  assumes 0 < p ∀x∈s'. x n = p
  shows ksimplex p n s' ⟷ (∃s a. ksimplex p (Suc n) s ∧ a ∈ s ∧ s' = s − {a})
  using assms
proof safe
  fix s a assume ksimplex p (Suc n) s and a: a ∈ s and na: ∀x∈s − {a}. x n =
p
  then show ksimplex p n (s − {a})
  proof cases
    case (ksimplex base upd)
    then interpret kuhn_simplex p Suc n base upd s .

    have a n < p
      using one_step[of a n p] na ‹a∈s› s_space by (auto simp: less_le)
    then have a = enum 0
      using ‹a ∈ s› na by (subst enum_0_bot) (auto simp: le_less intro!: less[of a _
n])
    then have s_eq: s − {a} = enum ‘ Suc ‘ {.. n}
      using s_eq by (simp add: atMost_Suc_eq_insert_0 insert_ident in_enum_image
subset_eq)
    then have enum 1 ∈ s − {a}
      by auto
    then have upd 0 = n
      using ‹a n < p› ‹a = enum 0› na[rule_format, of enum 1]
      by (auto simp: fun_eq_iff enum_Suc split: if_split_asm)
    then have bij_betw upd (Suc ‘ {..< n}) {..< n}
      using upd
      by (subst notIn_Un_bij_betw3[where b=0])
        (auto simp: lessThan_Suc[symmetric] lessThan_Suc_eq_insert_0)
    then have bij_betw (upd∘Suc) {..<n} {..<n}
      by (rule bij_betw_trans[rotated]) (auto simp: bij_betw_def)

    have a n = p − 1
```

```
    using enum_Suc[of 0] na[rule_format, OF ‹enum 1 ∈ s − {a}›] ‹a = enum
0› by (auto simp: ‹upd 0 = n›)

   show ?thesis
   proof (rule ksimplex.intros, standard)
     show bij_betw (upd∘Suc) {..< n} {..< n} by fact
     show base(n := p) ∈ {..<n} → {..<p} ⋀i. n≤i ⟹ (base(n := p)) i = p
       using base base_out by (auto simp: Pi_iff)

     have ⋀i. Suc ‘ {..< i} = {..< Suc i} − {0}
       by (auto simp: image_iff Ball_def) arith
     then have upd_Suc: ⋀i. i ≤ n ⟹ (upd∘Suc) ‘ {..< i} = upd ‘ {..< Suc i}
− {n}
       using ‹upd 0 = n› upd_inj by (auto simp add: image_iff less_Suc_eq_0_disj)
     have n_in_upd: ⋀i. n ∈ upd ‘ {..< Suc i}
       using ‹upd 0 = n› by auto

     define f′ where f′ i j =
      (if j ∈ (upd∘Suc)‘{..< i} then Suc ((base(n := p)) j) else (base(n := p)) j)
for i j
     { fix x i
       assume i [arith]: i ≤ n
       with upd_Suc have (upd ∘ Suc) ‘ {..<i} = upd ‘ {..<Suc i} − {n} .
       with ‹a n < p› ‹a = enum 0› ‹upd 0 = n› ‹a n = p − 1›
       have enum (Suc i) x = f′ i x
         by (auto simp add: f′_def enum_def) }
     then show s − {a} = f′ ‘ {.. n}
       unfolding s_eq image_comp by (intro image_cong) auto
   qed
  qed
next
  assume ksimplex p n s′ and ∗: ∀ x∈s′. x n = p
  then show ∃ s a. ksimplex p (Suc n) s ∧ a ∈ s ∧ s′ = s − {a}
  proof cases
   case (ksimplex base upd)
   then interpret kuhn_simplex p n base upd s′ .
   define b where b = base (n := p − 1)
   define u where u i = (case i of 0 ⇒ n | Suc i ⇒ upd i) for i

   have ksimplex p (Suc n) (s′ ∪ {b})
   proof (rule ksimplex.intros, standard)
     show b ∈ {..<Suc n} → {..<p}
       using base ‹0 < p› unfolding lessThan_Suc b_def by (auto simp: PiE_iff)
     show ⋀i. Suc n ≤ i ⟹ b i = p
       using base_out by (auto simp: b_def)

     have bij_betw u (Suc ‘ {..< n} ∪ {0}) ({..<n} ∪ {u 0})
       using upd
         by (intro notIn_Un_bij_betw) (auto simp: u_def bij_betw_def image_comp
```

```
comp_def inj_on_def)
    then show bij_betw u {..<Suc n} {..<Suc n}
      by (simp add: u_def lessThan_Suc[symmetric] lessThan_Suc_eq_insert_0)

    define f' where f' i j = (if j ∈ u`{..< i} then Suc (b j) else b j) for i j

    have u_eq: ⋀i. i ≤ n ⟹ u ` {..< Suc i} = upd ` {..< i} ∪ { n }
      by (auto simp: u_def image_iff upd_inj Ball_def split: nat.split) arith

    { fix x have x ≤ n ⟹ n ∉ upd ` {..<x}
        using upd_space by (simp add: image_iff neq_iff) }
    note n_not_upd = this

    have *: f' ` {.. Suc n} = f' ` (Suc ` {.. n} ∪ {0})
      unfolding atMost_Suc_eq_insert_0 by simp
    also have ... = (f' ∘ Suc) ` {.. n} ∪ {b}
      by (auto simp: f'_def)
    also have (f' ∘ Suc) ` {.. n} = s'
      using ‹0 < p› base_out[of n]
      unfolding s_eq enum_def[abs_def] f'_def[abs_def] upd_space
      by (intro image_cong) (simp_all add: u_eq b_def fun_eq_iff n_not_upd)
    finally show s' ∪ {b} = f' ` {.. Suc n} ..
  qed
  moreover have b ∉ s'
    using * ‹0 < p› by (auto simp: b_def)
  ultimately show ?thesis by auto
 qed
qed

lemma ksimplex_replace_0:
 assumes s: ksimplex p n s and a: a ∈ s
 assumes j: j < n and p: ∀x∈s − {a}. x j = 0
 shows card {s'. ksimplex p n s' ∧ (∃b∈s'. s' − {b} = s − {a})} = 1
 using s
proof cases
 case (ksimplex b_s u_s)

 { fix t b assume ksimplex p n t
  then obtain b_t u_t where kuhn_simplex p n b_t u_t t
    by (auto elim: ksimplex.cases)
  interpret kuhn_simplex_pair p n b_s u_s s b_t u_t t
    by intro_locales fact+

  assume b: b ∈ t t − {b} = s − {a}
  with a j p s.replace_0[of _ a] t.replace_0[of _ b] have s = t
    by (intro ksimplex_eq_top[of a b]) auto }
 then have {s'. ksimplex p n s' ∧ (∃b∈s'. s' − {b} = s − {a})} = {s}
  using s ‹a ∈ s› by auto
 then show ?thesis
```

```
  by simp
qed

lemma ksimplex_replace_1:
  assumes s: ksimplex p n s and a: a ∈ s
  assumes j: j < n and p: ∀x∈s − {a}. x j = p
  shows card {s'. ksimplex p n s' ∧ (∃b∈s'. s' − {b} = s − {a})} = 1
  using s
proof cases
  case (ksimplex b_s u_s)

  { fix t b assume ksimplex p n t
    then obtain b_t u_t where kuhn_simplex p n b_t u_t t
      by (auto elim: ksimplex.cases)
    interpret kuhn_simplex_pair p n b_s u_s s b_t u_t t
      by intro_locales fact+

    assume b: b ∈ t t − {b} = s − {a}
    with a j p s.replace_1[of _ a] t.replace_1[of _ b] have s = t
      by (intro ksimplex_eq_bot[of a b]) auto }
  then have {s'. ksimplex p n s' ∧ (∃b∈s'. s' − {b} = s − {a})} = {s}
    using s ‹a ∈ s› by auto
  then show ?thesis
    by simp
qed

lemma ksimplex_replace_2:
  assumes s: ksimplex p n s and a ∈ s and n ≠ 0
    and lb: ∀j<n. ∃x∈s − {a}. x j ≠ 0
    and ub: ∀j<n. ∃x∈s − {a}. x j ≠ p
  shows card {s'. ksimplex p n s' ∧ (∃b∈s'. s' − {b} = s − {a})} = 2
  using s
proof cases
  case (ksimplex base upd)
  then interpret kuhn_simplex p n base upd s .

  from ‹a ∈ s› obtain i where i ≤ n a = enum i
    unfolding s_eq by auto

  from ‹i ≤ n› have i = 0 ∨ i = n ∨ (0 < i ∧ i < n)
    by linarith
  then have ∃!s'. s' ≠ s ∧ ksimplex p n s' ∧ (∃b∈s'. s − {a} = s'− {b})
  proof (elim disjE conjE)
    assume i = 0
    define rot where [abs_def]: rot i = (if i + 1 = n then 0 else i + 1) for i
    let ?upd = upd ∘ rot

    have rot: bij_betw rot {..< n} {..< n}
      by (auto simp: bij_betw_def inj_on_def image_iff Ball_def rot_def)
```

```
      arith+
  from rot upd have bij_betw ?upd {..<n} {..<n}
    by (rule bij_betw_trans)

  define f' where [abs_def]: f' i j =
    (if j ∈ ?upd`{..< i} then Suc (enum (Suc 0) j) else enum (Suc 0) j) for i j

  interpret b: kuhn_simplex p n enum (Suc 0) upd ∘ rot f' ` {.. n}
  proof
    from ‹a = enum i› ub ‹n ≠ 0› ‹i = 0›
    obtain i' where i' ≤ n enum i' ≠ enum 0 enum i' (upd 0) ≠ p
      unfolding s_eq by (auto intro: upd_space simp: enum_inj)
    then have enum 1 ≤ enum i' enum i' (upd 0) < p
     using enum_le_p[of i' upd 0] by (auto simp: enum_inj enum_mono upd_space)
    then have enum 1 (upd 0) < p
      by (auto simp: le_fun_def intro: le_less_trans)
    then show enum (Suc 0) ∈ {..<n} → {..<p}
     using base ‹n ≠ 0› by (auto simp: enum_0 enum_Suc PiE_iff extensional_def
upd_space)

    { fix i assume n ≤ i then show enum (Suc 0) i = p
      using ‹n ≠ 0› by (auto simp: enum_eq_p) }
    show bij_betw ?upd {..<n} {..<n} by fact
  qed (simp add: f'_def)
  have ks_f': ksimplex p n (f' ` {.. n})
    by rule unfold_locales

  have b_enum: b.enum = f' unfolding f'_def b.enum_def[abs_def] ..
  with b.inj_enum have inj_f': inj_on f' {.. n} by simp

  have f'_eq_enum: f' j = enum (Suc j) if j < n for j
  proof -
    from that have rot ` {..< j} = {0 <..< Suc j}
      by (auto simp: rot_def image_Suc_lessThan cong: image_cong_simp)
    with that ‹n ≠ 0› show ?thesis
      by (simp only: f'_def enum_def fun_eq_iff image_comp [symmetric])
        (auto simp add: upd_inj)
  qed
  then have enum ` Suc ` {..< n} = f' ` {..< n}
    by (force simp: enum_inj)
  also have Suc ` {..< n} = {.. n} - {0}
    by (auto simp: image_iff Ball_def) arith
  also have {..< n} = {.. n} - {n}
    by auto
  finally have eq: s - {a} = f' ` {.. n} - {f' n}
    unfolding s_eq ‹a = enum i› ‹i = 0›
     by (simp add: inj_on_image_set_diff[OF inj_enum] inj_on_image_set_diff[OF
inj_f'])
```

have *enum 0* < *f' 0*
  using ‹*n* ≠ *0*› **by** (*simp add*: *enum_strict_mono f'_eq_enum*)
also have . . . < *f' n*
  using ‹*n* ≠ *0*› *b.enum_strict_mono*[*of 0 n*] **unfolding** *b_enum* **by** *simp*
finally have *a* ≠ *f' n*
  using ‹*a* = *enum i*› ‹*i* = *0*› **by** *auto*

{ fix *t c* **assume** *ksimplex p n t c* ∈ *t* **and** *eq_sma*: *s* − {*a*} = *t* − {*c*}
  obtain *b u* **where** *kuhn_simplex p n b u t*
    using ‹*ksimplex p n t*› **by** (*auto elim*: *ksimplex.cases*)
  then interpret *t*: *kuhn_simplex p n b u t* **.**

  { fix *x* **assume** *x* ∈ *s x* ≠ *a*
    then have *x* (*upd 0*) = *enum* (*Suc 0*) (*upd 0*)
      by (*auto simp*: ‹*a* = *enum i*› ‹*i* = *0*› *s_eq enum_def enum_inj*) **}**
  then have *eq_upd0*: ∀ *x*∈*t*−{*c*}. *x* (*upd 0*) = *enum* (*Suc 0*) (*upd 0*)
    unfolding *eq_sma*[*symmetric*] **by** *auto*
  then have *c* (*upd 0*) ≠ *enum* (*Suc 0*) (*upd 0*)
    using ‹*n* ≠ *0*› **by** (*intro t.one_step*[*OF* ‹*c*∈*t*›]) (*auto simp*: *upd_space*)
  then have *c* (*upd 0*) < *enum* (*Suc 0*) (*upd 0*) ∨ *c* (*upd 0*) > *enum* (*Suc 0*) (*upd 0*)
    by *auto*
  then have *t* = *s* ∨ *t* = *f'* ‘ {..*n*}
  proof (*elim disjE conjE*)
    assume ∗: *c* (*upd 0*) < *enum* (*Suc 0*) (*upd 0*)
    interpret *st*: *kuhn_simplex_pair p n base upd s b u t* **..**
    { fix *x* **assume** *x* ∈ *t* **with** ∗ ‹*c*∈*t*› *eq_upd0*[*rule_format*, *of x*] **have** *c* ≤ *x*
        by (*auto simp*: *le_less intro*!: *t.less*[*of _ _ upd 0*]) **}**
    note *top* = *this*
    have *s* = *t*
      using ‹*a* = *enum i*› ‹*i* = *0*› ‹*c* ∈ *t*›
      by (*intro st.ksimplex_eq_bot*[*OF _ _ _ _ eq_sma*])
         (*auto simp*: *s_eq enum_mono t.s_eq t.enum_mono top*)
    then show *?thesis* **by** *simp*
  next
    assume ∗: *c* (*upd 0*) > *enum* (*Suc 0*) (*upd 0*)
    interpret *st*: *kuhn_simplex_pair p n enum* (*Suc 0*) *upd* ∘ *rot f'* ‘ {.. *n*} *b u t* **..**
    have *eq*: *f'* ‘ {..*n*} − {*f' n*} = *t* − {*c*}
      using *eq_sma eq* **by** *simp*
    { fix *x* **assume** *x* ∈ *t* **with** ∗ ‹*c*∈*t*› *eq_upd0*[*rule_format*, *of x*] **have** *x* ≤ *c*
        by (*auto simp*: *le_less intro*!: *t.less*[*of _ _ upd 0*]) **}**
    note *top* = *this*
    have *f'* ‘ {..*n*} = *t*
      using ‹*a* = *enum i*› ‹*i* = *0*› ‹*c* ∈ *t*›
      by (*intro st.ksimplex_eq_top*[*OF _ _ _ _ eq*])
        (*auto simp*: *b.s_eq b.enum_mono t.s_eq t.enum_mono b_enum*[*symmetric*] *top*)
    then show *?thesis* **by** *simp*

```
    qed }
  with ks_f' eq ‹a ≠ f' n› ‹n ≠ 0› show ?thesis
    apply (intro ex1I[of _ f' ` {.. n}])
    apply auto []
    apply metis
    done
next
  assume i = n
  from ‹n ≠ 0› obtain n' where n': n = Suc n'
    by (cases n) auto

  define rot where rot i = (case i of 0 ⇒ n' | Suc i ⇒ i) for i
  let ?upd = upd ∘ rot

  have rot: bij_betw rot {..< n} {..< n}
      by (auto simp: bij_betw_def inj_on_def image_iff Bex_def rot_def n' split:
nat.splits)
      arith
  from rot upd have bij_betw ?upd {..<n} {..<n}
    by (rule bij_betw_trans)

  define b where b = base (upd n' := base (upd n') − 1)
  define f' where [abs_def]: f' i j = (if j ∈ ?upd`{..< i} then Suc (b j) else b
j) for i j

  interpret b: kuhn_simplex p n b upd ∘ rot f' ` {.. n}
  proof
    { fix i assume n ≤ i then show b i = p
        using base_out[of i] upd_space[of n'] by (auto simp: b_def n') }
    show b ∈ {..<n} → {..<p}
      using base ‹n ≠ 0› upd_space[of n']
      by (auto simp: b_def PiE_def Pi_iff Ball_def upd_space extensional_def n')

    show bij_betw ?upd {..<n} {..<n} by fact
  qed (simp add: f'_def)
  have f': b.enum = f' unfolding f'_def b.enum_def[abs_def] ..
  have ks_f': ksimplex p n (b.enum ` {.. n})
    unfolding f' by rule unfold_locales

  have 0 < n
    using ‹n ≠ 0› by auto

  { from ‹a = enum i› ‹n ≠ 0› ‹i = n› lb upd_space[of n']
    obtain i' where i' ≤ n enum i' ≠ enum n 0 < enum i' (upd n')
      unfolding s_eq by (auto simp: enum_inj n')
    moreover have enum i' (upd n') = base (upd n')
      unfolding enum_def using ‹i' ≤ n› ‹enum i' ≠ enum n› by (auto simp: n'
upd_inj enum_inj)
    ultimately have 0 < base (upd n')
```

by *auto* **}**
then have *benum1*: *b.enum* (*Suc 0*) = *base*
unfolding *b.enum_Suc*[*OF* ‹*0*<*n*›] *b.enum_0* **by** (*auto simp*: *b_def rot_def*)

have [*simp*]: ⋀*j*. *Suc j* < *n* ⟹ *rot* ‘ {..< *Suc j*} = {*n'*} ∪ {..< *j*}
by (*auto simp*: *rot_def image_iff Ball_def split*: *nat.splits*)
have *rot_simps*: ⋀*j*. *rot* (*Suc j*) = *j rot 0* = *n'*
by (*simp_all add*: *rot_def*)

{ fix *j* **assume** *j*: *Suc j* ≤ *n* **then have** *b.enum* (*Suc j*) = *enum j*
by (*induct j*) (*auto simp*: *benum1 enum_0 b.enum_Suc enum_Suc rot_simps*)
}
note *b_enum_eq_enum* = *this*
then have *enum* ‘ {..< *n*} = *b.enum* ‘ *Suc* ‘ {..< *n*}
by (*auto simp*: *image_comp intro*!: *image_cong*)
also have *Suc* ‘ {..< *n*} = {.. *n*} − {*0*}
by (*auto simp*: *image_iff Ball_def*) *arith*
also have {..< *n*} = {.. *n*} − {*n*}
by *auto*
finally have *eq*: *s* − {*a*} = *b.enum* ‘ {.. *n*} − {*b.enum 0*}
unfolding *s_eq* ‹*a* = *enum i*› ‹*i* = *n*›
using *inj_on_image_set_diff*[*OF inj_enum Diff_subset*, *of* {*n*}]
inj_on_image_set_diff[*OF b.inj_enum Diff_subset*, *of* {*0*}]
by (*simp add*: *comp_def*)

have *b.enum 0* ≤ *b.enum n*
by (*simp add*: *b.enum_mono*)
also have *b.enum n* < *enum n*
using ‹*n* ≠ *0*› **by** (*simp add*: *enum_strict_mono b_enum_eq_enum n'*)
finally have *a* ≠ *b.enum 0*
using ‹*a* = *enum i*› ‹*i* = *n*› **by** *auto*

{ fix *t c* **assume** *ksimplex p n t c* ∈ *t* **and** *eq_sma*: *s* − {*a*} = *t* − {*c*}
obtain *b' u* **where** *kuhn_simplex p n b' u t*
using ‹*ksimplex p n t*› **by** (*auto elim*: *ksimplex.cases*)
then interpret *t*: *kuhn_simplex p n b' u t* **.**

{ fix *x* **assume** *x* ∈ *s x* ≠ *a*
then have *x* (*upd n'*) = *enum n'* (*upd n'*)
by (*auto simp*: ‹*a* = *enum i*› *n'* ‹*i* = *n*› *s_eq enum_def enum_inj in_upd_image*) **}**
then have *eq_upd0*: ∀ *x*∈*t*−{*c*}. *x* (*upd n'*) = *enum n'* (*upd n'*)
unfolding *eq_sma*[*symmetric*] **by** *auto*
then have *c* (*upd n'*) ≠ *enum n'* (*upd n'*)
using ‹*n* ≠ *0*› **by** (*intro t.one_step*[*OF* ‹*c*∈*t*›]) (*auto simp*: *n' upd_space*[*unfolded n'*])
then have *c* (*upd n'*) < *enum n'* (*upd n'*) ∨ *c* (*upd n'*) > *enum n'* (*upd n'*)
by *auto*
then have *t* = *s* ∨ *t* = *b.enum* ‘ {..*n*}

```
    proof (elim disjE conjE)
      assume *: c (upd n') > enum n' (upd n')
      interpret st: kuhn_simplex_pair p n base upd s b' u t ..
      { fix x assume x ∈ t with * ‹c∈t› eq_upd0[rule_format, of x] have x ≤ c
          by (auto simp: le_less intro!: t.less[of _ _ upd n']) }
      note top = this
      have s = t
        using ‹a = enum i› ‹i = n› ‹c ∈ t›
        by (intro st.ksimplex_eq_top[OF _ _ _ _ eq_sma])
           (auto simp: s_eq enum_mono t.s_eq t.enum_mono top)
      then show ?thesis by simp
    next
      assume *: c (upd n') < enum n' (upd n')
      interpret st: kuhn_simplex_pair p n b upd ∘ rot f' ‘ {.. n} b' u t ..
      have eq: f' ‘ {..n} − {b.enum 0} = t − {c}
        using eq_sma eq f' by simp
      { fix x assume x ∈ t with * ‹c∈t› eq_upd0[rule_format, of x] have c ≤ x
          by (auto simp: le_less intro!: t.less[of _ _ upd n']) }
      note bot = this
      have f' ‘ {..n} = t
        using ‹a = enum i› ‹i = n› ‹c ∈ t›
        by (intro st.ksimplex_eq_bot[OF _ _ _ _ eq])
           (auto simp: b.s_eq b.enum_mono t.s_eq t.enum_mono bot)
      with f' show ?thesis by simp
    qed }
  with ks_f' eq ‹a ≠ b.enum 0› ‹n ≠ 0› show ?thesis
    apply (intro ex1I[of _ b.enum ‘ {.. n}])
    apply auto []
    apply metis
    done
next
  assume i: 0 < i i < n
  define i' where i' = i − 1
  with i have Suc i' < n
    by simp
  with i have Suc_i': Suc i' = i
    by (simp add: i'_def)

  let ?upd = Fun.swap i' i upd
  from i upd have bij_betw ?upd {..< n} {..< n}
    by (subst bij_betw_swap_iff) (auto simp: i'_def)

  define f' where [abs_def]: f' i j = (if j ∈ ?upd‘{..< i} then Suc (base j) else
base j)
    for i j
  interpret b: kuhn_simplex p n base ?upd f' ‘ {.. n}
  proof
    show base ∈ {..<n} → {..<p} by (rule base)
    { fix i assume n ≤ i then show base i = p by (rule base_out) }
```

show *bij_betw ?upd* {..<*n*} {..<*n*} **by** *fact*
qed (*simp add*: *f′_def*)
have *f′*: *b.enum* = *f′* **unfolding** *f′_def b.enum_def*[*abs_def*] **..**
have *ks_f′*: *ksimplex p n* (*b.enum* ‘ {.. *n*})
unfolding *f′* **by** *rule unfold_locales*

have {*i*} ⊆ {..*n*}
using *i* **by** *auto*
{ fix *j* **assume** *j* ≤ *n*
moreover have *j* < *i* ∨ *i* = *j* ∨ *i* < *j* **by** *arith*
moreover note *i*
ultimately have *enum j* = *b.enum j* ⟷ *j* ≠ *i*
unfolding *enum_def*[*abs_def*] *b.enum_def*[*abs_def*]
by (*auto simp*: *fun_eq_iff swap_image i′_def*
in_upd_image inj_on_image_set_diff[*OF inj_upd*]) **}**
note *enum_eq_benum* = *this*
then have *enum* ‘ ({.. *n*} − {*i*}) = *b.enum* ‘ ({.. *n*} − {*i*})
by (*intro image_cong*) *auto*
then have *eq*: *s* − {*a*} = *b.enum* ‘ {.. *n*} − {*b.enum i*}
unfolding *s_eq* ‹*a* = *enum i*›
using *inj_on_image_set_diff*[*OF inj_enum Diff_subset* ‹{*i*} ⊆ {..*n*}›]
inj_on_image_set_diff[*OF b.inj_enum Diff_subset* ‹{*i*} ⊆ {..*n*}›]
by (*simp add*: *comp_def*)

have *a* ≠ *b.enum i*
using ‹*a* = *enum i*› *enum_eq_benum i* **by** *auto*

{ fix *t c* **assume** *ksimplex p n t c* ∈ *t* **and** *eq_sma*: *s* − {*a*} = *t* − {*c*}
obtain *b′ u* **where** *kuhn_simplex p n b′ u t*
using ‹*ksimplex p n t*› **by** (*auto elim*: *ksimplex.cases*)
then interpret *t*: *kuhn_simplex p n b′ u t* **.**
have *enum i′* ∈ *s* − {*a*} *enum* (*i* + *1*) ∈ *s* − {*a*}
using ‹*a* = *enum i*› *i enum_in* **by** (*auto simp*: *enum_inj i′_def*)
then obtain *l k* **where**
l: *t.enum l* = *enum i′ l* ≤ *n t.enum l* ≠ *c* **and**
k: *t.enum k* = *enum* (*i* + *1*) *k* ≤ *n t.enum k* ≠ *c*
unfolding *eq_sma* **by** (*auto simp*: *t.s_eq*)
with *i* **have** *t.enum l* < *t.enum k*
by (*simp add*: *enum_strict_mono i′_def*)
with ‹*l* ≤ *n*› ‹*k* ≤ *n*› **have** *l* < *k*
by (*simp add*: *t.enum_strict_mono*)
{ assume *Suc l* = *k*
have *enum* (*Suc* (*Suc i′*)) = *t.enum* (*Suc l*)
using *i* **by** (*simp add*: *k* ‹*Suc l* = *k*› *i′_def*)
then have *False*
using ‹*l* < *k*› ‹*k* ≤ *n*› ‹*Suc i′* < *n*›
by (*auto simp*: *t.enum_Suc enum_Suc l upd_inj fun_eq_iff split*: *if_split_asm*)
(*metis Suc_lessD n_not_Suc_n upd_inj*) **}**
with ‹*l* < *k*› **have** *Suc l* < *k*

```
      by arith
    have c_eq: c = t.enum (Suc l)
    proof (rule ccontr)
      assume c ≠ t.enum (Suc l)
      then have t.enum (Suc l) ∈ s − {a}
        using ‹l < k› ‹k ≤ n› by (simp add: t.s_eq eq_sma)
      then obtain j where t.enum (Suc l) = enum j j ≤ n enum j ≠ enum i
        unfolding s_eq ‹a = enum i› by auto
      with i have t.enum (Suc l) ≤ t.enum l ∨ t.enum k ≤ t.enum (Suc l)
        by (auto simp: i'_def enum_mono enum_inj l k)
      with ‹Suc l < k› ‹k ≤ n› show False
        by (simp add: t.enum_mono)
    qed

    { have t.enum (Suc (Suc l)) ∈ s − {a}
          unfolding eq_sma c_eq t.s_eq using ‹Suc l < k› ‹k ≤ n› by (auto simp:
t.enum_inj)
      then obtain j where eq: t.enum (Suc (Suc l)) = enum j and j ≤ n j ≠ i
        by (auto simp: s_eq ‹a = enum i›)
      moreover have enum i' < t.enum (Suc (Suc l))
          unfolding l(1)[symmetric] using ‹Suc l < k› ‹k ≤ n› by (auto simp:
t.enum_strict_mono)
      ultimately have i' < j
        using i by (simp add: enum_strict_mono i'_def)
      with ‹j ≠ i› ‹j ≤ n› have t.enum k ≤ t.enum (Suc (Suc l))
        unfolding i'_def by (simp add: enum_mono k eq)
      then have k ≤ Suc (Suc l)
        using ‹k ≤ n› ‹Suc l < k› by (simp add: t.enum_mono) }
    with ‹Suc l < k› have Suc (Suc l) = k by simp
    then have enum (Suc (Suc i')) = t.enum (Suc (Suc l))
      using i by (simp add: k i'_def)
     also have ... = (enum i') (u l := Suc (enum i' (u l)), u (Suc l) := Suc
(enum i' (u (Suc l))))
      using ‹Suc l < k› ‹k ≤ n› by (simp add: t.enum_Suc l t.upd_inj)
    finally have (u l = upd i' ∧ u (Suc l) = upd (Suc i')) ∨
      (u l = upd (Suc i') ∧ u (Suc l) = upd i')
      using ‹Suc i' < n› by (auto simp: enum_Suc fun_eq_iff split: if_split_asm)

    then have t = s ∨ t = b.enum ‘ {..n}
    proof (elim disjE conjE)
      assume u: u l = upd i'
      have c = t.enum (Suc l) unfolding c_eq ..
      also have t.enum (Suc l) = enum (Suc i')
       using u ‹l < k› ‹k ≤ n› ‹Suc i' < n› by (simp add: enum_Suc t.enum_Suc
l)
      also have ... = a
        using ‹a = enum i› i by (simp add: i'_def)
      finally show ?thesis
        using eq_sma ‹a ∈ s› ‹c ∈ t› by auto
```

```
    next
      assume u: u l = upd (Suc i′)
      define B where B = b.enum ‘ {..n}
      have b.enum i′ = enum i′
        using enum_eq_benum[of i′] i by (auto simp: i′_def gr0_conv_Suc)
      have c = t.enum (Suc l) unfolding c_eq ..
      also have t.enum (Suc l) = b.enum (Suc i′)
        using u ‹l < k› ‹k ≤ n› ‹Suc i′ < n›
        by (simp_all add: enum_Suc t.enum_Suc l b.enum_Suc ‹b.enum i′ = enum
i′›)
           (simp add: Suc_i′)
      also have ... = b.enum i
        using i by (simp add: i′_def)
      finally have c = b.enum i .
      then have t − {c} = B − {c} c ∈ B
        unfolding eq_sma[symmetric] eq B_def using i by auto
      with ‹c ∈ t› have t = B
        by auto
      then show ?thesis
        by (simp add: B_def)
    qed }
  with ks_f′ eq ‹a ≠ b.enum i› ‹n ≠ 0› ‹i ≤ n› show ?thesis
    apply (intro ex1I[of _ b.enum ‘ {.. n}])
    apply auto []
    apply metis
    done
  qed
  then show ?thesis
    using s ‹a ∈ s› by (simp add: card_2_iff′ Ex1_def) metis
qed
```

Hence another step towards concreteness.

```
lemma kuhn_simplex_lemma:
  assumes ∀ s. ksimplex p (Suc n) s ⟶ rl ‘ s ⊆ {.. Suc n}
    and odd (card {f. ∃ s a. ksimplex p (Suc n) s ∧ a ∈ s ∧ (f = s − {a}) ∧
      rl ‘ f = {..n} ∧ ((∃ j≤n. ∀ x∈f. x j = 0) ∨ (∃ j≤n. ∀ x∈f. x j = p))})
  shows odd (card {s. ksimplex p (Suc n) s ∧ rl ‘ s = {..Suc n}})
proof (rule kuhn_complete_lemma[OF finite_ksimplexes refl, unfolded mem_Collect_eq,
    where bnd=λf. (∃ j∈{..n}. ∀ x∈f. x j = 0) ∨ (∃ j∈{..n}. ∀ x∈f. x j = p)],
    safe del: notI)

  have ∗: ⋀x y. x = y ⟹ odd (card x) ⟹ odd (card y)
    by auto
  show odd (card {f. (∃ s∈{s. ksimplex p (Suc n) s}. ∃ a∈s. f = s − {a}) ∧
    rl ‘ f = {..n} ∧ ((∃ j∈{..n}. ∀ x∈f. x j = 0) ∨ (∃ j∈{..n}. ∀ x∈f. x j = p))})
    apply (rule ∗[OF _ assms(2)])
    apply (auto simp: atLeast0AtMost)
    done
```

next

fix *s* **assume** *s*: *ksimplex p* (*Suc n*) *s*
then show *card s* = *n* + *2*
by (*simp add*: *ksimplex_card*)

fix *a* **assume** *a*: *a* ∈ *s* **then show** *rl a* ≤ *Suc n*
using *assms*(*1*) *s* **by** (*auto simp*: *subset_eq*)

let *?S* = {*t*. *ksimplex p* (*Suc n*) *t* ∧ (∃ *b*∈*t*. *s* − {*a*} = *t* − {*b*})}
{ **fix** *j* **assume** *j*: *j* ≤ *n* ∀ *x*∈*s* − {*a*}. *x j* = *0*
with *s a* **show** *card ?S* = *1*
using *ksimplex_replace_0*[*of p n* + *1 s a j*]
by (*subst eq_commute*) *simp* }

{ **fix** *j* **assume** *j*: *j* ≤ *n* ∀ *x*∈*s* − {*a*}. *x j* = *p*
with *s a* **show** *card ?S* = *1*
using *ksimplex_replace_1*[*of p n* + *1 s a j*]
by (*subst eq_commute*) *simp* }

{ **assume** *card ?S* ≠ *2* ¬ (∃ *j*∈{..*n*}. ∀ *x*∈*s* − {*a*}. *x j* = *p*)
with *s a* **show** ∃ *j*∈{..*n*}. ∀ *x*∈*s* − {*a*}. *x j* = *0*
using *ksimplex_replace_2*[*of p n* + *1 s a*]
by (*subst* (*asm*) *eq_commute*) *auto* }
qed

Reduced labelling

definition *reduced* :: *nat* ⇒ (*nat* ⇒ *nat*) ⇒ *nat* **where** *reduced n x* = (*LEAST k*. *k* = *n* ∨ *x k* ≠ *0*)

lemma *reduced_labelling*:
shows *reduced n x* ≤ *n*
and ∀ *i*<*reduced n x*. *x i* = *0*
and *reduced n x* = *n* ∨ *x* (*reduced n x*) ≠ *0*
proof −
show *reduced n x* ≤ *n*
unfolding *reduced_def* **by** (*rule LeastI2_wellorder*[**where** *a*=*n*]) *auto*
show ∀ *i*<*reduced n x*. *x i* = *0*
unfolding *reduced_def* **by** (*rule LeastI2_wellorder*[**where** *a*=*n*]) *fastforce*+
show *reduced n x* = *n* ∨ *x* (*reduced n x*) ≠ *0*
unfolding *reduced_def* **by** (*rule LeastI2_wellorder*[**where** *a*=*n*]) *fastforce*+
qed

lemma *reduced_labelling_unique*:
r ≤ *n* ⟹ ∀ *i*<*r*. *x i* = *0* ⟹ *r* = *n* ∨ *x r* ≠ *0* ⟹ *reduced n x* = *r*
unfolding *reduced_def* **by** (*rule LeastI2_wellorder*[**where** *a*=*n*]) (*metis le_less not_le*)+

lemma *reduced_labelling_zero*: $j < n \Longrightarrow x\ j = 0 \Longrightarrow reduced\ n\ x \neq j$
  using *reduced_labelling*[*of n x*] **by** *auto*

lemma *reduce_labelling_zero*[*simp*]: $reduced\ 0\ x = 0$
  by (*rule reduced_labelling_unique*) *auto*

lemma *reduced_labelling_nonzero*: $j < n \Longrightarrow x\ j \neq 0 \Longrightarrow reduced\ n\ x \leq j$
  using *reduced_labelling*[*of n x*] **by** (*elim allE*[**where** *x=j*]) *auto*

lemma *reduced_labelling_Suc*: $reduced\ (Suc\ n)\ x \neq Suc\ n \Longrightarrow reduced\ (Suc\ n)\ x = reduced\ n\ x$
  using *reduced_labelling*[*of Suc n x*]
  by (*intro reduced_labelling_unique*[*symmetric*]) *auto*

lemma *complete_face_top*:
  assumes $\forall x \in f.\ \forall j \leq n.\ x\ j = 0 \longrightarrow lab\ x\ j = 0$
    and $\forall x \in f.\ \forall j \leq n.\ x\ j = p \longrightarrow lab\ x\ j = 1$
    and *eq*: $(reduced\ (Suc\ n) \circ lab)$ ‘ $f = \{..n\}$
  shows $((\exists j \leq n.\ \forall x \in f.\ x\ j = 0) \vee (\exists j \leq n.\ \forall x \in f.\ x\ j = p)) \longleftrightarrow (\forall x \in f.\ x\ n = p)$
proof (*safe del*: *disjCI*)
  fix *x j* **assume** *j*: $j \leq n\ \forall x \in f.\ x\ j = 0$
  { fix *x* **assume** $x \in f$ **with** *assms j* **have** $reduced\ (Suc\ n)\ (lab\ x) \neq j$
      by (*intro reduced_labelling_zero*) *auto* **}**
  moreover have $j \in (reduced\ (Suc\ n) \circ lab)$ ‘ f
    using *j eq* **by** *auto*
  ultimately show $x\ n = p$
    by *force*
next
  fix *x j* **assume** *j*: $j \leq n\ \forall x \in f.\ x\ j = p$ **and** *x*: $x \in f$
  have $j = n$
  proof (*rule ccontr*)
    assume $\neg$ *?thesis*
    { fix *x* **assume** $x \in f$
      with *assms j* **have** $reduced\ (Suc\ n)\ (lab\ x) \leq j$
        by (*intro reduced_labelling_nonzero*) *auto*
      then have $reduced\ (Suc\ n)\ (lab\ x) \neq n$
        using ‹$j \neq n$› ‹$j \leq n$› **by** *simp* **}**
    moreover
    have $n \in (reduced\ (Suc\ n) \circ lab)$ ‘ f
      using *eq* **by** *auto*
    ultimately show *False*
      by *force*
  qed
  moreover have $j \in (reduced\ (Suc\ n) \circ lab)$ ‘ f
    using *j eq* **by** *auto*
  ultimately show $x\ n = p$
    using *j x* **by** *auto*
qed *auto*

Hence we get just about the nice induction.

lemma *kuhn_induction*:
  assumes $0 < p$
    and *lab_0*: $\forall x. \forall j{\le}n. (\forall j. x\ j \le p) \wedge x\ j = 0 \longrightarrow lab\ x\ j = 0$
    and *lab_1*: $\forall x. \forall j{\le}n. (\forall j. x\ j \le p) \wedge x\ j = p \longrightarrow lab\ x\ j = 1$
    and *odd*: $odd\ (card\ \{s.\ ksimplex\ p\ n\ s \wedge (reduced\ n{\circ}lab)\ `\ s = \{..n\}\})$
  shows $odd\ (card\ \{s.\ ksimplex\ p\ (Suc\ n)\ s \wedge (reduced\ (Suc\ n){\circ}lab)\ `\ s = \{..Suc\ n\}\})$
proof –
  let $?rl = reduced\ (Suc\ n) \circ lab$ **and** $?ext = \lambda f\ v.\ \exists j{\le}n.\ \forall x{\in}f.\ x\ j = v$
  let $?ext = \lambda s.\ (\exists j{\le}n.\ \forall x{\in}s.\ x\ j = 0) \vee (\exists j{\le}n.\ \forall x{\in}s.\ x\ j = p)$
  have $\forall s.\ ksimplex\ p\ (Suc\ n)\ s \longrightarrow ?rl\ `\ s \subseteq \{..Suc\ n\}$
    by (*simp add*: *reduced_labelling subset_eq*)
  moreover
  have $\{s.\ ksimplex\ p\ n\ s \wedge (reduced\ n \circ lab)\ `\ s = \{..n\}\} =$
      $\{f.\ \exists s\ a.\ ksimplex\ p\ (Suc\ n)\ s \wedge a \in s \wedge f = s - \{a\} \wedge ?rl\ `\ f = \{..n\} \wedge ?ext\ f\}$
  proof (*intro set_eqI*, *safe del*: *disjCI equalityI disjE*)
    fix *s* **assume** *s*: *ksimplex p n s* **and** *rl*: $(reduced\ n \circ lab)\ `\ s = \{..n\}$
    from *s* **obtain** *u b* **where** *kuhn_simplex p n u b s* **by** (*auto elim*: *ksimplex.cases*)
    then interpret *kuhn_simplex p n u b s* **.**
    have *all_eq_p*: $\forall x{\in}s.\ x\ n = p$
      by (*auto simp*: *out_eq_p*)
    moreover
    { fix *x* **assume** $x \in s$
      with *lab_1*[*rule_format*, *of n x*] *all_eq_p s_le_p*[*of x*]
      have $?rl\ x \le n$
        by (*auto intro*!: *reduced_labelling_nonzero*)
      then have $?rl\ x = reduced\ n\ (lab\ x)$
        by (*auto intro*!: *reduced_labelling_Suc*) **}**
    then have $?rl\ `\ s = \{..n\}$
      using *rl* **by** (*simp cong*: *image_cong*)
    moreover
    obtain *t a* **where** *ksimplex p (Suc n) t* $a \in t$ $s = t - \{a\}$
      using *s* **unfolding** *simplex_top_face*[*OF* ‹$0 < p$› *all_eq_p*] **by** *auto*
    ultimately
    show $\exists t\ a.\ ksimplex\ p\ (Suc\ n)\ t \wedge a \in t \wedge s = t - \{a\} \wedge ?rl\ `\ s = \{..n\} \wedge ?ext\ s$
      by *auto*
  next
    fix *x s a* **assume** *s*: *ksimplex p (Suc n) s* **and** *rl*: $?rl\ `\ (s - \{a\}) = \{..\ n\}$
      and *a*: $a \in s$ **and** $?ext\ (s - \{a\})$
     from *s* **obtain** *u b* **where** *kuhn_simplex p (Suc n) u b s* **by** (*auto elim*: *ksimplex.cases*)
    then interpret *kuhn_simplex p Suc n u b s* **.**
    have *all_eq_p*: $\forall x{\in}s.\ x\ (Suc\ n) = p$
      by (*auto simp*: *out_eq_p*)

    { fix *x* **assume** $x \in s - \{a\}$

```
    then have ?rl x ∈ ?rl ‘ (s − {a})
      by auto
    then have ?rl x ≤ n
      unfolding rl by auto
    then have ?rl x = reduced n (lab x)
      by (auto intro!: reduced_labelling_Suc) }
  then show rl': (reduced n∘lab) ‘ (s − {a}) = {..n}
    unfolding rl[symmetric] by (intro image_cong) auto

  from ‹?ext (s − {a})›
  have all_eq_p: ∀ x∈s − {a}. x n = p
  proof (elim disjE exE conjE)
    fix j assume j ≤ n ∀ x∈s − {a}. x j = 0
    with lab_0[rule_format, of j] all_eq_p s_le_p
    have ⋀x. x ∈ s − {a} ⟹ reduced (Suc n) (lab x) ≠ j
      by (intro reduced_labelling_zero) auto
    moreover have j ∈ ?rl ‘ (s − {a})
      using ‹j ≤ n› unfolding rl by auto
    ultimately show ?thesis
      by force
  next
    fix j assume j ≤ n and eq_p: ∀ x∈s − {a}. x j = p
    show ?thesis
    proof cases
      assume j = n with eq_p show ?thesis by simp
    next
      assume j ≠ n
      { fix x assume x: x ∈ s − {a}
        have reduced n (lab x) ≤ j
        proof (rule reduced_labelling_nonzero)
          show lab x j ≠ 0
            using lab_1[rule_format, of j x] x s_le_p[of x] eq_p ‹j ≤ n› by auto
          show j < n
            using ‹j ≤ n› ‹j ≠ n› by simp
        qed
        then have reduced n (lab x) ≠ n
          using ‹j ≤ n› ‹j ≠ n› by simp }
      moreover have n ∈ (reduced n∘lab) ‘ (s − {a})
        unfolding rl' by auto
      ultimately show ?thesis
        by force
    qed
  qed
  show ksimplex p n (s − {a})
    unfolding simplex_top_face[OF ‹0 < p› all_eq_p] using s a by auto
 qed
 ultimately show ?thesis
  using assms by (intro kuhn_simplex_lemma) auto
qed
```

And so we get the final combinatorial result.

lemma *ksimplex_0*: *ksimplex p 0 s* ⟷ *s* = {(λ*x*. *p*)}
proof
  assume *ksimplex p 0 s* **then show** *s* = {(λ*x*. *p*)}
    by (*blast dest*: *kuhn_simplex.ksimplex_0 elim*: *ksimplex.cases*)
next
  assume *s*: *s* = {(λ*x*. *p*)}
  show *ksimplex p 0 s*
  proof (*intro ksimplex*, *unfold_locales*)
    show (λ_. *p*) ∈ {..<*0*::*nat*} → {..<*p*} **by** *auto*
    show *bij_betw id* {..<*0*} {..<*0*}
      by *simp*
  qed (*auto simp*: *s*)
qed

lemma *kuhn_combinatorial*:
  assumes *0* < *p*
    and ∀ *x j*. (∀ *j*. *x j* ≤ *p*) ∧ *j* < *n* ∧ *x j* = *0* ⟶ *lab x j* = *0*
    and ∀ *x j*. (∀ *j*. *x j* ≤ *p*) ∧ *j* < *n* ∧ *x j* = *p* ⟶ *lab x j* = *1*
  shows *odd* (*card* {*s*. *ksimplex p n s* ∧ (*reduced n*∘*lab*) ‘ *s* = {..*n*}})
    (**is** *odd* (*card* (*?M n*)))
  using *assms*
proof (*induct n*)
  case *0* **then show** *?case*
    by (*simp add*: *ksimplex_0 cong*: *conj_cong*)
next
  case (*Suc n*)
  then have *odd* (*card* (*?M n*))
    by *force*
  with *Suc* **show** *?case*
    using *kuhn_induction*[*of p n*] **by** (*auto simp*: *comp_def*)
qed

lemma *kuhn_lemma*:
  fixes *n p* :: *nat*
  assumes *0* < *p*
    and ∀ *x*. (∀ *i*<*n*. *x i* ≤ *p*) ⟶ (∀ *i*<*n*. *label x i* = (*0*::*nat*) ∨ *label x i* = *1*)
    and ∀ *x*. (∀ *i*<*n*. *x i* ≤ *p*) ⟶ (∀ *i*<*n*. *x i* = *0* ⟶ *label x i* = *0*)
    and ∀ *x*. (∀ *i*<*n*. *x i* ≤ *p*) ⟶ (∀ *i*<*n*. *x i* = *p* ⟶ *label x i* = *1*)
  obtains *q* **where** ∀ *i*<*n*. *q i* < *p*
    and ∀ *i*<*n*. ∃ *r s*. (∀ *j*<*n*. *q j* ≤ *r j* ∧ *r j* ≤ *q j* + *1*) ∧ (∀ *j*<*n*. *q j* ≤ *s j* ∧ *s j* ≤ *q j* + *1*) ∧ *label r i* ≠ *label s i*
proof −
  let *?rl* = *reduced n* ∘ *label*
  let *?A* = {*s*. *ksimplex p n s* ∧ *?rl* ‘ *s* = {..*n*}}
  have *odd* (*card ?A*)
    using *assms* **by** (*intro kuhn_combinatorial*[*of p n label*]) *auto*
  then have *?A* ≠ {}
    by (*rule odd_card_imp_not_empty*)

then obtain *s b u* **where** *kuhn_simplex p n b u s* **and** *rl*: *?rl ‘ s* = {..*n*}
  by (*auto elim*: *ksimplex.cases*)
interpret *kuhn_simplex p n b u s* **by** *fact*

show *?thesis*
proof (*intro that*[*of b*] *allI impI*)
  fix *i*
  assume *i* < *n*
  then show *b i* < *p*
    using *base* **by** *auto*
next
  fix *i*
  assume *i* < *n*
  then have *i* ∈ {.. *n*} *Suc i* ∈ {.. *n*}
    by *auto*
  then obtain *u v* **where** *u*: *u* ∈ *s Suc i* = *?rl u* **and** *v*: *v* ∈ *s i* = *?rl v*
    unfolding *rl*[*symmetric*] **by** *blast*

  have *label u i* ≠ *label v i*
    using *reduced_labelling* [*of n label u*] *reduced_labelling* [*of n label v*]
      u(*2*)[*symmetric*] *v*(*2*)[*symmetric*] ‹*i* < *n*›
    by *auto*
  moreover
  have *b j* ≤ *u j u j* ≤ *b j* + *1 b j* ≤ *v j v j* ≤ *b j* + *1* **if** *j* < *n* **for** *j*
  using *that base_le*[*OF* ‹*u*∈*s*›] *le_Suc_base*[*OF* ‹*u*∈*s*›] *base_le*[*OF* ‹*v*∈*s*›] *le_Suc_base*[*OF* ‹*v*∈*s*›]
    by *auto*
  ultimately show ∃ *r s*. (∀ *j*<*n*. *b j* ≤ *r j* ∧ *r j* ≤ *b j* + *1*) ∧
      (∀ *j*<*n*. *b j* ≤ *s j* ∧ *s j* ≤ *b j* + *1*) ∧ *label r i* ≠ *label s i*
    by *blast*
qed
qed

Main result for the unit cube

lemma *kuhn_labelling_lemma'*:
  assumes (∀ *x*::*nat*⇒*real*. *P x* ⟶ *P* (*f x*))
    and ∀ *x*. *P x* ⟶ (∀ *i*::*nat*. *Q i* ⟶ *0* ≤ *x i* ∧ *x i* ≤ *1*)
  shows ∃ *l*. (∀ *x i*. *l x i* ≤ (*1*::*nat*)) ∧
      (∀ *x i*. *P x* ∧ *Q i* ∧ *x i* = *0* ⟶ *l x i* = *0*) ∧
      (∀ *x i*. *P x* ∧ *Q i* ∧ *x i* = *1* ⟶ *l x i* = *1*) ∧
      (∀ *x i*. *P x* ∧ *Q i* ∧ *l x i* = *0* ⟶ *x i* ≤ *f x i*) ∧
      (∀ *x i*. *P x* ∧ *Q i* ∧ *l x i* = *1* ⟶ *f x i* ≤ *x i*)
proof −
  have *and_forall_thm*: ⋀*P Q*. (∀ *x*. *P x*) ∧ (∀ *x*. *Q x*) ⟷ (∀ *x*. *P x* ∧ *Q x*)
    by *auto*
  have ∗: ∀ *x y*::*real*. *0* ≤ *x* ∧ *x* ≤ *1* ∧ *0* ≤ *y* ∧ *y* ≤ *1* ⟶ *x* ≠ *1* ∧ *x* ≤ *y* ∨ *x* ≠ *0* ∧ *y* ≤ *x*
    by *auto*

```
  show ?thesis
    unfolding and_forall_thm
    apply (subst choice_iff[symmetric])+
    apply rule
    apply rule
  proof –
    fix x x'
    let ?R = λy::nat.
      (P x ∧ Q x' ∧ x x' = 0 ⟶ y = 0) ∧
      (P x ∧ Q x' ∧ x x' = 1 ⟶ y = 1) ∧
      (P x ∧ Q x' ∧ y = 0 ⟶ x x' ≤ (f x) x') ∧
      (P x ∧ Q x' ∧ y = 1 ⟶ (f x) x' ≤ x x')
    have 0 ≤ f x x' ∧ f x x' ≤ 1 if P x Q x'
      using assms(2)[rule_format,of f x x'] that
      apply (drule_tac assms(1)[rule_format])
      apply auto
      done
    then have ?R 0 ∨ ?R 1
      by auto
    then show ∃y≤1. ?R y
      by auto
  qed
qed
```

6.31.3 Brouwer's fixed point theorem

We start proving Brouwer's fixed point theorem for the unit cube = *cbox 0 One*.

```
lemma brouwer_cube:
  fixes f :: 'a::euclidean_space ⇒ 'a
  assumes continuous_on (cbox 0 One) f
    and f ` cbox 0 One ⊆ cbox 0 One
  shows ∃x∈cbox 0 One. f x = x
proof (rule ccontr)
  define n where n = DIM('a)
  have n: 1 ≤ n 0 < n n ≠ 0
    unfolding n_def by (auto simp: Suc_le_eq)
  assume ¬ ?thesis
  then have ∗: ¬ (∃x∈cbox 0 One. f x − x = 0)
    by auto
  obtain d where
      d: d > 0 ⋀x. x ∈ cbox 0 One ⟹ d ≤ norm (f x − x)
    apply (rule brouwer_compactness_lemma[OF compact_cbox _ ∗])
    apply (rule continuous_intros assms)+
    apply blast
    done
  have ∗: ∀x. x ∈ cbox 0 One ⟶ f x ∈ cbox 0 One
   ∀x. x ∈ (cbox 0 One::'a set) ⟶ (∀i∈Basis. True ⟶ 0 ≤ x · i ∧ x · i ≤ 1)
    using assms(2)[unfolded image_subset_iff Ball_def]
```

```
  unfolding cbox_def
  by auto
obtain label :: 'a ⇒ 'a ⇒ nat where label [rule_format]:
  ∀ x. ∀ i∈Basis. label x i ≤ 1
  ∀ x. ∀ i∈Basis. x ∈ cbox 0 One ∧ x • i = 0 ⟶ label x i = 0
  ∀ x. ∀ i∈Basis. x ∈ cbox 0 One ∧ x • i = 1 ⟶ label x i = 1
  ∀ x. ∀ i∈Basis. x ∈ cbox 0 One ∧ label x i = 0 ⟶ x • i ≤ f x • i
  ∀ x. ∀ i∈Basis. x ∈ cbox 0 One ∧ label x i = 1 ⟶ f x • i ≤ x • i
  using kuhn_labelling_lemma[OF *] by auto
note label = this [rule_format]
have lem1: ∀ x∈cbox 0 One. ∀ y∈cbox 0 One. ∀ i∈Basis. label x i ≠ label y i ⟶
  |f x • i − x • i| ≤ norm (f y − f x) + norm (y − x)
proof safe
  fix x y :: 'a
  assume x: x ∈ cbox 0 One and y: y ∈ cbox 0 One
  fix i
  assume i: label x i ≠ label y i i ∈ Basis
  have *: ⋀x y fx fy :: real. x ≤ fx ∧ fy ≤ y ∨ fx ≤ x ∧ y ≤ fy ⟹
    |fx − x| ≤ |fy − fx| + |y − x| by auto
  have |(f x − x) • i| ≤ |(f y − f x)•i| + |(y − x)•i|
  proof (cases label x i = 0)
    case True
    then have fxy: ¬ f y • i ≤ y • i ⟹ f x • i ≤ x • i
      by (metis True i label(1) label(5) le_antisym less_one not_le_imp_less y)
    show ?thesis
    unfolding inner_simps
    by (rule *) (auto simp: True i label x y fxy)
  next
    case False
    then show ?thesis
      using label [OF ‹i ∈ Basis›] i(1) x y
      apply (auto simp: inner_diff_left le_Suc_eq)
      by (metis *)
  qed
  also have ... ≤ norm (f y − f x) + norm (y − x)
    by (simp add: add_mono i(2) norm_bound_Basis_le)
  finally show |f x • i − x • i| ≤ norm (f y − f x) + norm (y − x)
    unfolding inner_simps .
qed
have ∃ e>0. ∀ x∈cbox 0 One. ∀ y∈cbox 0 One. ∀ z∈cbox 0 One. ∀ i∈Basis.
  norm (x − z) < e ⟶ norm (y − z) < e ⟶ label x i ≠ label y i ⟶
    |(f(z) − z)•i| < d / (real n)
proof −
  have d': d / real n / 8 > 0
    using d(1) by (simp add: n_def)
  have *: uniformly_continuous_on (cbox 0 One) f
    by (rule compact_uniformly_continuous[OF assms(1) compact_cbox])
  obtain e where e:
      e > 0
```

```
      ⋀x x'. x ∈ cbox 0 One ⟹
        x' ∈ cbox 0 One ⟹
        norm (x' − x) < e ⟹
        norm (f x' − f x) < d / real n / 8
    using *[unfolded uniformly_continuous_on_def,rule_format,OF d']
    unfolding dist_norm
    by blast
  show ?thesis
  proof (intro exI conjI ballI impI)
    show 0 < min (e / 2) (d / real n / 8)
      using d' e by auto
    fix x y z i
    assume as:
      x ∈ cbox 0 One y ∈ cbox 0 One z ∈ cbox 0 One
      norm (x − z) < min (e / 2) (d / real n / 8)
      norm (y − z) < min (e / 2) (d / real n / 8)
      label x i ≠ label y i
    assume i: i ∈ Basis
    have *: ⋀z fz x fx n1 n2 n3 n4 d4 d :: real. |fx − x| ≤ n1 + n2 ⟹
      |fx − fz| ≤ n3 ⟹ |x − z| ≤ n4 ⟹
      n1 < d4 ⟹ n2 < 2 * d4 ⟹ n3 < d4 ⟹ n4 < d4 ⟹
      (8 * d4 = d) ⟹ |fz − z| < d
      by auto
    show |(f z − z) · i| < d / real n
      unfolding inner_simps
    proof (rule *)
      show |f x · i − x · i| ≤ norm (f y −f x) + norm (y − x)
        using as(1) as(2) as(6) i lem1 by blast
      show norm (f x − f z) < d / real n / 8
        using d' e as by auto
      show |f x · i − f z · i| ≤ norm (f x − f z) |x · i − z · i| ≤ norm (x − z)
        unfolding inner_diff_left[symmetric]
        by (rule Basis_le_norm[OF i])+
      have tria: norm (y − x) ≤ norm (y − z) + norm (x − z)
        using dist_triangle[of y x z, unfolded dist_norm]
        unfolding norm_minus_commute
        by auto
      also have ... < e / 2 + e / 2
        using as(4) as(5) by auto
      finally show norm (f y − f x) < d / real n / 8
        using as(1) as(2) e(2) by auto
      have norm (y − z) + norm (x − z) < d / real n / 8 + d / real n / 8
        using as(4) as(5) by auto
      with tria show norm (y − x) < 2 * (d / real n / 8)
        by auto
    qed (use as in auto)
  qed
qed
then
```

```
obtain e where e:
  e > 0
  ⋀x y z i. x ∈ cbox 0 One ⟹
    y ∈ cbox 0 One ⟹
    z ∈ cbox 0 One ⟹
    i ∈ Basis ⟹
    norm (x − z) < e ∧ norm (y − z) < e ∧ label x i ≠ label y i ⟹
    |(f z − z) · i| < d / real n
  by blast
obtain p :: nat where p: 1 + real n / e ≤ real p
  using real_arch_simple ..
have 1 + real n / e > 0
  using e(1) n by (simp add: add_pos_pos)
then have p > 0
  using p by auto

obtain b :: nat ⇒ 'a where b: bij_betw b {..< n} Basis
  by atomize_elim (auto simp: n_def intro!: finite_same_card_bij)
define b' where b' = inv_into {..< n} b
then have b': bij_betw b' Basis {..< n}
  using bij_betw_inv_into[OF b] by auto
then have b'_Basis: ⋀i. i ∈ Basis ⟹ b' i ∈ {..< n}
  unfolding bij_betw_def by (auto simp: set_eq_iff)
have bb'[simp]:⋀i. i ∈ Basis ⟹ b (b' i) = i
  unfolding b'_def
  using b
  by (auto simp: f_inv_into_f bij_betw_def)
have b'b[simp]:⋀i. i < n ⟹ b' (b i) = i
  unfolding b'_def
  using b
  by (auto simp: inv_into_f_eq bij_betw_def)
have *: ⋀x :: nat. x = 0 ∨ x = 1 ⟷ x ≤ 1
  by auto
have b'': ⋀j. j < n ⟹ b j ∈ Basis
  using b unfolding bij_betw_def by auto
have q1: 0 < p ∀x. (∀i<n. x i ≤ p) ⟶
  (∀i<n. (label (∑i∈Basis. (real (x (b' i)) / real p) *R i) ∘ b) i = 0 ∨
        (label (∑i∈Basis. (real (x (b' i)) / real p) *R i) ∘ b) i = 1)
  unfolding *
  using ‹p > 0› ‹n > 0›
  using label(1)[OF b'']
  by auto
{ fix x :: nat ⇒ nat and i assume ∀i<n. x i ≤ p i < n x i = p ∨ x i = 0
  then have (∑i∈Basis. (real (x (b' i)) / real p) *R i) ∈ (cbox 0 One::'a set)
    using b'_Basis
    by (auto simp: cbox_def bij_betw_def zero_le_divide_iff divide_le_eq_1) }
note cube = this
have q2: ∀x. (∀i<n. x i ≤ p) ⟶ (∀i<n. x i = 0 ⟶
    (label (∑i∈Basis. (real (x (b' i)) / real p) *R i) ∘ b) i = 0)
```

unfolding *o_def* **using** *cube* ‹*p* > *0*› **by** (*intro allI impI label*(*2*)) (*auto simp*: *b''*)
have *q3*: ∀ *x*. (∀ *i*<*n*. *x i* ≤ *p*) ⟶ (∀ *i*<*n*. *x i* = *p* ⟶
(*label* (∑ *i*∈*Basis*. (*real* (*x* (*b' i*)) / *real p*) $*_R$ *i*) ∘ *b*) *i* = *1*)
using *cube* ‹*p* > *0*› **unfolding** *o_def* **by** (*intro allI impI label*(*3*)) (*auto simp*: *b''*)
obtain *q* **where** *q*:
∀ *i*<*n*. *q i* < *p*
∀ *i*<*n*.
∃ *r s*. (∀ *j*<*n*. *q j* ≤ *r j* ∧ *r j* ≤ *q j* + *1*) ∧
(∀ *j*<*n*. *q j* ≤ *s j* ∧ *s j* ≤ *q j* + *1*) ∧
(*label* (∑ *i*∈*Basis*. (*real* (*r* (*b' i*)) / *real p*) $*_R$ *i*) ∘ *b*) *i* ≠
(*label* (∑ *i*∈*Basis*. (*real* (*s* (*b' i*)) / *real p*) $*_R$ *i*) ∘ *b*) *i*
by (*rule kuhn_lemma*[*OF q1 q2 q3*])
define *z* :: *'a* **where** *z* = (∑ *i*∈*Basis*. (*real* (*q* (*b' i*)) / *real p*) $*_R$ *i*)
have ∃ *i*∈*Basis*. *d* / *real n* ≤ |(*f z* − *z*)·*i*|
proof (*rule ccontr*)
have ∀ *i*∈*Basis*. *q* (*b' i*) ∈ {*0*..*p*}
using *q*(*1*) *b'*
by (*auto intro*: *less_imp_le simp*: *bij_betw_def*)
then have *z* ∈ *cbox 0 One*
unfolding *z_def cbox_def*
using *b'_Basis*
by (*auto simp*: *bij_betw_def zero_le_divide_iff divide_le_eq_1*)
then have *d_fz_z*: *d* ≤ *norm* (*f z* − *z*)
by (*rule d*)
assume ¬ *?thesis*
then have *as*: ∀ *i*∈*Basis*. |*f z* · *i* − *z* · *i*| < *d* / *real n*
using ‹*n* > *0*›
by (*auto simp*: *not_le inner_diff*)
have *norm* (*f z* − *z*) ≤ (∑ *i*∈*Basis*. |*f z* · *i* − *z* · *i*|)
unfolding *inner_diff_left*[*symmetric*]
by (*rule norm_le_l1*)
also have ... < (∑ (*i*::*'a*) ∈ *Basis*. *d* / *real n*)
by (*meson as finite_Basis nonempty_Basis sum_strict_mono*)
also have ... = *d*
using *DIM_positive*[**where** *'a*=*'a*] **by** (*auto simp*: *n_def*)
finally show *False*
using *d_fz_z* **by** *auto*
qed
then obtain *i* **where** *i*: *i* ∈ *Basis d* / *real n* ≤ |(*f z* − *z*) · *i*| **..**
have ∗: *b' i* < *n*
using *i* **and** *b'*[*unfolded bij_betw_def*]
by *auto*
obtain *r s* **where** *rs*:
⋀*j*. *j* < *n* ⟹ *q j* ≤ *r j* ∧ *r j* ≤ *q j* + *1*
⋀*j*. *j* < *n* ⟹ *q j* ≤ *s j* ∧ *s j* ≤ *q j* + *1*
(*label* (∑ *i*∈*Basis*. (*real* (*r* (*b' i*)) / *real p*) $*_R$ *i*) ∘ *b*) (*b' i*) ≠
(*label* (∑ *i*∈*Basis*. (*real* (*s* (*b' i*)) / *real p*) $*_R$ *i*) ∘ *b*) (*b' i*)

```
  using q(2)[rule_format,OF *] by blast
have b'_im: ⋀i. i ∈ Basis ⟹  b' i < n
  using b' unfolding bij_betw_def by auto
define r' ::'a where r' = (∑ i∈Basis. (real (r (b' i)) / real p) *R i)
have ⋀i. i ∈ Basis ⟹ r (b' i) ≤ p
  apply (rule order_trans)
  apply (rule rs(1)[OF b'_im,THEN conjunct2])
  using q(1)[rule_format,OF b'_im]
  apply (auto simp: Suc_le_eq)
  done
then have r' ∈ cbox 0 One
  unfolding r'_def cbox_def
  using b'_Basis
  by (auto simp: bij_betw_def zero_le_divide_iff divide_le_eq_1)
define s' :: 'a where s' = (∑ i∈Basis. (real (s (b' i)) / real p) *R i)
have ⋀i. i ∈ Basis ⟹ s (b' i) ≤ p
  using b'_im q(1) rs(2) by fastforce
then have s' ∈ cbox 0 One
  unfolding s'_def cbox_def
  using b'_Basis by (auto simp: bij_betw_def zero_le_divide_iff divide_le_eq_1)
have z ∈ cbox 0 One
  unfolding z_def cbox_def
  using b'_Basis q(1)[rule_format,OF b'_im] ‹p > 0›
  by (auto simp: bij_betw_def zero_le_divide_iff divide_le_eq_1 less_imp_le)
{
  have (∑ i∈Basis. |real (r (b' i)) − real (q (b' i))|) ≤ (∑ (i::'a)∈Basis. 1)
    by (rule sum_mono) (use rs(1)[OF b'_im] in force)
  also have ... < e * real p
    using p ‹e > 0› ‹p > 0›
    by (auto simp: field_simps n_def)
  finally have (∑ i∈Basis. |real (r (b' i)) − real (q (b' i))|) < e * real p .
}
moreover
{
  have (∑ i∈Basis. |real (s (b' i)) − real (q (b' i))|) ≤ (∑ (i::'a)∈Basis. 1)
    by (rule sum_mono) (use rs(2)[OF b'_im] in force)
  also have ... < e * real p
    using p ‹e > 0› ‹p > 0›
    by (auto simp: field_simps n_def)
  finally have (∑ i∈Basis. |real (s (b' i)) − real (q (b' i))|) < e * real p .
}
ultimately
have norm (r' − z) < e and norm (s' − z) < e
  unfolding r'_def s'_def z_def
  using ‹p > 0›
  apply (rule_tac[!] le_less_trans[OF norm_le_l1])
  apply (auto simp: field_simps sum_divide_distrib[symmetric] inner_diff_left)
  done
then have |(f z − z) · i| < d / real n
```

```
    using rs(3) i
    unfolding r'_def[symmetric] s'_def[symmetric] o_def bb'
    by (intro e(2)[OF ‹r'∈cbox 0 One› ‹s'∈cbox 0 One› ‹z∈cbox 0 One›]) auto
  then show False
    using i by auto
qed
```

Next step is to prove it for nonempty interiors.

```
lemma brouwer_weak:
  fixes f :: 'a::euclidean_space ⇒ 'a
  assumes compact S
    and convex S
    and interior S ≠ {}
    and continuous_on S f
    and f ` S ⊆ S
  obtains x where x ∈ S and f x = x
proof –
  let ?U = cbox 0 One :: 'a set
  have ∑ Basis /R 2 ∈ interior ?U
  proof (rule interiorI)
    let ?I = (⋂ i∈Basis. {x::'a. 0 < x · i} ∩ {x. x · i < 1})
    show open ?I
       by (intro open_INT finite_Basis ballI open_Int, auto intro: open_Collect_less
simp: continuous_on_inner)
    show ∑ Basis /R 2 ∈ ?I
      by simp
    show ?I ⊆ cbox 0 One
      unfolding cbox_def by force
  qed
  then have ∗: interior ?U ≠ {} by fast
  have ∗: ?U homeomorphic S
   using homeomorphic_convex_compact[OF convex_box(1) compact_cbox ∗ assms(2,1,3)]
.
  have ∀ f. continuous_on ?U f ∧ f ` ?U ⊆ ?U ⟶
    (∃ x∈?U. f x = x)
    using brouwer_cube by auto
  then show ?thesis
    unfolding homeomorphic_fixpoint_property[OF ∗]
    using assms
    by (auto intro: that)
qed
```

Then the particular case for closed balls.

```
lemma brouwer_ball:
  fixes f :: 'a::euclidean_space ⇒ 'a
  assumes e > 0
    and continuous_on (cball a e) f
    and f ` cball a e ⊆ cball a e
  obtains x where x ∈ cball a e and f x = x
```

using *brouwer_weak*[*OF compact_cball convex_cball*, *of a e f*]
unfolding *interior_cball ball_eq_empty*
using *assms* **by** *auto*

And finally we prove Brouwer's fixed point theorem in its general version.

theorem *brouwer*:
fixes *f* :: *'a::euclidean_space* ⇒ *'a*
assumes *S*: *compact S convex S S* ≠ {}
and *contf*: *continuous_on S f*
and *fim*: *f* ' *S* ⊆ *S*
obtains *x* **where** *x* ∈ *S* **and** *f x* = *x*
proof −
have ∃ *e*>*0*. *S* ⊆ *cball 0 e*
using *compact_imp_bounded*[*OF* ‹*compact S*›] **unfolding** *bounded_pos*
by *auto*
then obtain *e* **where** *e*: *e* > *0 S* ⊆ *cball 0 e*
by *blast*
have ∃ *x*∈ *cball 0 e*. (*f* ∘ *closest_point S*) *x* = *x*
proof (*rule_tac brouwer_ball*[*OF e*(*1*)])
show *continuous_on* (*cball 0 e*) (*f* ∘ *closest_point S*)
apply (*rule continuous_on_compose*)
using *S compact_eq_bounded_closed continuous_on_closest_point* **apply** *blast*
by (*meson S contf closest_point_in_set compact_imp_closed continuous_on_subset image_subsetI*)
show (*f* ∘ *closest_point S*) ' *cball 0 e* ⊆ *cball 0 e*
by *clarsimp* (*metis S fim closest_point_exists*(*1*) *compact_eq_bounded_closed e*(*2*) *image_subset_iff mem_cball_0 subsetCE*)
qed (*use assms* **in** *auto*)
then obtain *x* **where** *x*: *x* ∈ *cball 0 e* (*f* ∘ *closest_point S*) *x* = *x* **..**
have *x* ∈ *S*
by (*metis closest_point_in_set comp_apply compact_imp_closed fim image_eqI S*(*1*) *S*(*3*) *subset_iff x*(*2*))
then have ∗: *closest_point S x* = *x*
by (*rule closest_point_self*)
show *thesis*
proof
show *closest_point S x* ∈ *S*
by (*simp add*: ∗ ‹*x* ∈ *S*›)
show *f* (*closest_point S x*) = *closest_point S x*
using ∗ *x*(*2*) **by** *auto*
qed
qed

6.31.4 Applications

So we get the no-retraction theorem.

corollary *no_retraction_cball*:
fixes *a* :: *'a::euclidean_space*
assumes *e* > *0*

```
  shows ¬ (frontier (cball a e) retract_of (cball a e))
proof
  assume *: frontier (cball a e) retract_of (cball a e)
  have **: ⋀xa. a − (2 *R a − xa) = − (a − xa)
    using scaleR_left_distrib[of 1 1 a] by auto
  obtain x where x: x ∈ {x. norm (a − x) = e} 2 *R a − x = x
  proof (rule retract_fixpoint_property[OF *, of λx. scaleR 2 a − x])
    show continuous_on (frontier (cball a e)) ((−) (2 *R a))
      by (intro continuous_intros)
    show (−) (2 *R a) ‘ frontier (cball a e) ⊆ frontier (cball a e)
      by clarsimp (metis ** dist_norm norm_minus_cancel)
  qed (auto simp: dist_norm intro: brouwer_ball[OF assms])
  then have scaleR 2 a = scaleR 1 x + scaleR 1 x
    by (auto simp: algebra_simps)
  then have a = x
    unfolding scaleR_left_distrib[symmetric]
    by auto
  then show False
    using x assms by auto
qed

corollary contractible_sphere:
  fixes a :: 'a::euclidean_space
  shows contractible(sphere a r) ⟷ r ≤ 0
proof (cases 0 < r)
  case True
  then show ?thesis
    unfolding contractible_def nullhomotopic_from_sphere_extension
    using no_retraction_cball [OF True, of a]
    by (auto simp: retract_of_def retraction_def)
next
  case False
  then show ?thesis
    unfolding contractible_def nullhomotopic_from_sphere_extension
    using less_eq_real_def by auto
qed

corollary connected_sphere_eq:
  fixes a :: 'a :: euclidean_space
  shows connected(sphere a r) ⟷ 2 ≤ DIM('a) ∨ r ≤ 0
    (is ?lhs = ?rhs)
proof (cases r 0::real rule: linorder_cases)
  case less
  then show ?thesis by auto
next
  case equal
  then show ?thesis by auto
next
  case greater
```

show *?thesis*
proof
assume *L*: *?lhs*
have *False* **if** *1*: *DIM('a) = 1*
proof −
obtain *x y* **where** *xy*: *sphere a r = {x,y} x ≠ y*
using *sphere_1D_doubleton [OF 1 greater]*
by (*metis dist_self greater insertI1 less_add_same_cancel1 mem_sphere mult_2 not_le zero_le_dist*)
then have *finite* (*sphere a r*)
by *auto*
with *L* ‹*r > 0*› *xy* **show** *False*
using *connected_finite_iff_sing* **by** *auto*
qed
with *greater* **show** *?rhs*
by (*metis DIM_ge_Suc0 One_nat_def Suc_1 le_antisym not_less_eq_eq*)
next
assume *?rhs*
then show *?lhs*
using *connected_sphere greater* **by** *auto*
qed
qed

corollary *path_connected_sphere_eq*:
fixes *a* :: *'a* :: *euclidean_space*
shows *path_connected*(*sphere a r*) ⟷ *2* ≤ *DIM('a)* ∨ *r* ≤ *0*
(**is** *?lhs = ?rhs*)
proof
assume *?lhs*
then show *?rhs*
using *connected_sphere_eq path_connected_imp_connected* **by** *blast*
next
assume *R*: *?rhs*
then show *?lhs*
by (*auto simp*: *contractible_imp_path_connected contractible_sphere path_connected_sphere*)
qed

proposition *frontier_subset_retraction*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *bounded S* **and** *fros*: *frontier S ⊆ T*
and *contf*: *continuous_on* (*closure S*) *f*
and *fim*: *f ‘ S ⊆ T*
and *fid*: ⋀*x*. *x ∈ T* ⟹ *f x = x*
shows *S ⊆ T*
proof (*rule ccontr*)
assume ¬ *S ⊆ T*
then obtain *a* **where** *a ∈ S a ∉ T* **by** *blast*
define *g* **where** *g ≡ λz. if z ∈ closure S then f z else z*
have *continuous_on* (*closure S* ∪ *closure*(−*S*)) *g*

unfolding *g_def*
apply (*rule continuous_on_cases*)
using *fros fid frontier_closures* **by** (*auto simp*: *contf*)
moreover have *closure S* ∪ *closure*(− *S*) = *UNIV*
using *closure_Un* **by** *fastforce*
ultimately have *contg*: *continuous_on UNIV g* **by** *metis*
obtain *B* **where** *0* < *B* **and** *B*: *closure S* ⊆ *ball a B*
using ‹*bounded S*› *bounded_subset_ballD* **by** *blast*
have *notga*: *g x* ≠ *a* **for** *x*
unfolding *g_def* **using** *fros fim* ‹*a* ∉ *T*›
apply (*auto simp*: *frontier_def*)
using *fid interior_subset* **apply** *fastforce*
by (*simp add*: ‹*a* ∈ *S*› *closure_def*)
define *h* **where** *h* ≡ (λ*y*. *a* + (*B* / *norm*(*y* − *a*)) ∗$_R$ (*y* − *a*)) ∘ *g*
have ¬ (*frontier* (*cball a B*) *retract_of* (*cball a B*))
by (*metis no_retraction_cball* ‹*0* < *B*›)
then have ⋀*k*. ¬ *retraction* (*cball a B*) (*frontier* (*cball a B*)) *k*
by (*simp add*: *retract_of_def*)
moreover have *retraction* (*cball a B*) (*frontier* (*cball a B*)) *h*
unfolding *retraction_def*
proof (*intro conjI ballI*)
show *frontier* (*cball a B*) ⊆ *cball a B*
by *force*
show *continuous_on* (*cball a B*) *h*
unfolding *h_def*
by (*intro continuous_intros*) (*use contg continuous_on_subset notga* **in** *auto*)
show *h* ‘ *cball a B* ⊆ *frontier* (*cball a B*)
using ‹*0* < *B*› **by** (*auto simp*: *h_def notga dist_norm*)
show ⋀*x*. *x* ∈ *frontier* (*cball a B*) ⟹ *h x* = *x*
apply (*auto simp*: *h_def algebra_simps*)
apply (*simp add*: *vector_add_divide_simps notga*)
by (*metis* (*no_types*, *hide_lams*) *B add.commute dist_commute dist_norm g_def mem_ball not_less_iff_gr_or_eq subset_eq*)
qed
ultimately show *False* **by** *simp*
qed

Punctured affine hulls, etc

lemma *rel_frontier_deformation_retract_of_punctured_convex*:
fixes *S* :: *'a::euclidean_space set*
assumes *convex S convex T bounded S*
and *arelS*: *a* ∈ *rel_interior S*
and *relS*: *rel_frontier S* ⊆ *T*
and *affS*: *T* ⊆ *affine hull S*
obtains *r* **where** *homotopic_with_canon* (λ*x*. *True*) (*T* − {*a*}) (*T* − {*a*}) *id r*
retraction (*T* − {*a*}) (*rel_frontier S*) *r*
proof −
have ∃ *d*. *0* < *d* ∧ (*a* + *d* ∗$_R$ *l*) ∈ *rel_frontier S* ∧

$(\forall e.\ 0 \le e \wedge e < d \longrightarrow (a + e *_R l) \in rel\_interior\ S)$
if $(a + l) \in affine\ hull\ S\ l \neq 0$ **for** l
apply (*rule ray_to_rel_frontier* [*OF* ‹*bounded S*› *arelS*])
apply (*rule that*)+
by *metis*
then obtain *dd*
where *dd1*: $\bigwedge l.\ [\![(a + l) \in affine\ hull\ S;\ l \neq 0]\!] \Longrightarrow 0 < dd\ l \wedge (a + dd\ l *_R l) \in rel\_frontier\ S$
and *dd2*: $\bigwedge l\ e.\ [\![(a + l) \in affine\ hull\ S;\ e < dd\ l;\ 0 \le e;\ l \neq 0]\!] \Longrightarrow (a + e *_R l) \in rel\_interior\ S$
by *metis*+
have *aaffS*: $a \in affine\ hull\ S$
by (*meson arelS subsetD hull_inc rel_interior_subset*)
have $((\lambda z.\ z - a)\ `\ (affine\ hull\ S - \{a\})) = ((\lambda z.\ z - a)\ `\ (affine\ hull\ S)) - \{0\}$
by *auto*
moreover have $continuous\_on\ (((\lambda z.\ z - a)\ `\ (affine\ hull\ S)) - \{0\})\ (\lambda x.\ dd\ x *_R x)$
proof (*rule continuous_on_compact_surface_projection*)
show $compact\ (rel\_frontier\ ((\lambda z.\ z - a)\ `\ S))$
by (*simp add*: ‹*bounded S*› *bounded_translation_minus compact_rel_frontier_bounded*)
have *releq*: $rel\_frontier\ ((\lambda z.\ z - a)\ `\ S) = (\lambda z.\ z - a)\ `\ rel\_frontier\ S$
using *rel_frontier_translation* [*of* $-a$] *add.commute* **by** *simp*
also have $\ldots \subseteq (\lambda z.\ z - a)\ `\ (affine\ hull\ S) - \{0\}$
using *rel_frontier_affine_hull arelS rel_frontier_def* **by** *fastforce*
finally show $rel\_frontier\ ((\lambda z.\ z - a)\ `\ S) \subseteq (\lambda z.\ z - a)\ `\ (affine\ hull\ S) - \{0\}$ **.**
show $cone\ ((\lambda z.\ z - a)\ `\ (affine\ hull\ S))$
by (*rule subspace_imp_cone*)
(*use aaffS* **in** ‹*simp add*: *subspace_affine image_comp o_def affine_translation_aux* [*of a*]›)
show $(0 < k \wedge k *_R x \in rel\_frontier\ ((\lambda z.\ z - a)\ `\ S)) \longleftrightarrow (dd\ x = k)$
if x: $x \in (\lambda z.\ z - a)\ `\ (affine\ hull\ S) - \{0\}$ **for** $k\ x$
proof
show $dd\ x = k \Longrightarrow 0 < k \wedge k *_R x \in rel\_frontier\ ((\lambda z.\ z - a)\ `\ S)$
using *dd1* [*of x*] *that image_iff* **by** (*fastforce simp add*: *releq*)
next
assume k: $0 < k \wedge k *_R x \in rel\_frontier\ ((\lambda z.\ z - a)\ `\ S)$
have *False* **if** $dd\ x < k$
proof −
have $k \neq 0\ a + k *_R x \in closure\ S$
using k *closure_translation* [*of* $-a$]
by (*auto simp*: *rel_frontier_def cong*: *image_cong_simp*)
then have *segsub*: $open\_segment\ a\ (a + k *_R x) \subseteq rel\_interior\ S$
by (*metis rel_interior_closure_convex_segment* [*OF* ‹*convex S*› *arelS*])
have $x \neq 0$ **and** *xaffS*: $a + x \in affine\ hull\ S$
using x **by** *auto*
then have $0 < dd\ x$ **and** *inS*: $a + dd\ x *_R x \in rel\_frontier\ S$
using *dd1* **by** *auto*

moreover have *a* + *dd x* $*_R$ *x* ∈ *open_segment a* (*a* + *k* $*_R$ *x*)
using *k* ‹*x* ≠ *0*› ‹*0* < *dd x*›
apply (*simp add*: *in_segment*)
apply (*rule_tac x* = *dd x* / *k* **in** *exI*)
apply (*simp add*: *field_simps that*)
apply (*simp add*: *vector_add_divide_simps algebra_simps*)
done
ultimately show *?thesis*
using *segsub* **by** (*auto simp*: *rel_frontier_def*)
qed
moreover have *False* **if** *k* < *dd x*
using *x k that rel_frontier_def*
by (*fastforce simp*: *algebra_simps releq dest*!: *dd2*)
ultimately show *dd x* = *k*
by *fastforce*
qed
qed
ultimately have ∗: *continuous_on* ((λ*z*. *z* − *a*) ‘ (*affine hull S* − {*a*})) (λ*x*. *dd x* $*_R$ *x*)
by *auto*
have *continuous_on* (*affine hull S* − {*a*}) ((λ*x*. *a* + *dd x* $*_R$ *x*) ∘ (λ*z*. *z* − *a*))
by (*intro* ∗ *continuous_intros continuous_on_compose*)
with *affS* **have** *contdd*: *continuous_on* (*T* − {*a*}) ((λ*x*. *a* + *dd x* $*_R$ *x*) ∘ (λ*z*. *z* − *a*))
by (*blast intro*: *continuous_on_subset*)
show *?thesis*
proof
show *homotopic_with_canon* (λ*x*. *True*) (*T* − {*a*}) (*T* − {*a*}) *id* (λ*x*. *a* + *dd* (*x* − *a*) $*_R$ (*x* − *a*))
proof (*rule homotopic_with_linear*)
show *continuous_on* (*T* − {*a*}) *id*
by (*intro continuous_intros continuous_on_compose*)
show *continuous_on* (*T* − {*a*}) (λ*x*. *a* + *dd* (*x* − *a*) $*_R$ (*x* − *a*))
using *contdd* **by** (*simp add*: *o_def*)
show *closed_segment* (*id x*) (*a* + *dd* (*x* − *a*) $*_R$ (*x* − *a*)) ⊆ *T* − {*a*}
if *x* ∈ *T* − {*a*} **for** *x*
proof (*clarsimp simp*: *in_segment*, *intro conjI*)
fix *u*::*real* **assume** *u*: *0* ≤ *u u* ≤ *1*
have *a* + *dd* (*x* − *a*) $*_R$ (*x* − *a*) ∈ *T*
by (*metis DiffD1 DiffD2 add.commute add.right_neutral affS dd1 diff_add_cancel relS singletonI subsetCE that*)
then show (*1* − *u*) $*_R$ *x* + *u* $*_R$ (*a* + *dd* (*x* − *a*) $*_R$ (*x* − *a*)) ∈ *T*
using *convexD* [*OF* ‹*convex T*›] *that u* **by** *simp*
have *iff*: (*1* − *u*) $*_R$ *x* + *u* $*_R$ (*a* + *d* $*_R$ (*x* − *a*)) = *a* ⟷
(*1* − *u* + *u* ∗ *d*) $*_R$ (*x* − *a*) = *0* **for** *d*
by (*auto simp*: *algebra_simps*)
have *x* ∈ *T x* ≠ *a* **using** *that* **by** *auto*
then have *axa*: *a* + (*x* − *a*) ∈ *affine hull S*
by (*metis* (*no_types*) *add.commute affS diff_add_cancel rev_subsetD*)

```
    then have ¬ dd (x − a) ≤ 0 ∧ a + dd (x − a) *R (x − a) ∈ rel_frontier S
      using ‹x ≠ a› dd1 by fastforce
    with ‹x ≠ a› show (1 − u) *R x + u *R (a + dd (x − a) *R (x − a)) ≠ a
      apply (auto simp: iff)
      using less_eq_real_def mult_le_0_iff not_less u by fastforce
  qed
qed
show retraction (T − {a}) (rel_frontier S) (λx. a + dd (x − a) *R (x − a))
proof (simp add: retraction_def, intro conjI ballI)
  show rel_frontier S ⊆ T − {a}
    using arelS relS rel_frontier_def by fastforce
  show continuous_on (T − {a}) (λx. a + dd (x − a) *R (x − a))
    using contdd by (simp add: o_def)
  show (λx. a + dd (x − a) *R (x − a)) ` (T − {a}) ⊆ rel_frontier S
    apply (auto simp: rel_frontier_def)
   apply (metis Diff_subset add.commute affS dd1 diff_add_cancel eq_iff_diff_eq_0
rel_frontier_def subset_iff)
  by (metis DiffE add.commute affS dd1 diff_add_cancel eq_iff_diff_eq_0 rel_frontier_def
rev_subsetD)
  show a + dd (x − a) *R (x − a) = x if x: x ∈ rel_frontier S for x
  proof −
    have x ≠ a
      using that arelS by (auto simp: rel_frontier_def)
    have False if dd (x − a) < 1
    proof −
      have x ∈ closure S
        using x by (auto simp: rel_frontier_def)
      then have segsub: open_segment a x ⊆ rel_interior S
        by (metis rel_interior_closure_convex_segment [OF ‹convex S› arelS])
      have  xaffS: x ∈ affine hull S
        using affS relS x by auto
        then have 0 < dd (x − a) and inS: a + dd (x − a) *R (x − a) ∈
rel_frontier S
        using dd1 by (auto simp: ‹x ≠ a›)
      moreover have a + dd (x − a) *R (x − a) ∈ open_segment a x
        using  ‹x ≠ a› ‹0 < dd (x − a)›
        apply (simp add: in_segment)
        apply (rule_tac x = dd (x − a) in exI)
        apply (simp add: algebra_simps that)
        done
      ultimately show ?thesis
        using segsub by (auto simp: rel_frontier_def)
    qed
    moreover have False if 1 < dd (x − a)
      using x that dd2 [of x − a 1] ‹x ≠ a› closure_affine_hull
      by (auto simp: rel_frontier_def)
    ultimately have dd (x − a) = 1 — similar to another proof above
      by fastforce
    with that show ?thesis
```

```
        by (simp add: rel_frontier_def)
      qed
    qed
  qed
qed

corollary rel_frontier_retract_of_punctured_affine_hull:
  fixes S :: 'a::euclidean_space set
  assumes bounded S convex S a ∈ rel_interior S
    shows rel_frontier S retract_of (affine hull S − {a})
apply (rule rel_frontier_deformation_retract_of_punctured_convex [of S affine hull S
a])
apply (auto simp: affine_imp_convex rel_frontier_affine_hull retract_of_def assms)
done

corollary rel_boundary_retract_of_punctured_affine_hull:
  fixes S :: 'a::euclidean_space set
  assumes compact S convex S a ∈ rel_interior S
    shows (S − rel_interior S) retract_of (affine hull S − {a})
by (metis assms closure_closed compact_eq_bounded_closed rel_frontier_def
          rel_frontier_retract_of_punctured_affine_hull)

lemma homotopy_eqv_rel_frontier_punctured_convex:
  fixes S :: 'a::euclidean_space set
  assumes convex S bounded S a ∈ rel_interior S convex T rel_frontier S ⊆ T T
⊆ affine hull S
  shows (rel_frontier S) homotopy_eqv (T − {a})
  apply (rule rel_frontier_deformation_retract_of_punctured_convex [of S T])
  using assms
  apply auto
  using deformation_retract_imp_homotopy_eqv homotopy_equivalent_space_sym by
blast

lemma homotopy_eqv_rel_frontier_punctured_affine_hull:
  fixes S :: 'a::euclidean_space set
  assumes convex S bounded S a ∈ rel_interior S
    shows (rel_frontier S) homotopy_eqv (affine hull S − {a})
apply (rule homotopy_eqv_rel_frontier_punctured_convex)
  using assms rel_frontier_affine_hull  by force+

lemma path_connected_sphere_gen:
  assumes convex S bounded S aff_dim S ≠ 1
  shows path_connected(rel_frontier S)
proof (cases rel_interior S = {})
  case True
  then show ?thesis
    by (simp add: ‹convex S› convex_imp_path_connected rel_frontier_def)
next
  case False
```

then show *?thesis*
by (*metis aff_dim_affine_hull affine_affine_hull affine_imp_convex all_not_in_conv assms path_connected_punctured_convex rel_frontier_retract_of_punctured_affine_hull retract_of_path_connected*)
qed

lemma *connected_sphere_gen*:
assumes *convex S bounded S aff_dim S* $\neq$ *1*
shows *connected*(*rel_frontier S*)
by (*simp add*: *assms path_connected_imp_connected path_connected_sphere_gen*)

Borsuk-style characterization of separation

lemma *continuous_on_Borsuk_map*:
$a \notin s \Longrightarrow$ *continuous_on s* (λx. *inverse*(*norm* $(x - a)$) $*_R$ $(x - a)$)
by (*rule continuous_intros* | *force*)+

lemma *Borsuk_map_into_sphere*:
(λx. *inverse*(*norm* $(x - a)$) $*_R$ $(x - a)$) ' $s \subseteq$ *sphere 0 1* $\longleftrightarrow$ $(a \notin s)$
by *auto* (*metis eq_iff_diff_eq_0 left_inverse norm_eq_zero*)

lemma *Borsuk_maps_homotopic_in_path_component*:
assumes *path_component* $(- s)$ *a b*
shows *homotopic_with_canon* (λx. *True*) *s* (*sphere 0 1*)
(λx. *inverse*(*norm*$(x - a)$) $*_R$ $(x - a)$)
(λx. *inverse*(*norm*$(x - b)$) $*_R$ $(x - b)$)
proof −
obtain *g* **where** *path g path_image g* $\subseteq -s$ *pathstart g* $=$ *a pathfinish g* $=$ *b*
using *assms* **by** (*auto simp*: *path_component_def*)
then show *?thesis*
apply (*simp add*: *path_def path_image_def pathstart_def pathfinish_def homotopic_with_def*)
apply (*rule_tac x* $= \lambda z$. *inverse*(*norm*(*snd z* $-$ ($g \circ$ *fst*)*z*)) $*_R$ (*snd z* $-$ ($g \circ$ *fst*)*z*) **in** *exI*)
apply (*intro conjI continuous_intros*)
apply (*rule continuous_intros* | *erule continuous_on_subset* | *fastforce simp*: *divide_simps sphere_def*)+
done
qed

lemma *non_extensible_Borsuk_map*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *compact s* **and** *cin*: $c \in$ *components*$(- s)$ **and** *boc*: *bounded c* **and** $a \in c$
shows $\neg$ ($\exists g$. *continuous_on* $(s \cup c)$ *g* $\wedge$
g ' $(s \cup c) \subseteq$ *sphere 0 1* $\wedge$
($\forall x \in s$. *g x* $=$ *inverse*(*norm*$(x - a)$) $*_R$ $(x - a)$))
proof −
have *closed s* **using** *assms* **by** (*simp add*: *compact_imp_closed*)

```
 have c ⊆ −s
   using assms by (simp add: in_components_subset)
 with ‹a ∈ c› have a ∉ s by blast
 then have ceq: c = connected_component_set (− s) a
   by (metis ‹a ∈ c› cin components_iff connected_component_eq)
 then have bounded (s ∪ connected_component_set (− s) a)
   using ‹compact s› boc compact_imp_bounded by auto
 with bounded_subset_ballD obtain r where 0 < r and r: (s ∪ connected_component_set
(− s) a) ⊆ ball a r
   by blast
 { fix g
   assume continuous_on (s ∪ c) g
          g ‘ (s ∪ c) ⊆ sphere 0 1
      and [simp]: ⋀x. x ∈ s ⟹ g x = (x − a) /R norm (x − a)
   then have [simp]: ⋀x. x ∈ s ∪ c ⟹ norm (g x) = 1
     by force
   have cb_eq: cball a r = (s ∪ connected_component_set (− s) a) ∪
                     (cball a r − connected_component_set (− s) a)
     using ball_subset_cball [of a r] r by auto
   have cont1: continuous_on (s ∪ connected_component_set (− s) a)
                   (λx. a + r *R g x)
     apply (rule continuous_intros)+
     using ‹continuous_on (s ∪ c) g› ceq by blast
   have cont2: continuous_on (cball a r − connected_component_set (− s) a)
           (λx. a + r *R ((x − a) /R norm (x − a)))
     by (rule continuous_intros | force simp: ‹a ∉ s›)+
   have 1: continuous_on (cball a r)
            (λx. if connected_component (− s) a x
                 then a + r *R g x
                 else a + r *R ((x − a) /R norm (x − a)))
     apply (subst cb_eq)
     apply (rule continuous_on_cases [OF _ _ cont1 cont2])
       using ceq cin
     apply (auto intro: closed_Un_complement_component
                 simp: ‹closed s› open_Compl open_connected_component)
     done
   have 2: (λx. a + r *R g x) ‘ (cball a r ∩ connected_component_set (− s) a)
            ⊆ sphere a r
     using ‹0 < r› by (force simp: dist_norm ceq)
   have retraction (cball a r) (sphere a r)
           (λx. if x ∈ connected_component_set (− s) a
                then a + r *R g x
                else a + r *R ((x − a) /R norm (x − a)))
     using  ‹0 < r›
     apply (simp add: retraction_def dist_norm 1 2, safe)
     apply (force simp: dist_norm abs_if mult_less_0_iff divide_simps ‹a ∉ s›)
     using r
     by (auto simp: dist_norm norm_minus_commute)
   then have False
```

```
    using no_retraction_cball
          [OF ‹0 < r›, of a, unfolded retract_of_def, simplified, rule_format,
           of λx. if x ∈ connected_component_set (− s) a
                  then a + r *R g x
                  else a + r *R inverse(norm(x − a)) *R (x − a)]
    by blast
  }
  then show ?thesis
    by blast
qed
```

Proving surjectivity via Brouwer fixpoint theorem

```
lemma brouwer_surjective:
  fixes f :: 'n::euclidean_space ⇒ 'n
  assumes compact T
    and convex T
    and T ≠ {}
    and continuous_on T f
    and ⋀x y. ⟦x∈S; y∈T⟧ ⟹ x + (y − f y) ∈ T
    and x ∈ S
  shows ∃ y∈T. f y = x
proof −
  have *: ⋀x y. f y = x ⟷ x + (y − f y) = y
    by (auto simp add: algebra_simps)
  show ?thesis
    unfolding *
    apply (rule brouwer[OF assms(1−3), of λy. x + (y − f y)])
    apply (intro continuous_intros)
    using assms
    apply auto
    done
qed

lemma brouwer_surjective_cball:
  fixes f :: 'n::euclidean_space ⇒ 'n
  assumes continuous_on (cball a e) f
    and e > 0
    and x ∈ S
    and ⋀x y. ⟦x∈S; y∈cball a e⟧ ⟹ x + (y − f y) ∈ cball a e
  shows ∃ y∈cball a e. f y = x
  apply (rule brouwer_surjective)
  apply (rule compact_cball convex_cball)+
  unfolding cball_eq_empty
  using assms
  apply auto
  done
```

Inverse function theorem

See Sussmann: "Multidifferential calculus", Theorem 2.1.1

```
lemma sussmann_open_mapping:
  fixes f :: 'a::real_normed_vector ⇒ 'b::euclidean_space
  assumes open S
    and contf: continuous_on S f
    and x ∈ S
    and derf: (f has_derivative f') (at x)
    and bounded_linear g' f' ∘ g' = id
    and T ⊆ S
    and x: x ∈ interior T
  shows f x ∈ interior (f ` T)
proof −
  interpret f': bounded_linear f'
    using assms unfolding has_derivative_def by auto
  interpret g': bounded_linear g'
    using assms by auto
  obtain B where B: 0 < B ∀ x. norm (g' x) ≤ norm x ∗ B
    using bounded_linear.pos_bounded[OF assms(5)] by blast
  hence ∗: 1 / (2 ∗ B) > 0 by auto
  obtain e0 where e0:
      0 < e0
      ∀ y. norm (y − x) < e0 ⟶ norm (f y − f x − f' (y − x)) ≤ 1 / (2 ∗ B) ∗
norm (y − x)
    using derf unfolding has_derivative_at_alt
    using ∗ by blast
  obtain e1 where e1: 0 < e1 cball x e1 ⊆ T
    using mem_interior_cball x by blast
  have ∗: 0 < e0 / B 0 < e1 / B using e0 e1 B by auto
  obtain e where e: 0 < e e < e0 / B e < e1 / B
    using field_lbound_gt_zero[OF ∗] by blast
  have lem: ∃ y∈cball (f x) e. f (x + g' (y − f x)) = z if z∈cball (f x) (e / 2)
for z
  proof (rule brouwer_surjective_cball)
    have z: z ∈ S if as: y ∈cball (f x) e z = x + (g' y − g' (f x)) for y z
    proof−
      have dist x z = norm (g' (f x) − g' y)
        unfolding as(2) and dist_norm by auto
      also have ... ≤ norm (f x − y) ∗ B
        by (metis B(2) g'.diff)
      also have ... ≤ e ∗ B
        by (metis B(1) dist_norm mem_cball mult_le_cancel_iff1 that(1))
      also have ... ≤ e1
        using B(1) e(3) pos_less_divide_eq by fastforce
      finally have z ∈ cball x e1
        by force
      then show z ∈ S
        using e1 assms(7) by auto
```

```
  qed
  show continuous_on (cball (f x) e) (λy. f (x + g' (y − f x)))
    unfolding g'.diff
  proof (rule continuous_on_compose2 [OF _ _ order_refl, of _ _ f])
    show continuous_on ((λy. x + (g' y − g' (f x))) ‘ cball (f x) e) f
      by (rule continuous_on_subset[OF contf]) (use z in blast)
    show continuous_on (cball (f x) e) (λy. x + (g' y − g' (f x)))
      by (intro continuous_intros linear_continuous_on[OF ‹bounded_linear g'›])
  qed
next
  fix y z
  assume y: y ∈ cball (f x) (e / 2) and z: z ∈ cball (f x) e
  have norm (g' (z − f x)) ≤ norm (z − f x) ∗ B
    using B by auto
  also have ... ≤ e ∗ B
   by (metis B(1) z dist_norm mem_cball norm_minus_commute mult_le_cancel_iff1)
  also have ... < e0
    using B(1) e(2) pos_less_divide_eq by blast
  finally have ∗: norm (x + g' (z − f x) − x) < e0
    by auto
  have ∗∗: f x + f' (x + g' (z − f x) − x) = z
    using assms(6)[unfolded o_def id_def,THEN cong]
    by auto
  have norm (f x − (y + (z − f (x + g' (z − f x))))) ≤
        norm (f (x + g' (z − f x)) − z) + norm (f x − y)
    using norm_triangle_ineq[of f (x + g'(z − f x)) − z f x − y]
    by (auto simp add: algebra_simps)
  also have ... ≤ 1 / (B ∗ 2) ∗ norm (g' (z − f x)) + norm (f x − y)
    using e0(2)[rule_format, OF ∗]
    by (simp only: algebra_simps ∗∗) auto
  also have ... ≤ 1 / (B ∗ 2) ∗ norm (g' (z − f x)) + e/2
    using y by (auto simp: dist_norm)
  also have ... ≤ 1 / (B ∗ 2) ∗ B ∗ norm (z − f x) + e/2
    using ∗ B by (auto simp add: field_simps)
  also have ... ≤ 1 / 2 ∗ norm (z − f x) + e/2
    by auto
  also have ... ≤ e/2 + e/2
    using B(1) ‹norm (z − f x) ∗ B ≤ e ∗ B› by auto
  finally show y + (z − f (x + g' (z − f x))) ∈ cball (f x) e
    by (auto simp: dist_norm)
qed (use e that in auto)
show ?thesis
  unfolding mem_interior
proof (intro exI conjI subsetI)
  fix y
  assume y ∈ ball (f x) (e / 2)
  then have ∗: y ∈ cball (f x) (e / 2)
    by auto
  obtain z where z: z ∈ cball (f x) e f (x + g' (z − f x)) = y
```

```
      using lem * by blast
    then have norm (g' (z − f x)) ≤ norm (z − f x) * B
      using B
      by (auto simp add: field_simps)
    also have ... ≤ e * B
      by (metis B(1) dist_norm mem_cball norm_minus_commute mult_le_cancel_iff1
z(1))
    also have ... ≤ e1
      using e B unfolding less_divide_eq by auto
    finally have x + g'(z − f x) ∈ T
      by (metis add_diff_cancel diff_diff_add dist_norm e1(2) mem_cball norm_minus_commute
subset_eq)
    then show y ∈ f ` T
      using z by auto
  qed (use e in auto)
qed
```

Hence the following eccentric variant of the inverse function theorem. This has no continuity assumptions, but we do need the inverse function. We could put $f' \circ g = I$ but this happens to fit with the minimal linear algebra theory I've set up so far.

```
lemma has_derivative_inverse_strong:
  fixes f :: 'n::euclidean_space ⇒ 'n
  assumes open S
    and x ∈ S
    and contf: continuous_on S f
    and gf: ⋀x. x ∈ S ⟹ g (f x) = x
    and derf: (f has_derivative f') (at x)
    and id: f' ∘ g' = id
  shows (g has_derivative g') (at (f x))
proof −
  have linf: bounded_linear f'
    using derf unfolding has_derivative_def by auto
  then have ling: bounded_linear g'
    unfolding linear_conv_bounded_linear[symmetric]
    using id right_inverse_linear by blast
  moreover have g' ∘ f' = id
    using id linf ling
    unfolding linear_conv_bounded_linear[symmetric]
    using linear_inverse_left
    by auto
  moreover have *: ⋀T. ⟦T ⊆ S; x ∈ interior T⟧ ⟹ f x ∈ interior (f ` T)
    apply (rule sussmann_open_mapping)
    apply (rule assms ling)+
    apply auto
    done
  have continuous (at (f x)) g
    unfolding continuous_at Lim_at
  proof (rule, rule)
```

fix *e* :: *real*
assume *e* > *0*
then have *f x* ∈ *interior* (*f* ‘ (*ball x e* ∩ *S*))
using ∗[*rule_format*,*of ball x e* ∩ *S*] ‹*x* ∈ *S*›
by (*auto simp add*: *interior_open*[*OF open_ball*] *interior_open*[*OF assms*(*1*)])
then obtain *d* **where** *d*: *0* < *d ball* (*f x*) *d* ⊆ *f* ‘ (*ball x e* ∩ *S*)
unfolding *mem_interior* **by** *blast*
show ∃ *d*>*0*. ∀ *y*. *0* < *dist y* (*f x*) ∧ *dist y* (*f x*) < *d* ⟶ *dist* (*g y*) (*g* (*f x*)) < *e*
proof (*intro exI allI impI conjI*)
fix *y*
assume *0* < *dist y* (*f x*) ∧ *dist y* (*f x*) < *d*
then have *g y* ∈ *g* ‘ *f* ‘ (*ball x e* ∩ *S*)
by (*metis d*(*2*) *dist_commute mem_ball rev_image_eqI subset_iff*)
then show *dist* (*g y*) (*g* (*f x*)) < *e*
using *gf*[*OF* ‹*x* ∈ *S*›]
by (*simp add*: *assms*(*4*) *dist_commute image_iff*)
qed (*use d* **in** *auto*)
qed
moreover have *f x* ∈ *interior* (*f* ‘ *S*)
apply (*rule sussmann_open_mapping*)
apply (*rule assms ling*)+
using *interior_open*[*OF assms*(*1*)] **and** ‹*x* ∈ *S*›
apply *auto*
done
moreover have *f* (*g y*) = *y* **if** *y* ∈ *interior* (*f* ‘ *S*) **for** *y*
by (*metis gf imageE interiorE subsetD that*)
ultimately show *?thesis* **using** *assms*
by (*metis has_derivative_inverse_basic_x open_interior*)
qed

A rewrite based on the other domain.

lemma *has_derivative_inverse_strong_x*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*a*
assumes *open S*
and *g y* ∈ *S*
and *continuous_on S f*
and ⋀*x*. *x* ∈ *S* ⟹ *g* (*f x*) = *x*
and (*f has_derivative f* ′) (*at* (*g y*))
and *f* ′ ∘ *g* ′ = *id*
and *f* (*g y*) = *y*
shows (*g has_derivative g*′) (*at y*)
using *has_derivative_inverse_strong*[*OF assms*(*1*−*6*)]
unfolding *assms*(*7*)
by *simp*

On a region.

theorem *has_derivative_inverse_on*:
fixes *f* :: ′*n*::*euclidean_space* ⇒ ′*n*

assumes *open S*
and *derf*: $\bigwedge x.\ x \in S \Longrightarrow (f\ \textit{has\_derivative}\ f'(x))\ (\textit{at}\ x)$
and $\bigwedge x.\ x \in S \Longrightarrow g\ (f\ x) = x$
and $f'\ x \circ g'\ x = id$
and $x \in S$
shows $(g\ \textit{has\_derivative}\ g'(x))\ (\textit{at}\ (f\ x))$
proof (*rule has_derivative_inverse_strong*[**where** $g'=g'\ x$ **and** $f=f$])
show *continuous_on S f*
unfolding *continuous_on_eq_continuous_at*[*OF* ‹*open S*›]
using *derf has_derivative_continuous* **by** *blast*
qed (*use assms* **in** *auto*)

end

6.32 Fashoda Meet Theorem

theory *Fashoda_Theorem*
imports *Brouwer_Fixpoint Path_Connected Cartesian_Euclidean_Space*
begin

6.32.1 Bijections between intervals

definition *interval_bij* :: $'a \times 'a \Rightarrow 'a \times 'a \Rightarrow 'a \Rightarrow 'a$::*euclidean_space*
where *interval_bij* =
$(\lambda(a, b)\ (u, v)\ x.\ (\sum i{\in}Basis.\ (u{\cdot}i + (x{\cdot}i - a{\cdot}i) / (b{\cdot}i - a{\cdot}i) * (v{\cdot}i - u{\cdot}i))$
$*_R\ i))$

lemma *interval_bij_affine*:
interval_bij $(a,b)\ (u,v) = (\lambda x.\ (\sum i{\in}Basis.\ ((v{\cdot}i - u{\cdot}i) / (b{\cdot}i - a{\cdot}i) * (x{\cdot}i))$
$*_R\ i) +$
$(\sum i{\in}Basis.\ (u{\cdot}i - (v{\cdot}i - u{\cdot}i) / (b{\cdot}i - a{\cdot}i) * (a{\cdot}i)) *_R\ i))$
by (*auto simp add*: *interval_bij_def sum.distrib* [*symmetric*] *scaleR_add_left* [*symmetric*]
fun_eq_iff intro!: *sum.cong*)
(*simp add*: *algebra_simps diff_divide_distrib* [*symmetric*])

lemma *continuous_interval_bij*:
fixes *a b* :: *'a::euclidean_space*
shows *continuous* (*at x*) (*interval_bij* (*a*, *b*) (*u*, *v*))
by (*auto simp add*: *divide_inverse interval_bij_def intro*!: *continuous_sum continuous_intros*)

lemma *continuous_on_interval_bij*: *continuous_on s* (*interval_bij* (*a*, *b*) (*u*, *v*))
apply(*rule continuous_at_imp_continuous_on*)
apply (*rule*, *rule continuous_interval_bij*)
done

lemma *in_interval_interval_bij*:
fixes *a b u v x* :: *'a::euclidean_space*

assumes *x* ∈ *cbox a b*
and *cbox u v* ≠ {}
shows *interval_bij* (*a*, *b*) (*u*, *v*) *x* ∈ *cbox u v*
apply (*simp only*: *interval_bij_def split_conv mem_box inner_sum_left_Basis cong*: *ball_cong*)
apply *safe*
proof −
fix *i* :: ′*a*
assume *i*: *i* ∈ *Basis*
have *cbox a b* ≠ {}
using *assms* **by** *auto*
with *i* **have** ∗: *a*·*i* ≤ *b*·*i* *u*·*i* ≤ *v*·*i*
using *assms*(*2*) **by** (*auto simp add*: *box_eq_empty*)
have *x*: *a*·*i*≤*x*·*i* *x*·*i*≤*b*·*i*
using *assms*(*1*)[*unfolded mem_box*] **using** *i* **by** *auto*
have *0* ≤ (*x* · *i* − *a* · *i*) / (*b* · *i* − *a* · *i*) ∗ (*v* · *i* − *u* · *i*)
using ∗ *x* **by** *auto*
then show *u* · *i* ≤ *u* · *i* + (*x* · *i* − *a* · *i*) / (*b* · *i* − *a* · *i*) ∗ (*v* · *i* − *u* · *i*)
using ∗ **by** *auto*
have ((*x* · *i* − *a* · *i*) / (*b* · *i* − *a* · *i*)) ∗ (*v* · *i* − *u* · *i*) ≤ *1* ∗ (*v* · *i* − *u* · *i*)
apply (*rule mult_right_mono*)
unfolding *divide_le_eq_1*
using ∗ *x*
apply *auto*
done
then show *u* · *i* + (*x* · *i* − *a* · *i*) / (*b* · *i* − *a* · *i*) ∗ (*v* · *i* − *u* · *i*) ≤ *v* · *i*
using ∗ **by** *auto*
qed

lemma *interval_bij_bij*:
∀ (*i*::′*a*::*euclidean_space*)∈*Basis*. *a*·*i* < *b*·*i* ∧ *u*·*i* < *v*·*i* =⇒
interval_bij (*a*, *b*) (*u*, *v*) (*interval_bij* (*u*, *v*) (*a*, *b*) *x*) = *x*
by (*auto simp*: *interval_bij_def euclidean_eq_iff* [**where** ′*a*=′*a*])

lemma *interval_bij_bij_cart*: **fixes** *x*::*real*^′*n* **assumes** ∀ *i*. *a*\$*i* < *b*\$*i* ∧ *u*\$*i* < *v*\$*i*
shows *interval_bij* (*a*,*b*) (*u*,*v*) (*interval_bij* (*u*,*v*) (*a*,*b*) *x*) = *x*
using *assms* **by** (*intro interval_bij_bij*) (*auto simp*: *Basis_vec_def inner_axis*)

6.32.2 Fashoda meet theorem

lemma *infnorm_2*:
fixes *x* :: *real*^*2*
shows *infnorm x* = *max* |*x*\$*1*| |*x*\$*2*|
unfolding *infnorm_cart UNIV_2* **by** (*rule cSup_eq*) *auto*

lemma *infnorm_eq_1_2*:
fixes *x* :: *real*^*2*
shows *infnorm x* = *1* ←→
|*x*\$*1*| ≤ *1* ∧ |*x*\$*2*| ≤ *1* ∧ (*x*\$*1* = −*1* ∨ *x*\$*1* = *1* ∨ *x*\$*2* = −*1* ∨ *x*\$*2* = *1*)

```
  unfolding infnorm_2 by auto

lemma infnorm_eq_1_imp:
  fixes x :: real^2
  assumes infnorm x = 1
  shows |x$1| ≤ 1 and |x$2| ≤ 1
  using assms unfolding infnorm_eq_1_2 by auto

proposition fashoda_unit:
  fixes f g :: real ⇒ real^2
  assumes f ` {−1 .. 1} ⊆ cbox (−1) 1
    and g ` {−1 .. 1} ⊆ cbox (−1) 1
    and continuous_on {−1 .. 1} f
    and continuous_on {−1 .. 1} g
    and f (− 1)$1 = − 1
    and f 1$1 = 1 g (− 1) $2 = −1
    and g 1 $2 = 1
  shows ∃ s∈{−1 .. 1}. ∃ t∈{−1 .. 1}. f s = g t
proof (rule ccontr)
  assume ¬ ?thesis
  note as = this[unfolded bex_simps,rule_format]
  define sqprojection
    where [abs_def]: sqprojection z = (inverse (infnorm z)) *R z for z :: real^2
  define negatex :: real^2 ⇒ real^2
    where negatex x = (vector [−(x$1), x$2]) for x
  have lem1: ∀ z::real^2. infnorm (negatex z) = infnorm z
    unfolding negatex_def infnorm_2 vector_2 by auto
  have lem2: ∀ z. z ≠ 0 −→ infnorm (sqprojection z) = 1
    unfolding sqprojection_def infnorm_mul[unfolded scalar_mult_eq_scaleR]
    by (simp add: real_abs_infnorm infnorm_eq_0)
  let ?F = λw::real^2. (f ◦ (λx. x$1)) w − (g ◦ (λx. x$2)) w
  have ∗: ⋀i. (λx::real^2. x $ i) ` cbox (− 1) 1 = {−1..1}
  proof
    show (λx::real^2. x $ i) ` cbox (− 1) 1 ⊆ {−1..1} for i
      by (auto simp: mem_box_cart)
    show {−1..1} ⊆ (λx::real^2. x $ i) ` cbox (− 1) 1 for i
     by (clarsimp simp: image_iff mem_box_cart Bex_def) (metis (no_types, hide_lams)
vec_component)
  qed
  {
    fix x
     assume x ∈ (λw. (f ◦ (λx. x $ 1)) w − (g ◦ (λx. x $ 2)) w) ` (cbox (− 1)
(1::real^2))
    then obtain w :: real^2 where w:
        w ∈ cbox (− 1) 1
        x = (f ◦ (λx. x $ 1)) w − (g ◦ (λx. x $ 2)) w
      unfolding image_iff ..
    then have x ≠ 0
      using as[of w$1 w$2]
```

```
      unfolding mem_box_cart atLeastAtMost_iff
      by auto
  } note x0 = this
  have 1: box (- 1) (1::real^2) ≠ {}
    unfolding interval_eq_empty_cart by auto
  have negatex (x + y) $ i = (negatex x + negatex y) $ i ∧ negatex (c *R x) $ i
= (c *R negatex x) $ i
    for i x y c
    using exhaust_2 [of i] by (auto simp: negatex_def)
  then have bounded_linear negatex
    by (simp add: bounded_linearI' vec_eq_iff)
  then have 2: continuous_on (cbox (- 1) 1) (negatex ∘ sqprojection ∘ ?F)
    apply (intro continuous_intros continuous_on_component)
    unfolding * sqprojection_def
    apply (intro assms continuous_intros)+
     apply (simp_all add: infnorm_eq_0 x0 linear_continuous_on)
    done
  have 3: (negatex ∘ sqprojection ∘ ?F) ` cbox (-1) 1 ⊆ cbox (-1) 1
    unfolding subset_eq
  proof (rule, goal_cases)
    case (1 x)
    then obtain y :: real^2 where y:
        y ∈ cbox (- 1) 1
        x = (negatex ∘ sqprojection ∘ (λw. (f ∘ (λx. x $ 1)) w - (g ∘ (λx. x $ 2))
w)) y
      unfolding image_iff ..
    have ?F y ≠ 0
      by (rule x0) (use y in auto)
    then have *: infnorm (sqprojection (?F y)) = 1
      unfolding y o_def
      by - (rule lem2[rule_format])
    have inf1: infnorm x = 1
      unfolding *[symmetric] y o_def
      by (rule lem1[rule_format])
    show x ∈ cbox (-1) 1
      unfolding mem_box_cart interval_cbox_cart infnorm_2
    proof
      fix i
      show (- 1) $ i ≤ x $ i ∧ x $ i ≤ 1 $ i
        using exhaust_2 [of i] inf1 by (auto simp: infnorm_2)
    qed
  qed
  obtain x :: real^2 where x:
      x ∈ cbox (- 1) 1
      (negatex ∘ sqprojection ∘ (λw. (f ∘ (λx. x $ 1)) w - (g ∘ (λx. x $ 2)) w)) x
= x
     apply (rule brouwer_weak[of cbox (- 1) (1::real^2) negatex ∘ sqprojection ∘
?F])
    apply (rule compact_cbox convex_box)+
```

```
    unfolding interior_cbox
    apply (rule 1 2 3)+
    apply blast
    done
  have ?F x ≠ 0
    by (rule x0) (use x in auto)
  then have *: infnorm (sqprojection (?F x)) = 1
    unfolding o_def
    by (rule lem2[rule_format])
  have nx: infnorm x = 1
    apply (subst x(2)[symmetric])
    unfolding *[symmetric] o_def
    apply (rule lem1[rule_format])
    done
  have iff: 0 < sqprojection x$i ⟷ 0 < x$i sqprojection x$i < 0 ⟷ x$i < 0
if x ≠ 0 for x i
  proof −
    have inverse (infnorm x) > 0
      by (simp add: infnorm_pos_lt that)
    then show (0 < sqprojection x $ i) = (0 < x $ i)
      and (sqprojection x $ i < 0) = (x $ i < 0)
       unfolding sqprojection_def vector_component_simps vector_scaleR_component
real_scaleR_def
      unfolding zero_less_mult_iff mult_less_0_iff
      by (auto simp add: field_simps)
  qed
  have x1: x $ 1 ∈ {− 1..1::real} x $ 2 ∈ {− 1..1::real}
    using x(1) unfolding mem_box_cart by auto
  then have nz: f (x $ 1) − g (x $ 2) ≠ 0
    using as by auto
  consider x $ 1 = −1 | x $ 1 = 1 | x $ 2 = −1 | x $ 2 = 1
    using nx unfolding infnorm_eq_1_2 by auto
  then show False
  proof cases
    case 1
    then have *: f (x $ 1) $ 1 = − 1
      using assms(5) by auto
    have sqprojection (f (x$1) − g (x$2)) $ 1 > 0
      using x(2)[unfolded o_def vec_eq_iff,THEN spec[where x=1]]
      by (auto simp: negatex_def 1)
    moreover
    from x1 have g (x $ 2) ∈ cbox (−1) 1
      using assms(2) by blast
    ultimately show False
      unfolding iff[OF nz] vector_component_simps * mem_box_cart
      using not_le by auto
  next
    case 2
    then have *: f (x $ 1) $ 1 = 1
```

using *assms*(*6*) by *auto*
have *sqprojection* (*f* (*x*$*1*) − *g* (*x*$*2*)) $ *1* < *0*
using *x*(*2*)[*unfolded o_def vec_eq_iff* ,*THEN spec*[where *x*=*1*]] *2*
by (*auto simp*: *negatex_def*)
moreover have *g* (*x* $ *2*) ∈ *cbox* (−*1*) *1*
using *assms*(*2*) *x1* by *blast*
ultimately show *False*
unfolding *iff* [*OF nz*] *vector_component_simps* ∗ *mem_box_cart*
using *not_le* by *auto*
next
case *3*
then have ∗: *g* (*x* $ *2*) $ *2* = − *1*
using *assms*(*7*) by *auto*
have *sqprojection* (*f* (*x*$*1*) − *g* (*x*$*2*)) $ *2* < *0*
using *x*(*2*)[*unfolded o_def vec_eq_iff* ,*THEN spec*[where *x*=*2*]] *3* by (*auto simp*: *negatex_def*)
moreover
from *x1* have *f* (*x* $ *1*) ∈ *cbox* (−*1*) *1*
using *assms*(*1*) by *blast*
ultimately show *False*
unfolding *iff* [*OF nz*] *vector_component_simps* ∗ *mem_box_cart*
by (*erule_tac x*=*2* in *allE*) *auto*
next
case *4*
then have ∗: *g* (*x* $ *2*) $ *2* = *1*
using *assms*(*8*) by *auto*
have *sqprojection* (*f* (*x*$*1*) − *g* (*x*$*2*)) $ *2* > *0*
using *x*(*2*)[*unfolded o_def vec_eq_iff* ,*THEN spec*[where *x*=*2*]] *4* by (*auto simp*: *negatex_def*)
moreover
from *x1* have *f* (*x* $ *1*) ∈ *cbox* (−*1*) *1*
using *assms*(*1*) by *blast*
ultimately show *False*
unfolding *iff* [*OF nz*] *vector_component_simps* ∗ *mem_box_cart*
by (*erule_tac x*=*2* in *allE*) *auto*
qed
qed

proposition *fashoda_unit_path*:
fixes *f g* :: *real* ⇒ *real*ˆ*2*
assumes *path f*
and *path g*
and *path_image f* ⊆ *cbox* (−*1*) *1*
and *path_image g* ⊆ *cbox* (−*1*) *1*
and (*pathstart f*)$*1* = −*1*
and (*pathfinish f*)$*1* = *1*
and (*pathstart g*)$*2* = −*1*
and (*pathfinish g*)$*2* = *1*
obtains *z* where *z* ∈ *path_image f* and *z* ∈ *path_image g*

proof –
note *assms=assms[unfolded path_def pathstart_def pathfinish_def path_image_def]*
define *iscale* **where** *[abs_def]*: *iscale z = inverse 2 *$_R$ (z + 1)* **for** *z :: real*
have *isc*: *iscale ` {− 1..1} ⊆ {0..1}*
unfolding *iscale_def* **by** *auto*
have *∃ s∈{− 1..1}. ∃ t∈{− 1..1}. (f ∘ iscale) s = (g ∘ iscale) t*
proof *(rule fashoda_unit)*
show *(f ∘ iscale) ` {− 1..1} ⊆ cbox (− 1) 1 (g ∘ iscale) ` {− 1..1} ⊆ cbox (− 1) 1*
using *isc* **and** *assms(3−4)* **by** *(auto simp add: image_comp [symmetric])*
have *∗*: *continuous_on {− 1..1} iscale*
unfolding *iscale_def* **by** *(rule continuous_intros)+*
show *continuous_on {− 1..1} (f ∘ iscale) continuous_on {− 1..1} (g ∘ iscale)*
apply –
apply *(rule_tac[!] continuous_on_compose[OF ∗])*
apply *(rule_tac[!] continuous_on_subset[OF _ isc])*
apply *(rule assms)+*
done
have *∗*: *(1 / 2) *$_R$ (1 + (1::real^1)) = 1*
unfolding *vec_eq_iff* **by** *auto*
show *(f ∘ iscale) (− 1) $ 1 = − 1*
and *(f ∘ iscale) 1 $ 1 = 1*
and *(g ∘ iscale) (− 1) $ 2 = −1*
and *(g ∘ iscale) 1 $ 2 = 1*
unfolding *o_def iscale_def*
using *assms*
by *(auto simp add: ∗)*
qed
then obtain *s t* **where** *st*:
s ∈ {− 1..1}
t ∈ {− 1..1}
(f ∘ iscale) s = (g ∘ iscale) t
by *auto*
show *thesis*
apply *(rule_tac z = f (iscale s)* **in** *that)*
using *st*
unfolding *o_def path_image_def image_iff*
apply –
apply *(rule_tac x=iscale s* **in** *bexI)*
prefer *3*
apply *(rule_tac x=iscale t* **in** *bexI)*
using *isc[unfolded subset_eq, rule_format]*
apply *auto*
done
qed

theorem *fashoda*:
fixes *b :: real^2*
assumes *path f*

and *path g*
and *path_image f* ⊆ *cbox a b*
and *path_image g* ⊆ *cbox a b*
and (*pathstart f*)$*1* = *a*$*1*
and (*pathfinish f*)$*1* = *b*$*1*
and (*pathstart g*)$*2* = *a*$*2*
and (*pathfinish g*)$*2* = *b*$*2*
obtains *z* **where** *z* ∈ *path_image f* **and** *z* ∈ *path_image g*
proof −
fix *P Q S*
presume *P* ∨ *Q* ∨ *S P* ⟹ *thesis* **and** *Q* ⟹ *thesis* **and** *S* ⟹ *thesis*
then show *thesis*
by *auto*
next
have *cbox a b* ≠ {}
using *assms*(*3*) **using** *path_image_nonempty*[*of f*] **by** *auto*
then have *a* ≤ *b*
unfolding *interval_eq_empty_cart less_eq_vec_def* **by** (*auto simp add*: *not_less*)
then show *a*$*1* = *b*$*1* ∨ *a*$*2* = *b*$*2* ∨ (*a*$*1* < *b*$*1* ∧ *a*$*2* < *b*$*2*)
unfolding *less_eq_vec_def forall_2* **by** *auto*
next
assume *as*: *a*$*1* = *b*$*1*
have ∃ *z*∈*path_image g*. *z*$*2* = (*pathstart f*)$*2*
apply (*rule connected_ivt_component_cart*)
apply (*rule connected_path_image assms*)+
apply (*rule pathstart_in_path_image*)
apply (*rule pathfinish_in_path_image*)
unfolding *assms* **using** *assms*(*3*)[*unfolded path_image_def subset_eq*,*rule_format*,*of f 0*]
unfolding *pathstart_def*
apply (*auto simp add*: *less_eq_vec_def mem_box_cart*)
done
then obtain *z* :: *real*ˆ*2* **where** *z*: *z* ∈ *path_image g z* $ *2* = *pathstart f* $ *2* **..**
have *z* ∈ *cbox a b*
using *z*(*1*) *assms*(*4*)
unfolding *path_image_def*
by *blast*
then have *z* = *f 0*
unfolding *vec_eq_iff forall_2*
unfolding *z*(*2*) *pathstart_def*
using *assms*(*3*)[*unfolded path_image_def subset_eq mem_box_cart*,*rule_format*,*of f 0 1*]
unfolding *mem_box_cart*
apply (*erule_tac x*=*1* **in** *allE*)
using *as*
apply *auto*
done
then show *thesis*
apply −

apply (*rule that*[*OF* _ *z*(*1*)])
unfolding *path_image_def*
apply *auto*
done
next
assume *as*: *a*$*2* = *b*$*2*
have ∃ *z*∈*path_image f*. *z*$*1* = (*pathstart g*)$*1*
apply (*rule connected_ivt_component_cart*)
apply (*rule connected_path_image assms*)+
apply (*rule pathstart_in_path_image*)
apply (*rule pathfinish_in_path_image*)
unfolding *assms*
using *assms*(*4*)[*unfolded path_image_def subset_eq*,*rule_format*,*of g 0*]
unfolding *pathstart_def*
apply (*auto simp add*: *less_eq_vec_def mem_box_cart*)
done
then obtain *z* **where** *z*: *z* ∈ *path_image f z* $ *1* = *pathstart g* $ *1* **..**
have *z* ∈ *cbox a b*
using *z*(*1*) *assms*(*3*)
unfolding *path_image_def*
by *blast*
then have *z* = *g 0*
unfolding *vec_eq_iff forall_2*
unfolding *z*(*2*) *pathstart_def*
using *assms*(*4*)[*unfolded path_image_def subset_eq mem_box_cart*,*rule_format*,*of g 0 2*]
unfolding *mem_box_cart*
apply (*erule_tac x*=*2* **in** *allE*)
using *as*
apply *auto*
done
then show *thesis*
apply −
apply (*rule that*[*OF z*(*1*)])
unfolding *path_image_def*
apply *auto*
done
next
assume *as*: *a* $ *1* < *b* $ *1* ∧ *a* $ *2* < *b* $ *2*
have *int_nem*: *cbox* (−*1*) (*1*::*real*^*2*) ≠ {}
unfolding *interval_eq_empty_cart* **by** *auto*
obtain *z* :: *real*^*2* **where** *z*:
z ∈ (*interval_bij* (*a*, *b*) (− *1*, *1*) ∘ *f*) ‘ {*0*..*1*}
z ∈ (*interval_bij* (*a*, *b*) (− *1*, *1*) ∘ *g*) ‘ {*0*..*1*}
apply (*rule fashoda_unit_path*[*of interval_bij* (*a*,*b*) (− *1*,*1*) ∘ *f interval_bij* (*a*,*b*) (− *1*,*1*) ∘ *g*])
unfolding *path_def path_image_def pathstart_def pathfinish_def*
apply (*rule_tac*[*1*−*2*] *continuous_on_compose*)
apply (*rule assms*[*unfolded path_def*] *continuous_on_interval_bij*)+

```
  unfolding subset_eq
  apply(rule_tac[1-2] ballI)
proof -
  fix x
  assume x ∈ (interval_bij (a, b) (- 1, 1) ∘ f) ` {0..1}
  then obtain y where y:
      y ∈ {0..1}
      x = (interval_bij (a, b) (- 1, 1) ∘ f) y
    unfolding image_iff ..
  show x ∈ cbox (- 1) 1
    unfolding y o_def
    apply (rule in_interval_interval_bij)
    using y(1)
    using assms(3)[unfolded path_image_def subset_eq] int_nem
    apply auto
    done
next
  fix x
  assume x ∈ (interval_bij (a, b) (- 1, 1) ∘ g) ` {0..1}
  then obtain y where y:
      y ∈ {0..1}
      x = (interval_bij (a, b) (- 1, 1) ∘ g) y
    unfolding image_iff ..
  show x ∈ cbox (- 1) 1
    unfolding y o_def
    apply (rule in_interval_interval_bij)
    using y(1)
    using assms(4)[unfolded path_image_def subset_eq] int_nem
    apply auto
    done
next
  show (interval_bij (a, b) (- 1, 1) ∘ f) 0 $ 1 = -1
    and (interval_bij (a, b) (- 1, 1) ∘ f) 1 $ 1 = 1
    and (interval_bij (a, b) (- 1, 1) ∘ g) 0 $ 2 = -1
    and (interval_bij (a, b) (- 1, 1) ∘ g) 1 $ 2 = 1
    using assms as
   by (simp_all add: cart_eq_inner_axis pathstart_def pathfinish_def interval_bij_def)
      (simp_all add: inner_axis)
qed
from z(1) obtain zf where zf:
    zf ∈ {0..1}
    z = (interval_bij (a, b) (- 1, 1) ∘ f) zf
  unfolding image_iff ..
from z(2) obtain zg where zg:
    zg ∈ {0..1}
    z = (interval_bij (a, b) (- 1, 1) ∘ g) zg
  unfolding image_iff ..
have *: ∀ i. (- 1) $ i < (1::real^2) $ i ∧ a $ i < b $ i
  unfolding forall_2
```

```
    using as
    by auto
  show thesis
  proof (rule_tac z=interval_bij (− 1,1) (a,b) z in that)
    show interval_bij (− 1, 1) (a, b) z ∈ path_image f
      using zf by (simp add: interval_bij_bij_cart[OF ∗] path_image_def)
    show interval_bij (− 1, 1) (a, b) z ∈ path_image g
      using zg by (simp add: interval_bij_bij_cart[OF ∗] path_image_def)
  qed
qed
```

6.32.3 Some slightly ad hoc lemmas I use below

```
lemma segment_vertical:
  fixes a :: real^2
  assumes a$1 = b$1
  shows x ∈ closed_segment a b ⟷
    x$1 = a$1 ∧ x$1 = b$1 ∧ (a$2 ≤ x$2 ∧ x$2 ≤ b$2 ∨ b$2 ≤ x$2 ∧ x$2 ≤
a$2)
  (is _ = ?R)
proof −
  let ?L = ∃ u. (x $ 1 = (1 − u) ∗ a $ 1 + u ∗ b $ 1 ∧ x $ 2 = (1 − u) ∗ a $ 2
+ u ∗ b $ 2) ∧ 0 ≤ u ∧ u ≤ 1
  {
    presume ?L ⟹ ?R and ?R ⟹ ?L
    then show ?thesis
      unfolding closed_segment_def mem_Collect_eq
    unfolding vec_eq_iff forall_2 scalar_mult_eq_scaleR[symmetric] vector_component_simps
      by blast
  }
  {
    assume ?L
    then obtain u where u:
        x $ 1 = (1 − u) ∗ a $ 1 + u ∗ b $ 1
        x $ 2 = (1 − u) ∗ a $ 2 + u ∗ b $ 2
        0 ≤ u
        u ≤ 1
      by blast
    { fix b a
      assume b + u ∗ a > a + u ∗ b
      then have (1 − u) ∗ b > (1 − u) ∗ a
        by (auto simp add:field_simps)
      then have b ≥ a
        apply (drule_tac mult_left_less_imp_less)
        using u
        apply auto
        done
      then have u ∗ a ≤ u ∗ b
        apply −
```

```
      apply (rule mult_left_mono[OF _ u(3)])
      using u(3−4)
      apply (auto simp add: field_simps)
      done
    } note ∗ = this
    {
      fix a b
      assume u ∗ b > u ∗ a
      then have (1 − u) ∗ a ≤ (1 − u) ∗ b
        apply −
        apply (rule mult_left_mono)
        apply (drule mult_left_less_imp_less)
        using u
        apply auto
        done
      then have a + u ∗ b ≤ b + u ∗ a
        by (auto simp add: field_simps)
    } note ∗∗ = this
    then show ?R
      unfolding u assms
      using u
      by (auto simp add:field_simps not_le intro: ∗ ∗∗)
  }
  {
    assume ?R
    then show ?L
    proof (cases x$2 = b$2)
      case True
      then show ?L
        apply (rule_tac x=(x$2 − a$2) / (b$2 − a$2) in exI)
        unfolding assms True using ‹?R› apply (auto simp add: field_simps)
        done
    next
      case False
      then show ?L
        apply (rule_tac x=1 − (x$2 − b$2) / (a$2 − b$2) in exI)
        unfolding assms using ‹?R› apply (auto simp add: field_simps)
        done
    qed
  }
qed

lemma segment_horizontal:
  fixes a :: real^2
  assumes a$2 = b$2
  shows x ∈ closed_segment a b ⟷
    x$2 = a$2 ∧ x$2 = b$2 ∧ (a$1 ≤ x$1 ∧ x$1 ≤ b$1 ∨ b$1 ≤ x$1 ∧ x$1 ≤
a$1)
  (is _ = ?R)
```

```
proof –
 let ?L = ∃ u. (x $ 1 = (1 − u) ∗ a $ 1 + u ∗ b $ 1 ∧ x $ 2 = (1 − u) ∗ a $ 2
+ u ∗ b $ 2) ∧ 0 ≤ u ∧ u ≤ 1
 {
   presume ?L ⟹ ?R and ?R ⟹ ?L
   then show ?thesis
     unfolding closed_segment_def mem_Collect_eq
   unfolding vec_eq_iff forall_2 scalar_mult_eq_scaleR[symmetric] vector_component_simps
     by blast
 }
 {
   assume ?L
   then obtain u where u:
       x $ 1 = (1 − u) ∗ a $ 1 + u ∗ b $ 1
       x $ 2 = (1 − u) ∗ a $ 2 + u ∗ b $ 2
       0 ≤ u
       u ≤ 1
     by blast
   {
     fix b a
     assume b + u ∗ a > a + u ∗ b
     then have (1 − u) ∗ b > (1 − u) ∗ a
       by (auto simp add: field_simps)
     then have b ≥ a
       apply (drule_tac mult_left_less_imp_less)
       using u
       apply auto
       done
     then have u ∗ a ≤ u ∗ b
       apply –
       apply (rule mult_left_mono[OF _ u(3)])
       using u(3−4)
       apply (auto simp add: field_simps)
       done
   } note ∗ = this
   {
     fix a b
     assume u ∗ b > u ∗ a
     then have (1 − u) ∗ a ≤ (1 − u) ∗ b
       apply –
       apply (rule mult_left_mono)
       apply (drule mult_left_less_imp_less)
       using u
       apply auto
       done
     then have a + u ∗ b ≤ b + u ∗ a
       by (auto simp add: field_simps)
   } note ∗∗ = this
   then show ?R
```

```
    unfolding u assms
    using u
    by (auto simp add: field_simps not_le intro: * **)
  }
  {
   assume ?R
   then show ?L
   proof (cases x$1 = b$1)
    case True
    then show ?L
     apply (rule_tac x=(x$1 − a$1) / (b$1 − a$1) in exI)
     unfolding assms True
     using ‹?R›
     apply (auto simp add: field_simps)
     done
   next
    case False
    then show ?L
     apply (rule_tac x=1 − (x$1 − b$1) / (a$1 − b$1) in exI)
     unfolding assms
     using ‹?R›
     apply (auto simp add: field_simps)
     done
   qed
  }
qed
```

6.32.4 Useful Fashoda corollary pointed out to me by Tom Hales

```
corollary fashoda_interlace:
 fixes a :: real^2
 assumes path f
  and path g
  and paf: path_image f ⊆ cbox a b
  and pag: path_image g ⊆ cbox a b
  and (pathstart f)$2 = a$2
  and (pathfinish f)$2 = a$2
  and (pathstart g)$2 = a$2
  and (pathfinish g)$2 = a$2
  and (pathstart f)$1 < (pathstart g)$1
  and (pathstart g)$1 < (pathfinish f)$1
  and (pathfinish f)$1 < (pathfinish g)$1
 obtains z where z ∈ path_image f and z ∈ path_image g
proof −
 have cbox a b ≠ {}
  using path_image_nonempty[of f] using assms(3) by auto
 note ab=this[unfolded interval_eq_empty_cart not_ex forall_2 not_less]
 have pathstart f ∈ cbox a b
```

and *pathfinish f* ∈ *cbox a b*
and *pathstart g* ∈ *cbox a b*
and *pathfinish g* ∈ *cbox a b*
using *pathstart_in_path_image pathfinish_in_path_image*
using *assms(3−4)*
by *auto*
note *startfin = this[unfolded mem_box_cart forall_2]*
let *?P1 = linepath (vector[a$1 − 2, a$2 − 2]) (vector[(pathstart f)$1,a$2 − 2]) +++*
linepath(vector[(pathstart f)$1,a$2 − 2])(pathstart f) +++ f +++
linepath(pathfinish f)(vector[(pathfinish f)$1,a$2 − 2]) +++
linepath(vector[(pathfinish f)$1,a$2 − 2])(vector[b$1 + 2,a$2 − 2])
let *?P2 = linepath(vector[(pathstart g)$1, (pathstart g)$2 − 3])(pathstart g) +++ g +++*
linepath(pathfinish g)(vector[(pathfinish g)$1,a$2 − 1]) +++
linepath(vector[(pathfinish g)$1,a$2 − 1])(vector[b$1 + 1,a$2 − 1]) +++
linepath(vector[b$1 + 1,a$2 − 1])(vector[b$1 + 1,b$2 + 3])
let *?a = vector[a$1 − 2, a$2 − 3]*
let *?b = vector[b$1 + 2, b$2 + 3]*
have *P1P2: path_image ?P1 = path_image (linepath (vector[a$1 − 2, a$2 − 2]) (vector[(pathstart f)$1,a$2 − 2])) ∪*
path_image (linepath(vector[(pathstart f)$1,a$2 − 2])(pathstart f)) ∪ path_image f ∪
path_image (linepath(pathfinish f)(vector[(pathfinish f)$1,a$2 − 2])) ∪
path_image (linepath(vector[(pathfinish f)$1,a$2 − 2])(vector[b$1 + 2,a$2 − 2]))
path_image ?P2 = path_image(linepath(vector[(pathstart g)$1, (pathstart g)$2 − 3])(pathstart g)) ∪ path_image g ∪
path_image(linepath(pathfinish g)(vector[(pathfinish g)$1,a$2 − 1])) ∪
path_image(linepath(vector[(pathfinish g)$1,a$2 − 1])(vector[b$1 + 1,a$2 − 1])) ∪
path_image(linepath(vector[b$1 + 1,a$2 − 1])(vector[b$1 + 1,b$2 + 3]))
using *assms(1−2)*
by(*auto simp add: path_image_join*)
have *abab: cbox a b ⊆ cbox ?a ?b*
unfolding *interval_cbox_cart[symmetric]*
by (*auto simp add:less_eq_vec_def forall_2*)
obtain *z* **where**
z ∈ *path_image*
(*linepath (vector [a $ 1 − 2, a $ 2 − 2]) (vector [pathstart f $ 1, a $ 2 − 2]) +++*
linepath (vector [pathstart f $ 1, a $ 2 − 2]) (pathstart f) +++
f +++
linepath (pathfinish f) (vector [pathfinish f $ 1, a $ 2 − 2]) +++
linepath (vector [pathfinish f $ 1, a $ 2 − 2]) (vector [b $ 1 + 2, a $ 2 − 2]))
z ∈ *path_image*
(*linepath (vector [pathstart g $ 1, pathstart g $ 2 − 3]) (pathstart g) +++*
g +++

linepath (*pathfinish g*) (*vector* [*pathfinish g* $ *1*, *a* $ *2* − *1*]) +++
linepath (*vector* [*pathfinish g* $ *1*, *a* $ *2* − *1*]) (*vector* [*b* $ *1* + *1*, *a* $ *2* − *1*]) +++
linepath (*vector* [*b* $ *1* + *1*, *a* $ *2* − *1*]) (*vector* [*b* $ *1* + *1*, *b* $ *2* + *3*]))
apply (*rule fashoda*[*of ?P1 ?P2 ?a ?b*])
unfolding *pathstart_join pathfinish_join pathstart_linepath pathfinish_linepath vector_2*
proof −
show *path ?P1* **and** *path ?P2*
using *assms* **by** *auto*
show *path_image ?P1* ⊆ *cbox ?a ?b path_image ?P2* ⊆ *cbox ?a ?b*
unfolding *P1P2 path_image_linepath* **using** *startfin paf pag*
by (*auto simp*: *mem_box_cart segment_horizontal segment_vertical forall_2*)
show *a* $ *1* − *2* = *a* $ *1* − *2*
and *b* $ *1* + *2* = *b* $ *1* + *2*
and *pathstart g* $ *2* − *3* = *a* $ *2* − *3*
and *b* $ *2* + *3* = *b* $ *2* + *3*
by (*auto simp add*: *assms*)
qed
note *z*=*this*[*unfolded P1P2 path_image_linepath*]
show *thesis*
proof (*rule that*[*of z*])
have (*z* ∈ *closed_segment* (*vector* [*a* $ *1* − *2*, *a* $ *2* − *2*]) (*vector* [*pathstart f* $ *1*, *a* $ *2* − *2*]) ∨
z ∈ *closed_segment* (*vector* [*pathstart f* $ *1*, *a* $ *2* − *2*]) (*pathstart f*)) ∨
z ∈ *closed_segment* (*pathfinish f*) (*vector* [*pathfinish f* $ *1*, *a* $ *2* − *2*]) ∨
z ∈ *closed_segment* (*vector* [*pathfinish f* $ *1*, *a* $ *2* − *2*]) (*vector* [*b* $ *1* + *2*, *a* $ *2* − *2*]) ⟹
(((*z* ∈ *closed_segment* (*vector* [*pathstart g* $ *1*, *pathstart g* $ *2* − *3*]) (*pathstart g*)) ∨
z ∈ *closed_segment* (*pathfinish g*) (*vector* [*pathfinish g* $ *1*, *a* $ *2* − *1*])) ∨
z ∈ *closed_segment* (*vector* [*pathfinish g* $ *1*, *a* $ *2* − *1*]) (*vector* [*b* $ *1* + *1*, *a* $ *2* − *1*])) ∨
z ∈ *closed_segment* (*vector* [*b* $ *1* + *1*, *a* $ *2* − *1*]) (*vector* [*b* $ *1* + *1*, *b* $ *2* + *3*]) ⟹ *False*
proof (*simp only*: *segment_vertical segment_horizontal vector_2*, *goal_cases*)
case *prems*: *1*
have *pathfinish f* ∈ *cbox a b*
using *assms*(*3*) *pathfinish_in_path_image*[*of f*] **by** *auto*
then have *1* + *b* $ *1* ≤ *pathfinish f* $ *1* ⟹ *False*
unfolding *mem_box_cart forall_2* **by** *auto*
then have *z*$*1* ≠ *pathfinish f*$*1*
using *prems*(*2*)
using *assms ab*
by (*auto simp add*: *field_simps*)
moreover have *pathstart f* ∈ *cbox a b*
using *assms*(*3*) *pathstart_in_path_image*[*of f*]
by *auto*
then have *1* + *b* $ *1* ≤ *pathstart f* $ *1* ⟹ *False*

unfolding *mem_box_cart forall_2*
by *auto*
then have $z\$1 \neq pathstart\ f\$1$
using *prems*(*2*) **using** *assms ab*
by (*auto simp add*: *field_simps*)
ultimately have ∗: $z\$2 = a\$2 - 2$
using *prems*(*1*) **by** *auto*
have $z\$1 \neq pathfinish\ g\$1$
using *prems*(*2*) *assms ab*
by (*auto simp add*: *field_simps* ∗)
moreover have $pathstart\ g \in cbox\ a\ b$
using *assms*(*4*) *pathstart_in_path_image*[*of g*]
by *auto*
note *this*[*unfolded mem_box_cart forall_2*]
then have $z\$1 \neq pathstart\ g\$1$
using *prems*(*1*) *assms ab*
by (*auto simp add*: *field_simps* ∗)
ultimately have $a \$ 2 - 1 \leq z \$ 2 \wedge z \$ 2 \leq b \$ 2 + 3 \vee b \$ 2 + 3 \leq z \$ 2 \wedge z \$ 2 \leq a \$ 2 - 1$
using *prems*(*2*) **unfolding** ∗ *assms* **by** (*auto simp add*: *field_simps*)
then show *False*
unfolding ∗ **using** *ab* **by** *auto*
qed
then have $z \in path\_image\ f \vee z \in path\_image\ g$
using *z* **unfolding** *Un_iff* **by** *blast*
then have *z'*: $z \in cbox\ a\ b$
using *assms*(*3*−*4*) **by** *auto*
have $a \$ 2 = z \$ 2 \Longrightarrow (z \$ 1 = pathstart\ f \$ 1 \vee z \$ 1 = pathfinish\ f \$ 1) \Longrightarrow z = pathstart\ f \vee z = pathfinish\ f$
unfolding *vec_eq_iff forall_2 assms*
by *auto*
with *z'* **show** $z \in path\_image\ f$
using *z*(*1*)
unfolding *Un_iff mem_box_cart forall_2*
by (*simp only*: *segment_vertical segment_horizontal vector_2*) (*auto simp*: *assms*)
have $a \$ 2 = z \$ 2 \Longrightarrow (z \$ 1 = pathstart\ g \$ 1 \vee z \$ 1 = pathfinish\ g \$ 1) \Longrightarrow z = pathstart\ g \vee z = pathfinish\ g$
unfolding *vec_eq_iff forall_2 assms*
by *auto*
with *z'* **show** $z \in path\_image\ g$
using *z*(*2*)
unfolding *Un_iff mem_box_cart forall_2*
by (*simp only*: *segment_vertical segment_horizontal vector_2*) (*auto simp*: *assms*)
qed
qed

end

6.33 Vector Cross Products in 3 Dimensions

theory *Cross3*
  imports *Determinants Cartesian_Euclidean_Space*
begin

context includes *no_Set_Product_syntax*
begin — locally disable syntax for set product, to avoid warnings

definition *cross3* :: [*real^3*, *real^3*] ⇒ *real^3* (**infixr** × *80*)
  where *a* × *b* ≡
    vector [*a*\$*2* ∗ *b*\$*3* − *a*\$*3* ∗ *b*\$*2*,
        a\$*3* ∗ *b*\$*1* − *a*\$*1* ∗ *b*\$*3*,
        a\$*1* ∗ *b*\$*2* − *a*\$*2* ∗ *b*\$*1*]

end

bundle *cross3_syntax* **begin**
notation *cross3* (**infixr** × *80*)
no_notation *Product_Type.Times* (**infixr** × *80*)
end

bundle *no_cross3_syntax* **begin**
no_notation *cross3* (**infixr** × *80*)
notation *Product_Type.Times* (**infixr** × *80*)
end

unbundle *cross3_syntax*

6.33.1 Basic lemmas

lemmas *cross3_simps* = *cross3_def inner_vec_def sum_3 det_3 vec_eq_iff vector_def algebra_simps*

lemma *dot_cross_self*: *x* · (*x* × *y*) = *0* *x* · (*y* × *x*) = *0* (*x* × *y*) · *y* = *0* (*y* × *x*) · *y* = *0*
  by (*simp_all add*: *orthogonal_def cross3_simps*)

lemma *orthogonal_cross*: *orthogonal* (*x* × *y*) *x* *orthogonal* (*x* × *y*) *y*
  orthogonal *y* (*x* × *y*) *orthogonal* (*x* × *y*) *x*
  by (*simp_all add*: *orthogonal_def dot_cross_self*)

lemma *cross_zero_left* [*simp*]: *0* × *x* = *0* **and** *cross_zero_right* [*simp*]: *x* × *0* = *0*
for *x*::*real^3*
  by (*simp_all add*: *cross3_simps*)

lemma *cross_skew*: $(x \times y) = -(y \times x)$ **for** *x::real^3*
by (*simp add*: *cross3_simps*)

lemma *cross_refl* [*simp*]: $x \times x = 0$ **for** *x::real^3*
by (*simp add*: *cross3_simps*)

lemma *cross_add_left*: $(x + y) \times z = (x \times z) + (y \times z)$ **for** *x::real^3*
by (*simp add*: *cross3_simps*)

lemma *cross_add_right*: $x \times (y + z) = (x \times y) + (x \times z)$ **for** *x::real^3*
by (*simp add*: *cross3_simps*)

lemma *cross_mult_left*: $(c *_R x) \times y = c *_R (x \times y)$ **for** *x::real^3*
by (*simp add*: *cross3_simps*)

lemma *cross_mult_right*: $x \times (c *_R y) = c *_R (x \times y)$ **for** *x::real^3*
by (*simp add*: *cross3_simps*)

lemma *cross_minus_left* [*simp*]: $(-x) \times y = - (x \times y)$ **for** *x::real^3*
by (*simp add*: *cross3_simps*)

lemma *cross_minus_right* [*simp*]: $x \times -y = - (x \times y)$ **for** *x::real^3*
by (*simp add*: *cross3_simps*)

lemma *left_diff_distrib*: $(x - y) \times z = x \times z - y \times z$ **for** *x::real^3*
by (*simp add*: *cross3_simps*)

lemma *right_diff_distrib*: $x \times (y - z) = x \times y - x \times z$ **for** *x::real^3*
by (*simp add*: *cross3_simps*)

hide_fact (**open**) *left_diff_distrib right_diff_distrib*

proposition *Jacobi*: $x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0$ **for** *x::real^3*
by (*simp add*: *cross3_simps*)

proposition *Lagrange*: $x \times (y \times z) = (x \cdot z) *_R y - (x \cdot y) *_R z$
by (*simp add*: *cross3_simps*) (*metis* (*full_types*) *exhaust_3*)

proposition *cross_triple*: $(x \times y) \cdot z = (y \times z) \cdot x$
by (*simp add*: *cross3_def inner_vec_def sum_3 vec_eq_iff algebra_simps*)

lemma *cross_components*:
$(x \times y)\$1 = x\$2 * y\$3 - y\$2 * x\$3$ $(x \times y)\$2 = x\$3 * y\$1 - y\$3 * x\$1$ $(x \times y)\$3 = x\$1 * y\$2 - y\$1 * x\$2$
by (*simp_all add*: *cross3_def inner_vec_def sum_3 vec_eq_iff algebra_simps*)

lemma *cross_basis*: $(axis\ 1\ 1) \times (axis\ 2\ 1) = axis\ 3\ 1$ $(axis\ 2\ 1) \times (axis\ 1\ 1) = -(axis\ 3\ 1)$
$(axis\ 2\ 1) \times (axis\ 3\ 1) = axis\ 1\ 1$ $(axis\ 3\ 1) \times (axis\ 2\ 1) = -(axis$

1 1)
(*axis 3 1*) × (*axis 1 1*) = *axis 2 1* (*axis 1 1*) × (*axis 3 1*) = −(*axis 2 1*)
using *exhaust_3*
by (*force simp add*: *axis_def cross3_simps*)+

lemma *cross_basis_nonzero*:
u ≠ *0* ⟹ *u* × *axis 1 1* ≠ *0* ∨ *u* × *axis 2 1* ≠ *0* ∨ *u* × *axis 3 1* ≠ *0*
by (*clarsimp simp add*: *axis_def cross3_simps*) (*metis exhaust_3*)

lemma *cross_dot_cancel*:
fixes *x*::*real^3*
assumes *deq*: *x* · *y* = *x* · *z* **and** *veq*: *x* × *y* = *x* × *z* **and** *x*: *x* ≠ *0*
shows *y* = *z*
proof −
have *x* · *x* ≠ *0*
by (*simp add*: *x*)
then have *y* − *z* = *0*
using *veq*
by (*metis* (*no_types*, *lifting*) *Cross3*.*right_diff_distrib Lagrange deq eq_iff_diff_eq_0 inner_diff_right scale_eq_0_iff*)
then show *?thesis*
using *eq_iff_diff_eq_0* **by** *blast*
qed

lemma *norm_cross_dot*: (*norm* (*x* × *y*))^2 + (*x* · *y*)^2 = (*norm x* ∗ *norm y*)^2
unfolding *power2_norm_eq_inner power_mult_distrib*
by (*simp add*: *cross3_simps power2_eq_square*)

lemma *dot_cross_det*: *x* · (*y* × *z*) = *det*(*vector*[*x*,*y*,*z*])
by (*simp add*: *cross3_simps*)

lemma *cross_cross_det*: (*w* × *x*) × (*y* × *z*) = *det*(*vector*[*w*,*x*,*z*]) $*_R$ *y* − *det*(*vector*[*w*,*x*,*y*]) $*_R$ *z*
using *exhaust_3* **by** (*force simp add*: *cross3_simps*)

proposition *dot_cross*: (*w* × *x*) · (*y* × *z*) = (*w* · *y*) ∗ (*x* · *z*) − (*w* · *z*) ∗ (*x* · *y*)
by (*force simp add*: *cross3_simps*)

proposition *norm_cross*: (*norm* (*x* × *y*))^2 = (*norm x*)^2 ∗ (*norm y*)^2 − (*x* · *y*)^2
unfolding *power2_norm_eq_inner power_mult_distrib*
by (*simp add*: *cross3_simps power2_eq_square*)

lemma *cross_eq_0*: *x* × *y* = *0* ⟷ *collinear*{*0*,*x*,*y*}
proof −
have *x* × *y* = *0* ⟷ *norm* (*x* × *y*) = *0*
by *simp*
also have ... ⟷ (*norm x* ∗ *norm y*)^2 = (*x* · *y*)^2
using *norm_cross* [*of x y*] **by** (*auto simp*: *power_mult_distrib*)

also have ... ⟷ $|x \cdot y| = norm\ x * norm\ y$
  using *power2_eq_iff*
 by (*metis* (*mono_tags*, *hide_lams*) *abs_minus abs_norm_cancel abs_power2 norm_mult power_abs real_norm_def*)
 also have ... ⟷ *collinear* $\{0, x, y\}$
  by (*rule norm_cauchy_schwarz_equal*)
 finally show *?thesis* **.**
qed

lemma *cross_eq_self*: $x \times y = x \longleftrightarrow x = 0$ $x \times y = y \longleftrightarrow y = 0$
 apply (*metis cross_zero_left dot_cross_self*(*1*) *inner_eq_zero_iff*)
 by (*metis cross_zero_right dot_cross_self*(*2*) *inner_eq_zero_iff*)

lemma *norm_and_cross_eq_0*:
 $x \cdot y = 0 \wedge x \times y = 0 \longleftrightarrow x = 0 \vee y = 0$ (**is** *?lhs* = *?rhs*)
proof
 assume *?lhs*
 then show *?rhs*
  by (*metis cross_dot_cancel cross_zero_right inner_zero_right*)
qed *auto*

lemma *bilinear_cross*: *bilinear*(×)
 apply (*auto simp add*: *bilinear_def linear_def*)
 apply *unfold_locales*
 apply (*simp add*: *cross_add_right*)
 apply (*simp add*: *cross_mult_right*)
 apply (*simp add*: *cross_add_left*)
 apply (*simp add*: *cross_mult_left*)
 done

6.33.2 Preservation by rotation, or other orthogonal transformation up to sign

lemma *cross_matrix_mult*: *transpose* $A *v ((A *v x) \times (A *v y)) = det\ A *_R (x \times y)$
 apply (*simp add*: *vec_eq_iff*)
 apply (*simp add*: *vector_matrix_mult_def matrix_vector_mult_def forall_3 cross3_simps*)
 done

lemma *cross_orthogonal_matrix*:
 assumes *orthogonal_matrix A*
 shows $(A *v x) \times (A *v y) = det\ A *_R (A *v (x \times y))$
proof −
 have *mat 1* = *transpose* (*A* ∗∗ *transpose A*)
  by (*metis* (*no_types*) *assms orthogonal_matrix_def transpose_mat*)
 then show *?thesis*
  by (*metis* (*no_types*) *vector_matrix_mul_rid vector_transpose_matrix cross_matrix_mult matrix_vector_mul_assoc matrix_vector_mult_scaleR*)
qed

lemma *cross_rotation_matrix*: *rotation_matrix* $A \Longrightarrow (A *v\ x) \times (A *v\ y) = A *v\ (x \times y)$
by (*simp add*: *rotation_matrix_def cross_orthogonal_matrix*)

lemma *cross_rotoinversion_matrix*: *rotoinversion_matrix* $A \Longrightarrow (A *v\ x) \times (A *v\ y) = - A *v\ (x \times y)$
by (*simp add*: *rotoinversion_matrix_def cross_orthogonal_matrix scaleR_matrix_vector_assoc*)

lemma *cross_orthogonal_transformation*:
assumes *orthogonal_transformation f*
shows $(f\ x) \times (f\ y) = det(matrix\ f) *_R f(x \times y)$
proof −
have *orth*: *orthogonal_matrix* (*matrix f*)
using *assms orthogonal_transformation_matrix* **by** *blast*
have *matrix f* $*v$ *z* = *f z* **for** *z*
using *assms orthogonal_transformation_matrix* **by** *force*
with *cross_orthogonal_matrix* [*OF orth*] **show** *?thesis*
by *simp*
qed

lemma *cross_linear_image*:
$[\![linear\ f;\ \bigwedge x.\ norm(f\ x) = norm\ x;\ det(matrix\ f) = 1]\!]$
$\Longrightarrow (f\ x) \times (f\ y) = f(x \times y)$
by (*simp add*: *cross_orthogonal_transformation orthogonal_transformation*)

6.33.3 Continuity

lemma *continuous_cross*: $[\![continuous\ F\ f;\ continuous\ F\ g]\!] \Longrightarrow continuous\ F\ (\lambda x.\ (f\ x) \times (g\ x))$
apply (*subst continuous_componentwise*)
apply (*clarsimp simp add*: *cross3_simps*)
apply (*intro continuous_intros*; *simp*)
done

lemma *continuous_on_cross*:
fixes f :: *'a::t2_space* ⇒ *real^3*
shows $[\![continuous\_on\ S\ f;\ continuous\_on\ S\ g]\!] \Longrightarrow continuous\_on\ S\ (\lambda x.\ (f\ x) \times (g\ x))$
by (*simp add*: *continuous_on_eq_continuous_within continuous_cross*)

unbundle *no_cross3_syntax*

end

6.34 Bounded Continuous Functions

theory *Bounded_Continuous_Function*
imports

Topology_Euclidean_Space
Uniform_Limit
begin

6.34.1 Definition

definition *bcontfun* = {*f*. *continuous_on UNIV f* ∧ *bounded* (*range f*)}

typedef (**overloaded**) ('*a*, '*b*) *bcontfun* ((_ ⇒$_C$ /_) [*22*, *21*] *21*) =
bcontfun::('*a*::*topological_space* ⇒ '*b*::*metric_space*) *set*
morphisms *apply_bcontfun Bcontfun*
by (*auto intro*: *continuous_intros simp*: *bounded_def bcontfun_def*)

declare [[*coercion apply_bcontfun* :: ('*a*::*topological_space* ⇒$_C$'*b*::*metric_space*) ⇒ '*a* ⇒ '*b*]]

setup_lifting *type_definition_bcontfun*

lemma *continuous_on_apply_bcontfun*[*intro*, *simp*]: *continuous_on T* (*apply_bcontfun x*)
and *bounded_apply_bcontfun*[*intro*, *simp*]: *bounded* (*range* (*apply_bcontfun x*))
using *apply_bcontfun*[*of x*]
by (*auto simp*: *bcontfun_def intro*: *continuous_on_subset*)

lemma *bcontfun_eqI*: (⋀*x*. *apply_bcontfun f x* = *apply_bcontfun g x*) ⟹ *f* = *g*
by *transfer auto*

lemma *bcontfunE*:
assumes *f* ∈ *bcontfun*
obtains *g* **where** *f* = *apply_bcontfun g*
by (*blast intro*: *apply_bcontfun_cases assms*)

lemma *const_bcontfun*: (λ*x*. *b*) ∈ *bcontfun*
by (*auto simp*: *bcontfun_def image_def*)

lift_definition *const_bcontfun*::'*b*::*metric_space* ⇒ ('*a*::*topological_space* ⇒$_C$ '*b*) **is** λ*c* _. *c*
by (*rule const_bcontfun*)

instantiation *bcontfun* :: (*topological_space*, *metric_space*) *metric_space*
begin

lift_definition *dist_bcontfun* :: '*a* ⇒$_C$ '*b* ⇒ '*a* ⇒$_C$ '*b* ⇒ *real*
is λ*f g*. (*SUP x*. *dist* (*f x*) (*g x*)) **.**

definition *uniformity_bcontfun* :: ('*a* ⇒$_C$ '*b* × '*a* ⇒$_C$ '*b*) *filter*
where *uniformity_bcontfun* = (*INF e*∈{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

definition *open_bcontfun* :: ('a ⇒$_C$ 'b) set ⇒ bool
  where *open_bcontfun S* = (∀ x∈S. ∀$_F$ (x', y) in uniformity. x' = x ⟶ y ∈ S)

lemma *bounded_dist_le_SUP_dist*:
  bounded (*range f*) ⟹ *bounded* (*range g*) ⟹ *dist* (*f x*) (*g x*) ≤ (*SUP x*. *dist* (*f x*) (*g x*))
  by (*auto intro*!: *cSUP_upper bounded_imp_bdd_above bounded_dist_comp*)

lemma *dist_bounded*:
  fixes *f g* :: 'a ⇒$_C$ 'b
  shows *dist* (*f x*) (*g x*) ≤ *dist f g*
  by *transfer* (*auto intro*!: *bounded_dist_le_SUP_dist simp*: *bcontfun_def*)

lemma *dist_bound*:
  fixes *f g* :: 'a ⇒$_C$ 'b
  assumes ⋀x. *dist* (*f x*) (*g x*) ≤ *b*
  shows *dist f g* ≤ *b*
  using *assms*
  by *transfer* (*auto intro*!: *cSUP_least*)

lemma *dist_fun_lt_imp_dist_val_lt*:
  fixes *f g* :: 'a ⇒$_C$ 'b
  assumes *dist f g* < *e*
  shows *dist* (*f x*) (*g x*) < *e*
  using *dist_bounded assms* **by** (*rule le_less_trans*)

instance
proof
  fix *f g h* :: 'a ⇒$_C$ 'b
  show *dist f g* = *0* ⟷ *f* = *g*
  proof
    have ⋀x. *dist* (*f x*) (*g x*) ≤ *dist f g*
      by (*rule dist_bounded*)
    also assume *dist f g* = *0*
    finally show *f* = *g*
      by (*auto simp*: *apply_bcontfun_inject*[*symmetric*])
  qed (*auto simp*: *dist_bcontfun_def intro*!: *cSup_eq*)
  show *dist f g* ≤ *dist f h* + *dist g h*
  proof (*rule dist_bound*)
    fix *x*
    have *dist* (*f x*) (*g x*) ≤ *dist* (*f x*) (*h x*) + *dist* (*g x*) (*h x*)
      by (*rule dist_triangle2*)
    also have *dist* (*f x*) (*h x*) ≤ *dist f h*
      by (*rule dist_bounded*)
    also have *dist* (*g x*) (*h x*) ≤ *dist g h*
      by (*rule dist_bounded*)
    finally show *dist* (*f x*) (*g x*) ≤ *dist f h* + *dist g h*
      by *simp*
  qed

qed (*rule open_bcontfun_def uniformity_bcontfun_def*)+

end

lift_definition *PiC*::'*a*::*topological_space set* ⇒ ('*a* ⇒ '*b set*) ⇒ ('*a* ⇒$_C$ '*b*::*metric_space*) *set*
  is *λI X. Pi I X* ∩ *bcontfun*
  by *auto*

lemma *mem_PiC_iff*: *x* ∈ *PiC I X* ⟷ *apply_bcontfun x* ∈ *Pi I X*
  by *transfer simp*

lemmas *mem_PiCD* = *mem_PiC_iff*[*THEN iffD1*]
  and *mem_PiCI* = *mem_PiC_iff*[*THEN iffD2*]

lemma *tendsto_bcontfun_uniform_limit*:
  fixes *f*::'*i* ⇒ '*a*::*topological_space* ⇒$_C$ '*b*::*metric_space*
  assumes (*f* ⟶ *l*) *F*
  shows *uniform_limit UNIV f l F*
proof (*rule uniform_limitI*)
  fix *e*::*real* **assume** *e* > *0*
  from *tendstoD*[*OF assms this*] **have** ∀$_F$ *x in F. dist* (*f x*) *l* < *e* **.**
  then show ∀$_F$ *n in F.* ∀*x*∈*UNIV. dist* ((*f n*) *x*) (*l x*) < *e*
    by *eventually_elim* (*auto simp*: *dist_fun_lt_imp_dist_val_lt*)
qed

lemma *uniform_limit_tendsto_bcontfun*:
  fixes *f*::'*i* ⇒ '*a*::*topological_space* ⇒$_C$ '*b*::*metric_space*
    and *l*::'*a*::*topological_space* ⇒$_C$ '*b*::*metric_space*
  assumes *uniform_limit UNIV f l F*
  shows (*f* ⟶ *l*) *F*
proof (*rule tendstoI*)
  fix *e*::*real* **assume** *e* > *0*
  then have *e* / *2* > *0* **by** *simp*
  from *uniform_limitD*[*OF assms this*]
  have ∀$_F$ *i in F.* ∀*x. dist* (*f i x*) (*l x*) < *e* / *2* **by** *simp*
  then have ∀$_F$ *x in F. dist* (*f x*) *l* ≤ *e* / *2*
    by *eventually_elim* (*blast intro*: *dist_bound less_imp_le*)
  then show ∀$_F$ *x in F. dist* (*f x*) *l* < *e*
    by *eventually_elim* (*use* ‹*0* < *e*› **in** *auto*)
qed

lemma *uniform_limit_bcontfunE*:
  fixes *f*::'*i* ⇒ '*a*::*topological_space* ⇒$_C$ '*b*::*metric_space*
    and *l*::'*a*::*topological_space* ⇒ '*b*::*metric_space*
  assumes *uniform_limit UNIV f l F F* ≠ *bot*
  obtains *l'*::'*a*::*topological_space* ⇒$_C$ '*b*::*metric_space*
  where *l* = *l'* (*f* ⟶ *l'*) *F*
 by (*metis* (*mono_tags*, *lifting*) *always_eventually apply_bcontfun apply_bcontfun_cases*

assms
bcontfun_def mem_Collect_eq uniform_limit_bounded uniform_limit_tendsto_bcontfun uniform_limit_theorem)

lemma *closed_PiC*:
fixes *I* :: *'a*::*metric_space set*
and *X* :: *'a* ⇒ *'b*::*complete_space set*
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *closed* (*X i*)
shows *closed* (*PiC I X*)
unfolding *closed_sequential_limits*
proof *safe*
fix *f l*
assume *seq*: ∀ *n*. *f n* ∈ *PiC I X* **and** *lim*: *f* ⟶ *l*
show *l* ∈ *PiC I X*
proof (*safe intro*!: *mem_PiCI*)
fix *x* **assume** *x* ∈ *I*
then have *closed* (*X x*)
using *assms* **by** *simp*
moreover have *eventually* (λ*i*. *f i x* ∈ *X x*) *sequentially*
using *seq* ‹*x* ∈ *I*›
by (*auto intro*!: *eventuallyI dest*!: *mem_PiCD simp*: *Pi_iff*)
moreover note *sequentially_bot*
moreover have (λ*n*. (*f n*) *x*) ⟶ *l x*
using *tendsto_bcontfun_uniform_limit*[*OF lim*]
by (*rule tendsto_uniform_limitI*) *simp*
ultimately show *l x* ∈ *X x*
by (*rule Lim_in_closed_set*)
qed
qed

6.34.2 Complete Space

instance *bcontfun* :: (*metric_space*, *complete_space*) *complete_space*
proof
fix *f* :: *nat* ⇒ (*'a*, *'b*) *bcontfun*
assume *Cauchy f* — Cauchy equals uniform convergence
then obtain *g* **where** *uniform_limit UNIV f g sequentially*
using *uniformly_convergent_eq_cauchy*[*of* λ_. *True f*]
unfolding *Cauchy_def uniform_limit_sequentially_iff*
by (*metis dist_fun_lt_imp_dist_val_lt*)

from *uniform_limit_bcontfunE*[*OF this sequentially_bot*]
obtain *l'* **where** *g* = *apply_bcontfun l'* (*f* ⟶ *l'*) **by** *metis*
then show *convergent f*
by (*intro convergentI*)
qed

6.34.3 Supremum norm for a normed vector space

instantiation *bcontfun* :: (*topological_space*, *real_normed_vector*) *real_vector*

begin

lemma *uminus_cont*: $f \in bcontfun \Longrightarrow (\lambda x. - f\ x) \in bcontfun$ **for** $f::'a \Rightarrow 'b$
by (*auto simp*: *bcontfun_def intro*!: *continuous_intros*)

lemma *plus_cont*: $f \in bcontfun \Longrightarrow g \in bcontfun \Longrightarrow (\lambda x. f\ x + g\ x) \in bcontfun$ **for** $f\ g::'a \Rightarrow 'b$
by (*auto simp*: *bcontfun_def intro*!: *continuous_intros bounded_plus_comp*)

lemma *minus_cont*: $f \in bcontfun \Longrightarrow g \in bcontfun \Longrightarrow (\lambda x. f\ x - g\ x) \in bcontfun$ **for** $f\ g::'a \Rightarrow 'b$
by (*auto simp*: *bcontfun_def intro*!: *continuous_intros bounded_minus_comp*)

lemma *scaleR_cont*: $f \in bcontfun \Longrightarrow (\lambda x.\ a *_R f\ x) \in bcontfun$ **for** $f :: 'a \Rightarrow 'b$
by (*auto simp*: *bcontfun_def intro*!: *continuous_intros bounded_scaleR_comp*)

lemma *bcontfun_normI*: *continuous_on UNIV f* $\Longrightarrow (\bigwedge x.\ norm\ (f\ x) \le b) \Longrightarrow f \in bcontfun$
by (*auto simp*: *bcontfun_def intro*: *boundedI*)

lift_definition *uminus_bcontfun*::$('a \Rightarrow_C 'b) \Rightarrow 'a \Rightarrow_C 'b$ **is** $\lambda f\ x. - f\ x$
by (*rule uminus_cont*)

lift_definition *plus_bcontfun*::$('a \Rightarrow_C 'b) \Rightarrow ('a \Rightarrow_C 'b) \Rightarrow 'a \Rightarrow_C 'b$ **is** $\lambda f\ g\ x.\ f\ x + g\ x$
by (*rule plus_cont*)

lift_definition *minus_bcontfun*::$('a \Rightarrow_C 'b) \Rightarrow ('a \Rightarrow_C 'b) \Rightarrow 'a \Rightarrow_C 'b$ **is** $\lambda f\ g\ x.\ f\ x - g\ x$
by (*rule minus_cont*)

lift_definition *zero_bcontfun*::$'a \Rightarrow_C 'b$ **is** $\lambda\_.\ 0$
by (*rule const_bcontfun*)

lemma *const_bcontfun_0_eq_0*[*simp*]: *const_bcontfun* $0 = 0$
by *transfer simp*

lift_definition *scaleR_bcontfun*::*real* $\Rightarrow ('a \Rightarrow_C 'b) \Rightarrow 'a \Rightarrow_C 'b$ **is** $\lambda r\ g\ x.\ r *_R g\ x$
by (*rule scaleR_cont*)

lemmas [*simp*] =
const_bcontfun.rep_eq
uminus_bcontfun.rep_eq
plus_bcontfun.rep_eq
minus_bcontfun.rep_eq
zero_bcontfun.rep_eq
scaleR_bcontfun.rep_eq

instance
  by *standard* (*auto intro*!: *bcontfun_eqI simp*: *algebra_simps*)

end

lemma *bounded_norm_le_SUP_norm*:
  bounded (*range f*) $\Longrightarrow$ *norm* (*f x*) $\leq$ (*SUP x*. *norm* (*f x*))
  by (*auto intro*!: *cSUP_upper bounded_imp_bdd_above simp*: *bounded_norm_comp*)

instantiation *bcontfun* :: (*topological_space*, *real_normed_vector*) *real_normed_vector*
begin

definition *norm_bcontfun* :: ('*a*, '*b*) *bcontfun* $\Rightarrow$ *real*
  where *norm_bcontfun f* = *dist f 0*

definition *sgn* (*f*::('*a*,'*b*) *bcontfun*) = *f* $/_R$ *norm f*

instance
proof
  fix *a* :: *real*
  fix *f g* :: ('*a*, '*b*) *bcontfun*
  show *dist f g* = *norm* (*f* − *g*)
    unfolding *norm_bcontfun_def*
    by *transfer* (*simp add*: *dist_norm*)
  show *norm* (*f* + *g*) $\leq$ *norm f* + *norm g*
    unfolding *norm_bcontfun_def*
    by *transfer*
    (*auto intro*!: *cSUP_least norm_triangle_le add_mono bounded_norm_le_SUP_norm*
      simp: *dist_norm bcontfun_def*)
  show *norm* (*a* $*_R$ *f*) = |*a*| ∗ *norm f*
    unfolding *norm_bcontfun_def*
    apply *transfer*
    by (*rule trans*[*OF _ continuous_at_Sup_mono*[*symmetric*]])
      (*auto intro*!: *monoI mult_left_mono continuous_intros bounded_imp_bdd_above*
        simp: *bounded_norm_comp bcontfun_def image_comp*)
qed (*auto simp*: *norm_bcontfun_def sgn_bcontfun_def*)

end

lemma *norm_bounded*:
  fixes *f* :: ('*a*::*topological_space*, '*b*::*real_normed_vector*) *bcontfun*
  shows *norm* (*apply_bcontfun f x*) $\leq$ *norm f*
  using *dist_bounded*[*of f x 0*]
  by (*simp add*: *dist_norm*)

lemma *norm_bound*:
  fixes *f* :: ('*a*::*topological_space*, '*b*::*real_normed_vector*) *bcontfun*
  assumes $\bigwedge$*x*. *norm* (*apply_bcontfun f x*) $\leq$ *b*

```
  shows norm f ≤ b
  using dist_bound[of f 0 b] assms
  by (simp add: dist_norm)
```

6.34.4 (bounded) continuous extenstion

```
lemma continuous_on_cbox_bcontfunE:
  fixes f::'a::euclidean_space ⇒ 'b::metric_space
  assumes continuous_on (cbox a b) f
  obtains g::'a ⇒C 'b where
    ⋀x. x ∈ cbox a b ⟹ g x = f x
    ⋀x. g x = f (clamp a b x)
proof -
  define g where g ≡ ext_cont f a b
  have g ∈ bcontfun
    using assms
    by (auto intro!: continuous_on_ext_cont simp: g_def bcontfun_def)
      (auto simp: g_def ext_cont_def
        intro!: clamp_bounded compact_imp_bounded[OF compact_continuous_image]
assms)
  then obtain h where h: g = apply_bcontfun h by (rule bcontfunE)
  then have h x = f x if x ∈ cbox a b for x
    by (auto simp: h[symmetric] g_def that)
  moreover
  have h x = f (clamp a b x) for x
    by (auto simp: h[symmetric] g_def ext_cont_def)
  ultimately show ?thesis ..
qed

lifting_update bcontfun.lifting
lifting_forget bcontfun.lifting

end
```

6.35 Lindelöf spaces

```
theory Lindelof_Spaces
imports T1_Spaces
begin

definition Lindelof_space where
  Lindelof_space X ≡
       ∀𝒰. (∀ U ∈ 𝒰. openin X U) ∧ ⋃𝒰 = topspace X
          ⟶ (∃𝒱. countable 𝒱 ∧ 𝒱 ⊆ 𝒰 ∧ ⋃𝒱 = topspace X)

lemma Lindelof_spaceD:
  ⟦Lindelof_space X; ⋀U. U ∈ 𝒰 ⟹ openin X U; ⋃𝒰 = topspace X⟧
  ⟹ ∃𝒱. countable 𝒱 ∧ 𝒱 ⊆ 𝒰 ∧ ⋃𝒱 = topspace X
  by (auto simp: Lindelof_space_def)
```

lemma *Lindelof_space_alt*:
  Lindelof_space X $\longleftrightarrow$
    $(\forall \mathcal{U}.\ (\forall\ U \in \mathcal{U}.\ openin\ X\ U) \wedge topspace\ X \subseteq \bigcup \mathcal{U}$
      $\longrightarrow (\exists \mathcal{V}.\ countable\ \mathcal{V} \wedge \mathcal{V} \subseteq \mathcal{U} \wedge topspace\ X \subseteq \bigcup \mathcal{V}))$
  unfolding *Lindelof_space_def*
  using *openin_subset* **by** *fastforce*

lemma *compact_imp_Lindelof_space*:
  compact_space X $\Longrightarrow$ *Lindelof_space X*
  unfolding *Lindelof_space_def compact_space*
  by (*meson uncountable_infinite*)

lemma *Lindelof_space_topspace_empty*:
  topspace X = {} $\Longrightarrow$ *Lindelof_space X*
  using *compact_imp_Lindelof_space compact_space_topspace_empty* **by** *blast*

lemma *Lindelof_space_Union*:
  assumes $\mathcal{U}$: *countable* $\mathcal{U}$ **and** *lin*: $\bigwedge U.\ U \in \mathcal{U} \Longrightarrow$ *Lindelof_space* (*subtopology X U*)
  shows *Lindelof_space* (*subtopology X* ($\bigcup \mathcal{U}$))
proof −
  have $\exists \mathcal{V}.\ countable\ \mathcal{V} \wedge \mathcal{V} \subseteq \mathcal{F} \wedge \bigcup \mathcal{U} \cap \bigcup \mathcal{V} = topspace\ X \cap \bigcup \mathcal{U}$
    if $\mathcal{F}$: $\mathcal{F} \subseteq$ *Collect* (*openin X*) **and** *UF*: $\bigcup \mathcal{U} \cap \bigcup \mathcal{F} = topspace\ X \cap \bigcup \mathcal{U}$
    for $\mathcal{F}$
  proof −
    have $\bigwedge U.\ [\![U \in \mathcal{U};\ U \cap \bigcup \mathcal{F} = topspace\ X \cap U]\!]$
        $\Longrightarrow \exists \mathcal{V}.\ countable\ \mathcal{V} \wedge \mathcal{V} \subseteq \mathcal{F} \wedge U \cap \bigcup \mathcal{V} = topspace\ X \cap U$
      using *lin* $\mathcal{F}$
     unfolding *Lindelof_space_def openin_subtopology_alt Ball_def subset_iff* [*symmetric*]
      by (*simp add*: *all_subset_image imp_conjL ex_countable_subset_image*)
    then obtain *g* **where** *g*: $\bigwedge U.\ [\![U \in \mathcal{U};\ U \cap \bigcup \mathcal{F} = topspace\ X \cap U]\!]$
        $\Longrightarrow countable\ (g\ U) \wedge (g\ U) \subseteq \mathcal{F} \wedge U \cap \bigcup (g\ U) = topspace\ X \cap U$
      by *metis*
    show *?thesis*
    proof (*intro exI conjI*)
      show *countable* ($\bigcup (g\ `\ \mathcal{U})$)
        using *Int_commute UF g* **by** (*fastforce intro*: *countable_UN* [*OF* $\mathcal{U}$])
      show $\bigcup (g\ `\ \mathcal{U}) \subseteq \mathcal{F}$
        using *g UF* **by** *blast*
      show $\bigcup \mathcal{U} \cap \bigcup (\bigcup (g\ `\ \mathcal{U})) = topspace\ X \cap \bigcup \mathcal{U}$
      proof
        show $\bigcup \mathcal{U} \cap \bigcup (\bigcup (g\ `\ \mathcal{U})) \subseteq topspace\ X \cap \bigcup \mathcal{U}$
          using *g UF* **by** *blast*
        show $topspace\ X \cap \bigcup \mathcal{U} \subseteq \bigcup \mathcal{U} \cap \bigcup (\bigcup (g\ `\ \mathcal{U}))$
        proof *clarsimp*
          show $\exists\ y \in \mathcal{U}.\ \exists\ W \in g\ y.\ x \in W$
            if $x \in topspace\ X\ x \in V\ V \in \mathcal{U}$ **for** *x V*

```
          proof -
            have V ∩ ⋃ℱ = topspace X ∩ V
              using UF ‹V ∈ 𝒰› by blast
            with that g [OF ‹V ∈ 𝒰›] show ?thesis by blast
          qed
        qed
      qed
    qed
  qed
  then show ?thesis
    unfolding Lindelof_space_def openin_subtopology_alt Ball_def subset_iff [symmetric]
      by (simp add: all_subset_image imp_conjL ex_countable_subset_image)
qed

lemma countable_imp_Lindelof_space:
  assumes countable(topspace X)
  shows Lindelof_space X
proof -
  have Lindelof_space (subtopology X (⋃x ∈ topspace X. {x}))
  proof (rule Lindelof_space_Union)
    show countable ((λx. {x}) ` topspace X)
      using assms by blast
    show Lindelof_space (subtopology X U)
      if U ∈ (λx. {x}) ` topspace X for U
    proof -
      have compactin X U
        using that by force
      then show ?thesis
        by (meson compact_imp_Lindelof_space compact_space_subtopology)
    qed
  qed
  then show ?thesis
    by simp
qed
lemma Lindelof_space_subtopology:
   Lindelof_space(subtopology X S) ⟷
       (∀𝒰. (∀U ∈ 𝒰. openin X U) ∧ topspace X ∩ S ⊆ ⋃𝒰
         ⟶ (∃V. countable V ∧ V ⊆ 𝒰 ∧ topspace X ∩ S ⊆ ⋃V))
proof -
  have *: (S ∩ ⋃𝒰 = topspace X ∩ S) = (topspace X ∩ S ⊆ ⋃𝒰)
    if ⋀x. x ∈ 𝒰 ⟹ openin X x for 𝒰
    by (blast dest: openin_subset [OF that])
  moreover have (𝒱 ⊆ 𝒰 ∧ S ∩ ⋃𝒱 = topspace X ∩ S) = (𝒱 ⊆ 𝒰 ∧ topspace X
∩ S ⊆ ⋃𝒱)
    if ∀x. x ∈ 𝒰 ⟶ openin X x topspace X ∩ S ⊆ ⋃𝒰 countable 𝒱 for 𝒰 𝒱
    using that * by blast
  ultimately show ?thesis
    unfolding Lindelof_space_def openin_subtopology_alt Ball_def
     apply (simp add: all_subset_image imp_conjL ex_countable_subset_image flip:
```

subset_iff)
  apply (*intro all_cong1 imp_cong ex_cong*, *auto*)
  done
qed

lemma *Lindelof_space_subtopology_subset*:
  $S \subseteq$ *topspace* X
    $\Longrightarrow$ (*Lindelof_space*(*subtopology* X S) $\longleftrightarrow$
      ($\forall\mathcal{U}$. ($\forall$ $U \in \mathcal{U}$. *openin* X U) $\wedge$ $S \subseteq \bigcup\mathcal{U}$
        $\longrightarrow$ ($\exists$ V. *countable* V $\wedge$ $V \subseteq \mathcal{U}$ $\wedge$ $S \subseteq \bigcup V$)))
 by (*metis Lindelof_space_subtopology topspace_subtopology topspace_subtopology_subset*)

lemma *Lindelof_space_closedin_subtopology*:
 assumes X: *Lindelof_space* X **and** *clo*: *closedin* X S
 shows *Lindelof_space* (*subtopology* X S)
proof −
 have $S \subseteq$ *topspace* X
  by (*simp add*: *clo closedin_subset*)
 then show *?thesis*
 proof (*clarsimp simp add*: *Lindelof_space_subtopology_subset*)
  show $\exists$ V. *countable* V $\wedge$ $V \subseteq \mathcal{F}$ $\wedge$ $S \subseteq \bigcup$ V
   if $\forall$ $U{\in}\mathcal{F}$. *openin* X U **and** $S \subseteq \bigcup\mathcal{F}$ **for** $\mathcal{F}$
  proof −
   have $\exists\mathcal{V}$. *countable* $\mathcal{V}$ $\wedge$ $\mathcal{V} \subseteq$ *insert* (*topspace* X − S) $\mathcal{F}$ $\wedge$ $\bigcup\mathcal{V}$ = *topspace* X
   proof (*rule Lindelof_spaceD* [*OF* X, *of insert* (*topspace* X − S) $\mathcal{F}$])
    show *openin* X U
     if $U \in$ *insert* (*topspace* X − S) $\mathcal{F}$ **for** U
     using *that* ‹$\forall$ $U{\in}\mathcal{F}$. *openin* X U› *clo* **by** *blast*
    show $\bigcup$(*insert* (*topspace* X − S) $\mathcal{F}$) = *topspace* X
     apply *auto*
     apply (*meson in_mono openin_closedin_eq that*(*1*))
     using *UnionE* ‹$S \subseteq \bigcup\mathcal{F}$› **by** *auto*
   qed
   then obtain $\mathcal{V}$ **where** *countable* $\mathcal{V}$ $\mathcal{V} \subseteq$ *insert* (*topspace* X − S) $\mathcal{F}$ $\bigcup\mathcal{V}$ = *topspace* X
    by *metis*
   with ‹$S \subseteq$ *topspace* X›
   show *?thesis*
    by (*rule_tac x*=($\mathcal{V}$ − {*topspace* X − S}) **in** *exI*) *auto*
  qed
 qed
qed

lemma *Lindelof_space_continuous_map_image*:
  assumes X: *Lindelof_space* X **and** f: *continuous_map* X Y f **and** *fim*: f ‘ (*topspace* X) = *topspace* Y
 shows *Lindelof_space* Y
proof −
 have $\exists\mathcal{V}$. *countable* $\mathcal{V}$ $\wedge$ $\mathcal{V} \subseteq \mathcal{U}$ $\wedge$ $\bigcup\mathcal{V}$ = *topspace* Y

```
    if 𝒰: ⋀U. U ∈ 𝒰 ⟹ openin Y U and UU: ⋃𝒰 = topspace Y for 𝒰
  proof −
    define 𝒱 where 𝒱 ≡ (λU. {x ∈ topspace X. f x ∈ U}) ` 𝒰
    have ⋀V. V ∈ 𝒱 ⟹ openin X V
      unfolding 𝒱_def using 𝒰 continuous_map f by fastforce
    moreover have ⋃𝒱 = topspace X
      unfolding 𝒱_def using UU fim by fastforce
    ultimately have ∃𝒲. countable 𝒲 ∧ 𝒲 ⊆ 𝒱 ∧ ⋃𝒲 = topspace X
      using X by (simp add: Lindelof_space_def)
    then obtain 𝒞 where countable 𝒞 𝒞 ⊆ 𝒰 and 𝒞: (⋃U∈𝒞. {x ∈ topspace X. f
x ∈ U}) = topspace X
      by (metis (no_types, lifting) 𝒱_def countable_subset_image)
    moreover have ⋃𝒞 = topspace Y
    proof
      show ⋃𝒞 ⊆ topspace Y
        using UU 𝒞 ‹𝒞 ⊆ 𝒰› by fastforce
      have y ∈ ⋃𝒞 if y ∈ topspace Y for y
      proof −
        obtain x where x ∈ topspace X y = f x
          using that fim by (metis ‹y ∈ topspace Y› imageE)
        with 𝒞 show ?thesis by auto
      qed
      then show topspace Y ⊆ ⋃𝒞 by blast
    qed
    ultimately show ?thesis
      by blast
  qed
  then show ?thesis
    unfolding Lindelof_space_def
    by auto
qed

lemma Lindelof_space_quotient_map_image:
  ⟦quotient_map X Y q; Lindelof_space X⟧ ⟹ Lindelof_space Y
  by (meson Lindelof_space_continuous_map_image quotient_imp_continuous_map
quotient_imp_surjective_map)

lemma Lindelof_space_retraction_map_image:
  ⟦retraction_map X Y r; Lindelof_space X⟧ ⟹ Lindelof_space Y
 using Abstract_Topology.retraction_imp_quotient_map Lindelof_space_quotient_map_image
by blast

lemma locally_finite_cover_of_Lindelof_space:
 assumes X: Lindelof_space X and UU: topspace X ⊆ ⋃𝒰 and fin: locally_finite_in
X 𝒰
  shows countable 𝒰
proof −
  have UU_eq: ⋃𝒰 = topspace X
    by (meson UU fin locally_finite_in_def subset_antisym)
```

obtain *T* **where** *T*: $\bigwedge x.\ x \in topspace\ X \Longrightarrow openin\ X\ (T\ x) \wedge x \in T\ x \wedge finite$ $\{U \in \mathcal{U}.\ U \cap T\ x \neq \{\}\}$
  using *fin* **unfolding** *locally_finite_in_def* **by** *metis*
 then obtain *I* **where** *countable I* $I \subseteq topspace\ X$ **and** *I*: $topspace\ X \subseteq \bigcup(T\ `\ I)$
  using *X* **unfolding** *Lindelof_space_alt*
  by (*drule_tac x=image T* (*topspace X*) **in** *spec*) (*auto simp*: *ex_countable_subset_image*)
 show *?thesis*
 proof (*rule countable_subset*)
  have $\bigwedge i.\ i \in I \Longrightarrow countable\ \{U \in \mathcal{U}.\ U \cap T\ i \neq \{\}\}$
   using *T*
   by (*meson* ‹$I \subseteq topspace\ X$› *in_mono uncountable_infinite*)
  then show *countable* (*insert* {} ($\bigcup i{\in}I.\ \{U \in \mathcal{U}.\ U \cap T\ i \neq \{\}\}$))
   by (*simp add*: ‹*countable I*›)
 qed (*use UU_eq I* **in** *auto*)
qed

lemma *Lindelof_space_proper_map_preimage*:
 assumes *f*: *proper_map X Y f* **and** *Y*: *Lindelof_space Y*
 shows *Lindelof_space X*
proof (*clarsimp simp*: *Lindelof_space_alt*)
 show $\exists \mathcal{V}.\ countable\ \mathcal{V} \wedge \mathcal{V} \subseteq \mathcal{U} \wedge topspace\ X \subseteq \bigcup \mathcal{V}$
  if $\mathcal{U}$: $\forall U{\in}\mathcal{U}.\ openin\ X\ U$ **and** *sub_UU*: $topspace\ X \subseteq \bigcup \mathcal{U}$ **for** $\mathcal{U}$
 proof −
  have $\exists \mathcal{V}.\ finite\ \mathcal{V} \wedge \mathcal{V} \subseteq \mathcal{U} \wedge \{x \in topspace\ X.\ f\ x = y\} \subseteq \bigcup \mathcal{V}$ **if** $y \in topspace$ Y **for** *y*
  proof (*rule compactinD*)
   show *compactin X* $\{x \in topspace\ X.\ f\ x = y\}$
    using *f proper_map_def that* **by** *fastforce*
  qed (*use sub_UU* $\mathcal{U}$ **in** *auto*)
  then obtain $\mathcal{V}$ **where** $\mathcal{V}$: $\bigwedge y.\ y \in topspace\ Y \Longrightarrow finite\ (\mathcal{V}\ y) \wedge \mathcal{V}\ y \subseteq \mathcal{U} \wedge$ $\{x \in topspace\ X.\ f\ x = y\} \subseteq \bigcup(\mathcal{V}\ y)$
   by *meson*
  define $\mathcal{W}$ **where** $\mathcal{W} \equiv (\lambda y.\ topspace\ Y - image\ f\ (topspace\ X - \bigcup(\mathcal{V}\ y)))\ `$ $topspace\ Y$
  have $\forall U \in \mathcal{W}.\ openin\ Y\ U$
   using *f* $\mathcal{U}$ $\mathcal{V}$ **unfolding** *$\mathcal{W}$_def proper_map_def closed_map_def*
   by (*simp add*: *closedin_diff openin_Union openin_diff subset_iff*)
  moreover have $topspace\ Y \subseteq \bigcup \mathcal{W}$
   using $\mathcal{V}$ **unfolding** *$\mathcal{W}$_def* **by** *clarsimp fastforce*
  ultimately have $\exists \mathcal{V}.\ countable\ \mathcal{V} \wedge \mathcal{V} \subseteq \mathcal{W} \wedge topspace\ Y \subseteq \bigcup \mathcal{V}$
   using *Y* **by** (*simp add*: *Lindelof_space_alt*)
  then obtain *I* **where** *countable I* $I \subseteq topspace\ Y$
   and *I*: $topspace\ Y \subseteq (\bigcup i{\in}I.\ topspace\ Y - f\ `\ (topspace\ X - \bigcup(\mathcal{V}\ i)))$
   unfolding *$\mathcal{W}$_def ex_countable_subset_image* **by** *metis*
  show *?thesis*
  proof (*intro exI conjI*)
   have $\bigwedge i.\ i \in I \Longrightarrow countable\ (\mathcal{V}\ i)$

```
        by (meson 𝒱 ‹I ⊆ topspace Y› in_mono uncountable_infinite)
      with ‹countable I› show countable (⋃(𝒱 ‘ I))
        by auto
      show ⋃(𝒱 ‘ I) ⊆ 𝒰
        using 𝒱 ‹I ⊆ topspace Y› by fastforce
      show topspace X ⊆ ⋃(⋃(𝒱 ‘ I))
      proof
        show x ∈ ⋃ (⋃ (𝒱 ‘ I)) if x ∈ topspace X for x
        proof -
          have f x ∈ topspace Y
            by (meson f image_subset_iff proper_map_imp_subset_topspace that)
          then show ?thesis
            using that I by auto
        qed
      qed
    qed
  qed
qed

lemma Lindelof_space_perfect_map_image:
   ⟦Lindelof_space X; perfect_map X Y f⟧ ⟹ Lindelof_space Y
  using Lindelof_space_quotient_map_image perfect_imp_quotient_map by blast

lemma Lindelof_space_perfect_map_image_eq:
   perfect_map X Y f ⟹ Lindelof_space X ⟷ Lindelof_space Y
  using Lindelof_space_perfect_map_image Lindelof_space_proper_map_preimage per-
fect_map_def by blast

end
```

6.36 Infinite Products

```
theory Infinite_Products
  imports Topology_Euclidean_Space Complex_Transcendental
begin
```

6.36.1 Preliminaries

```
lemma sum_le_prod:
  fixes f :: 'a ⇒ 'b :: linordered_semidom
  assumes ⋀x. x ∈ A ⟹ f x ≥ 0
  shows   sum f A ≤ (∏x∈A. 1 + f x)
  using assms
proof (induction A rule: infinite_finite_induct)
  case (insert x A)
  from insert.hyps have sum f A + f x * (∏x∈A. 1) ≤ (∏x∈A. 1 + f x) + f x
* (∏x∈A. 1 + f x)
   by (intro add_mono insert mult_left_mono prod_mono) (auto intro: insert.prems)
  with insert.hyps show ?case by (simp add: algebra_simps)
```

qed *simp_all*

lemma *prod_le_exp_sum*:
fixes f :: $'a \Rightarrow real$
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x \geq 0$
shows *prod* $(\lambda x.\ 1 + f\ x)\ A \leq exp\ (sum\ f\ A)$
using *assms*
proof (*induction A rule*: *infinite_finite_induct*)
case (*insert x A*)
have $(1 + f\ x) * (\prod x{\in}A.\ 1 + f\ x) \leq exp\ (f\ x) * exp\ (sum\ f\ A)$
using *insert.prems* **by** (*intro mult_mono insert prod_nonneg exp_ge_add_one_self*) *auto*
with *insert.hyps* **show** *?case* **by** (*simp add*: *algebra_simps exp_add*)
qed *simp_all*

lemma *lim_ln_1_plus_x_over_x_at_0*: $(\lambda x{::}real.\ ln\ (1 + x)\ /\ x)\ {-0\to}\ 1$
proof (*rule lhopital*)
show $(\lambda x{::}real.\ ln\ (1 + x))\ {-0\to}\ 0$
by (*rule tendsto_eq_intros refl* | *simp*)+
have *eventually* $(\lambda x{::}real.\ x \in \{-1/2{<}..{<}1/2\})$ (*nhds 0*)
by (*rule eventually_nhds_in_open*) *auto*
hence ∗: *eventually* $(\lambda x{::}real.\ x \in \{-1/2{<}..{<}1/2\})$ (*at 0*)
by (*rule filter_leD* [*rotated*]) (*simp_all add*: *at_within_def*)
show *eventually* $(\lambda x{::}real.\ ((\lambda x.\ ln\ (1 + x))$ *has_field_derivative inverse* $(1 + x))$ (*at x*)) (*at 0*)
using ∗ **by** *eventually_elim* (*auto intro*!: *derivative_eq_intros simp*: *field_simps*)
show *eventually* $(\lambda x{::}real.\ ((\lambda x.\ x)$ *has_field_derivative 1*) (*at x*)) (*at 0*)
using ∗ **by** *eventually_elim* (*auto intro*!: *derivative_eq_intros simp*: *field_simps*)
show $\forall_F\ x$ *in at 0*. $x \neq 0$ **by** (*auto simp*: *at_within_def eventually_inf_principal*)
show $(\lambda x{::}real.\ inverse\ (1 + x)\ /\ 1)\ {-0\to}\ 1$
by (*rule tendsto_eq_intros refl* | *simp*)+
qed *auto*

6.36.2 Definitions and basic properties

definition *raw_has_prod* :: $[nat \Rightarrow 'a{::}\{t2\_space,\ comm\_semiring\_1\},\ nat,\ 'a] \Rightarrow bool$
where *raw_has_prod f M p* $\equiv (\lambda n.\ \prod i{\leq}n.\ f\ (i{+}M)) \longrightarrow p \wedge p \neq 0$

The nonzero and zero cases, as in *Complex Analysis* by Joseph Bak and Donald J.Newman, page 241

definition
has_prod :: $(nat \Rightarrow 'a{::}\{t2\_space,\ comm\_semiring\_1\}) \Rightarrow 'a \Rightarrow bool$ (**infixr** *has'_prod* *80*)
where *f has_prod p* $\equiv$ *raw_has_prod f 0 p* $\vee (\exists\ i\ q.\ p = 0 \wedge f\ i = 0 \wedge$ *raw_has_prod f* (*Suc i*) *q*)

definition *convergent_prod* :: $(nat \Rightarrow 'a :: \{t2\_space,comm\_semiring\_1\}) \Rightarrow bool$
where

convergent_prod f ≡ ∃ *M p*. *raw_has_prod f M p*

definition *prodinf* :: (*nat* ⇒ *'a*::{*t2_space*, *comm_semiring_1*}) ⇒ *'a*
(**binder** ∏ *10*)
where *prodinf f* = (*THE p*. *f has_prod p*)

lemmas *prod_defs* = *raw_has_prod_def has_prod_def convergent_prod_def prodinf_def*

lemma *has_prod_subst*[*trans*]: *f* = *g* ⟹ *g has_prod z* ⟹ *f has_prod z*
by *simp*

lemma *has_prod_cong*: (⋀*n*. *f n* = *g n*) ⟹ *f has_prod c* ⟷ *g has_prod c*
by *presburger*

lemma *raw_has_prod_nonzero* [*simp*]: ¬ *raw_has_prod f M 0*
by (*simp add*: *raw_has_prod_def*)

lemma *raw_has_prod_eq_0*:
fixes *f* :: *nat* ⇒ *'a*::{*semidom*,*t2_space*}
assumes *p*: *raw_has_prod f m p* **and** *i*: *f i* = *0 i* ≥ *m*
shows *p* = *0*
proof −
have *eq0*: (∏*k*≤*n*. *f* (*k*+*m*)) = *0* **if** *i* − *m* ≤ *n* **for** *n*
proof −
have ∃ *k*≤*n*. *f* (*k* + *m*) = *0*
using *i that* **by** *auto*
then show *?thesis*
by *auto*
qed
have (λ*n*. ∏*i*≤*n*. *f* (*i* + *m*)) ⟶ *0*
by (*rule LIMSEQ_offset* [**where** *k* = *i*−*m*]) (*simp add*: *eq0*)
with *p* **show** *?thesis*
unfolding *raw_has_prod_def*
using *LIMSEQ_unique* **by** *blast*
qed

lemma *raw_has_prod_Suc*:
raw_has_prod f (*Suc M*) *a* ⟷ *raw_has_prod* (λ*n*. *f* (*Suc n*)) *M a*
unfolding *raw_has_prod_def* **by** *auto*

lemma *has_prod_0_iff*: *f has_prod 0* ⟷ (∃ *i*. *f i* = *0* ∧ (∃ *p*. *raw_has_prod f* (*Suc i*) *p*))
by (*simp add*: *has_prod_def*)

lemma *has_prod_unique2*:
fixes *f* :: *nat* ⇒ *'a*::{*semidom*,*t2_space*}
assumes *f has_prod a f has_prod b* **shows** *a* = *b*
using *assms*
by (*auto simp*: *has_prod_def raw_has_prod_eq_0*) (*meson raw_has_prod_def sequen-*

tially_bot tendsto_unique)

lemma *has_prod_unique*:
  fixes *f* :: *nat* ⇒ *'a* :: {*semidom*,*t2_space*}
  shows *f has_prod s* ⟹ *s* = *prodinf f*
  by (*simp add*: *has_prod_unique2 prodinf_def the_equality*)

lemma *convergent_prod_altdef*:
  fixes *f* :: *nat* ⇒ *'a* :: {*t2_space*,*comm_semiring_1*}
  shows *convergent_prod f* ⟷ (∃ *M L*. (∀ *n*≥*M*. *f n* ≠ *0*) ∧ (λ*n*. ∏ *i*≤*n*. *f* (*i*+*M*)) ⟶ *L* ∧ *L* ≠ *0*)
proof
  assume *convergent_prod f*
  then obtain *M L* **where** ∗: (λ*n*. ∏ *i*≤*n*. *f* (*i*+*M*)) ⟶ *L L* ≠ *0*
    by (*auto simp*: *prod_defs*)
  have *f i* ≠ *0* **if** *i* ≥ *M* **for** *i*
  proof
    assume *f i* = *0*
    have ∗∗: *eventually* (λ*n*. (∏ *i*≤*n*. *f* (*i*+*M*)) = *0*) *sequentially*
      using *eventually_ge_at_top*[*of i* − *M*]
    proof *eventually_elim*
      case (*elim n*)
      with ‹*f i* = *0*› **and** ‹*i* ≥ *M*› **show** *?case*
        by (*auto intro*!: *bexI*[*of* _ *i* − *M*] *prod_zero*)
    qed
    have (λ*n*. (∏ *i*≤*n*. *f* (*i*+*M*))) ⟶ *0*
      unfolding *filterlim_iff*
      by (*auto dest*!: *eventually_nhds_x_imp_x intro*!: *eventually_mono*[*OF* ∗∗])
    from *tendsto_unique*[*OF* _ *this* ∗(*1*)] **and** ∗(*2*)
      show *False* **by** *simp*
  qed
  with ∗ **show** (∃ *M L*. (∀ *n*≥*M*. *f n* ≠ *0*) ∧ (λ*n*. ∏ *i*≤*n*. *f* (*i*+*M*)) ⟶ *L* ∧ *L* ≠ *0*)
    by *blast*
qed (*auto simp*: *prod_defs*)

6.36.3 Absolutely convergent products

definition *abs_convergent_prod* :: (*nat* ⇒ _) ⇒ *bool* **where**
  abs_convergent_prod f ⟷ *convergent_prod* (λ*i*. *1* + *norm* (*f i* − *1*))

lemma *abs_convergent_prodI*:
  assumes *convergent* (λ*n*. ∏ *i*≤*n*. *1* + *norm* (*f i* − *1*))
  shows *abs_convergent_prod f*
proof −
  from *assms* **obtain** *L* **where** *L*: (λ*n*. ∏ *i*≤*n*. *1* + *norm* (*f i* − *1*)) ⟶ *L*
    by (*auto simp*: *convergent_def*)
  have *L* ≥ *1*
  proof (*rule tendsto_le*)

```
    show eventually (λn. (∏i≤n. 1 + norm (f i − 1)) ≥ 1) sequentially
    proof (intro always_eventually allI)
      fix n
      have (∏i≤n. 1 + norm (f i − 1)) ≥ (∏i≤n. 1)
        by (intro prod_mono) auto
      thus (∏i≤n. 1 + norm (f i − 1)) ≥ 1 by simp
    qed
  qed (use L in simp_all)
  hence L ≠ 0 by auto
  with L show ?thesis unfolding abs_convergent_prod_def prod_defs
    by (intro exI[of _ 0::nat] exI[of _ L]) auto
qed

lemma
  fixes f :: nat ⇒ 'a :: {topological_semigroup_mult,t2_space,idom}
  assumes convergent_prod f
  shows   convergent_prod_imp_convergent:     convergent (λn. ∏i≤n. f i)
    and   convergent_prod_to_zero_iff [simp]: (λn. ∏i≤n. f i) ⟶ 0 ⟷ (∃i.
f i = 0)
proof −
  from assms obtain M L
    where M: ⋀n. n ≥ M ⟹ f n ≠ 0 and (λn. ∏i≤n. f (i + M)) ⟶ L
and L ≠ 0
    by (auto simp: convergent_prod_altdef)
  note this(2)
  also have (λn. ∏i≤n. f (i + M)) = (λn. ∏i=M..M+n. f i)
    by (intro ext prod.reindex_bij_witness[of _ λn. n − M λn. n + M]) auto
  finally have (λn. (∏i<M. f i) ∗ (∏i=M..M+n. f i)) ⟶ (∏i<M. f i) ∗
L
    by (intro tendsto_mult tendsto_const)
 also have (λn. (∏i<M. f i) ∗ (∏i=M..M+n. f i)) = (λn. (∏i∈{..<M}∪{M..M+n}.
f i))
    by (subst prod.union_disjoint) auto
  also have (λn. {..<M} ∪ {M..M+n}) = (λn. {..n+M}) by auto
  finally have lim: (λn. prod f {..n}) ⟶ prod f {..<M} ∗ L
    by (rule LIMSEQ_offset)
  thus convergent (λn. ∏i≤n. f i)
    by (auto simp: convergent_def)

  show (λn. ∏i≤n. f i) ⟶ 0 ⟷ (∃i. f i = 0)
  proof
    assume ∃i. f i = 0
    then obtain i where f i = 0 by auto
    moreover with M have i < M by (cases i < M) auto
    ultimately have (∏i<M. f i) = 0 by auto
    with lim show (λn. ∏i≤n. f i) ⟶ 0 by simp
  next
    assume (λn. ∏i≤n. f i) ⟶ 0
    from tendsto_unique[OF _ this lim] and ‹L ≠ 0›
```

show $\exists$ *i*. *f i* = *0* **by** *auto*
qed
qed

lemma *convergent_prod_iff_nz_lim*:
fixes *f* :: *nat* ⇒ *'a* :: {*topological_semigroup_mult*,*t2_space*,*idom*}
assumes ⋀*i*. *f i* ≠ *0*
shows *convergent_prod f* ⟷ ($\exists$ *L*. (λ*n*. ∏ *i*≤*n*. *f i*) ⟶ *L* ∧ *L* ≠ *0*)
(**is** *?lhs* ⟷ *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
using *assms convergentD convergent_prod_imp_convergent convergent_prod_to_zero_iff*
by *blast*
next
assume *?rhs* **then show** *?lhs*
unfolding *prod_defs*
by (*rule_tac x=0* **in** *exI*) *auto*
qed

lemma *convergent_prod_iff_convergent*:
fixes *f* :: *nat* ⇒ *'a* :: {*topological_semigroup_mult*,*t2_space*,*idom*}
assumes ⋀*i*. *f i* ≠ *0*
shows *convergent_prod f* ⟷ *convergent* (λ*n*. ∏ *i*≤*n*. *f i*) ∧ *lim* (λ*n*. ∏ *i*≤*n*. *f*
i) ≠ *0*
by (*force simp*: *convergent_prod_iff_nz_lim assms convergent_def limI*)

lemma *bounded_imp_convergent_prod*:
fixes *a* :: *nat* ⇒ *real*
assumes *1*: ⋀*n*. *a n* ≥ *1* **and** *bounded*: ⋀*n*. (∏ *i*≤*n*. *a i*) ≤ *B*
shows *convergent_prod a*
proof −
have *bdd_above* (*range*(λ*n*. ∏ *i*≤*n*. *a i*))
by (*meson bdd_aboveI2 bounded*)
moreover have *incseq* (λ*n*. ∏ *i*≤*n*. *a i*)
unfolding *mono_def* **by** (*metis 1 prod_mono2 atMost_subset_iff dual_order.trans*
finite_atMost zero_le_one)
ultimately obtain *p* **where** *p*: (λ*n*. ∏ *i*≤*n*. *a i*) ⟶ *p*
using *LIMSEQ_incseq_SUP* **by** *blast*
then have *p* ≠ *0*
by (*metis 1 not_one_le_zero prod_ge_1 LIMSEQ_le_const*)
with *1 p* **show** *?thesis*
by (*metis convergent_prod_iff_nz_lim not_one_le_zero*)
qed

lemma *abs_convergent_prod_altdef*:
fixes *f* :: *nat* ⇒ *'a* :: {*one*,*real_normed_vector*}
shows *abs_convergent_prod f* ⟷ *convergent* (λ*n*. ∏ *i*≤*n*. *1* + *norm* (*f i* − *1*))
proof

```
  assume abs_convergent_prod f
  thus convergent (λn. ∏i≤n. 1 + norm (f i − 1))
    by (auto simp: abs_convergent_prod_def intro!: convergent_prod_imp_convergent)
qed (auto intro: abs_convergent_prodI)

lemma Weierstrass_prod_ineq:
  fixes f :: 'a ⇒ real
  assumes ⋀x. x ∈ A ⟹ f x ∈ {0..1}
  shows  1 − sum f A ≤ (∏x∈A. 1 − f x)
  using assms
proof (induction A rule: infinite_finite_induct)
  case (insert x A)
  from insert.hyps and insert.prems
    have 1 − sum f A + f x ∗ (∏x∈A. 1 − f x) ≤ (∏x∈A. 1 − f x) + f x ∗
(∏x∈A. 1)
    by (intro insert.IH add_mono mult_left_mono prod_mono) auto
  with insert.hyps show ?case by (simp add: algebra_simps)
qed simp_all

lemma norm_prod_minus1_le_prod_minus1:
  fixes f :: nat ⇒ 'a :: {real_normed_div_algebra,comm_ring_1}
  shows norm (prod (λn. 1 + f n) A − 1) ≤ prod (λn. 1 + norm (f n)) A − 1
proof (induction A rule: infinite_finite_induct)
  case (insert x A)
  from insert.hyps have
    norm ((∏n∈insert x A. 1 + f n) − 1) =
      norm ((∏n∈A. 1 + f n) − 1 + f x ∗ (∏n∈A. 1 + f n))
    by (simp add: algebra_simps)
  also have . . . ≤ norm ((∏n∈A. 1 + f n) − 1) + norm (f x ∗ (∏n∈A. 1 + f
n))
    by (rule norm_triangle_ineq)
  also have norm (f x ∗ (∏n∈A. 1 + f n)) = norm (f x) ∗ (∏x∈A. norm (1 +
f x))
    by (simp add: prod_norm norm_mult)
  also have (∏x∈A. norm (1 + f x)) ≤ (∏x∈A. norm (1::'a) + norm (f x))
    by (intro prod_mono norm_triangle_ineq ballI conjI) auto
  also have norm (1::'a) = 1 by simp
  also note insert.IH
  also have (∏n∈A. 1 + norm (f n)) − 1 + norm (f x) ∗ (∏x∈A. 1 + norm (f
x)) =
            (∏n∈insert x A. 1 + norm (f n)) − 1
    using insert.hyps by (simp add: algebra_simps)
  finally show ?case by − (simp_all add: mult_left_mono)
qed simp_all

lemma convergent_prod_imp_ev_nonzero:
  fixes f :: nat ⇒ 'a :: {t2_space,comm_semiring_1}
  assumes convergent_prod f
  shows  eventually (λn. f n ≠ 0) sequentially
```

using *assms* **by** (*auto simp*: *eventually_at_top_linorder convergent_prod_altdef*)

lemma *convergent_prod_imp_LIMSEQ*:
fixes *f* :: *nat* ⇒ *'a* :: {*real_normed_field*}
assumes *convergent_prod f*
shows *f* ⟶ *1*
proof −
from *assms* **obtain** *M L* **where** *L*: (*λn*. ∏ *i*≤*n*. *f* (*i*+*M*)) ⟶ *L* ⋀*n*. *n* ≥ *M* ⟹ *f n* ≠ *0 L* ≠ *0*
by (*auto simp*: *convergent_prod_altdef*)
hence *L'*: (*λn*. ∏ *i*≤*Suc n*. *f* (*i*+*M*)) ⟶ *L* **by** (*subst filterlim_sequentially_Suc*)
have (*λn*. (∏ *i*≤*Suc n*. *f* (*i*+*M*)) / (∏ *i*≤*n*. *f* (*i*+*M*))) ⟶ *L* / *L*
using *L L'* **by** (*intro tendsto_divide*) *simp_all*
also from *L* **have** *L* / *L* = *1* **by** *simp*
also have (*λn*. (∏ *i*≤*Suc n*. *f* (*i*+*M*)) / (∏ *i*≤*n*. *f* (*i*+*M*))) = (*λn*. *f* (*n* + *Suc M*))
using *assms L* **by** (*auto simp*: *fun_eq_iff atMost_Suc*)
finally show *?thesis* **by** (*rule LIMSEQ_offset*)
qed

lemma *abs_convergent_prod_imp_summable*:
fixes *f* :: *nat* ⇒ *'a* :: *real_normed_div_algebra*
assumes *abs_convergent_prod f*
shows *summable* (*λi*. *norm* (*f i* − *1*))
proof −
from *assms* **have** *convergent* (*λn*. ∏ *i*≤*n*. *1* + *norm* (*f i* − *1*))
unfolding *abs_convergent_prod_def* **by** (*rule convergent_prod_imp_convergent*)
then obtain *L* **where** *L*: (*λn*. ∏ *i*≤*n*. *1* + *norm* (*f i* − *1*)) ⟶ *L*
unfolding *convergent_def* **by** *blast*
have *convergent* (*λn*. ∑ *i*≤*n*. *norm* (*f i* − *1*))
proof (*rule Bseq_monoseq_convergent*)
have *eventually* (*λn*. (∏ *i*≤*n*. *1* + *norm* (*f i* − *1*)) < *L* + *1*) *sequentially*
using *L*(*1*) **by** (*rule order_tendstoD*) *simp_all*
hence ∀$_F$ *x in sequentially*. *norm* (∑ *i*≤*x*. *norm* (*f i* − *1*)) ≤ *L* + *1*
proof *eventually_elim*
case (*elim n*)
have *norm* (∑ *i*≤*n*. *norm* (*f i* − *1*)) = (∑ *i*≤*n*. *norm* (*f i* − *1*))
unfolding *real_norm_def* **by** (*intro abs_of_nonneg sum_nonneg*) *simp_all*
also have ... ≤ (∏ *i*≤*n*. *1* + *norm* (*f i* − *1*)) **by** (*rule sum_le_prod*) *auto*
also have ... < *L* + *1* **by** (*rule elim*)
finally show *?case* **by** *simp*
qed
thus *Bseq* (*λn*. ∑ *i*≤*n*. *norm* (*f i* − *1*)) **by** (*rule BfunI*)
next
show *monoseq* (*λn*. ∑ *i*≤*n*. *norm* (*f i* − *1*))
by (*rule mono_SucI1*) *auto*
qed
thus *summable* (*λi*. *norm* (*f i* − *1*)) **by** (*simp add*: *summable_iff_convergent'*)
qed

lemma *summable_imp_abs_convergent_prod*:
  fixes *f* :: *nat* ⇒ *'a* :: *real_normed_div_algebra*
  assumes *summable* (λ*i*. *norm* (*f i* − *1*))
  shows *abs_convergent_prod f*
proof (*intro abs_convergent_prodI Bseq_monoseq_convergent*)
  show *monoseq* (λ*n*. ∏*i*≤*n*. *1* + *norm* (*f i* − *1*))
    by (*intro mono_SucI1*)
     (*auto simp*: *atMost_Suc algebra_simps intro*!: *mult_nonneg_nonneg prod_nonneg*)
next
  show *Bseq* (λ*n*. ∏*i*≤*n*. *1* + *norm* (*f i* − *1*))
  proof (*rule Bseq_eventually_mono*)
    show *eventually* (λ*n*. *norm* (∏*i*≤*n*. *1* + *norm* (*f i* − *1*)) ≤
          norm (*exp* (∑*i*≤*n*. *norm* (*f i* − *1*)))) *sequentially*
     by (*intro always_eventually allI*) (*auto simp*: *abs_prod exp_sum intro*!: *prod_mono*)
  next
    from *assms* **have** (λ*n*. ∑*i*≤*n*. *norm* (*f i* − *1*)) ⟶ (∑*i*. *norm* (*f i* − *1*))
      using *sums_def_le* **by** *blast*
    hence (λ*n*. *exp* (∑*i*≤*n*. *norm* (*f i* − *1*))) ⟶ *exp* (∑*i*. *norm* (*f i* − *1*))
      by (*rule tendsto_exp*)
    hence *convergent* (λ*n*. *exp* (∑*i*≤*n*. *norm* (*f i* − *1*)))
      by (*rule convergentI*)
    thus *Bseq* (λ*n*. *exp* (∑*i*≤*n*. *norm* (*f i* − *1*)))
      by (*rule convergent_imp_Bseq*)
  qed
qed

theorem *abs_convergent_prod_conv_summable*:
  fixes *f* :: *nat* ⇒ *'a* :: *real_normed_div_algebra*
  shows *abs_convergent_prod f* ⟷ *summable* (λ*i*. *norm* (*f i* − *1*))
 by (*blast intro*: *abs_convergent_prod_imp_summable summable_imp_abs_convergent_prod*)

lemma *abs_convergent_prod_imp_LIMSEQ*:
  fixes *f* :: *nat* ⇒ *'a* :: {*comm_ring_1*,*real_normed_div_algebra*}
  assumes *abs_convergent_prod f*
  shows *f* ⟶ *1*
proof −
  from *assms* **have** *summable* (λ*n*. *norm* (*f n* − *1*))
    by (*rule abs_convergent_prod_imp_summable*)
  from *summable_LIMSEQ_zero*[*OF this*] **have** (λ*n*. *f n* − *1*) ⟶ *0*
    by (*simp add*: *tendsto_norm_zero_iff*)
  from *tendsto_add*[*OF this tendsto_const*[*of 1*]] **show** *?thesis* **by** *simp*
qed

lemma *abs_convergent_prod_imp_ev_nonzero*:
  fixes *f* :: *nat* ⇒ *'a* :: {*comm_ring_1*,*real_normed_div_algebra*}
  assumes *abs_convergent_prod f*
  shows *eventually* (λ*n*. *f n* ≠ *0*) *sequentially*
proof −

from *assms* **have** $f \longrightarrow 1$
by (*rule abs_convergent_prod_imp_LIMSEQ*)
hence *eventually* ($\lambda n.$ *dist* ($f\ n$) $1 < 1$) *at_top*
by (*auto simp*: *tendsto_iff*)
thus *?thesis* **by** *eventually_elim auto*
qed

6.36.4 Ignoring initial segments

lemma *convergent_prod_offset*:
assumes *convergent_prod* ($\lambda n.\ f\ (n + m)$)
shows *convergent_prod f*
proof −
from *assms* **obtain** $M\ L$ **where** ($\lambda n. \prod k{\le}n.\ f\ (k + (M + m))$) $\longrightarrow L\ L \neq 0$
by (*auto simp*: *prod_defs add.assoc*)
thus *convergent_prod f*
unfolding *prod_defs* **by** *blast*
qed

lemma *abs_convergent_prod_offset*:
assumes *abs_convergent_prod* ($\lambda n.\ f\ (n + m)$)
shows *abs_convergent_prod f*
using *assms* **unfolding** *abs_convergent_prod_def* **by** (*rule convergent_prod_offset*)

lemma *raw_has_prod_ignore_initial_segment*:
fixes f :: $nat \Rightarrow 'a$ :: *real_normed_field*
assumes *raw_has_prod f M p* $N \ge M$
obtains q **where** *raw_has_prod f N q*
proof −
have p: ($\lambda n. \prod k{\le}n.\ f\ (k + M)$) $\longrightarrow p$ **and** $p \neq 0$
using *assms* **by** (*auto simp*: *raw_has_prod_def*)
then have nz: $\bigwedge n.\ n \ge M \Longrightarrow f\ n \neq 0$
using *assms* **by** (*auto simp*: *raw_has_prod_eq_0*)
define C **where** $C = (\prod k{<}N{-}M.\ f\ (k + M))$
from *nz* **have** [*simp*]: $C \neq 0$
by (*auto simp*: *C_def*)

from p **have** ($\lambda i. \prod k{\le}i + (N{-}M).\ f\ (k + M)$) $\longrightarrow p$
by (*rule LIMSEQ_ignore_initial_segment*)
also have ($\lambda i. \prod k{\le}i + (N{-}M).\ f\ (k + M)$) = ($\lambda n.\ C * (\prod k{\le}n.\ f\ (k + N))$)
proof (*rule ext*, *goal_cases*)
case (*1 n*)
have $\{..n{+}(N{-}M)\} = \{..{<}(N{-}M)\} \cup \{(N{-}M)..n{+}(N{-}M)\}$ **by** *auto*
also have ($\prod k{\in}\dots.\ f\ (k + M)$) = $C * (\prod k{=}(N{-}M)..n{+}(N{-}M).\ f\ (k + M))$
unfolding *C_def* **by** (*rule prod.union_disjoint*) *auto*
also have ($\prod k{=}(N{-}M)..n{+}(N{-}M).\ f\ (k + M)$) = ($\prod k{\le}n.\ f\ (k + (N{-}M)$

+ M))
by (intro ext prod.reindex_bij_witness[of _ λk. k + (N−M) λk. k − (N−M)]) auto
finally show ?case
using ‹N ≥ M› by (simp add: add_ac)
qed
finally have (λn. C ∗ (∏k≤n. f (k + N)) / C) ⟶ p / C
by (intro tendsto_divide tendsto_const) auto
hence (λn. ∏k≤n. f (k + N)) ⟶ p / C by simp
moreover from ‹p ≠ 0› have p / C ≠ 0 by simp
ultimately show ?thesis
using raw_has_prod_def that by blast
qed

corollary *convergent_prod_ignore_initial_segment*:
fixes f :: nat ⇒ 'a :: real_normed_field
assumes convergent_prod f
shows convergent_prod (λn. f (n + m))
using assms
unfolding convergent_prod_def
apply clarify
apply (erule_tac N=M+m **in** raw_has_prod_ignore_initial_segment)
apply (auto simp add: raw_has_prod_def add_ac)
done

corollary *convergent_prod_ignore_nonzero_segment*:
fixes f :: nat ⇒ 'a :: real_normed_field
assumes f: convergent_prod f **and** nz: ⋀i. i ≥ M ⟹ f i ≠ 0
shows ∃p. raw_has_prod f M p
using convergent_prod_ignore_initial_segment [OF f]
by (metis convergent_LIMSEQ_iff convergent_prod_iff_convergent le_add_same_cancel2 nz prod_defs(1) zero_order(1))

corollary *abs_convergent_prod_ignore_initial_segment*:
assumes abs_convergent_prod f
shows abs_convergent_prod (λn. f (n + m))
using assms **unfolding** abs_convergent_prod_def
by (rule convergent_prod_ignore_initial_segment)

6.36.5 More elementary properties

theorem *abs_convergent_prod_imp_convergent_prod*:
fixes f :: nat ⇒ 'a :: {real_normed_div_algebra,complete_space,comm_ring_1}
assumes abs_convergent_prod f
shows convergent_prod f
proof −
from assms **have** eventually (λn. f n ≠ 0) sequentially
by (rule abs_convergent_prod_imp_ev_nonzero)
then obtain N **where** N: f n ≠ 0 **if** n ≥ N **for** n

by (*auto simp*: *eventually_at_top_linorder*)
let *?P* = λ*n*. ∏*i*≤*n*. *f* (*i* + *N*) **and** *?Q* = λ*n*. ∏*i*≤*n*. *1* + *norm* (*f* (*i* + *N*) − *1*)

have *Cauchy ?P*
proof (*rule CauchyI′*, *goal_cases*)
case (*1* ε)
from *assms* **have** *abs_convergent_prod* (λ*n*. *f* (*n* + *N*))
by (*rule abs_convergent_prod_ignore_initial_segment*)
hence *Cauchy ?Q*
unfolding *abs_convergent_prod_def*
by (*intro convergent_Cauchy convergent_prod_imp_convergent*)
from *CauchyD*[*OF this 1*] **obtain** *M* **where** *M*: *norm* (*?Q m* − *?Q n*) < ε **if** *m* ≥ *M n* ≥ *M* **for** *m n*
by *blast*
show *?case*
proof (*rule exI*[*of* _ *M*], *safe*, *goal_cases*)
case (*1 m n*)
have *dist* (*?P m*) (*?P n*) = *norm* (*?P n* − *?P m*)
by (*simp add*: *dist_norm norm_minus_commute*)
also from *1* **have** {..*n*} = {..*m*} ∪ {*m*<..*n*} **by** *auto*
hence *norm* (*?P n* − *?P m*) = *norm* (*?P m* ∗ (∏*k*∈{*m*<..*n*}. *f* (*k* + *N*)) − *?P m*)
by (*subst prod.union_disjoint* [*symmetric*]) (*auto simp*: *algebra_simps*)
also have ... = *norm* (*?P m* ∗ ((∏*k*∈{*m*<..*n*}. *f* (*k* + *N*)) − *1*))
by (*simp add*: *algebra_simps*)
also have ... = (∏*k*≤*m*. *norm* (*f* (*k* + *N*))) ∗ *norm* ((∏*k*∈{*m*<..*n*}. *f* (*k* + *N*)) − *1*)
by (*simp add*: *norm_mult prod_norm*)
also have ... ≤ *?Q m* ∗ ((∏*k*∈{*m*<..*n*}. *1* + *norm* (*f* (*k* + *N*) − *1*)) − *1*)
using *norm_prod_minus1_le_prod_minus1*[*of* λ*k*. *f* (*k* + *N*) − *1* {*m*<..*n*}]
norm_triangle_ineq[*of 1 f k* − *1* **for** *k*]
by (*intro mult_mono prod_mono ballI conjI norm_prod_minus1_le_prod_minus1 prod_nonneg*) *auto*
also have ... = *?Q m* ∗ (∏*k*∈{*m*<..*n*}. *1* + *norm* (*f* (*k* + *N*) − *1*)) − *?Q m*
by (*simp add*: *algebra_simps*)
also have *?Q m* ∗ (∏*k*∈{*m*<..*n*}. *1* + *norm* (*f* (*k* + *N*) − *1*)) =
(∏*k*∈{..*m*}∪{*m*<..*n*}. *1* + *norm* (*f* (*k* + *N*) − *1*))
by (*rule prod.union_disjoint* [*symmetric*]) *auto*
also from *1* **have** {..*m*}∪{*m*<..*n*} = {..*n*} **by** *auto*
also have *?Q n* − *?Q m* ≤ *norm* (*?Q n* − *?Q m*) **by** *simp*
also from *1* **have** ... < ε **by** (*intro M*) *auto*
finally show *?case* **.**
qed
qed
hence *conv*: *convergent ?P* **by** (*rule Cauchy_convergent*)
then obtain *L* **where** *L*: *?P* ⟶ *L*
by (*auto simp*: *convergent_def*)

```
  have L ≠ 0
  proof
    assume [simp]: L = 0
    from tendsto_norm[OF L] have limit: (λn. ∏ k≤n. norm (f (k + N))) ⟶
0
      by (simp add: prod_norm)

    from assms have (λn. f (n + N)) ⟶ 1
    by (intro abs_convergent_prod_imp_LIMSEQ abs_convergent_prod_ignore_initial_segment)
    hence eventually (λn. norm (f (n + N) − 1) < 1) sequentially
      by (auto simp: tendsto_iff dist_norm)
    then obtain M0 where M0: norm (f (n + N) − 1) < 1 if n ≥ M0 for n
      by (auto simp: eventually_at_top_linorder)

    {
      fix M assume M: M ≥ M0
      with M0 have M: norm (f (n + N) − 1) < 1 if n ≥ M for n using that
by simp

      have (λn. ∏ k≤n. 1 − norm (f (k+M+N) − 1)) ⟶ 0
      proof (rule tendsto_sandwich)
          show eventually (λn. (∏ k≤n. 1 − norm (f (k+M+N) − 1)) ≥ 0)
sequentially
          using M by (intro always_eventually prod_nonneg allI ballI) (auto intro:
less_imp_le)
        have norm (1::'a) − norm (f (i + M + N) − 1) ≤ norm (f (i + M +
N)) for i
          using norm_triangle_ineq3[of f (i + M + N) 1] by simp
        thus eventually (λn. (∏ k≤n. 1 − norm (f (k+M+N) − 1)) ≤ (∏ k≤n.
norm (f (k+M+N)))) at_top
          using M by (intro always_eventually allI prod_mono ballI conjI) (auto
intro: less_imp_le)

        define C where C = (∏ k<M. norm (f (k + N)))
        from N have [simp]: C ≠ 0 by (auto simp: C_def)
        from L have (λn. norm (∏ k≤n+M. f (k + N))) ⟶ 0
        by (intro LIMSEQ_ignore_initial_segment) (simp add: tendsto_norm_zero_iff)
        also have (λn. norm (∏ k≤n+M. f (k + N))) = (λn. C * (∏ k≤n. norm
(f (k + M + N))))
        proof (rule ext, goal_cases)
          case (1 n)
          have {..n+M} = {..<M} ∪ {M..n+M} by auto
          also have norm (∏ k∈.... f (k + N)) = C * norm (∏ k=M..n+M. f (k
+ N))
            unfolding C_def by (subst prod.union_disjoint) (auto simp: norm_mult
prod_norm)
          also have (∏ k=M..n+M. f (k + N)) = (∏ k≤n. f (k + N + M))
            by (intro prod.reindex_bij_witness[of _ λi. i + M λi. i − M]) auto
```

finally show *?case* **by** (*simp add*: *add_ac prod_norm*)
qed
finally have ($\lambda n.$ $C * (\prod k \leq n.$ *norm* $(f\ (k + M + N))) / C) \longrightarrow 0 /$
C
by (*intro tendsto_divide tendsto_const*) *auto*
thus ($\lambda n.$ $\prod k \leq n.$ *norm* $(f\ (k + M + N))) \longrightarrow 0$ **by** *simp*
qed *simp_all*

have $1 - (\sum i.$ *norm* $(f\ (i + M + N) - 1)) \leq 0$
proof (*rule tendsto_le*)
show *eventually* ($\lambda n.$ $1 - (\sum k \leq n.$ *norm* $(f\ (k{+}M{+}N) - 1)) \leq$
$(\prod k \leq n.$ $1 -$ *norm* $(f\ (k{+}M{+}N) - 1)))$ *at_top*
using M **by** (*intro always_eventually allI Weierstrass_prod_ineq*) (*auto*
intro: *less_imp_le*)
show ($\lambda n.$ $\prod k \leq n.$ $1 -$ *norm* $(f\ (k{+}M{+}N) - 1)) \longrightarrow 0$ **by** *fact*
show ($\lambda n.$ $1 - (\sum k \leq n.$ *norm* $(f\ (k + M + N) - 1)))$
$\longrightarrow 1 - (\sum i.$ *norm* $(f\ (i + M + N) - 1))$
by (*intro tendsto_intros summable_LIMSEQ' summable_ignore_initial_segment*

abs_convergent_prod_imp_summable assms)
qed *simp_all*
hence $(\sum i.$ *norm* $(f\ (i + M + N) - 1)) \geq 1$ **by** *simp*
also have $\ldots + (\sum i{<}M.$ *norm* $(f\ (i + N) - 1)) = (\sum i.$ *norm* $(f\ (i + N)$
$- 1))$
by (*intro suminf_split_initial_segment* [*symmetric*] *summable_ignore_initial_segment*
abs_convergent_prod_imp_summable assms)
finally have $1 + (\sum i{<}M.$ *norm* $(f\ (i + N) - 1)) \leq (\sum i.$ *norm* $(f\ (i +$
$N) - 1))$ **by** *simp*
} **note** $*$ = *this*

have $1 + (\sum i.$ *norm* $(f\ (i + N) - 1)) \leq (\sum i.$ *norm* $(f\ (i + N) - 1))$
proof (*rule tendsto_le*)
show ($\lambda M.$ $1 + (\sum i{<}M.$ *norm* $(f\ (i + N) - 1))) \longrightarrow 1 + (\sum i.$ *norm*
$(f\ (i + N) - 1))$
by (*intro tendsto_intros summable_LIMSEQ summable_ignore_initial_segment*

abs_convergent_prod_imp_summable assms)
show *eventually* ($\lambda M.$ $1 + (\sum i{<}M.$ *norm* $(f\ (i + N) - 1)) \leq (\sum i.$ *norm*
$(f\ (i + N) - 1)))$ *at_top*
using *eventually_ge_at_top*[*of M0*] **by** *eventually_elim* (*use* $*$ **in** *auto*)
qed *simp_all*
thus *False* **by** *simp*
qed
with L **show** *?thesis* **by** (*auto simp*: *prod_defs*)
qed

lemma *raw_has_prod_cases*:
fixes f :: *nat* $\Rightarrow$ *'a* :: {*idom*,*topological_semigroup_mult*,*t2_space*}
assumes *raw_has_prod f M p*

```
  obtains i where i<M f i = 0 | p where raw_has_prod f 0 p
proof –
  have (λn. ∏ i≤n. f (i + M)) ⟶ p p ≠ 0
    using assms unfolding raw_has_prod_def by blast+
  then have (λn. prod f {..<M} * (∏ i≤n. f (i + M))) ⟶ prod f {..<M} *
p
    by (metis tendsto_mult_left)
  moreover have prod f {..<M} * (∏ i≤n. f (i + M)) = prod f {..n+M} for n
  proof –
    have {..n+M} = {..<M} ∪ {M..n+M}
      by auto
    then have prod f {..n+M} = prod f {..<M} * prod f {M..n+M}
      by simp (subst prod.union_disjoint; force)
    also have ... = prod f {..<M} * (∏ i≤n. f (i + M))
     by (metis (mono_tags, lifting) add.left_neutral atMost_atLeast0 prod.shift_bounds_cl_nat_ivl)
    finally show ?thesis by metis
  qed
  ultimately have (λn. prod f {..n}) ⟶ prod f {..<M} * p
    by (auto intro: LIMSEQ_offset [where k=M])
  then have raw_has_prod f 0 (prod f {..<M} * p) if ∀ i<M. f i ≠ 0
    using ‹p ≠ 0› assms that by (auto simp: raw_has_prod_def)
  then show thesis
    using that by blast
qed

corollary convergent_prod_offset_0:
  fixes f :: nat ⇒ 'a :: {idom,topological_semigroup_mult,t2_space}
  assumes convergent_prod f ⋀i. f i ≠ 0
  shows ∃ p. raw_has_prod f 0 p
  using assms convergent_prod_def raw_has_prod_cases by blast

lemma prodinf_eq_lim:
  fixes f :: nat ⇒ 'a :: {idom,topological_semigroup_mult,t2_space}
  assumes convergent_prod f ⋀i. f i ≠ 0
  shows prodinf f = lim (λn. ∏ i≤n. f i)
  using assms convergent_prod_offset_0 [OF assms]
 by (simp add: prod_defs lim_def) (metis (no_types) assms(1) convergent_prod_to_zero_iff)

lemma has_prod_one[simp, intro]: (λn. 1) has_prod 1
  unfolding prod_defs by auto

lemma convergent_prod_one[simp, intro]: convergent_prod (λn. 1)
  unfolding prod_defs by auto

lemma prodinf_cong: (⋀n. f n = g n) ⟹ prodinf f = prodinf g
  by presburger

lemma convergent_prod_cong:
  fixes f g :: nat ⇒ 'a::{field,topological_semigroup_mult,t2_space}
```

assumes *ev*: *eventually* $(\lambda x.\ f\ x = g\ x)$ *sequentially* **and** *f*: $\bigwedge i.\ f\ i \neq 0$ **and** *g*: $\bigwedge i.\ g\ i \neq 0$
shows *convergent_prod f* = *convergent_prod g*
proof −
from *assms* **obtain** *N* **where** *N*: $\forall\, n \geq N.\ f\ n = g\ n$
by (*auto simp*: *eventually_at_top_linorder*)
define *C* **where** $C = (\prod k<N.\ f\ k\ /\ g\ k)$
with *g* **have** $C \neq 0$
by (*simp add*: *f*)
have ∗: *eventually* (λ*n*. *prod f* {..*n*} = *C* ∗ *prod g* {..*n*}) *sequentially*
using *eventually_ge_at_top*[*of N*]
proof *eventually_elim*
case (*elim n*)
then have {..*n*} = {..<*N*} ∪ {*N*..*n*}
by *auto*
also have *prod f* ... = *prod f* {..<*N*} ∗ *prod f* {*N*..*n*}
by (*intro prod.union_disjoint*) *auto*
also from *N* **have** *prod f* {*N*..*n*} = *prod g* {*N*..*n*}
by (*intro prod.cong*) *simp_all*
also have *prod f* {..<*N*} ∗ *prod g* {*N*..*n*} = *C* ∗ (*prod g* {..<*N*} ∗ *prod g* {*N*..*n*})
unfolding *C_def* **by** (*simp add*: *g prod_dividef*)
also have *prod g* {..<*N*} ∗ *prod g* {*N*..*n*} = *prod g* ({..<*N*} ∪ {*N*..*n*})
by (*intro prod.union_disjoint* [*symmetric*]) *auto*
also from *elim* **have** {..<*N*} ∪ {*N*..*n*} = {..*n*}
by *auto*
finally show *prod f* {..*n*} = *C* ∗ *prod g* {..*n*} **.**
qed
then have *cong*: *convergent* (λ*n*. *prod f* {..*n*}) = *convergent* (λ*n*. *C* ∗ *prod g* {..*n*})
by (*rule convergent_cong*)
show *?thesis*
proof
assume *cf*: *convergent_prod f*
then have ¬ (λ*n*. *prod g* {..*n*}) ⟶ *0*
using *tendsto_mult_left* ∗ *convergent_prod_to_zero_iff f filterlim_cong* **by** *fastforce*
then show *convergent_prod g*
by (*metis convergent_mult_const_iff* ‹*C* ≠ *0*› *cong cf convergent_LIMSEQ_iff convergent_prod_iff_convergent convergent_prod_imp_convergent g*)
next
assume *cg*: *convergent_prod g*
have ∃ *a*. *C* ∗ *a* ≠ *0* ∧ (λ*n*. *prod g* {..*n*}) ⟶ *a*
by (*metis* (*no_types*) ‹*C* ≠ *0*› *cg convergent_prod_iff_nz_lim divide_eq_0_iff g nonzero_mult_div_cancel_right*)
then show *convergent_prod f*
using ∗ *tendsto_mult_left filterlim_cong*
by (*fastforce simp add*: *convergent_prod_iff_nz_lim f*)
qed
qed

```
lemma has_prod_finite:
  fixes f :: nat ⇒ 'a::{semidom,t2_space}
  assumes [simp]: finite N
    and f: ⋀n. n ∉ N ⟹ f n = 1
  shows f has_prod (∏n∈N. f n)
proof –
  have eq: prod f {..n + Suc (Max N)} = prod f N for n
  proof (rule prod.mono_neutral_right)
    show N ⊆ {..n + Suc (Max N)}
      by (auto simp: le_Suc_eq trans_le_add2)
    show ∀i∈{..n + Suc (Max N)} – N. f i = 1
      using f by blast
  qed auto
  show ?thesis
  proof (cases ∀n∈N. f n ≠ 0)
    case True
    then have prod f N ≠ 0
      by simp
    moreover have (λn. prod f {..n}) ⟶ prod f N
      by (rule LIMSEQ_offset[of _ Suc (Max N)]) (simp add: eq atLeast0LessThan
del: add_Suc_right)
    ultimately show ?thesis
      by (simp add: raw_has_prod_def has_prod_def)
  next
    case False
    then obtain k where k ∈ N f k = 0
      by auto
    let ?Z = {n ∈ N. f n = 0}
    have maxge: Max ?Z ≥ n if f n = 0 for n
      using Max_ge [of ?Z] ‹finite N› ‹f n = 0›
       by (metis (mono_tags) Collect_mem_eq f finite_Collect_conjI mem_Collect_eq
zero_neq_one)
    let ?q = prod f {Suc (Max ?Z)..Max N}
    have [simp]: ?q ≠ 0
      using maxge Suc_n_not_le_n le_trans by force
    have eq: (∏i≤n + Max N. f (Suc (i + Max ?Z))) = ?q for n
    proof –
      have (∏i≤n + Max N. f (Suc (i + Max ?Z))) = prod f {Suc (Max ?Z)..n
+ Max N + Suc (Max ?Z)}
        proof (rule prod.reindex_cong [where l = λi. i + Suc (Max ?Z), THEN
sym])
         show {Suc (Max ?Z)..n + Max N + Suc (Max ?Z)} = (λi. i + Suc (Max
?Z)) ` {..n + Max N}
           using le_Suc_ex by fastforce
      qed (auto simp: inj_on_def)
      also have ... = ?q
        by (rule prod.mono_neutral_right)
           (use Max.coboundedI [OF ‹finite N›] f in ‹force+›)
```

finally show *?thesis* **.**
qed
have *q*: *raw_has_prod f* (*Suc* (*Max ?Z*)) *?q*
proof (*simp add*: *raw_has_prod_def*)
show ($\lambda n. \prod i \leq n.$ *f* (*Suc* (*i* + *Max ?Z*))) ⟶ *?q*
by (*rule LIMSEQ_offset*[*of* _ (*Max N*)]) (*simp add*: *eq*)
qed
show *?thesis*
unfolding *has_prod_def*
proof (*intro disjI2 exI conjI*)
show *prod f N* = *0*
using ‹*f k* = *0*› ‹*k* ∈ *N*› ‹*finite N*› *prod_zero* **by** *blast*
show *f* (*Max ?Z*) = *0*
using *Max_in* [*of ?Z*] ‹*finite N*› ‹*f k* = *0*› ‹*k* ∈ *N*› **by** *auto*
qed (*use q* **in** *auto*)
qed
qed

corollary *has_prod_0*:
fixes *f* :: *nat* ⇒ *'a*::{*semidom*,*t2_space*}
assumes ⋀*n*. *f n* = *1*
shows *f has_prod 1*
by (*simp add*: *assms has_prod_cong*)

lemma *prodinf_zero*[*simp*]: *prodinf* (λ*n*. *1*::*'a*::*real_normed_field*) = *1*
using *has_prod_unique* **by** *force*

lemma *convergent_prod_finite*:
fixes *f* :: *nat* ⇒ *'a*::{*idom*,*t2_space*}
assumes *finite N* ⋀*n*. *n* ∉ *N* ⟹ *f n* = *1*
shows *convergent_prod f*
proof −
have ∃ *n p*. *raw_has_prod f n p*
using *assms has_prod_def has_prod_finite* **by** *blast*
then show *?thesis*
by (*simp add*: *convergent_prod_def*)
qed

lemma *has_prod_If_finite_set*:
fixes *f* :: *nat* ⇒ *'a*::{*idom*,*t2_space*}
shows *finite A* ⟹ (λ*r*. *if r* ∈ *A then f r else 1*) *has_prod* ($\prod r \in A.$ *f r*)
using *has_prod_finite*[*of A* (λ*r*. *if r* ∈ *A then f r else 1*)]
by *simp*

lemma *has_prod_If_finite*:
fixes *f* :: *nat* ⇒ *'a*::{*idom*,*t2_space*}
shows *finite* {*r*. *P r*} ⟹ (λ*r*. *if P r then f r else 1*) *has_prod* ($\prod r \mid P\ r.$ *f r*)
using *has_prod_If_finite_set*[*of* {*r*. *P r*}] **by** *simp*

```
lemma convergent_prod_If_finite_set[simp, intro]:
  fixes f :: nat ⇒ 'a::{idom,t2_space}
  shows finite A ⟹ convergent_prod (λr. if r ∈ A then f r else 1)
  by (simp add: convergent_prod_finite)

lemma convergent_prod_If_finite[simp, intro]:
  fixes f :: nat ⇒ 'a::{idom,t2_space}
  shows finite {r. P r} ⟹ convergent_prod (λr. if P r then f r else 1)
  using convergent_prod_def has_prod_If_finite has_prod_def by fastforce

lemma has_prod_single:
  fixes f :: nat ⇒ 'a::{idom,t2_space}
  shows (λr. if r = i then f r else 1) has_prod f i
  using has_prod_If_finite[of λr. r = i] by simp

context
  fixes f :: nat ⇒ 'a :: real_normed_field
begin

lemma convergent_prod_imp_has_prod:
  assumes convergent_prod f
  shows ∃p. f has_prod p
proof –
  obtain M p where p: raw_has_prod f M p
    using assms convergent_prod_def by blast
  then have p ≠ 0
    using raw_has_prod_nonzero by blast
  with p have fnz: f i ≠ 0 if i ≥ M for i
    using raw_has_prod_eq_0 that by blast
  define C where C = (∏n<M. f n)
  show ?thesis
  proof (cases ∀n≤M. f n ≠ 0)
    case True
    then have C ≠ 0
      by (simp add: C_def)
    then show ?thesis
     by (meson True assms convergent_prod_offset_0 fnz has_prod_def nat_le_linear)
  next
    case False
    let ?N = GREATEST n. f n = 0
    have 0: f ?N = 0
      using fnz False
      by (metis (mono_tags, lifting) GreatestI_ex_nat nat_le_linear)
    have f i ≠ 0 if i > ?N for i
      by (metis (mono_tags, lifting) Greatest_le_nat fnz leD linear that)
    then have ∃p. raw_has_prod f (Suc ?N) p
      using assms by (auto simp: intro!: convergent_prod_ignore_nonzero_segment)
    then show ?thesis
      unfolding has_prod_def using 0 by blast
```

qed
qed

lemma *convergent_prod_has_prod* [*intro*]:
shows *convergent_prod f* $\Longrightarrow$ *f has_prod* (*prodinf f*)
unfolding *prodinf_def*
by (*metis convergent_prod_imp_has_prod has_prod_unique theI'*)

lemma *convergent_prod_LIMSEQ*:
shows *convergent_prod f* $\Longrightarrow$ ($\lambda n.\ \prod i{\le}n.\ f\ i$) $\longrightarrow$ *prodinf f*
by (*metis convergent_LIMSEQ_iff convergent_prod_has_prod convergent_prod_imp_convergent*

convergent_prod_to_zero_iff raw_has_prod_eq_0 has_prod_def prodinf_eq_lim zero_le)

theorem *has_prod_iff*: *f has_prod x* $\longleftrightarrow$ *convergent_prod f* $\wedge$ *prodinf f* = *x*
proof
assume *f has_prod x*
then show *convergent_prod f* $\wedge$ *prodinf f* = *x*
apply *safe*
using *convergent_prod_def has_prod_def* **apply** *blast*
using *has_prod_unique* **by** *blast*
qed *auto*

lemma *convergent_prod_has_prod_iff*: *convergent_prod f* $\longleftrightarrow$ *f has_prod prodinf f*
by (*auto simp*: *has_prod_iff convergent_prod_has_prod*)

lemma *prodinf_finite*:
assumes *N*: *finite N*
and *f*: $\bigwedge n.\ n \notin N \Longrightarrow f\ n = 1$
shows *prodinf f* = ($\prod n{\in}N.\ f\ n$)
using *has_prod_finite*[*OF assms*, *THEN has_prod_unique*] **by** *simp*

end

6.36.6 Infinite products on ordered topological monoids

lemma *LIMSEQ_prod_0*:
fixes *f* :: *nat* $\Rightarrow$ *'a*::{*semidom*,*topological_space*}
assumes *f i* = *0*
shows ($\lambda n.$ *prod f* {..*n*}) $\longrightarrow$ *0*
proof (*subst tendsto_cong*)
show $\forall_F$ *n in sequentially*. *prod f* {..*n*} = *0*
proof
show *prod f* {..*n*} = *0* **if** $n \ge i$ **for** *n*
using *that assms* **by** *auto*
qed
qed *auto*

lemma *LIMSEQ_prod_nonneg*:

```
  fixes f :: nat ⇒ 'a::{linordered_semidom,linorder_topology}
  assumes 0: ⋀n. 0 ≤ f n and a: (λn. prod f {..n}) ⟶ a
  shows a ≥ 0
  by (simp add: 0 prod_nonneg LIMSEQ_le_const [OF a])


context
  fixes f :: nat ⇒ 'a::{linordered_semidom,linorder_topology}
begin

lemma has_prod_le:
  assumes f: f has_prod a and g: g has_prod b and le: ⋀n. 0 ≤ f n ∧ f n ≤ g n
  shows a ≤ b
proof (cases a=0 ∨ b=0)
  case True
  then show ?thesis
  proof
    assume [simp]: a=0
    have b ≥ 0
    proof (rule LIMSEQ_prod_nonneg)
      show (λn. prod g {..n}) ⟶ b
        using g by (auto simp: has_prod_def raw_has_prod_def LIMSEQ_prod_0)
    qed (use le order_trans in auto)
    then show ?thesis
      by auto
  next
    assume [simp]: b=0
    then obtain i where g i = 0
      using g by (auto simp: prod_defs)
    then have f i = 0
      using antisym le by force
    then have a=0
      using f by (auto simp: prod_defs LIMSEQ_prod_0 LIMSEQ_unique)
    then show ?thesis
      by auto
  qed
next
  case False
  then show ?thesis
    using assms
    unfolding has_prod_def raw_has_prod_def
    by (force simp: LIMSEQ_prod_0 intro!: LIMSEQ_le prod_mono)
qed

lemma prodinf_le:
  assumes f: f has_prod a and g: g has_prod b and le: ⋀n. 0 ≤ f n ∧ f n ≤ g n
  shows prodinf f ≤ prodinf g
  using has_prod_le [OF assms] has_prod_unique f g by blast
```

```
end

lemma prod_le_prodinf:
  fixes f :: nat ⇒ 'a::{linordered_idom,linorder_topology}
  assumes f has_prod a ⋀i. 0 ≤ f i ⋀i. i≥n ⟹ 1 ≤ f i
  shows prod f {..<n} ≤ prodinf f
   by(rule has_prod_le[OF has_prod_If_finite_set]) (use assms has_prod_unique in
auto)

lemma prodinf_nonneg:
  fixes f :: nat ⇒ 'a::{linordered_idom,linorder_topology}
  assumes f has_prod a ⋀i. 1 ≤ f i
  shows 1 ≤ prodinf f
  using prod_le_prodinf[of f a 0] assms
  by (metis order_trans prod_ge_1 zero_le_one)

lemma prodinf_le_const:
  fixes f :: nat ⇒ real
  assumes convergent_prod f ⋀n. prod f {..<n} ≤ x
  shows prodinf f ≤ x
 by (metis lessThan_Suc_atMost assms convergent_prod_LIMSEQ LIMSEQ_le_const2)

lemma prodinf_eq_one_iff [simp]:
  fixes f :: nat ⇒ real
  assumes f: convergent_prod f and ge1: ⋀n. 1 ≤ f n
  shows prodinf f = 1 ⟷ (∀n. f n = 1)
proof
  assume prodinf f = 1
  then have (λn. ∏i<n. f i) ⟶ 1
   using convergent_prod_LIMSEQ[of f] assms by (simp add: LIMSEQ_lessThan_iff_atMost)
  then have ⋀i. (∏n∈{i}. f n) ≤ 1
  proof (rule LIMSEQ_le_const)
    have 1 ≤ prod f n for n
      by (simp add: ge1 prod_ge_1)
    have prod f {..<n} = 1 for n
     by (metis ‹⋀n. 1 ≤ prod f n› ‹prodinf f = 1› antisym f convergent_prod_has_prod
ge1 order_trans prod_le_prodinf zero_le_one)
    then have (∏n∈{i}. f n) ≤ prod f {..<n} if n ≥ Suc i for i n
     by (metis mult.left_neutral order_refl prod.cong prod.neutral_const prod.lessThan_Suc)
    then show ∃N. ∀n≥N. (∏n∈{i}. f n) ≤ prod f {..<n} for i
      by blast
  qed
  with ge1 show ∀n. f n = 1
    by (auto intro!: antisym)
qed (metis prodinf_zero fun_eq_iff)

lemma prodinf_pos_iff:
  fixes f :: nat ⇒ real
```

assumes *convergent_prod f* $\bigwedge n.$ *1* $\leq$ *f n*
shows *1* < *prodinf f* $\longleftrightarrow$ ($\exists$ *i*. *1* < *f i*)
using *prod_le_prodinf*[*of f 1*] *prodinf_eq_one_iff*
by (*metis convergent_prod_has_prod assms less_le prodinf_nonneg*)

lemma *less_1_prodinf2*:
fixes *f* :: *nat* ⇒ *real*
assumes *convergent_prod f* $\bigwedge n.$ *1* ≤ *f n* *1* < *f i*
shows *1* < *prodinf f*
proof –
have *1* < ($\prod$ *n*<*Suc i*. *f n*)
using *assms* **by** (*intro less_1_prod2*[**where** *i*=*i*]) *auto*
also have ... ≤ *prodinf f*
by (*intro prod_le_prodinf*) (*use assms order_trans zero_le_one* **in** ‹*blast*+›)
finally show *?thesis* **.**
qed

lemma *less_1_prodinf*:
fixes *f* :: *nat* ⇒ *real*
shows ⟦*convergent_prod f*; $\bigwedge n.$ *1* < *f n*⟧ ⟹ *1* < *prodinf f*
by (*intro less_1_prodinf2*[**where** *i*=*1*]) (*auto intro*: *less_imp_le*)

lemma *prodinf_nonzero*:
fixes *f* :: *nat* ⇒ *'a* :: {*idom*,*topological_semigroup_mult*,*t2_space*}
assumes *convergent_prod f* $\bigwedge i.$ *f i* ≠ *0*
shows *prodinf f* ≠ *0*
by (*metis assms convergent_prod_offset_0 has_prod_unique raw_has_prod_def has_prod_def*)

lemma *less_0_prodinf*:
fixes *f* :: *nat* ⇒ *real*
assumes *f*: *convergent_prod f* **and** *0*: $\bigwedge i.$ *f i* > *0*
shows *0* < *prodinf f*
proof –
have *prodinf f* ≠ *0*
by (*metis assms less_irrefl prodinf_nonzero*)
moreover have *0* < ($\prod$ *n*<*i*. *f n*) **for** *i*
by (*simp add*: *0 prod_pos*)
then have *prodinf f* ≥ *0*
using *convergent_prod_LIMSEQ* [*OF f*] *LIMSEQ_prod_nonneg 0 less_le* **by** *blast*
ultimately show *?thesis*
by *auto*
qed

lemma *prod_less_prodinf2*:
fixes *f* :: *nat* ⇒ *real*
assumes *f*: *convergent_prod f* **and** *1*: $\bigwedge m.$ *m*≥*n* ⟹ *1* ≤ *f m* **and** *0*: $\bigwedge m.$ *0* < *f m* **and** *i*: *n* ≤ *i* *1* < *f i*
shows *prod f* {..<*n*} < *prodinf f*
proof –

have *prod f* {..<*n*} ≤ *prod f* {..<*i*}
  by (*rule prod_mono2*) (*use assms less_le* **in** *auto*)
then have *prod f* {..<*n*} < *f i* ∗ *prod f* {..<*i*}
  using *mult_less_le_imp_less*[*of 1 f i prod f* {..<*n*} *prod f* {..<*i*}] *assms*
  by (*simp add*: *prod_pos*)
moreover have *prod f* {..<*Suc i*} ≤ *prodinf f*
  using *prod_le_prodinf*[*of f* _ *Suc i*]
 by (*meson 0 1 Suc_leD convergent_prod_has_prod f* ‹*n* ≤ *i*› *le_trans less_eq_real_def*)
ultimately show *?thesis*
  by (*metis le_less_trans mult.commute not_le prod.lessThan_Suc*)
qed

lemma *prod_less_prodinf*:
  fixes *f* :: *nat* ⇒ *real*
  assumes *f*: *convergent_prod f* **and** *1*: ⋀*m*. *m*≥*n* ⟹ *1* < *f m* **and** *0*: ⋀*m*. *0* < *f m*
  shows *prod f* {..<*n*} < *prodinf f*
  by (*meson 0 1 f le_less prod_less_prodinf2*)

lemma *raw_has_prodI_bounded*:
  fixes *f* :: *nat* ⇒ *real*
  assumes *pos*: ⋀*n*. *1* ≤ *f n*
    and *le*: ⋀*n*. (∏*i*<*n*. *f i*) ≤ *x*
  shows ∃*p*. *raw_has_prod f 0 p*
  unfolding *raw_has_prod_def add_0_right*
proof (*rule exI LIMSEQ_incseq_SUP conjI*)+
  show *bdd_above* (*range* (λ*n*. *prod f* {..*n*}))
    by (*metis bdd_aboveI2 le lessThan_Suc_atMost*)
  then have (*SUP i*. *prod f* {..*i*}) > *0*
    by (*metis UNIV_I cSUP_upper less_le_trans pos prod_pos zero_less_one*)
  then show (*SUP i*. *prod f* {..*i*}) ≠ *0*
    by *auto*
  show *incseq* (λ*n*. *prod f* {..*n*})
   using *pos order_trans* [*OF zero_le_one*] **by** (*auto simp*: *mono_def intro*!: *prod_mono2*)
qed

lemma *convergent_prodI_nonneg_bounded*:
  fixes *f* :: *nat* ⇒ *real*
  assumes ⋀*n*. *1* ≤ *f n* ⋀*n*. (∏*i*<*n*. *f i*) ≤ *x*
  shows *convergent_prod f*
  using *convergent_prod_def raw_has_prodI_bounded* [*OF assms*] **by** *blast*

6.36.7 Infinite products on topological spaces

context
  fixes *f g* :: *nat* ⇒ ′*a*::{*t2_space*,*topological_semigroup_mult*,*idom*}
begin

lemma *raw_has_prod_mult*: ⟦*raw_has_prod f M a*; *raw_has_prod g M b*⟧ ⟹ *raw_has_prod*

```
(λn. f n ∗ g n) M (a ∗ b)
  by (force simp add: prod.distrib tendsto_mult raw_has_prod_def)

lemma has_prod_mult_nz: [[f has_prod a; g has_prod b; a ≠ 0; b ≠ 0]] =⇒ (λn. f n
∗ g n) has_prod (a ∗ b)
  by (simp add: raw_has_prod_mult has_prod_def)

end


context
  fixes f g :: nat ⇒ 'a::real_normed_field
begin

lemma has_prod_mult:
  assumes f: f has_prod a and g: g has_prod b
  shows (λn. f n ∗ g n) has_prod (a ∗ b)
  using f [unfolded has_prod_def]
proof (elim disjE exE conjE)
  assume f0: raw_has_prod f 0 a
  show ?thesis
    using g [unfolded has_prod_def]
  proof (elim disjE exE conjE)
    assume g0: raw_has_prod g 0 b
    with f0 show ?thesis
      by (force simp add: has_prod_def prod.distrib tendsto_mult raw_has_prod_def)
  next
    fix j q
    assume b = 0 and g j = 0 and q: raw_has_prod g (Suc j) q
    obtain p where p: raw_has_prod f (Suc j) p
      using f0 raw_has_prod_ignore_initial_segment by blast
    then have Ex (raw_has_prod (λn. f n ∗ g n) (Suc j))
      using q raw_has_prod_mult by blast
    then show ?thesis
      using ‹b = 0› ‹g j = 0› has_prod_0_iff by fastforce
  qed
next
  fix i p
  assume a = 0 and f i = 0 and p: raw_has_prod f (Suc i) p
  show ?thesis
    using g [unfolded has_prod_def]
  proof (elim disjE exE conjE)
    assume g0: raw_has_prod g 0 b
    obtain q where q: raw_has_prod g (Suc i) q
      using g0 raw_has_prod_ignore_initial_segment by blast
    then have Ex (raw_has_prod (λn. f n ∗ g n) (Suc i))
      using raw_has_prod_mult p by blast
    then show ?thesis
      using ‹a = 0› ‹f i = 0› has_prod_0_iff by fastforce
```

next
fix *j q*
assume *b = 0* **and** *g j = 0* **and** *q*: *raw_has_prod g* (*Suc j*) *q*
obtain *p′* **where** *p′*: *raw_has_prod f* (*Suc* (*max i j*)) *p′*
by (*metis raw_has_prod_ignore_initial_segment max_Suc_Suc max_def p*)
moreover
obtain *q′* **where** *q′*: *raw_has_prod g* (*Suc* (*max i j*)) *q′*
by (*metis raw_has_prod_ignore_initial_segment max.cobounded2 max_Suc_Suc q*)
ultimately show *?thesis*
using ‹*b = 0*› **by** (*simp add*: *has_prod_def*) (*metis* ‹*f i = 0*› ‹*g j = 0*› *raw_has_prod_mult max_def*)
qed
qed

lemma *convergent_prod_mult*:
assumes *f*: *convergent_prod f* **and** *g*: *convergent_prod g*
shows *convergent_prod* (*λn. f n ∗ g n*)
unfolding *convergent_prod_def*
proof −
obtain *M p N q* **where** *p*: *raw_has_prod f M p* **and** *q*: *raw_has_prod g N q*
using *convergent_prod_def f g* **by** *blast+*
then obtain *p′ q′* **where** *p′*: *raw_has_prod f* (*max M N*) *p′* **and** *q′*: *raw_has_prod g* (*max M N*) *q′*
by (*meson raw_has_prod_ignore_initial_segment max.cobounded1 max.cobounded2*)
then show ∃ *M p. raw_has_prod* (*λn. f n ∗ g n*) *M p*
using *raw_has_prod_mult* **by** *blast*
qed

lemma *prodinf_mult*: *convergent_prod f* ⟹ *convergent_prod g* ⟹ *prodinf f ∗ prodinf g* = (∏ *n. f n ∗ g n*)
by (*intro has_prod_unique has_prod_mult convergent_prod_has_prod*)

end

context
fixes *f* :: *′i* ⇒ *nat* ⇒ *′a*::*real_normed_field*
and *I* :: *′i set*
begin

lemma *has_prod_prod*: (⋀*i. i* ∈ *I* ⟹ (*f i*) *has_prod* (*x i*)) ⟹ (*λn.* ∏ *i*∈*I. f i n*) *has_prod* (∏ *i*∈*I. x i*)
by (*induct I rule*: *infinite_finite_induct*) (*auto intro*!: *has_prod_mult*)

lemma *prodinf_prod*: (⋀*i. i* ∈ *I* ⟹ *convergent_prod* (*f i*)) ⟹ (∏ *n.* ∏ *i*∈*I. f i n*) = (∏ *i*∈*I.* ∏ *n. f i n*)
using *has_prod_unique*[*OF has_prod_prod*, *OF convergent_prod_has_prod*] **by** *simp*

lemma *convergent_prod_prod*: (⋀*i. i* ∈ *I* ⟹ *convergent_prod* (*f i*)) ⟹ *conver-*

gent_prod $(\lambda n. \prod i\in I. f\ i\ n)$
  using *convergent_prod_has_prod_iff has_prod_prod prodinf_prod* **by** *force*

end

6.36.8 Infinite summability on real normed fields

context
  fixes f :: *nat* ⇒ *'a::real_normed_field*
begin

lemma *raw_has_prod_Suc_iff*: *raw_has_prod f M* $(a * f\ M) \longleftrightarrow$ *raw_has_prod* $(\lambda n. f\ (Suc\ n))\ M\ a \wedge f\ M \neq 0$
proof –
  have *raw_has_prod f M* $(a * f\ M) \longleftrightarrow (\lambda i. \prod j\le Suc\ i. f\ (j+M)) \longrightarrow a * f\ M \wedge a * f\ M \neq 0$
    by (*subst filterlim_sequentially_Suc*) (*simp add*: *raw_has_prod_def*)
  also have ... $\longleftrightarrow (\lambda i. (\prod j\le i. f\ (Suc\ j + M)) * f\ M) \longrightarrow a * f\ M \wedge a * f\ M \neq 0$
   by (*simp add*: *ac_simps atMost_Suc_eq_insert_0 image_Suc_atMost prod.atLeast1_atMost_eq lessThan_Suc_atMost*
                  del: *prod.cl_ivl_Suc*)
  also have ... $\longleftrightarrow$ *raw_has_prod* $(\lambda n. f\ (Suc\ n))\ M\ a \wedge f\ M \neq 0$
  proof *safe*
    assume *tends*: $(\lambda i. (\prod j\le i. f\ (Suc\ j + M)) * f\ M) \longrightarrow a * f\ M$ **and** *0*: $a * f\ M \neq 0$
    with *tendsto_divide*[*OF tends tendsto_const*, *of f M*]
    show *raw_has_prod* $(\lambda n. f\ (Suc\ n))\ M\ a$
      by (*simp add*: *raw_has_prod_def*)
  qed (*auto intro*: *tendsto_mult_right simp*: *raw_has_prod_def*)
  finally show *?thesis* **.**
qed

lemma *has_prod_Suc_iff*:
  assumes $f\ 0 \neq 0$ **shows** $(\lambda n. f\ (Suc\ n))$ *has_prod* $a \longleftrightarrow f$ *has_prod* $(a * f\ 0)$
proof (*cases* $a = 0$)
  case *True*
  then show *?thesis*
  proof (*simp add*: *has_prod_def*, *safe*)
    fix *i x*
    assume $f\ (Suc\ i) = 0$ **and** *raw_has_prod* $(\lambda n. f\ (Suc\ n))\ (Suc\ i)\ x$
    then obtain *y* **where** *raw_has_prod f* $(Suc\ (Suc\ i))\ y$
       by (*metis* (*no_types*) *raw_has_prod_eq_0 Suc_n_not_le_n raw_has_prod_Suc_iff raw_has_prod_ignore_initial_segment raw_has_prod_nonzero linear*)
    then show $\exists i. f\ i = 0 \wedge Ex$ (*raw_has_prod f* $(Suc\ i)$)
      using ‹$f\ (Suc\ i) = 0$› **by** *blast*
  next
    fix *i x*
    assume $f\ i = 0$ **and** *x*: *raw_has_prod f* $(Suc\ i)\ x$

then obtain *j* **where** *j*: $i = Suc\ j$
by (*metis assms not0_implies_Suc*)
moreover have $\exists\ y.\ raw\_has\_prod\ (\lambda n.\ f\ (Suc\ n))\ i\ y$
using *x* **by** (*auto simp*: *raw_has_prod_def*)
then show $\exists i.\ f\ (Suc\ i) = 0 \wedge Ex\ (raw\_has\_prod\ (\lambda n.\ f\ (Suc\ n))\ (Suc\ i))$
using ‹$f\ i = 0$› *j* **by** *blast*
qed
next
case *False*
then show *?thesis*
by (*auto simp*: *has_prod_def raw_has_prod_Suc_iff assms*)
qed

lemma *convergent_prod_Suc_iff* [*simp*]:
shows $convergent\_prod\ (\lambda n.\ f\ (Suc\ n)) = convergent\_prod\ f$
proof
assume *convergent_prod f*
then obtain *M L* **where** *M_nz*:$\forall n \geq M.\ f\ n \neq 0$ **and**
M_L:$(\lambda n.\ \prod i \leq n.\ f\ (i + M)) \longrightarrow L$ **and** $L \neq 0$
unfolding *convergent_prod_altdef* **by** *auto*
have $(\lambda n.\ \prod i \leq n.\ f\ (Suc\ (i + M))) \longrightarrow L\ /\ f\ M$
proof −
have $(\lambda n.\ \prod i \in \{0..Suc\ n\}.\ f\ (i + M)) \longrightarrow L$
using *M_L*
apply (*subst* (*asm*) *filterlim_sequentially_Suc*[*symmetric*])
using *atLeast0AtMost* **by** *auto*
then have $(\lambda n.\ f\ M * (\prod i \in \{0..n\}.\ f\ (Suc\ (i + M)))) \longrightarrow L$
apply (*subst* (*asm*) *prod.atLeast0_atMost_Suc_shift*)
by *simp*
then have $(\lambda n.\ (\prod i \in \{0..n\}.\ f\ (Suc\ (i + M)))) \longrightarrow L/f\ M$
apply (*drule_tac tendsto_divide*)
using *M_nz*[*rule_format*,*of M*,*simplified*] **by** *auto*
then show *?thesis* **unfolding** *atLeast0AtMost* **.**
qed
then show $convergent\_prod\ (\lambda n.\ f\ (Suc\ n))$ **unfolding** *convergent_prod_altdef*
apply (*rule_tac exI*[**where** $x=M$])
apply (*rule_tac exI*[**where** $x=L/f\ M$])
using *M_nz* ‹$L \neq 0$› **by** *auto*
next
assume $convergent\_prod\ (\lambda n.\ f\ (Suc\ n))$
then obtain *M* **where** $\exists L.\ (\forall n \geq M.\ f\ (Suc\ n) \neq 0) \wedge (\lambda n.\ \prod i \leq n.\ f\ (Suc\ (i + M))) \longrightarrow L \wedge L \neq 0$
unfolding *convergent_prod_altdef* **by** *auto*
then show *convergent_prod f* **unfolding** *convergent_prod_altdef*
apply (*rule_tac exI*[**where** $x=Suc\ M$])
using *Suc_le_D* **by** *auto*
qed

lemma *raw_has_prod_inverse*:

assumes *raw_has_prod f M a* **shows** *raw_has_prod* (*λn. inverse* (*f n*)) *M* (*inverse a*)
using *assms* **unfolding** *raw_has_prod_def* **by** (*auto dest*: *tendsto_inverse simp*: *prod_inversef* [*symmetric*])

lemma *has_prod_inverse*:
assumes *f has_prod a* **shows** (*λn. inverse* (*f n*)) *has_prod* (*inverse a*)
using *assms raw_has_prod_inverse* **unfolding** *has_prod_def* **by** *auto*

lemma *convergent_prod_inverse*:
assumes *convergent_prod f*
shows *convergent_prod* (*λn. inverse* (*f n*))
using *assms* **unfolding** *convergent_prod_def* **by** (*blast intro*: *raw_has_prod_inverse elim*:)

end

context
fixes *f* :: *nat* ⇒ *'a*::*real_normed_field*
begin

lemma *raw_has_prod_Suc_iff'*: *raw_has_prod f M a* ⟷ *raw_has_prod* (*λn. f* (*Suc n*)) *M* (*a* / *f M*) ∧ *f M* ≠ *0*
by (*metis raw_has_prod_eq_0 add.commute add.left_neutral raw_has_prod_Suc_iff raw_has_prod_nonzero le_add1 nonzero_mult_div_cancel_right times_divide_eq_left*)

lemma *has_prod_divide*: *f has_prod a* ⟹ *g has_prod b* ⟹ (*λn. f n* / *g n*) *has_prod* (*a* / *b*)
unfolding *divide_inverse* **by** (*intro has_prod_inverse has_prod_mult*)

lemma *convergent_prod_divide*:
assumes *f*: *convergent_prod f* **and** *g*: *convergent_prod g*
shows *convergent_prod* (*λn. f n* / *g n*)
using *f g has_prod_divide has_prod_iff* **by** *blast*

lemma *prodinf_divide*: *convergent_prod f* ⟹ *convergent_prod g* ⟹ *prodinf f* / *prodinf g* = (∏ *n. f n* / *g n*)
by (*intro has_prod_unique has_prod_divide convergent_prod_has_prod*)

lemma *prodinf_inverse*: *convergent_prod f* ⟹ (∏ *n. inverse* (*f n*)) = *inverse* (∏ *n. f n*)
by (*intro has_prod_unique* [*symmetric*] *has_prod_inverse convergent_prod_has_prod*)

lemma *has_prod_Suc_imp*:
assumes (*λn. f* (*Suc n*)) *has_prod a*
shows *f has_prod* (*a* ∗ *f 0*)
proof −
have *f has_prod* (*a* ∗ *f 0*) **when** *raw_has_prod* (*λn. f* (*Suc n*)) *0 a*
apply (*cases f 0*=*0*)

using *that* **unfolding** *has_prod_def raw_has_prod_Suc*
by (*auto simp add*: *raw_has_prod_Suc_iff*)
moreover have *f has_prod* (*a* ∗ *f 0*) **when**
(∃ *i q*. *a* = *0* ∧ *f* (*Suc i*) = *0* ∧ *raw_has_prod* (λ*n*. *f* (*Suc n*)) (*Suc i*) *q*)
proof −
from *that*
obtain *i q* **where** *a* = *0 f* (*Suc i*) = *0 raw_has_prod* (λ*n*. *f* (*Suc n*)) (*Suc i*) *q*
by *auto*
then show *?thesis* **unfolding** *has_prod_def*
by (*auto intro*!:*exI*[**where** *x*=*Suc i*] *simp*:*raw_has_prod_Suc*)
qed
ultimately show *f has_prod* (*a* ∗ *f 0*) **using** *assms* **unfolding** *has_prod_def* **by** *auto*
qed

lemma *has_prod_iff_shift*:
assumes ⋀*i*. *i* < *n* ⟹ *f i* ≠ *0*
shows (λ*i*. *f* (*i* + *n*)) *has_prod a* ⟷ *f has_prod* (*a* ∗ (∏ *i*<*n*. *f i*))
using *assms*
proof (*induct n arbitrary*: *a*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*)
then have (λ*i*. *f* (*Suc i* + *n*)) *has_prod a* ⟷ (λ*i*. *f* (*i* + *n*)) *has_prod* (*a* ∗ *f n*)
by (*subst has_prod_Suc_iff*) *auto*
with *Suc* **show** *?case*
by (*simp add*: *ac_simps*)
qed

corollary *has_prod_iff_shift'*:
assumes ⋀*i*. *i* < *n* ⟹ *f i* ≠ *0*
shows (λ*i*. *f* (*i* + *n*)) *has_prod* (*a* / (∏ *i*<*n*. *f i*)) ⟷ *f has_prod a*
by (*simp add*: *assms has_prod_iff_shift*)

lemma *has_prod_one_iff_shift*:
assumes ⋀*i*. *i* < *n* ⟹ *f i* = *1*
shows (λ*i*. *f* (*i*+*n*)) *has_prod a* ⟷ (λ*i*. *f i*) *has_prod a*
by (*simp add*: *assms has_prod_iff_shift*)

lemma *convergent_prod_iff_shift* [*simp*]:
shows *convergent_prod* (λ*i*. *f* (*i* + *n*)) ⟷ *convergent_prod f*
apply *safe*
using *convergent_prod_offset* **apply** *blast*
using *convergent_prod_ignore_initial_segment convergent_prod_def* **by** *blast*

lemma *has_prod_split_initial_segment*:
assumes *f has_prod a* ⋀*i*. *i* < *n* ⟹ *f i* ≠ *0*

shows (λ*i*. *f* (*i* + *n*)) *has_prod* (*a* / (∏*i*<*n*. *f i*))
using *assms has_prod_iff_shift'* **by** *blast*

lemma *prodinf_divide_initial_segment*:
assumes *convergent_prod f* ⋀*i*. *i* < *n* ⟹ *f i* ≠ *0*
shows (∏*i*. *f* (*i* + *n*)) = (∏*i*. *f i*) / (∏*i*<*n*. *f i*)
by (*rule has_prod_unique*[*symmetric*]) (*auto simp*: *assms has_prod_iff_shift*)

lemma *prodinf_split_initial_segment*:
assumes *convergent_prod f* ⋀*i*. *i* < *n* ⟹ *f i* ≠ *0*
shows *prodinf f* = (∏*i*. *f* (*i* + *n*)) ∗ (∏*i*<*n*. *f i*)
by (*auto simp add*: *assms prodinf_divide_initial_segment*)

lemma *prodinf_split_head*:
assumes *convergent_prod f f 0* ≠ *0*
shows (∏*n*. *f* (*Suc n*)) = *prodinf f* / *f 0*
using *prodinf_split_initial_segment*[*of 1*] *assms* **by** *simp*

end

context
fixes *f* :: *nat* ⇒ *'a*::*real_normed_field*
begin

lemma *convergent_prod_inverse_iff* [*simp*]: *convergent_prod* (λ*n*. *inverse* (*f n*)) ⟷ *convergent_prod f*
by (*auto dest*: *convergent_prod_inverse*)

lemma *convergent_prod_const_iff* [*simp*]:
fixes *c* :: *'a* :: {*real_normed_field*}
shows *convergent_prod* (λ_. *c*) ⟷ *c* = *1*
proof
assume *convergent_prod* (λ_. *c*)
then show *c* = *1*
using *convergent_prod_imp_LIMSEQ LIMSEQ_unique* **by** *blast*
next
assume *c* = *1*
then show *convergent_prod* (λ_. *c*)
by *auto*
qed

lemma *has_prod_power*: *f has_prod a* ⟹ (λ*i*. *f i* ^ *n*) *has_prod* (*a* ^ *n*)
by (*induction n*) (*auto simp*: *has_prod_mult*)

lemma *convergent_prod_power*: *convergent_prod f* ⟹ *convergent_prod* (λ*i*. *f i* ^ *n*)
by (*induction n*) (*auto simp*: *convergent_prod_mult*)

lemma *prodinf_power*: *convergent_prod f* ⟹ *prodinf* (λ*i*. *f i* ^ *n*) = *prodinf f* ^ *n*

by (*metis has_prod_unique convergent_prod_imp_has_prod has_prod_power*)

end

6.36.9 Exponentials and logarithms

context
fixes *f* :: *nat* ⇒ *'a*::{*real_normed_field*,*banach*}
begin

lemma *sums_imp_has_prod_exp*:
assumes *f sums s*
shows *raw_has_prod* (λ*i. exp* (*f i*)) *0* (*exp s*)
using *assms continuous_on_exp* [*of UNIV* λ*x*::*'a. x*]
using *continuous_on_tendsto_compose* [*of UNIV exp* (λ*n. sum f* {..*n*}) *s*]
by (*simp add*: *prod_defs sums_def_le exp_sum*)

lemma *convergent_prod_exp*:
assumes *summable f*
shows *convergent_prod* (λ*i. exp* (*f i*))
using *sums_imp_has_prod_exp assms* **unfolding** *summable_def convergent_prod_def*
by *blast*

lemma *prodinf_exp*:
assumes *summable f*
shows *prodinf* (λ*i. exp* (*f i*)) = *exp* (*suminf f*)
proof −
have *f sums suminf f*
using *assms* **by** *blast*
then have (λ*i. exp* (*f i*)) *has_prod exp* (*suminf f*)
by (*simp add*: *has_prod_def sums_imp_has_prod_exp*)
then show *?thesis*
by (*rule has_prod_unique* [*symmetric*])
qed

end

theorem *convergent_prod_iff_summable_real*:
fixes *a* :: *nat* ⇒ *real*
assumes ⋀*n. a n* > *0*
shows *convergent_prod* (λ*k. 1* + *a k*) ⟷ *summable a* (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then obtain *p* **where** *raw_has_prod* (λ*k. 1* + *a k*) *0 p*
by (*metis assms add_less_same_cancel2 convergent_prod_offset_0 not_one_less_zero*)
then have *to_p*: (λ*n.* ∏*k*≤*n. 1* + *a k*) ⟶ *p*
by (*auto simp*: *raw_has_prod_def*)
moreover have *le*: (∑*k*≤*n. a k*) ≤ (∏*k*≤*n. 1* + *a k*) **for** *n*
by (*rule sum_le_prod*) (*use assms less_le* **in** *force*)

have ($\prod k \le n.\ 1 + a\ k$) $\le$ *p* **for** *n*
proof (*rule incseq_le* [*OF* _ *to_p*])
show *incseq* ($\lambda n.\ \prod k \le n.\ 1 + a\ k$)
using *assms* **by** (*auto simp*: *mono_def order.strict_implies_order intro*!: *prod_mono2*)
qed
with *le* **have** ($\sum k \le n.\ a\ k$) $\le$ *p* **for** *n*
by (*metis order_trans*)
with *assms bounded_imp_summable* **show** *?rhs*
by (*metis not_less order.asym*)
next
assume *R*: *?rhs*
have ($\prod k \le n.\ 1 + a\ k$) $\le$ *exp* (*suminf a*) **for** *n*
proof –
have ($\prod k \le n.\ 1 + a\ k$) $\le$ *exp* ($\sum k \le n.\ a\ k$) **for** *n*
by (*rule prod_le_exp_sum*) (*use assms less_le* **in** *force*)
moreover have *exp* ($\sum k \le n.\ a\ k$) $\le$ *exp* (*suminf a*) **for** *n*
unfolding *exp_le_cancel_iff*
by (*meson sum_le_suminf R assms finite_atMost less_eq_real_def*)
ultimately show *?thesis*
by (*meson order_trans*)
qed
then obtain *L* **where** *L*: ($\lambda n.\ \prod k \le n.\ 1 + a\ k$) $\longrightarrow$ *L*
by (*metis assms bounded_imp_convergent_prod convergent_prod_iff_nz_lim le_add_same_cancel1 le_add_same_cancel2 less_le not_le zero_le_one*)
moreover have $L \neq 0$
proof
assume $L = 0$
with *L* **have** ($\lambda n.\ \prod k \le n.\ 1 + a\ k$) $\longrightarrow$ *0*
by *simp*
moreover have ($\prod k \le n.\ 1 + a\ k$) $>$ *1* **for** *n*
by (*simp add*: *assms less_1_prod*)
ultimately show *False*
by (*meson Lim_bounded2 not_one_le_zero less_imp_le*)
qed
ultimately show *?lhs*
using *assms convergent_prod_iff_nz_lim*
by (*metis add_less_same_cancel1 less_le not_le zero_less_one*)
qed

lemma *exp_suminf_prodinf_real*:
fixes *f* :: *nat* $\Rightarrow$ *real*
assumes *ge0*: $\bigwedge n.\ f\ n \ge 0$ **and** *ac*: *abs_convergent_prod* ($\lambda n.$ *exp* (*f n*))
shows *prodinf* ($\lambda i.$ *exp* (*f i*)) = *exp* (*suminf f*)
proof –
have *summable f*
using *ac* **unfolding** *abs_convergent_prod_conv_summable*
proof (*elim summable_comparison_test'*)
fix *n*

```
    have |f n| = f n
      by (simp add: ge0)
    also have ... ≤ exp (f n) − 1
      by (metis diff_diff_add exp_ge_add_one_self ge_iff_diff_ge_0)
    finally show norm (f n) ≤ norm (exp (f n) − 1)
      by simp
  qed
  then show ?thesis
    by (simp add: prodinf_exp)
qed

lemma has_prod_imp_sums_ln_real:
  fixes f :: nat ⇒ real
  assumes raw_has_prod f 0 p and 0: ⋀x. f x > 0
  shows (λi. ln (f i)) sums (ln p)
proof −
  have p > 0
   using assms unfolding prod_defs by (metis LIMSEQ_prod_nonneg less_eq_real_def)
  then show ?thesis
  using assms continuous_on_ln [of {0<..} λx. x]
  using continuous_on_tendsto_compose [of {0<..} ln (λn. prod f {..n}) p]
  by (auto simp: prod_defs sums_def_le ln_prod order_tendstoD)
qed

lemma summable_ln_real:
  fixes f :: nat ⇒ real
  assumes f: convergent_prod f and 0: ⋀x. f x > 0
  shows summable (λi. ln (f i))
proof −
  obtain M p where raw_has_prod f M p
    using f convergent_prod_def by blast
  then consider i where i<M f i = 0 | p where raw_has_prod f 0 p
    using raw_has_prod_cases by blast
  then show ?thesis
  proof cases
    case 1
    with 0 show ?thesis
      by (metis less_irrefl)
  next
    case 2
    then show ?thesis
      using 0 has_prod_imp_sums_ln_real summable_def by blast
  qed
qed

lemma suminf_ln_real:
  fixes f :: nat ⇒ real
  assumes f: convergent_prod f and 0: ⋀x. f x > 0
  shows suminf (λi. ln (f i)) = ln (prodinf f)
```

proof −
  have *f has_prod prodinf f*
    by (*simp add*: *f has_prod_iff*)
  then have *raw_has_prod f 0* (*prodinf f*)
    by (*metis 0 has_prod_def less_irrefl*)
  then have (λ*i*. *ln* (*f i*)) *sums ln* (*prodinf f*)
    using *0 has_prod_imp_sums_ln_real* **by** *blast*
  then show *?thesis*
    by (*rule sums_unique* [*symmetric*])
qed

lemma *prodinf_exp_real*:
  fixes *f* :: *nat* ⇒ *real*
  assumes *f*: *convergent_prod f* **and** *0*: ⋀*x*. *f x* > *0*
  shows *prodinf f* = *exp* (*suminf* (λ*i*. *ln* (*f i*)))
  by (*simp add*: *0 f less_0_prodinf suminf_ln_real*)

theorem *Ln_prodinf_complex*:
  fixes *z* :: *nat* ⇒ *complex*
  assumes *z*: ⋀*j*. *z j* ≠ *0* **and** ξ: ξ ≠ *0*
  shows ((λ*n*. ∏*j*≤*n*. *z j*) ⟶ ξ) ⟷ (∃ *k*. (λ*n*. (∑*j*≤*n*. *Ln* (*z j*))) ⟶ *Ln* ξ + *of_int k* ∗ (*of_real*(*2*∗*pi*) ∗ i)) (**is** *?lhs* = *?rhs*)
proof
  assume *L*: *?lhs*
  have *pnz*: (∏*j*≤*n*. *z j*) ≠ *0* **for** *n*
    using *z* **by** *auto*
  define Θ **where** Θ ≡ *Arg* ξ + *2*∗*pi*
  then have Θ > *pi*
    using *Arg_def mpi_less_Im_Ln* **by** *fastforce*
  have ξ*_eq*: ξ = *cmod* ξ ∗ *exp* (i ∗ Θ)
    using *Arg_def Arg_eq* ξ **unfolding** Θ*_def* **by** (*simp add*: *algebra_simps exp_add*)
  define ϑ **where** ϑ ≡ λ*n*. *THE t*. *is_Arg* (∏*j*≤*n*. *z j*) *t* ∧ *t* ∈ {Θ−*pi*<..Θ+*pi*}
  have *uniq*: ∃!*s*. *is_Arg* (∏*j*≤*n*. *z j*) *s* ∧ *s* ∈ {Θ−*pi*<..Θ+*pi*} **for** *n*
    using *Argument_exists_unique* [*OF pnz*] **by** *metis*
  have ϑ: *is_Arg* (∏*j*≤*n*. *z j*) (ϑ *n*) **and** ϑ*_interval*: ϑ *n* ∈ {Θ−*pi*<..Θ+*pi*} **for** *n*
    unfolding ϑ*_def*
    using *theI'* [*OF uniq*] **by** *metis+*
  have ϑ*_pos*: ⋀*j*. ϑ *j* > *0*
    using ϑ*_interval* ‹Θ > *pi*› **by** *simp* (*meson diff_gt_0_iff_gt less_trans*)
  have (∏*j*≤*n*. *z j*) = *cmod* (∏*j*≤*n*. *z j*) ∗ *exp* (i ∗ ϑ *n*) **for** *n*
    using ϑ **by** (*auto simp*: *is_Arg_def*)
  then have *eq*: (λ*n*. ∏*j*≤*n*. *z j*) = (λ*n*. *cmod* (∏*j*≤*n*. *z j*) ∗ *exp* (i ∗ ϑ *n*))
    by *simp*
  then have (λ*n*. (*cmod* (∏*j*≤*n*. *z j*)) ∗ *exp* (i ∗ (ϑ *n*))) ⟶ ξ
    using *L* **by** *force*
  then obtain *k* **where** *k*: (λ*j*. ϑ *j* − *of_int* (*k j*) ∗ (*2* ∗ *pi*)) ⟶ Θ
    using *L* **by** (*subst* (*asm*) ξ*_eq*) (*auto simp add*: *eq z* ξ *polar_convergence*)
  moreover have ∀$_F$ *n in sequentially*. *k n* = *0*

proof −
have ∗: *kj* = *0* **if** *dist* (*vj* − *real_of_int kj* ∗ *2*) *V* < *1 vj* ∈ {*V* − *1*<..*V* + *1*} **for** *kj vj V*
using *that* **by** (*auto simp*: *dist_norm*)
have $\forall_F$ *j in sequentially*. *dist* (ϑ *j* − *of_int* (*k j*) ∗ (*2* ∗ *pi*)) Θ < *pi*
using *tendstoD* [*OF k*] *pi_gt_zero* **by** *blast*
then show *?thesis*
proof (*rule eventually_mono*)
fix *j*
assume *d*: *dist* (ϑ *j* − *real_of_int* (*k j*) ∗ (*2* ∗ *pi*)) Θ < *pi*
show *k j* = *0*
by (*rule* ∗ [*of* ϑ *j*/*pi* _ Θ/*pi*])
(*use* ϑ*_interval* [*of j*] *d* **in** ‹*simp_all add*: *divide_simps dist_norm*›)
qed
qed
ultimately have ϑ*to*Θ: ϑ ⟶ Θ
apply (*simp only*: *tendsto_def*)
apply (*erule all_forward imp_forward asm_rl*)+
apply (*drule* (*1*) *eventually_conj*)
apply (*auto elim*: *eventually_mono*)
done
then have *to0*: (λ*n*. |ϑ (*Suc n*) − ϑ *n*|) ⟶ *0*
by (*metis* (*full_types*) *diff_self filterlim_sequentially_Suc tendsto_diff tendsto_rabs_zero*)
have ∃ *k*. *Im* (∑*j*≤*n*. *Ln* (*z j*)) − *of_int k* ∗ (*2*∗*pi*) = ϑ *n* **for** *n*
proof (*rule is_Arg_exp_diff_2pi*)
show *is_Arg* (*exp* (∑*j*≤*n*. *Ln* (*z j*))) (ϑ *n*)
using *pnz* ϑ **by** (*simp add*: *is_Arg_def exp_sum prod_norm*)
qed
then have ∃ *k*. (∑*j*≤*n*. *Im* (*Ln* (*z j*))) = ϑ *n* + *of_int k* ∗ (*2*∗*pi*) **for** *n*
by (*simp add*: *algebra_simps*)
then obtain *k* **where** *k*: ⋀*n*. (∑*j*≤*n*. *Im* (*Ln* (*z j*))) = ϑ *n* + *of_int* (*k n*) ∗ (*2*∗*pi*)
by *metis*
obtain *K* **where** $\forall_F$ *n in sequentially*. *k n* = *K*
proof −
have *k_le*: (*2*∗*pi*) ∗ |*k* (*Suc n*) − *k n*| ≤ |ϑ (*Suc n*) − ϑ *n*| + |*Im* (*Ln* (*z* (*Suc n*)))| **for** *n*
proof −
have (∑*j*≤*Suc n*. *Im* (*Ln* (*z j*))) − (∑*j*≤*n*. *Im* (*Ln* (*z j*))) = *Im* (*Ln* (*z* (*Suc n*)))
by *simp*
then show *?thesis*
using *k* [*of Suc n*] *k* [*of n*] **by** (*auto simp*: *abs_if algebra_simps*)
qed
have *z* ⟶ *1*
using *L* ξ *convergent_prod_iff_nz_lim z* **by** (*blast intro*: *convergent_prod_imp_LIMSEQ*)
with *z* **have** (λ*n*. *Ln* (*z n*)) ⟶ *Ln 1*
using *isCont_tendsto_compose* [*OF continuous_at_Ln*] *nonpos_Reals_one_I* **by** *blast*

```
    then have (λn. Ln (z n)) ⟶ 0
      by simp
    then have (λn. |Im (Ln (z (Suc n)))|) ⟶ 0
     by (metis LIMSEQ_unique ‹z ⟶ 1› continuous_at_Ln filterlim_sequentially_Suc
isCont_tendsto_compose nonpos_Reals_one_I tendsto_Im tendsto_rabs_zero_iff zero_complex.simps(2))
    then have ∀F n in sequentially. |Im (Ln (z (Suc n)))| < 1
      by (simp add: order_tendsto_iff)
    moreover have ∀F n in sequentially. |ϑ (Suc n) − ϑ n| < 1
      using to0 by (simp add: order_tendsto_iff)
    ultimately have ∀F n in sequentially. (2*pi) * |k (Suc n) − k n| < 1 + 1
    proof (rule eventually_elim2)
      fix n
      assume |Im (Ln (z (Suc n)))| < 1 and |ϑ (Suc n) − ϑ n| < 1
      with k_le [of n] show 2 * pi * real_of_int |k (Suc n) − k n| < 1 + 1
        by linarith
    qed
    then have ∀F n in sequentially. real_of_int|k (Suc n) − k n| < 1
    proof (rule eventually_mono)
      fix n :: nat
      assume 2 * pi * |k (Suc n) − k n| < 1 + 1
      then have |k (Suc n) − k n| < 2 / (2*pi)
        by (simp add: field_simps)
      also have ... < 1
        using pi_ge_two by auto
      finally show real_of_int |k (Suc n) − k n| < 1 .
    qed
  then obtain N where N: ⋀n. n≥N ⟹ |k (Suc n) − k n| = 0
    using eventually_sequentially less_irrefl of_int_abs by fastforce
  have k (N+i) = k N for i
  proof (induction i)
    case (Suc i)
    with N [of N+i] show ?case
      by auto
  qed simp
  then have ⋀n. n≥N ⟹ k n = k N
    using le_Suc_ex by auto
  then show ?thesis
    by (force simp add: eventually_sequentially intro: that)
  qed
  with ϑtoΘ have (λn. (∑j≤n. Im (Ln (z j)))) ⟶ Θ + of_int K * (2*pi)
    by (simp add: k tendsto_add tendsto_mult tendsto_eventually)
  moreover have (λn. (∑k≤n. Re (Ln (z k)))) ⟶ Re (Ln ξ)
    using assms continuous_imp_tendsto [OF isCont_ln tendsto_norm [OF L]]
    by (simp add: o_def flip: prod_norm ln_prod)
  ultimately show ?rhs
     by (rule_tac x=K+1 in exI) (auto simp: tendsto_complex_iff Θ_def Arg_def
assms algebra_simps)
next
  assume ?rhs
```

then obtain r **where** r: $(\lambda n.\ (\sum k{\leq}n.\ Ln\ (z\ k))) \longrightarrow Ln\ \xi + of\_int\ r * (of\_real(2{*}pi) * \mathrm{i})$ **..**
have $(\lambda n.\ exp\ (\sum k{\leq}n.\ Ln\ (z\ k))) \longrightarrow \xi$
using *assms continuous_imp_tendsto* [*OF isCont_exp r*] *exp_integer_2pi* [*of r*]
by (*simp add*: *o_def exp_add algebra_simps*)
moreover have $exp\ (\sum k{\leq}n.\ Ln\ (z\ k)) = (\prod k{\leq}n.\ z\ k)$ **for** n
by (*simp add*: *exp_sum add_eq_0_iff assms*)
ultimately show *?lhs*
by *auto*
qed

Prop 17.2 of Bak and Newman, Complex Analysis, p.242

proposition *convergent_prod_iff_summable_complex*:
fixes z :: *nat* ⇒ *complex*
assumes $\bigwedge k.\ z\ k \neq 0$
shows *convergent_prod* $(\lambda k.\ z\ k) \longleftrightarrow$ *summable* $(\lambda k.\ Ln\ (z\ k))$ (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then obtain p **where** p: $(\lambda n.\ \prod k{\leq}n.\ z\ k) \longrightarrow p$ **and** $p \neq 0$
using *convergent_prod_LIMSEQ prodinf_nonzero add_eq_0_iff assms* **by** *fastforce*
then show *?rhs*
using *Ln_prodinf_complex assms*
by (*auto simp*: *prodinf_nonzero summable_def sums_def_le*)
next
assume *R*: *?rhs*
have $(\prod k{\leq}n.\ z\ k) = exp\ (\sum k{\leq}n.\ Ln\ (z\ k))$ **for** n
by (*simp add*: *exp_sum add_eq_0_iff assms*)
then have $(\lambda n.\ \prod k{\leq}n.\ z\ k) \longrightarrow exp\ (suminf\ (\lambda k.\ Ln\ (z\ k)))$
using *continuous_imp_tendsto* [*OF isCont_exp summable_LIMSEQ'* [*OF R*]] **by** (*simp add*: *o_def*)
then show *?lhs*
by (*subst convergent_prod_iff_convergent*) (*auto simp*: *convergent_def tendsto_Lim assms add_eq_0_iff*)
qed

Prop 17.3 of Bak and Newman, Complex Analysis

proposition *summable_imp_convergent_prod_complex*:
fixes z :: *nat* ⇒ *complex*
assumes *z*: *summable* $(\lambda k.\ norm\ (z\ k))$ **and** *non0*: $\bigwedge k.\ z\ k \neq -1$
shows *convergent_prod* $(\lambda k.\ 1 + z\ k)$
proof −
note *if_cong* [*cong*] *power_Suc* [*simp del*]
obtain N **where** N: $\bigwedge k.\ k{\geq}N \Longrightarrow norm\ (z\ k) < 1/2$
using *summable_LIMSEQ_zero* [*OF z*]
by (*metis diff_zero dist_norm half_gt_zero_iff less_numeral_extra*(*1*) *lim_sequentially tendsto_norm_zero_iff*)
have $norm\ (Ln\ (1 + z\ k)) \leq 2 * norm\ (z\ k)$ **if** $k \geq N$ **for** k
proof (*cases* $z\ k = 0$)
case *False*

```
    let ?f = λi. cmod ((− 1) ^ i ∗ z k ^ i / of_nat (Suc i))
    have normf: norm (?f n) ≤ (1 / 2) ^ n for n
    proof −
      have norm (?f n) = cmod (z k) ^ n / cmod (1 + of_nat n)
        by (auto simp: norm_divide norm_mult norm_power)
      also have . . . ≤ cmod (z k) ^ n
        by (auto simp: field_split_simps mult_le_cancel_left1 in_Reals_norm)
      also have . . . ≤ (1 / 2) ^ n
        using N [OF that] by (simp add: power_mono)
      finally show norm (?f n) ≤ (1 / 2) ^ n .
    qed
    have summablef: summable ?f
     by (intro normf summable_comparison_test' [OF summable_geometric [of 1/2]])
auto
    have (λn. (− 1) ^ Suc n / of_nat n ∗ z k ^ n) sums Ln (1 + z k)
      using Ln_series [of z k] N that by fastforce
    then have ∗: (λi. z k ∗ (((− 1) ^ i ∗ z k ^ i) / (Suc i))) sums Ln (1 + z k)
       using sums_split_initial_segment [where n= 1]  by (force simp: power_Suc
mult_ac)
    then have norm (Ln (1 + z k)) = norm (suminf (λi. z k ∗ (((− 1) ^ i ∗ z k
^ i) / (Suc i))))
      using sums_unique by force
    also have . . . = norm (z k ∗ suminf (λi. ((− 1) ^ i ∗ z k ^ i) / (Suc i)))
      apply (subst suminf_mult)
      using ∗ False
      by (auto simp: sums_summable intro: summable_mult_D [of z k])
    also have . . . = norm (z k) ∗ norm (suminf (λi. ((− 1) ^ i ∗ z k ^ i) / (Suc
i)))
      by (simp add: norm_mult)
    also have . . . ≤ norm (z k) ∗ suminf (λi. norm (((− 1) ^ i ∗ z k ^ i) / (Suc
i)))
      by (intro mult_left_mono summable_norm summablef) auto
    also have . . . ≤ norm (z k) ∗ suminf (λi. (1/2) ^ i)
     by (intro mult_left_mono suminf_le) (use summable_geometric [of 1/2] summablef
normf in auto)
    also have . . . ≤ norm (z k) ∗ 2
      using suminf_geometric [of 1/2::real] by simp
    finally show ?thesis
      by (simp add: mult_ac)
  qed simp
  then have summable (λk. Ln (1 + z k))
    by (metis summable_comparison_test summable_mult z)
  with non0 show ?thesis
    by (simp add: add_eq_0_iff convergent_prod_iff_summable_complex)
qed

lemma summable_Ln_complex:
  fixes z :: nat ⇒ complex
  assumes convergent_prod z ⋀k. z k ≠ 0
```

shows *summable* (λ*k*. *Ln* (*z k*))
using *convergent_prod_def assms convergent_prod_iff_summable_complex* **by** *blast*

6.36.10 Embeddings from the reals into some complete real normed field

lemma *tendsto_eq_of_real_lim*:
assumes (λ*n*. *of_real* (*f n*) :: '*a*::{*complete_space*,*real_normed_field*}) ⟶ *q*
shows *q* = *of_real* (*lim f*)
proof −
have *convergent* (λ*n*. *of_real* (*f n*) :: '*a*)
using *assms convergent_def* **by** *blast*
then have *convergent f*
unfolding *convergent_def*
by (*simp add*: *convergent_eq_Cauchy Cauchy_def*)
then show *?thesis*
by (*metis LIMSEQ_unique assms convergentD sequentially_bot tendsto_Lim tendsto_of_real*)
qed

lemma *tendsto_eq_of_real*:
assumes (λ*n*. *of_real* (*f n*) :: '*a*::{*complete_space*,*real_normed_field*}) ⟶ *q*
obtains *r* **where** *q* = *of_real r*
using *tendsto_eq_of_real_lim assms* **by** *blast*

lemma *has_prod_of_real_iff* [*simp*]:
(λ*n*. *of_real* (*f n*) :: '*a*::{*complete_space*,*real_normed_field*}) *has_prod of_real c* ⟷ *f has_prod c*
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
apply (*auto simp*: *prod_defs LIMSEQ_prod_0 tendsto_of_real_iff simp flip*: *of_real_prod*)
using *tendsto_eq_of_real*
by (*metis of_real_0 tendsto_of_real_iff*)
next
assume *?rhs*
with *tendsto_of_real_iff* **show** *?lhs*
by (*fastforce simp*: *prod_defs simp flip*: *of_real_prod*)
qed

end

6.37 Sums over Infinite Sets

theory *Infinite_Set_Sum*
imports *Set_Integral*
begin

lemma *sets_eq_countable*:
  assumes *countable A space M = A* $\bigwedge x.\ x \in A \Longrightarrow \{x\} \in$ *sets M*
  shows *sets M = Pow A*
proof (*intro equalityI subsetI*)
  fix *X* **assume** *X* ∈ *Pow A*
  hence ($\bigcup x \in X.\ \{x\}$) ∈ *sets M*
    by (*intro sets.countable_UN′ countable_subset*[*OF _ assms*(*1*)]) (*auto intro*!: *assms*(*3*))
  also have ($\bigcup x \in X.\ \{x\}$) = *X* **by** *auto*
  finally show *X* ∈ *sets M* **.**
next
  fix *X* **assume** *X* ∈ *sets M*
  from *sets.sets_into_space*[*OF this*] **and** *assms*
    show *X* ∈ *Pow A* **by** *simp*
qed

lemma *measure_eqI_countable′*:
  assumes *spaces*: *space M = A space N = A*
  assumes *sets*: $\bigwedge x.\ x \in A \Longrightarrow \{x\} \in$ *sets M* $\bigwedge x.\ x \in A \Longrightarrow \{x\} \in$ *sets N*
  assumes *A*: *countable A*
  assumes *eq*: $\bigwedge a.\ a \in A \Longrightarrow$ *emeasure M* {*a*} = *emeasure N* {*a*}
  shows *M = N*
proof (*rule measure_eqI_countable*)
  show *sets M = Pow A*
    by (*intro sets_eq_countable assms*)
  show *sets N = Pow A*
    by (*intro sets_eq_countable assms*)
qed *fact*+

lemma *count_space_PiM_finite*:
  fixes *B* :: ′*a* ⇒ ′*b set*
  assumes *finite A* $\bigwedge i.$ *countable* (*B i*)
  shows *PiM A* (λ*i*. *count_space* (*B i*)) = *count_space* (*PiE A B*)
proof (*rule measure_eqI_countable′*)
  show *space* (*PiM A* (λ*i*. *count_space* (*B i*))) = *PiE A B*
    by (*simp add*: *space_PiM*)
  show *space* (*count_space* (*PiE A B*)) = *PiE A B* **by** *simp*
next
  fix *f* **assume** *f*: *f* ∈ *PiE A B*
  hence *PiE A* (λ*x*. {*f x*}) ∈ *sets* (Pi_M *A* (λ*i*. *count_space* (*B i*)))
    by (*intro sets_PiM_I_finite assms*) *auto*
  also from *f* **have** *PiE A* (λ*x*. {*f x*}) = {*f*}
    by (*intro PiE_singleton*) (*auto simp*: *PiE_def*)
  finally show {*f*} ∈ *sets* (Pi_M *A* (λ*i*. *count_space* (*B i*))) **.**
next
  interpret *product_sigma_finite* (λ*i*. *count_space* (*B i*))
    by (*intro product_sigma_finite.intro sigma_finite_measure_count_space_countable assms*)

thm *sigma_finite_measure_count_space*
fix *f* **assume** *f*: *f* ∈ *PiE A B*
hence {*f*} = *PiE A* (λ*x*. {*f x*})
by (*intro PiE_singleton* [*symmetric*]) (*auto simp*: *PiE_def*)
also have *emeasure* (Pi_M *A* (λ*i*. *count_space* (*B i*))) ... =
(∏ *i*∈*A*. *emeasure* (*count_space* (*B i*)) {*f i*})
using *f assms* **by** (*subst emeasure_PiM*) *auto*
also have ... = (∏ *i*∈*A*. *1*)
by (*intro prod.cong refl*, *subst emeasure_count_space_finite*) (*use f* **in** *auto*)
also have ... = *emeasure* (*count_space* (*PiE A B*)) {*f*}
using *f* **by** (*subst emeasure_count_space_finite*) *auto*
finally show *emeasure* (Pi_M *A* (λ*i*. *count_space* (*B i*))) {*f*} =
emeasure (*count_space* (Pi_E *A B*)) {*f*} **.**
qed (*simp_all add*: *countable_PiE assms*)

definition *abs_summable_on* ::
(′*a* ⇒ ′*b* :: {*banach*, *second_countable_topology*}) ⇒ ′*a set* ⇒ *bool*
(**infix** *abs′_summable′_on 50*)
where
f abs_summable_on A ⟷ *integrable* (*count_space A*) *f*

definition *infsetsum* ::
(′*a* ⇒ ′*b* :: {*banach*, *second_countable_topology*}) ⇒ ′*a set* ⇒ ′*b*
where
infsetsum f A = *lebesgue_integral* (*count_space A*) *f*

syntax (*ASCII*)
_infsetsum :: *pttrn* ⇒ ′*a set* ⇒ ′*b* ⇒ ′*b*::{*banach*, *second_countable_topology*}
((*3INFSETSUM* _:_./ _) [*0*, *51*, *10*] *10*)
syntax
_infsetsum :: *pttrn* ⇒ ′*a set* ⇒ ′*b* ⇒ ′*b*::{*banach*, *second_countable_topology*}
((*2*∑ $_a$_∈_./ _) [*0*, *51*, *10*] *10*)
translations — Beware of argument permutation!
∑ $_a$*i*∈*A*. *b* ⇌ *CONST infsetsum* (λ*i*. *b*) *A*

syntax (*ASCII*)
_uinfsetsum :: *pttrn* ⇒ ′*a set* ⇒ ′*b* ⇒ ′*b*::{*banach*, *second_countable_topology*}
((*3INFSETSUM* _:_./ _) [*0*, *51*, *10*] *10*)
syntax
_uinfsetsum :: *pttrn* ⇒ ′*b* ⇒ ′*b*::{*banach*, *second_countable_topology*}
((*2*∑ $_a$_./ _) [*0*, *10*] *10*)
translations — Beware of argument permutation!
∑ $_a$*i*. *b* ⇌ *CONST infsetsum* (λ*i*. *b*) (*CONST UNIV*)

syntax (*ASCII*)
_qinfsetsum :: *pttrn* ⇒ *bool* ⇒ ′*a* ⇒ ′*a*::{*banach*, *second_countable_topology*}

((*3INFSETSUM* _ |/ _./ _) [*0*, *0*, *10*] *10*)
syntax
 _qinfsetsum :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a*::{*banach*, *second_countable_topology*}
 ((*2*$\sum_{a}$ _ | (_)./ _) [*0*, *0*, *10*] *10*)
translations
 $\sum_{a}x|P.\ t$ => *CONST infsetsum* (λ*x*. *t*) {*x*. *P*}

print_translation ‹
let
 fun sum_tr' [*Abs* (*x*, *Tx*, *t*), *Const* (***const_syntax***‹*Collect*›, _) $ *Abs* (*y*, *Ty*, *P*)]
=
 if x <> y then raise Match
 else
 let
 val x' = Syntax_Trans.mark_bound_body (*x*, *Tx*);
 val t' = subst_bound (*x'*, *t*);
 val P' = subst_bound (*x'*, *P*);
 in
 Syntax.const ***syntax_const***‹*_qinfsetsum*› $ *Syntax_Trans.mark_bound_abs*
(*x*, *Tx*) $ *P'* $ *t'*
 end
 | *sum_tr'* _ = *raise Match*;
in [(***const_syntax***‹*infsetsum*›, *K sum_tr'*)] *end*
›

lemma *restrict_count_space_subset*:
 A ⊆ *B* ⟹ *restrict_space* (*count_space B*) *A* = *count_space A*
 by (*subst restrict_count_space*) (*simp_all add*: *Int_absorb2*)

lemma *abs_summable_on_restrict*:
 fixes *f* :: *'a* ⇒ *'b* :: {*banach*, *second_countable_topology*}
 assumes *A* ⊆ *B*
 shows *f abs_summable_on A* ⟷ (λ*x*. *indicator A x* $*_R$ *f x*) *abs_summable_on*
B
proof −
 have *count_space A* = *restrict_space* (*count_space B*) *A*
 by (*rule restrict_count_space_subset* [*symmetric*]) *fact+*
 also have *integrable* ... *f* ⟷ *set_integrable* (*count_space B*) *A f*
 by (*simp add*: *integrable_restrict_space set_integrable_def*)
 finally show *?thesis*
 unfolding *abs_summable_on_def set_integrable_def* **.**
qed

lemma *abs_summable_on_altdef*: *f abs_summable_on A* ⟷ *set_integrable* (*count_space*
UNIV) *A f*
 unfolding *abs_summable_on_def set_integrable_def*
 by (*metis* (*no_types*) *inf_top.right_neutral integrable_restrict_space restrict_count_space*
sets_UNIV)

lemma *abs_summable_on_altdef'*:
$A \subseteq B \Longrightarrow$ *f abs_summable_on* $A \longleftrightarrow$ *set_integrable* (*count_space B*) *A f*
unfolding *abs_summable_on_def set_integrable_def*
by (*metis* (*no_types*) *Pow_iff abs_summable_on_def inf.orderE integrable_restrict_space restrict_count_space_subset sets_count_space space_count_space*)

lemma *abs_summable_on_norm_iff* [*simp*]:
($\lambda x.$ *norm* (*f x*)) *abs_summable_on* $A \longleftrightarrow$ *f abs_summable_on A*
by (*simp add*: *abs_summable_on_def integrable_norm_iff*)

lemma *abs_summable_on_normI*: *f abs_summable_on* $A \Longrightarrow$ ($\lambda x.$ *norm* (*f x*)) *abs_summable_on A*
by *simp*

lemma *abs_summable_complex_of_real* [*simp*]: ($\lambda n.$ *complex_of_real* (*f n*)) *abs_summable_on* $A \longleftrightarrow$ *f abs_summable_on A*
by (*simp add*: *abs_summable_on_def complex_of_real_integrable_eq*)

lemma *abs_summable_on_comparison_test*:
assumes *g abs_summable_on A*
assumes $\bigwedge x.\ x \in A \Longrightarrow norm\ (f\ x) \leq norm\ (g\ x)$
shows *f abs_summable_on A*
using *assms Bochner_Integration.integrable_bound*[*of count_space A g f*]
unfolding *abs_summable_on_def* **by** (*auto simp*: *AE_count_space*)

lemma *abs_summable_on_comparison_test'*:
assumes *g abs_summable_on A*
assumes $\bigwedge x.\ x \in A \Longrightarrow norm\ (f\ x) \leq g\ x$
shows *f abs_summable_on A*
proof (*rule abs_summable_on_comparison_test*[*OF assms*(*1*), *of f*])
fix *x* **assume** $x \in A$
with *assms*(*2*) **have** $norm\ (f\ x) \leq g\ x$ **.**
also have $\ldots \leq norm\ (g\ x)$ **by** *simp*
finally show $norm\ (f\ x) \leq norm\ (g\ x)$ **.**
qed

lemma *abs_summable_on_cong* [*cong*]:
($\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x$) $\Longrightarrow A = B \Longrightarrow$ (*f abs_summable_on A*) $\longleftrightarrow$ (*g abs_summable_on B*)
unfolding *abs_summable_on_def* **by** (*intro integrable_cong*) *auto*

lemma *abs_summable_on_cong_neutral*:
assumes $\bigwedge x.\ x \in A - B \Longrightarrow f\ x = 0$
assumes $\bigwedge x.\ x \in B - A \Longrightarrow g\ x = 0$
assumes $\bigwedge x.\ x \in A \cap B \Longrightarrow f\ x = g\ x$
shows *f abs_summable_on* $A \longleftrightarrow$ *g abs_summable_on B*
unfolding *abs_summable_on_altdef set_integrable_def* **using** *assms*
by (*intro Bochner_Integration.integrable_cong refl*)

(*auto simp*: *indicator_def split*: *if_splits*)

lemma *abs_summable_on_restrict'*:
fixes *f* :: *'a* ⇒ *'b* :: {*banach*, *second_countable_topology*}
assumes $A \subseteq B$
shows *f abs_summable_on A* ⟷ (λ*x*. *if x* ∈ *A then f x else 0*) *abs_summable_on B*
by (*subst abs_summable_on_restrict*[*OF assms*]) (*intro abs_summable_on_cong*, *auto*)

lemma *abs_summable_on_nat_iff*:
f abs_summable_on (*A* :: *nat set*) ⟷ *summable* (λ*n*. *if n* ∈ *A then norm* (*f n*) *else 0*)
proof –
have *f abs_summable_on A* ⟷ *summable* (λ*x*. *norm* (*if x* ∈ *A then f x else 0*))
by (*subst abs_summable_on_restrict'*[*of _ UNIV*])
(*simp_all add*: *abs_summable_on_def integrable_count_space_nat_iff*)
also have (λ*x*. *norm* (*if x* ∈ *A then f x else 0*)) = (λ*x*. *if x* ∈ *A then norm* (*f x*) *else 0*)
by *auto*
finally show *?thesis* **.**
qed

lemma *abs_summable_on_nat_iff'*:
f abs_summable_on (*UNIV* :: *nat set*) ⟷ *summable* (λ*n*. *norm* (*f n*))
by (*subst abs_summable_on_nat_iff*) *auto*

lemma *nat_abs_summable_on_comparison_test*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *second_countable_topology*}
assumes *g abs_summable_on I*
assumes ⋀*n*. ⟦*n*≥*N*; *n* ∈ *I*⟧ ⟹ *norm* (*f n*) ≤ *g n*
shows *f abs_summable_on I*
using *assms* **by** (*fastforce simp add*: *abs_summable_on_nat_iff intro*: *summable_comparison_test'*)

lemma *abs_summable_comparison_test_ev*:
assumes *g abs_summable_on I*
assumes *eventually* (λ*x*. *x* ∈ *I* ⟶ *norm* (*f x*) ≤ *g x*) *sequentially*
shows *f abs_summable_on I*
by (*metis* (*no_types*, *lifting*) *nat_abs_summable_on_comparison_test eventually_at_top_linorder assms*)

lemma *abs_summable_on_Cauchy*:
f abs_summable_on (*UNIV* :: *nat set*) ⟷ (∀ *e*>*0*. ∃ *N*. ∀ *m*≥*N*. ∀ *n*. (∑ *x* = *m*..<*n*. *norm* (*f x*)) < *e*)
by (*simp add*: *abs_summable_on_nat_iff' summable_Cauchy sum_nonneg*)

lemma *abs_summable_on_finite* [*simp*]: *finite A* ⟹ *f abs_summable_on A*
unfolding *abs_summable_on_def* **by** (*rule integrable_count_space*)

lemma *abs_summable_on_empty* [*simp*, *intro*]: *f abs_summable_on* {}
by *simp*

lemma *abs_summable_on_subset*:
assumes *f abs_summable_on B* **and** $A \subseteq B$
shows *f abs_summable_on A*
unfolding *abs_summable_on_altdef*
by (*rule set_integrable_subset*) (*insert assms*, *auto simp*: *abs_summable_on_altdef*)

lemma *abs_summable_on_union* [*intro*]:
assumes *f abs_summable_on A* **and** *f abs_summable_on B*
shows *f abs_summable_on* $(A \cup B)$
using *assms* **unfolding** *abs_summable_on_altdef* **by** (*intro set_integrable_Un*) *auto*

lemma *abs_summable_on_insert_iff* [*simp*]:
f abs_summable_on insert x A $\longleftrightarrow$ *f abs_summable_on A*
proof *safe*
assume *f abs_summable_on insert x A*
thus *f abs_summable_on A*
by (*rule abs_summable_on_subset*) *auto*
next
assume *f abs_summable_on A*
from *abs_summable_on_union*[*OF this*, *of* {*x*}]
show *f abs_summable_on insert x A* **by** *simp*
qed

lemma *abs_summable_sum*:
assumes $\bigwedge x.\ x \in A \Longrightarrow$ *f x abs_summable_on B*
shows $(\lambda y.\ \sum x{\in}A.\ f\ x\ y)$ *abs_summable_on B*
using *assms* **unfolding** *abs_summable_on_def* **by** (*intro Bochner_Integration.integrable_sum*)

lemma *abs_summable_Re*: *f abs_summable_on A* $\Longrightarrow$ $(\lambda x.\ Re\ (f\ x))$ *abs_summable_on A*
by (*simp add*: *abs_summable_on_def*)

lemma *abs_summable_Im*: *f abs_summable_on A* $\Longrightarrow$ $(\lambda x.\ Im\ (f\ x))$ *abs_summable_on A*
by (*simp add*: *abs_summable_on_def*)

lemma *abs_summable_on_finite_diff*:
assumes *f abs_summable_on A* $A \subseteq B$ *finite* $(B - A)$
shows *f abs_summable_on B*
proof −
have *f abs_summable_on* $(A \cup (B - A))$
by (*intro abs_summable_on_union assms abs_summable_on_finite*)
also from *assms* **have** $A \cup (B - A) = B$ **by** *blast*
finally show *?thesis* **.**
qed

lemma *abs_summable_on_reindex_bij_betw*:
  assumes *bij_betw g A B*
  shows (λ*x*. *f* (*g x*)) *abs_summable_on A* ⟷ *f abs_summable_on B*
proof −
  have ∗: *count_space B* = *distr* (*count_space A*) (*count_space B*) *g*
    by (*rule distr_bij_count_space* [*symmetric*]) *fact*
  show *?thesis* **unfolding** *abs_summable_on_def*
    by (*subst* ∗, *subst integrable_distr_eq*[*of* _ _ *count_space B*])
      (*insert assms*, *auto simp*: *bij_betw_def*)
qed

lemma *abs_summable_on_reindex*:
  assumes (λ*x*. *f* (*g x*)) *abs_summable_on A*
  shows *f abs_summable_on* (*g* ‘ *A*)
proof −
  define *g'* **where** *g'* = *inv_into A g*
  from *assms* **have** (λ*x*. *f* (*g x*)) *abs_summable_on* (*g'* ‘ *g* ‘ *A*)
    by (*rule abs_summable_on_subset*) (*auto simp*: *g'_def inv_into_into*)
  also have *?this* ⟷ (λ*x*. *f* (*g* (*g' x*))) *abs_summable_on* (*g* ‘ *A*) **unfolding** *g'_def*
    by (*intro abs_summable_on_reindex_bij_betw* [*symmetric*] *inj_on_imp_bij_betw inj_on_inv_into*) *auto*
  also have ... ⟷ *f abs_summable_on* (*g* ‘ *A*)
    by (*intro abs_summable_on_cong refl*) (*auto simp*: *g'_def f_inv_into_f*)
  finally show *?thesis* **.**
qed

lemma *abs_summable_on_reindex_iff*:
  inj_on g A ⟹ (λ*x*. *f* (*g x*)) *abs_summable_on A* ⟷ *f abs_summable_on* (*g* ‘ *A*)
  by (*intro abs_summable_on_reindex_bij_betw inj_on_imp_bij_betw*)

lemma *abs_summable_on_Sigma_project2*:
  fixes *A* :: *'a set* **and** *B* :: *'a* ⇒ *'b set*
  assumes *f abs_summable_on* (*Sigma A B*) *x* ∈ *A*
  shows (λ*y*. *f* (*x*, *y*)) *abs_summable_on* (*B x*)
proof −
  from *assms*(*2*) **have** *f abs_summable_on* (*Sigma* {*x*} *B*)
    by (*intro abs_summable_on_subset* [*OF assms*(*1*)]) *auto*
  also have *?this* ⟷ (λ*z*. *f* (*x*, *snd z*)) *abs_summable_on* (*Sigma* {*x*} *B*)
    by (*rule abs_summable_on_cong*) *auto*
  finally have (λ*y*. *f* (*x*, *y*)) *abs_summable_on* (*snd* ‘ *Sigma* {*x*} *B*)
    by (*rule abs_summable_on_reindex*)
  also have *snd* ‘ *Sigma* {*x*} *B* = *B x*
    using *assms* **by** (*auto simp*: *image_iff*)
  finally show *?thesis* **.**
qed

lemma *abs_summable_on_Times_swap*:
  f abs_summable_on A × *B* ⟷ (λ(*x*,*y*). *f* (*y*,*x*)) *abs_summable_on B* × *A*

proof −
  have *bij*: *bij_betw* (λ(*x*,*y*). (*y*,*x*)) (*B* × *A*) (*A* × *B*)
    by (*auto simp*: *bij_betw_def inj_on_def*)
  show *?thesis*
    by (*subst abs_summable_on_reindex_bij_betw*[*OF bij*, *of f*, *symmetric*])
      (*simp_all add*: *case_prod_unfold*)
qed

lemma *abs_summable_on_0* [*simp*, *intro*]: (λ_. *0*) *abs_summable_on A*
  by (*simp add*: *abs_summable_on_def*)

lemma *abs_summable_on_uminus* [*intro*]:
  f abs_summable_on A ⟹ (λ*x*. −*f x*) *abs_summable_on A*
  unfolding *abs_summable_on_def* **by** (*rule Bochner_Integration.integrable_minus*)

lemma *abs_summable_on_add* [*intro*]:
  assumes *f abs_summable_on A* **and** *g abs_summable_on A*
  shows (λ*x*. *f x* + *g x*) *abs_summable_on A*
  using *assms* **unfolding** *abs_summable_on_def* **by** (*rule Bochner_Integration.integrable_add*)

lemma *abs_summable_on_diff* [*intro*]:
  assumes *f abs_summable_on A* **and** *g abs_summable_on A*
  shows (λ*x*. *f x* − *g x*) *abs_summable_on A*
  using *assms* **unfolding** *abs_summable_on_def* **by** (*rule Bochner_Integration.integrable_diff*)

lemma *abs_summable_on_scaleR_left* [*intro*]:
  assumes *c* ≠ *0* ⟹ *f abs_summable_on A*
  shows (λ*x*. *f x* $*_R$ *c*) *abs_summable_on A*
  using *assms* **unfolding** *abs_summable_on_def* **by** (*intro Bochner_Integration.integrable_scaleR_left*)

lemma *abs_summable_on_scaleR_right* [*intro*]:
  assumes *c* ≠ *0* ⟹ *f abs_summable_on A*
  shows (λ*x*. *c* $*_R$ *f x*) *abs_summable_on A*
  using *assms* **unfolding** *abs_summable_on_def* **by** (*intro Bochner_Integration.integrable_scaleR_right*)

lemma *abs_summable_on_cmult_right* [*intro*]:
  fixes *f* :: '*a* ⇒ '*b* :: {*banach*, *real_normed_algebra*, *second_countable_topology*}
  assumes *c* ≠ *0* ⟹ *f abs_summable_on A*
  shows (λ*x*. *c* ∗ *f x*) *abs_summable_on A*
  using *assms* **unfolding** *abs_summable_on_def* **by** (*intro Bochner_Integration.integrable_mult_right*)

lemma *abs_summable_on_cmult_left* [*intro*]:
  fixes *f* :: '*a* ⇒ '*b* :: {*banach*, *real_normed_algebra*, *second_countable_topology*}
  assumes *c* ≠ *0* ⟹ *f abs_summable_on A*
  shows (λ*x*. *f x* ∗ *c*) *abs_summable_on A*
  using *assms* **unfolding** *abs_summable_on_def* **by** (*intro Bochner_Integration.integrable_mult_left*)

lemma *abs_summable_on_prod_PiE*:
  fixes *f* :: '*a* ⇒ '*b* ⇒ '*c* :: {*real_normed_field*,*banach*,*second_countable_topology*}

assumes *finite*: *finite A* **and** *countable*: $\bigwedge x.\ x \in A \Longrightarrow$ *countable* (*B x*)
assumes *summable*: $\bigwedge x.\ x \in A \Longrightarrow$ *f x abs_summable_on B x*
shows ($\lambda g.\ \prod x{\in}A.\ f\ x\ (g\ x)$) *abs_summable_on PiE A B*
proof −
define *B′* **where** *B′* = ($\lambda x.$ *if* $x \in A$ *then B x else* {})
from *assms* **have** [*simp*]: *countable* (*B′ x*) **for** *x*
by (*auto simp*: *B′_def*)
then interpret *product_sigma_finite count_space* ∘ *B′*
unfolding *o_def* **by** (*intro product_sigma_finite.intro sigma_finite_measure_count_space_countable*)
from *assms* **have** *integrable* (*PiM A* (*count_space* ∘ *B′*)) ($\lambda g.\ \prod x{\in}A.\ f\ x\ (g\ x)$)
by (*intro product_integrable_prod*) (*auto simp*: *abs_summable_on_def B′_def*)
also have *PiM A* (*count_space* ∘ *B′*) = *count_space* (*PiE A B′*)
unfolding *o_def* **using** *finite* **by** (*intro count_space_PiM_finite*) *simp_all*
also have *PiE A B′* = *PiE A B* **by** (*intro PiE_cong*) (*simp_all add*: *B′_def*)
finally show *?thesis* **by** (*simp add*: *abs_summable_on_def*)
qed

lemma *not_summable_infsetsum_eq*:
¬*f abs_summable_on A* ⟹ *infsetsum f A* = *0*
by (*simp add*: *abs_summable_on_def infsetsum_def not_integrable_integral_eq*)

lemma *infsetsum_altdef*:
infsetsum f A = *set_lebesgue_integral* (*count_space UNIV*) *A f*
unfolding *set_lebesgue_integral_def*
by (*subst integral_restrict_space* [*symmetric*])
(*auto simp*: *restrict_count_space_subset infsetsum_def*)

lemma *infsetsum_altdef′*:
A ⊆ *B* ⟹ *infsetsum f A* = *set_lebesgue_integral* (*count_space B*) *A f*
unfolding *set_lebesgue_integral_def*
by (*subst integral_restrict_space* [*symmetric*])
(*auto simp*: *restrict_count_space_subset infsetsum_def*)

lemma *nn_integral_conv_infsetsum*:
assumes *f abs_summable_on A* $\bigwedge x.\ x \in A \Longrightarrow f\ x \geq 0$
shows *nn_integral* (*count_space A*) *f* = *ennreal* (*infsetsum f A*)
using *assms* **unfolding** *infsetsum_def abs_summable_on_def*
by (*subst nn_integral_eq_integral*) *auto*

lemma *infsetsum_conv_nn_integral*:
assumes *nn_integral* (*count_space A*) *f* ≠ ∞ $\bigwedge x.\ x \in A \Longrightarrow f\ x \geq 0$
shows *infsetsum f A* = *enn2real* (*nn_integral* (*count_space A*) *f*)
unfolding *infsetsum_def* **using** *assms*
by (*subst integral_eq_nn_integral*) *auto*

lemma *infsetsum_cong* [*cong*]:
($\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x$) ⟹ *A* = *B* ⟹ *infsetsum f A* = *infsetsum g B*

unfolding *infsetsum_def* **by** (*intro Bochner_Integration.integral_cong*) *auto*

lemma *infsetsum_0* [*simp*]: *infsetsum* (λ_. *0*) *A* = *0*
by (*simp add*: *infsetsum_def*)

lemma *infsetsum_all_0*: (⋀*x*. *x* ∈ *A* ⟹ *f x* = *0*) ⟹ *infsetsum f A* = *0*
by *simp*

lemma *infsetsum_nonneg*: (⋀*x*. *x* ∈ *A* ⟹ *f x* ≥ (*0*::*real*)) ⟹ *infsetsum f A* ≥ *0*
unfolding *infsetsum_def* **by** (*rule Bochner_Integration.integral_nonneg*) *auto*

lemma *sum_infsetsum*:
assumes ⋀*x*. *x* ∈ *A* ⟹ *f x abs_summable_on B*
shows (∑*x*∈*A*. ∑$_a$*y*∈*B*. *f x y*) = (∑$_a$*y*∈*B*. ∑*x*∈*A*. *f x y*)
using *assms* **by** (*simp add*: *infsetsum_def abs_summable_on_def Bochner_Integration.integral_sum*)

lemma *Re_infsetsum*: *f abs_summable_on A* ⟹ *Re* (*infsetsum f A*) = (∑$_a$*x*∈*A*.
Re (*f x*))
by (*simp add*: *infsetsum_def abs_summable_on_def*)

lemma *Im_infsetsum*: *f abs_summable_on A* ⟹ *Im* (*infsetsum f A*) = (∑$_a$*x*∈*A*.
Im (*f x*))
by (*simp add*: *infsetsum_def abs_summable_on_def*)

lemma *infsetsum_of_real*:
shows *infsetsum* (λ*x*. *of_real* (*f x*)
:: ′*a* :: {*real_normed_algebra_1*,*banach*,*second_countable_topology*,*real_inner*})
A =
of_real (*infsetsum f A*)
unfolding *infsetsum_def*
by (*rule integral_bounded_linear*′[*OF bounded_linear_of_real bounded_linear_inner_left*[*of*
1]]) *auto*

lemma *infsetsum_finite* [*simp*]: *finite A* ⟹ *infsetsum f A* = (∑*x*∈*A*. *f x*)
by (*simp add*: *infsetsum_def lebesgue_integral_count_space_finite*)

lemma *infsetsum_nat*:
assumes *f abs_summable_on A*
shows *infsetsum f A* = (∑*n*. *if n* ∈ *A then f n else 0*)
proof −
from *assms* **have** *infsetsum f A* = (∑*n*. *indicator A n* $*_R$ *f n*)
unfolding *infsetsum_altdef abs_summable_on_altdef set_lebesgue_integral_def set_integrable_def*
by (*subst integral_count_space_nat*) *auto*
also have (λ*n*. *indicator A n* $*_R$ *f n*) = (λ*n*. *if n* ∈ *A then f n else 0*)
by *auto*
finally show *?thesis* **.**
qed

lemma *infsetsum_nat*′:

assumes *f abs_summable_on UNIV*
shows *infsetsum f UNIV* = ($\sum$ *n*. *f n*)
using *assms* **by** (*subst infsetsum_nat*) *auto*

lemma *sums_infsetsum_nat*:
assumes *f abs_summable_on A*
shows (λ*n*. *if n* ∈ *A then f n else 0*) *sums infsetsum f A*
proof −
from *assms* **have** *summable* (λ*n*. *if n* ∈ *A then norm* (*f n*) *else 0*)
by (*simp add*: *abs_summable_on_nat_iff*)
also have (λ*n*. *if n* ∈ *A then norm* (*f n*) *else 0*) = (λ*n*. *norm* (*if n* ∈ *A then f n else 0*))
by *auto*
finally have *summable* (λ*n*. *if n* ∈ *A then f n else 0*)
by (*rule summable_norm_cancel*)
with *assms* **show** *?thesis*
by (*auto simp*: *sums_iff infsetsum_nat*)
qed

lemma *sums_infsetsum_nat′*:
assumes *f abs_summable_on UNIV*
shows *f sums infsetsum f UNIV*
using *sums_infsetsum_nat* [*OF assms*] **by** *simp*

lemma *infsetsum_Un_disjoint*:
assumes *f abs_summable_on A f abs_summable_on B A* ∩ *B* = {}
shows *infsetsum f* (*A* ∪ *B*) = *infsetsum f A* + *infsetsum f B*
using *assms* **unfolding** *infsetsum_altdef abs_summable_on_altdef*
by (*subst set_integral_Un*) *auto*

lemma *infsetsum_Diff*:
assumes *f abs_summable_on B A* ⊆ *B*
shows *infsetsum f* (*B* − *A*) = *infsetsum f B* − *infsetsum f A*
proof −
have *infsetsum f* ((*B* − *A*) ∪ *A*) = *infsetsum f* (*B* − *A*) + *infsetsum f A*
using *assms*(*2*) **by** (*intro infsetsum_Un_disjoint abs_summable_on_subset*[*OF assms*(*1*)]) *auto*
also from *assms*(*2*) **have** (*B* − *A*) ∪ *A* = *B*
by *auto*
ultimately show *?thesis*
by (*simp add*: *algebra_simps*)
qed

lemma *infsetsum_Un_Int*:
assumes *f abs_summable_on* (*A* ∪ *B*)
shows *infsetsum f* (*A* ∪ *B*) = *infsetsum f A* + *infsetsum f B* − *infsetsum f* (*A* ∩ *B*)
proof −
have *A* ∪ *B* = *A* ∪ (*B* − *A* ∩ *B*)

by *auto*
also have *infsetsum f ... = infsetsum f A + infsetsum f (B − A ∩ B)*
by (*intro infsetsum_Un_disjoint abs_summable_on_subset*[*OF assms*]) *auto*
also have *infsetsum f (B − A ∩ B) = infsetsum f B − infsetsum f (A ∩ B)*
by (*intro infsetsum_Diff abs_summable_on_subset*[*OF assms*]) *auto*
finally show *?thesis*
by (*simp add*: *algebra_simps*)
qed

lemma *infsetsum_reindex_bij_betw*:
assumes *bij_betw g A B*
shows *infsetsum (λx. f (g x)) A = infsetsum f B*
proof −
have ∗: *count_space B = distr (count_space A) (count_space B) g*
by (*rule distr_bij_count_space* [*symmetric*]) *fact*
show *?thesis* **unfolding** *infsetsum_def*
by (*subst* ∗, *subst integral_distr*[*of _ _ count_space B*])
(*insert assms*, *auto simp*: *bij_betw_def*)
qed

theorem *infsetsum_reindex*:
assumes *inj_on g A*
shows *infsetsum f (g ‘ A) = infsetsum (λx. f (g x)) A*
by (*intro infsetsum_reindex_bij_betw* [*symmetric*] *inj_on_imp_bij_betw assms*)

lemma *infsetsum_cong_neutral*:
assumes $\bigwedge x.\ x \in A - B \Longrightarrow f\ x = 0$
assumes $\bigwedge x.\ x \in B - A \Longrightarrow g\ x = 0$
assumes $\bigwedge x.\ x \in A \cap B \Longrightarrow f\ x = g\ x$
shows *infsetsum f A = infsetsum g B*
unfolding *infsetsum_altdef set_lebesgue_integral_def* **using** *assms*
by (*intro Bochner_Integration.integral_cong refl*)
(*auto simp*: *indicator_def split*: *if_splits*)

lemma *infsetsum_mono_neutral*:
fixes *f g* :: *'a ⇒ real*
assumes *f abs_summable_on A* **and** *g abs_summable_on B*
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x \le g\ x$
assumes $\bigwedge x.\ x \in A - B \Longrightarrow f\ x \le 0$
assumes $\bigwedge x.\ x \in B - A \Longrightarrow g\ x \ge 0$
shows *infsetsum f A ≤ infsetsum g B*
using *assms* **unfolding** *infsetsum_altdef set_lebesgue_integral_def abs_summable_on_altdef set_integrable_def*
by (*intro Bochner_Integration.integral_mono*) (*auto simp*: *indicator_def*)

lemma *infsetsum_mono_neutral_left*:
fixes *f g* :: *'a ⇒ real*
assumes *f abs_summable_on A* **and** *g abs_summable_on B*
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x \le g\ x$

assumes $A \subseteq B$
assumes $\bigwedge x.\ x \in B - A \Longrightarrow g\ x \geq 0$
shows *infsetsum f A* $\leq$ *infsetsum g B*
using ‹$A \subseteq B$› **by** (*intro infsetsum_mono_neutral assms*) *auto*

lemma *infsetsum_mono_neutral_right*:
fixes *f g* :: *'a* ⇒ *real*
assumes *f abs_summable_on A* **and** *g abs_summable_on B*
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x \leq g\ x$
assumes $B \subseteq A$
assumes $\bigwedge x.\ x \in A - B \Longrightarrow f\ x \leq 0$
shows *infsetsum f A* $\leq$ *infsetsum g B*
using ‹$B \subseteq A$› **by** (*intro infsetsum_mono_neutral assms*) *auto*

lemma *infsetsum_mono*:
fixes *f g* :: *'a* ⇒ *real*
assumes *f abs_summable_on A* **and** *g abs_summable_on A*
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x \leq g\ x$
shows *infsetsum f A* $\leq$ *infsetsum g A*
by (*intro infsetsum_mono_neutral assms*) *auto*

lemma *norm_infsetsum_bound*:
norm (*infsetsum f A*) $\leq$ *infsetsum* ($\lambda x.$ *norm* (*f x*)) *A*
unfolding *abs_summable_on_def infsetsum_def*
by (*rule Bochner_Integration.integral_norm_bound*)

theorem *infsetsum_Sigma*:
fixes *A* :: *'a set* **and** *B* :: *'a* ⇒ *'b set*
assumes [*simp*]: *countable A* **and** $\bigwedge i.$ *countable* (*B i*)
assumes *summable*: *f abs_summable_on* (*Sigma A B*)
shows *infsetsum f* (*Sigma A B*) = *infsetsum* ($\lambda x.$ *infsetsum* ($\lambda y.$ *f* (*x*, *y*)) (*B x*)) *A*
proof −
define *B'* **where** *B'* = ($\bigcup i \in A.$ *B i*)
have [*simp*]: *countable B'*
unfolding *B'_def* **by** (*intro countable_UN assms*)
interpret *pair_sigma_finite count_space A count_space B'*
by (*intro pair_sigma_finite.intro sigma_finite_measure_count_space_countable*) *fact*+

have *integrable* (*count_space* (*A* × *B'*)) ($\lambda z.$ *indicator* (*Sigma A B*) *z* $*_R$ *f z*)
using *summable*
by (*metis* (*mono_tags*, *lifting*) *abs_summable_on_altdef abs_summable_on_def integrable_cong integrable_mult_indicator set_integrable_def sets_UNIV*)
also have *?this* ⟷ *integrable* (*count_space A* $\bigotimes_M$ *count_space B'*) (λ(*x*, *y*). *indicator* (*B x*) *y* $*_R$ *f* (*x*, *y*))
by (*intro Bochner_Integration.integrable_cong*)
(*auto simp*: *pair_measure_countable indicator_def split*: *if_splits*)
finally have *integrable*: . . . .

have *infsetsum* (λ*x*. *infsetsum* (λ*y*. *f* (*x*, *y*)) (*B x*)) *A* =
(∫ *x*. *infsetsum* (λ*y*. *f* (*x*, *y*)) (*B x*) ∂*count_space A*)
unfolding *infsetsum_def* **by** *simp*
also have ... = (∫ *x*. ∫ *y*. *indicator* (*B x*) *y* $*_R$ *f* (*x*, *y*) ∂*count_space B′* ∂*count_space A*)
proof (*rule Bochner_Integration.integral_cong* [*OF refl*])
show ⋀*x*. *x* ∈ *space* (*count_space A*) ⟹
(∑$_a$*y*∈*B x*. *f* (*x*, *y*)) = *LINT y*|*count_space B′*. *indicat_real* (*B x*) *y* $*_R$ *f* (*x*, *y*)
using *infsetsum_altdef′*[*of* _ *B′*]
unfolding *set_lebesgue_integral_def B′_def*
by *auto*
qed
also have ... = (∫ (*x*,*y*). *indicator* (*B x*) *y* $*_R$ *f* (*x*, *y*) ∂(*count_space A* ⨂$_M$ *count_space B′*))
by (*subst integral_fst* [*OF integrable*]) *auto*
also have ... = (∫ *z*. *indicator* (*Sigma A B*) *z* $*_R$ *f z* ∂*count_space* (*A* × *B′*))
by (*intro Bochner_Integration.integral_cong*)
(*auto simp*: *pair_measure_countable indicator_def split*: *if_splits*)
also have ... = *infsetsum f* (*Sigma A B*)
unfolding *set_lebesgue_integral_def* [*symmetric*]
by (*rule infsetsum_altdef′* [*symmetric*]) (*auto simp*: *B′_def*)
finally show *?thesis* **..**
qed

lemma *infsetsum_Sigma′*:
fixes *A* :: *′a set* **and** *B* :: *′a* ⇒ *′b set*
assumes [*simp*]: *countable A* **and** ⋀*i*. *countable* (*B i*)
assumes *summable*: (λ(*x*,*y*). *f x y*) *abs_summable_on* (*Sigma A B*)
shows *infsetsum* (λ*x*. *infsetsum* (λ*y*. *f x y*) (*B x*)) *A* = *infsetsum* (λ(*x*,*y*). *f x y*) (*Sigma A B*)
using *assms* **by** (*subst infsetsum_Sigma*) *auto*

lemma *infsetsum_Times*:
fixes *A* :: *′a set* **and** *B* :: *′b set*
assumes [*simp*]: *countable A* **and** *countable B*
assumes *summable*: *f abs_summable_on* (*A* × *B*)
shows *infsetsum f* (*A* × *B*) = *infsetsum* (λ*x*. *infsetsum* (λ*y*. *f* (*x*, *y*)) *B*) *A*
using *assms* **by** (*subst infsetsum_Sigma*) *auto*

lemma *infsetsum_Times′*:
fixes *A* :: *′a set* **and** *B* :: *′b set*
fixes *f* :: *′a* ⇒ *′b* ⇒ *′c* :: {*banach*, *second_countable_topology*}
assumes [*simp*]: *countable A* **and** [*simp*]: *countable B*
assumes *summable*: (λ(*x*,*y*). *f x y*) *abs_summable_on* (*A* × *B*)
shows *infsetsum* (λ*x*. *infsetsum* (λ*y*. *f x y*) *B*) *A* = *infsetsum* (λ(*x*,*y*). *f x y*) (*A* × *B*)
using *assms* **by** (*subst infsetsum_Times*) *auto*

```
lemma infsetsum_swap:
  fixes A :: 'a set and B :: 'b set
  fixes f :: 'a ⇒ 'b ⇒ 'c :: {banach, second_countable_topology}
  assumes [simp]: countable A and [simp]: countable B
  assumes summable: (λ(x,y). f x y) abs_summable_on A × B
  shows   infsetsum (λx. infsetsum (λy. f x y) B) A = infsetsum (λy. infsetsum
(λx. f x y) A) B
proof –
  from summable have summable': (λ(x,y). f y x) abs_summable_on B × A
    by (subst abs_summable_on_Times_swap) auto
  have bij: bij_betw (λ(x, y). (y, x)) (B × A) (A × B)
    by (auto simp: bij_betw_def inj_on_def)
  have infsetsum (λx. infsetsum (λy. f x y) B) A = infsetsum (λ(x,y). f x y) (A
× B)
    using summable by (subst infsetsum_Times) auto
  also have ... = infsetsum (λ(x,y). f y x) (B × A)
    by (subst infsetsum_reindex_bij_betw[OF bij, of λ(x,y). f x y, symmetric])
       (simp_all add: case_prod_unfold)
  also have ... = infsetsum (λy. infsetsum (λx. f x y) A) B
    using summable' by (subst infsetsum_Times) auto
  finally show ?thesis .
qed

theorem abs_summable_on_Sigma_iff:
  assumes [simp]: countable A and ⋀x. x ∈ A ⟹ countable (B x)
  shows   f abs_summable_on Sigma A B ⟷
            (∀x∈A. (λy. f (x, y)) abs_summable_on B x) ∧
            ((λx. infsetsum (λy. norm (f (x, y))) (B x)) abs_summable_on A)
proof safe
  define B' where B' = (⋃x∈A. B x)
  have [simp]: countable B'
    unfolding B'_def using assms by auto
  interpret pair_sigma_finite count_space A count_space B'
      by (intro pair_sigma_finite.intro sigma_finite_measure_count_space_countable)
fact+
  {
    assume *: f abs_summable_on Sigma A B
    thus (λy. f (x, y)) abs_summable_on B x if x ∈ A for x
      using that by (rule abs_summable_on_Sigma_project2)

    have set_integrable (count_space (A × B')) (Sigma A B) (λz. norm (f z))
      using abs_summable_on_normI[OF *]
      by (subst abs_summable_on_altdef' [symmetric]) (auto simp: B'_def)
    also have count_space (A × B') = count_space A ⨂M count_space B'
      by (simp add: pair_measure_countable)
    finally have integrable (count_space A)
                   (λx. lebesgue_integral (count_space B')
                     (λy. indicator (Sigma A B) (x, y) *R norm (f (x, y))))
```

unfolding *set_integrable_def* **by** (*rule integrable_fst'*)
also have *?this* ⟷ *integrable* (*count_space A*)
(λ*x*. *lebesgue_integral* (*count_space B'*)
(λ*y*. *indicator* (*B x*) *y* $*_R$ *norm* (*f* (*x*, *y*))))
by (*intro integrable_cong refl*) (*simp_all add*: *indicator_def*)
also have ... ⟷ *integrable* (*count_space A*) (λ*x*. *infsetsum* (λ*y*. *norm* (*f* (*x*, *y*))) (*B x*))
unfolding *set_lebesgue_integral_def* [*symmetric*]
by (*intro integrable_cong refl infsetsum_altdef'* [*symmetric*]) (*auto simp*: *B'_def*)
also have ... ⟷ (λ*x*. *infsetsum* (λ*y*. *norm* (*f* (*x*, *y*))) (*B x*)) *abs_summable_on A*
by (*simp add*: *abs_summable_on_def*)
finally show ... **.**
}
{
assume ∗: ∀ *x*∈*A*. (λ*y*. *f* (*x*, *y*)) *abs_summable_on B x*
assume (λ*x*. $\sum_a$ *y*∈*B x*. *norm* (*f* (*x*, *y*))) *abs_summable_on A*
also have *?this* ⟷ (λ*x*. ∫ *y*∈*B x*. *norm* (*f* (*x*, *y*)) ∂*count_space B'*) *abs_summable_on A*
by (*intro abs_summable_on_cong refl infsetsum_altdef'*) (*auto simp*: *B'_def*)
also have ... ⟷ (λ*x*. ∫ *y*. *indicator* (*Sigma A B*) (*x*, *y*) $*_R$ *norm* (*f* (*x*, *y*)) ∂*count_space B'*)
abs_summable_on A (**is** _ ⟷ *?h abs_summable_on* _)
unfolding *set_lebesgue_integral_def*
by (*intro abs_summable_on_cong*) (*auto simp*: *indicator_def*)
also have ... ⟷ *integrable* (*count_space A*) *?h*
by (*simp add*: *abs_summable_on_def*)
finally have ∗∗: ... **.**

have *integrable* (*count_space A* $\bigotimes_M$ *count_space B'*) (λ*z*. *indicator* (*Sigma A B*) *z* $*_R$ *f z*)
proof (*rule Fubini_integrable*, *goal_cases*)
case *3*
{
fix *x* **assume** *x*: *x* ∈ *A*
with ∗ **have** (λ*y*. *f* (*x*, *y*)) *abs_summable_on B x*
by *blast*
also have *?this* ⟷ *integrable* (*count_space B'*)
(λ*y*. *indicator* (*B x*) *y* $*_R$ *f* (*x*, *y*))
unfolding *set_integrable_def* [*symmetric*]
using *x* **by** (*intro abs_summable_on_altdef'*) (*auto simp*: *B'_def*)
also have (λ*y*. *indicator* (*B x*) *y* $*_R$ *f* (*x*, *y*)) =
(λ*y*. *indicator* (*Sigma A B*) (*x*, *y*) $*_R$ *f* (*x*, *y*))
using *x* **by** (*auto simp*: *indicator_def*)
finally have *integrable* (*count_space B'*)
(λ*y*. *indicator* (*Sigma A B*) (*x*, *y*) $*_R$ *f* (*x*, *y*)) **.**
}
thus *?case* **by** (*auto simp*: *AE_count_space*)

```
    qed (insert **, auto simp: pair_measure_countable)
    moreover have count_space A ⨂M count_space B' = count_space (A × B')
      by (simp add: pair_measure_countable)
    moreover have set_integrable (count_space (A × B')) (Sigma A B) f ⟷
                f abs_summable_on Sigma A B
      by (rule abs_summable_on_altdef' [symmetric]) (auto simp: B'_def)
    ultimately show f abs_summable_on Sigma A B
      by (simp add: set_integrable_def)
  }
qed

lemma abs_summable_on_Sigma_project1:
  assumes (λ(x,y). f x y) abs_summable_on Sigma A B
  assumes [simp]: countable A and ⋀x. x ∈ A ⟹ countable (B x)
  shows   (λx. infsetsum (λy. norm (f x y)) (B x)) abs_summable_on A
  using assms by (subst (asm) abs_summable_on_Sigma_iff) auto

lemma abs_summable_on_Sigma_project1':
  assumes (λ(x,y). f x y) abs_summable_on Sigma A B
  assumes [simp]: countable A and ⋀x. x ∈ A ⟹ countable (B x)
  shows   (λx. infsetsum (λy. f x y) (B x)) abs_summable_on A
  by (intro abs_summable_on_comparison_test' [OF abs_summable_on_Sigma_project1 [OF
assms]]
        norm_infsetsum_bound)

theorem infsetsum_prod_PiE:
  fixes f :: 'a ⇒ 'b ⇒ 'c :: {real_normed_field,banach,second_countable_topology}
  assumes finite: finite A and countable: ⋀x. x ∈ A ⟹ countable (B x)
  assumes summable: ⋀x. x ∈ A ⟹ f x abs_summable_on B x
  shows   infsetsum (λg. ∏x∈A. f x (g x)) (PiE A B) = (∏x∈A. infsetsum (f
x) (B x))
proof -
  define B' where B' = (λx. if x ∈ A then B x else {})
  from assms have [simp]: countable (B' x) for x
    by (auto simp: B'_def)
  then interpret product_sigma_finite count_space ∘ B'
   unfolding o_def by (intro product_sigma_finite.intro sigma_finite_measure_count_space_countable)
  have infsetsum (λg. ∏x∈A. f x (g x)) (PiE A B) =
          (∫ g. (∏x∈A. f x (g x)) ∂count_space (PiE A B))
    by (simp add: infsetsum_def)
  also have PiE A B = PiE A B'
    by (intro PiE_cong) (simp_all add: B'_def)
  hence count_space (PiE A B) = count_space (PiE A B')
    by simp
  also have ... = PiM A (count_space ∘ B')
     unfolding o_def using finite by (intro count_space_PiM_finite [symmetric])
simp_all
  also have (∫ g. (∏x∈A. f x (g x)) ∂...) = (∏x∈A. infsetsum (f x) (B' x))
    by (subst product_integral_prod)
```

(*insert summable finite*, *simp_all add*: *infsetsum_def B′_def abs_summable_on_def*)
also have ... = ($\prod$ *x*∈*A*. *infsetsum* (*f x*) (*B x*))
by (*intro prod.cong refl*) (*simp_all add*: *B′_def*)
finally show *?thesis* **.**
qed

lemma *infsetsum_uminus*: *infsetsum* (λ*x*. −*f x*) *A* = −*infsetsum f A*
unfolding *infsetsum_def abs_summable_on_def*
by (*rule Bochner_Integration.integral_minus*)

lemma *infsetsum_add*:
assumes *f abs_summable_on A* **and** *g abs_summable_on A*
shows *infsetsum* (λ*x*. *f x* + *g x*) *A* = *infsetsum f A* + *infsetsum g A*
using *assms* **unfolding** *infsetsum_def abs_summable_on_def*
by (*rule Bochner_Integration.integral_add*)

lemma *infsetsum_diff*:
assumes *f abs_summable_on A* **and** *g abs_summable_on A*
shows *infsetsum* (λ*x*. *f x* − *g x*) *A* = *infsetsum f A* − *infsetsum g A*
using *assms* **unfolding** *infsetsum_def abs_summable_on_def*
by (*rule Bochner_Integration.integral_diff*)

lemma *infsetsum_scaleR_left*:
assumes *c* ≠ *0* ⟹ *f abs_summable_on A*
shows *infsetsum* (λ*x*. *f x* $*_R$ *c*) *A* = *infsetsum f A* $*_R$ *c*
using *assms* **unfolding** *infsetsum_def abs_summable_on_def*
by (*rule Bochner_Integration.integral_scaleR_left*)

lemma *infsetsum_scaleR_right*:
infsetsum (λ*x*. *c* $*_R$ *f x*) *A* = *c* $*_R$ *infsetsum f A*
unfolding *infsetsum_def abs_summable_on_def*
by (*subst Bochner_Integration.integral_scaleR_right*) *auto*

lemma *infsetsum_cmult_left*:
fixes *f* :: ′*a* ⇒ ′*b* :: {*banach*, *real_normed_algebra*, *second_countable_topology*}
assumes *c* ≠ *0* ⟹ *f abs_summable_on A*
shows *infsetsum* (λ*x*. *f x* ∗ *c*) *A* = *infsetsum f A* ∗ *c*
using *assms* **unfolding** *infsetsum_def abs_summable_on_def*
by (*rule Bochner_Integration.integral_mult_left*)

lemma *infsetsum_cmult_right*:
fixes *f* :: ′*a* ⇒ ′*b* :: {*banach*, *real_normed_algebra*, *second_countable_topology*}
assumes *c* ≠ *0* ⟹ *f abs_summable_on A*
shows *infsetsum* (λ*x*. *c* ∗ *f x*) *A* = *c* ∗ *infsetsum f A*
using *assms* **unfolding** *infsetsum_def abs_summable_on_def*
by (*rule Bochner_Integration.integral_mult_right*)

lemma *infsetsum_cdiv*:
fixes *f* :: ′*a* ⇒ ′*b* :: {*banach*, *real_normed_field*, *second_countable_topology*}

assumes $c \neq 0 \Longrightarrow f$ *abs_summable_on* A
shows *infsetsum* $(\lambda x.\ f\ x\ /\ c)\ A =$ *infsetsum* $f\ A\ /\ c$
using *assms* **unfolding** *infsetsum_def abs_summable_on_def* **by** *auto*

lemma
fixes f :: $'a \Rightarrow 'c$ :: {*banach*, *real_normed_field*, *second_countable_topology*}
assumes [*simp*]: *countable* A **and** [*simp*]: *countable* B
assumes f *abs_summable_on* A **and** g *abs_summable_on* B
shows *abs_summable_on_product*: $(\lambda(x,y).\ f\ x * g\ y)$ *abs_summable_on* $A \times B$
and *infsetsum_product*: *infsetsum* $(\lambda(x,y).\ f\ x * g\ y)\ (A \times B) =$
infsetsum $f\ A$ $*$ *infsetsum* $g\ B$
proof –
from *assms* **show** $(\lambda(x,y).\ f\ x * g\ y)$ *abs_summable_on* $A \times B$
by (*subst abs_summable_on_Sigma_iff*)
(*auto intro*!: *abs_summable_on_cmult_right simp*: *norm_mult infsetsum_cmult_right*)
with *assms* **show** *infsetsum* $(\lambda(x,y).\ f\ x * g\ y)\ (A \times B) =$ *infsetsum* $f\ A$ $*$ *infsetsum* $g\ B$
by (*subst infsetsum_Sigma*)
(*auto simp*: *infsetsum_cmult_left infsetsum_cmult_right*)
qed

end

6.38 Faces, Extreme Points, Polytopes, Polyhedra etc

Ported from HOL Light by L C Paulson

theory *Polytope*
imports *Cartesian_Euclidean_Space Path_Connected*
begin

6.38.1 Faces of a (usually convex) set

definition *face_of* :: [$'a$::*real_vector set*, $'a$ *set*] $\Rightarrow$ *bool* (**infixr** (*face'_of*) *50*)
where
T *face_of* $S \longleftrightarrow$
$T \subseteq S \wedge$ *convex* $T \wedge$
$(\forall a \in S.\ \forall b \in S.\ \forall x \in T.\ x \in$ *open_segment* $a\ b \longrightarrow a \in T \wedge b \in T)$

lemma *face_ofD*: $[\![T$ *face_of* $S;\ x \in$ *open_segment* $a\ b;\ a \in S;\ b \in S;\ x \in T]\!] \Longrightarrow a \in T \wedge b \in T$
unfolding *face_of_def* **by** *blast*

lemma *face_of_translation_eq* [*simp*]:
$((+)\ a$ ` $T$ *face_of* $(+)\ a$ ` $S) \longleftrightarrow T$ *face_of* S

proof −
  have ∗: $\bigwedge a\ T\ S.\ T$ *face_of* $S \Longrightarrow ((+)\ a$ ‘ T *face_of* $(+)\ a$ ‘ $S)$
    by (*simp add*: *face_of_def*)
  show *?thesis*
    by (*force simp*: *image_comp o_def dest*: ∗ [**where** $a = -a$] *intro*: ∗)
qed

lemma *face_of_linear_image*:
  assumes *linear f inj f*
    shows (*f* ‘ *c face_of f* ‘ *S*) ⟷ *c face_of S*
by (*simp add*: *face_of_def inj_image_subset_iff inj_image_mem_iff open_segment_linear_image assms*)

lemma *face_of_refl*: *convex S* ⟹ *S face_of S*
  by (*auto simp*: *face_of_def*)

lemma *face_of_refl_eq*: *S face_of S* ⟷ *convex S*
  by (*auto simp*: *face_of_def*)

lemma *empty_face_of* [*iff*]: {} *face_of S*
  by (*simp add*: *face_of_def*)

lemma *face_of_empty* [*simp*]: *S face_of* {} ⟷ $S = \{\}$
  by (*meson empty_face_of face_of_def subset_empty*)

lemma *face_of_trans* [*trans*]: ⟦*S face_of T*; *T face_of u*⟧ ⟹ *S face_of u*
  unfolding *face_of_def* **by** (*safe*; *blast*)

lemma *face_of_face*: *T face_of S* ⟹ (*f face_of T* ⟷ *f face_of S* ∧ $f \subseteq T$)
  unfolding *face_of_def* **by** (*safe*; *blast*)

lemma *face_of_subset*: ⟦*F face_of S*; $F \subseteq T$; $T \subseteq S$⟧ ⟹ *F face_of T*
  unfolding *face_of_def* **by** (*safe*; *blast*)

lemma *face_of_slice*: ⟦*F face_of S*; *convex T*⟧ ⟹ $(F \cap T)$ *face_of* $(S \cap T)$
  unfolding *face_of_def* **by** (*blast intro*: *convex_Int*)

lemma *face_of_Int*: ⟦*t1 face_of S*; *t2 face_of S*⟧ ⟹ (*t1* ∩ *t2*) *face_of S*
  unfolding *face_of_def* **by** (*blast intro*: *convex_Int*)

lemma *face_of_Inter*: ⟦$A \neq \{\}$; $\bigwedge T.\ T \in A \Longrightarrow T$ *face_of S*⟧ ⟹ $(\bigcap A)$ *face_of S*
  unfolding *face_of_def* **by** (*blast intro*: *convex_Inter*)

lemma *face_of_Int_Int*: ⟦*F face_of T*; *F′ face_of t′*⟧ ⟹ $(F \cap F')$ *face_of* $(T \cap t')$
  unfolding *face_of_def* **by** (*blast intro*: *convex_Int*)

lemma *face_of_imp_subset*: *T face_of S* ⟹ $T \subseteq S$
  unfolding *face_of_def* **by** *blast*

proposition *face_of_imp_eq_affine_Int*:
fixes *S* :: *'a::euclidean_space set*
assumes *S*: *convex S* **and** *T*: *T face_of S*
shows *T* = (*affine hull T*) ∩ *S*
proof –
have *convex T* **using** *T* **by** (*simp add*: *face_of_def*)
have ∗: *False* **if** *x*: *x* ∈ *affine hull T* **and** *x* ∈ *S x* ∉ *T* **and** *y*: *y* ∈ *rel_interior T* **for** *x y*
proof –
obtain *e* **where** *e*>*0* **and** *e*: *cball y e* ∩ *affine hull T* ⊆ *T*
using *y* **by** (*auto simp*: *rel_interior_cball*)
have *y* ≠ *x y* ∈ *S y* ∈ *T*
using *face_of_imp_subset rel_interior_subset T that* **by** *blast*+
then have *zne*: ⋀*u*. ⟦*u* ∈ {*0*<..<*1*}; (*1* − *u*) ∗*R* *y* + *u* ∗*R* *x* ∈ *T*⟧ ⟹ *False*
using ‹*x* ∈ *S*› ‹*x* ∉ *T*› ‹*T face_of S*› **unfolding** *face_of_def*
by (*meson greaterThanLessThan_iff in_segment*(*2*))
have *in01*: *min* (*1*/*2*) (*e* / *norm* (*x* − *y*)) ∈ {*0*<..<*1*}
using ‹*y* ≠ *x*› ‹*e* > *0*› **by** *simp*
have §: *norm* (*min* (*1*/*2*) (*e* / *norm* (*x* − *y*)) ∗*R* *y* − *min* (*1*/*2*) (*e* / *norm* (*x* − *y*)) ∗*R* *x*) ≤ *e*
using ‹*e* > *0*›
by (*simp add*: *scaleR_diff_right* [*symmetric*] *norm_minus_commute min_mult_distrib_right*)
show *False*
apply (*rule zne* [*OF in01 e* [*THEN subsetD*]])
using ‹*y* ∈ *T*›
apply (*simp add*: *hull_inc mem_affine x*)
by (*simp add*: *dist_norm algebra_simps* §)
qed
show *?thesis*
proof (*rule subset_antisym*)
show *T* ⊆ *affine hull T* ∩ *S*
using *assms* **by** (*simp add*: *hull_subset face_of_imp_subset*)
show *affine hull T* ∩ *S* ⊆ *T*
using ∗ ‹*convex T*› *rel_interior_eq_empty* **by** *fastforce*
qed
qed

lemma *face_of_imp_closed*:
fixes *S* :: *'a::euclidean_space set*
assumes *convex S closed S T face_of S* **shows** *closed T*
by (*metis affine_affine_hull affine_closed closed_Int face_of_imp_eq_affine_Int assms*)

lemma *face_of_Int_supporting_hyperplane_le_strong*:
assumes *convex*(*S* ∩ {*x*. *a* · *x* = *b*}) **and** *aleb*: ⋀*x*. *x* ∈ *S* ⟹ *a* · *x* ≤ *b*
shows (*S* ∩ {*x*. *a* · *x* = *b*}) *face_of S*
proof –
have ∗: *a* · *u* = *a* · *x* **if** *x* ∈ *open_segment u v u* ∈ *S v* ∈ *S* **and** *b*: *b* = *a* · *x*
for *u v x*
proof (*rule antisym*)

show $a \cdot u \le a \cdot x$
using *aleb* ‹$u \in S$› ‹$b = a \cdot x$› **by** *blast*
next
obtain ξ **where** $b = a \cdot ((1 - \xi) *_R u + \xi *_R v)$ $0 < \xi$ $\xi < 1$
using ‹$b = a \cdot x$› ‹$x \in$ *open_segment u v*› *in_segment*
by (*auto simp*: *open_segment_image_interval split*: *if_split_asm*)
then have $b + \xi * (a \cdot u) \le a \cdot u + \xi * b$
using *aleb* [*OF* ‹$v \in S$›] **by** (*simp add*: *algebra_simps*)
then have $(1 - \xi) * b \le (1 - \xi) * (a \cdot u)$
by (*simp add*: *algebra_simps*)
then have $b \le a \cdot u$
using ‹$\xi < 1$› **by** *auto*
with *b* **show** $a \cdot x \le a \cdot u$ **by** *simp*
qed
show *?thesis*
using $*$ *open_segment_commute* **by** (*fastforce simp add*: *face_of_def assms*)
qed

lemma *face_of_Int_supporting_hyperplane_ge_strong*:
⟦*convex*($S \cap \{x.\ a \cdot x = b\}$); $\bigwedge x.\ x \in S \Longrightarrow a \cdot x \ge b$⟧
$\Longrightarrow$ ($S \cap \{x.\ a \cdot x = b\}$) *face_of S*
using *face_of_Int_supporting_hyperplane_le_strong* [*of S* $-a$ $-b$] **by** *simp*

lemma *face_of_Int_supporting_hyperplane_le*:
⟦*convex S*; $\bigwedge x.\ x \in S \Longrightarrow a \cdot x \le b$⟧ $\Longrightarrow$ ($S \cap \{x.\ a \cdot x = b\}$) *face_of S*
by (*simp add*: *convex_Int convex_hyperplane face_of_Int_supporting_hyperplane_le_strong*)

lemma *face_of_Int_supporting_hyperplane_ge*:
⟦*convex S*; $\bigwedge x.\ x \in S \Longrightarrow a \cdot x \ge b$⟧ $\Longrightarrow$ ($S \cap \{x.\ a \cdot x = b\}$) *face_of S*
by (*simp add*: *convex_Int convex_hyperplane face_of_Int_supporting_hyperplane_ge_strong*)

lemma *face_of_imp_convex*: *T face_of S* $\Longrightarrow$ *convex T*
using *face_of_def* **by** *blast*

lemma *face_of_imp_compact*:
fixes *S* :: *'a::euclidean_space set*
shows ⟦*convex S*; *compact S*; *T face_of S*⟧ $\Longrightarrow$ *compact T*
by (*meson bounded_subset compact_eq_bounded_closed face_of_imp_closed face_of_imp_subset*)

lemma *face_of_Int_subface*:
⟦$A \cap B$ *face_of A*; $A \cap B$ *face_of B*; *C face_of A*; *D face_of B*⟧
$\Longrightarrow$ ($C \cap D$) *face_of C* $\wedge$ ($C \cap D$) *face_of D*
by (*meson face_of_Int_Int face_of_face inf_le1 inf_le2*)

lemma *subset_of_face_of*:
fixes *S* :: *'a::real_normed_vector set*
assumes *T face_of S* $u \subseteq S$ $T \cap$ (*rel_interior u*) $\ne \{\}$
shows $u \subseteq T$
proof

fix *c*
assume *c* ∈ *u*
obtain *b* **where** *b* ∈ *T b* ∈ *rel_interior u* **using** *assms* **by** *auto*
then obtain *e* **where** *e*>*0 b* ∈ *u* **and** *e*: *cball b e* ∩ *affine hull u* ⊆ *u*
by (*auto simp*: *rel_interior_cball*)
show *c* ∈ *T*
proof (*cases b*=*c*)
case *True* **with** ‹*b* ∈ *T*› **show** *?thesis* **by** *blast*
next
case *False*
define *d* **where** *d* = *b* + (*e* / *norm*(*b* − *c*)) $*_R$ (*b* − *c*)
have *d* ∈ *cball b e* ∩ *affine hull u*
using ‹*e* > *0*› ‹*b* ∈ *u*› ‹*c* ∈ *u*›
by (*simp add*: *d_def dist_norm hull_inc mem_affine_3_minus False*)
with *e* **have** *d* ∈ *u* **by** *blast*
have *nbc*: *norm* (*b* − *c*) + *e* > *0* **using** ‹*e* > *0*›
by (*metis add.commute le_less_trans less_add_same_cancel2 norm_ge_zero*)
then have [*simp*]: *d* ≠ *c* **using** *False scaleR_cancel_left* [*of 1* + (*e* / *norm* (*b* − *c*)) *b c*]
by (*simp add*: *algebra_simps d_def*) (*simp add*: *field_split_simps*)
have [*simp*]: ((*e* − *e* ∗ *e* / (*e* + *norm* (*b* − *c*))) / *norm* (*b* − *c*)) = (*e* / (*e* + *norm* (*b* − *c*)))
using *False nbc*
by (*simp add*: *divide_simps*) (*simp add*: *algebra_simps*)
have *b* ∈ *open_segment d c*
apply (*simp add*: *open_segment_image_interval*)
apply (*simp add*: *d_def algebra_simps image_def*)
apply (*rule_tac x*=*e* / (*e* + *norm* (*b* − *c*)) **in** *bexI*)
using *False nbc* ‹*0* < *e*› **by** (*auto simp*: *algebra_simps*)
then have *d* ∈ *T* ∧ *c* ∈ *T*
by (*meson* ‹*b* ∈ *T*› ‹*c* ∈ *u*› ‹*d* ∈ *u*› *assms face_ofD subset_iff*)
then show *?thesis* **..**
qed
qed

lemma *face_of_eq*:
fixes *S* :: *'a::real_normed_vector set*
assumes *T face_of S U face_of S* (*rel_interior T*) ∩ (*rel_interior U*) ≠ {}
shows *T* = *U*
using *assms*
unfolding *disjoint_iff_not_equal*
by (*metis IntI empty_iff face_of_imp_subset mem_rel_interior_ball subset_antisym subset_of_face_of*)

lemma *face_of_disjoint_rel_interior*:
fixes *S* :: *'a::real_normed_vector set*
assumes *T face_of S T* ≠ *S*
shows *T* ∩ *rel_interior S* = {}
by (*meson assms subset_of_face_of face_of_imp_subset order_refl subset_antisym*)

lemma *face_of_disjoint_interior*:
fixes *S* :: *'a::real_normed_vector set*
assumes *T face_of S T* ≠ *S*
shows *T* ∩ *interior S* = {}
proof –
have *T* ∩ *interior S* ⊆ *rel_interior S*
by (*meson inf_sup_ord*(*2*) *interior_subset_rel_interior order.trans*)
thus *?thesis*
by (*metis* (*no_types*) *Int_greatest assms face_of_disjoint_rel_interior inf_sup_ord*(*1*) *subset_empty*)
qed

lemma *face_of_subset_rel_boundary*:
fixes *S* :: *'a::real_normed_vector set*
assumes *T face_of S T* ≠ *S*
shows *T* ⊆ (*S* − *rel_interior S*)
by (*meson DiffI assms disjoint_iff_not_equal face_of_disjoint_rel_interior face_of_imp_subset rev_subsetD subsetI*)

lemma *face_of_subset_rel_frontier*:
fixes *S* :: *'a::real_normed_vector set*
assumes *T face_of S T* ≠ *S*
shows *T* ⊆ *rel_frontier S*
using *assms closure_subset face_of_disjoint_rel_interior face_of_imp_subset rel_frontier_def*
by *fastforce*

lemma *face_of_aff_dim_lt*:
fixes *S* :: *'a::euclidean_space set*
assumes *convex S T face_of S T* ≠ *S*
shows *aff_dim T* < *aff_dim S*
proof –
have *aff_dim T* ≤ *aff_dim S*
by (*simp add*: *face_of_imp_subset aff_dim_subset assms*)
moreover have *aff_dim T* ≠ *aff_dim S*
proof (*cases T* = {})
case *True* **then show** *?thesis*
by (*metis aff_dim_empty* ‹*T* ≠ *S*›)
next case *False* **then show** *?thesis*
by (*metis Set.set_insert assms convex_rel_frontier_aff_dim dual_order.irrefl face_of_imp_convex face_of_subset_rel_frontier insert_not_empty subsetI*)
qed
ultimately show *?thesis*
by *simp*
qed

lemma *subset_of_face_of_affine_hull*:
fixes *S* :: *'a::euclidean_space set*
assumes *T*: *T face_of S* **and** *convex S U* ⊆ *S* **and** *dis*: ¬ *disjnt* (*affine hull T*)

(*rel_interior U*)
  shows $U \subseteq T$
proof (*rule subset_of_face_of* [*OF T* ‹$U \subseteq S$›])
  show $T \cap$ *rel_interior* $U \neq \{\}$
    using *face_of_imp_eq_affine_Int* [*OF* ‹*convex S*› *T*] *rel_interior_subset* [*of U*] *dis*
‹$U \subseteq S$› *disjnt_def*
    by *fastforce*
qed

lemma *affine_hull_face_of_disjoint_rel_interior*:
    fixes *S* :: *'a::euclidean_space set*
  assumes *convex S F face_of S* $F \neq S$
  shows *affine hull F* $\cap$ *rel_interior S* $= \{\}$
 by (*metis assms disjnt_def face_of_imp_subset order_refl subset_antisym subset_of_face_of_affine_hull*)

lemma *affine_diff_divide*:
    assumes *affine S* $k \neq 0$ $k \neq 1$ **and** *xy*: $x \in S$ $y /_R (1 - k) \in S$
      shows $(x - y) /_R k \in S$
proof –
  have *inverse*$(k) *_R (x - y) = (1 - $ *inverse* $k) *_R$ *inverse*$(1 - k) *_R y +$
inverse$(k) *_R x$
    using *assms*
  by (*simp add*: *algebra_simps*) (*simp add*: *scaleR_left_distrib* [*symmetric*] *field_split_simps*)
  then show *?thesis*
    using ‹*affine S*› *xy* **by** (*auto simp*: *affine_alt*)
qed

proposition *face_of_convex_hulls*:
      assumes *S*: *finite S* $T \subseteq S$ **and** *disj*: *affine hull T* $\cap$ *convex hull* $(S - T) =$
$\{\}$
      shows (*convex hull T*) *face_of* (*convex hull S*)
proof –
  have *fin*: *finite T finite* $(S - T)$ **using** *assms*
    by (*auto simp*: *finite_subset*)
  have $*$: $x \in$ *convex hull T*
        if *x*: $x \in$ *convex hull S* **and** *y*: $y \in$ *convex hull S* **and** *w*: $w \in$ *convex hull*
T $w \in$ *open_segment x y*
          for *x y w*
  proof –
    have *waff*: $w \in$ *affine hull T*
      using *convex_hull_subset_affine_hull w* **by** *blast*
    obtain *a b* **where** *a*: $\bigwedge i.\ i \in S \Longrightarrow 0 \leq a\ i$ **and** *asum*: *sum a S* $= 1$ **and**
aeqx: $(\sum i\in S.\ a\ i *_R i) = x$
                and *b*: $\bigwedge i.\ i \in S \Longrightarrow 0 \leq b\ i$ **and** *bsum*: *sum b S* $= 1$ **and** *beqy*:
$(\sum i\in S.\ b\ i *_R i) = y$
      using *x y* **by** (*auto simp*: *assms convex_hull_finite*)
    obtain *u* **where** $(1 - u) *_R x + u *_R y \in$ *convex hull T* $x \neq y$ **and** *weq*: w
$= (1 - u) *_R x + u *_R y$
              and *u01*: $0 < u$ $u < 1$

using *w* **by** (*auto simp*: *open_segment_image_interval split*: *if_split_asm*)
define *c* **where** *c i* = (*1* − *u*) ∗ *a i* + *u* ∗ *b i* **for** *i*
have *cge0*: $\bigwedge i.\ i \in S \Longrightarrow 0 \le c\ i$
using *a b u01* **by** (*simp add*: *c_def*)
have *sumc1*: *sum c S* = *1*
by (*simp add*: *c_def sum.distrib sum_distrib_left* [*symmetric*] *asum bsum*)
have *sumci_xy*: $(\sum i{\in}S.\ c\ i *_R i) = (1 - u) *_R x + u *_R y$
apply (*simp add*: *c_def sum.distrib scaleR_left_distrib*)
by (*simp only*: *scaleR_scaleR* [*symmetric*] *Real_Vector_Spaces.scaleR_right.sum* [*symmetric*] *aeqx beqy*)
show *?thesis*
proof (*cases sum c* (*S* − *T*) = *0*)
case *True*
have *ci0*: $\bigwedge i.\ i \in (S - T) \Longrightarrow c\ i = 0$
using *True cge0 fin*(*2*) *sum_nonneg_eq_0_iff* **by** *auto*
have *a0*: *a i* = *0* **if** *i* ∈ (*S* − *T*) **for** *i*
using *ci0* [*OF that*] *u01 a* [*of i*] *b* [*of i*] *that*
by (*simp add*: *c_def Groups.ordered_comm_monoid_add_class.add_nonneg_eq_0_iff*)
have [*simp*]: *sum a T* = *1*
using *assms* **by** (*metis sum.mono_neutral_cong_right a0 asum*)
show *?thesis*
apply (*simp add*: *convex_hull_finite* ‹*finite T*›)
apply (*rule_tac x*=*a* **in** *exI*)
using *a0 assms*
apply (*auto simp*: *cge0 a aeqx* [*symmetric*] *sum.mono_neutral_right*)
done
next
case *False*
define *k* **where** *k* = *sum c* (*S* − *T*)
have *k* > *0* **using** *False*
unfolding *k_def* **by** (*metis DiffD1 antisym_conv cge0 sum_nonneg not_less*)
have *weq_sumsum*: $w = sum\ (\lambda x.\ c\ x *_R x)\ T + sum\ (\lambda x.\ c\ x *_R x)\ (S - T)$
by (*metis* (*no_types*) *add.commute S*(*1*) *S*(*2*) *sum.subset_diff sumci_xy weq*)
show *?thesis*
proof (*cases k* = *1*)
case *True*
then have *sum c T* = *0*
by (*simp add*: *S k_def sum_diff sumc1*)
then have [*simp*]: *sum c* (*S* − *T*) = *1*
by (*simp add*: *S sum_diff sumc1*)
have *ci0*: $\bigwedge i.\ i \in T \Longrightarrow c\ i = 0$
by (*meson* ‹*finite T*› ‹*sum c T* = *0*› ‹*T* ⊆ *S*› *cge0 sum_nonneg_eq_0_iff subsetCE*)
then have [*simp*]: $(\sum i{\in}S{-}T.\ c\ i *_R i) = w$
by (*simp add*: *weq_sumsum*)
have *w* ∈ *convex hull* (*S* − *T*)
apply (*simp add*: *convex_hull_finite fin*)
apply (*rule_tac x*=*c* **in** *exI*)

```
        apply (auto simp: cge0 weq True k_def)
        done
      then show ?thesis
        using disj waff by blast
    next
      case False
      then have sumcf: sum c T = 1 − k
        by (simp add: S k_def sum_diff sumc1)
      have ge0: ⋀x. x ∈ T ⟹ 0 ≤ inverse (1 − k) ∗ c x
     by (metis ‹T ⊆ S› cge0 inverse_nonnegative_iff_nonnegative mult_nonneg_nonneg
subsetD sum_nonneg sumcf)
      have eq1: (∑x∈T. inverse (1 − k) ∗ c x) = 1
        by (metis False eq_iff_diff_eq_0 mult.commute right_inverse sum_distrib_left
sumcf)
      have (∑i∈T. c i ∗R i) /R (1 − k) ∈ convex hull T
        apply (simp add: convex_hull_finite fin)
        apply (rule_tac x=λi. inverse (1−k) ∗ c i in exI)
          by (metis (mono_tags, lifting) eq1 ge0 scaleR_scaleR scale_sum_right
sum.cong)
      with ‹0 < k› have inverse(k) ∗R (w − sum (λi. c i ∗R i) T) ∈ affine hull
T
          by (simp add: affine_diff_divide [OF affine_affine_hull] False waff con-
vex_hull_subset_affine_hull [THEN subsetD])
      moreover have inverse(k) ∗R (w − sum (λx. c x ∗R x) T) ∈ convex hull
(S − T)
        apply (simp add: weq_sumsum convex_hull_finite fin)
        apply (rule_tac x=λi. inverse k ∗ c i in exI)
        using ‹k > 0› cge0
          apply (auto simp: scaleR_right.sum sum_distrib_left [symmetric] k_def
[symmetric])
        done
      ultimately show ?thesis
        using disj by blast
    qed
  qed
 qed
 have [simp]: convex hull T ⊆ convex hull S
   by (simp add: ‹T ⊆ S› hull_mono)
 show ?thesis
   using open_segment_commute by (auto simp: face_of_def intro: ∗)
qed

proposition face_of_convex_hull_insert:
 assumes finite S a ∉ affine hull S and T: T face_of convex hull S
 shows T face_of convex hull insert a S
proof −
 have convex hull S face_of convex hull insert a S
   by (simp add: assms face_of_convex_hulls insert_Diff_if subset_insertI)
 then show ?thesis
```

using *T face_of_trans* **by** *blast*
qed

proposition *face_of_affine_trivial*:
assumes *affine S T face_of S*
shows $T = \{\} \vee T = S$
proof (*rule ccontr*, *clarsimp*)
assume $T \neq \{\}$ $T \neq S$
then obtain *a* **where** $a \in T$ **by** *auto*
then have $a \in S$
using ‹*T face_of S*› *face_of_imp_subset* **by** *blast*
have $S \subseteq T$
proof
fix *b* **assume** $b \in S$
show $b \in T$
proof (*cases* $a = b$)
case *True* **with** ‹$a \in T$› **show** *?thesis* **by** *auto*
next
case *False*
then have $a \neq 2 *_R a - b$
by (*simp add*: *scaleR_2*)
with *False* **have** $a \in$ *open_segment* $(2 *_R a - b)$ *b*
apply (*clarsimp simp*: *open_segment_def closed_segment_def*)
apply (*rule_tac x=1/2* **in** *exI*)
by (*simp add*: *algebra_simps*)
moreover have $2 *_R a - b \in S$
by (*rule mem_affine* [*OF* ‹*affine S*› ‹$a \in S$› ‹$b \in S$›, *of 2 −1*, *simplified*])
moreover note ‹$b \in S$› ‹$a \in T$›
ultimately show *?thesis*
by (*rule face_ofD* [*OF* ‹*T face_of S*›, *THEN conjunct2*])
qed
qed
then show *False*
using ‹$T \neq S$› ‹*T face_of S*› *face_of_imp_subset* **by** *blast*
qed

lemma *face_of_affine_eq*:
affine S $\Longrightarrow$ (*T face_of S* $\longleftrightarrow$ $T = \{\} \vee T = S$)
using *affine_imp_convex face_of_affine_trivial face_of_refl* **by** *auto*

proposition *Inter_faces_finite_altbound*:
fixes *T* :: *'a::euclidean_space set set*
assumes *cfaI*: $\bigwedge c.\ c \in T \Longrightarrow c$ *face_of S*
shows $\exists F'.$ *finite* $F' \wedge F' \subseteq T \wedge$ *card* $F' \leq DIM('a) + 2 \wedge \bigcap F' = \bigcap T$
proof (*cases* $\forall F'.$ *finite* $F' \wedge F' \subseteq T \wedge$ *card* $F' \leq DIM('a) + 2 \longrightarrow (\exists c.\ c \in T$ $\wedge\ c \cap (\bigcap F') \subset (\bigcap F')))$
case *True*

then obtain *c* **where** *c*:
$\bigwedge F'.$ [*finite* F'; $F' \subseteq T$; *card* $F' \le DIM('a) + 2$] $\Longrightarrow c\ F' \in T \wedge c\ F' \cap (\bigcap F') \subset (\bigcap F')$
by *metis*
define *d* **where** $d = rec\_nat\ \{c\{\}\}\ (\lambda n\ r.\ insert\ (c\ r)\ r)$
have [*simp*]: $d\ 0 = \{c\ \{\}\}$
by (*simp add*: *d_def*)
have *dSuc* [*simp*]: $\bigwedge n.\ d\ (Suc\ n) = insert\ (c\ (d\ n))\ (d\ n)$
by (*simp add*: *d_def*)
have *dn_notempty*: $d\ n \neq \{\}$ **for** *n*
by (*induction n*) *auto*
have *dn_le_Suc*: $d\ n \subseteq T \wedge finite(d\ n) \wedge card(d\ n) \le Suc\ n$ **if** $n \le DIM('a) + 2$ **for** *n*
using *that*
proof (*induction n*)
case *0*
then show *?case* **by** (*simp add*: *c*)
next
case (*Suc n*)
then show *?case* **by** (*auto simp*: *c card_insert_if*)
qed
have *aff_dim_le*: $aff\_dim(\bigcap(d\ n)) \le DIM('a) - int\ n$ **if** $n \le DIM('a) + 2$ **for** *n*
using *that*
proof (*induction n*)
case *0*
then show *?case*
by (*simp add*: *aff_dim_le_DIM*)
next
case (*Suc n*)
have *fs*: $\bigcap(d\ (Suc\ n))$ *face_of S*
by (*meson Suc.prems cfaI dn_le_Suc dn_notempty face_of_Inter subsetCE*)
have *condn*: *convex* $(\bigcap(d\ n))$
using *Suc.prems nat_le_linear not_less_eq_eq*
by (*blast intro*: *face_of_imp_convex cfaI convex_Inter dest*: *dn_le_Suc*)
have *fdn*: $\bigcap(d\ (Suc\ n))$ *face_of* $\bigcap(d\ n)$
by (*metis* (*no_types*, *lifting*) *Inter_anti_mono Suc.prems dSuc cfaI dn_le_Suc dn_notempty face_of_Inter face_of_imp_subset face_of_subset subset_iff subset_insertI*)
have *ne*: $\bigcap(d\ (Suc\ n)) \neq \bigcap(d\ n)$
by (*metis* (*no_types*, *lifting*) *Suc.prems Suc_leD c complete_lattice_class.Inf_insert dSuc dn_le_Suc less_irrefl order.trans*)
have $*$: $\bigwedge m{::}int.\ \bigwedge d.\ \bigwedge d'{::}int.\ d < d' \wedge d' \le m - n \Longrightarrow d \le m - of\_nat(n+1)$
by *arith*
have $aff\_dim\ (\bigcap(d\ (Suc\ n))) < aff\_dim\ (\bigcap(d\ n))$
by (*rule face_of_aff_dim_lt* [*OF condn fdn ne*])
moreover have $aff\_dim\ (\bigcap(d\ n)) \le int\ (DIM('a)) - int\ n$
using *Suc* **by** *auto*
ultimately
have $aff\_dim\ (\bigcap(d\ (Suc\ n))) \le int\ (DIM('a)) - (n+1)$ **by** *arith*

then show *?case* **by** *linarith*
qed
have *aff_dim* $(\bigcap(d\ (DIM('a) + 2))) \leq -2$
using *aff_dim_le* [*OF order_refl*] **by** *simp*
with *aff_dim_geq* [*of* $\bigcap(d\ (DIM('a) + 2))$] **show** *?thesis*
using *order.trans* **by** *fastforce*
next
case *False*
then show *?thesis*
apply *simp*
apply (*erule ex_forward*)
by *blast*
qed

lemma *faces_of_translation*:
$\{F.\ F$ *face_of image* $(\lambda x.\ a + x)\ S\} =$ *image* (*image* $(\lambda x.\ a + x))\ \{F.\ F$ *face_of* $S\}$
proof −
have $\bigwedge F.\ F$ *face_of* $(+)\ a$ ` $S \Longrightarrow \exists\, G.\ G$ *face_of* $S \wedge F = (+)\ a$ ` G
by (*metis face_of_imp_subset face_of_translation_eq subset_imageE*)
then show *?thesis*
by (*auto simp*: *image_iff*)
qed

proposition *face_of_Times*:
assumes F *face_of* S **and** F' *face_of* S'
shows $(F \times F')$ *face_of* $(S \times S')$
proof −
have $F \times F' \subseteq S \times S'$
using *assms* [*unfolded face_of_def*] **by** *blast*
moreover
have *convex* $(F \times F')$
using *assms* [*unfolded face_of_def*] **by** (*blast intro*: *convex_Times*)
moreover
have $a \in F \wedge a' \in F' \wedge b \in F \wedge b' \in F'$
if $a \in S\ b \in S\ a' \in S'\ b' \in S'\ x \in F \times F'\ x \in$ *open_segment* $(a,a')\ (b,b')$
for $a\ b\ a'\ b'\ x$
proof (*cases* $b{=}a \vee b'{=}a'$)
case *True* **with** *that* **show** *?thesis*
using *assms*
by (*force simp*: *in_segment dest*: *face_ofD*)
next
case *False* **with** *assms* [*unfolded face_of_def*] *that* **show** *?thesis*
by (*blast dest*!: *open_segment_PairD*)
qed
ultimately show *?thesis*
unfolding *face_of_def* **by** *blast*
qed

```
corollary face_of_Times_decomp:
  fixes S :: 'a::euclidean_space set and S' :: 'b::euclidean_space set
  shows C face_of (S × S') ⟷ (∃ F F'. F face_of S ∧ F' face_of S' ∧ C = F
× F')
   (is ?lhs = ?rhs)
proof
 assume C: ?lhs
 show ?rhs
 proof (cases C = {})
  case True then show ?thesis by auto
 next
  case False
  have 1: fst ` C ⊆ S snd ` C ⊆ S'
   using C face_of_imp_subset by fastforce+
  have convex C
   using C by (metis face_of_imp_convex)
  have conv: convex (fst ` C) convex (snd ` C)
   by (simp_all add: ‹convex C› convex_linear_image linear_fst linear_snd)
  have fstab: a ∈ fst ` C ∧ b ∈ fst ` C
       if a ∈ S b ∈ S x ∈ open_segment a b (x,x') ∈ C for a b x x'
  proof -
   have *: (x,x') ∈ open_segment (a,x') (b,x')
    using that by (auto simp: in_segment)
   show ?thesis
    using face_ofD [OF C *] that face_of_imp_subset [OF C] by force
  qed
  have fst: fst ` C face_of S
   by (force simp: face_of_def 1 conv fstab)
  have sndab: a' ∈ snd ` C ∧ b' ∈ snd ` C
   if a' ∈ S' b' ∈ S' x' ∈ open_segment a' b' (x,x') ∈ C for a' b' x x'
  proof -
   have *: (x,x') ∈ open_segment (x,a') (x,b')
    using that by (auto simp: in_segment)
   show ?thesis
    using face_ofD [OF C *] that face_of_imp_subset [OF C] by force
  qed
  have snd: snd ` C face_of S'
   by (force simp: face_of_def 1 conv sndab)
  have cc: rel_interior C ⊆ rel_interior (fst ` C) × rel_interior (snd ` C)
    by (force simp: face_of_Times rel_interior_Times conv fst snd ‹convex C›
linear_fst linear_snd rel_interior_convex_linear_image [symmetric])
  have C = fst ` C × snd ` C
  proof (rule face_of_eq [OF C])
   show fst ` C × snd ` C face_of S × S'
    by (simp add: face_of_Times rel_interior_Times conv fst snd)
   show rel_interior C ∩ rel_interior (fst ` C × snd ` C) ≠ {}
    using False rel_interior_eq_empty ‹convex C› cc
    by (auto simp: face_of_Times rel_interior_Times conv fst)
  qed
```

with *fst snd* **show** *?thesis* **by** *metis*
qed
next
assume *?rhs* **with** *face_of_Times* **show** *?lhs* **by** *auto*
qed

lemma *face_of_Times_eq*:
fixes S :: $'a$::*euclidean_space set* **and** S' :: $'b$::*euclidean_space set*
shows $(F \times F')$ *face_of* $(S \times S') \longleftrightarrow$
$F = \{\} \vee F' = \{\} \vee F$ *face_of* $S \wedge F'$ *face_of* S'
by (*auto simp*: *face_of_Times_decomp times_eq_iff*)

lemma *hyperplane_face_of_halfspace_le*: $\{x.\ a \cdot x = b\}$ *face_of* $\{x.\ a \cdot x \leq b\}$
proof −
have $\{x.\ a \cdot x \leq b\} \cap \{x.\ a \cdot x = b\} = \{x.\ a \cdot x = b\}$
by *auto*
with *face_of_Int_supporting_hyperplane_le* [*OF convex_halfspace_le* [*of a b*], *of a b*]
show *?thesis* **by** *auto*
qed

lemma *hyperplane_face_of_halfspace_ge*: $\{x.\ a \cdot x = b\}$ *face_of* $\{x.\ a \cdot x \geq b\}$
proof −
have $\{x.\ a \cdot x \geq b\} \cap \{x.\ a \cdot x = b\} = \{x.\ a \cdot x = b\}$
by *auto*
with *face_of_Int_supporting_hyperplane_ge* [*OF convex_halfspace_ge* [*of b a*], *of b a*]
show *?thesis* **by** *auto*
qed

lemma *face_of_halfspace_le*:
fixes a :: $'n$::*euclidean_space*
shows F *face_of* $\{x.\ a \cdot x \leq b\} \longleftrightarrow$
$F = \{\} \vee F = \{x.\ a \cdot x = b\} \vee F = \{x.\ a \cdot x \leq b\}$
(**is** *?lhs* = *?rhs*)
proof (*cases* $a = 0$)
case *True* **then show** *?thesis*
using *face_of_affine_eq affine_UNIV* **by** *auto*
next
case *False*
then have *ine*: *interior* $\{x.\ a \cdot x \leq b\} \neq \{\}$
using *halfspace_eq_empty_lt interior_halfspace_le* **by** *blast*
show *?thesis*
proof
assume L: *?lhs*
have F *face_of* $\{x.\ a \cdot x = b\}$ **if** $F \neq \{x.\ a \cdot x \leq b\}$
proof −
have F *face_of rel_frontier* $\{x.\ a \cdot x \leq b\}$
proof (*rule face_of_subset* [*OF L*])
show $F \subseteq$ *rel_frontier* $\{x.\ a \cdot x \leq b\}$

```
          by (simp add: L face_of_subset_rel_frontier that)
        qed (force simp: rel_frontier_def closed_halfspace_le)
        then show ?thesis
          using False
          by (simp add: frontier_halfspace_le rel_frontier_nonempty_interior [OF ine])
      qed
      with L show ?rhs
        using affine_hyperplane face_of_affine_eq by blast
  next
    assume ?rhs
    then show ?lhs
     by (metis convex_halfspace_le empty_face_of face_of_refl hyperplane_face_of_halfspace_le)
  qed
qed

lemma face_of_halfspace_ge:
  fixes a :: 'n::euclidean_space
  shows F face_of {x. a · x ≥ b} ⟷
        F = {} ∨ F = {x. a · x = b} ∨ F = {x. a · x ≥ b}
using face_of_halfspace_le [of F −a −b] by simp
```

6.38.2 Exposed faces

That is, faces that are intersection with supporting hyperplane

```
definition exposed_face_of :: ['a::euclidean_space set, 'a set] ⇒ bool
                              (infixr (exposed'_face'_of) 50)
  where T exposed_face_of S ⟷
        T face_of S ∧ (∃ a b. S ⊆ {x. a · x ≤ b} ∧ T = S ∩ {x. a · x = b})

lemma empty_exposed_face_of [iff]: {} exposed_face_of S
  apply (simp add: exposed_face_of_def)
  apply (rule_tac x=0 in exI)
  apply (rule_tac x=1 in exI, force)
  done

lemma exposed_face_of_refl_eq [simp]: S exposed_face_of S ⟷ convex S
proof
  assume S: convex S
  have S ⊆ {x. 0 · x ≤ 0} ∧ S = S ∩ {x. 0 · x = 0}
    by auto
  with S show S exposed_face_of S
    using exposed_face_of_def face_of_refl_eq by blast
qed (simp add: exposed_face_of_def face_of_refl_eq)

lemma exposed_face_of_refl: convex S ⟹ S exposed_face_of S
  by simp

lemma exposed_face_of:
    T exposed_face_of S ⟷
```

```
    T face_of S ∧
    (T = {} ∨ T = S ∨
     (∃ a b. a ≠ 0 ∧ S ⊆ {x. a · x ≤ b} ∧ T = S ∩ {x. a · x = b}))
proof (cases T = {})
  case True then show ?thesis
    by simp
next
  case False
  show ?thesis
  proof (cases T = S)
    case True then show ?thesis
      by (simp add: face_of_refl_eq)
  next
    case False
    with ‹T ≠ {}› show ?thesis
      apply (auto simp: exposed_face_of_def)
      apply (metis inner_zero_left)
      done
  qed
qed

lemma exposed_face_of_Int_supporting_hyperplane_le:
   ⟦convex S; ⋀x. x ∈ S ⟹ a · x ≤ b⟧ ⟹ (S ∩ {x. a · x = b}) exposed_face_of
S
by (force simp: exposed_face_of_def face_of_Int_supporting_hyperplane_le)

lemma exposed_face_of_Int_supporting_hyperplane_ge:
   ⟦convex S; ⋀x. x ∈ S ⟹ a · x ≥ b⟧ ⟹ (S ∩ {x. a · x = b}) exposed_face_of
S
using exposed_face_of_Int_supporting_hyperplane_le [of S −a −b] by simp

proposition exposed_face_of_Int:
  assumes T exposed_face_of S
      and u exposed_face_of S
    shows (T ∩ u) exposed_face_of S
proof −
  obtain a b where T: S ∩ {x. a · x = b} face_of S
             and S: S ⊆ {x. a · x ≤ b}
             and teq: T = S ∩ {x. a · x = b}
    using assms by (auto simp: exposed_face_of_def)
  obtain a' b' where u: S ∩ {x. a' · x = b'} face_of S
               and s': S ⊆ {x. a' · x ≤ b'}
               and ueq: u = S ∩ {x. a' · x = b'}
    using assms by (auto simp: exposed_face_of_def)
  have tu: T ∩ u face_of S
    using T teq u ueq by (simp add: face_of_Int)
  have ss: S ⊆ {x. (a + a') · x ≤ b + b'}
    using S s' by (force simp: inner_left_distrib)
  show ?thesis
```

apply (*simp add*: *exposed_face_of_def tu*)
apply (*rule_tac x*=*a*+*a'* **in** *exI*)
apply (*rule_tac x*=*b*+*b'* **in** *exI*)
using *S s'*
apply (*fastforce simp*: *ss inner_left_distrib teq ueq*)
done
qed

proposition *exposed_face_of_Inter*:
fixes *P* :: *'a::euclidean_space set set*
assumes $P \neq \{\}$
and $\bigwedge T.\ T \in P \Longrightarrow T$ *exposed_face_of S*
shows $\bigcap P$ *exposed_face_of S*
proof −
obtain *Q* **where** *finite Q* **and** *QsubP*: $Q \subseteq P$ *card Q* $\leq$ *DIM*('a) + *2* **and** *IntQ*: $\bigcap Q = \bigcap P$
using *Inter_faces_finite_altbound* [*of P S*] *assms* [*unfolded exposed_face_of*]
by *force*
show *?thesis*
proof (*cases Q* = {})
case *True* **then show** *?thesis*
by (*metis IntQ Inter_UNIV_conv*(*2*) *assms*(*1*) *assms*(*2*) *ex_in_conv*)
next
case *False*
have $Q \subseteq \{T.\ T$ *exposed_face_of S*}
using *QsubP assms* **by** *blast*
moreover have $Q \subseteq \{T.\ T$ *exposed_face_of S*} $\Longrightarrow \bigcap Q$ *exposed_face_of S*
using ‹*finite Q*› *False*
by (*induction Q rule*: *finite_induct*; *use exposed_face_of_Int* **in** *fastforce*)
ultimately show *?thesis*
by (*simp add*: *IntQ*)
qed
qed

proposition *exposed_face_of_sums*:
assumes *convex S* **and** *convex T*
and *F exposed_face_of* $\{x + y \mid x\ y.\ x \in S \wedge y \in T\}$
(**is** *F exposed_face_of ?ST*)
obtains *k l*
where *k exposed_face_of S l exposed_face_of T*
$F = \{x + y \mid x\ y.\ x \in k \wedge y \in l\}$
proof (*cases F* = {})
case *True* **then show** *?thesis*
using *that* **by** *blast*
next
case *False*
show *?thesis*
proof (*cases F* = *?ST*)
case *True* **then show** *?thesis*

using *assms exposed_face_of_refl_eq that* **by** *blast*
next
case *False*
obtain *p* **where** $p \in F$ **using** ‹$F \neq \{\}$› **by** *blast*
moreover
obtain *u z* **where** *T*: *?ST* $\cap$ {*x*. $u \cdot x = z$} *face_of ?ST*
and *S*: *?ST* $\subseteq$ {*x*. $u \cdot x \leq z$}
and *feq*: $F =$ *?ST* $\cap$ {*x*. $u \cdot x = z$}
using *assms* **by** (*auto simp*: *exposed_face_of_def*)
ultimately obtain *a0 b0*
where *p*: $p = a0 + b0$ **and** $a0 \in S$ $b0 \in T$ **and** *z*: $u \cdot p = z$
by *auto*
have *lez*: $u \cdot (x + y) \leq z$ **if** $x \in S$ $y \in T$ **for** *x y*
using *S that* **by** *auto*
have *sef*: $S \cap \{x.\ u \cdot x = u \cdot a0\}$ *exposed_face_of S*
proof (*rule exposed_face_of_Int_supporting_hyperplane_le* [*OF* ‹*convex S*›])
show $\bigwedge x.\ x \in S \Longrightarrow u \cdot x \leq u \cdot a0$
by (*metis p z add_le_cancel_right inner_right_distrib lez* [*OF* _ ‹$b0 \in T$›])
qed
have *tef*: $T \cap \{x.\ u \cdot x = u \cdot b0\}$ *exposed_face_of T*
proof (*rule exposed_face_of_Int_supporting_hyperplane_le* [*OF* ‹*convex T*›])
show $\bigwedge x.\ x \in T \Longrightarrow u \cdot x \leq u \cdot b0$
by (*metis p z add.commute add_le_cancel_right inner_right_distrib lez* [*OF* ‹$a0 \in S$›])
qed
have $\{x + y \mid x\ y.\ x \in S \wedge u \cdot x = u \cdot a0 \wedge y \in T \wedge u \cdot y = u \cdot b0\} \subseteq F$
by (*auto simp*: *feq*) (*metis inner_right_distrib p z*)
moreover have $F \subseteq \{x + y \mid x\ y.\ x \in S \wedge u \cdot x = u \cdot a0 \wedge y \in T \wedge u \cdot y = u \cdot b0\}$
proof −
have $\bigwedge x\ y.\ [\![z = u \cdot (x + y);\ x \in S;\ y \in T]\!]$
$\Longrightarrow u \cdot x = u \cdot a0 \wedge u \cdot y = u \cdot b0$
using *z p* ‹$a0 \in S$› ‹$b0 \in T$›
apply (*simp add*: *inner_right_distrib*)
apply (*metis add_le_cancel_right antisym lez* [*unfolded inner_right_distrib*] *add.commute*)
done
then show *?thesis*
using *feq* **by** *blast*
qed
ultimately have $F = \{x + y \mid x\ y.\ x \in S \cap \{x.\ u \cdot x = u \cdot a0\} \wedge y \in T \cap \{x.\ u \cdot x = u \cdot b0\}\}$
by *blast*
then show *?thesis*
by (*rule that* [*OF sef tef*])
qed
qed

proposition *exposed_face_of_parallel*:

```
  T exposed_face_of S ⟷
      T face_of S ∧
      (∃ a b. S ⊆ {x. a · x ≤ b} ∧ T = S ∩ {x. a · x = b} ∧
           (T ≠ {} ⟶ T ≠ S ⟶ a ≠ 0) ∧
           (T ≠ S ⟶ (∀ w ∈ affine hull S. (w + a) ∈ affine hull S)))
 (is ?lhs = ?rhs)
proof
 assume ?lhs then show ?rhs
 proof (clarsimp simp: exposed_face_of_def)
   fix a b
   assume faceS: S ∩ {x. a · x = b} face_of S and Ssub: S ⊆ {x. a · x ≤ b}
   show ∃ c d. S ⊆ {x. c · x ≤ d} ∧
              S ∩ {x. a · x = b} = S ∩ {x. c · x = d} ∧
             (S ∩ {x. a · x = b} ≠ {} ⟶ S ∩ {x. a · x = b} ≠ S ⟶ c ≠ 0) ∧
              (S ∩ {x. a · x = b} ≠ S ⟶ (∀ w ∈ affine hull S. w + c ∈ affine
hull S))
    proof (cases affine hull S ∩ {x. -a · x ≤ -b} = {} ∨ affine hull S ⊆ {x. −
a · x ≤ − b})
      case True
      then show ?thesis
      proof
        assume affine hull S ∩ {x. − a · x ≤ − b} = {}
       then show ?thesis
         apply (rule_tac x=0 in exI)
         apply (rule_tac x=1 in exI)
         using hull_subset by fastforce
   next
      assume affine hull S ⊆ {x. − a · x ≤ − b}
      then show ?thesis
         apply (rule_tac x=0 in exI)
         apply (rule_tac x=0 in exI)
        using Ssub hull_subset by fastforce
   qed
 next
   case False
   then obtain a' b' where a' ≠ 0
    and le: affine hull S ∩ {x. a' · x ≤ b'} = affine hull S ∩ {x. − a · x ≤ − b}
    and eq: affine hull S ∩ {x. a' · x = b'} = affine hull S ∩ {x. − a · x = − b}
     and mem: ⋀w. w ∈ affine hull S ⟹ w + a' ∈ affine hull S
     using affine_parallel_slice affine_affine_hull by metis
   show ?thesis
   proof (intro conjI impI allI ballI exI)
     have *: S ⊆ − (affine hull S ∩ {x. P x}) ∪ affine hull S ∩ {x. Q x} ⟹ S
⊆ {x. ¬ P x ∨ Q x}
        for P Q
        using hull_subset by fastforce
      have S ⊆ {x. ¬ (a' · x ≤ b') ∨ a' · x = b'}
        by (rule *) (use le eq Ssub in auto)
      then show S ⊆ {x. − a' · x ≤ − b'}
```

        by *auto*
      show $S \cap \{x.\ a \cdot x = b\} = S \cap \{x.\ -\ a' \cdot x = -\ b'\}$
        using *eq hull_subset* [*of S affine*] **by** *force*
      show $[\![S \cap \{x.\ a \cdot x = b\} \neq \{\};\ S \cap \{x.\ a \cdot x = b\} \neq S]\!] \Longrightarrow -\ a' \neq 0$
        using ‹$a' \neq 0$› **by** *auto*
      show $w + -\ a' \in$ *affine hull S*
        if $S \cap \{x.\ a \cdot x = b\} \neq S$ $w \in$ *affine hull S* **for** *w*
      proof −
        have $w + 1 *_R (w - (w + a')) \in$ *affine hull S*
          using *affine_affine_hull mem mem_affine_3_minus that*(*2*) **by** *blast*
        then show *?thesis* **by** *simp*
      qed
    qed
  qed
qed
next
  assume *?rhs* **then show** *?lhs*
    unfolding *exposed_face_of_def* **by** *blast*
qed

6.38.3 Extreme points of a set: its singleton faces

definition *extreme_point_of* :: ['*a*::*real_vector*, '*a set*] ⇒ *bool*
                (**infixr** (*extreme'_point'_of*) *50*)
  where *x extreme_point_of S* ⟷
      $x \in S \wedge (\forall a \in S.\ \forall b \in S.\ x \notin$ *open_segment a b*)

lemma *extreme_point_of_stillconvex*:
  convex S ⟹ (*x extreme_point_of S* ⟷ $x \in S \wedge$ *convex*$(S - \{x\})$)
 by (*fastforce simp add*: *convex_contains_segment extreme_point_of_def open_segment_def*)

lemma *face_of_singleton*:
  $\{x\}$ *face_of S* ⟷ *x extreme_point_of S*
by (*fastforce simp add*: *extreme_point_of_def face_of_def*)

lemma *extreme_point_not_in_REL_INTERIOR*:
   fixes *S* :: '*a*::*real_normed_vector set*
   shows [[*x extreme_point_of S*; $S \neq \{x\}$]] ⟹ $x \notin$ *rel_interior S*
  by (*metis disjoint_iff face_of_disjoint_rel_interior face_of_singleton insertI1*)

lemma *extreme_point_not_in_interior*:
  fixes *S* :: '*a*::{*real_normed_vector*, *perfect_space*} *set*
  assumes *x extreme_point_of S* **shows** $x \notin$ *interior S*
proof (*cases* $S = \{x\}$)
  case *False*
  then show *?thesis*
   by (*meson assms subsetD extreme_point_not_in_REL_INTERIOR interior_subset_rel_interior*)
qed (*simp add*: *empty_interior_finite*)

lemma *extreme_point_of_face*:
F face_of S $\Longrightarrow$ *v extreme_point_of F* $\longleftrightarrow$ *v extreme_point_of S* $\wedge$ *v* $\in$ *F*
by (*meson empty_subsetI face_of_face face_of_singleton insert_subset*)

lemma *extreme_point_of_convex_hull*:
x extreme_point_of (*convex hull S*) $\Longrightarrow$ *x* $\in$ *S*
using *hull_minimal* [*of S* (*convex hull S*) $-$ {*x*} *convex*]
using *hull_subset* [*of S convex*]
by (*force simp add*: *extreme_point_of_stillconvex*)

proposition *extreme_points_of_convex_hull*:
{*x*. *x extreme_point_of* (*convex hull S*)} $\subseteq$ *S*
using *extreme_point_of_convex_hull* **by** *auto*

lemma *extreme_point_of_empty* [*simp*]: $\neg$ (*x extreme_point_of* {})
by (*simp add*: *extreme_point_of_def*)

lemma *extreme_point_of_singleton* [*iff*]: *x extreme_point_of* {*a*} $\longleftrightarrow$ *x* = *a*
using *extreme_point_of_stillconvex* **by** *auto*

lemma *extreme_point_of_translation_eq*:
(*a* + *x*) *extreme_point_of* (*image* (λ*x*. *a* + *x*) *S*) $\longleftrightarrow$ *x extreme_point_of S*
by (*auto simp*: *extreme_point_of_def*)

lemma *extreme_points_of_translation*:
{*x*. *x extreme_point_of* (*image* (λ*x*. *a* + *x*) *S*)} =
(λ*x*. *a* + *x*) ‘ {*x*. *x extreme_point_of S*}
using *extreme_point_of_translation_eq*
by *auto* (*metis* (*no_types*, *lifting*) *image_iff mem_Collect_eq minus_add_cancel*)

lemma *extreme_points_of_translation_subtract*:
{*x*. *x extreme_point_of* (*image* (λ*x*. *x* $-$ *a*) *S*)} =
(λ*x*. *x* $-$ *a*) ‘ {*x*. *x extreme_point_of S*}
using *extreme_points_of_translation* [*of* $-$ *a S*]
by *simp*

lemma *extreme_point_of_Int*:
$[\![$*x extreme_point_of S*; *x extreme_point_of T*$]\!]$ $\Longrightarrow$ *x extreme_point_of* (*S* $\cap$ *T*)
by (*simp add*: *extreme_point_of_def*)

lemma *extreme_point_of_Int_supporting_hyperplane_le*:
$[\![$*S* $\cap$ {*x*. *a* $\cdot$ *x* = *b*} = {*c*}; $\bigwedge$*x*. *x* $\in$ *S* $\Longrightarrow$ *a* $\cdot$ *x* $\le$ *b*$]\!]$ $\Longrightarrow$ *c extreme_point_of S*
by (*metis convex_singleton face_of_Int_supporting_hyperplane_le_strong face_of_singleton*)

lemma *extreme_point_of_Int_supporting_hyperplane_ge*:
$[\![$*S* $\cap$ {*x*. *a* $\cdot$ *x* = *b*} = {*c*}; $\bigwedge$*x*. *x* $\in$ *S* $\Longrightarrow$ *a* $\cdot$ *x* $\ge$ *b*$]\!]$ $\Longrightarrow$ *c extreme_point_of S*
using *extreme_point_of_Int_supporting_hyperplane_le* [*of S* $-$*a* $-$*b c*]
by *simp*

lemma *exposed_point_of_Int_supporting_hyperplane_le*:
$[\![S \cap \{x.\ a \cdot x = b\} = \{c\}; \bigwedge x.\ x \in S \Longrightarrow a \cdot x \le b]\!] \Longrightarrow \{c\}$ *exposed_face_of S*
unfolding *exposed_face_of_def*
by (*force simp*: *face_of_singleton extreme_point_of_Int_supporting_hyperplane_le*)

lemma *exposed_point_of_Int_supporting_hyperplane_ge*:
$[\![S \cap \{x.\ a \cdot x = b\} = \{c\}; \bigwedge x.\ x \in S \Longrightarrow a \cdot x \ge b]\!] \Longrightarrow \{c\}$ *exposed_face_of S*
using *exposed_point_of_Int_supporting_hyperplane_le* [*of S* $-a$ $-b$ *c*]
by *simp*

lemma *extreme_point_of_convex_hull_insert*:
assumes *finite S a* $\notin$ *convex hull S*
shows *a extreme_point_of* (*convex hull* (*insert a S*))
proof (*cases a* $\in$ *S*)
case *False*
then show *?thesis*
using *face_of_convex_hulls* [*of insert a S* {*a*}] *assms*
by (*auto simp*: *face_of_singleton hull_same*)
qed (*use assms* **in** ‹*simp add*: *hull_inc*›)

6.38.4 Facets

definition *facet_of* :: [*'a*::*euclidean_space set*, *'a set*] ⇒ *bool*
(**infixr** (*facet'_of*) *50*)
where *F facet_of S* ⟷ *F face_of S* ∧ *F* ≠ {} ∧ *aff_dim F* = *aff_dim S* − *1*

lemma *facet_of_empty* [*simp*]: ¬ *S facet_of* {}
by (*simp add*: *facet_of_def*)

lemma *facet_of_irrefl* [*simp*]: ¬ *S facet_of S*
by (*simp add*: *facet_of_def*)

lemma *facet_of_imp_face_of*: *F facet_of S* ⟹ *F face_of S*
by (*simp add*: *facet_of_def*)

lemma *facet_of_imp_subset*: *F facet_of S* ⟹ *F* ⊆ *S*
by (*simp add*: *face_of_imp_subset facet_of_def*)

lemma *hyperplane_facet_of_halfspace_le*:
$a \neq 0 \Longrightarrow \{x.\ a \cdot x = b\}$ *facet_of* $\{x.\ a \cdot x \le b\}$
unfolding *facet_of_def hyperplane_eq_empty*
by (*auto simp*: *hyperplane_face_of_halfspace_ge hyperplane_face_of_halfspace_le Suc_leI of_nat_diff aff_dim_halfspace_le*)

lemma *hyperplane_facet_of_halfspace_ge*:
$a \neq 0 \Longrightarrow \{x.\ a \cdot x = b\}$ *facet_of* $\{x.\ a \cdot x \ge b\}$
unfolding *facet_of_def hyperplane_eq_empty*
by (*auto simp*: *hyperplane_face_of_halfspace_le hyperplane_face_of_halfspace_ge Suc_leI of_nat_diff aff_dim_halfspace_ge*)

lemma *facet_of_halfspace_le*:
  F facet_of {*x*. *a* · *x* ≤ *b*} ⟷ *a* ≠ *0* ∧ *F* = {*x*. *a* · *x* = *b*}
  (**is** *?lhs* = *?rhs*)
proof
 assume *c*: *?lhs*
 with *c facet_of_irrefl* **show** *?rhs*
  by (*force simp*: *aff_dim_halfspace_le facet_of_def face_of_halfspace_le cong*: *conj_cong split*: *if_split_asm*)
next
 assume *?rhs* **then show** *?lhs*
  by (*simp add*: *hyperplane_facet_of_halfspace_le*)
qed

lemma *facet_of_halfspace_ge*:
  F facet_of {*x*. *a* · *x* ≥ *b*} ⟷ *a* ≠ *0* ∧ *F* = {*x*. *a* · *x* = *b*}
using *facet_of_halfspace_le* [*of F* −*a* −*b*] **by** *simp*

6.38.5 Edges: faces of affine dimension 1

definition *edge_of* :: ['*a*::*euclidean_space set*, '*a set*] ⇒ *bool* (**infixr** (*edge*'*_of*) *50*)
 where *e edge_of S* ⟷ *e face_of S* ∧ *aff_dim e* = *1*

lemma *edge_of_imp_subset*:
 S edge_of T ⟹ *S* ⊆ *T*
by (*simp add*: *edge_of_def face_of_imp_subset*)

6.38.6 Existence of extreme points

proposition *different_norm_3_collinear_points*:
 fixes *a* :: '*a*::*euclidean_space*
 assumes *x* ∈ *open_segment a b norm*(*a*) = *norm*(*b*) *norm*(*x*) = *norm*(*b*)
 shows *False*
proof −
 obtain *u* **where** *norm* ((*1* − *u*) $*_R$ *a* + *u* $*_R$ *b*) = *norm b*
     and *a* ≠ *b*
     and *u01*: *0* < *u u* < *1*
  using *assms* **by** (*auto simp*: *open_segment_image_interval if_splits*)
 then have (*1* − *u*) $*_R$ *a* · (*1* − *u*) $*_R$ *a* + ((*1* − *u*) * *2*) $*_R$ *a* · *u* $*_R$ *b* =
     (*1* − *u* * *u*) $*_R$ (*a* · *a*)
  using *assms* **by** (*simp add*: *norm_eq algebra_simps inner_commute*)
 then have (*1* − *u*) $*_R$ ((*1* − *u*) $*_R$ *a* · *a* + (*2* * *u*) $*_R$ *a* · *b*) =
     (*1* − *u*) $*_R$ ((*1* + *u*) $*_R$ (*a* · *a*))
  by (*simp add*: *algebra_simps*)
 then have (*1* − *u*) $*_R$ (*a* · *a*) + (*2* * *u*) $*_R$ (*a* · *b*) = (*1* + *u*) $*_R$ (*a* · *a*)
  using *u01* **by** *auto*
 then have *a* · *b* = *a* · *a*
  using *u01* **by** (*simp add*: *algebra_simps*)
 then have *a* = *b*
  using ‹*norm*(*a*) = *norm*(*b*)› *norm_eq vector_eq* **by** *fastforce*

then show *?thesis*
using ‹$a \neq b$› **by** *force*
qed

proposition *extreme_point_exists_convex*:
fixes S :: *'a::euclidean_space set*
assumes *compact S convex S* $S \neq \{\}$
obtains x **where** *x extreme_point_of S*
proof –
obtain x **where** $x \in S$ **and** *xsup*: $\bigwedge y.\ y \in S \Longrightarrow$ *norm y* $\leq$ *norm x*
using *distance_attains_sup* [*of S 0*] *assms* **by** *auto*
have *False* **if** $a \in S$ $b \in S$ **and** *x*: $x \in$ *open_segment a b* **for** *a b*
proof –
have *noax*: *norm a* $\leq$ *norm x* **and** *nobx*: *norm b* $\leq$ *norm x* **using** *xsup that* **by** *auto*
have $a \neq b$
using *empty_iff open_segment_idem x* **by** *auto*
show *False*
by (*metis dist_0_norm dist_decreases_open_segment noax nobx not_le x*)
qed
then show *?thesis*
by (*meson* ‹$x \in S$› *extreme_point_of_def that*)
qed

6.38.7 Krein-Milman, the weaker form

proposition *Krein_Milman*:
fixes S :: *'a::euclidean_space set*
assumes *compact S convex S*
shows S = *closure*(*convex hull* {*x. x extreme_point_of S*})
proof (*cases* $S = \{\}$)
case *True* **then show** *?thesis* **by** *simp*
next
case *False*
have *closed S*
by (*simp add*: ‹*compact S*› *compact_imp_closed*)
have *closure* (*convex hull* {*x. x extreme_point_of S*}) $\subseteq S$
by (*simp add*: ‹*closed S*› *assms closure_minimal extreme_point_of_def hull_minimal*)
moreover have $u \in$ *closure* (*convex hull* {*x. x extreme_point_of S*})
if $u \in S$ **for** u
proof (*rule ccontr*)
assume *unot*: $u \notin$ *closure*(*convex hull* {*x. x extreme_point_of S*})
then obtain *a b* **where** $a \cdot u < b$
and *ab*: $\bigwedge x.\ x \in$ *closure*(*convex hull* {*x. x extreme_point_of S*}) $\Longrightarrow b < a \cdot x$
using *separating_hyperplane_closed_point* [*of closure*(*convex hull* {*x. x extreme_point_of S*})]
by *blast*
have *continuous_on S* $((\cdot)\ a)$

by (*rule continuous_intros*)+
then obtain *m* **where** $m \in S$ **and** *m*: $\bigwedge y.\ y \in S \Longrightarrow a \cdot m \le a \cdot y$
using *continuous_attains_inf* [*of S* $\lambda x.\ a \cdot x$] ‹*compact S*› ‹$u \in S$›
by *auto*
define *T* **where** $T = S \cap \{x.\ a \cdot x = a \cdot m\}$
have $m \in T$
by (*simp add*: *T_def* ‹$m \in S$›)
moreover have *compact T*
by (*simp add*: *T_def compact_Int_closed* [*OF* ‹*compact S*› *closed_hyperplane*])
moreover have *convex T*
by (*simp add*: *T_def convex_Int* [*OF* ‹*convex S*› *convex_hyperplane*])
ultimately obtain *v* **where** *v*: *v extreme_point_of T*
using *extreme_point_exists_convex* [*of T*] **by** *auto*
then have $\{v\}$ *face_of T*
by (*simp add*: *face_of_singleton*)
also have *T face_of S*
by (*simp add*: *T_def m face_of_Int_supporting_hyperplane_ge* [*OF* ‹*convex S*›])
finally have *v extreme_point_of S*
by (*simp add*: *face_of_singleton*)
then have $b < a \cdot v$
using *closure_subset* **by** (*simp add*: *closure_hull hull_inc ab*)
then show *False*
using ‹$a \cdot u < b$› ‹$\{v\}$ *face_of T*› *face_of_imp_subset m T_def that* **by** *fastforce*
qed
ultimately show *?thesis*
by *blast*
qed

Now the sharper form.

lemma *Krein_Milman_Minkowski_aux*:
fixes *S* :: *'a::euclidean_space set*
assumes *n*: *dim S = n* **and** *S*: *compact S convex S* $0 \in S$
shows $0 \in$ *convex hull* $\{x.\ x$ *extreme_point_of S*$\}$
using *n S*
proof (*induction n arbitrary*: *S rule*: *less_induct*)
case (*less n S*) **show** *?case*
proof (*cases* $0 \in$ *rel_interior S*)
case *True* **with** *Krein_Milman less.prems*
show *?thesis*
by (*metis subsetD convex_convex_hull convex_rel_interior_closure rel_interior_subset*)
next
case *False*
have *rel_interior S* $\neq \{\}$
by (*simp add*: *rel_interior_convex_nonempty_aux less*)
then obtain *c* **where** *c*: $c \in$ *rel_interior S* **by** *blast*
obtain *a* **where** $a \neq 0$
and *le_ay*: $\bigwedge y.\ y \in S \Longrightarrow a \cdot 0 \le a \cdot y$
and *less_ay*: $\bigwedge y.\ y \in$ *rel_interior S* $\Longrightarrow a \cdot 0 < a \cdot y$
by (*blast intro*: *supporting_hyperplane_rel_boundary intro*!: *less False*)

```
    have face: S ∩ {x. a · x = 0} face_of S
      using face_of_Int_supporting_hyperplane_ge le_ay ‹convex S› by auto
    then have co: compact (S ∩ {x. a · x = 0}) convex (S ∩ {x. a · x = 0})
      using less.prems by (blast intro: face_of_imp_compact face_of_imp_convex)+
    have a · y = 0 if y ∈ span (S ∩ {x. a · x = 0}) for y
    proof −
      have y ∈ span {x. a · x = 0}
        by (metis inf.cobounded2 span_mono subsetCE that)
      then show ?thesis
        by (blast intro: span_induct [OF _ subspace_hyperplane])
    qed
    then have dim (S ∩ {x. a · x = 0}) < n
      by (metis (no_types) less_ay c subsetD dim_eq_span inf.strict_order_iff
          inf_le1 ‹dim S = n› not_le rel_interior_subset span_0 span_base)
    then have 0 ∈ convex hull {x. x extreme_point_of (S ∩ {x. a · x = 0})}
      by (rule less.IH) (auto simp: co less.prems)
    then show ?thesis
        by (metis (mono_tags, lifting) Collect_mono_iff face extreme_point_of_face
hull_mono subset_iff)
  qed
qed


theorem Krein_Milman_Minkowski:
  fixes S :: 'a::euclidean_space set
  assumes compact S convex S
    shows S = convex hull {x. x extreme_point_of S}
proof
  show S ⊆ convex hull {x. x extreme_point_of S}
  proof
    fix a assume [simp]: a ∈ S
    have 1: compact ((+) (− a) ‘ S)
     by (simp add: ‹compact S› compact_translation_subtract cong: image_cong_simp)
    have 2: convex ((+) (− a) ‘ S)
      by (simp add: ‹convex S› compact_translation_subtract)
    show a_invex: a ∈ convex hull {x. x extreme_point_of S}
      using Krein_Milman_Minkowski_aux [OF refl 1 2]
            convex_hull_translation [of −a]
       by (auto simp: extreme_points_of_translation_subtract translation_assoc cong:
image_cong_simp)
    qed
next
  show convex hull {x. x extreme_point_of S} ⊆ S
  proof −
    have {a. a extreme_point_of S} ⊆ S
      using extreme_point_of_def by blast
    then show ?thesis
      by (simp add: ‹convex S› hull_minimal)
  qed
```

qed

6.38.8 Applying it to convex hulls of explicitly indicated finite sets

corollary *Krein_Milman_polytope*:
fixes *S* :: *'a::euclidean_space set*
shows
finite S
⟹ *convex hull S* =
convex hull {*x*. *x extreme_point_of* (*convex hull S*)}
by (*simp add*: *Krein_Milman_Minkowski finite_imp_compact_convex_hull*)

lemma *extreme_points_of_convex_hull_eq*:
fixes *S* :: *'a::euclidean_space set*
shows
⟦*compact S*; ⋀*T*. *T* ⊂ *S* ⟹ *convex hull T* ≠ *convex hull S*⟧
⟹ {*x*. *x extreme_point_of* (*convex hull S*)} = *S*
by (*metis* (*full_types*) *Krein_Milman_Minkowski compact_convex_hull convex_convex_hull extreme_points_of_convex_hull psubsetI*)

lemma *extreme_point_of_convex_hull_eq*:
fixes *S* :: *'a::euclidean_space set*
shows
⟦*compact S*; ⋀*T*. *T* ⊂ *S* ⟹ *convex hull T* ≠ *convex hull S*⟧
⟹ (*x extreme_point_of* (*convex hull S*) ⟷ *x* ∈ *S*)
using *extreme_points_of_convex_hull_eq* **by** *auto*

lemma *extreme_point_of_convex_hull_convex_independent*:
fixes *S* :: *'a::euclidean_space set*
assumes *compact S* **and** *S*: ⋀*a*. *a* ∈ *S* ⟹ *a* ∉ *convex hull* (*S* − {*a*})
shows (*x extreme_point_of* (*convex hull S*) ⟷ *x* ∈ *S*)
proof −
have *convex hull T* ≠ *convex hull S* **if** *T* ⊂ *S* **for** *T*
proof −
obtain *a* **where** *T* ⊆ *S* *a* ∈ *S* *a* ∉ *T* **using** ‹*T* ⊂ *S*› **by** *blast*
then show *?thesis*
by (*metis* (*full_types*) *Diff_eq_empty_iff Diff_insert0 S hull_mono hull_subset insert_Diff_single subsetCE*)
qed
then show *?thesis*
by (*rule extreme_point_of_convex_hull_eq* [*OF* ‹*compact S*›])
qed

lemma *extreme_point_of_convex_hull_affine_independent*:
fixes *S* :: *'a::euclidean_space set*
shows
¬ *affine_dependent S*

$\Longrightarrow$ (*x extreme_point_of* (*convex hull S*) $\longleftrightarrow$ $x \in S$)
by (*metis aff_independent_finite affine_dependent_def affine_hull_convex_hull extreme_point_of_convex_hull_
finite_imp_compact hull_inc)

Elementary proofs exist, not requiring Euclidean spaces and all this development

lemma *extreme_point_of_convex_hull_2*:
fixes *x* :: *'a::euclidean_space*
shows *x extreme_point_of* (*convex hull* {*a*,*b*}) $\longleftrightarrow$ $x = a \lor x = b$
proof –
have *x extreme_point_of* (*convex hull* {*a*,*b*}) $\longleftrightarrow$ $x \in \{a,b\}$
by (*intro extreme_point_of_convex_hull_affine_independent affine_independent_2*)
then show *?thesis*
by *simp*
qed

lemma *extreme_point_of_segment*:
fixes *x* :: *'a::euclidean_space*
shows
x extreme_point_of closed_segment a b $\longleftrightarrow$ $x = a \lor x = b$
by (*simp add*: *extreme_point_of_convex_hull_2 segment_convex_hull*)

lemma *face_of_convex_hull_subset*:
fixes *S* :: *'a::euclidean_space set*
assumes *compact S* **and** *T*: *T face_of* (*convex hull S*)
obtains *s'* **where** $s' \subseteq S$ T = *convex hull s'*
proof
show {*x*. *x extreme_point_of T*} $\subseteq S$
using *T extreme_point_of_convex_hull extreme_point_of_face* **by** *blast*
show *T* = *convex hull* {*x*. *x extreme_point_of T*}
proof (*rule Krein_Milman_Minkowski*)
show *compact T*
using *T assms compact_convex_hull face_of_imp_compact* **by** *auto*
show *convex T*
using *T face_of_imp_convex* **by** *blast*
qed
qed

lemma *face_of_convex_hull_aux*:
assumes *eq*: $x *_R p = u *_R a + v *_R b + w *_R c$
and *x*: $u + v + w = x$ $x \neq 0$ **and** *S*: *affine S* $a \in S$ $b \in S$ $c \in S$
shows $p \in S$
proof –
have $p = (u *_R a + v *_R b + w *_R c) /_R x$
by (*metis* ‹$x \neq 0$› *eq mult.commute right_inverse scaleR_one scaleR_scaleR*)
moreover have *affine hull* {*a*,*b*,*c*} $\subseteq S$
by (*simp add*: *S hull_minimal*)
moreover have $(u *_R a + v *_R b + w *_R c) /_R x \in$ *affine hull* {*a*,*b*,*c*}

```
    apply (simp add: affine_hull_3)
    apply (rule_tac x=u/x in exI)
    apply (rule_tac x=v/x in exI)
    apply (rule_tac x=w/x in exI)
    using x apply (auto simp: field_split_simps)
    done
  ultimately show ?thesis by force
qed

proposition face_of_convex_hull_insert_eq:
  fixes a :: 'a :: euclidean_space
  assumes finite S and a: a ∉ affine hull S
  shows (F face_of (convex hull (insert a S)) ⟷
          F face_of (convex hull S) ∨
          (∃F'. F' face_of (convex hull S) ∧ F = convex hull (insert a F')))
         (is F face_of ?CAS ⟷ _)
proof safe
  assume F: F face_of ?CAS
    and *: ∄F'. F' face_of convex hull S ∧ F = convex hull insert a F'
  obtain T where T: T ⊆ insert a S and FeqT: F = convex hull T
   by (metis F ‹finite S› compact_insert finite_imp_compact face_of_convex_hull_subset)
  show F face_of convex hull S
  proof (cases a ∈ T)
    case True
    have F = convex hull insert a (convex hull T ∩ convex hull S)
    proof
      have T ⊆ insert a (convex hull T ∩ convex hull S)
        using T hull_subset by fastforce
      then show F ⊆ convex hull insert a (convex hull T ∩ convex hull S)
        by (simp add: FeqT hull_mono)
      show convex hull insert a (convex hull T ∩ convex hull S) ⊆ F
        by (simp add: FeqT True hull_inc hull_minimal)
    qed
    moreover have convex hull T ∩ convex hull S face_of convex hull S
       by (metis F FeqT convex_convex_hull face_of_slice hull_mono inf.absorb_iff2
subset_insertI)
    ultimately show ?thesis
      using * by force
  next
    case False
    then show ?thesis
      by (metis FeqT F T face_of_subset hull_mono subset_insert subset_insertI)
  qed
next
  assume F face_of convex hull S
  show F face_of ?CAS
    by (simp add: ‹F face_of convex hull S› a face_of_convex_hull_insert ‹finite S›)
next
  fix F
```

assume *F*: *F face_of convex hull S*
show *convex hull insert a F face_of ?CAS*
proof (*cases S* = {})
case *True*
then show *?thesis*
using *F face_of_affine_eq* **by** *auto*
next
case *False*
have *anotc*: $a \notin$ *convex hull S*
by (*metis* (*no_types*) *a affine_hull_convex_hull hull_inc*)
show *?thesis*
proof (*cases F* = {})
case *True* **show** *?thesis*
using *anotc* **by** (*simp add*: ‹*F* = {}› ‹*finite S*› *extreme_point_of_convex_hull_insert face_of_singleton*)
next
case *False*
have *convex hull insert a F* ⊆ *?CAS*
by (*simp add*: *F a* ‹*finite S*› *convex_hull_subset face_of_convex_hull_insert face_of_imp_subset hull_inc*)
moreover
have $(\exists y\ v.\ (1 - ub) *_R a + ub *_R b = (1 - v) *_R a + v *_R y \wedge$
$0 \le v \wedge v \le 1 \wedge y \in F) \wedge$
$(\exists x\ u.\ (1 - uc) *_R a + uc *_R c = (1 - u) *_R a + u *_R x \wedge$
$0 \le u \wedge u \le 1 \wedge x \in F)$
if ∗: $(1 - ux) *_R a + ux *_R x$
$\in$ *open_segment* $((1 - ub) *_R a + ub *_R b)\ ((1 - uc) *_R a + uc *_R c)$
and $0 \le ub\ ub \le 1\ 0 \le uc\ uc \le 1\ 0 \le ux\ ux \le 1$
and *b*: $b \in$ *convex hull S* **and** *c*: $c \in$ *convex hull S* **and** $x \in F$
for *b c ub uc ux x*
proof −
have *xah*: $x \in$ *affine hull S*
using *F convex_hull_subset_affine_hull face_of_imp_subset* ‹$x \in F$› **by** *blast*
have *ah*: $b \in$ *affine hull S* $c \in$ *affine hull S*
using *b c convex_hull_subset_affine_hull* **by** *blast+*
obtain *v* **where** *ne*: $(1 - ub) *_R a + ub *_R b \ne (1 - uc) *_R a + uc *_R c$
and *eq*: $(1 - ux) *_R a + ux *_R x =$
$(1 - v) *_R ((1 - ub) *_R a + ub *_R b) + v *_R ((1 - uc) *_R a + uc *_R c)$
and $0 < v\ v < 1$
using ∗ **by** (*auto simp*: *in_segment*)
then have *0*: $((1 - ux) - ((1 - v) * (1 - ub) + v * (1 - uc))) *_R a +$
$(ux *_R x - (((1 - v) * ub) *_R b + (v * uc) *_R c)) = 0$
by (*auto simp*: *algebra_simps*)
then have $((1 - ux) - ((1 - v) * (1 - ub) + v * (1 - uc))) *_R a =$
$((1 - v) * ub) *_R b + (v * uc) *_R c + (-ux) *_R x$
by (*auto simp*: *algebra_simps*)
then have $a \in$ *affine hull S* **if** $1 - ux - ((1 - v) * (1 - ub) + v * (1 -$

```
uc)) ≠ 0
            by (rule face_of_convex_hull_aux) (use b c xah ah that in ‹auto simp:
algebra_simps›)
        then have 1 − ux − ((1 − v) ∗ (1 − ub) + v ∗ (1 − uc)) = 0
          using a by blast
        with 0 have equx: (1 − v) ∗ ub + v ∗ uc = ux
          and uxx: ux ∗R x = (((1 − v) ∗ ub) ∗R b + (v ∗ uc) ∗R c)
          by auto (auto simp: algebra_simps)
        show ?thesis
        proof (cases uc = 0)
          case True
          then show ?thesis
            using equx ‹0 ≤ ub› ‹ub ≤ 1› ‹v < 1› uxx ‹x ∈ F› by force
        next
          case False
          show ?thesis
          proof (cases ub = 0)
            case True
            then show ?thesis
              using equx ‹0 ≤ uc› ‹uc ≤ 1› ‹0 < v› uxx ‹x ∈ F› by force
          next
            case False
            then have 0 < ub 0 < uc
              using ‹uc ≠ 0› ‹0 ≤ ub› ‹0 ≤ uc› by auto
            then have (1 − v) ∗ ub > 0 v ∗ uc > 0
              by (simp_all add: ‹0 < uc› ‹0 < v› ‹v < 1›)
            then have ux ≠ 0
              using equx ‹0 < v› by auto
            have b ∈ F ∧ c ∈ F
            proof (cases b = c)
              case True
              then show ?thesis
                 by (metis ‹ux ≠ 0› equx real_vector.scale_cancel_left scaleR_add_left
uxx ‹x ∈ F›)
            next
              case False
              have x = (((1 − v) ∗ ub) ∗R b + (v ∗ uc) ∗R c) /R ux
                    by (metis ‹ux ≠ 0› uxx mult.commute right_inverse scaleR_one
scaleR_scaleR)
              also have ... = (1 − v ∗ uc / ux) ∗R b + (v ∗ uc / ux) ∗R c
                using ‹ux ≠ 0› equx apply (auto simp: field_split_simps)
                 by (metis add.commute add_diff_eq add_divide_distrib diff_add_cancel
scaleR_add_left)
              finally have x = (1 − v ∗ uc / ux) ∗R b + (v ∗ uc / ux) ∗R c .
              then have x ∈ open_segment b c
                apply (simp add: in_segment ‹b ≠ c›)
                apply (rule_tac x=(v ∗ uc) / ux in exI)
                using ‹0 ≤ ux› ‹ux ≠ 0› ‹0 < uc› ‹0 < v› ‹0 < ub› ‹v < 1› equx
                apply (force simp: field_split_simps)
```

```
            done
          then show ?thesis
            by (rule face_ofD [OF F _ b c ‹x ∈ F›])
        qed
        with ‹0 ≤ ub› ‹ub ≤ 1› ‹0 ≤ uc› ‹uc ≤ 1› show ?thesis by blast
      qed
    qed
  qed
  moreover have convex hull F = F
    by (meson F convex_hull_eq face_of_imp_convex)
  ultimately show ?thesis
    unfolding face_of_def by (fastforce simp: convex_hull_insert_alt ‹S ≠ {}› ‹F
≠ {}›)
  qed
 qed
qed

lemma face_of_convex_hull_insert2:
  fixes a :: 'a :: euclidean_space
  assumes S: finite S and a: a ∉ affine hull S and F: F face_of convex hull S
  shows convex hull (insert a F) face_of convex hull (insert a S)
  by (metis F face_of_convex_hull_insert_eq [OF S a])

proposition face_of_convex_hull_affine_independent:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent S
    shows (T face_of (convex hull S) ⟷ (∃ c. c ⊆ S ∧ T = convex hull c))
        (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
   by (meson ‹T face_of convex hull S› aff_independent_finite assms face_of_convex_hull_subset
finite_imp_compact)
next
  assume ?rhs
  then obtain c where c ⊆ S and T: T = convex hull c
    by blast
  have affine hull c ∩ affine hull (S − c) = {}
    by (intro disjoint_affine_hull [OF assms ‹c ⊆ S›], auto)
  then have affine hull c ∩ convex hull (S − c) = {}
    using convex_hull_subset_affine_hull by fastforce
  then show ?lhs
    by (metis face_of_convex_hulls ‹c ⊆ S› aff_independent_finite assms T)
qed

lemma facet_of_convex_hull_affine_independent:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent S
    shows T facet_of (convex hull S) ⟷
```

```
        T ≠ {} ∧ (∃u. u ∈ S ∧ T = convex hull (S − {u}))
      (is ?lhs = ?rhs)
proof
  assume ?lhs
  then have T face_of (convex hull S) T ≠ {}
      and afft: aff_dim T = aff_dim (convex hull S) − 1
    by (auto simp: facet_of_def)
  then obtain c where c ⊆ S and c: T = convex hull c
    by (auto simp: face_of_convex_hull_affine_independent [OF assms])
  then have affs: aff_dim S = aff_dim c + 1
    by (metis aff_dim_convex_hull afft eq_diff_eq)
  have ¬ affine_dependent c
    using ‹c ⊆ S› affine_dependent_subset assms by blast
  with affs have card (S − c) = 1
    apply (simp add: aff_dim_affine_independent [symmetric] aff_dim_convex_hull)
    by (metis aff_dim_affine_independent aff_independent_finite One_nat_def ‹c ⊆
S› add.commute
                  add_diff_cancel_right' assms card_Diff_subset card_mono of_nat_1
of_nat_diff of_nat_eq_iff)
  then obtain u where u: u ∈ S − c
   by (metis DiffI ‹c ⊆ S› aff_independent_finite assms cancel_comm_monoid_add_class.diff_cancel
              card_Diff_subset subsetI subset_antisym zero_neq_one)
  then have u: S = insert u c
    by (metis Diff_subset ‹c ⊆ S› ‹card (S − c) = 1› card_1_singletonE double_diff
insert_Diff insert_subset singletonD)
  have T = convex hull (c − {u})
     by (metis Diff_empty Diff_insert0 ‹T facet_of convex hull S› c facet_of_irrefl
insert_absorb u)
  with ‹T ≠ {}› show ?rhs
    using c u by auto
next
  assume ?rhs
  then obtain u where T ≠ {} u ∈ S and u: T = convex hull (S − {u})
    by (force simp: facet_of_def)
  then have ¬ S ⊆ {u}
    using ‹T ≠ {}› u by auto
  have aff_dim (S − {u}) = aff_dim S − 1
    using assms ‹u ∈ S›
    unfolding affine_dependent_def
    by (metis add_diff_cancel_right' aff_dim_insert insert_Diff [of u S])
  then have aff_dim (convex hull (S − {u})) = aff_dim (convex hull S) − 1
    by (simp add: aff_dim_convex_hull)
  then show ?lhs
     by (metis Diff_subset ‹T ≠ {}› assms face_of_convex_hull_affine_independent
facet_of_def u)
qed

lemma facet_of_convex_hull_affine_independent_alt:
  fixes S :: 'a::euclidean_space set
```

assumes ¬ affine_dependent S
shows (T facet_of (convex hull S) ⟷ 2 ≤ card S ∧ (∃ u. u ∈ S ∧ T = convex hull (S − {u})))
(is ?lhs = ?rhs)
proof
assume L: ?lhs
then obtain x where
x ∈ S and x: T = convex hull (S − {x}) and finite S
using assms facet_of_convex_hull_affine_independent aff_independent_finite by blast
moreover have Suc (Suc 0) ≤ card S
using L x ‹x ∈ S› ‹finite S›
by (metis Suc_leI assms card.remove convex_hull_eq_empty card_gt_0_iff facet_of_convex_hull_affine_indep
finite_Diff not_less_eq_eq)
ultimately show ?rhs
by auto
next
assume ?rhs then show ?lhs
using assms
by (auto simp: facet_of_convex_hull_affine_independent Set.subset_singleton_iff)
qed

lemma segment_face_of:
assumes (closed_segment a b) face_of S
shows a extreme_point_of S b extreme_point_of S
proof −
have as: {a} face_of S
by (metis (no_types) assms convex_hull_singleton empty_iff extreme_point_of_convex_hull_insert face_of_face face_of_singleton finite.emptyI finite.insertI insert_absorb insert_iff segment_convex_hull)
moreover have {b} face_of S
proof −
have b ∈ convex hull {a} ∨ b extreme_point_of convex hull {b, a}
by (meson extreme_point_of_convex_hull_insert finite.emptyI finite.insertI)
moreover have closed_segment a b = convex hull {b, a}
using closed_segment_commute segment_convex_hull by blast
ultimately show ?thesis
by (metis as assms face_of_face convex_hull_singleton empty_iff face_of_singleton insertE)
qed
ultimately show a extreme_point_of S b extreme_point_of S
using face_of_singleton by blast+
qed

proposition Krein_Milman_frontier:
fixes S :: 'a::euclidean_space set
assumes convex S compact S
shows S = convex hull (frontier S)

(**is** *?lhs* = *?rhs*)
proof
have *?lhs* ⊆ *convex hull* {*x*. *x extreme_point_of S*}
using *Krein_Milman_Minkowski assms* **by** *blast*
also have ... ⊆ *?rhs*
proof (*rule hull_mono*)
show {*x*. *x extreme_point_of S*} ⊆ *frontier S*
using *closure_subset*
by (*auto simp*: *frontier_def extreme_point_not_in_interior extreme_point_of_def*)
qed
finally show *?lhs* ⊆ *?rhs* .
next
have *?rhs* ⊆ *convex hull S*
by (*metis Diff_subset* ‹*compact S*› *closure_closed compact_eq_bounded_closed frontier_def hull_mono*)
also have ... ⊆ *?lhs*
by (*simp add*: ‹*convex S*› *hull_same*)
finally show *?rhs* ⊆ *?lhs* .
qed

6.38.9 Polytopes

definition *polytope* **where**
polytope S ≡ ∃ *v*. *finite v* ∧ *S* = *convex hull v*

lemma *polytope_translation_eq*: *polytope* (*image* (λ*x*. *a* + *x*) *S*) ⟷ *polytope S*
proof −
have ⋀*a A*. *polytope A* ⟹ *polytope* ((+) *a* ` *A*)
by (*metis* (*no_types*) *convex_hull_translation finite_imageI polytope_def*)
then show *?thesis*
by (*metis* (*no_types*) *add.left_inverse image_add_0 translation_assoc*)
qed

lemma *polytope_linear_image*: ⟦*linear f*; *polytope p*⟧ ⟹ *polytope*(*image f p*)
unfolding *polytope_def* **using** *convex_hull_linear_image* **by** *blast*

lemma *polytope_empty*: *polytope* {}
using *convex_hull_empty polytope_def* **by** *blast*

lemma *polytope_convex_hull*: *finite S* ⟹ *polytope*(*convex hull S*)
using *polytope_def* **by** *auto*

lemma *polytope_Times*: ⟦*polytope S*; *polytope T*⟧ ⟹ *polytope*(*S* × *T*)
unfolding *polytope_def*
by (*metis finite_cartesian_product convex_hull_Times*)

lemma *face_of_polytope_polytope*:
fixes *S* :: ′*a*::*euclidean_space set*
shows ⟦*polytope S*; *F face_of S*⟧ ⟹ *polytope F*

unfolding *polytope_def*
by (*meson face_of_convex_hull_subset finite_imp_compact finite_subset*)

lemma *finite_polytope_faces*:
  fixes *S* :: *'a::euclidean_space set*
  assumes *polytope S*
  shows *finite* {*F*. *F face_of S*}
proof −
  obtain *v* **where** *finite v S* = *convex hull v*
    using *assms polytope_def* **by** *auto*
  have *finite* ((*hull*) *convex* ‘ {*T*. *T* ⊆ *v*})
    by (*simp add*: ‹*finite v*›)
  moreover have {*F*. *F face_of S*} ⊆ ((*hull*) *convex* ‘ {*T*. *T* ⊆ *v*})
   by (*metis* (*no_types*, *lifting*) ‹*finite v*› ‹*S* = *convex hull v*› *face_of_convex_hull_subset finite_imp_compact image_eqI mem_Collect_eq subsetI*)
  ultimately show *?thesis*
    by (*blast intro*: *finite_subset*)
qed

lemma *finite_polytope_facets*:
  assumes *polytope S*
  shows *finite* {*T*. *T facet_of S*}
by (*simp add*: *assms facet_of_def finite_polytope_faces*)

lemma *polytope_scaling*:
  assumes *polytope S* **shows** *polytope* (*image* (*λx*. *c* $*_R$ *x*) *S*)
by (*simp add*: *assms polytope_linear_image*)

lemma *polytope_imp_compact*:
  fixes *S* :: *'a::real_normed_vector set*
  shows *polytope S* ⟹ *compact S*
by (*metis finite_imp_compact_convex_hull polytope_def*)

lemma *polytope_imp_convex*: *polytope S* ⟹ *convex S*
  by (*metis convex_convex_hull polytope_def*)

lemma *polytope_imp_closed*:
  fixes *S* :: *'a::real_normed_vector set*
  shows *polytope S* ⟹ *closed S*
by (*simp add*: *compact_imp_closed polytope_imp_compact*)

lemma *polytope_imp_bounded*:
  fixes *S* :: *'a::real_normed_vector set*
  shows *polytope S* ⟹ *bounded S*
by (*simp add*: *compact_imp_bounded polytope_imp_compact*)

lemma *polytope_interval*: *polytope*(*cbox a b*)
  unfolding *polytope_def* **by** (*meson closed_interval_as_convex_hull*)

lemma *polytope_sing*: *polytope* {*a*}
  using *polytope_def* **by** *force*

lemma *face_of_polytope_insert*:
  ⟦*polytope S*; *a* ∉ *affine hull S*; *F face_of S*⟧ ⟹ *F face_of convex hull* (*insert a S*)
  by (*metis* (*no_types*, *lifting*) *affine_hull_convex_hull face_of_convex_hull_insert hull_insert polytope_def*)

proposition *face_of_polytope_insert2*:
  fixes *a* :: ′*a* :: *euclidean_space*
  assumes *polytope S a* ∉ *affine hull S F face_of S*
  shows *convex hull* (*insert a F*) *face_of convex hull* (*insert a S*)
proof −
  obtain *V* **where** *finite V S* = *convex hull V*
    using *assms* **by** (*auto simp*: *polytope_def*)
  then have *convex hull* (*insert a F*) *face_of convex hull* (*insert a V*)
    using *affine_hull_convex_hull assms face_of_convex_hull_insert2* **by** *blast*
  then show *?thesis*
    by (*metis* ‹*S* = *convex hull V*› *hull_insert*)
qed

6.38.10 Polyhedra

definition *polyhedron* **where**
 polyhedron S ≡
    ∃ *F*. *finite F* ∧
      S = ⋂ *F* ∧
      (∀ *h* ∈ *F*. ∃ *a b*. *a* ≠ *0* ∧ *h* = {*x*. *a* · *x* ≤ *b*})

lemma *polyhedron_Int* [*intro*,*simp*]:
  ⟦*polyhedron S*; *polyhedron T*⟧ ⟹ *polyhedron* (*S* ∩ *T*)
  apply (*clarsimp simp add*: *polyhedron_def*)
  subgoal for *F G*
    by (*rule_tac x*=*F* ∪ *G* **in** *exI*, *auto*)
  done

lemma *polyhedron_UNIV* [*iff*]: *polyhedron UNIV*
  unfolding *polyhedron_def*
  by (*rule_tac x*={} **in** *exI*) *auto*

lemma *polyhedron_Inter* [*intro*,*simp*]:
  ⟦*finite F*; ⋀*S*. *S* ∈ *F* ⟹ *polyhedron S*⟧ ⟹ *polyhedron*(⋂ *F*)
by (*induction F rule*: *finite_induct*) *auto*

lemma *polyhedron_empty* [*iff*]: *polyhedron* ({} :: ′*a* :: *euclidean_space set*)
proof −
  define *i*::′*a* **where** (*i* ≡ *SOME i*. *i* ∈ *Basis*)

have $\exists a.\ a \neq 0 \wedge (\exists b.\ \{x.\ i \cdot x \leq -1\} = \{x.\ a \cdot x \leq b\})$
  by (*rule_tac x=i* **in** *exI*) (*force simp*: *i_def SOME_Basis nonzero_Basis*)
moreover have $\exists a\ b.\ a \neq 0 \wedge \{x.\ -i \cdot x \leq -1\} = \{x.\ a \cdot x \leq b\}$
    apply (*rule_tac x=−i* **in** *exI*)
    apply (*rule_tac x=−1* **in** *exI*)
    apply (*simp add*: *i_def SOME_Basis nonzero_Basis*)
    done
ultimately show *?thesis*
  unfolding *polyhedron_def*
  by (*rule_tac x*=$\{\{x.\ i \cdot x \leq -1\}, \{x.\ -i \cdot x \leq -1\}\}$ **in** *exI*) *force*
qed

lemma *polyhedron_halfspace_le*:
  fixes *a* :: *'a* :: *euclidean_space*
  shows *polyhedron* $\{x.\ a \cdot x \leq b\}$
proof (*cases a = 0*)
  case *True* **then show** *?thesis* **by** *auto*
next
  case *False*
  then show *?thesis*
    unfolding *polyhedron_def*
    by (*rule_tac x*=$\{\{x.\ a \cdot x \leq b\}\}$ **in** *exI*) *auto*
qed

lemma *polyhedron_halfspace_ge*:
  fixes *a* :: *'a* :: *euclidean_space*
  shows *polyhedron* $\{x.\ a \cdot x \geq b\}$
using *polyhedron_halfspace_le* [*of* −*a* −*b*] **by** *simp*

lemma *polyhedron_hyperplane*:
  fixes *a* :: *'a* :: *euclidean_space*
  shows *polyhedron* $\{x.\ a \cdot x = b\}$
proof −
  have $\{x.\ a \cdot x = b\} = \{x.\ a \cdot x \leq b\} \cap \{x.\ a \cdot x \geq b\}$
    by *force*
  then show *?thesis*
    by (*simp add*: *polyhedron_halfspace_ge polyhedron_halfspace_le*)
qed

lemma *affine_imp_polyhedron*:
  fixes *S* :: *'a* :: *euclidean_space set*
  shows *affine S* ⟹ *polyhedron S*
by (*metis affine_hull_eq polyhedron_Inter polyhedron_hyperplane affine_hull_finite_intersection_hyperplanes* [*of S*])

lemma *polyhedron_imp_closed*:
  fixes *S* :: *'a* :: *euclidean_space set*
  shows *polyhedron S* ⟹ *closed S*
  by (*metis closed_Inter closed_halfspace_le polyhedron_def*)

lemma *polyhedron_imp_convex*:
  fixes S :: *'a* :: *euclidean_space set*
  shows *polyhedron* $S \Longrightarrow$ *convex* S
  by (*metis convex_Inter convex_halfspace_le polyhedron_def*)

lemma *polyhedron_affine_hull*:
  fixes S :: *'a* :: *euclidean_space set*
  shows *polyhedron*(*affine hull* S)
by (*simp add*: *affine_imp_polyhedron*)

6.38.11 Canonical polyhedron representation making facial structure explicit

proposition *polyhedron_Int_affine*:
  fixes S :: *'a* :: *euclidean_space set*
  shows *polyhedron* $S \longleftrightarrow$
    $(\exists F.\ finite\ F \wedge S = (affine\ hull\ S) \cap \bigcap F \wedge$
      $(\forall h \in F.\ \exists a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\}))$
    (**is** *?lhs* = *?rhs*)
proof
  assume *?lhs* **then show** *?rhs*
    using *hull_subset polyhedron_def* **by** *fastforce*
next
  assume *?rhs* **then show** *?lhs*
   by (*metis polyhedron_Int polyhedron_Inter polyhedron_affine_hull polyhedron_halfspace_le*)
qed

proposition *rel_interior_polyhedron_explicit*:
  assumes *finite* F
    and *seq*: $S = affine\ hull\ S \cap \bigcap F$
    and *faceq*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
    and *psub*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset affine\ hull\ S \cap \bigcap F'$
   shows $rel\_interior\ S = \{x \in S.\ \forall h \in F.\ a\ h \cdot x < b\ h\}$
proof −
  have *rels*: $\bigwedge x.\ x \in rel\_interior\ S \Longrightarrow x \in S$
   by (*meson IntE mem_rel_interior*)
  moreover have $a\ i \cdot x < b\ i$ **if** x: $x \in rel\_interior\ S$ **and** $i \in F$ **for** $x\ i$
  proof −
   have *fif*: $F - \{i\} \subset F$
    using ‹$i \in F$› *Diff_insert_absorb Diff_subset set_insert psubsetI* **by** *blast*
   then have $S \subset affine\ hull\ S \cap \bigcap (F - \{i\})$
    by (*rule psub*)
   then obtain z **where** *ssub*: $S \subseteq \bigcap (F - \{i\})$ **and** *zint*: $z \in \bigcap (F - \{i\})$
          and $z \notin S$ **and** *zaff*: $z \in affine\ hull\ S$
    by *auto*
   have $z \neq x$
    using ‹$z \notin S$› *rels* x **by** *blast*
   have $z \notin affine\ hull\ S \cap \bigcap F$

using ‹$z \notin S$› *seq* **by** *auto*
then have *aiz*: $a\ i \cdot z > b\ i$
using *faceq zint zaff* **by** *fastforce*
obtain e **where** $e > 0$ $x \in S$ **and** e: $ball\ x\ e \cap affine\ hull\ S \subseteq S$
using x **by** (*auto simp*: *mem_rel_interior_ball*)
then have *ins*: $\bigwedge y.\ [\![norm\ (x - y) < e;\ y \in affine\ hull\ S]\!] \Longrightarrow y \in S$
by (*metis IntI subsetD dist_norm mem_ball*)
define ξ **where** $\xi = min\ (1/2)\ (e\ /\ 2\ /\ norm(z - x))$
have $norm\ (\xi *_R x - \xi *_R z) = norm\ (\xi *_R (x - z))$
by (*simp add*: *ξ_def algebra_simps norm_mult*)
also have ... $= \xi * norm\ (x - z)$
using ‹$e > 0$› **by** (*simp add*: *ξ_def*)
also have ... $< e$
using ‹$z \neq x$› ‹$e > 0$› **by** (*simp add*: *ξ_def min_def field_split_simps norm_minus_commute*)
finally have *lte*: $norm\ (\xi *_R x - \xi *_R z) < e$ **.**
have *ξ_aff*: $\xi *_R z + (1 - \xi) *_R x \in affine\ hull\ S$
by (*metis* ‹$x \in S$› *add.commute affine_affine_hull diff_add_cancel hull_inc mem_affine zaff*)
have $\xi *_R z + (1 - \xi) *_R x \in S$
using *ins* [*OF* _ *ξ_aff*] **by** (*simp add*: *algebra_simps lte*)
then obtain l **where** l: $0 < l$ $l < 1$ **and** *ls*: $(l *_R z + (1 - l) *_R x) \in S$
using ‹$e > 0$› ‹$z \neq x$›
by (*rule_tac* $l = \xi$ **in** *that*) (*auto simp*: *ξ_def*)
then have i: $l *_R z + (1 - l) *_R x \in i$
using *seq* ‹$i \in F$› **by** *auto*
have $b\ i * l + (a\ i \cdot x) * (1 - l) < a\ i \cdot (l *_R z + (1 - l) *_R x)$
using l **by** (*simp add*: *algebra_simps aiz*)
also have $\ldots \le b\ i$ **using** $i\ l$
using *faceq mem_Collect_eq* ‹$i \in F$› **by** *blast*
finally have $(a\ i \cdot x) * (1 - l) < b\ i * (1 - l)$
by (*simp add*: *algebra_simps*)
with l **show** *?thesis*
by *simp*
qed
moreover have $x \in rel\_interior\ S$
if $x \in S$ **and** *less*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \cdot x < b\ h$ **for** x
proof −
have *1*: $\bigwedge h.\ h \in F \Longrightarrow x \in interior\ h$
by (*metis interior_halfspace_le mem_Collect_eq less faceq*)
have *2*: $\bigwedge y.\ [\![\forall h \in F.\ y \in interior\ h;\ y \in affine\ hull\ S]\!] \Longrightarrow y \in S$
by (*metis IntI Inter_iff subsetD interior_subset seq*)
show *?thesis*
apply (*simp add*: *rel_interior* ‹$x \in S$›)
apply (*rule_tac* $x = \bigcap h \in F.\ interior\ h$ **in** *exI*)
apply (*auto simp*: ‹*finite F*› *open_INT 1 2*)
done
qed
ultimately show *?thesis* **by** *blast*
qed

lemma *polyhedron_Int_affine_parallel*:
fixes S :: $'a$ :: *euclidean_space set*
shows *polyhedron* $S \longleftrightarrow$
$(\exists F.$ *finite* $F \wedge$
$S = (\textit{affine hull } S) \cap (\bigcap F) \wedge$
$(\forall h \in F. \exists a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\} \wedge$
$(\forall x \in \textit{affine hull } S.\ (x + a) \in \textit{affine hull } S)))$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then obtain F **where** *finite* F **and** *seq*: $S = (\textit{affine hull } S) \cap \bigcap F$
and *faces*: $\bigwedge h.\ h \in F \Longrightarrow \exists a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\}$
by (*fastforce simp add*: *polyhedron_Int_affine*)
then obtain $a\ b$ **where** *ab*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
by *metis*
show *?rhs*
proof –
have $\exists a'\ b'.\ a' \neq 0 \wedge$
$\textit{affine hull } S \cap \{x.\ a' \cdot x \leq b'\} = \textit{affine hull } S \cap h \wedge$
$(\forall w \in \textit{affine hull } S.\ (w + a') \in \textit{affine hull } S)$
if $h \in F$ $\neg(\textit{affine hull } S \subseteq h)$ **for** h
proof –
have $a\ h \neq 0$ **and** $h = \{x.\ a\ h \cdot x \leq b\ h\}$ $h \cap \bigcap F = \bigcap F$
using ‹$h \in F$› *ab* **by** *auto*
then have $(\textit{affine hull } S) \cap \{x.\ a\ h \cdot x \leq b\ h\} \neq \{\}$
by (*metis* (*no_types*) *affine_hull_eq_empty inf.absorb_iff2 inf_assoc inf_bot_right inf_commute seq that*(*2*))
moreover have $\neg$ $(\textit{affine hull } S \subseteq \{x.\ a\ h \cdot x \leq b\ h\})$
using ‹$h = \{x.\ a\ h \cdot x \leq b\ h\}$› *that*(*2*) **by** *blast*
ultimately show *?thesis*
using *affine_parallel_slice* [*of affine hull S*]
by (*metis* ‹$h = \{x.\ a\ h \cdot x \leq b\ h\}$› *affine_affine_hull*)
qed
then obtain $a\ b$
where *ab*: $\bigwedge h.\ [\![h \in F; \neg\ (\textit{affine hull } S \subseteq h)]\!]$
$\Longrightarrow a\ h \neq 0 \wedge$
$\textit{affine hull } S \cap \{x.\ a\ h \cdot x \leq b\ h\} = \textit{affine hull } S \cap h \wedge$
$(\forall w \in \textit{affine hull } S.\ (w + a\ h) \in \textit{affine hull } S)$
by *metis*
have *seq2*: $S = \textit{affine hull } S \cap (\bigcap h\in\{h \in F.\ \neg\ \textit{affine hull } S \subseteq h\}.\ \{x.\ a\ h \cdot x \leq b\ h\})$
by (*subst seq*) (*auto simp*: *ab INT_extend_simps*)
show *?thesis*
apply (*rule_tac* $x = (\lambda h.\ \{x.\ a\ h \cdot x \leq b\ h\})$ ‘ $\{h.\ h \in F \wedge \neg(\textit{affine hull } S \subseteq h)\}$ **in** *exI*)
apply (*intro conjI seq2*)
using ‹*finite* F› **apply** *force*

using *ab* **apply** *blast*
done
qed
next
assume *?rhs* **then show** *?lhs*
by (*metis polyhedron_Int_affine*)
qed

proposition *polyhedron_Int_affine_parallel_minimal*:
fixes S :: $'a$ :: *euclidean_space set*
shows *polyhedron* $S \longleftrightarrow$
$(\exists F.\ finite\ F \wedge$
$S = (affine\ hull\ S) \cap (\bigcap F) \wedge$
$(\forall h \in F.\ \exists a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\} \wedge$
$(\forall x \in affine\ hull\ S.\ (x + a) \in affine\ hull\ S)) \wedge$
$(\forall F'.\ F' \subset F \longrightarrow S \subset (affine\ hull\ S) \cap (\bigcap F')))$
(**is** $?lhs = ?rhs$)
proof
assume *?lhs*
then obtain *f0*
where *f0*: *finite f0*
$S = (affine\ hull\ S) \cap (\bigcap f0)$
(**is** *?P f0*)
$\forall h \in f0.\ \exists a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\} \wedge$
$(\forall x \in affine\ hull\ S.\ (x + a) \in affine\ hull\ S)$
(**is** *?Q f0*)
by (*force simp*: *polyhedron_Int_affine_parallel*)
define n **where** $n = (LEAST\ n.\ \exists F.\ card\ F = n \wedge finite\ F \wedge ?P\ F \wedge ?Q\ F)$
have *nf*: $\exists F.\ card\ F = n \wedge finite\ F \wedge ?P\ F \wedge ?Q\ F$
apply (*simp add*: *n_def*)
apply (*rule LeastI* [**where** $k = card\ f0$])
using *f0* **apply** *auto*
done
then obtain F **where** F: $card\ F = n$ *finite* F **and** *seq*: *?P F* **and** *aff*: *?Q F*
by *blast*
then have $\neg$ (*finite g* $\wedge$ *?P g* $\wedge$ *?Q g*) **if** $card\ g < n$ **for** g
using *that* **by** (*auto simp*: *n_def dest*!: *not_less_Least*)
then have $*$: $\neg$ (*?P g* $\wedge$ *?Q g*) **if** $g \subset F$ **for** g
using *that* ⟨*finite F*⟩ *psubset_card_mono* ⟨$card\ F = n$⟩
by (*metis finite_Int inf.strict_order_iff*)
have *1*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subseteq affine\ hull\ S \cap \bigcap F'$
by (*subst seq*) *blast*
have *2*: $S \neq affine\ hull\ S \cap \bigcap F'$ **if** $F' \subset F$ **for** F'
using $*$ [*OF that*] **by** (*metis IntE aff inf.strict_order_iff that*)
show *?rhs*
by (*metis* ⟨*finite F*⟩ *seq aff psubsetI 1 2*)
next
assume *?rhs* **then show** *?lhs*

by (*auto simp*: *polyhedron_Int_affine_parallel*)
qed

lemma *polyhedron_Int_affine_minimal*:
fixes S :: *'a* :: *euclidean_space set*
shows *polyhedron* $S \longleftrightarrow$
$(\exists F.\ \mathit{finite}\ F \wedge S = (\mathit{affine\ hull}\ S) \cap \bigcap F \wedge$
$(\forall h \in F.\ \exists a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\}) \wedge$
$(\forall F'.\ F' \subset F \longrightarrow S \subset (\mathit{affine\ hull}\ S) \cap \bigcap F'))$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*force simp*: *polyhedron_Int_affine_parallel_minimal elim*!: *ex_forward*)
qed (*auto simp*: *polyhedron_Int_affine elim*!: *ex_forward*)

proposition *facet_of_polyhedron_explicit*:
assumes *finite* F
and *seq*: $S = \mathit{affine\ hull}\ S \cap \bigcap F$
and *faceq*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
and *psub*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset \mathit{affine\ hull}\ S \cap \bigcap F'$
shows C *facet_of* $S \longleftrightarrow (\exists h.\ h \in F \wedge C = S \cap \{x.\ a\ h \cdot x = b\ h\})$
proof (*cases* $S = \{\}$)
case *True* **with** *psub* **show** *?thesis* **by** *force*
next
case *False*
have *polyhedron* S
unfolding *polyhedron_Int_affine* **by** (*metis* ‹*finite* F› *faceq seq*)
then have *convex* S
by (*rule polyhedron_imp_convex*)
with *False rel_interior_eq_empty* **have** *rel_interior* $S \neq \{\}$ **by** *blast*
then obtain x **where** $x \in$ *rel_interior* S **by** *auto*
then obtain T **where** *open* T $x \in T$ $x \in S$ $T \cap \mathit{affine\ hull}\ S \subseteq S$
by (*force simp*: *mem_rel_interior*)
then have *xaff*: $x \in \mathit{affine\ hull}\ S$ **and** *xint*: $x \in \bigcap F$
using *seq hull_inc* **by** *auto*
have *rel_interior* $S = \{x \in S.\ \forall h{\in}F.\ a\ h \cdot x < b\ h\}$
by (*rule rel_interior_polyhedron_explicit* [*OF* ‹*finite* F› *seq faceq psub*])
with ‹$x \in$ *rel_interior* S›
have [*simp*]: $\bigwedge h.\ h{\in}F \Longrightarrow a\ h \cdot x < b\ h$ **by** *blast*
have $*$: $(S \cap \{x.\ a\ h \cdot x = b\ h\})$ *facet_of* S **if** $h \in F$ **for** h
proof −
have $S \subset \mathit{affine\ hull}\ S \cap \bigcap(F - \{h\})$
using *psub that* **by** (*metis Diff_disjoint Diff_subset insert_disjoint*(*2*) *psubsetI*)
then obtain z **where** *zaff*: $z \in \mathit{affine\ hull}\ S$ **and** *zint*: $z \in \bigcap(F - \{h\})$ **and**
$z \notin S$
by *force*
then have $z \neq x$ $z \notin h$ **using** *seq* ‹$x \in S$› **by** *auto*

have $x \in h$ **using** *that xint* **by** *auto*
then have *able*: $a\ h \cdot x \le b\ h$
using *faceq that* **by** *blast*
also have ... $< a\ h \cdot z$ **using** ‹$z \notin h$› *faceq* [*OF that*] *xint* **by** *auto*
finally have *xltz*: $a\ h \cdot x < a\ h \cdot z$ **.**
define l **where** $l = (b\ h - a\ h \cdot x) / (a\ h \cdot z - a\ h \cdot x)$
define w **where** $w = (1 - l) *_R x + l *_R z$
have $0 < l\ l < 1$
using *able xltz* ‹$b\ h < a\ h \cdot z$› ‹$h \in F$›
by (*auto simp*: *l_def field_split_simps*)
have *awlt*: $a\ i \cdot w < b\ i$ **if** $i \in F\ i \neq h$ **for** i
proof −
have $(1 - l) * (a\ i \cdot x) < (1 - l) * b\ i$
by (*simp add*: ‹$l < 1$› ‹$i \in F$›)
moreover have $l * (a\ i \cdot z) \le l * b\ i$
proof (*rule mult_left_mono*)
show $a\ i \cdot z \le b\ i$
by (*metis Diff_insert_absorb Inter_iff Set.set_insert* ‹$h \in F$› *faceq insertE mem_Collect_eq that zint*)
qed (*use* ‹$0 < l$› **in** *auto*)
ultimately show *?thesis* **by** (*simp add*: *w_def algebra_simps*)
qed
have *weq*: $a\ h \cdot w = b\ h$
using *xltz* **unfolding** *w_def l_def*
by (*simp add*: *algebra_simps*) (*simp add*: *field_simps*)
have *faceS*: $S \cap \{x.\ a\ h \cdot x = b\ h\}$ *face_of* S
proof (*rule face_of_Int_supporting_hyperplane_le*)
show $\bigwedge x.\ x \in S \Longrightarrow a\ h \cdot x \le b\ h$
using *faceq seq that* **by** *fastforce*
qed *fact*
have $w \in$ *affine hull* S
by (*simp add*: *w_def mem_affine xaff zaff*)
moreover have $w \in \bigcap F$
using ‹$a\ h \cdot w = b\ h$› *awlt faceq less_eq_real_def* **by** *blast*
ultimately have $w \in S$
using *seq* **by** *blast*
with *weq* **have** *ne*: $S \cap \{x.\ a\ h \cdot x = b\ h\} \neq \{\}$ **by** *blast*
moreover have *affine hull* $(S \cap \{x.\ a\ h \cdot x = b\ h\}) = ($*affine hull* $S) \cap \{x.\ a\ h \cdot x = b\ h\}$
proof
show *affine hull* $(S \cap \{x.\ a\ h \cdot x = b\ h\}) \subseteq$ *affine hull* $S \cap \{x.\ a\ h \cdot x = b\ h\}$
apply (*intro Int_greatest hull_mono Int_lower1*)
apply (*metis affine_hull_eq affine_hyperplane hull_mono inf_le2*)
done
next
show *affine hull* $S \cap \{x.\ a\ h \cdot x = b\ h\} \subseteq$ *affine hull* $(S \cap \{x.\ a\ h \cdot x = b\ h\})$
proof

```
    fix y
    assume yaff: y ∈ affine hull S ∩ {y. a h · y = b h}
    obtain T where 0 < T
          and T: ⋀j. ⟦j ∈ F; j ≠ h⟧ ⟹ T * (a j · y − a j · w) ≤ b j − a j
· w
    proof (cases F − {h} = {})
      case True then show ?thesis
        by (rule_tac T=1 in that) auto
    next
      case False
      then obtain h' where h': h' ∈ F − {h} by auto
      let ?body = (λj. if 0 < a j · y − a j · w
          then (b j − a j · w) / (a j · y − a j · w) else 1) ` (F − {h})
      define inff where inff = Inf ?body
      from ‹finite F› have finite ?body
        by blast
      moreover from h' have ?body ≠ {}
        by blast
      moreover have j > 0 if j ∈ ?body for j
      proof −
        from that obtain x where x ∈ F and x ≠ h and *: j =
          (if 0 < a x · y − a x · w
            then (b x − a x · w) / (a x · y − a x · w) else 1)
          by blast
        with awlt [of x] have a x · w < b x
          by simp
        with * show ?thesis
          by simp
      qed
      ultimately have 0 < inff
        by (simp_all add: finite_less_Inf_iff inff_def)
      moreover have inff * (a j · y − a j · w) ≤ b j − a j · w
                  if j ∈ F j ≠ h for j
      proof (cases a j · w < a j · y)
        case True
        then have inff ≤ (b j − a j · w) / (a j · y − a j · w)
          unfolding inff_def
            using ‹finite F› by (auto intro: cInf_le_finite simp add: that split:
if_split_asm)
        then show ?thesis
          using ‹0 < inff› awlt [OF that] mult_strict_left_mono
          by (fastforce simp add: field_split_simps split: if_split_asm)
      next
        case False
        with ‹0 < inff› have inff * (a j · y − a j · w) ≤ 0
          by (simp add: mult_le_0_iff)
        also have ... < b j − a j · w
          by (simp add: awlt that)
        finally show ?thesis by simp
```

qed
ultimately show *?thesis*
by (*blast intro*: *that*)
qed
define *C* **where** *C* = (*1* − *T*) $*_R$ *w* + *T* $*_R$ *y*
have (*1* − *T*) $*_R$ *w* + *T* $*_R$ *y* ∈ *j* **if** *j* ∈ *F* **for** *j*
proof (*cases j* = *h*)
case *True*
have (*1* − *T*) $*_R$ *w* + *T* $*_R$ *y* ∈ {*x*. *a h* · *x* ≤ *b h*}
using *weq yaff* **by** (*auto simp*: *algebra_simps*)
with *True faceq* [*OF that*] **show** *?thesis* **by** *metis*
next
case *False*
with *T that* **have** (*1* − *T*) $*_R$ *w* + *T* $*_R$ *y* ∈ {*x*. *a j* · *x* ≤ *b j*}
by (*simp add*: *algebra_simps*)
with *faceq* [*OF that*] **show** *?thesis* **by** *simp*
qed
moreover have (*1* − *T*) $*_R$ *w* + *T* $*_R$ *y* ∈ *affine hull S*
using *yaff* ‹*w* ∈ *affine hull S*› *affine_affine_hull affine_alt* **by** *blast*
ultimately have *C* ∈ *S*
using *seq* **by** (*force simp*: *C_def*)
moreover have *a h* · *C* = *b h*
using *yaff* **by** (*force simp*: *C_def algebra_simps weq*)
ultimately have *caff*: *C* ∈ *affine hull* (*S* ∩ {*y*. *a h* · *y* = *b h*})
by (*simp add*: *hull_inc*)
have *waff*: *w* ∈ *affine hull* (*S* ∩ {*y*. *a h* · *y* = *b h*})
using ‹*w* ∈ *S*› *weq* **by** (*blast intro*: *hull_inc*)
have *yeq*: *y* = (*1* − *inverse T*) $*_R$ *w* + *C* $/_R$ *T*
using ‹*0* < *T*› **by** (*simp add*: *C_def algebra_simps*)
show *y* ∈ *affine hull* (*S* ∩ {*y*. *a h* · *y* = *b h*})
by (*metis yeq affine_affine_hull* [*simplified affine_alt*, *rule_format*, *OF waff caff*])
qed
qed
ultimately have *aff_dim* (*affine hull* (*S* ∩ {*x*. *a h* · *x* = *b h*})) = *aff_dim S* − *1*
using ‹*b h* < *a h* · *z*› *zaff* **by** (*force simp*: *aff_dim_affine_Int_hyperplane*)
then show *?thesis*
by (*simp add*: *ne faceS facet_of_def*)
qed
show *?thesis*
proof
show ∃ *h*. *h* ∈ *F* ∧ *C* = *S* ∩ {*x*. *a h* · *x* = *b h*} ⟹ *C facet_of S*
using ∗ **by** *blast*
next
assume *C facet_of S*
then have *C face_of S convex C C* ≠ {} **and** *affc*: *aff_dim C* = *aff_dim S* − *1*
by (*auto simp*: *facet_of_def face_of_imp_convex*)
then obtain *x* **where** *x*: *x* ∈ *rel_interior C*

```
      by (force simp: rel_interior_eq_empty)
    then have x ∈ C
      by (meson subsetD rel_interior_subset)
    then have x ∈ S
      using ‹C facet_of S› facet_of_imp_subset by blast
    have rels: rel_interior S = {x ∈ S. ∀h∈F. a h · x < b h}
      by (rule rel_interior_polyhedron_explicit [OF assms])
    have C ≠ S
      using ‹C facet_of S› facet_of_irrefl by blast
    then have x ∉ rel_interior S
     by (metis IntI empty_iff ‹x ∈ C› ‹C ≠ S› ‹C face_of S› face_of_disjoint_rel_interior)
    with rels ‹x ∈ S› obtain i where i ∈ F and i: a i · x ≥ b i
      by force
    have x ∈ {u. a i · u ≤ b i}
      by (metis IntD2 InterE ‹i ∈ F› ‹x ∈ S› faceq seq)
    then have a i · x ≤ b i by simp
    then have a i · x = b i using i by auto
    have C ⊆ S ∩ {x. a i · x = b i}
    proof (rule subset_of_face_of [of _ S])
      show S ∩ {x. a i · x = b i} face_of S
        by (simp add: * ‹i ∈ F› facet_of_imp_face_of)
      show C ⊆ S
        by (simp add: ‹C face_of S› face_of_imp_subset)
      show S ∩ {x. a i · x = b i} ∩ rel_interior C ≠ {}
        using ‹a i · x = b i› ‹x ∈ S› x by blast
    qed
    then have cface: C face_of (S ∩ {x. a i · x = b i})
      by (meson ‹C face_of S› face_of_subset inf_le1)
    have con: convex (S ∩ {x. a i · x = b i})
      by (simp add: ‹convex S› convex_Int convex_hyperplane)
    show ∃h. h ∈ F ∧ C = S ∩ {x. a h · x = b h}
      apply (rule_tac x=i in exI)
       by (metis (no_types) * ‹i ∈ F› affc facet_of_def less_irrefl face_of_aff_dim_lt
[OF con cface])
  qed
qed


lemma face_of_polyhedron_subset_explicit:
  fixes S :: 'a :: euclidean_space set
  assumes finite F
      and seq: S = affine hull S ∩ ⋂F
      and faceq: ⋀h. h ∈ F ⟹ a h ≠ 0 ∧ h = {x. a h · x ≤ b h}
      and psub: ⋀F'. F' ⊂ F ⟹ S ⊂ affine hull S ∩ ⋂F'
      and C: C face_of S and C ≠ {} C ≠ S
   obtains h where h ∈ F C ⊆ S ∩ {x. a h · x = b h}
proof -
  have C ⊆ S using ‹C face_of S›
    by (simp add: face_of_imp_subset)
```

have *polyhedron S*
by (*metis* ‹*finite F*› *faceq polyhedron_Int polyhedron_Inter polyhedron_affine_hull polyhedron_halfspace_le seq*)
then have *convex S*
by (*simp add*: *polyhedron_imp_convex*)
then have ∗: $(S \cap \{x.\ a\ h \cdot x = b\ h\})$ *face_of S* **if** $h \in F$ **for** *h*
using *faceq seq face_of_Int_supporting_hyperplane_le that* **by** *fastforce*
have *rel_interior* $C \neq \{\}$
using *C* ‹$C \neq \{\}$› *face_of_imp_convex rel_interior_eq_empty* **by** *blast*
then obtain *x* **where** $x \in$ *rel_interior C* **by** *auto*
have *rels*: *rel_interior* $S = \{x \in S.\ \forall h{\in}F.\ a\ h \cdot x < b\ h\}$
by (*rule rel_interior_polyhedron_explicit* [*OF* ‹*finite F*› *seq faceq psub*])
then have *xnot*: $x \notin$ *rel_interior S*
by (*metis IntI* ‹$x \in$ *rel_interior C*› *C* ‹$C \neq S$› *contra_subsetD empty_iff face_of_disjoint_rel_interior rel_interior_subset*)
then have $x \in S$
using ‹$C \subseteq S$› ‹$x \in$ *rel_interior C*› *rel_interior_subset* **by** *auto*
then have *xint*: $x \in \bigcap F$
using *seq* **by** *blast*
have $F \neq \{\}$ **using** *assms*
by (*metis affine_Int affine_Inter affine_affine_hull ex_in_conv face_of_affine_trivial*)
then obtain *i* **where** $i \in F$ $\neg$ $(a\ i \cdot x < b\ i)$
using ‹$x \in S$› *rels xnot* **by** *auto*
with *xint* **have** $a\ i \cdot x = b\ i$
by (*metis eq_iff mem_Collect_eq not_le Inter_iff faceq*)
have *face*: $S \cap \{x.\ a\ i \cdot x = b\ i\}$ *face_of S*
by (*simp add*: ∗ ‹$i \in F$›)
show *?thesis*
proof
show $C \subseteq S \cap \{x.\ a\ i \cdot x = b\ i\}$
using *subset_of_face_of* [*OF face* ‹$C \subseteq S$›] ‹$a\ i \cdot x = b\ i$› ‹$x \in$ *rel_interior C*› ‹$x \in S$› **by** *blast*
qed *fact*
qed

Initial part of proof duplicates that above

proposition *face_of_polyhedron_explicit*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *finite F*
and *seq*: $S =$ *affine hull* $S \cap \bigcap F$
and *faceq*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \land h = \{x.\ a\ h \cdot x \le b\ h\}$
and *psub*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset$ *affine hull* $S \cap \bigcap F'$
and *C*: *C face_of S* **and** $C \neq \{\}$ $C \neq S$
shows $C = \bigcap\{S \cap \{x.\ a\ h \cdot x = b\ h\} \mid h.\ h \in F \land C \subseteq S \cap \{x.\ a\ h \cdot x = b\ h\}\}$
proof −
let $?ab = \lambda h.\ \{x.\ a\ h \cdot x = b\ h\}$
have $C \subseteq S$ **using** ‹*C face_of S*›
by (*simp add*: *face_of_imp_subset*)

have *polyhedron S*
  by (*metis* ‹*finite F*› *faceq polyhedron_Int polyhedron_Inter polyhedron_affine_hull polyhedron_halfspace_le seq*)
then have *convex S*
  by (*simp add*: *polyhedron_imp_convex*)
then have ∗: $(S \cap ?ab\ h)$ *face_of S* **if** $h \in F$ **for** *h*
  using *faceq seq face_of_Int_supporting_hyperplane_le that* **by** *fastforce*
have *rel_interior C* $\neq \{\}$
  using *C* ‹$C \neq \{\}$› *face_of_imp_convex rel_interior_eq_empty* **by** *blast*
then obtain *z* **where** *z*: $z \in$ *rel_interior C* **by** *auto*
have *rels*: *rel_interior S* $= \{z \in S.\ \forall h \in F.\ a\ h \cdot z < b\ h\}$
  by (*rule rel_interior_polyhedron_explicit* [*OF* ‹*finite F*› *seq faceq psub*])
then have *xnot*: $z \notin$ *rel_interior S*
  by (*metis IntI* ‹$z \in$ *rel_interior C*› *C* ‹$C \neq S$› *contra_subsetD empty_iff face_of_disjoint_rel_interior rel_interior_subset*)
then have $z \in S$
  using ‹$C \subseteq S$› ‹$z \in$ *rel_interior C*› *rel_interior_subset* **by** *auto*
with *seq* **have** *xint*: $z \in \bigcap F$ **by** *blast*
have *open* $(\bigcap h\in\{h \in F.\ a\ h \cdot z < b\ h\}.\ \{w.\ a\ h \cdot w < b\ h\})$
  by (*auto simp*: ‹*finite F*› *open_halfspace_lt open_INT*)
then obtain *e* **where** $0 < e$
    ball z e $\subseteq (\bigcap h\in\{h \in F.\ a\ h \cdot z < b\ h\}.\ \{w.\ a\ h \cdot w < b\ h\})$
  by (*auto intro*: *openE* [*of* _ *z*])
then have *e*: $\bigwedge h.\ [\![h \in F;\ a\ h \cdot z < b\ h]\!] \Longrightarrow$ *ball z e* $\subseteq \{w.\ a\ h \cdot w < b\ h\}$
  by *blast*
have $C \subseteq (S \cap ?ab\ h) \longleftrightarrow z \in S \cap ?ab\ h$ **if** $h \in F$ **for** *h*
proof
  show $z \in S \cap ?ab\ h \Longrightarrow C \subseteq S \cap ?ab\ h$
    by (*metis* ∗ *Collect_cong IntI* ‹$C \subseteq S$› *empty_iff subset_of_face_of that z*)
next
  show $C \subseteq S \cap ?ab\ h \Longrightarrow z \in S \cap ?ab\ h$
    using ‹$z \in$ *rel_interior C*› *rel_interior_subset* **by** *force*
qed
then have ∗∗: $\{S \cap ?ab\ h \mid h.\ h \in F \wedge C \subseteq S \wedge C \subseteq ?ab\ h\} =$
    $\{S \cap ?ab\ h \mid h.\ h \in F \wedge z \in S \cap ?ab\ h\}$
  by *blast*
have *bsub*: *ball z e* $\cap$ *affine hull* $\bigcap\{S \cap ?ab\ h \mid h.\ h \in F \wedge a\ h \cdot z = b\ h\}$
    $\subseteq$ *affine hull S* $\cap \bigcap F \cap \bigcap\{?ab\ h \mid h.\ h \in F \wedge a\ h \cdot z = b\ h\}$
    if $i \in F$ **and** *i*: $a\ i \cdot z = b\ i$ **for** *i*
proof −
  have *sub*: *ball z e* $\cap \bigcap\{?ab\ h \mid h.\ h \in F \wedge a\ h \cdot z = b\ h\} \subseteq j$
      if $j \in F$ **for** *j*
  proof −
    have $a\ j \cdot z \le b\ j$ **using** *faceq that xint* **by** *auto*
    then consider $a\ j \cdot z < b\ j \mid a\ j \cdot z = b\ j$ **by** *linarith*
    then have $\exists G.\ G \in \{?ab\ h \mid h.\ h \in F \wedge a\ h \cdot z = b\ h\} \wedge$ *ball z e* $\cap G \subseteq j$
    proof *cases*
      assume $a\ j \cdot z < b\ j$
      then have *ball z e* $\cap \{x.\ a\ i \cdot x = b\ i\} \subseteq j$

using *e* [*OF* ‹*j* ∈ *F*›] *faceq that*
by (*fastforce simp*: *ball_def*)
then show *?thesis*
by (*rule_tac x*={*x*. *a i* · *x* = *b i*} **in** *exI*) (*force simp*: ‹*i* ∈ *F*› *i*)
next
assume *eq*: *a j* · *z* = *b j*
with *faceq that* **show** *?thesis*
by (*rule_tac x*={*x*. *a j* · *x* = *b j*} **in** *exI*) (*fastforce simp add*: ‹*j* ∈ *F*›)
qed
then show *?thesis* **by** *blast*
qed
have *1*: *affine hull* ⋂{*S* ∩ *?ab h* |*h*. *h* ∈ *F* ∧ *a h* · *z* = *b h*} ⊆ *affine hull S*
using *that* ‹*z* ∈ *S*› **by** (*intro hull_mono*) *auto*
have *2*: *affine hull* ⋂{*S* ∩ *?ab h* |*h*. *h* ∈ *F* ∧ *a h* · *z* = *b h*}
⊆ ⋂{*?ab h* |*h*. *h* ∈ *F* ∧ *a h* · *z* = *b h*}
by (*rule hull_minimal*) (*auto intro*: *affine_hyperplane*)
have *3*: *ball z e* ∩ ⋂{*?ab h* |*h*. *h* ∈ *F* ∧ *a h* · *z* = *b h*} ⊆ ⋂*F*
by (*iprover intro*: *sub Inter_greatest*)
have ∗: ⟦*A* ⊆ (*B* :: *'a set*); *A* ⊆ *C*; *E* ∩ *C* ⊆ *D*⟧ ⟹ *E* ∩ *A* ⊆ (*B* ∩ *D*) ∩ *C*
for *A B C D E* **by** *blast*
show *?thesis* **by** (*intro* ∗ *1 2 3*)
qed
have ∃*h*. *h* ∈ *F* ∧ *C* ⊆ *?ab h*
using *assms*
by (*metis face_of_polyhedron_subset_explicit* [*OF* ‹*finite F*› *seq faceq psub*] *le_inf_iff*)
then have *fac*: ⋂{*S* ∩ *?ab h* |*h*. *h* ∈ *F* ∧ *C* ⊆ *S* ∩ *?ab h*} *face_of S*
using ∗ **by** (*force simp*: ‹*C* ⊆ *S*› *intro*: *face_of_Inter*)
have *red*: (⋀*a*. *P a* ⟹ *T* ⊆ *S* ∩ ⋂{*F X* |*X*. *P X*}) ⟹ *T* ⊆ ⋂{*S* ∩ *F X* |*X*::*'a set*. *P X*} **for** *P T F*
by *blast*
have *ball z e* ∩ *affine hull* ⋂{*S* ∩ *?ab h* |*h*. *h* ∈ *F* ∧ *a h* · *z* = *b h*}
⊆ ⋂{*S* ∩ *?ab h* |*h*. *h* ∈ *F* ∧ *a h* · *z* = *b h*}
by (*rule red*) (*metis seq bsub*)
with ‹*0* < *e*› **have** *zinrel*: *z* ∈ *rel_interior*
(⋂{*S* ∩ *?ab h* |*h*. *h* ∈ *F* ∧ *z* ∈ *S* ∧ *a h* · *z* = *b h*})
by (*auto simp*: *mem_rel_interior_ball* ‹*z* ∈ *S*›)
show *?thesis*
using *z zinrel*
by (*intro face_of_eq* [*OF C fac*]) (*force simp*: ∗∗)
qed

6.38.12 More general corollaries from the explicit representation

corollary *facet_of_polyhedron*:
assumes *polyhedron S* **and** *C facet_of S*
obtains *a b* **where** *a* ≠ *0 S* ⊆ {*x*. *a* · *x* ≤ *b*} *C* = *S* ∩ {*x*. *a* · *x* = *b*}
proof −
obtain *F* **where** *finite F* **and** *seq*: *S* = *affine hull S* ∩ ⋂*F*

and *faces*: $\bigwedge h.\ h \in F \Longrightarrow \exists\, a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\}$
and *min*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset (\mathit{affine\ hull}\ S) \cap \bigcap F'$
using *assms* **by** (*simp add*: *polyhedron_Int_affine_minimal*) *meson*
then obtain *a b* **where** *ab*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
by *metis*
obtain *i* **where** $i \in F$ **and** *C*: $C = S \cap \{x.\ a\ i \cdot x = b\ i\}$
using *facet_of_polyhedron_explicit* [*OF* ‹*finite F*› *seq ab min*] *assms*
by *force*
moreover have *ssub*: $S \subseteq \{x.\ a\ i \cdot x \leq b\ i\}$
using ‹$i \in F$› *ab* **by** (*subst seq*) *auto*
ultimately show *?thesis*
by (*rule_tac* $a = a\ i$ **and** $b = b\ i$ **in** *that*) (*simp_all add*: *ab*)
qed

corollary *face_of_polyhedron*:
assumes *polyhedron S* **and** *C face_of S* **and** $C \neq \{\}$ **and** $C \neq S$
shows $C = \bigcap\{F.\ F\ \mathit{facet\_of}\ S \wedge C \subseteq F\}$
proof −
obtain *F* **where** *finite F* **and** *seq*: $S = \mathit{affine\ hull}\ S \cap \bigcap F$
and *faces*: $\bigwedge h.\ h \in F \Longrightarrow \exists\, a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\}$
and *min*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset (\mathit{affine\ hull}\ S) \cap \bigcap F'$
using *assms* **by** (*simp add*: *polyhedron_Int_affine_minimal*) *meson*
then obtain *a b* **where** *ab*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
by *metis*
show *?thesis*
apply (*subst face_of_polyhedron_explicit* [*OF* ‹*finite F*› *seq ab min*])
apply (*auto simp*: *assms facet_of_polyhedron_explicit* [*OF* ‹*finite F*› *seq ab min*]
cong: *Collect_cong*)
done
qed

lemma *face_of_polyhedron_subset_facet*:
assumes *polyhedron S* **and** *C face_of S* **and** $C \neq \{\}$ **and** $C \neq S$
obtains *F* **where** *F facet_of S* $C \subseteq F$
using *face_of_polyhedron assms*
by (*metis* (*no_types*, *lifting*) *Inf_greatest antisym_conv face_of_imp_subset mem_Collect_eq*)

lemma *exposed_face_of_polyhedron*:
assumes *polyhedron S*
shows *F exposed_face_of S* $\longleftrightarrow$ *F face_of S*
proof
show *F exposed_face_of S* $\Longrightarrow$ *F face_of S*
by (*simp add*: *exposed_face_of_def*)
next
assume *F face_of S*
show *F exposed_face_of S*
proof (*cases* $F = \{\} \vee F = S$)
case *True* **then show** *?thesis*

using ‹*F face_of S*› *exposed_face_of* **by** *blast*
next
case *False*
then have $\{g.\ g\ facet\_of\ S \wedge F \subseteq g\} \neq \{\}$
by (*metis Collect_empty_eq_bot* ‹*F face_of S*› *assms empty_def face_of_polyhedron_subset_facet*)
moreover have $\bigwedge T.\ [\![T\ facet\_of\ S;\ F \subseteq T]\!] \Longrightarrow T\ exposed\_face\_of\ S$
by (*metis assms exposed_face_of facet_of_imp_face_of facet_of_polyhedron*)
ultimately have $\bigcap\{G.\ G\ facet\_of\ S \wedge F \subseteq G\}\ exposed\_face\_of\ S$
by (*metis* (*no_types*, *lifting*) *mem_Collect_eq exposed_face_of_Inter*)
then show *?thesis*
using *False* ‹*F face_of S*› *assms face_of_polyhedron* **by** *fastforce*
qed
qed

lemma *face_of_polyhedron_polyhedron*:
fixes $S :: 'a :: euclidean\_space\ set$
assumes *polyhedron S c face_of S* **shows** *polyhedron c*
by (*metis assms face_of_imp_eq_affine_Int polyhedron_Int polyhedron_affine_hull polyhedron_imp_convex*)

lemma *finite_polyhedron_faces*:
fixes $S :: 'a :: euclidean\_space\ set$
assumes *polyhedron S*
shows *finite* $\{F.\ F\ face\_of\ S\}$
proof −
obtain *F* **where** *finite F* **and** *seq*: $S = affine\ hull\ S \cap \bigcap F$
and *faces*: $\bigwedge h.\ h \in F \Longrightarrow \exists\, a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\}$
and *min*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset (affine\ hull\ S) \cap \bigcap F'$
using *assms* **by** (*simp add*: *polyhedron_Int_affine_minimal*) *meson*
then obtain *a b* **where** *ab*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
by *metis*
have *finite* $\{\bigcap\{S \cap \{x.\ a\ h \cdot x = b\ h\}\ |h.\ h \in F'\}|\ F'.\ F' \in Pow\ F\}$
by (*simp add*: ‹*finite F*›)
moreover have $\{F.\ F\ face\_of\ S\} - \{\{\}, S\} \subseteq \{\bigcap\{S \cap \{x.\ a\ h \cdot x = b\ h\}\ |h.\ h \in F'\}|\ F'.\ F' \in Pow\ F\}$
apply *clarify*
apply (*rename_tac c*)
apply (*drule face_of_polyhedron_explicit* [*OF* ‹*finite F*› *seq ab min*, *simplified*], *simp_all*)
apply (*rule_tac x*={*h* ∈ *F*. *c* ⊆ *S* ∩ {*x*. *a h* · *x* = *b h*}} **in** *exI*, *auto*)
done
ultimately show *?thesis*
by (*meson finite.emptyI finite.insertI finite_Diff2 finite_subset*)
qed

lemma *finite_polyhedron_exposed_faces*:
polyhedron S ⟹ *finite* $\{F.\ F\ exposed\_face\_of\ S\}$
using *exposed_face_of_polyhedron finite_polyhedron_faces* **by** *fastforce*

lemma *finite_polyhedron_extreme_points*:
  fixes *S* :: *'a* :: *euclidean_space set*
  assumes *polyhedron S* **shows** *finite* {*v*. *v extreme_point_of S*}
proof −
  have *finite* {*v*. {*v*} *face_of S*}
   using *assms* **by** (*intro finite_subset* [*OF* _ *finite_vimageI* [*OF finite_polyhedron_faces*]], *auto*)
  then show *?thesis*
    by (*simp add*: *face_of_singleton*)
qed

lemma *finite_polyhedron_facets*:
  fixes *S* :: *'a* :: *euclidean_space set*
  shows *polyhedron S* ⟹ *finite* {*F*. *F facet_of S*}
  unfolding *facet_of_def*
  by (*blast intro*: *finite_subset* [*OF* _ *finite_polyhedron_faces*])

proposition *rel_interior_of_polyhedron*:
  fixes *S* :: *'a* :: *euclidean_space set*
  assumes *polyhedron S*
    shows *rel_interior S* = *S* − ⋃{*F*. *F facet_of S*}
proof −
  obtain *F* **where** *finite F* **and** *seq*: *S* = *affine hull S* ∩ ⋂ *F*
      and *faces*: ⋀*h*. *h* ∈ *F* ⟹ ∃ *a b*. *a* ≠ *0* ∧ *h* = {*x*. *a* · *x* ≤ *b*}
      and *min*: ⋀*F'*. *F'* ⊂ *F* ⟹ *S* ⊂ (*affine hull S*) ∩ ⋂ *F'*
    using *assms* **by** (*simp add*: *polyhedron_Int_affine_minimal*) *meson*
  then obtain *a b* **where** *ab*: ⋀*h*. *h* ∈ *F* ⟹ *a h* ≠ *0* ∧ *h* = {*x*. *a h* · *x* ≤ *b h*}
    by *metis*
  have *facet*: (*c facet_of S*) ⟷ (∃ *h*. *h* ∈ *F* ∧ *c* = *S* ∩ {*x*. *a h* · *x* = *b h*}) **for** *c*
    by (*rule facet_of_polyhedron_explicit* [*OF* ‹*finite F*› *seq ab min*])
  have *rel*: *rel_interior S* = {*x* ∈ *S*. ∀ *h*∈*F*. *a h* · *x* < *b h*}
    by (*rule rel_interior_polyhedron_explicit* [*OF* ‹*finite F*› *seq ab min*])
  have *a h* · *x* < *b h* **if** *x* ∈ *S h* ∈ *F* **and** *xnot*: *x* ∉ ⋃{*F*. *F facet_of S*} **for** *x h*
  proof −
    have *x* ∈ ⋂ *F* **using** *seq that* **by** *force*
    with ‹*h* ∈ *F*› *ab* **have** *a h* · *x* ≤ *b h* **by** *auto*
    then consider *a h* · *x* < *b h* | *a h* · *x* = *b h* **by** *linarith*
    then show *?thesis*
    proof *cases*
      case *1* **then show** *?thesis* **.**
    next
      case *2*
      have *Collect* ((∈) *x*) ∉ *Collect* ((∈) (⋃{*A*. *A facet_of S*}))
        using *xnot* **by** *fastforce*
      then have *F* ∉ *Collect* ((∈) *h*)
        using *2* ‹*x* ∈ *S*› *facet* **by** *blast*
      with *2 that* ‹*x* ∈ ⋂ *F*› **show** *?thesis*
        by *blast*

```
    qed
  qed
  moreover have ∃h∈F. a h · x ≥ b h if x ∈ ⋃{F. F facet_of S} for x
    using that by (force simp: facet)
  ultimately show ?thesis
    by (force simp: rel)
qed

lemma rel_boundary_of_polyhedron:
  fixes S :: 'a :: euclidean_space set
  assumes polyhedron S
    shows S − rel_interior S = ⋃ {F. F facet_of S}
using facet_of_imp_subset by (fastforce simp add: rel_interior_of_polyhedron assms)

lemma rel_frontier_of_polyhedron:
  fixes S :: 'a :: euclidean_space set
  assumes polyhedron S
    shows rel_frontier S = ⋃ {F. F facet_of S}
by (simp add: assms rel_frontier_def polyhedron_imp_closed rel_boundary_of_polyhedron)

lemma rel_frontier_of_polyhedron_alt:
  fixes S :: 'a :: euclidean_space set
  assumes polyhedron S
  shows rel_frontier S = ⋃ {F. F face_of S ∧ F ≠ S}
proof
  show rel_frontier S ⊆ ⋃ {F. F face_of S ∧ F ≠ S}
    by (force simp: rel_frontier_of_polyhedron facet_of_def assms)
qed (use face_of_subset_rel_frontier in fastforce)
```

A characterization of polyhedra as having finitely many faces

```
proposition polyhedron_eq_finite_exposed_faces:
  fixes S :: 'a :: euclidean_space set
  shows polyhedron S ⟷ closed S ∧ convex S ∧ finite {F. F exposed_face_of S}
        (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
   by (auto simp: polyhedron_imp_closed polyhedron_imp_convex finite_polyhedron_exposed_faces)
next
  assume ?rhs
  then have closed S convex S and fin: finite {F. F exposed_face_of S} by auto
  show ?lhs
  proof (cases S = {})
    case True then show ?thesis by auto
  next
    case False
    define F where F = {h. h exposed_face_of S ∧ h ≠ {} ∧ h ≠ S}
    have finite F by (simp add: fin F_def)
    have hface: h face_of S
```

and ∃ *a b*. *a* ≠ *0* ∧ *S* ⊆ {*x*. *a* · *x* ≤ *b*} ∧ *h* = *S* ∩ {*x*. *a* · *x* = *b*}
if *h* ∈ *F* **for** *h*
using *exposed_face_of F_def that* **by** *blast+*
then obtain *a b* **where** *ab*:
⋀*h*. *h* ∈ *F* ⟹ *a h* ≠ *0* ∧ *S* ⊆ {*x*. *a h* · *x* ≤ *b h*} ∧ *h* = *S* ∩ {*x*. *a h* · *x* = *b h*}
by *metis*
have ∗: *False*
if *paff*: *p* ∈ *affine hull S* **and** *p* ∉ *S*
and *pint*: *p* ∈ ⋂{{*x*. *a h* · *x* ≤ *b h*} |*h*. *h* ∈ *F*} **for** *p*
proof −
have *rel_interior S* ≠ {}
by (*simp add*: ‹*S* ≠ {}› ‹*convex S*› *rel_interior_eq_empty*)
then obtain *c* **where** *c*: *c* ∈ *rel_interior S* **by** *auto*
with *rel_interior_subset* **have** *c* ∈ *S* **by** *blast*
have *ccp*: *closed_segment c p* ⊆ *affine hull S*
by (*meson affine_affine_hull affine_imp_convex c closed_segment_subset hull_subset paff rel_interior_subset subsetCE*)
have *oS*: *openin* (*top_of_set* (*closed_segment c p*)) (*closed_segment c p* ∩ *rel_interior S*)
by (*force simp*: *openin_rel_interior openin_Int intro*: *openin_subtopology_Int_subset* [*OF _ ccp*])
obtain *x* **where** *xcl*: *x* ∈ *closed_segment c p* **and** *x* ∈ *S* **and** *xnot*: *x* ∉ *rel_interior S*
using *connected_openin* [*of closed_segment c p*]
apply *simp*
apply (*drule_tac x=closed_segment c p* ∩ *rel_interior S* **in** *spec*)
apply (*drule mp* [*OF _ oS*])
apply (*drule_tac x=closed_segment c p* ∩ (− *S*) **in** *spec*)
using *rel_interior_subset* ‹*closed S*› *c* ‹*p* ∉ *S*› **apply** *blast*
done
then obtain *μ* **where** *0* ≤ *μ μ* ≤ *1* **and** *xeq*: *x* = (*1* − *μ*) $*_R$ *c* + *μ* $*_R$ *p*
by (*auto simp*: *in_segment*)
show *False*
proof (*cases μ=0* ∨ *μ=1*)
case *True* **with** *xeq c xnot* ‹*x* ∈ *S*› ‹*p* ∉ *S*›
show *False* **by** *auto*
next
case *False*
then have *xos*: *x* ∈ *open_segment c p*
using ‹*x* ∈ *S*› *c open_segment_def that*(*2*) *xcl xnot* **by** *auto*
have *xclo*: *x* ∈ *closure S*
using ‹*x* ∈ *S*› *closure_subset* **by** *blast*
obtain *d* **where** *d* ≠ *0*
and *dle*: ⋀*y*. *y* ∈ *closure S* ⟹ *d* · *x* ≤ *d* · *y*
and *dless*: ⋀*y*. *y* ∈ *rel_interior S* ⟹ *d* · *x* < *d* · *y*
by (*metis supporting_hyperplane_relative_frontier* [*OF* ‹*convex S*› *xclo xnot*])
have *sex*: *S* ∩ {*y*. *d* · *y* = *d* · *x*} *exposed_face_of S*
by (*simp add*: ‹*closed S*› *dle exposed_face_of_Int_supporting_hyperplane_ge*

[*OF* ‹*convex S*›])
have *sne*: $S \cap \{y.\ d \cdot y = d \cdot x\} \neq \{\}$
using ‹$x \in S$› **by** *blast*
have *sns*: $S \cap \{y.\ d \cdot y = d \cdot x\} \neq S$
by (*metis* (*mono_tags*) *Int_Collect c subsetD dless not_le order_refl rel_interior_subset*)
obtain *h* **where** $h \in F$ $x \in h$
using *F_def* ‹$x \in S$› *sex sns* **by** *blast*
have *abface*: $\{y.\ a\ h \cdot y = b\ h\}$ *face_of* $\{y.\ a\ h \cdot y \leq b\ h\}$
using *hyperplane_face_of_halfspace_le* **by** *blast*
then have $c \in h$
using *face_ofD* [*OF abface xos*] ‹$c \in S$› ‹$h \in F$› *ab pint* ‹$x \in h$› **by** *blast*
with *c* **have** $h \cap$ *rel_interior* $S \neq \{\}$ **by** *blast*
then show *False*
using ‹$h \in F$› *F_def face_of_disjoint_rel_interior hface* **by** *auto*
qed
qed
have $S \subseteq$ *affine hull* $S \cap \bigcap\{\{x.\ a\ h \cdot x \leq b\ h\}\ |h.\ h \in F\}$
using *ab* **by** (*auto simp*: *hull_subset*)
moreover have *affine hull* $S \cap \bigcap\{\{x.\ a\ h \cdot x \leq b\ h\}\ |h.\ h \in F\} \subseteq S$
using ∗ **by** *blast*
ultimately have $S =$ *affine hull* $S \cap \bigcap\ \{\{x.\ a\ h \cdot x \leq b\ h\}\ |h.\ h \in F\}$ **..**
then show *?thesis*
apply (*rule ssubst*)
apply (*force intro*: *polyhedron_affine_hull polyhedron_halfspace_le simp*: ‹*finite F*›)
done
qed
qed

corollary *polyhedron_eq_finite_faces*:
fixes *S* :: *'a* :: *euclidean_space set*
shows *polyhedron S* ⟷ *closed S* ∧ *convex S* ∧ *finite* {*F*. *F face_of S*}
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*simp add*: *finite_polyhedron_faces polyhedron_imp_closed polyhedron_imp_convex*)
next
assume *?rhs*
then show *?lhs*
by (*force simp*: *polyhedron_eq_finite_exposed_faces exposed_face_of intro*: *finite_subset*)
qed

lemma *polyhedron_linear_image_eq*:
fixes *h* :: *'a* :: *euclidean_space* ⇒ *'b* :: *euclidean_space*
assumes *linear h bij h*
shows *polyhedron* (*h* ‘ *S*) ⟷ *polyhedron S*
proof −

have ∗: {f. P f} = (image h) ' {f. P (h ' f)} **for** P
  apply *safe*
  apply (*rule_tac x=inv h ' x* **in** *image_eqI*)
  apply (*auto simp*: ‹*bij h*› *bij_is_surj image_f_inv_f*)
  done
have *inj h* **using** *bij_is_inj assms* **by** *blast*
then have *injim*: *inj_on* ((') h) A **for** A
  by (*simp add*: *inj_on_def inj_image_eq_iff*)
show *?thesis*
  using ‹*linear h*› ‹*inj h*›
  apply (*simp add*: *polyhedron_eq_finite_faces closed_injective_linear_image_eq*)
  apply (*simp add*: ∗ *face_of_linear_image* [*of h _ S*, *symmetric*] *finite_image_iff injim*)
  done
qed

lemma *polyhedron_negations*:
  fixes S :: 'a :: *euclidean_space set*
  shows *polyhedron S* ⟹ *polyhedron*(*image uminus S*)
 by (*subst polyhedron_linear_image_eq*) (*auto simp*: *bij_uminus intro*!: *linear_uminus*)

6.38.13 Relation between polytopes and polyhedra

proposition *polytope_eq_bounded_polyhedron*:
  fixes S :: 'a :: *euclidean_space set*
  shows *polytope S* ⟷ *polyhedron S* ∧ *bounded S*
    (**is** *?lhs* = *?rhs*)
proof
  assume *?lhs*
  then show *?rhs*
    by (*simp add*: *finite_polytope_faces polyhedron_eq_finite_faces polytope_imp_closed polytope_imp_convex polytope_imp_bounded*)
next
  assume R: *?rhs*
  then have *finite* {v. v *extreme_point_of* S}
    by (*simp add*: *finite_polyhedron_extreme_points*)
  moreover have S = *convex hull* {v. v *extreme_point_of* S}
    using R **by** (*simp add*: *Krein_Milman_Minkowski compact_eq_bounded_closed polyhedron_imp_closed polyhedron_imp_convex*)
  ultimately show *?lhs*
    unfolding *polytope_def* **by** *blast*
qed

lemma *polytope_Int*:
  fixes S :: 'a :: *euclidean_space set*
  shows ⟦*polytope S*; *polytope T*⟧ ⟹ *polytope*(S ∩ T)
by (*simp add*: *polytope_eq_bounded_polyhedron bounded_Int*)

lemma *polytope_Int_polyhedron*:
  fixes *S* :: ′*a* :: *euclidean_space set*
  shows ⟦*polytope S*; *polyhedron T*⟧ ⟹ *polytope*(*S* ∩ *T*)
  by (*simp add*: *bounded_Int polytope_eq_bounded_polyhedron*)

lemma *polyhedron_Int_polytope*:
  fixes *S* :: ′*a* :: *euclidean_space set*
  shows ⟦*polyhedron S*; *polytope T*⟧ ⟹ *polytope*(*S* ∩ *T*)
  by (*simp add*: *bounded_Int polytope_eq_bounded_polyhedron*)

lemma *polytope_imp_polyhedron*:
  fixes *S* :: ′*a* :: *euclidean_space set*
  shows *polytope S* ⟹ *polyhedron S*
  by (*simp add*: *polytope_eq_bounded_polyhedron*)

lemma *polytope_facet_exists*:
  fixes *p* :: ′*a* :: *euclidean_space set*
  assumes *polytope p 0* < *aff_dim p*
  obtains *F* **where** *F facet_of p*
proof (*cases p* = {})
  case *True* **with** *assms* **show** *?thesis* **by** *auto*
next
  case *False*
  then obtain *v* **where** *v extreme_point_of p*
    using *extreme_point_exists_convex*
    by (*blast intro*: ‹*polytope p*› *polytope_imp_compact polytope_imp_convex*)
  then
  show *?thesis*
    by (*metis face_of_polyhedron_subset_facet polytope_imp_polyhedron aff_dim_sing all_not_in_conv assms face_of_singleton less_irrefl singletonI that*)
qed

lemma *polyhedron_interval* [*iff*]: *polyhedron*(*cbox a b*)
by (*metis polytope_imp_polyhedron polytope_interval*)

lemma *polyhedron_convex_hull*:
  fixes *S* :: ′*a* :: *euclidean_space set*
  shows *finite S* ⟹ *polyhedron*(*convex hull S*)
by (*simp add*: *polytope_convex_hull polytope_imp_polyhedron*)

6.38.14 Relative and absolute frontier of a polytope

lemma *rel_boundary_of_convex_hull*:
    fixes *S* :: ′*a*::*euclidean_space set*
    assumes ¬ *affine_dependent S*
      shows (*convex hull S*) − *rel_interior*(*convex hull S*) = (⋃ *a*∈*S*. *convex hull* (*S* − {*a*}))
proof −
  have *finite S* **by** (*metis assms aff_independent_finite*)

then consider *card S = 0 | card S = 1 | 2 ≤ card S* **by** *arith*
then show *?thesis*
proof *cases*
case *1* **then have** *S = {}* **by** (*simp add*: ‹*finite S*›)
then show *?thesis* **by** *simp*
next
case *2* **show** *?thesis*
by (*auto intro*: *card_1_singletonE* [*OF* ‹*card S = 1*›])
next
case *3*
with *assms* **show** *?thesis*
by (*auto simp*: *polyhedron_convex_hull rel_boundary_of_polyhedron facet_of_convex_hull_affine_independent_alt* ‹*finite S*›)
qed
qed

proposition *frontier_of_convex_hull*:
fixes *S* :: *'a::euclidean_space set*
assumes *card S = Suc* (*DIM*(*'a*))
shows *frontier*(*convex hull S*) = ⋃ {*convex hull* (*S* − {*a*}) | *a*. *a* ∈ *S*}
proof (*cases affine_dependent S*)
case *True*
have [*iff*]: *finite S*
using *assms* **using** *card.infinite* **by** *force*
then have *ccs*: *closed* (*convex hull S*)
by (*simp add*: *compact_imp_closed finite_imp_compact_convex_hull*)
{ **fix** *x T*
assume *int* (*card T*) ≤ *aff_dim S* + *1* *finite T T* ⊆ *Sx* ∈ *convex hull T*
then have *S* ≠ *T*
using *True* ‹*finite S*› *aff_dim_le_card affine_independent_iff_card* **by** *fastforce*
then obtain *a* **where** *a* ∈ *S a* ∉ *T*
using ‹*T* ⊆ *S*› **by** *blast*
then have ∃ *y*∈*S*. *x* ∈ *convex hull* (*S* − {*y*})
using *True affine_independent_iff_card* [*of S*]
by (*metis* (*no_types*, *hide_lams*) *Diff_eq_empty_iff Diff_insert0* ‹*a* ∉ *T*› ‹*T* ⊆ *S*› ‹*x* ∈ *convex hull T*› *hull_mono insert_Diff_single subsetCE*)
} **note** ∗ = *this*
have *1*: *convex hull S* ⊆ (⋃ *a*∈*S*. *convex hull* (*S* − {*a*}))
by (*subst caratheodory_aff_dim*) (*blast dest*: ∗)
have *2*: ⋃((λ*a*. *convex hull* (*S* − {*a*})) ‘ *S*) ⊆ *convex hull S*
by (*rule Union_least*) (*metis* (*no_types*, *lifting*) *Diff_subset hull_mono imageE*)
show *?thesis* **using** *True*
apply (*simp add*: *segment_convex_hull frontier_def*)
using *interior_convex_hull_eq_empty* [*OF assms*]
apply (*simp add*: *closure_closed* [*OF ccs*])
using *1 2* **by** *auto*
next
case *False*
then have *frontier* (*convex hull S*) = *closure* (*convex hull S*) − *interior* (*convex

hull S)
  by (*simp add*: *rel_boundary_of_convex_hull frontier_def*)
 also have ... = (*convex hull S*) − *rel_interior*(*convex hull S*)
  by (*metis False aff_independent_finite assms closure_convex_hull finite_imp_compact_convex_hull hull_hull interior_convex_hull_eq_empty rel_interior_nonempty_interior*)
 also have ... = ⋃{*convex hull* (*S* − {*a*}) |*a*. *a* ∈ *S*}
 proof −
  have *convex hull S* − *rel_interior* (*convex hull S*) = *rel_frontier* (*convex hull S*)
   by (*simp add*: *False aff_independent_finite polyhedron_convex_hull rel_boundary_of_polyhedron rel_frontier_of_polyhedron*)
  then show *?thesis*
   by (*simp add*: *False rel_frontier_convex_hull_cases*)
 qed
 finally show *?thesis* .
qed

6.38.15 Special case of a triangle

proposition *frontier_of_triangle*:
  fixes *a* :: *'a::euclidean_space*
  assumes *DIM*(*'a*) = *2*
  shows *frontier*(*convex hull* {*a*,*b*,*c*}) = *closed_segment a b* ∪ *closed_segment b c* ∪ *closed_segment c a*
    (**is** *?lhs* = *?rhs*)
proof (*cases b* = *a* ∨ *c* = *a* ∨ *c* = *b*)
 case *True* **then show** *?thesis*
  by (*auto simp*: *assms segment_convex_hull frontier_def empty_interior_convex_hull insert_commute card_insert_le_m1 hull_inc insert_absorb*)
next
 case *False* **then have** [*simp*]: *card* {*a*, *b*, *c*} = *Suc* (*DIM*(*'a*))
  by (*simp add*: *card.insert_remove Set.insert_Diff_if assms*)
 show *?thesis*
 proof
  show *?lhs* ⊆ *?rhs*
   using *False*
   by (*force simp*: *segment_convex_hull frontier_of_convex_hull insert_Diff_if insert_commute split*: *if_split_asm*)
  show *?rhs* ⊆ *?lhs*
   using *False*
   apply (*simp add*: *frontier_of_convex_hull segment_convex_hull*)
   apply (*intro conjI subsetI*)
   apply (*rule_tac X*=*convex hull* {*a*,*b*} **in** *UnionI*; *force simp*: *Set.insert_Diff_if*)
   apply (*rule_tac X*=*convex hull* {*b*,*c*} **in** *UnionI*; *force*)
   apply (*rule_tac X*=*convex hull* {*a*,*c*} **in** *UnionI*; *force simp*: *insert_commute Set.insert_Diff_if*)
   done
 qed
qed

corollary *inside_of_triangle*:
  fixes *a* :: *'a*::*euclidean_space*
  assumes *DIM*(*'a*) = *2*
  shows *inside* (*closed_segment a b* ∪ *closed_segment b c* ∪ *closed_segment c a*) = *interior*(*convex hull* {*a*,*b*,*c*})
by (*metis assms frontier_of_triangle bounded_empty bounded_insert convex_convex_hull inside_frontier_eq_interior bounded_convex_hull*)

corollary *interior_of_triangle*:
  fixes *a* :: *'a*::*euclidean_space*
  assumes *DIM*(*'a*) = *2*
  shows *interior*(*convex hull* {*a*,*b*,*c*}) =
      convex hull {*a*,*b*,*c*} − (*closed_segment a b* ∪ *closed_segment b c* ∪ *closed_segment c a*)
 using *interior_subset*
 by (*force simp*: *frontier_of_triangle* [*OF assms*, *symmetric*] *frontier_def Diff_Diff_Int*)

6.38.16 Subdividing a cell complex

lemma *subdivide_interval*:
 fixes *x*::*real*
 assumes *a* < |*x* − *y*| *0* < *a*
 obtains *n* **where** *n* ∈ ℤ *x* < *n* ∗ *a* ∧ *n* ∗ *a* < *y* ∨ *y* < *n* ∗ *a* ∧ *n* ∗ *a* < *x*
proof −
 consider *a* + *x* < *y* | *a* + *y* < *x*
  using *assms* **by** *linarith*
 then show *?thesis*
 proof *cases*
  case *1*
  let *?n* = *of_int* (*floor* (*x*/*a*)) + *1*
  have *x*: *x* < *?n* ∗ *a*
   by (*meson* ‹*0* < *a*› *divide_less_eq floor_eq_iff*)
  have *?n* ∗ *a* ≤ *a* + *x*
   apply (*simp add*: *algebra_simps*)
   by (*metis assms*(*2*) *floor_divide_lower mult.commute*)
  also have ... < *y*
   by (*rule 1*)
  finally have *?n* ∗ *a* < *y* **.**
  with *x* **show** *?thesis*
   using *Ints_1 Ints_add Ints_of_int that* **by** *blast*
 next
  case *2*
  let *?n* = *of_int* (*floor* (*y*/*a*)) + *1*
  have *y*: *y* < *?n* ∗ *a*
   by (*meson* ‹*0* < *a*› *divide_less_eq floor_eq_iff*)
  have *?n* ∗ *a* ≤ *a* + *y*
   apply (*simp add*: *algebra_simps*)
   by (*metis assms*(*2*) *floor_divide_lower mult.commute*)
  also have ... < *x*

```
    by (rule 2)
   finally have ?n ∗ a < x .
   then show ?thesis
    using Ints_1 Ints_add Ints_of_int that y by blast
  qed
qed

lemma cell_subdivision_lemma:
  assumes finite ℱ
    and ⋀X. X ∈ ℱ ⟹ polytope X
    and ⋀X. X ∈ ℱ ⟹ aff_dim X ≤ d
    and ⋀X Y. ⟦X ∈ ℱ; Y ∈ ℱ⟧ ⟹ (X ∩ Y) face_of X
    and finite I
   shows ∃𝒢. ⋃𝒢 = ⋃ℱ ∧
            finite 𝒢 ∧
            (∀ C ∈ 𝒢. ∃ D. D ∈ ℱ ∧ C ⊆ D) ∧
            (∀ C ∈ ℱ. ∀ x ∈ C. ∃ D. D ∈ 𝒢 ∧ x ∈ D ∧ D ⊆ C) ∧
            (∀ X ∈ 𝒢. polytope X) ∧
            (∀ X ∈ 𝒢. aff_dim X ≤ d) ∧
            (∀ X ∈ 𝒢. ∀ Y ∈ 𝒢. X ∩ Y face_of X) ∧
            (∀ X ∈ 𝒢. ∀ x ∈ X. ∀ y ∈ X. ∀ a b.
                (a,b) ∈ I ⟶ a · x ≤ b ∧ a · y ≤ b ∨
                        a · x ≥ b ∧ a · y ≥ b)
  using ‹finite I›
proof induction
  case empty
  then show ?case
   by (rule_tac x=ℱ in exI) (auto simp: assms)
next
  case (insert ab I)
  then obtain 𝒢 where eq: ⋃𝒢 = ⋃ℱ and finite 𝒢
           and sub1: ⋀C. C ∈ 𝒢 ⟹ ∃ D. D ∈ ℱ ∧ C ⊆ D
           and sub2: ⋀C x. C ∈ ℱ ∧ x ∈ C ⟹ ∃ D. D ∈ 𝒢 ∧ x ∈ D ∧ D
⊆ C
           and poly: ⋀X. X ∈ 𝒢 ⟹ polytope X
           and aff: ⋀X. X ∈ 𝒢 ⟹ aff_dim X ≤ d
           and face: ⋀X Y. ⟦X ∈ 𝒢; Y ∈ 𝒢⟧ ⟹ X ∩ Y face_of X
           and I: ⋀X x y a b. ⟦X ∈ 𝒢; x ∈ X; y ∈ X; (a,b) ∈ I⟧ ⟹
                   a · x ≤ b ∧ a · y ≤ b ∨ a · x ≥ b ∧ a · y ≥ b
   by (auto simp: that)
  obtain a b where ab = (a,b)
   by fastforce
  let ?𝒢 = (λX. X ∩ {x. a · x ≤ b}) ‘ 𝒢 ∪ (λX. X ∩ {x. a · x ≥ b}) ‘ 𝒢
  have eqInt: (S ∩ Collect P) ∩ (T ∩ Collect Q) = (S ∩ T) ∩ (Collect P ∩ Collect
Q) for S T::'a set and P Q
   by blast
  show ?case
  proof (intro conjI exI)
   show ⋃ ?𝒢 = ⋃ℱ
```

```
      by (force simp: eq [symmetric])
    show finite ?𝒢
      using ‹finite 𝒢› by force
    show ∀ X ∈ ?𝒢. polytope X
       by (force simp: poly polytope_Int_polyhedron polyhedron_halfspace_le polyhe-
dron_halfspace_ge)
    show ∀ X ∈ ?𝒢. aff_dim X ≤ d
      by (auto; metis order_trans aff aff_dim_subset inf_le1)
    show ∀ X ∈ ?𝒢. ∀ x ∈ X. ∀ y ∈ X. ∀ a b.
                         (a,b) ∈ insert ab I ⟶ a · x ≤ b ∧ a · y ≤ b ∨
                                                  a · x ≥ b ∧ a · y ≥ b
      using ‹ab = (a, b)› I by fastforce
    show ∀ X ∈ ?𝒢. ∀ Y ∈ ?𝒢. X ∩ Y face_of X
      by (auto simp: eqInt halfspace_Int_eq face_of_Int_Int face face_of_halfspace_le
face_of_halfspace_ge)
    show ∀ C ∈ ?𝒢. ∃ D. D ∈ ℱ ∧ C ⊆ D
      using sub1 by force
    show ∀ C∈ℱ. ∀ x∈C. ∃ D. D ∈ ?𝒢 ∧ x ∈ D ∧ D ⊆ C
    proof (intro ballI)
      fix C z
      assume C ∈ ℱ z ∈ C
      with sub2 obtain D where D: D ∈ 𝒢 z ∈ D D ⊆ C by blast
      have D ∈ 𝒢 ∧ z ∈ D ∩ {x. a · x ≤ b} ∧ D ∩ {x. a · x ≤ b} ⊆ C ∨
           D ∈ 𝒢 ∧ z ∈ D ∩ {x. a · x ≥ b} ∧ D ∩ {x. a · x ≥ b} ⊆ C
        using linorder_class.linear [of a · z b] D by blast
      then show ∃ D. D ∈ ?𝒢 ∧ z ∈ D ∧ D ⊆ C
        by blast
    qed
  qed
qed


proposition cell_complex_subdivision_exists:
  fixes ℱ :: 'a::euclidean_space set set
  assumes 0 < e finite ℱ
      and poly: ⋀X. X ∈ ℱ ⟹ polytope X
      and aff: ⋀X. X ∈ ℱ ⟹ aff_dim X ≤ d
      and face: ⋀X Y. ⟦X ∈ ℱ; Y ∈ ℱ⟧ ⟹ X ∩ Y face_of X
  obtains ℱ' where finite ℱ' ⋃ℱ' = ⋃ℱ ⋀X. X ∈ ℱ' ⟹ diameter X < e
                 ⋀X. X ∈ ℱ' ⟹ polytope X ⋀X. X ∈ ℱ' ⟹ aff_dim X ≤ d
                 ⋀X Y. ⟦X ∈ ℱ'; Y ∈ ℱ'⟧ ⟹ X ∩ Y face_of X
                 ⋀C. C ∈ ℱ' ⟹ ∃ D. D ∈ ℱ ∧ C ⊆ D
                 ⋀C x. C ∈ ℱ ∧ x ∈ C ⟹ ∃ D. D ∈ ℱ' ∧ x ∈ D ∧ D ⊆ C
proof −
  have bounded(⋃ℱ)
    by (simp add: ‹finite ℱ› poly bounded_Union polytope_imp_bounded)
  then obtain B where B > 0 and B: ⋀x. x ∈ ⋃ℱ ⟹ norm x < B
    by (meson bounded_pos_less)
  define C where C ≡ {z ∈ ℤ. |z ∗ e / 2 / real DIM('a)| ≤ B}
```

define I **where** $I \equiv \bigcup i \in Basis. \bigcup j \in C.\ \{ (i::'a, j * e / 2 / DIM('a)) \}$
have $C \subseteq \{x \in \mathbb{Z}. - B / (e / 2 / real\ DIM('a)) \leq x \wedge x \leq B / (e / 2 / real\ DIM('a))\}$
using ‹$0 < e$› **by** (*auto simp*: *field_split_simps C_def*)
then have *finite C*
using *finite_int_segment finite_subset* **by** *blast*
then have *finite I*
by (*simp add*: *I_def*)
obtain $\mathcal{F}'$ **where** *eq*: $\bigcup \mathcal{F}' = \bigcup \mathcal{F}$ **and** *finite* $\mathcal{F}'$
and *poly*: $\bigwedge X.\ X \in \mathcal{F}' \Longrightarrow$ *polytope* X
and *aff*: $\bigwedge X.\ X \in \mathcal{F}' \Longrightarrow$ *aff_dim* $X \leq d$
and *face*: $\bigwedge X\ Y.\ [\![X \in \mathcal{F}'; Y \in \mathcal{F}']\!] \Longrightarrow X \cap Y$ *face_of* X
and *I*: $\bigwedge X\ x\ y\ a\ b.\ [\![X \in \mathcal{F}'; x \in X; y \in X; (a,b) \in I]\!] \Longrightarrow$
$a \cdot x \leq b \wedge a \cdot y \leq b \vee a \cdot x \geq b \wedge a \cdot y \geq b$
and *sub1*: $\bigwedge C.\ C \in \mathcal{F}' \Longrightarrow \exists D.\ D \in \mathcal{F} \wedge C \subseteq D$
and *sub2*: $\bigwedge C\ x.\ C \in \mathcal{F} \wedge x \in C \Longrightarrow \exists D.\ D \in \mathcal{F}' \wedge x \in D \wedge D \subseteq C$
apply (*rule exE* [*OF cell_subdivision_lemma*])
using *assms* ‹*finite I*› **by** *auto*
show *?thesis*
proof (*rule_tac* $\mathcal{F}'$=$\mathcal{F}'$ **in** *that*)
show *diameter* $X < e$ **if** $X \in \mathcal{F}'$ **for** X
proof −
have *diameter* $X \leq e/2$
proof (*rule diameter_le*)
show *norm* $(x - y) \leq e / 2$ **if** $x \in X$ $y \in X$ **for** $x\ y$
proof −
have *norm* $x < B$ *norm* $y < B$
using B ‹$X \in \mathcal{F}'$› *eq that* **by** *blast+*
have *norm* $(x - y) \leq (\sum b{\in}Basis.\ |(x-y) \cdot b|)$
by (*rule norm_le_l1*)
also have ... $\leq$ *of_nat* $(DIM('a)) * (e / 2 / DIM('a))$
proof (*rule sum_bounded_above*)
fix $i::'a$
assume $i \in Basis$
then have I': $\bigwedge z\ b.\ [\![z \in C; b = z * e / (2 * real\ DIM('a))]\!] \Longrightarrow i \cdot x \leq b \wedge i \cdot y \leq b \vee i \cdot x \geq b \wedge i \cdot y \geq b$
using I[*of X x y*] ‹$X \in \mathcal{F}'$› *that* **unfolding** *I_def* **by** *auto*
show $|(x - y) \cdot i| \leq e / 2 / real\ DIM('a)$
proof (*rule ccontr*)
assume $\neg\ |(x - y) \cdot i| \leq e / 2 / real\ DIM('a)$
then have *xyi*: $|i \cdot x - i \cdot y| > e / 2 / real\ DIM('a)$
by (*simp add*: *inner_commute inner_diff_right*)
obtain n **where** $n \in \mathbb{Z}$ **and** n: $i \cdot x < n * (e / 2 / real\ DIM('a)) \wedge n * (e / 2 / real\ DIM('a)) < i \cdot y \vee i \cdot y < n * (e / 2 / real\ DIM('a)) \wedge n * (e / 2 / real\ DIM('a)) < i \cdot x$
using *subdivide_interval* [*OF xyi*] *DIM_positive* ‹$0 < e$›
by (*auto simp*: *zero_less_divide_iff*)
have $|i \cdot x| < B$
by (*metis* ‹$i \in Basis$› ‹*norm* $x < B$› *inner_commute norm_bound_Basis_lt*)

```
            have |i · y| < B
          by (metis ‹i ∈ Basis› ‹norm y < B› inner_commute norm_bound_Basis_lt)
            have *: |n * e| ≤ B * (2 * real DIM('a))
                    if |ix| < B |iy| < B
                       and ix: ix * (2 * real DIM('a)) < n * e
                       and iy: n * e < iy * (2 * real DIM('a)) for ix iy
            proof (rule abs_leI)
              have iy * (2 * real DIM('a)) ≤ B * (2 * real DIM('a))
                by (rule mult_right_mono) (use ‹|iy| < B› in linarith)+
              then show n * e ≤ B * (2 * real DIM('a))
                using iy by linarith
            next
              have − ix * (2 * real DIM('a)) ≤ B * (2 * real DIM('a))
                by (rule mult_right_mono) (use ‹|ix| < B› in linarith)+
              then show − (n * e) ≤ B * (2 * real DIM('a))
                using ix by linarith
            qed
            have n ∈ C
             using ‹n ∈ ℤ› n  by (auto simp: C_def divide_simps intro: * ‹|i · x|
< B› ‹|i · y| < B›)
            show False
              using  I' [OF ‹n ∈ C› refl] n  by auto
          qed
        qed
        also have ... = e / 2
          by simp
        finally show ?thesis .
      qed
    qed (use ‹0 < e› in force)
    also have ... < e
      by (simp add: ‹0 < e›)
    finally show ?thesis .
  qed
 qed (auto simp: eq poly aff face sub1 sub2 ‹finite ℱ'›)
qed
```

6.38.17 Simplexes

The notion of n-simplex for integer $-(1::'a) \leq n$

```
definition simplex :: int ⇒ 'a::euclidean_space set ⇒ bool (infix simplex 50)
  where n simplex S ≡ ∃ C. ¬ affine_dependent C ∧ int(card C) = n + 1 ∧ S =
convex hull C

lemma simplex:
    n simplex S ⟷ (∃ C. finite C ∧
                   ¬ affine_dependent C ∧
                   int(card C) = n + 1 ∧
                   S = convex hull C)
  by (auto simp add: simplex_def intro: aff_independent_finite)
```

lemma *simplex_convex_hull*:
  ¬ *affine_dependent C* ∧ *int*(*card C*) = *n* + *1* ⟹ *n simplex* (*convex hull C*)
  by (*auto simp add*: *simplex_def*)

lemma *convex_simplex*: *n simplex S* ⟹ *convex S*
  by (*metis convex_convex_hull simplex_def*)

lemma *compact_simplex*: *n simplex S* ⟹ *compact S*
  unfolding *simplex*
  using *finite_imp_compact_convex_hull* **by** *blast*

lemma *closed_simplex*: *n simplex S* ⟹ *closed S*
  by (*simp add*: *compact_imp_closed compact_simplex*)

lemma *simplex_imp_polytope*:
  n simplex S ⟹ *polytope S*
  unfolding *simplex_def polytope_def*
  using *aff_independent_finite* **by** *blast*

lemma *simplex_imp_polyhedron*:
  n simplex S ⟹ *polyhedron S*
  by (*simp add*: *polytope_imp_polyhedron simplex_imp_polytope*)

lemma *simplex_dim_ge*: *n simplex S* ⟹ −*1* ≤ *n*
 by (*metis* (*no_types*, *hide_lams*) *aff_dim_geq affine_independent_iff_card diff_add_cancel diff_diff_eq2 simplex_def*)

lemma *simplex_empty* [*simp*]: *n simplex* {} ⟷ *n* = −*1*
proof
  assume *n simplex* {}
  then show *n* = −*1*
    unfolding *simplex* **by** (*metis card.empty convex_hull_eq_empty diff_0 diff_eq_eq of_nat_0*)
next
  assume *n* = −*1* **then show** *n simplex* {}
    by (*fastforce simp*: *simplex*)
qed

lemma *simplex_minus_1* [*simp*]: −*1 simplex S* ⟷ *S* = {}
 by (*metis simplex cancel_comm_monoid_add_class.diff_cancel card_0_eq diff_minus_eq_add of_nat_eq_0_iff simplex_empty*)

lemma *aff_dim_simplex*:
  n simplex S ⟹ *aff_dim S* = *n*
 by (*metis simplex add.commute add_diff_cancel_left′ aff_dim_convex_hull affine_independent_iff_card*)

lemma *zero_simplex_sing*: *0 simplex* {*a*}

apply (*simp add*: *simplex_def*)
using *affine_independent_1 card_1_singleton_iff convex_hull_singleton* **by** *blast*

lemma *simplex_sing* [*simp*]: *n simplex* {*a*} ⟷ *n* = *0*
using *aff_dim_simplex aff_dim_sing zero_simplex_sing* **by** *blast*

lemma *simplex_zero*: *0 simplex S* ⟷ (∃ *a*. *S* = {*a*})
by (*metis aff_dim_eq_0 aff_dim_simplex simplex_sing*)

lemma *one_simplex_segment*: *a* ≠ *b* ⟹ *1 simplex closed_segment a b*
unfolding *simplex_def*
by (*rule_tac x*={*a*,*b*} **in** *exI*) (*auto simp*: *segment_convex_hull*)

lemma *simplex_segment_cases*:
(*if a* = *b then 0 else 1*) *simplex closed_segment a b*
by (*auto simp*: *one_simplex_segment*)

lemma *simplex_segment*:
∃ *n*. *n simplex closed_segment a b*
using *simplex_segment_cases* **by** *metis*

lemma *polytope_lowdim_imp_simplex*:
assumes *polytope P aff_dim P* ≤ *1*
obtains *n* **where** *n simplex P*
proof (*cases P* = {})
case *True*
then show *?thesis*
by (*simp add*: *that*)
next
case *False*
then show *?thesis*
by (*metis assms compact_convex_collinear_segment collinear_aff_dim polytope_imp_compact polytope_imp_convex simplex_segment_cases that*)
qed

lemma *simplex_insert_dimplus1*:
fixes *n*::*int*
assumes *n simplex S* **and** *a*: *a* ∉ *affine hull S*
shows (*n*+*1*) *simplex* (*convex hull* (*insert a S*))
proof −
obtain *C* **where** *C*: *finite C* ¬ *affine_dependent C int*(*card C*) = *n*+*1* **and** *S*: *S* = *convex hull C*
using *assms* **unfolding** *simplex* **by** *force*
show *?thesis*
unfolding *simplex*
proof (*intro exI conjI*)
have *aff_dim S* = *n*
using *aff_dim_simplex assms*(*1*) **by** *blast*
moreover have *a* ∉ *affine hull C*

using *S a affine_hull_convex_hull* **by** *blast*
moreover have $a \notin C$
using *S a hull_inc* **by** *fastforce*
ultimately show ¬ *affine_dependent* (*insert a C*)
by (*simp add*: *C S aff_dim_convex_hull aff_dim_insert affine_independent_iff_card*)
next
have $a \notin C$
using *S a hull_inc* **by** *fastforce*
then show *int* (*card* (*insert a C*)) = *n* + *1* + *1*
by (*simp add*: *C*)
next
show *convex hull insert a S* = *convex hull* (*insert a C*)
by (*simp add*: *S convex_hull_insert_segments*)
qed (*use C* **in** *auto*)
qed

6.38.18 Simplicial complexes and triangulations

definition *simplicial_complex* **where**
simplicial_complex $\mathcal{C}$ ≡
finite $\mathcal{C}$ ∧
($\forall S \in \mathcal{C}$. $\exists n$. *n simplex S*) ∧
($\forall F\ S$. $S \in \mathcal{C} \wedge F$ *face_of S* $\longrightarrow F \in \mathcal{C}$) ∧
($\forall S\ S'$. $S \in \mathcal{C} \wedge S' \in \mathcal{C} \longrightarrow (S \cap S')$ *face_of S*)

definition *triangulation* **where**
triangulation $\mathcal{T}$ ≡
finite $\mathcal{T}$ ∧
($\forall T \in \mathcal{T}$. $\exists n$. *n simplex T*) ∧
($\forall T\ T'$. $T \in \mathcal{T} \wedge T' \in \mathcal{T} \longrightarrow (T \cap T')$ *face_of T*)

6.38.19 Refining a cell complex to a simplicial complex

proposition *convex_hull_insert_Int_eq*:
fixes *z* :: *'a* :: *euclidean_space*
assumes *z*: $z \in$ *rel_interior S*
and *T*: $T \subseteq$ *rel_frontier S*
and *U*: $U \subseteq$ *rel_frontier S*
and *convex S convex T convex U*
shows *convex hull* (*insert z T*) ∩ *convex hull* (*insert z U*) = *convex hull* (*insert z* (*T* ∩ *U*))
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
proof (*cases T*={} ∨ *U*={})
case *True* **then show** *?thesis* **by** *auto*
next
case *False*
then have *T* ≠ {} *U* ≠ {} **by** *auto*
have *TU*: *convex* (*T* ∩ *U*)

```
    by (simp add: ‹convex T› ‹convex U› convex_Int)
  have (⋃x∈T. closed_segment z x) ∩ (⋃x∈U. closed_segment z x)
        ⊆ (if T ∩ U = {} then {z} else ⋃((closed_segment z) ` (T ∩ U))) (is _
⊆ ?IF)
  proof clarify
    fix x t u
    assume xt: x ∈ closed_segment z t
      and xu: x ∈ closed_segment z u
      and t ∈ T u ∈ U
    then have ne: t ≠ z u ≠ z
      using T U z unfolding rel_frontier_def by blast+
    show x ∈ ?IF
    proof (cases x = z)
      case True then show ?thesis by auto
    next
      case False
      have t: t ∈ closure S
        using T ‹t ∈ T› rel_frontier_def by auto
      have u: u ∈ closure S
        using U ‹u ∈ U› rel_frontier_def by auto
      show ?thesis
      proof (cases t = u)
        case True
        then show ?thesis
          using ‹t ∈ T› ‹u ∈ U› xt by auto
      next
        case False
        have tnot: t ∉ closed_segment u z
        proof -
          have t ∈ closure S − rel_interior S
            using T ‹t ∈ T› rel_frontier_def by blast
          then have t ∉ open_segment z u
            by (meson DiffD2 rel_interior_closure_convex_segment [OF ‹convex S›
z u] subsetD)
          then show ?thesis
           by (simp add: ‹t ≠ u› ‹t ≠ z› open_segment_commute open_segment_def)
        qed
        moreover have u ∉ closed_segment z t
         using rel_interior_closure_convex_segment [OF ‹convex S› z t] ‹u ∈ U› ‹u
≠ z›
            U [unfolded rel_frontier_def] tnot
          by (auto simp: closed_segment_eq_open)
        ultimately
        have ¬(between (t,u) z | between (u,z) t | between (z,t) u) if x ≠ z
          using that xt xu
              by (meson between_antisym between_mem_segment between_trans_2
ends_in_segment(2))
        then have ¬ collinear {t, z, u} if x ≠ z
          by (auto simp: that collinear_between_cases between_commute)
```

```
        moreover have collinear {t, z, x}
          by (metis closed_segment_commute collinear_2 collinear_closed_segment
collinear_triples ends_in_segment(1) insert_absorb insert_absorb2 xt)
        moreover have collinear {z, x, u}
          by (metis closed_segment_commute collinear_2 collinear_closed_segment
collinear_triples ends_in_segment(1) insert_absorb insert_absorb2 xu)
        ultimately have False
          using collinear_3_trans [of t z x u] ‹x ≠ z› by blast
        then show ?thesis by metis
      qed
    qed
  qed
  then show ?thesis
    using False ‹convex T› ‹convex U› TU
    by (simp add: convex_hull_insert_segments hull_same split: if_split_asm)
 qed
 show ?rhs ⊆ ?lhs
   by (metis inf_greatest hull_mono inf.cobounded1 inf.cobounded2 insert_mono)
qed

lemma simplicial_subdivision_aux:
  assumes finite 𝓜
    and ⋀C. C ∈ 𝓜 ⟹ polytope C
    and ⋀C. C ∈ 𝓜 ⟹ aff_dim C ≤ of_nat n
    and ⋀C F. ⟦C ∈ 𝓜; F face_of C⟧ ⟹ F ∈ 𝓜
    and ⋀C1 C2. ⟦C1 ∈ 𝓜; C2 ∈ 𝓜⟧ ⟹ C1 ∩ C2 face_of C1
  shows ∃𝒯. simplicial_complex 𝒯 ∧
           (∀K ∈ 𝒯. aff_dim K ≤ of_nat n) ∧
           ⋃𝒯 = ⋃𝓜 ∧
           (∀C ∈ 𝓜. ∃F. finite F ∧ F ⊆ 𝒯 ∧ C = ⋃F) ∧
           (∀K ∈ 𝒯. ∃C. C ∈ 𝓜 ∧ K ⊆ C)
  using assms
proof (induction n arbitrary: 𝓜 rule: less_induct)
  case (less n)
  then have poly𝓜: ⋀C. C ∈ 𝓜 ⟹ polytope C
    and aff𝓜:    ⋀C. C ∈ 𝓜 ⟹ aff_dim C ≤ of_nat n
    and face𝓜:   ⋀C F. ⟦C ∈ 𝓜; F face_of C⟧ ⟹ F ∈ 𝓜
    and intface𝓜: ⋀C1 C2. ⟦C1 ∈ 𝓜; C2 ∈ 𝓜⟧ ⟹ C1 ∩ C2 face_of C1
    by metis+
  show ?case
  proof (cases n ≤ 1)
    case True
    have ⋀s. ⟦n ≤ 1; s ∈ 𝓜⟧ ⟹ ∃m. m simplex s
      using poly𝓜 aff𝓜 by (force intro: polytope_lowdim_imp_simplex)
    then show ?thesis
      unfolding simplicial_complex_def using True
      by (rule_tac x=𝓜 in exI) (auto simp: less.prems)
  next
    case False
```

define $\mathcal{S}$ **where** $\mathcal{S} \equiv \{C \in \mathcal{M}.\ aff\_dim\ C < n\}$
have *finite* $\mathcal{S}$ $\bigwedge C.\ C \in \mathcal{S} \Longrightarrow polytope\ C$ $\bigwedge C.\ C \in \mathcal{S} \Longrightarrow aff\_dim\ C \leq int\ (n - 1)$
$\bigwedge C1\ C2.\ [\![C1 \in \mathcal{S};\ C2 \in \mathcal{S}]\!] \Longrightarrow C1 \cap C2\ face\_of\ C1$
using *less.prems* **by** (*auto simp*: $\mathcal{S}$*_def*)
moreover have §: $\bigwedge C\ F.\ [\![C \in \mathcal{S};\ F\ face\_of\ C]\!] \Longrightarrow F \in \mathcal{S}$
using *less.prems* **unfolding** $\mathcal{S}$*_def*
by (*metis* (*no_types*, *lifting*) *mem_Collect_eq aff_dim_subset face_of_imp_subset less_le not_le*)
ultimately obtain $\mathcal{U}$ **where** *simplicial_complex* $\mathcal{U}$
and *aff_dim*$\mathcal{U}$: $\bigwedge K.\ K \in \mathcal{U} \Longrightarrow aff\_dim\ K \leq int\ (n - 1)$
and $\bigcup\mathcal{U} = \bigcup\mathcal{S}$
and *fin*$\mathcal{U}$: $\bigwedge C.\ C \in \mathcal{S} \Longrightarrow \exists F.\ finite\ F \wedge F \subseteq \mathcal{U} \wedge C = \bigcup F$
and *C*$\mathcal{U}$: $\bigwedge K.\ K \in \mathcal{U} \Longrightarrow \exists C.\ C \in \mathcal{S} \wedge K \subseteq C$
using *less.IH* [*of* $n{-}1$ $\mathcal{S}$] *False* **by** *auto*
then have *finite* $\mathcal{U}$
and *simpl*$\mathcal{U}$: $\bigwedge S.\ S \in \mathcal{U} \Longrightarrow \exists n.\ n\ simplex\ S$
and *face*$\mathcal{U}$: $\bigwedge F\ S.\ [\![S \in \mathcal{U};\ F\ face\_of\ S]\!] \Longrightarrow F \in \mathcal{U}$
and *faceI*$\mathcal{U}$: $\bigwedge S\ S'.\ [\![S \in \mathcal{U};\ S' \in \mathcal{U}]\!] \Longrightarrow (S \cap S')\ face\_of\ S$
by (*auto simp*: *simplicial_complex_def*)
define $\mathcal{N}$ **where** $\mathcal{N} \equiv \{C \in \mathcal{M}.\ aff\_dim\ C = n\}$
have *finite* $\mathcal{N}$
by (*simp add*: $\mathcal{N}$*_def less.prems*(*1*))
have *poly*$\mathcal{N}$: $\bigwedge C.\ C \in \mathcal{N} \Longrightarrow polytope\ C$
and *convex*$\mathcal{N}$: $\bigwedge C.\ C \in \mathcal{N} \Longrightarrow convex\ C$
and *closed*$\mathcal{N}$: $\bigwedge C.\ C \in \mathcal{N} \Longrightarrow closed\ C$
by (*auto simp*: $\mathcal{N}$*_def poly*$\mathcal{M}$ *polytope_imp_convex polytope_imp_closed*)
have *in_rel_interior*: (*SOME* $z.\ z \in rel\_interior\ C) \in rel\_interior\ C$ **if** $C \in \mathcal{N}$ **for** C
using *that poly*$\mathcal{M}$ *polytope_imp_convex rel_interior_aff_dim some_in_eq* **by** (*fastforce simp*: $\mathcal{N}$*_def*)
have $*$: $\exists T.\ \neg\ affine\_dependent\ T \wedge card\ T \leq n \wedge aff\_dim\ K < n \wedge K = convex\ hull\ T$
if $K \in \mathcal{U}$ **for** K
proof −
obtain r **where** r: r *simplex* K
using ‹$K \in \mathcal{U}$› *simpl*$\mathcal{U}$ **by** *blast*
have $r = aff\_dim\ K$
using ‹r *simplex* K› *aff_dim_simplex* **by** *blast*
with r
show *?thesis*
unfolding *simplex_def*
using *False* ‹$\bigwedge K.\ K \in \mathcal{U} \Longrightarrow aff\_dim\ K \leq int\ (n - 1)$› *that* **by** *fastforce*
qed
have *ahK_C_disjoint*: *affine hull* $K \cap rel\_interior\ C = \{\}$
if $C \in \mathcal{N}$ $K \in \mathcal{U}$ $K \subseteq rel\_frontier\ C$ **for** $C\ K$
proof −
have *convex* C *closed* C
by (*auto simp*: *convex*$\mathcal{N}$ *closed*$\mathcal{N}$ ‹$C \in \mathcal{N}$›)

obtain F **where** F: F *face_of* C **and** $F \neq C$ $K \subseteq F$
proof −
obtain L **where** $L \in \mathcal{S}$ $K \subseteq L$
using ‹$K \in \mathcal{U}$› $C\mathcal{U}$ **by** *blast*
have $K \leq$ *rel_frontier* C
by (*simp add*: ‹$K \subseteq$ *rel_frontier* C›)
also have ... $\leq C$
by (*simp add*: ‹*closed* C› *rel_frontier_def subset_iff*)
finally have $K \subseteq C$ **.**
have $L \cap C$ *face_of* C
using $\mathcal{N}$*_def* $\mathcal{S}$*_def* ‹$C \in \mathcal{N}$› ‹$L \in \mathcal{S}$› *intfaceM* **by** (*simp add*: *inf_commute*)
moreover have $L \cap C \neq C$
using ‹$C \in \mathcal{N}$› ‹$L \in \mathcal{S}$›
by (*metis* (*mono_tags*, *lifting*) $\mathcal{N}$*_def* $\mathcal{S}$*_def* *intfaceM* *mem_Collect_eq* *not_le order_refl* §)
moreover have $K \subseteq L \cap C$
using ‹$C \in \mathcal{N}$› ‹$L \in \mathcal{S}$› ‹$K \subseteq C$› ‹$K \subseteq L$› **by** (*auto simp*: $\mathcal{N}$*_def* $\mathcal{S}$*_def*)
ultimately show *?thesis* **using** *that* **by** *metis*
qed
have *affine hull* $F \cap$ *rel_interior* $C = \{\}$
by (*rule affine_hull_face_of_disjoint_rel_interior* [*OF* ‹*convex* C› F ‹$F \neq C$›])
with *hull_mono* [*OF* ‹$K \subseteq F$›]
show *affine hull* $K \cap$ *rel_interior* $C = \{\}$
by *fastforce*
qed
let $?\mathcal{T} = (\bigcup C \in \mathcal{N}. \bigcup K \in \mathcal{U} \cap Pow\ (rel\_frontier\ C).$
$\{convex\ hull\ (insert\ (SOME\ z.\ z \in rel\_interior\ C)\ K)\})$
have $\exists \mathcal{T}.$ *simplicial_complex* $\mathcal{T} \wedge$
$(\forall K \in \mathcal{T}.\ aff\_dim\ K \leq of\_nat\ n) \wedge$
$(\forall C \in \mathcal{M}.\ \exists F.\ F \subseteq \mathcal{T} \wedge C = \bigcup F) \wedge$
$(\forall K \in \mathcal{T}.\ \exists C.\ C \in \mathcal{M} \wedge K \subseteq C)$
proof (*rule exI*, *intro conjI ballI*)
show *simplicial_complex* $(\mathcal{U} \cup ?\mathcal{T})$
unfolding *simplicial_complex_def*
proof (*intro conjI impI ballI allI*)
show *finite* $(\mathcal{U} \cup ?\mathcal{T})$
using ‹*finite* $\mathcal{U}$› ‹*finite* $\mathcal{N}$› **by** *simp*
show $\exists n.$ n *simplex* S **if** $S \in \mathcal{U} \cup ?\mathcal{T}$ **for** S
using *that ahK_C_disjoint in_rel_interior simplU simplex_insert_dimplus1* **by** *fastforce*
show $F \in \mathcal{U} \cup ?\mathcal{T}$ **if** S: $S \in \mathcal{U} \cup ?\mathcal{T} \wedge F$ *face_of* S **for** F S
proof −
have $F \in \mathcal{U}$ **if** $S \in \mathcal{U}$
using S *faceU that* **by** *blast*
moreover have $F \in \mathcal{U} \cup ?\mathcal{T}$
if F *face_of* S $C \in \mathcal{N}$ $K \in \mathcal{U}$ **and** $K \subseteq$ *rel_frontier* C
and S: $S =$ *convex hull insert* (*SOME* $z.\ z \in$ *rel_interior* C) K **for** C K
proof −

let *?z* = *SOME z*. *z* ∈ *rel_interior C*
have *?z* ∈ *rel_interior C*
by (*simp add*: *in_rel_interior* ‹*C* ∈ 𝒩›)
moreover
obtain *I* **where** ¬ *affine_dependent I card I* ≤ *n aff_dim K* < *int n K* = *convex hull I*
using ∗ [*OF* ‹*K* ∈ 𝒰›] **by** *auto*
ultimately have *?z* ∉ *affine hull I*
using *ahK_C_disjoint affine_hull_convex_hull that* **by** *blast*
have *compact I finite I*
by (*auto simp*: ‹¬ *affine_dependent I*› *aff_independent_finite finite_imp_compact*)
moreover have *F face_of convex hull insert ?z I*
by (*metis S* ‹*F face_of S*› ‹*K* = *convex hull I*› *convex_hull_eq_empty convex_hull_insert_segments hull_hull*)
ultimately obtain *J* **where** *J* ⊆ *insert ?z I F* = *convex hull J*
using *face_of_convex_hull_subset* [*of insert ?z I F*] **by** *auto*
show *?thesis*
proof (*cases ?z* ∈ *J*)
case *True*
have *F* ∈ (⋃*K*∈𝒰 ∩ *Pow* (*rel_frontier C*). {*convex hull insert ?z K*})
proof
have *convex hull* (*J* − {*?z*}) *face_of K*
by (*metis True* ‹*J* ⊆ *insert ?z I*› ‹*K* = *convex hull I*› ‹¬ *affine_dependent I*› *face_of_convex_hull_affine_independent subset_insert_iff*)
then have *convex hull* (*J* − {*?z*}) ∈ 𝒰
by (*rule face*𝒰 [*OF* ‹*K* ∈ 𝒰›])
moreover
have ⋀*x*. *x* ∈ *convex hull* (*J* − {*?z*}) ⟹ *x* ∈ *rel_frontier C*
by (*metis True* ‹*J* ⊆ *insert ?z I*› ‹*K* = *convex hull I*› *subsetD hull_mono subset_insert_iff that*(*4*))
ultimately show *convex hull* (*J* − {*?z*}) ∈ 𝒰 ∩ *Pow* (*rel_frontier C*) **by** *auto*
let *?F* = *convex hull insert ?z* (*convex hull* (*J* − {*?z*}))
have *F* ⊆ *?F*
apply (*clarsimp simp*: ‹*F* = *convex hull J*›)
by (*metis True subsetD hull_mono hull_subset subset_insert_iff*)
moreover have *?F* ⊆ *F*
apply (*clarsimp simp*: ‹*F* = *convex hull J*›)
by (*metis* (*no_types*, *lifting*) *True convex_hull_eq_empty convex_hull_insert_segments hull_hull insert_Diff*)
ultimately
show *F* ∈ {*?F*} **by** *auto*
qed
with ‹*C*∈𝒩› **show** *?thesis* **by** *auto*
next
case *False*
then have *F* ∈ 𝒰
using *face_of_convex_hull_affine_independent* [*OF* ‹¬ *affine_dependent*

I›]
by (*metis Int_absorb2 Int_insert_right_if0* ‹*F* = *convex hull J*› ‹*J* ⊆ *insert ?z I*› ‹*K* = *convex hull I*› *face𝒰 inf_le2* ‹*K* ∈ 𝒰›)
then show *F* ∈ 𝒰 ∪ *?𝒯*
by *blast*
qed
qed
ultimately show *?thesis*
using *that* **by** *auto*
qed
have §: *X* ∩ *Y face_of X* ∧ *X* ∩ *Y face_of Y*
if *XY*: *X* ∈ 𝒰 *Y* ∈ *?𝒯* **for** *X Y*
proof −
obtain *C K*
where *C* ∈ 𝒩 *K* ∈ 𝒰 *K* ⊆ *rel_frontier C*
and *Y*: *Y* = *convex hull insert* (*SOME z. z* ∈ *rel_interior C*) *K*
using *XY* **by** *blast*
have *convex C*
by (*simp add*: ‹*C* ∈ 𝒩› *convex𝒩*)
have *K* ⊆ *C*
by (*metis DiffE* ‹*C* ∈ 𝒩› ‹*K* ⊆ *rel_frontier C*› *closed𝒩 closure_closed rel_frontier_def subset_iff*)
let *?z* = (*SOME z. z* ∈ *rel_interior C*)
have *z*: *?z* ∈ *rel_interior C*
using ‹*C* ∈ 𝒩› *in_rel_interior* **by** *blast*
obtain *D* **where** *D* ∈ 𝒮 *X* ⊆ *D*
using *C𝒰* ‹*X* ∈ 𝒰› **by** *blast*
have *D* ∩ *rel_interior C* = (*C* ∩ *D*) ∩ *rel_interior C*
using *rel_interior_subset* **by** *blast*
also have (*C* ∩ *D*) ∩ *rel_interior C* = {}
proof (*rule face_of_disjoint_rel_interior*)
show *C* ∩ *D face_of C*
using *𝒩_def 𝒮_def* ‹*C* ∈ 𝒩› ‹*D* ∈ 𝒮› *intface𝓜* **by** *blast*
show *C* ∩ *D* ≠ *C*
by (*metis* (*mono_tags, lifting*) *Int_lower2 𝒩_def 𝒮_def* ‹*C* ∈ 𝒩› ‹*D* ∈ 𝒮› *aff_dim_subset mem_Collect_eq not_le*)
qed
finally have *DC*: *D* ∩ *rel_interior C* = {} **.**
have *eq*: *X* ∩ *convex hull* (*insert ?z K*) = *X* ∩ *convex hull K*
proof (*rule Int_convex_hull_insert_rel_exterior* [*OF* ‹*convex C*› ‹*K* ⊆ *C*› *z*])
show *disjnt X* (*rel_interior C*)
using *DC* **by** (*meson* ‹*X* ⊆ *D*› *disjnt_def disjnt_subset1*)
qed
obtain *I* **where** *I*: ¬ *affine_dependent I*
and *Keq*: *K* = *convex hull I* **and** [*simp*]: *convex hull K* = *K*
using ∗ ‹*K* ∈ 𝒰› **by** *force*
then have *?z* ∉ *affine hull I*
using *ahK_C_disjoint* ‹*C* ∈ 𝒩› ‹*K* ∈ 𝒰› ‹*K* ⊆ *rel_frontier C*› *affine_hull_convex_hull*

```
z by blast
          have X ∩ K face_of K
            by (simp add: XY(1) ‹K ∈ 𝒰› faceI𝒰 inf_commute)
          also have ... face_of convex hull insert ?z K
         by (metis I Keq ‹?z ∉ affine hull I› aff_independent_finite convex_convex_hull
face_of_convex_hull_insert face_of_refl hull_insert)
          finally have X ∩ K face_of convex hull insert ?z K .
          then show ?thesis
            by (simp add: XY(1) Y ‹K ∈ 𝒰› eq faceI𝒰)
        qed

        show S ∩ S' face_of S
          if S ∈ 𝒰 ∪ ?𝒯 ∧ S' ∈ 𝒰 ∪ ?𝒯 for S S'
          using that
        proof (elim conjE UnE)
          fix X Y
          assume X ∈ 𝒰 and Y ∈ 𝒰
          then show X ∩ Y face_of X
            by (simp add: faceI𝒰)
        next
          fix X Y
          assume XY: X ∈ 𝒰 Y ∈ ?𝒯
          then show X ∩ Y face_of X Y ∩ X face_of Y
            using § [OF XY] by (auto simp: Int_commute)
        next
          fix X Y
          assume XY: X ∈ ?𝒯 Y ∈ ?𝒯
          show X ∩ Y face_of X
          proof –
            obtain C K D L
              where C ∈ 𝒩 K ∈ 𝒰 K ⊆ rel_frontier C
                and X: X = convex hull insert (SOME z. z ∈ rel_interior C) K
                and D ∈ 𝒩 L ∈ 𝒰 L ⊆ rel_frontier D
                and Y: Y = convex hull insert (SOME z. z ∈ rel_interior D) L
              using XY by blast
            let ?z = (SOME z. z ∈ rel_interior C)
            have z: ?z ∈ rel_interior C
              using ‹C ∈ 𝒩› in_rel_interior by blast
            have convex C
              by (simp add: ‹C ∈ 𝒩› convex𝒩)
            have convex K
              using ∗ ‹K ∈ 𝒰› by blast
            have convex L
              by (meson ‹L ∈ 𝒰› convex_simplex simpl𝒰)
            show ?thesis
            proof (cases D=C)
              case True
              then have L ⊆ rel_frontier C
                using ‹L ⊆ rel_frontier D› by auto
```

have *convex hull insert* (*SOME z. z* ∈ *rel_interior C*) (*K* ∩ *L*) *face_of* *convex hull insert* (*SOME z. z* ∈ *rel_interior C*) *K*
by (*metis face_of_polytope_insert2* ∗ *IntI* ‹*C* ∈ 𝒩› *aff_independent_finite* *ahK_C_disjoint empty_iff faceI𝒰 polytope_def z* ‹*K* ∈ 𝒰› ‹*L* ∈ 𝒰›‹*K* ⊆ *rel_frontier* *C*›)
then show *?thesis*
using *True X Y* ‹*K* ⊆ *rel_frontier C*› ‹*L* ⊆ *rel_frontier C*› ‹*convex C*› ‹*convex K*› ‹*convex L*› *convex_hull_insert_Int_eq z* **by** *force*
next
case *False*
have *convex D*
by (*simp add*: ‹*D* ∈ 𝒩› *convex*𝒩)
have *K* ⊆ *C*
by (*metis DiffE* ‹*C* ∈ 𝒩› ‹*K* ⊆ *rel_frontier C*› *closed*𝒩 *closure_closed* *rel_frontier_def subset_eq*)
have *L* ⊆ *D*
by (*metis DiffE* ‹*D* ∈ 𝒩› ‹*L* ⊆ *rel_frontier D*› *closed*𝒩 *closure_closed* *rel_frontier_def subset_eq*)
let *?w* = (*SOME w. w* ∈ *rel_interior D*)
have *w*: *?w* ∈ *rel_interior D*
using ‹*D* ∈ 𝒩› *in_rel_interior* **by** *blast*
have *C* ∩ *rel_interior D* = (*D* ∩ *C*) ∩ *rel_interior D*
using *rel_interior_subset* **by** *blast*
also have (*D* ∩ *C*) ∩ *rel_interior D* = {}
proof (*rule face_of_disjoint_rel_interior*)
show *D* ∩ *C face_of D*
using 𝒩*_def* ‹*C* ∈ 𝒩› ‹*D* ∈ 𝒩› *intface*ℳ **by** *blast*
have *D* ∈ ℳ ∧ *aff_dim D* = *int n*
using 𝒩*_def* ‹*D* ∈ 𝒩› **by** *blast*
moreover have *C* ∈ ℳ ∧ *aff_dim C* = *int n*
using 𝒩*_def* ‹*C* ∈ 𝒩› **by** *blast*
ultimately show *D* ∩ *C* ≠ *D*
by (*metis Int_commute False face_of_aff_dim_lt inf.idem inf_le1* *intface*ℳ *not_le poly*ℳ *polytope_imp_convex*)
qed
finally have *CD*: *C* ∩ (*rel_interior D*) = {} **.**
have *zKC*: (*convex hull insert ?z K*) ⊆ *C*
by (*metis DiffE* ‹*C* ∈ 𝒩› ‹*K* ⊆ *rel_frontier C*› *closed*𝒩 *closure_closed* *convex*𝒩 *hull_minimal insert_subset rel_frontier_def rel_interior_subset subset_iff z*)
have *disjnt* (*convex hull insert* (*SOME z. z* ∈ *rel_interior C*) *K*) (*rel_interior D*)
using *zKC CD* **by** (*force simp*: *disjnt_def*)
then have *eq*: *convex hull* (*insert ?z K*) ∩ *convex hull* (*insert ?w L*) = *convex hull* (*insert ?z K*) ∩ *convex hull L*
by (*rule Int_convex_hull_insert_rel_exterior* [*OF* ‹*convex D*› ‹*L* ⊆ *D*› *w*])
have *ch_id*: *convex hull K* = *K convex hull L* = *L*
using ∗ ‹*K* ∈ 𝒰› ‹*L* ∈ 𝒰› *hull_same* **by** *auto*
have *convex C*

by (*simp add*: ‹$C \in \mathcal{N}$› *convex*$\mathcal{N}$)
have *convex hull* (*insert ?z K*) ∩ *L* = *L* ∩ *convex hull* (*insert ?z K*)
by *blast*
also have ... = *convex hull K* ∩ *L*
proof (*subst Int_convex_hull_insert_rel_exterior* [*OF* ‹*convex C*› ‹$K \subseteq C$› *z*])
have ($C \cap D$) ∩ *rel_interior C* = {}
proof (*rule face_of_disjoint_rel_interior*)
show *C* ∩ *D face_of C*
using $\mathcal{N}$*_def* ‹$C \in \mathcal{N}$› ‹$D \in \mathcal{N}$› *intface*$\mathcal{M}$ **by** *blast*
have $D \in \mathcal{M}$ *aff_dim D* = *int n*
using $\mathcal{N}$*_def* ‹$D \in \mathcal{N}$› **by** *fastforce*+
moreover have $C \in \mathcal{M}$ *aff_dim C* = *int n*
using $\mathcal{N}$*_def* ‹$C \in \mathcal{N}$› **by** *fastforce*+
ultimately have *aff_dim D* + − *1* ∗ *aff_dim C* ≤ *0*
by *fastforce*
then have ¬ *C face_of D*
using *False* ‹*convex D*› *face_of_aff_dim_lt* **by** *fastforce*
show *C* ∩ *D* ≠ *C*
by (*metis inf_commute* ‹$C \in \mathcal{M}$› ‹$D \in \mathcal{M}$› ‹¬ *C face_of D*› *intface*$\mathcal{M}$)
qed
then have *D* ∩ *rel_interior C* = {}
by (*metis inf.absorb_iff2 inf_assoc inf_sup_aci*(*1*) *rel_interior_subset*)
then show *disjnt L* (*rel_interior C*)
by (*meson* ‹$L \subseteq D$› *disjnt_def disjnt_subset1*)
next
show *L* ∩ *convex hull K* = *convex hull K* ∩ *L*
by *force*
qed
finally have *chKL*: *convex hull* (*insert ?z K*) ∩ *L* = *convex hull K* ∩ *L* .
have *convex hull insert ?z K* ∩ *convex hull L face_of K*
by (*simp add*: ‹$K \in \mathcal{U}$› ‹$L \in \mathcal{U}$› *ch_id chKL faceI*$\mathcal{U}$)
also have ... *face_of convex hull insert ?z K*
proof −
obtain *I* **where** *I*: ¬ *affine_dependent I K* = *convex hull I*
using ∗ [*OF* ‹$K \in \mathcal{U}$›] **by** *auto*
then have ⋀*a*. *a* ∉ *rel_interior C* ∨ *a* ∉ *affine hull I*
using *ahK_C_disjoint* ‹$C \in \mathcal{N}$› ‹$K \in \mathcal{U}$› ‹$K \subseteq$ *rel_frontier C*› *affine_hull_convex_hull* **by** *blast*
then show *?thesis*
by (*metis I affine_independent_insert face_of_convex_hull_affine_independent hull_insert subset_insertI z*)
qed
finally have *1*: *convex hull insert ?z K* ∩ *convex hull L face_of convex hull insert ?z K* .
have *convex hull insert ?z K* ∩ *convex hull L face_of L*
by (*metis* ‹$K \in \mathcal{U}$› ‹$L \in \mathcal{U}$› *chKL ch_id faceI*$\mathcal{U}$ *inf_commute*)

also have ... *face_of convex hull insert ?w L*
proof −
obtain *I* **where** *I*: ¬ *affine_dependent I L* = *convex hull I*
using ∗ [*OF* ‹*L* ∈ *𝒰*›] **by** *auto*
then have $\bigwedge$*a*. *a* ∉ *rel_interior D* ∨ *a* ∉ *affine hull I*
using ‹*D* ∈ *𝒩*› ‹*L* ∈ *𝒰*› ‹*L* ⊆ *rel_frontier D*› *affine_hull_convex_hull ahK_C_disjoint* **by** *blast*
then show *?thesis*
by (*metis I aff_independent_finite convex_convex_hull face_of_convex_hull_insert face_of_refl hull_insert w*)
qed
finally have *2*: *convex hull insert ?z K* ∩ *convex hull L face_of convex hull insert ?w L* **.**
show *?thesis*
by (*simp add*: *X Y eq 1 2*)
qed
qed
qed
qed
show ∃ *F* ⊆ *𝒰* ∪ *?𝒯*. *C* = ⋃ *F* **if** *C* ∈ *ℳ* **for** *C*
proof (*cases C* ∈ *𝒮*)
case *True*
then show *?thesis*
by (*meson UnCI fin𝒰 subsetD subsetI*)
next
case *False*
then have *C* ∈ *𝒩*
by (*simp add*: *𝒩_def 𝒮_def affℳ less_le that*)
let *?z* = *SOME z*. *z* ∈ *rel_interior C*
have *z*: *?z* ∈ *rel_interior C*
using ‹*C* ∈ *𝒩*› *in_rel_interior* **by** *blast*
let *?F* = ⋃ *K* ∈ *𝒰* ∩ *Pow* (*rel_frontier C*). {*convex hull* (*insert ?z K*)}
have *?F* ⊆ *?𝒯*
using ‹*C* ∈ *𝒩*› **by** *blast*
moreover have *C* ⊆ ⋃ *?F*
proof
fix *x*
assume *x* ∈ *C*
have *convex C*
using ‹*C* ∈ *𝒩*› *convex𝒩* **by** *blast*
have *bounded C*
using ‹*C* ∈ *𝒩*› **by** (*simp add*: *polyℳ polytope_imp_bounded that*)
have *polytope C*
using ‹*C* ∈ *𝒩*› *poly𝒩* **by** *auto*
have ¬ (*?z* = *x* ∧ *C* = {*?z*})
using ‹*C* ∈ *𝒩*› *aff_dim_sing* [*of ?z*] ‹¬ *n* ≤ *1*› **by** (*force simp*: *𝒩_def*)
then obtain *y* **where** *y*: *y* ∈ *rel_frontier C* **and** *xzy*: *x* ∈ *closed_segment ?z y*
and *sub*: *open_segment ?z y* ⊆ *rel_interior C*

by (*blast intro*: *segment_to_rel_frontier* [*OF* ‹*convex C*› ‹*bounded C*› *z* ‹*x* ∈ *C*›])
then obtain *F* **where** *y* ∈ *F F face_of C F* ≠ *C*
by (*auto simp*: *rel_frontier_of_polyhedron_alt* [*OF polytope_imp_polyhedron* [*OF* ‹*polytope C*›]])
then obtain $\mathcal{G}$ **where** *finite* $\mathcal{G}$ $\mathcal{G} \subseteq \mathcal{U}$ $F = \bigcup \mathcal{G}$
by (*metis* (*mono_tags*, *lifting*) $\mathcal{S}$*_def* ‹*C* ∈ $\mathcal{M}$› ‹*convex C*› *aff*$\mathcal{M}$ *face*$\mathcal{M}$ *face_of_aff_dim_lt fin*$\mathcal{U}$ *le_less_trans mem_Collect_eq not_less*)
then obtain *K* **where** *y* ∈ *K K* ∈ $\mathcal{G}$
using ‹*y* ∈ *F*› **by** *blast*
moreover have *x*: *x* ∈ *convex hull* {*?z*,*y*}
using *segment_convex_hull xzy* **by** *auto*
moreover have *convex hull* {*?z*,*y*} ⊆ *convex hull insert ?z K*
by (*metis* (*full_types*) ‹*y* ∈ *K*› *hull_mono empty_subsetI insertCI insert_subset*)
moreover have *K* ∈ $\mathcal{U}$
using ‹*K* ∈ $\mathcal{G}$› ‹$\mathcal{G} \subseteq \mathcal{U}$› **by** *blast*
moreover have *K* ⊆ *rel_frontier C*
using ‹$F = \bigcup \mathcal{G}$› ‹*F* ≠ *C*› ‹*F face_of C*› ‹*K* ∈ $\mathcal{G}$› *face_of_subset_rel_frontier* **by** *fastforce*
ultimately show *x* ∈ ⋃ *?F*
by *force*
qed
moreover
have *convex hull insert* (*SOME z*. *z* ∈ *rel_interior C*) *K* ⊆ *C*
if *K* ∈ $\mathcal{U}$ *K* ⊆ *rel_frontier C* **for** *K*
proof (*rule hull_minimal*)
show *insert* (*SOME z*. *z* ∈ *rel_interior C*) *K* ⊆ *C*
using *that* ‹*C* ∈ $\mathcal{N}$› *in_rel_interior rel_interior_subset*
by (*force simp*: *closure_eq rel_frontier_def closed*$\mathcal{N}$)
show *convex C*
by (*simp add*: ‹*C* ∈ $\mathcal{N}$› *convex*$\mathcal{N}$)
qed
then have ⋃ *?F* ⊆ *C*
by *auto*
ultimately show *?thesis*
by *blast*
qed
have (∃ *C*. *C* ∈ $\mathcal{M}$ ∧ *L* ⊆ *C*) ∧ *aff_dim L* ≤ *int n* **if** *L* ∈ $\mathcal{U}$ ∪ *?*$\mathcal{T}$ **for** *L*
using *that*
proof
assume *L* ∈ $\mathcal{U}$
then show *?thesis*
using *C*$\mathcal{U}$ $\mathcal{S}$*_def* ∗ **by** *fastforce*
next
assume *L* ∈ *?*$\mathcal{T}$
then obtain *C K* **where** *C* ∈ $\mathcal{N}$
and *L*: *L* = *convex hull insert* (*SOME z*. *z* ∈ *rel_interior C*) *K*
and *K*: *K* ∈ $\mathcal{U}$ *K* ⊆ *rel_frontier C*

by *auto*
then have *convex hull C = C*
by (*meson convex𝒩 convex_hull_eq*)
then have *convex C*
by (*metis* (*no_types*) *convex_convex_hull*)
have *rel_frontier C ⊆ C*
by (*metis DiffE closed𝒩* ‹*C* ∈ 𝒩› *closure_closed rel_frontier_def subsetI*)
have *K ⊆ C*
using *K* ‹*rel_frontier C ⊆ C*› **by** *blast*
have *C* ∈ ℳ
using 𝒩*_def* ‹*C* ∈ 𝒩› **by** *auto*
moreover have *L ⊆ C*
using *K L* ‹*C* ∈ 𝒩›
by (*metis* ‹*K ⊆ C*› ‹*convex hull C = C*› *contra_subsetD hull_mono in_rel_interior insert_subset rel_interior_subset*)
ultimately show *?thesis*
using ‹*rel_frontier C ⊆ C*› ‹*L ⊆ C*› *aff* ℳ *aff_dim_subset* ‹*C* ∈ ℳ› *dual_order.trans* **by** *blast*
qed
then show ∃ *C*. *C* ∈ ℳ ∧ *L ⊆ C aff_dim L ≤ int n* **if** *L* ∈ 𝒰 ∪ *?*𝒯 **for** *L*
using *that* **by** *auto*
qed
then show *?thesis*
apply (*rule ex_forward*, *safe*)
apply (*meson Union_iff subsetCE*, *fastforce*)
by (*meson infinite_super simplicial_complex_def*)
qed
qed

lemma *simplicial_subdivision_of_cell_complex_lowdim*:
assumes *finite* ℳ
and *poly*: ⋀*C*. *C* ∈ ℳ ⟹ *polytope C*
and *face*: ⋀*C1 C2*. ⟦*C1* ∈ ℳ; *C2* ∈ ℳ⟧ ⟹ *C1* ∩ *C2 face_of C1*
and *aff*: ⋀*C*. *C* ∈ ℳ ⟹ *aff_dim C ≤ d*
obtains 𝒯 **where** *simplicial_complex* 𝒯 ⋀*K*. *K* ∈ 𝒯 ⟹ *aff_dim K ≤ d*
⋃𝒯 = ⋃ℳ
⋀*C*. *C* ∈ ℳ ⟹ ∃ *F*. *finite F* ∧ *F* ⊆ 𝒯 ∧ *C* = ⋃ *F*
⋀*K*. *K* ∈ 𝒯 ⟹ ∃ *C*. *C* ∈ ℳ ∧ *K ⊆ C*
proof (*cases d ≥ 0*)
case *True*
then obtain *n* **where** *n*: *d = of_nat n*
using *zero_le_imp_eq_int* **by** *blast*
have ∃𝒯. *simplicial_complex* 𝒯 ∧
(∀ *K*∈𝒯. *aff_dim K ≤ int n*) ∧
⋃𝒯 = ⋃(⋃ *C*∈ℳ. {*F*. *F face_of C*}) ∧
(∀ *C*∈⋃ *C*∈ℳ. {*F*. *F face_of C*}.
∃ *F*. *finite F* ∧ *F* ⊆ 𝒯 ∧ *C* = ⋃ *F*) ∧
(∀ *K*∈𝒯. ∃ *C*. *C* ∈ (⋃ *C*∈ℳ. {*F*. *F face_of C*}) ∧ *K ⊆ C*)

```
  proof (rule simplicial_subdivision_aux)
    show finite (⋃C∈ℳ. {F. F face_of C})
       using ‹finite ℳ› poly polyhedron_eq_finite_faces polytope_imp_polyhedron by
fastforce
    show polytope F if F ∈ (⋃C∈ℳ. {F. F face_of C}) for F
      using poly that face_of_polytope_polytope by blast
    show aff_dim F ≤ int n if F ∈ (⋃C∈ℳ. {F. F face_of C}) for F
      using that
      by clarify (metis n aff_dim_subset aff face_of_imp_subset order_trans)
    show F ∈ (⋃C∈ℳ. {F. F face_of C})
      if G ∈ (⋃C∈ℳ. {F. F face_of C}) and F face_of G for F G
      using that face_of_trans by blast
  next
    fix F1 F2
    assume F1 ∈ (⋃C∈ℳ. {F. F face_of C}) and F2 ∈ (⋃C∈ℳ. {F. F face_of
C})
    then obtain C1 C2 where C1 ∈ ℳ C2 ∈ ℳ and F: F1 face_of C1 F2 face_of
C2
      by auto
    show F1 ∩ F2 face_of F1
      using face_of_Int_subface [OF _ _ F]
      by (metis ‹C1 ∈ ℳ› ‹C2 ∈ ℳ› face inf_commute)
  qed
  moreover
  have ⋃(⋃C∈ℳ. {F. F face_of C}) = ⋃ℳ
    using face_of_imp_subset face by blast
  ultimately show ?thesis
    using face_of_imp_subset n
    by (fastforce intro!: that simp add: poly face_of_refl polytope_imp_convex)
next
  case False
  then have m1: ⋀C. C ∈ ℳ ⟹ aff_dim C = -1
    by (metis aff aff_dim_empty_eq aff_dim_negative_iff dual_order.trans not_less)
  then have faceℳ: ⋀F S. ⟦S ∈ ℳ; F face_of S⟧ ⟹ F ∈ ℳ
    by (metis aff_dim_empty face_of_empty)
  show ?thesis
  proof
    have ⋀S. S ∈ ℳ ⟹ ∃n. n simplex S
      by (metis (no_types) m1 aff_dim_empty simplex_minus_1)
    then show simplicial_complex ℳ
      by (auto simp: simplicial_complex_def ‹finite ℳ› face intro: faceℳ)
    show aff_dim K ≤ d if K ∈ ℳ for K
      by (simp add: that aff)
    show ∃F. finite F ∧ F ⊆ ℳ ∧ C = ⋃F if C ∈ ℳ for C
      using ‹C ∈ ℳ› equals0I by auto
    show ∃C. C ∈ ℳ ∧ K ⊆ C if K ∈ ℳ for K
      using ‹K ∈ ℳ› by blast
  qed auto
qed
```

proposition *simplicial_subdivision_of_cell_complex*:
assumes *finite* $\mathcal{M}$
and *poly*: $\bigwedge C.\ C \in \mathcal{M} \Longrightarrow$ *polytope* C
and *face*: $\bigwedge C1\ C2.\ [\![C1 \in \mathcal{M};\ C2 \in \mathcal{M}]\!] \Longrightarrow C1 \cap C2$ *face_of* $C1$
obtains $\mathcal{T}$ **where** *simplicial_complex* $\mathcal{T}$
$\bigcup\mathcal{T} = \bigcup\mathcal{M}$
$\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \exists F.$ *finite* $F \wedge F \subseteq \mathcal{T} \wedge C = \bigcup F$
$\bigwedge K.\ K \in \mathcal{T} \Longrightarrow \exists C.\ C \in \mathcal{M} \wedge K \subseteq C$
by (*blast intro*: *simplicial_subdivision_of_cell_complex_lowdim* [*OF assms aff_dim_le_DIM*])

corollary *fine_simplicial_subdivision_of_cell_complex*:
assumes $0 < e$ *finite* $\mathcal{M}$
and *poly*: $\bigwedge C.\ C \in \mathcal{M} \Longrightarrow$ *polytope* C
and *face*: $\bigwedge C1\ C2.\ [\![C1 \in \mathcal{M};\ C2 \in \mathcal{M}]\!] \Longrightarrow C1 \cap C2$ *face_of* $C1$
obtains $\mathcal{T}$ **where** *simplicial_complex* $\mathcal{T}$
$\bigwedge K.\ K \in \mathcal{T} \Longrightarrow$ *diameter* $K < e$
$\bigcup\mathcal{T} = \bigcup\mathcal{M}$
$\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \exists F.$ *finite* $F \wedge F \subseteq \mathcal{T} \wedge C = \bigcup F$
$\bigwedge K.\ K \in \mathcal{T} \Longrightarrow \exists C.\ C \in \mathcal{M} \wedge K \subseteq C$
proof −
obtain $\mathcal{N}$ **where** $\mathcal{N}$: *finite* $\mathcal{N}$ $\bigcup\mathcal{N} = \bigcup\mathcal{M}$
and *diapoly*: $\bigwedge X.\ X \in \mathcal{N} \Longrightarrow$ *diameter* $X < e$ $\bigwedge X.\ X \in \mathcal{N} \Longrightarrow$ *polytope* X
and $\bigwedge X\ Y.\ [\![X \in \mathcal{N};\ Y \in \mathcal{N}]\!] \Longrightarrow X \cap Y$ *face_of* X
and $\mathcal{N}$*covers*: $\bigwedge C\ x.\ C \in \mathcal{M} \wedge x \in C \Longrightarrow \exists D.\ D \in \mathcal{N} \wedge x \in D \wedge D \subseteq C$
and $\mathcal{N}$*covered*: $\bigwedge C.\ C \in \mathcal{N} \Longrightarrow \exists D.\ D \in \mathcal{M} \wedge C \subseteq D$
by (*blast intro*: *cell_complex_subdivision_exists* [*OF* ‹$0 < e$› ‹*finite* $\mathcal{M}$› *poly aff_dim_le_DIM face*])
then obtain $\mathcal{T}$ **where** $\mathcal{T}$: *simplicial_complex* $\mathcal{T}$ $\bigcup\mathcal{T} = \bigcup\mathcal{N}$
and $\mathcal{T}$*covers*: $\bigwedge C.\ C \in \mathcal{N} \Longrightarrow \exists F.$ *finite* $F \wedge F \subseteq \mathcal{T} \wedge C = \bigcup F$
and $\mathcal{T}$*covered*: $\bigwedge K.\ K \in \mathcal{T} \Longrightarrow \exists C.\ C \in \mathcal{N} \wedge K \subseteq C$
using *simplicial_subdivision_of_cell_complex* [*OF* ‹*finite* $\mathcal{N}$›] **by** *metis*
show *?thesis*
proof
show *simplicial_complex* $\mathcal{T}$
by (*rule* $\mathcal{T}$)
show *diameter* $K < e$ **if** $K \in \mathcal{T}$ **for** K
by (*metis le_less_trans diapoly* $\mathcal{T}$*covered diameter_subset polytope_imp_bounded that*)
show $\bigcup\mathcal{T} = \bigcup\mathcal{M}$
by (*simp add*: $\mathcal{N}(2)$ ‹$\bigcup\mathcal{T} = \bigcup\mathcal{N}$›)
show $\exists F.$ *finite* $F \wedge F \subseteq \mathcal{T} \wedge C = \bigcup F$ **if** $C \in \mathcal{M}$ **for** C
proof −
{ **fix** x
assume $x \in C$
then obtain D **where** $D \in \mathcal{T}$ $x \in D$ $D \subseteq C$
using $\mathcal{N}$*covers* ‹$C \in \mathcal{M}$› $\mathcal{T}$*covers* **by** *force*

```
      then have ∃X∈𝒯 ∩ Pow C. x ∈ X
        using ‹D ∈ 𝒯› ‹D ⊆ C› ‹x ∈ D› by blast
    }
    moreover
    have finite (𝒯 ∩ Pow C)
      using ‹simplicial_complex 𝒯› simplicial_complex_def by auto
    ultimately show ?thesis
      by (rule_tac x=(𝒯 ∩ Pow C) in exI) auto
  qed
  show ∃C. C ∈ ℳ ∧ K ⊆ C if K ∈ 𝒯 for K
    by (meson 𝒩covered 𝒯covered order_trans that)
 qed
qed
```

6.38.20 Some results on cell division with full-dimensional cells only

```
lemma convex_Union_fulldim_cells:
 assumes finite 𝒮 and clo: ⋀C. C ∈ 𝒮 ⟹ closed C and con: ⋀C. C ∈ 𝒮 ⟹
convex C
   and eq: ⋃𝒮 = U and convex U
 shows ⋃{C ∈ 𝒮. aff_dim C = aff_dim U} = U (is ?lhs = U)
proof −
 have closed U
   using ‹finite 𝒮› clo eq by blast
 have ?lhs ⊆ U
   using eq by blast
 moreover have U ⊆ ?lhs
 proof (cases ∀C ∈ 𝒮. aff_dim C = aff_dim U)
   case True
   then show ?thesis
     using eq by blast
 next
   case False
   have closed ?lhs
     by (simp add: ‹finite 𝒮› clo closed_Union)
   moreover have U ⊆ closure ?lhs
   proof −
     have U ⊆ closure(⋂{U − C |C. C ∈ 𝒮 ∧ aff_dim C < aff_dim U})
     proof (rule Baire [OF ‹closed U›])
       show countable {U − C |C. C ∈ 𝒮 ∧ aff_dim C < aff_dim U}
         using ‹finite 𝒮› uncountable_infinite by fastforce
       have ⋀C. C ∈ 𝒮 ⟹ openin (top_of_set U) (U−C)
      by (metis Sup_upper clo closed_limpt closedin_limpt eq openin_diff openin_subtopology_self)
       then show openin (top_of_set U) T ∧ U ⊆ closure T
         if T ∈ {U − C |C. C ∈ 𝒮 ∧ aff_dim C < aff_dim U} for T
        using that dense_complement_convex_closed ‹closed U› ‹convex U› by auto
     qed
     also have ... ⊆ closure ?lhs
```

proof −
obtain C **where** $C \in \mathcal{S}$ *aff_dim* $C <$ *aff_dim* U
by (*metis False Sup_upper aff_dim_subset eq eq_iff not_le*)
have $\exists X.\ X \in \mathcal{S} \wedge$ *aff_dim* $X =$ *aff_dim* $U \wedge x \in X$
if $\bigwedge V.\ (\exists C.\ V = U - C \wedge C \in \mathcal{S} \wedge$ *aff_dim* $C <$ *aff_dim* $U) \Longrightarrow x \in V$ **for** x
proof −
have $x \in U \wedge x \in \bigcup\mathcal{S}$
using ‹$C \in \mathcal{S}$› ‹*aff_dim* $C <$ *aff_dim* U› *eq that* **by** *blast*
then show *?thesis*
by (*metis Diff_iff Sup_upper Union_iff aff_dim_subset dual_order.order_iff_strict eq that*)
qed
then show *?thesis*
by (*auto intro*!: *closure_mono*)
qed
finally show *?thesis* **.**
qed
ultimately show *?thesis*
using *closure_subset_eq* **by** *blast*
qed
ultimately show *?thesis* **by** *blast*
qed

proposition *fine_triangular_subdivision_of_cell_complex*:
assumes $0 < e$ *finite* $\mathcal{M}$
and *poly*: $\bigwedge C.\ C \in \mathcal{M} \Longrightarrow$ *polytope* C
and *aff*: $\bigwedge C.\ C \in \mathcal{M} \Longrightarrow$ *aff_dim* $C = d$
and *face*: $\bigwedge C1\ C2.\ [\![C1 \in \mathcal{M}; C2 \in \mathcal{M}]\!] \Longrightarrow C1 \cap C2$ *face_of* $C1$
obtains $\mathcal{T}$ **where** *triangulation* $\mathcal{T}$ $\bigwedge k.\ k \in \mathcal{T} \Longrightarrow$ *diameter* $k < e$
$\bigwedge k.\ k \in \mathcal{T} \Longrightarrow$ *aff_dim* $k = d$ $\bigcup\mathcal{T} = \bigcup\mathcal{M}$
$\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \exists f.$ *finite* $f \wedge f \subseteq \mathcal{T} \wedge C = \bigcup f$
$\bigwedge k.\ k \in \mathcal{T} \Longrightarrow \exists C.\ C \in \mathcal{M} \wedge k \subseteq C$
proof −
obtain $\mathcal{T}$ **where** *simplicial_complex* $\mathcal{T}$
and *dia*$\mathcal{T}$: $\bigwedge K.\ K \in \mathcal{T} \Longrightarrow$ *diameter* $K < e$
and $\bigcup\mathcal{T} = \bigcup\mathcal{M}$
and *in*$\mathcal{M}$: $\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \exists F.$ *finite* $F \wedge F \subseteq \mathcal{T} \wedge C = \bigcup F$
and *in*$\mathcal{T}$: $\bigwedge K.\ K \in \mathcal{T} \Longrightarrow \exists C.\ C \in \mathcal{M} \wedge K \subseteq C$
by (*blast intro*: *fine_simplicial_subdivision_of_cell_complex* [*OF* ‹$e > 0$› ‹*finite* $\mathcal{M}$› *poly face*])
let $?\mathcal{T} = \{K \in \mathcal{T}.\ $ *aff_dim* $K = d\}$
show *thesis*
proof
show *triangulation* $?\mathcal{T}$
using ‹*simplicial_complex* $\mathcal{T}$› **by** (*auto simp*: *triangulation_def simplicial_complex_def*)
show *diameter* $L < e$ **if** $L \in \{K \in \mathcal{T}.\ $ *aff_dim* $K = d\}$ **for** L
using *that* **by** (*auto simp*: *dia*$\mathcal{T}$)
show *aff_dim* $L = d$ **if** $L \in \{K \in \mathcal{T}.\ $ *aff_dim* $K = d\}$ **for** L

using *that* **by** *auto*
show $\exists F.$ *finite* $F \wedge F \subseteq \{K \in \mathcal{T}.$ *aff_dim* $K = d\} \wedge C = \bigcup F$ **if** $C \in \mathcal{M}$ **for** C
proof −
obtain F **where** *finite* F $F \subseteq \mathcal{T}$ $C = \bigcup F$
using *in$\mathcal{M}$* [*OF* ‹$C \in \mathcal{M}$›] **by** *auto*
show *?thesis*
proof (*intro exI conjI*)
show *finite* $\{K \in F.$ *aff_dim* $K = d\}$
by (*simp add*: ‹*finite* F›)
show $\{K \in F.$ *aff_dim* $K = d\} \subseteq \{K \in \mathcal{T}.$ *aff_dim* $K = d\}$
using ‹$F \subseteq \mathcal{T}$› **by** *blast*
have $d =$ *aff_dim* C
by (*simp add*: *aff that*)
moreover have $\bigwedge K.\ K \in F \Longrightarrow$ *closed* $K \wedge$ *convex* K
using ‹*simplicial_complex* $\mathcal{T}$› ‹$F \subseteq \mathcal{T}$›
unfolding *simplicial_complex_def* **by** (*metis subsetCE* ‹$F \subseteq \mathcal{T}$› *closed_simplex convex_simplex*)
moreover have *convex* $(\bigcup F)$
using ‹$C = \bigcup F$› *poly polytope_imp_convex that* **by** *blast*
ultimately show $C = \bigcup\{K \in F.$ *aff_dim* $K = d\}$
by (*simp add*: *convex_Union_fulldim_cells* ‹$C = \bigcup F$› ‹*finite* F›)
qed
qed
then show $\bigcup\{K \in \mathcal{T}.$ *aff_dim* $K = d\} = \bigcup \mathcal{M}$
by *auto* (*meson in$\mathcal{T}$ subsetCE*)
show $\exists C.\ C \in \mathcal{M} \wedge L \subseteq C$
if $L \in \{K \in \mathcal{T}.$ *aff_dim* $K = d\}$ **for** L
using *that* **by** (*auto simp*: *in$\mathcal{T}$*)
qed
qed

end

6.39 Arcwise-Connected Sets

theory *Arcwise_Connected*
imports *Path_Connected Ordered_Euclidean_Space HOL−Computational_Algebra.Primes*
begin

lemma *path_connected_interval* [*simp*]:
fixes *a b*::*'a*::*ordered_euclidean_space*
shows *path_connected* $\{a..b\}$
using *is_interval_cc is_interval_path_connected* **by** *blast*

lemma *segment_to_closest_point*:
fixes S :: *'a* :: *euclidean_space set*
shows $[\![$*closed* S; $S \neq \{\}]\!] \Longrightarrow$ *open_segment* a (*closest_point* S a) $\cap S = \{\}$
unfolding *disjoint_iff*

by (*metis closest_point_le dist_commute dist_in_open_segment not_le*)

lemma *segment_to_point_exists*:
fixes S :: $'a$ :: *euclidean_space set*
assumes *closed* S $S \neq \{\}$
obtains b **where** $b \in S$ *open_segment* a $b \cap S = \{\}$
by (*metis assms segment_to_closest_point closest_point_exists that*)

6.39.1 The Brouwer reduction theorem

theorem *Brouwer_reduction_theorem_gen*:
fixes S :: $'a$::*euclidean_space set*
assumes *closed* S φ S
and φ: $\bigwedge F.$ $[\![\bigwedge n.$ *closed*$(F\ n)$; $\bigwedge n.$ $\varphi(F\ n)$; $\bigwedge n.$ $F(Suc\ n) \subseteq F\ n]\!] \Longrightarrow \varphi(\bigcap(range\ F))$
obtains T **where** $T \subseteq S$ *closed* T φ T $\bigwedge U.$ $[\![U \subseteq S;$ *closed* $U;$ φ $U]\!] \Longrightarrow \neg (U \subset T)$
proof −
obtain B :: $nat \Rightarrow 'a$ *set*
where *inj* B $\bigwedge n.$ *open*$(B\ n)$ **and** *open_cov*: $\bigwedge S.$ *open* $S \Longrightarrow \exists K.$ $S = \bigcup(B$ ‘ $K)$
by (*metis Setcompr_eq_image that univ_second_countable_sequence*)
define A **where** $A \equiv$ *rec_nat* S $(\lambda n\ a.$ *if* $\exists U.$ $U \subseteq a \wedge$ *closed* $U \wedge \varphi\ U \wedge U \cap (B\ n) = \{\}$
then SOME $U.$ $U \subseteq a \wedge$ *closed* $U \wedge \varphi\ U \wedge U \cap (B\ n) = \{\}$
else $a)$
have [*simp*]: $A\ 0 = S$
by (*simp add*: *A_def*)
have *ASuc*: $A(Suc\ n) = ($*if* $\exists U.$ $U \subseteq A\ n \wedge$ *closed* $U \wedge \varphi\ U \wedge U \cap (B\ n) = \{\}$
then SOME $U.$ $U \subseteq A\ n \wedge$ *closed* $U \wedge \varphi\ U \wedge U \cap (B\ n) = \{\}$
else $A\ n)$ **for** n
by (*auto simp*: *A_def*)
have *sub*: $\bigwedge n.$ $A(Suc\ n) \subseteq A\ n$
by (*auto simp*: *ASuc dest*!: *someI_ex*)
have *subS*: $A\ n \subseteq S$ **for** n
by (*induction n*) (*use sub* **in** *auto*)
have *clo*: *closed* $(A\ n) \wedge \varphi\ (A\ n)$ **for** n
by (*induction n*) (*auto simp*: *assms ASuc dest*!: *someI_ex*)
show *?thesis*
proof
show $\bigcap(range\ A) \subseteq S$
using ‹$\bigwedge n.$ $A\ n \subseteq S$› **by** *blast*
show *closed* $(\bigcap(A$ ‘ $UNIV))$
using *clo* **by** *blast*
show φ $(\bigcap(A$ ‘ $UNIV))$
by (*simp add*: *clo* φ *sub*)
show $\neg$ $U \subset \bigcap(A$ ‘ $UNIV)$ **if** $U \subseteq S$ *closed* U φ U **for** U

```
    proof –
      have ∃y. x ∉ A y if x ∉ U and Usub: U ⊆ (⋂x. A x) for x
      proof –
        obtain e where e > 0 and e: ball x e ⊆ −U
          using ‹closed U› ‹x ∉ U› openE [of −U] by blast
        moreover obtain K where K: ball x e = ⋃(B ` K)
          using open_cov [of ball x e] by auto
        ultimately have ⋃(B ` K) ⊆ −U
          by blast
        have K ≠ {}
          using ‹0 < e› ‹ball x e = ⋃(B ` K)› by auto
        then obtain n where n ∈ K x ∈ B n
          by (metis K UN_E ‹0 < e› centre_in_ball)
        then have U ∩ B n = {}
          using K e by auto
        show ?thesis
        proof (cases ∃U⊆A n. closed U ∧ φ U ∧ U ∩ B n = {})
          case True
          then show ?thesis
            apply (rule_tac x=Suc n in exI)
            apply (simp add: ASuc)
            apply (erule someI2_ex)
            using ‹x ∈ B n› by blast
        next
          case False
          then show ?thesis
             by (meson Inf_lower Usub ‹U ∩ B n = {}› ‹φ U› ‹closed U› range_eqI
subset_trans)
        qed
      qed
      with that show ?thesis
        by (meson Inter_iff psubsetE rangeI subsetI)
    qed
  qed
qed

corollary Brouwer_reduction_theorem:
  fixes S :: 'a::euclidean_space set
  assumes compact S φ S S ≠ {}
      and φ: ⋀F. ⟦⋀n. compact(F n); ⋀n. F n ≠ {}; ⋀n. φ(F n); ⋀n. F(Suc n)
⊆ F n⟧ ⟹ φ(⋂(range F))
  obtains T where T ⊆ S compact T T ≠ {} φ T
                  ⋀U. ⟦U ⊆ S; closed U; U ≠ {}; φ U⟧ ⟹ ¬ (U ⊂ T)
proof (rule Brouwer_reduction_theorem_gen [of S λT. T ≠ {} ∧ T ⊆ S ∧ φ T])
  fix F
  assume cloF: ⋀n. closed (F n)
     and F: ⋀n. F n ≠ {} ∧ F n ⊆ S ∧ φ (F n) and Fsub: ⋀n. F (Suc n) ⊆ F n
  show ⋂(F ` UNIV) ≠ {} ∧ ⋂(F ` UNIV) ⊆ S ∧ φ (⋂(F ` UNIV))
  proof (intro conjI)
```

show $\bigcap(F$ ‘ $UNIV) \neq \{\}$
by (*metis F Fsub* ‹*compact S*› *cloF closed_Int_compact compact_nest inf.orderE lift_Suc_antimono_le*)
show $\bigcap(F$ ‘ $UNIV) \subseteq S$
using *F* **by** *blast*
show φ $(\bigcap(F$ ‘ $UNIV))$
by (*metis F Fsub* φ ‹*compact S*› *cloF closed_Int_compact inf.orderE*)
qed
next
show $S \neq \{\} \wedge S \subseteq S \wedge \varphi\ S$
by (*simp add*: *assms*)
qed (*meson assms compact_imp_closed seq_compact_closed_subset seq_compact_eq_compact*)+

6.39.2 Arcwise Connections

6.39.3 Density of points with dyadic rational coordinates

proposition *closure_dyadic_rationals*:
closure ($\bigcup k.\ \bigcup f \in Basis \to \mathbb{Z}.$
$\{ \sum i :: 'a :: euclidean\_space \in Basis.\ (f\ i\ /\ 2\hat{}k) *_R i \}) = UNIV$
proof −
have $x \in closure\ (\bigcup k.\ \bigcup f \in Basis \to \mathbb{Z}.\ \{\sum i \in Basis.\ (f\ i\ /\ 2\hat{}k) *_R i\})$ **for** $x{::}'a$
proof (*clarsimp simp*: *closure_approachable*)
fix $e{::}real$
assume $e > 0$
then obtain k **where** k: $(1/2)\hat{}k < e/DIM('a)$
by (*meson DIM_positive divide_less_eq_1_pos of_nat_0_less_iff one_less_numeral_iff real_arch_pow_inv semiring_norm*(*76*) *zero_less_divide_iff zero_less_numeral*)
have $dist\ (\sum i{\in}Basis.\ (real\_of\_int\ \lfloor 2\hat{}k*(x \cdot i)\rfloor\ /\ 2\hat{}k) *_R i)\ x =$
$dist\ (\sum i{\in}Basis.\ (real\_of\_int\ \lfloor 2\hat{}k*(x \cdot i)\rfloor\ /\ 2\hat{}k) *_R i)\ (\sum i{\in}Basis.\ (x \cdot i) *_R i)$
by (*simp add*: *euclidean_representation*)
also have ... $= norm\ ((\sum i{\in}Basis.\ (real\_of\_int\ \lfloor 2\hat{}k*(x \cdot i)\rfloor\ /\ 2\hat{}k) *_R i - (x \cdot i) *_R i))$
by (*simp add*: *dist_norm sum_subtractf*)
also have ... $\leq DIM('a)*((1/2)\hat{}k)$
proof (*rule sum_norm_bound*, *simp add*: *algebra_simps*)
fix $i{::}'a$
assume $i \in Basis$
then have $norm\ ((real\_of\_int\ \lfloor x \cdot i*2\hat{}k\rfloor\ /\ 2\hat{}k) *_R i - (x \cdot i) *_R i) =$
$|real\_of\_int\ \lfloor x \cdot i*2\hat{}k\rfloor\ /\ 2\hat{}k - x \cdot i|$
by (*simp add*: *scaleR_left_diff_distrib* [*symmetric*])
also have ... $\leq (1/2)\ \hat{}\ k$
by (*simp add*: *divide_simps*) *linarith*
finally show $norm\ ((real\_of\_int\ \lfloor x \cdot i*2\hat{}k\rfloor\ /\ 2\hat{}k) *_R i - (x \cdot i) *_R i) \leq (1/2)\ \hat{}\ k$ **.**
qed
also have ... $< DIM('a)*(e/DIM('a))$
using *DIM_positive k linordered_comm_semiring_strict_class.comm_mult_strict_left_mono*

```
of_nat_0_less_iff by blast
    also have ... = e
      by simp
    finally have dist (∑ i∈Basis. (⌊2^k*(x · i)⌋ / 2^k) *R i) x < e .
    with Ints_of_int
    show ∃ k. ∃ f ∈ Basis → ℤ. dist (∑ b∈Basis. (f b / 2^k) *R b) x < e
      by fastforce
  qed
  then show ?thesis by auto
qed

corollary closure_rational_coordinates:
    closure (⋃ f ∈ Basis → ℚ. { ∑ i :: 'a :: euclidean_space ∈ Basis. f i *R i })
= UNIV
proof -
  have *: (⋃ k. ⋃ f ∈ Basis → ℤ. { ∑ i::'a ∈ Basis. (f i / 2^k) *R i })
          ⊆ (⋃ f ∈ Basis → ℚ. { ∑ i ∈ Basis. f i *R i })
  proof clarsimp
    fix k and f :: 'a ⇒ real
    assume f: f ∈ Basis → ℤ
    show ∃ x ∈ Basis → ℚ. (∑ i ∈ Basis. (f i / 2^k) *R i) = (∑ i ∈ Basis. x i
*R i)
      apply (rule_tac x=λi. f i / 2^k in bexI)
      using Ints_subset_Rats f by auto
  qed
  show ?thesis
    using closure_dyadic_rationals closure_mono [OF *] by blast
qed

lemma closure_dyadic_rationals_in_convex_set:
  ⟦convex S; interior S ≠ {}⟧
     ⟹ closure(S ∩
              (⋃ k. ⋃ f ∈ Basis → ℤ.
             { ∑ i :: 'a :: euclidean_space ∈ Basis. (f i / 2^k) *R i })) =
          closure S
  by (simp add: closure_dyadic_rationals closure_convex_Int_superset)

lemma closure_rationals_in_convex_set:
  ⟦convex S; interior S ≠ {}⟧
    ⟹ closure(S ∩ (⋃ f ∈ Basis → ℚ. { ∑ i :: 'a :: euclidean_space ∈ Basis. f i
*R i })) =
        closure S
  by (simp add: closure_rational_coordinates closure_convex_Int_superset)
```

Every path between distinct points contains an arc, and hence path connection is equivalent to arcwise connection for distinct points. The proof is based on Whyburn's "Topological Analysis".

```
lemma closure_dyadic_rationals_in_convex_set_pos_1:
  fixes S :: real set
```

assumes *convex S* **and** *intnz*: *interior S* ≠ {} **and** *pos*: ⋀*x*. *x* ∈ *S* ⟹ *0* ≤ *x*
shows *closure*(*S* ∩ (⋃*k m*. {*of_nat m* / *2^k*})) = *closure S*
proof −
have ∃*m*. *f 1*/*2^k* = *real m* / *2^k* **if** (*f 1*) / *2^k* ∈ *S f 1* ∈ ℤ **for** *k* **and** *f* :: *real* ⇒ *real*
using *that* **by** (*force simp*: *Ints_def zero_le_divide_iff power_le_zero_eq dest*: *pos zero_le_imp_eq_int*)
then have *S* ∩ (⋃*k m*. {*real m* / *2^k*}) = *S* ∩
(⋃*k*. ⋃*f*∈*Basis* → ℤ. {∑*i*∈*Basis*. (*f i* / *2^k*) $*_R$ *i*})
by *force*
then show *?thesis*
using *closure_dyadic_rationals_in_convex_set* [*OF* ‹*convex S*› *intnz*] **by** *simp*
qed

definition *dyadics* :: *'a::field_char_0 set* **where** *dyadics* ≡ ⋃*k m*. {*of_nat m* / *2^k*}

lemma *real_in_dyadics* [*simp*]: *real m* ∈ *dyadics*
by (*simp add*: *dyadics_def*) (*metis divide_numeral_1 numeral_One power_0*)

lemma *nat_neq_4k1*: *of_nat m* ≠ (*4* ∗ *of_nat k* + *1*) / (*2* ∗ *2^n* :: *'a::field_char_0*)
proof
assume *of_nat m* = (*4* ∗ *of_nat k* + *1*) / (*2* ∗ *2^n* :: *'a*)
then have *of_nat* (*m* ∗ (*2* ∗ *2^n*)) = (*of_nat* (*Suc* (*4* ∗ *k*)) :: *'a*)
by (*simp add*: *field_split_simps*)
then have *m* ∗ (*2* ∗ *2^n*) = *Suc* (*4* ∗ *k*)
using *of_nat_eq_iff* **by** *blast*
then have *odd* (*m* ∗ (*2* ∗ *2^n*))
by *simp*
then show *False*
by *simp*
qed

lemma *nat_neq_4k3*: *of_nat m* ≠ (*4* ∗ *of_nat k* + *3*) / (*2* ∗ *2^n* :: *'a::field_char_0*)
proof
assume *of_nat m* = (*4* ∗ *of_nat k* + *3*) / (*2* ∗ *2^n* :: *'a*)
then have *of_nat* (*m* ∗ (*2* ∗ *2^n*)) = (*of_nat* (*4* ∗ *k* + *3*) :: *'a*)
by (*simp add*: *field_split_simps*)
then have *m* ∗ (*2* ∗ *2^n*) = (*4* ∗ *k*) + *3*
using *of_nat_eq_iff* **by** *blast*
then have *odd* (*m* ∗ (*2* ∗ *2^n*))
by *simp*
then show *False*
by *simp*
qed

lemma *iff_4k*:
assumes *r* = *real k odd k*

```
  shows (4 * real m + r) / (2 * 2^n) = (4 * real m' + r) / (2 * 2 ^ n') ⟷
m=m' ∧ n=n'
proof -
  { assume (4 * real m + r) / (2 * 2^n) = (4 * real m' + r) / (2 * 2 ^ n')
   then have real ((4 * m + k) * (2 * 2 ^ n')) = real ((4 * m' + k) * (2 * 2^n))
     using assms by (auto simp: field_simps)
    then have (4 * m + k) * (2 * 2 ^ n') = (4 * m' + k) * (2 * 2^n)
     using of_nat_eq_iff by blast
    then have (4 * m + k) * (2 ^ n') = (4 * m' + k) * (2^n)
     by linarith
    then obtain 4*m + k = 4*m' + k n=n'
     using prime_power_cancel2 [OF two_is_prime_nat] assms
     by (metis even_mult_iff even_numeral odd_add)
    then have m=m' n=n'
     by auto
  }
  then show ?thesis by blast
qed

lemma neq_4k1_k43: (4 * real m + 1) / (2 * 2^n) ≠ (4 * real m' + 3) / (2 * 2
^ n')
proof
  assume (4 * real m + 1) / (2 * 2^n) = (4 * real m' + 3) / (2 * 2 ^ n')
  then have real (Suc (4 * m) * (2 * 2 ^ n')) = real ((4 * m' + 3) * (2 * 2^n))
    by (auto simp: field_simps)
  then have Suc (4 * m) * (2 * 2 ^ n') = (4 * m' + 3) * (2 * 2^n)
    using of_nat_eq_iff by blast
  then have Suc (4 * m) * (2 ^ n') = (4 * m' + 3) * (2^n)
    by linarith
  then have Suc (4 * m) = (4 * m' + 3)
    by (rule prime_power_cancel2 [OF two_is_prime_nat]) auto
  then have 1 + 2 * m' = 2 * m
    using ‹Suc (4 * m) = 4 * m' + 3› by linarith
  then show False
    using even_Suc by presburger
qed

lemma dyadic_413_cases:
  obtains (of_nat m::'a::field_char_0) / 2^k ∈ Nats
  | m' k' where k' < k (of_nat m:: 'a) / 2^k = of_nat (4*m' + 1) / 2^Suc k'
  | m' k' where k' < k (of_nat m:: 'a) / 2^k = of_nat (4*m' + 3) / 2^Suc k'
proof (cases m>0)
  case False
  then have m=0 by simp
  with that show ?thesis by auto
next
  case True
  obtain k' m' where m': odd m' and k': m = m' * 2^k'
    using prime_power_canonical [OF two_is_prime_nat True] by blast
```

then obtain *q r* **where** *q*: $m' = 4*q + r$ **and** *r*: $r < 4$
by (*metis not_add_less2 split_div zero_neq_numeral*)
show *?thesis*
proof (*cases* $k \le k'$)
case *True*
have (*of_nat m*:: *'a*) / *2^k* = *of_nat m'* ∗ (*2* ^ *k'* / *2^k*)
using *k'* **by** (*simp add*: *field_simps*)
also have ... = (*of_nat m'*::*'a*) ∗ *2* ^ (*k'−k*)
using *k' True* **by** (*simp add*: *power_diff*)
also have ... ∈ ℕ
by (*metis Nats_mult of_nat_in_Nats of_nat_numeral of_nat_power*)
finally show *?thesis* **by** (*auto simp*: *that*)
next
case *False*
then obtain *kd* **where** *kd*: *Suc kd* = *k* − *k'*
using *Suc_diff_Suc not_less* **by** *blast*
have (*of_nat m*:: *'a*) / *2^k* = *of_nat m'* ∗ (*2* ^ *k'* / *2^k*)
using *k'* **by** (*simp add*: *field_simps*)
also have ... = (*of_nat m'*::*'a*) / *2* ^ (*k−k'*)
using *k' False* **by** (*simp add*: *power_diff*)
also have ... = ((*of_nat r* + *4* ∗ *of_nat q*)::*'a*) / *2* ^ (*k−k'*)
using *q* **by** *force*
finally have *meq*: (*of_nat m*:: *'a*) / *2^k* = (*of_nat r* + *4* ∗ *of_nat q*) / *2* ^ (*k* − *k'*) **.**
have *r* ≠ *0* *r* ≠ *2*
using *q m'* **by** *presburger+*
with *r* **consider** *r* = *1* | *r* = *3*
by *linarith*
then show *?thesis*
proof *cases*
assume *r* = *1*
with *meq kd that*(*2*) [*of kd q*] **show** *?thesis*
by *simp*
next
assume *r* = *3*
with *meq kd that*(*3*) [*of kd q*] **show** *?thesis*
by *simp*
qed
qed
qed

lemma *dyadics_iff*:
(*dyadics* :: *'a::field_char_0 set*) =
Nats ∪ (⋃*k m*. {*of_nat* (*4*∗*m* + *1*) / *2^Suc k*}) ∪ (⋃*k m*. {*of_nat* (*4*∗*m* + *3*) / *2^Suc k*})
(**is** _ = *?rhs*)
proof
show *dyadics* ⊆ *?rhs*

```
    unfolding dyadics_def
    apply clarify
    apply (rule dyadic_413_cases, force+)
    done
next
  have range of_nat ⊆ (⋃k m. {(of_nat m::'a) / 2 ^ k})
    by clarsimp (metis divide_numeral_1 numeral_One power_0)
  moreover have ⋀k m. ∃k' m'. ((1::'a) + 4 * of_nat m) / 2 ^ Suc k = of_nat
m' / 2 ^ k'
    by (metis (no_types) of_nat_Suc of_nat_mult of_nat_numeral)
  moreover have ⋀k m. ∃k' m'. (4 * of_nat m + (3::'a)) / 2 ^ Suc k = of_nat
m' / 2 ^ k'
    by (metis of_nat_add of_nat_mult of_nat_numeral)
  ultimately show ?rhs ⊆ dyadics
    by (auto simp: dyadics_def Nats_def)
qed


function (domintros) dyad_rec :: [nat ⇒ 'a, 'a⇒'a, 'a⇒'a, real] ⇒ 'a where
    dyad_rec b l r (real m) = b m
  | dyad_rec b l r ((4 * real m + 1) / 2 ^ (Suc n)) = l (dyad_rec b l r ((2*m + 1)
/ 2^n))
  | dyad_rec b l r ((4 * real m + 3) / 2 ^ (Suc n)) = r (dyad_rec b l r ((2*m +
1) / 2^n))
  | x ∉ dyadics ⟹ dyad_rec b l r x = undefined
  using iff_4k [of _ 1] iff_4k [of _ 3]
           apply (simp_all add: nat_neq_4k1 nat_neq_4k3 neq_4k1_k43 dyadics_iff
Nats_def)
  by (fastforce simp: field_simps)+

lemma dyadics_levels: dyadics = (⋃K. ⋃k<K. ⋃ m. {of_nat m / 2^k})
  unfolding dyadics_def by auto

lemma dyad_rec_level_termination:
  assumes k < K
  shows dyad_rec_dom(b, l, r, real m / 2^k)
  using assms
proof (induction K arbitrary: k m)
  case 0
  then show ?case by auto
next
  case (Suc K)
  then consider k = K | k < K
    using less_antisym by blast
  then show ?case
  proof cases
    assume k = K
    show ?case
    proof (rule dyadic_413_cases [of m k, where 'a=real])
```

```
    show real m / 2^k ∈ ℕ ⟹ dyad_rec_dom (b, l, r, real m / 2^k)
     by (force simp: Nats_def nat_neq_4k1 nat_neq_4k3 intro: dyad_rec.domintros)
    show ?case if k' < k and eq: real m / 2^k = real (4 * m' + 1) / 2^Suc k'
for m' k'
    proof -
      have dyad_rec_dom (b, l, r, (4 * real m' + 1) / 2^Suc k')
      proof (rule dyad_rec.domintros)
        fix m n
        assume (4 * real m' + 1) / (2 * 2 ^ k') = (4 * real m + 1) / (2 * 2^n)
        then have m' = m k' = n using iff_4k [of _ 1]
          by auto
        have dyad_rec_dom (b, l, r, real (2 * m + 1) / 2 ^ k')
          using Suc.IH ‹k = K› ‹k' < k› by blast
        then show dyad_rec_dom (b, l, r, (2 * real m + 1) / 2^n)
          using ‹k' = n› by (auto simp: algebra_simps)
      next
        fix m n
        assume (4 * real m' + 1) / (2 * 2 ^ k') = (4 * real m + 3) / (2 * 2^n)
        then have False
          by (metis neq_4k1_k43)
        then show dyad_rec_dom (b, l, r, (2 * real m + 1) / 2^n) ..
      qed
      then show ?case by (simp add: eq add_ac)
    qed
    show ?case if k' < k and eq: real m / 2^k = real (4 * m' + 3) / 2^Suc k'
for m' k'
    proof -
      have dyad_rec_dom (b, l, r, (4 * real m' + 3) / 2^Suc k')
      proof (rule dyad_rec.domintros)
        fix m n
        assume (4 * real m' + 3) / (2 * 2 ^ k') = (4 * real m + 1) / (2 * 2^n)
        then have False
          by (metis neq_4k1_k43)
        then show dyad_rec_dom (b, l, r, (2 * real m + 1) / 2^n) ..
      next
        fix m n
        assume (4 * real m' + 3) / (2 * 2 ^ k') = (4 * real m + 3) / (2 * 2^n)
        then have m' = m k' = n using iff_4k [of _ 3]
          by auto
        have dyad_rec_dom (b, l, r, real (2 * m + 1) / 2 ^ k')
          using Suc.IH ‹k = K› ‹k' < k› by blast
        then show dyad_rec_dom (b, l, r, (2 * real m + 1) / 2^n)
          using ‹k' = n› by (auto simp: algebra_simps)
      qed
      then show ?case by (simp add: eq add_ac)
    qed
  qed
 next
  assume k < K
```

then show *?case*
using *Suc.IH* **by** *blast*
qed
qed

lemma *dyad_rec_termination*: $x \in dyadics \Longrightarrow dyad\_rec\_dom(b,l,r,x)$
by (*auto simp*: *dyadics_levels intro*: *dyad_rec_level_termination*)

lemma *dyad_rec_of_nat* [*simp*]: *dyad_rec b l r* (*real m*) = *b m*
by (*simp add*: *dyad_rec.psimps dyad_rec_termination*)

lemma *dyad_rec_41* [*simp*]: *dyad_rec b l r* ((*4* ∗ *real m* + *1*) / *2* ^ (*Suc n*)) = *l* (*dyad_rec b l r* ((*2*∗*m* + *1*) / *2*^*n*))
proof (*rule dyad_rec.psimps*)
show *dyad_rec_dom* (*b*, *l*, *r*, (*4* ∗ *real m* + *1*) / *2* ^ *Suc n*)
by (*metis add.commute dyad_rec_level_termination lessI of_nat_Suc of_nat_mult of_nat_numeral*)
qed

lemma *dyad_rec_43* [*simp*]: *dyad_rec b l r* ((*4* ∗ *real m* + *3*) / *2* ^ (*Suc n*)) = *r* (*dyad_rec b l r* ((*2*∗*m* + *1*) / *2*^*n*))
proof (*rule dyad_rec.psimps*)
show *dyad_rec_dom* (*b*, *l*, *r*, (*4* ∗ *real m* + *3*) / *2* ^ *Suc n*)
by (*metis dyad_rec_level_termination lessI of_nat_add of_nat_mult of_nat_numeral*)
qed

lemma *dyad_rec_41_times2*:
assumes *n* > *0*
shows *dyad_rec b l r* (*2* ∗ ((*4* ∗ *real m* + *1*) / *2*^*Suc n*)) = *l* (*dyad_rec b l r* (*2* ∗ (*2* ∗ *real m* + *1*) / *2*^*n*))
proof −
obtain *n'* **where** *n'*: *n* = *Suc n'*
using *assms not0_implies_Suc* **by** *blast*
have *dyad_rec b l r* (*2* ∗ ((*4* ∗ *real m* + *1*) / *2*^*Suc n*)) = *dyad_rec b l r* ((*2* ∗ (*4* ∗ *real m* + *1*)) / (*2* ∗ *2*^*n*))
by *auto*
also have ... = *dyad_rec b l r* ((*4* ∗ *real m* + *1*) / *2*^*n*)
by (*subst mult_divide_mult_cancel_left*) *auto*
also have ... = *l* (*dyad_rec b l r* ((*2* ∗ *real m* + *1*) / *2* ^ *n'*))
by (*simp add*: *add.commute* [*of 1*] *n'* *del*: *power_Suc*)
also have ... = *l* (*dyad_rec b l r* ((*2* ∗ (*2* ∗ *real m* + *1*)) / (*2* ∗ *2* ^ *n'*)))
by (*subst mult_divide_mult_cancel_left*) *auto*
also have ... = *l* (*dyad_rec b l r* (*2* ∗ (*2* ∗ *real m* + *1*) / *2*^*n*))
by (*simp add*: *add.commute n'*)
finally show *?thesis* **.**
qed

lemma *dyad_rec_43_times2*:

assumes *n* > *0*
shows *dyad_rec b l r* (*2* ∗ ((*4* ∗ *real m* + *3*) / *2*ˆ*Suc n*)) = *r* (*dyad_rec b l r* (*2* ∗ (*2* ∗ *real m* + *1*) / *2*ˆ*n*))
proof −
obtain *n′* **where** *n′*: *n* = *Suc n′*
using *assms not0_implies_Suc* **by** *blast*
have *dyad_rec b l r* (*2* ∗ ((*4* ∗ *real m* + *3*) / *2*ˆ*Suc n*)) = *dyad_rec b l r* ((*2* ∗ (*4* ∗ *real m* + *3*)) / (*2* ∗ *2*ˆ*n*))
by *auto*
also have ... = *dyad_rec b l r* ((*4* ∗ *real m* + *3*) / *2*ˆ*n*)
by (*subst mult_divide_mult_cancel_left*) *auto*
also have ... = *r* (*dyad_rec b l r* ((*2* ∗ *real m* + *1*) / *2* ˆ *n′*))
by (*simp add*: *n′ del*: *power_Suc*)
also have ... = *r* (*dyad_rec b l r* ((*2* ∗ (*2* ∗ *real m* + *1*)) / (*2* ∗ *2* ˆ *n′*)))
by (*subst mult_divide_mult_cancel_left*) *auto*
also have ... = *r* (*dyad_rec b l r* (*2* ∗ (*2* ∗ *real m* + *1*) / *2*ˆ*n*))
by (*simp add*: *n′*)
finally show *?thesis* **.**
qed

definition *dyad_rec2*
where *dyad_rec2 u v lc rc x* =
dyad_rec (λ*z*. (*u*,*v*)) (λ(*a*,*b*). (*a*, *lc a b* (*midpoint a b*))) (λ(*a*,*b*). (*rc a b* (*midpoint a b*), *b*)) (*2*∗*x*)

abbreviation *leftrec* **where** *leftrec u v lc rc x* ≡ *fst* (*dyad_rec2 u v lc rc x*)
abbreviation *rightrec* **where** *rightrec u v lc rc x* ≡ *snd* (*dyad_rec2 u v lc rc x*)

lemma *leftrec_base*: *leftrec u v lc rc* (*real m* / *2*) = *u*
by (*simp add*: *dyad_rec2_def*)

lemma *leftrec_41*: *n* > *0* =⇒ *leftrec u v lc rc* ((*4* ∗ *real m* + *1*) / *2* ˆ (*Suc n*)) = *leftrec u v lc rc* ((*2* ∗ *real m* + *1*) / *2*ˆ*n*)
unfolding *dyad_rec2_def dyad_rec_41_times2*
by (*simp add*: *case_prod_beta*)

lemma *leftrec_43*: *n* > *0* =⇒
leftrec u v lc rc ((*4* ∗ *real m* + *3*) / *2* ˆ (*Suc n*)) =
rc (*leftrec u v lc rc* ((*2* ∗ *real m* + *1*) / *2*ˆ*n*)) (*rightrec u v lc rc* ((*2* ∗ *real m* + *1*) / *2*ˆ*n*))
(*midpoint* (*leftrec u v lc rc* ((*2* ∗ *real m* + *1*) / *2*ˆ*n*)) (*rightrec u v lc rc* ((*2* ∗ *real m* + *1*) / *2*ˆ*n*)))
unfolding *dyad_rec2_def dyad_rec_43_times2*
by (*simp add*: *case_prod_beta*)

lemma *rightrec_base*: *rightrec u v lc rc* (*real m* / *2*) = *v*
by (*simp add*: *dyad_rec2_def*)

lemma *rightrec_41*: *n* > *0* =⇒

```
        rightrec u v lc rc ((4 * real m + 1) / 2 ^ (Suc n)) =
          lc (leftrec u v lc rc ((2 * real m + 1) / 2^n)) (rightrec u v lc rc ((2
* real m + 1) / 2^n))
          (midpoint (leftrec u v lc rc ((2 * real m + 1) / 2^n)) (rightrec u v lc
rc ((2 * real m + 1) / 2^n)))
  unfolding dyad_rec2_def dyad_rec_41_times2
  by (simp add: case_prod_beta)

lemma rightrec_43: n > 0 ⟹ rightrec u v lc rc ((4 * real m + 3) / 2 ^ (Suc n))
= rightrec u v lc rc ((2 * real m + 1) / 2^n)
  unfolding dyad_rec2_def dyad_rec_43_times2
  by (simp add: case_prod_beta)

lemma dyadics_in_open_unit_interval:
   {0<..<1} ∩ (⋃k m. {real m / 2^k}) = (⋃k. ⋃m ∈ {0<..<2^k}. {real m /
2^k})
  by (auto simp: field_split_simps)



theorem homeomorphic_monotone_image_interval:
  fixes f :: real ⇒ 'a::{real_normed_vector,complete_space}
  assumes cont_f: continuous_on {0..1} f
     and conn: ⋀y. connected ({0..1} ∩ f −` {y})
     and f_1not0: f 1 ≠ f 0
   shows (f ` {0..1}) homeomorphic {0..1::real}
proof −
  have ∃c d. a ≤ c ∧ c ≤ m ∧ m ≤ d ∧ d ≤ b ∧
            (∀x ∈ {c..d}. f x = f m) ∧
            (∀x ∈ {a..<c}. (f x ≠ f m)) ∧
            (∀x ∈ {d<..b}. (f x ≠ f m)) ∧
            (∀x ∈ {a..<c}. ∀y ∈ {d<..b}. f x ≠ f y)
    if m: m ∈ {a..b} and ab01: {a..b} ⊆ {0..1} for a b m
  proof −
    have comp: compact (f −` {f m} ∩ {0..1})
     by (simp add: compact_eq_bounded_closed bounded_Int closed_vimage_Int cont_f)
    obtain c0 d0 where cd0: {0..1} ∩ f −` {f m} = {c0..d0}
      using connected_compact_interval_1 [of {0..1} ∩ f −` {f m}] conn comp
      by (metis Int_commute)
    with that have m ∈ cbox c0 d0
      by auto
    obtain c d where cd: {a..b} ∩ f −` {f m} = {c..d}
      using ab01 cd0
      by (rule_tac c=max a c0 and d=min b d0 in that) auto
    then have cdab: {c..d} ⊆ {a..b}
      by blast
    show ?thesis
    proof (intro exI conjI ballI)
      show a ≤ c d ≤ b
```

using *cdab cd m* **by** *auto*
show $c \leq m$ $m \leq d$
using *cd m* **by** *auto*
show $\bigwedge x.\ x \in \{c..d\} \Longrightarrow f\ x = f\ m$
using *cd* **by** *blast*
show $f\ x \neq f\ m$ **if** $x \in \{a..<c\}$ **for** *x*
using *that m cd* [*THEN equalityD1*, *THEN subsetD*] ‹$c \leq m$› **by** *force*
show $f\ x \neq f\ m$ **if** $x \in \{d<..b\}$ **for** *x*
using *that m cd* [*THEN equalityD1*, *THEN subsetD*, *of x*] ‹$m \leq d$› **by** *force*
show $f\ x \neq f\ y$ **if** $x \in \{a..<c\}$ $y \in \{d<..b\}$ **for** *x y*
proof (*cases* $f\ x = f\ m \lor f\ y = f\ m$)
case *True*
then show *?thesis*
using ‹$\bigwedge x.\ x \in \{a..<c\} \Longrightarrow f\ x \neq f\ m$› *that* **by** *auto*
next
case *False*
have *False* **if** $f\ x = f\ y$
proof −
have $x \leq m$ $m \leq y$
using ‹$c \leq m$› ‹$x \in \{a..<c\}$› ‹$m \leq d$› ‹$y \in \{d<..b\}$› **by** *auto*
then have $x \in (\{0..1\} \cap f -` \{f\ y\})$ $y \in (\{0..1\} \cap f -` \{f\ y\})$
using ‹$x \in \{a..<c\}$› ‹$y \in \{d<..b\}$› *ab01* **by** (*auto simp*: *that*)
then have $m \in (\{0..1\} \cap f -` \{f\ y\})$
by (*meson* ‹$m \leq y$› ‹$x \leq m$› *is_interval_connected_1 conn* [*of f y*] *is_interval_1*)
with *False* **show** *False* **by** *auto*
qed
then show *?thesis* **by** *auto*
qed
qed
qed
then obtain *leftcut rightcut* **where** *LR*:
$\bigwedge a\ b\ m.\ [\![m \in \{a..b\}; \{a..b\} \subseteq \{0..1\}]\!] \Longrightarrow$
($a \leq leftcut\ a\ b\ m \land leftcut\ a\ b\ m \leq m \land m \leq rightcut\ a\ b\ m \land rightcut\ a\ b\ m \leq b\ \land$
$(\forall x \in \{leftcut\ a\ b\ m..rightcut\ a\ b\ m\}.\ f\ x = f\ m) \land$
$(\forall x \in \{a..<leftcut\ a\ b\ m\}.\ f\ x \neq f\ m) \land$
$(\forall x \in \{rightcut\ a\ b\ m<..b\}.\ f\ x \neq f\ m) \land$
$(\forall x \in \{a..<leftcut\ a\ b\ m\}.\ \forall y \in \{rightcut\ a\ b\ m<..b\}.\ f\ x \neq f\ y)$)
apply *atomize*
apply (*clarsimp simp only*: *imp_conjL* [*symmetric*] *choice_iff choice_iff'*)
apply (*rule that*, *blast*)
done
then have *left_right*: $\bigwedge a\ b\ m.\ [\![m \in \{a..b\}; \{a..b\} \subseteq \{0..1\}]\!] \Longrightarrow a \leq leftcut\ a\ b\ m \land rightcut\ a\ b\ m \leq b$
and *left_right_m*: $\bigwedge a\ b\ m.\ [\![m \in \{a..b\}; \{a..b\} \subseteq \{0..1\}]\!] \Longrightarrow leftcut\ a\ b\ m \leq m \land m \leq rightcut\ a\ b\ m$
by *auto*
have *left_neq*: $[\![a \leq x; x < leftcut\ a\ b\ m; a \leq m; m \leq b; \{a..b\} \subseteq \{0..1\}]\!] \Longrightarrow$

f x ≠ *f m*
and *right_neq*: ⟦*rightcut a b m* < *x*; *x* ≤ *b*; *a* ≤ *m*; *m* ≤ *b*; {*a*..*b*} ⊆ {*0*..*1*}⟧ ⟹ *f x* ≠ *f m*
and *left_right_neq*: ⟦*a* ≤ *x*; *x* < *leftcut a b m*; *rightcut a b m* < *y*; *y* ≤ *b*; *a* ≤ *m*; *m* ≤ *b*; {*a*..*b*} ⊆ {*0*..*1*}⟧ ⟹ *f x* ≠ *f m*
and *feqm*: ⟦*leftcut a b m* ≤ *x*; *x* ≤ *rightcut a b m*; *a* ≤ *m*; *m* ≤ *b*; {*a*..*b*} ⊆ {*0*..*1*}⟧
⟹ *f x* = *f m* **for** *a b m x y*
by (*meson atLeastAtMost_iff greaterThanAtMost_iff atLeastLessThan_iff LR*)+
have *f_eqI*: ⋀*a b m x y*. ⟦*leftcut a b m* ≤ *x*; *x* ≤ *rightcut a b m*; *leftcut a b m* ≤ *y*; *y* ≤ *rightcut a b m*;
a ≤ *m*; *m* ≤ *b*; {*a*..*b*} ⊆ {*0*..*1*}⟧ ⟹ *f x* = *f y*
by (*metis feqm*)
define *u* **where** *u* ≡ *rightcut 0 1 0*
have *lc*[*simp*]: *leftcut 0 1 0* = *0* **and** *u01*: *0* ≤ *u u* ≤ *1*
using *LR* [*of 0 0 1*] **by** (*auto simp*: *u_def*)
have *f0u*: ⋀*x*. *x* ∈ {*0*..*u*} ⟹ *f x* = *f 0*
using *LR* [*of 0 0 1*] **unfolding** *u_def* [*symmetric*]
by (*metis* ‹*leftcut 0 1 0* = *0*› *atLeastAtMost_iff order_refl zero_le_one*)
have *fu1*: ⋀*x*. *x* ∈ {*u*<..*1*} ⟹ *f x* ≠ *f 0*
using *LR* [*of 0 0 1*] **unfolding** *u_def* [*symmetric*] **by** *fastforce*
define *v* **where** *v* ≡ *leftcut u 1 1*
have *rc*[*simp*]: *rightcut u 1 1* = *1* **and** *v01*: *u* ≤ *v v* ≤ *1*
using *LR* [*of 1 u 1*] *u01* **by** (*auto simp*: *v_def*)
have *fuv*: ⋀*x*. *x* ∈ {*u*..<*v*} ⟹ *f x* ≠ *f 1*
using *LR* [*of 1 u 1*] *u01 v_def* **by** *fastforce*
have *f0v*: ⋀*x*. *x* ∈ {*0*..<*v*} ⟹ *f x* ≠ *f 1*
by (*metis f_1not0 atLeastAtMost_iff atLeastLessThan_iff f0u fuv linear*)
have *fv1*: ⋀*x*. *x* ∈ {*v*..*1*} ⟹ *f x* = *f 1*
using *LR* [*of 1 u 1*] *u01 v_def* **by** (*metis atLeastAtMost_iff atLeastatMost_subset_iff order_refl rc*)
define *a* **where** *a* ≡ *leftrec u v leftcut rightcut*
define *b* **where** *b* ≡ *rightrec u v leftcut rightcut*
define *c* **where** *c* ≡ λ*x*. *midpoint* (*a x*) (*b x*)
have *a_real* [*simp*]: *a* (*real j*) = *u* **for** *j*
using *a_def leftrec_base*
by (*metis nonzero_mult_div_cancel_right of_nat_mult of_nat_numeral zero_neq_numeral*)
have *b_real* [*simp*]: *b* (*real j*) = *v* **for** *j*
using *b_def rightrec_base*
by (*metis nonzero_mult_div_cancel_right of_nat_mult of_nat_numeral zero_neq_numeral*)
have *a41*: *a* ((*4* ∗ *real m* + *1*) / *2*ˆ*Suc n*) = *a* ((*2* ∗ *real m* + *1*) / *2*ˆ*n*) **if** *n* > *0* **for** *m n*
using *that a_def leftrec_41* **by** *blast*
have *b41*: *b* ((*4* ∗ *real m* + *1*) / *2*ˆ*Suc n*) =
leftcut (*a* ((*2* ∗ *real m* + *1*) / *2*ˆ*n*))
(*b* ((*2* ∗ *real m* + *1*) / *2*ˆ*n*))
(*c* ((*2* ∗ *real m* + *1*) / *2*ˆ*n*)) **if** *n* > *0* **for** *m n*
using *that a_def b_def c_def rightrec_41* **by** *blast*
have *a43*: *a* ((*4* ∗ *real m* + *3*) / *2*ˆ*Suc n*) =

rightcut (*a* ((*2* ∗ *real m* + *1*) / *2*ˆ*n*))
(*b* ((*2* ∗ *real m* + *1*) / *2*ˆ*n*))
(*c* ((*2* ∗ *real m* + *1*) / *2*ˆ*n*)) **if** *n* > *0* **for** *m n*
using *that a_def b_def c_def leftrec_43* **by** *blast*
have *b43*: *b* ((*4* ∗ *real m* + *3*) / *2*ˆ*Suc n*) = *b* ((*2* ∗ *real m* + *1*) / *2*ˆ*n*) **if** *n* > *0* **for** *m n*
using *that b_def rightrec_43* **by** *blast*
have *uabv*: *u* ≤ *a* (*real m* / *2* ˆ *n*) ∧ *a* (*real m* / *2* ˆ *n*) ≤ *b* (*real m* / *2* ˆ *n*) ∧ *b* (*real m* / *2* ˆ *n*) ≤ *v* **for** *m n*
proof (*induction n arbitrary*: *m*)
case *0*
then show *?case* **by** (*simp add*: *v01*)
next
case (*Suc n p*)
show *?case*
proof (*cases even p*)
case *True*
then obtain *m* **where** *p* = *2*∗*m* **by** (*metis evenE*)
then show *?thesis*
by (*simp add*: *Suc.IH*)
next
case *False*
then obtain *m* **where** *m*: *p* = *2*∗*m* + *1* **by** (*metis oddE*)
show *?thesis*
proof (*cases n*)
case *0*
then show *?thesis*
by (*simp add*: *a_def b_def leftrec_base rightrec_base v01*)
next
case (*Suc n'*)
then have *n* > *0* **by** *simp*
have *a_le_c*: *a* (*real m* / *2*ˆ*n*) ≤ *c* (*real m* / *2*ˆ*n*) **for** *m*
unfolding *c_def* **by** (*metis Suc.IH ge_midpoint_1*)
have *c_le_b*: *c* (*real m* / *2*ˆ*n*) ≤ *b* (*real m* / *2*ˆ*n*) **for** *m*
unfolding *c_def* **by** (*metis Suc.IH le_midpoint_1*)
have *c_ge_u*: *c* (*real m* / *2*ˆ*n*) ≥ *u* **for** *m*
using *Suc.IH a_le_c order_trans* **by** *blast*
have *c_le_v*: *c* (*real m* / *2*ˆ*n*) ≤ *v* **for** *m*
using *Suc.IH c_le_b order_trans* **by** *blast*
have *a_ge_0*: *0* ≤ *a* (*real m* / *2*ˆ*n*) **for** *m*
using *Suc.IH order_trans u01*(*1*) **by** *blast*
have *b_le_1*: *b* (*real m* / *2*ˆ*n*) ≤ *1* **for** *m*
using *Suc.IH order_trans v01*(*2*) **by** *blast*
have *left_le*: *leftcut* (*a* ((*real m*) / *2*ˆ*n*)) (*b* ((*real m*) / *2*ˆ*n*)) (*c* ((*real m*) / *2*ˆ*n*)) ≤ *c* ((*real m*) / *2*ˆ*n*) **for** *m*
by (*simp add*: *LR a_ge_0 a_le_c b_le_1 c_le_b*)
have *right_ge*: *rightcut* (*a* ((*real m*) / *2*ˆ*n*)) (*b* ((*real m*) / *2*ˆ*n*)) (*c* ((*real m*) / *2*ˆ*n*)) ≥ *c* ((*real m*) / *2*ˆ*n*) **for** *m*
by (*simp add*: *LR a_ge_0 a_le_c b_le_1 c_le_b*)

```
        show ?thesis
        proof (cases even m)
          case True
          then obtain r where r: m = 2*r by (metis evenE)
          show ?thesis
            using order_trans [OF left_le c_le_v, of 1+2*r] Suc.IH [of m+1]
            using a_le_c [of m+1] c_le_b [of m+1] a_ge_0 [of m+1] b_le_1 [of m+1]
left_right ‹n > 0›
            by (simp_all add: r m add.commute [of 1]  a41 b41 del: power_Suc)
        next
          case False
          then obtain r where r: m = 2*r + 1 by (metis oddE)
          show ?thesis
            using order_trans [OF c_ge_u right_ge, of 1+2*r] Suc.IH [of m]
            using a_le_c [of m] c_le_b [of m] a_ge_0 [of m] b_le_1 [of m] left_right ‹n
> 0›
            apply (simp_all add: r m add.commute [of 3] a43 b43 del: power_Suc)
            by (simp add: add.commute)
        qed
      qed
    qed
  qed
  have a_ge_0 [simp]: 0 ≤ a(m / 2^n) and b_le_1 [simp]: b(m / 2^n) ≤ 1 for
m::nat and n
    using uabv order_trans u01 v01 by blast+
  then have b_ge_0 [simp]: 0 ≤ b(m / 2^n) and a_le_1 [simp]: a(m / 2^n) ≤ 1
for m::nat and n
    using uabv order_trans by blast+
  have alec [simp]: a(m / 2^n) ≤ c(m / 2^n) and cleb [simp]: c(m / 2^n) ≤ b(m
/ 2^n) for m::nat and n
    by (auto simp: c_def ge_midpoint_1 le_midpoint_1 uabv)
  have c_ge_0 [simp]: 0 ≤ c(m / 2^n) and c_le_1 [simp]: c(m / 2^n) ≤ 1 for
m::nat and n
    using a_ge_0 alec b_le_1 cleb order_trans by blast+
  have ⟦d = m−n; odd j; |real i / 2^m − real j / 2^n| < 1/2 ^ n⟧
       ⟹ (a(j / 2^n)) ≤ (c(i / 2^m)) ∧ (c(i / 2^m)) ≤ (b(j / 2^n)) for d i j m
n
  proof (induction d arbitrary: j n rule: less_induct)
    case (less d j n)
    show ?case
    proof (cases m ≤ n)
      case True
      have |2^n| * |real i / 2^m − real j / 2^n| = 0
      proof (rule Ints_nonzero_abs_less1)
        have (real i * 2^n − real j * 2^m) / 2^m = (real i * 2^n) / 2^m − (real j
* 2^m) / 2^m
          using diff_divide_distrib by blast
        also have ... = (real i * 2 ^ (n−m)) − (real j)
          using True by (auto simp: power_diff field_simps)
```

```
      also have ... ∈ ℤ
        by simp
      finally have (real i ∗ 2^n − real j ∗ 2^m) / 2^m ∈ ℤ .
      with True Ints_abs show |2^n| ∗ |real i / 2^m − real j / 2^n| ∈ ℤ
        by (fastforce simp: field_split_simps)
      show ||2^n| ∗ |real i / 2^m − real j / 2^n|| < 1
        using less.prems by (auto simp: field_split_simps)
    qed
    then have real i / 2^m = real j / 2^n
      by auto
    then show ?thesis
      by auto
  next
    case False
    then have n < m by auto
    obtain k where k: j = Suc (2∗k)
      using ‹odd j› oddE by fastforce
    show ?thesis
    proof (cases n > 0)
      case False
      then have a (real j / 2^n) = u
        by simp
      also have ... ≤ c (real i / 2^m)
        using alec uabv by (blast intro: order_trans)
      finally have ac: a (real j / 2^n) ≤ c (real i / 2^m) .
      have c (real i / 2^m) ≤ v
        using cleb uabv by (blast intro: order_trans)
      also have ... = b (real j / 2^n)
        using False by simp
      finally show ?thesis
        by (auto simp: ac)
    next
      case True show ?thesis
      proof (cases i / 2^m j / 2^n rule: linorder_cases)
        case less
        moreover have real (4 ∗ k + 1) / 2 ^ Suc n + 1 / (2 ^ Suc n) = real j
/ 2 ^ n
          using k by (force simp: field_split_simps)
        moreover have |real i / 2 ^ m − j / 2 ^ n| < 2 / (2 ^ Suc n)
          using less.prems by simp
        ultimately have closer: |real i / 2 ^ m − real (4 ∗ k + 1) / 2 ^ Suc n|
< 1 / (2 ^ Suc n)
          using less.prems by linarith
        have a (real (4 ∗ k + 1) / 2 ^ Suc n) ≤ c (i / 2 ^ m) ∧
              c (real i / 2 ^ m) ≤ b (real (4 ∗ k + 1) / 2 ^ Suc n)
        proof (rule less.IH [OF _ refl])
          show m − Suc n < d
            using ‹n < m› diff_less_mono2 less.prems(1) lessI by presburger
          show |real i / 2 ^ m − real (4 ∗ k + 1) / 2 ^ Suc n| < 1 / 2 ^ Suc n
```

using *closer* ‹*n* < *m*› ‹*d* = *m* − *n*› **by** (*auto simp*: *field_split_simps* ‹*n* < *m*› *diff_less_mono2*)
qed *auto*
then show *?thesis*
using *LR* [*of c*((2∗*k* + 1) / 2^*n*) *a*((2∗*k* + 1) / 2^*n*) *b*((2∗*k* + 1) / 2^*n*)]
using *alec* [*of* 2∗*k*+1] *cleb* [*of* 2∗*k*+1] *a_ge_0* [*of* 2∗*k*+1] *b_le_1* [*of* 2∗*k*+1]
using *k a41 b41* ‹*0* < *n*›
by (*simp add*: *add.commute*)
next
case *equal* **then show** *?thesis* **by** *simp*
next
case *greater*
moreover have *real* (4 ∗ *k* + 3) / 2 ^ *Suc n* − 1 / (2 ^ *Suc n*) = *real j* / 2 ^ *n*
using *k* **by** (*force simp*: *field_split_simps*)
moreover have |*real i* / 2 ^ *m* − *real j* / 2 ^ *n*| < 2 ∗ 1 / (2 ^ *Suc n*)
using *less.prems* **by** *simp*
ultimately have *closer*: |*real i* / 2 ^ *m* − *real* (4 ∗ *k* + 3) / 2 ^ *Suc n*| < 1 / (2 ^ *Suc n*)
using *less.prems* **by** *linarith*
have *a* (*real* (4 ∗ *k* + 3) / 2 ^ *Suc n*) ≤ *c* (*real i* / 2 ^ *m*) ∧
c (*real i* / 2 ^ *m*) ≤ *b* (*real* (4 ∗ *k* + 3) / 2 ^ *Suc n*)
proof (*rule less.IH* [*OF* _ *refl*])
show *m* − *Suc n* < *d*
using ‹*n* < *m*› *diff_less_mono2 less.prems*(1) **by** *blast*
show |*real i* / 2 ^ *m* − *real* (4 ∗ *k* + 3) / 2 ^ *Suc n*| < 1 / 2 ^ *Suc n*
using *closer* ‹*n* < *m*› ‹*d* = *m* − *n*› **by** (*auto simp*: *field_split_simps* ‹*n* < *m*› *diff_less_mono2*)
qed *auto*
then show *?thesis*
using *LR* [*of c*((2∗*k* + 1) / 2^*n*) *a*((2∗*k* + 1) / 2^*n*) *b*((2∗*k* + 1) / 2^*n*)]
using *alec* [*of* 2∗*k*+1] *cleb* [*of* 2∗*k*+1] *a_ge_0* [*of* 2∗*k*+1] *b_le_1* [*of* 2∗*k*+1]
using *k a43 b43* ‹*0* < *n*›
by (*simp add*: *add.commute*)
qed
qed
qed
qed
then have *aj_le_ci*: *a* (*real j* / 2 ^ *n*) ≤ *c* (*real i* / 2 ^ *m*)
and *ci_le_bj*: *c* (*real i* / 2 ^ *m*) ≤ *b* (*real j* / 2 ^ *n*) **if** *odd j* |*real i* / 2^*m* − *real j* / 2^*n*| < 1/2 ^ *n* **for** *i j m n*
using *that* **by** *blast*+
have *close_ab*: *odd m* ⟹ |*a* (*real m* / 2 ^ *n*) − *b* (*real m* / 2 ^ *n*)| ≤ 2 / 2^*n* **for** *m n*
proof (*induction n arbitrary*: *m*)

case *0*
with *u01 v01* **show** *?case* **by** *auto*
next
case (*Suc n m*)
with *oddE* **obtain** *k* **where** *k*: *m* = *Suc* (*2***k*) **by** *fastforce*
show *?case*
proof (*cases n* > *0*)
case *False*
with *u01 v01* **show** *?thesis*
by (*simp add*: *a_def b_def leftrec_base rightrec_base*)
next
case *True*
show *?thesis*
proof (*cases even k*)
case *True*
then obtain *j* **where** *j*: *k* = *2***j* **by** (*metis evenE*)
have |*a* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*) − (*b* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*))| ≤ *2*/*2* ˆ *n*
proof −
have *odd* (*Suc k*)
using *True* **by** *auto*
then show *?thesis*
by (*metis* (*no_types*) *Groups.add_ac*(*2*) *Suc.IH j of_nat_Suc of_nat_mult of_nat_numeral*)
qed
moreover have *a* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*) ≤
leftcut (*a* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*)) (*b* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*)) (*c* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*))
using *alec* [*of 2***j*+*1*] *cleb* [*of 2***j*+*1*] *a_ge_0* [*of 2***j*+*1*] *b_le_1* [*of 2***j*+*1*]
by (*auto simp*: *add.commute left_right*)
moreover have *leftcut* (*a* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*)) (*b* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*)) (*c* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*)) ≤
c ((*2* ∗ *real j* + *1*) / *2* ˆ *n*)
using *alec* [*of 2***j*+*1*] *cleb* [*of 2***j*+*1*] *a_ge_0* [*of 2***j*+*1*] *b_le_1* [*of 2***j*+*1*]
by (*auto simp*: *add.commute left_right_m*)
ultimately have |*a* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*) −
leftcut (*a* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*)) (*b* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*)) (*c* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*))|
≤ *2*/*2* ˆ *Suc n*
by (*simp add*: *c_def midpoint_def*)
with *j k* ‹*n* > *0*› **show** *?thesis*
by (*simp add*: *add.commute* [*of 1*] *a41 b41 del*: *power_Suc*)
next
case *False*
then obtain *j* **where** *j*: *k* = *2***j* + *1* **by** (*metis oddE*)
have |*a* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*) − (*b* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*))| ≤ *2*/*2* ˆ *n*
using *Suc.IH* [*OF False*] *j* **by** (*auto simp*: *algebra_simps*)
moreover have *c* ((*2* ∗ *real j* + *1*) / *2* ˆ *n*) ≤

rightcut (a ((2 ∗ real j + 1) / 2 ^ n)) (b ((2 ∗ real j + 1) / 2 ^ n)) (c ((2 ∗ real j + 1) / 2 ^ n))
using alec [of 2∗j+1] cleb [of 2∗j+1] a_ge_0 [of 2∗j+1] b_le_1 [of 2∗j+1]
by (auto simp: add.commute left_right_m)
moreover have rightcut (a ((2 ∗ real j + 1) / 2 ^ n)) (b ((2 ∗ real j + 1) / 2 ^ n)) (c ((2 ∗ real j + 1) / 2 ^ n)) ≤
b ((2 ∗ real j + 1) / 2 ^ n)
using alec [of 2∗j+1] cleb [of 2∗j+1] a_ge_0 [of 2∗j+1] b_le_1 [of 2∗j+1]
by (auto simp: add.commute left_right)
ultimately have |rightcut (a ((2 ∗ real j + 1) / 2 ^ n)) (b ((2 ∗ real j + 1) / 2 ^ n)) (c ((2 ∗ real j + 1) / 2 ^ n)) −
b ((2 ∗ real j + 1) / 2 ^ n)| ≤ 2/2 ^ Suc n
by (simp add: c_def midpoint_def)
with j k ‹n > 0› show ?thesis
by (simp add: add.commute [of 3] a43 b43 del: power_Suc)
qed
qed
qed
have m1_to_3: 4 ∗ real k − 1 = real (4 ∗ (k−1)) + 3 if 0 < k for k
using that by auto
have fb_eq_fa: [[0 < j; 2∗j < 2 ^ n]] =⇒ f(b((2 ∗ real j − 1) / 2^n)) = f(a((2 ∗ real j + 1) / 2^n)) for n j
proof (induction n arbitrary: j)
case 0
then show ?case by auto
next
case (Suc n j) show ?case
proof (cases n > 0)
case False
with Suc.prems show ?thesis by auto
next
case True
show ?thesis proof (cases even j)
case True
then obtain k where k: j = 2∗k by (metis evenE)
with ‹0 < j› have k > 0 2 ∗ k < 2 ^ n
using Suc.prems(2) k by auto
with k ‹0 < n› Suc.IH [of k] show ?thesis
by (simp add: m1_to_3 a41 b43 del: power_Suc) (auto simp: of_nat_diff)
next
case False
then obtain k where k: j = 2∗k + 1 by (metis oddE)
have f (leftcut (a ((2 ∗ k + 1) / 2^n)) (b ((2 ∗ k + 1) / 2^n)) (c ((2 ∗ k + 1) / 2^n)))
= f (c ((2 ∗ k + 1) / 2^n))
f (c ((2 ∗ k + 1) / 2^n))
= f (rightcut (a ((2 ∗ k + 1) / 2^n)) (b ((2 ∗ k + 1) / 2^n)) (c ((2 ∗ k + 1) / 2^n)))
using alec [of 2∗k+1 n] cleb [of 2∗k+1 n] a_ge_0 [of 2∗k+1 n] b_le_1 [of

2∗k+1 n] *k*
using *left_right_m* [*of c*((*2∗k* + *1*) / *2*ˆ*n*) *a*((*2∗k* + *1*) / *2*ˆ*n*) *b*((*2∗k* + *1*) / *2*ˆ*n*)]
by (*auto simp*: *add.commute feqm* [*OF order_refl*] *feqm* [*OF* _ *order_refl*, *symmetric*])
then
show *?thesis*
by (*simp add*: *k add.commute* [*of 1*] *add.commute* [*of 3*] *a43 b41*‹*0* < *n*› *del*: *power_Suc*)
qed
qed
qed
have *f_eq_fc*: ⟦*0* < *j*; *j* < *2* ˆ *n*⟧
⟹ *f*(*b*((*2∗j* − *1*) / *2* ˆ (*Suc n*))) = *f*(*c*(*j* / *2*ˆ*n*)) ∧
f(*a*((*2∗j* + *1*) / *2* ˆ (*Suc n*))) = *f*(*c*(*j* / *2*ˆ*n*)) **for** *n* **and** *j*::*nat*
proof (*induction n arbitrary*: *j*)
case *0*
then show *?case* **by** *auto*
next
case (*Suc n*)
show *?case*
proof (*cases even j*)
case *True*
then obtain *k* **where** *k*: *j* = *2∗k* **by** (*metis evenE*)
then have *less2n*: *k* < *2* ˆ *n*
using *Suc.prems*(*2*) **by** *auto*
have *0* < *k* **using** ‹*0* < *j*› *k* **by** *linarith*
then have *m1_to_3*: *real* (*4* ∗ *k* − *Suc 0*) = *real* (*4* ∗ (*k*−*1*)) + *3*
by *auto*
then show *?thesis*
using *Suc.IH* [*of k*] *k* ‹*0* < *k*›
by (*simp add*: *less2n add.commute* [*of 1*] *m1_to_3 a41 b43 del*: *power_Suc*) (*auto simp*: *of_nat_diff*)
next
case *False*
then obtain *k* **where** *k*: *j* = *2∗k* + *1* **by** (*metis oddE*)
with *Suc.prems* **have** *k* < *2*ˆ*n* **by** *auto*
show *?thesis*
using *alec* [*of 2∗k+1 Suc n*] *cleb* [*of 2∗k+1 Suc n*] *a_ge_0* [*of 2∗k+1 Suc n*] *b_le_1* [*of 2∗k+1 Suc n*] *k*
using *left_right_m* [*of c*((*2∗k* + *1*) / *2* ˆ *Suc n*) *a*((*2∗k* + *1*) / *2* ˆ *Suc n*) *b*((*2∗k* + *1*) / *2* ˆ *Suc n*)]
apply (*simp add*: *add.commute* [*of 1*] *add.commute* [*of 3*] *m1_to_3 b41 a43 del*: *power_Suc*)
apply (*force intro*: *feqm*)
done
qed
qed
define *D01* **where** *D01* ≡ {*0*<..<*1*} ∩ (⋃*k m*. {*real m* / *2*ˆ*k*})

```
  have cloD01 [simp]: closure D01 = {0..1}
    unfolding D01_def
    by (subst closure_dyadic_rationals_in_convex_set_pos_1) auto
  have uniformly_continuous_on D01 (f ∘ c)
  proof (clarsimp simp: uniformly_continuous_on_def)
    fix e::real
    assume 0 < e
    have ucontf: uniformly_continuous_on {0..1} f
      by (simp add: compact_uniformly_continuous [OF cont_f])
    then obtain d where 0 < d and d: ⋀x x'. ⟦x ∈ {0..1}; x' ∈ {0..1}; norm
(x' − x) < d⟧ ⟹ norm (f x' − f x) < e/2
      unfolding uniformly_continuous_on_def dist_norm
      by (metis ‹0 < e› less_divide_eq_numeral1(1) mult_zero_left)
    obtain n where n: 1/2^n < min d 1
     by (metis ‹0 < d› divide_less_eq_1 less_numeral_extra(1) min_def one_less_numeral_iff
power_one_over real_arch_pow_inv semiring_norm(76) zero_less_numeral)
    with gr0I have n > 0
      by (force simp: field_split_simps)
    show ∃d>0. ∀x∈D01. ∀x'∈D01. dist x' x < d ⟶ dist (f (c x')) (f (c x))
< e
    proof (intro exI ballI impI conjI)
      show (0::real) < 1/2^n by auto
    next
      have dist_fc_close: dist (f(c(real i / 2^m))) (f(c(real j / 2^n))) < e/2
        if i: 0 < i i < 2 ^ m and j: 0 < j j < 2 ^ n and clo: abs(i / 2^m − j /
2^n) < 1/2 ^ n for i j m
      proof −
        have abs3: |x − a| < e ⟹ x = a ∨ |x − (a − e/2)| < e/2 ∨ |x − (a +
e/2)| < e/2 for x a e::real
          by linarith
        consider i / 2 ^ m = j / 2 ^ n
          | |i / 2 ^ m − (2 * j − 1) / 2 ^ Suc n| < 1/2 ^ Suc n
          | |i / 2 ^ m − (2 * j + 1) / 2 ^ Suc n| < 1/2 ^ Suc n
          using abs3 [OF clo] j by (auto simp: field_simps of_nat_diff)
        then show ?thesis
        proof cases
          case 1 with ‹0 < e› show ?thesis by auto
        next
          case 2
          have *: abs(a − b) ≤ 1/2 ^ n ∧ 1/2 ^ n < d ∧ a ≤ c ∧ c ≤ b ⟹ b −
c < d for a b c
            by auto
          have norm (c (real i / 2 ^ m) − b (real (2 * j − 1) / 2 ^ Suc n)) < d
            using 2 j n close_ab [of 2*j−1 Suc n]
            using b_ge_0 [of 2*j−1 Suc n] b_le_1 [of 2*j−1 Suc n]
            using aj_le_ci [of 2*j−1 i m Suc n]
            using ci_le_bj [of 2*j−1 i m Suc n]
            apply (simp add: divide_simps of_nat_diff del: power_Suc)
            apply (auto simp: divide_simps intro!: *)
```

done
moreover have *f*(*c*(*j* / *2*ˆ*n*)) = *f*(*b* ((*2*∗*j* − *1*) / *2* ˆ (*Suc n*)))
using *f_eq_fc* [*OF j*] **by** *metis*
ultimately show *?thesis*
by (*metis dist_norm atLeastAtMost_iff b_ge_0 b_le_1 c_ge_0 c_le_1 d*)
next
case *3*
have ∗: *abs*(*a* − *b*) ≤ *1*/*2* ˆ *n* ∧ *1*/*2* ˆ *n* < *d* ∧ *a* ≤ *c* ∧ *c* ≤ *b* ⟹ *c* − *a* < *d* **for** *a b c*
by *auto*
have *norm* (*c* (*real i* / *2* ˆ *m*) − *a* (*real* (*2* ∗ *j* + *1*) / *2* ˆ *Suc n*)) < *d*
using *3 j n close_ab* [*of 2*∗*j*+*1 Suc n*]
using *b_ge_0* [*of 2*∗*j*+*1 Suc n*] *b_le_1* [*of 2*∗*j*+*1 Suc n*]
using *aj_le_ci* [*of 2*∗*j*+*1 i m Suc n*]
using *ci_le_bj* [*of 2*∗*j*+*1 i m Suc n*]
apply (*simp add*: *divide_simps of_nat_diff del*: *power_Suc*)
apply (*auto simp*: *divide_simps intro*!: ∗)
done
moreover have *f*(*c*(*j* / *2*ˆ*n*)) = *f*(*a* ((*2*∗*j* + *1*) / *2* ˆ (*Suc n*)))
using *f_eq_fc* [*OF j*] **by** *metis*
ultimately show *?thesis*
by (*metis dist_norm a_ge_0 atLeastAtMost_iff a_ge_0 a_le_1 c_ge_0 c_le_1 d*)
qed
qed
show *dist* (*f* (*c x*′)) (*f* (*c x*)) < *e*
if *x* ∈ *D01 x*′ ∈ *D01 dist x*′ *x* < *1*/*2*ˆ*n* **for** *x x*′
using *that* **unfolding** *D01_def dyadics_in_open_unit_interval*
proof *clarsimp*
fix *i k*::*nat* **and** *m p*
assume *i*: *0* < *i i* < *2* ˆ *m* **and** *k*: *0*<*k k* < *2* ˆ *p*
assume *clo*: *dist* (*real k* / *2* ˆ *p*) (*real i* / *2* ˆ *m*) < *1*/*2* ˆ *n*
obtain *j*::*nat* **where** *0* < *j j* < *2* ˆ *n*
and *clo_ij*: *abs*(*i* / *2*ˆ*m* − *j* / *2*ˆ*n*) < *1*/*2* ˆ *n*
and *clo_kj*: *abs*(*k* / *2*ˆ*p* − *j* / *2*ˆ*n*) < *1*/*2* ˆ *n*
proof −
have *max* (*2*ˆ*n* ∗ *i* / *2*ˆ*m*) (*2*ˆ*n* ∗ *k* / *2*ˆ*p*) ≥ *0*
by (*auto simp*: *le_max_iff_disj*)
then obtain *j* **where** *floor* (*max* (*2*ˆ*n*∗*i* / *2*ˆ*m*) (*2*ˆ*n*∗*k* / *2*ˆ*p*)) = *int j*
using *zero_le_floor zero_le_imp_eq_int* **by** *blast*
then have *j_le*: *real j* ≤ *max* (*2*ˆ*n* ∗ *i* / *2*ˆ*m*) (*2*ˆ*n* ∗ *k* / *2*ˆ*p*)
and *less_j1*: *max* (*2*ˆ*n* ∗ *i* / *2*ˆ*m*) (*2*ˆ*n* ∗ *k* / *2*ˆ*p*) < *real j* + *1*
using *floor_correct* [*of max* (*2*ˆ*n* ∗ *i* / *2*ˆ*m*) (*2*ˆ*n* ∗ *k* / *2*ˆ*p*)] **by** *linarith*+
show *thesis*
proof (*cases j* = *0*)
case *True*
show *thesis*
proof
show (*1*::*nat*) < *2* ˆ *n*

```
        by (metis Suc_1 ‹0 < n› lessI one_less_power)
      show |real i / 2 ^ m − real 1/2 ^ n| < 1/2 ^ n
        using i less_j1 by (simp add: dist_norm field_simps True)
      show |real k / 2 ^ p − real 1/2 ^ n| < 1/2 ^ n
        using k less_j1 by (simp add: dist_norm field_simps True)
    qed simp
  next
    case False
    have 1: real j ∗ 2 ^ m < real i ∗ 2 ^ n
      if j: real j ∗ 2 ^ p ≤ real k ∗ 2 ^ n and k: real k ∗ 2 ^ m < real i ∗ 2
^ p
      for i k m p
    proof −
      have real j ∗ 2 ^ p ∗ 2 ^ m ≤ real k ∗ 2 ^ n ∗ 2 ^ m
        using j by simp
      moreover have real k ∗ 2 ^ m ∗ 2 ^ n < real i ∗ 2 ^ p ∗ 2 ^ n
        using k by simp
      ultimately have real j ∗ 2 ^ p ∗ 2 ^ m < real i ∗ 2 ^ p ∗ 2 ^ n
        by (simp only: mult_ac)
      then show ?thesis
        by simp
    qed
    have 2: real j ∗ 2 ^ m < 2 ^ m + real i ∗ 2 ^ n
      if j: real j ∗ 2 ^ p ≤ real k ∗ 2 ^ n and k: real k ∗ (2 ^ m ∗ 2 ^ n) <
2 ^ m ∗ 2 ^ p + real i ∗ (2 ^ n ∗ 2 ^ p)
      for i k m p
    proof −
      have real j ∗ 2 ^ p ∗ 2 ^ m ≤ real k ∗ (2 ^ m ∗ 2 ^ n)
        using j by simp
      also have ... < 2 ^ m ∗ 2 ^ p + real i ∗ (2 ^ n ∗ 2 ^ p)
        by (rule k)
     finally have (real j ∗ 2 ^ m) ∗ 2 ^ p < (2 ^ m + real i ∗ 2 ^ n) ∗ 2 ^ p
        by (simp add: algebra_simps)
      then show ?thesis
        by simp
    qed
    have 3: real j ∗ 2 ^ p < 2 ^ p + real k ∗ 2 ^ n
     if j: real j ∗ 2 ^ m ≤ real i ∗ 2 ^ n and i: real i ∗ 2 ^ p ≤ real k ∗ 2 ^
m
    proof −
      have real j ∗ 2 ^ m ∗ 2 ^ p ≤ real i ∗ 2 ^ n ∗ 2 ^ p
        using j by simp
      moreover have real i ∗ 2 ^ p ∗ 2 ^ n ≤ real k ∗ 2 ^ m ∗ 2 ^ n
        using i by simp
      ultimately have real j ∗ 2 ^ m ∗ 2 ^ p ≤ real k ∗ 2 ^ m ∗ 2 ^ n
        by (simp only: mult_ac)
      then have real j ∗ 2 ^ p ≤ real k ∗ 2 ^ n
        by simp
      also have ... < 2 ^ p + real k ∗ 2 ^ n
```

by *auto*
finally show *?thesis* **by** *simp*
qed
show *?thesis*
proof
have *2 ˆ n ∗ real i / 2 ˆ m < 2 ˆ n 2 ˆ n ∗ real k / 2 ˆ p < 2 ˆ n*
using *i k* **by** (*auto simp*: *field_simps*)
then have *max* (*2ˆn ∗ i / 2ˆm*) (*2ˆn ∗ k / 2ˆp*) *< 2ˆn*
by *simp*
with *j_le* **have** *real j < 2 ˆ n* **by** *linarith*
then show *j < 2 ˆ n*
by *auto*
have |*real i ∗ 2 ˆ n − real j ∗ 2 ˆ m*| *< 2 ˆ m*
using *clo less_j1 j_le*
by (*auto simp*: *le_max_iff_disj field_split_simps dist_norm abs_if split*: *if_split_asm dest*: *1 2*)
then show |*real i / 2 ˆ m − real j / 2 ˆ n*| *< 1/2 ˆ n*
by (*auto simp*: *field_split_simps*)
have |*real k ∗ 2 ˆ n − real j ∗ 2 ˆ p*| *< 2 ˆ p*
using *clo less_j1 j_le*
by (*auto simp*: *le_max_iff_disj field_split_simps dist_norm abs_if split*: *if_split_asm dest*: *3 2*)
then show |*real k / 2 ˆ p − real j / 2 ˆ n*| *< 1/2 ˆ n*
by (*auto simp*: *le_max_iff_disj field_split_simps dist_norm*)
qed (*use False* **in** *simp*)
qed
qed
show *dist* (*f* (*c* (*real k / 2 ˆ p*))) (*f* (*c* (*real i / 2 ˆ m*))) *< e*
proof (*rule dist_triangle_half_l*)
show *dist* (*f* (*c* (*real k / 2 ˆ p*))) (*f*(*c*(*j / 2ˆn*))) *< e/2*
using ‹*0 < j*› ‹*j < 2 ˆ n*› *k clo_kj*
by (*intro dist_fc_close*) *auto*
show *dist* (*f* (*c* (*real i / 2 ˆ m*))) (*f* (*c* (*real j / 2 ˆ n*))) *< e/2*
using ‹*0 < j*› ‹*j < 2 ˆ n*› *i clo_ij*
by (*intro dist_fc_close*) *auto*
qed
qed
qed
qed
then obtain *h* **where** *ucont_h*: *uniformly_continuous_on* {*0..1*} *h*
and *fc_eq*: ⋀*x. x ∈ D01 =⇒* (*f ∘ c*) *x = h x*
proof (*rule uniformly_continuous_on_extension_on_closure* [*of D01 f ∘ c*])
qed (*use closure_subset* [*of D01*] **in** ‹*auto intro*!: *that*›)
then have *cont_h*: *continuous_on* {*0..1*} *h*
using *uniformly_continuous_imp_continuous* **by** *blast*
have *h_eq*: *h* (*real k / 2 ˆ m*) *= f* (*c* (*real k / 2 ˆ m*)) **if** *0 < k k < 2ˆm* **for** *k m*
using *fc_eq that* **by** (*force simp*: *D01_def*)
have *h ‘* {*0..1*} *= f ‘* {*0..1*}
proof

```
have h ` (closure D01) ⊆ f ` {0..1}
proof (rule image_closure_subset)
  show continuous_on (closure D01) h
    using cont_h by simp
  show closed (f ` {0..1})
    using compact_continuous_image [OF cont_f] compact_imp_closed by blast
  show h ` D01 ⊆ f ` {0..1}
    by (force simp: dyadics_in_open_unit_interval D01_def h_eq)
qed
with cloD01 show h ` {0..1} ⊆ f ` {0..1} by simp
have a12 [simp]: a (1/2) = u
  by (metis a_def leftrec_base numeral_One of_nat_numeral)
have b12 [simp]: b (1/2) = v
  by (metis b_def rightrec_base numeral_One of_nat_numeral)
have f ` {0..1} ⊆ closure(h ` D01)
proof (clarsimp simp: closure_approachable dyadics_in_open_unit_interval D01_def)
  fix x e::real
  assume 0 ≤ x x ≤ 1 0 < e
  have ucont_f: uniformly_continuous_on {0..1} f
    using compact_uniformly_continuous cont_f by blast
  then obtain δ where δ > 0
    and δ: ⋀x x'. ⟦x ∈ {0..1}; x' ∈ {0..1}; dist x' x < δ⟧ ⟹ norm (f x' − f
x) < e
    using ‹0 < e› by (auto simp: uniformly_continuous_on_def dist_norm)
  have *: ∃m::nat. ∃y. odd m ∧ 0 < m ∧ m < 2 ^ n ∧ y ∈ {a(m / 2^n) ..
b(m / 2^n)} ∧ f y = f x
    if n ≠ 0 for n
    using that
  proof (induction n)
    case 0 then show ?case by auto
  next
    case (Suc n)
    show ?case
    proof (cases n=0)
      case True
      consider x ∈ {0..u} | x ∈ {u..v} | x ∈ {v..1}
        using ‹0 ≤ x› ‹x ≤ 1› by force
      then have ∃y≥a (real 1/2). y ≤ b (real 1/2) ∧ f y = f x
      proof cases
        case 1
        then show ?thesis
          using uabv [of 1 1] f0u [of u] f0u [of x] by force
      next
        case 2
        then show ?thesis
          by (rule_tac x=x in exI) auto
      next
        case 3
        then show ?thesis
```

```
        using uabv [of 1 1] fv1 [of v] fv1 [of x] by force
      qed
      with ‹n=0› show ?thesis
        by (rule_tac x=1 in exI) auto
    next
      case False
      with Suc obtain m y
        where odd m 0 < m and mless: m < 2 ^ n
          and y: y ∈ {a (real m / 2 ^ n)..b (real m / 2 ^ n)} and feq: f y = f x
        by metis
      then obtain j where j: m = 2*j + 1 by (metis oddE)
      have j4: 4 * j + 1 < 2 ^ Suc n
        using mless j by (simp add: algebra_simps)

      consider y ∈ {a((2*j + 1) / 2^n) .. b((4*j + 1) / 2 ^ (Suc n))}
        | y ∈ {b((4*j + 1) / 2 ^ (Suc n)) .. a((4*j + 3) / 2 ^ (Suc n))}
        | y ∈ {a((4*j + 3) / 2 ^ (Suc n)) .. b((2*j + 1) / 2^n)}
        using y j by force
      then show ?thesis
      proof cases
        case 1
        show ?thesis
        proof (intro exI conjI)
          show y ∈ {a (real (4 * j + 1) / 2 ^ Suc n)..b (real (4 * j + 1) / 2 ^
Suc n)}
            using mless j ‹n ≠ 0› 1 by (simp add: a41 b41 add.commute [of 1]
del: power_Suc)
        qed (use feq j4 in auto)
      next
        case 2
        show ?thesis
        proof (intro exI conjI)
          show b (real (4 * j + 1) / 2 ^ Suc n) ∈ {a (real (4 * j + 1) / 2 ^
Suc n)..b (real (4 * j + 1) / 2 ^ Suc n)}
            using ‹n ≠ 0› alec [of 2*j+1 n] cleb [of 2*j+1 n] a_ge_0 [of 2*j+1
n] b_le_1 [of 2*j+1 n]
              using left_right [of c((2*j + 1) / 2^n) a((2*j + 1) / 2^n) b((2*j
+ 1) / 2^n)]
            by (simp add: a41 b41 add.commute [of 1] del: power_Suc)
          show f (b (real (4 * j + 1) / 2 ^ Suc n)) = f x
            using ‹n ≠ 0› 2
           using alec [of 2*j+1 n] cleb [of 2*j+1 n] a_ge_0 [of 2*j+1 n]  b_le_1
[of 2*j+1 n]
           by (force simp add: b41 a43 add.commute [of 1] feq [symmetric] simp
del: power_Suc intro: f_eqI)
        qed (use j4 in auto)
      next
        case 3
        show ?thesis
```

```
          proof (intro exI conjI)
            show 4 * j + 3 < 2 ^ Suc n
              using mless j by simp
            show f y = f x
              by fact
            show y ∈ {a (real (4 * j + 3) / 2 ^ Suc n) .. b (real (4 * j + 3) / 2
^ Suc n)}
              using 3 False b43 [of n j] by (simp add: add.commute)
          qed (use 3 in auto)
        qed
      qed
    qed
    obtain n where n: 1/2^n < min (δ / 2) 1
      by (metis ‹0 < δ› divide_less_eq_1 less_numeral_extra(1) min_less_iff_conj
one_less_numeral_iff power_one_over real_arch_pow_inv semiring_norm(76) zero_less_divide_iff
zero_less_numeral)
    with gr0I have n ≠ 0
      by fastforce
    with * obtain m::nat and y
      where odd m 0 < m and mless: m < 2 ^ n
        and y: a(m / 2^n) ≤ y ∧ y ≤ b(m / 2^n) and feq: f x = f y
      by (metis atLeastAtMost_iff)
    then have 0 ≤ y y ≤ 1
      by (meson a_ge_0 b_le_1 order.trans)+
    moreover have y < δ + c (real m / 2 ^ n) c (real m / 2 ^ n) < δ + y
      using y alec [of m n] cleb [of m n] n field_sum_of_halves close_ab [OF ‹odd
m›, of n]
      by linarith+
    moreover note ‹0 < m› mless ‹0 ≤ x› ‹x ≤ 1›
    ultimately have dist (h (real m / 2 ^ n)) (f x) < e
      by (auto simp: dist_norm h_eq feq δ)
    then show ∃k. ∃m∈{0<..<2 ^ k}. dist (h (real m / 2 ^ k)) (f x) < e
      using ‹0 < m› greaterThanLessThan_iff mless by blast
  qed
  also have ... ⊆ h ` {0..1}
  proof (rule closure_minimal)
    show h ` D01 ⊆ h ` {0..1}
      using cloD01 closure_subset by blast
    show closed (h ` {0..1})
      using compact_continuous_image [OF cont_h] compact_imp_closed by auto
  qed
  finally show f ` {0..1} ⊆ h ` {0..1} .
qed
moreover have inj_on h {0..1}
proof -
  have u < v
    by (metis atLeastAtMost_iff f0u f_1not0 fv1 order.not_eq_order_implies_strict
u01(1) u01(2) v01(1))
  have f_not_fu: ⋀x. ⟦u < x; x ≤ v⟧ ⟹ f x ≠ f u
```

by (*metis atLeastAtMost_iff f0u fu1 greaterThanAtMost_iff order_refl order_trans u01*(*1*) *v01*(*2*))
have *f_not_fv*: $\bigwedge x.\ [\![u \le x;\ x < v]\!] \Longrightarrow f\ x \ne f\ v$
by (*metis atLeastAtMost_iff order_refl order_trans v01*(*2*) *atLeastLessThan_iff fuv fv1*)
have *a_less_b*:
$a(j\ /\ 2\hat{\ }n) < b(j\ /\ 2\hat{\ }n) \wedge$
$(\forall x.\ a(j\ /\ 2\hat{\ }n) < x \longrightarrow x \le b(j\ /\ 2\hat{\ }n) \longrightarrow f\ x \ne f(a(j\ /\ 2\hat{\ }n))) \wedge$
$(\forall x.\ a(j\ /\ 2\hat{\ }n) \le x \longrightarrow x < b(j\ /\ 2\hat{\ }n) \longrightarrow f\ x \ne f(b(j\ /\ 2\hat{\ }n)))$ **for** *n* **and** *j*::*nat*
proof (*induction n arbitrary*: *j*)
case *0* **then show** *?case*
by (*simp add*: ‹$u < v$› *f_not_fu f_not_fv*)
next
case (*Suc n j*) **show** *?case*
proof (*cases* $n > 0$)
case *False* **then show** *?thesis*
by (*auto simp*: *a_def b_def leftrec_base rightrec_base* ‹$u < v$› *f_not_fu f_not_fv*)
next
case *True* **show** *?thesis*
proof (*cases even j*)
case *True*
with ‹$0 < n$› *Suc.IH* **show** *?thesis*
by (*auto elim*!: *evenE*)
next
case *False*
then obtain *k* **where** *k*: $j = 2{*}k + 1$ **by** (*metis oddE*)
then show *?thesis*
proof (*cases even k*)
case *True*
then obtain *m* **where** *m*: $k = 2{*}m$ **by** (*metis evenE*)
have *fleft*: $f\ (leftcut\ (a\ ((2{*}m + 1)\ /\ 2\hat{\ }n))\ (b\ ((2{*}m + 1)\ /\ 2\hat{\ }n))\ (c\ ((2{*}m + 1)\ /\ 2\hat{\ }n))) =$
$f\ (c((2{*}m + 1)\ /\ 2\hat{\ }n))$
using *alec* [*of 2*∗*m*+*1 n*] *cleb* [*of 2*∗*m*+*1 n*] *a_ge_0* [*of 2*∗*m*+*1 n*] *b_le_1* [*of 2*∗*m*+*1 n*]
using *left_right_m* [*of* $c((2{*}m + 1)\ /\ 2\hat{\ }n)$ $a((2{*}m + 1)\ /\ 2\hat{\ }n)$ $b((2{*}m + 1)\ /\ 2\hat{\ }n)$]
by (*auto intro*: *f_eqI*)
show *?thesis*
proof (*intro conjI impI notI allI*)
have *False* **if** $b\ (real\ j\ /\ 2\ \hat{}\ Suc\ n) \le a\ (real\ j\ /\ 2\ \hat{}\ Suc\ n)$
proof −
have $f\ (c\ ((1 + real\ m * 2)\ /\ 2\ \hat{}\ n)) = f\ (a\ ((1 + real\ m * 2)\ /\ 2\ \hat{}\ n))$
using *k m* ‹$0 < n$› *fleft that a41* [*of n m*] *b41* [*of n m*]
using *alec* [*of 2*∗*m*+*1 n*] *cleb* [*of 2*∗*m*+*1 n*] *a_ge_0* [*of 2*∗*m*+*1 n*] *b_le_1* [*of 2*∗*m*+*1 n*]
using *left_right* [*of* $c((2{*}m + 1)\ /\ 2\hat{\ }n)$ $a((2{*}m + 1)\ /\ 2\hat{\ }n)$ $b((2{*}m$

+ 1) / 2^n)]
by (auto simp: algebra_simps)
moreover have a (real (1 + m * 2) / 2 ^ n) < c (real (1 + m * 2) / 2 ^ n)
using Suc.IH [of 1 + m * 2] by (simp add: c_def midpoint_def)
moreover have c (real (1 + m * 2) / 2 ^ n) ≤ b (real (1 + m * 2) / 2 ^ n)
using cleb by blast
ultimately show ?thesis
using Suc.IH [of 1 + m * 2] by force
qed
then show a (real j / 2 ^ Suc n) < b (real j / 2 ^ Suc n) by force
next
fix x
assume a (real j / 2 ^ Suc n) < x x ≤ b (real j / 2 ^ Suc n) f x = f (a (real j / 2 ^ Suc n))
then show False
using Suc.IH [of 1 + m * 2, THEN conjunct2, THEN conjunct1]
using k m ‹0 < n› a41 [of n m] b41 [of n m]
using alec [of 2*m+1 n] cleb [of 2*m+1 n] a_ge_0 [of 2*m+1 n] b_le_1 [of 2*m+1 n]
using left_right_m [of c((2*m + 1) / 2^n) a((2*m + 1) / 2^n) b((2*m + 1) / 2^n)]
by (auto simp: algebra_simps)
next
fix x
assume a (real j / 2 ^ Suc n) ≤ x x < b (real j / 2 ^ Suc n) f x = f (b (real j / 2 ^ Suc n))
then show False
using k m ‹0 < n› a41 [of n m] b41 [of n m] fleft left_neq
using alec [of 2*m+1 n] cleb [of 2*m+1 n] a_ge_0 [of 2*m+1 n] b_le_1 [of 2*m+1 n]
by (auto simp: algebra_simps)
qed
next
case False
with oddE obtain m where m: k = Suc (2*m) by fastforce
have fright: f (rightcut (a ((2*m + 1) / 2^n)) (b ((2*m + 1) / 2^n)) (c ((2*m + 1) / 2^n))) = f (c((2*m + 1) / 2^n))
using alec [of 2*m+1 n] cleb [of 2*m+1 n] a_ge_0 [of 2*m+1 n] b_le_1 [of 2*m+1 n]
using left_right_m [of c((2*m + 1) / 2^n) a((2*m + 1) / 2^n) b((2*m + 1) / 2^n)]
by (auto intro: f_eqI [OF _ order_refl])
show ?thesis
proof (intro conjI impI notI allI)
have False if b (real j / 2 ^ Suc n) ≤ a (real j / 2 ^ Suc n)
proof −
have f (c ((1 + real m * 2) / 2 ^ n)) = f (b ((1 + real m * 2) / 2

```
^ n))
        using k m ‹0 < n› fright that a43 [of n m] b43 [of n m]
         using alec [of 2*m+1 n] cleb [of 2*m+1 n] a_ge_0 [of 2*m+1 n]
b_le_1 [of 2*m+1 n]
       using left_right [of c((2*m + 1) / 2^n) a((2*m + 1) / 2^n) b((2*m
+ 1) / 2^n)]
        by (auto simp: algebra_simps)
       moreover have a (real (1 + m * 2) / 2 ^ n) ≤ c (real (1 + m *
2) / 2 ^ n)
        using alec by blast
       moreover have c (real (1 + m * 2) / 2 ^ n) < b (real (1 + m *
2) / 2 ^ n)
        using Suc.IH [of 1 + m * 2] by (simp add: c_def midpoint_def)
      ultimately show ?thesis
        using Suc.IH [of 1 + m * 2] by force
     qed
     then show a (real j / 2 ^ Suc n) < b (real j / 2 ^ Suc n) by force
    next
     fix x
     assume a (real j / 2 ^ Suc n) < x x ≤ b (real j / 2 ^ Suc n) f x = f
(a (real j / 2 ^ Suc n))
     then show False
      using k m ‹0 < n› a43 [of n m] b43 [of n m] fright right_neq
        using alec [of 2*m+1 n] cleb [of 2*m+1 n] a_ge_0 [of 2*m+1 n]
b_le_1 [of 2*m+1 n]
      by (auto simp: algebra_simps)
    next
     fix x
     assume a (real j / 2 ^ Suc n) ≤ x x < b (real j / 2 ^ Suc n) f x = f
(b (real j / 2 ^ Suc n))
     then show False
      using Suc.IH [of 1 + m * 2, THEN conjunct2, THEN conjunct2]
      using k m ‹0 < n› a43 [of n m] b43 [of n m]
        using alec [of 2*m+1 n] cleb [of 2*m+1 n] a_ge_0 [of 2*m+1 n]
b_le_1 [of 2*m+1 n]
         using left_right_m [of c((2*m + 1) / 2^n) a((2*m + 1) / 2^n)
b((2*m + 1) / 2^n)]
      by (auto simp: algebra_simps fright simp del: power_Suc)
    qed
   qed
  qed
 qed
qed
have c_gt_0 [simp]: 0 < c(m / 2^n) and c_less_1 [simp]: c(m / 2^n) < 1 for
m::nat and n
  using a_less_b [of m n] apply (simp_all add: c_def midpoint_def)
  using a_ge_0 [of m n] b_le_1 [of m n] by linarith+
have approx: ∃j n. odd j ∧ n ≠ 0 ∧
          real i / 2^m ≤ real j / 2^n ∧
```

```
                   real j / 2^n ≤ real k / 2^p ∧
                   |real i / 2 ^ m − real j / 2 ^ n| < 1/2^n ∧
                   |real k / 2 ^ p − real j / 2 ^ n| < 1/2^n
    if 0 < i i < 2 ^ m 0 < k k < 2 ^ p i / 2^m < k / 2^p m + p = N for N m
p i k
    using that
  proof (induction N arbitrary: m p i k rule: less_induct)
    case (less N)
    then consider i / 2^m ≤ 1/2 1/2 ≤ k / 2^p | k / 2^p < 1/2 | k / 2^p ≥
1/2 1/2 < i / 2^m
      by linarith
    then show ?case
    proof cases
      case 1
      with less.prems show ?thesis
        by (rule_tac x=1 in exI)+ (fastforce simp: field_split_simps)
    next
      case 2 show ?thesis
      proof (cases m)
        case 0 with less.prems show ?thesis
          by auto
      next
        case (Suc m') show ?thesis
        proof (cases p)
          case 0 with less.prems show ?thesis by auto
        next
          case (Suc p')
          have §: False if real i * 2 ^ p' < real k * 2 ^ m' k < 2 ^ p' 2 ^ m' ≤ i
          proof −
            have real k * 2 ^ m' < 2 ^ p' * 2 ^ m'
              using that by simp
            then have real i * 2 ^ p' < 2 ^ p' * 2 ^ m'
              using that by linarith
            with that show ?thesis by simp
          qed
          moreover have *: real i / 2 ^ m' < real k / 2^p'  k < 2 ^ p'
          using less.prems ‹m = Suc m'› 2 Suc by (force simp: field_split_simps)+
          moreover have i < 2 ^ m'
             using § * by (clarsimp simp: divide_simps linorder_not_le) (meson
linorder_not_le)
          ultimately show ?thesis
            using less.IH [of m'+p' i m' k p'] less.prems ‹m = Suc m'› 2 Suc
            by (force simp: field_split_simps)
        qed
      qed
    next
      case 3 show ?thesis
      proof (cases m)
        case 0 with less.prems show ?thesis
```

```
          by auto
        next
          case (Suc m′) show ?thesis
          proof (cases p)
            case 0 with less.prems show ?thesis by auto
          next
            case (Suc p′)
            have real (i − 2 ^ m′) / 2 ^ m′ < real (k − 2 ^ p′) / 2 ^ p′
                using less.prems ‹m = Suc m′› Suc 3  by (auto simp: field_simps
of_nat_diff)
            moreover have k − 2 ^ p′ < 2 ^ p′ i − 2 ^ m′ < 2 ^ m′
              using less.prems Suc ‹m = Suc m′› by auto
            moreover
            have 2 ^ p′ ≤ k 2 ^ p′ ≠ k
              using less.prems ‹m = Suc m′› Suc 3 by auto
            then have 2 ^ p′ < k
              by linarith
            ultimately show ?thesis
              using less.IH [of m′+p′ i − 2^m′ m′ k − 2 ^ p′ p′] less.prems ‹m =
Suc m′› Suc 3
              apply (clarsimp simp: field_simps of_nat_diff)
              apply (rule_tac x=2 ^ n + j in exI, simp)
              apply (rule_tac x=Suc n in exI)
              apply (auto simp: field_simps)
              done
          qed
        qed
      qed
    qed
    have clec: c(real i / 2^m) ≤ c(real j / 2^n)
      if i: 0 < i i < 2 ^ m and j: 0 < j j < 2 ^ n and ij: i / 2^m < j / 2^n for
m i n j
    proof −
      obtain j′ n′ where odd j′ n′ ≠ 0
        and i_le_j: real i / 2 ^ m ≤ real j′ / 2 ^ n′
        and j_le_j: real j′ / 2 ^ n′ ≤ real j / 2 ^ n
        and clo_ij: |real i / 2 ^ m − real j′ / 2 ^ n′| < 1/2 ^ n′
        and clo_jj: |real j / 2 ^ n − real j′ / 2 ^ n′| < 1/2 ^ n′
        using approx [of i m j n m+n] that i j ij by auto
      with oddE obtain q where q: j′ = Suc (2∗q) by fastforce
      have c (real i / 2 ^ m) ≤ c((2∗q + 1) / 2^n′)
      proof (cases i / 2^m = (2∗q + 1) / 2^n′)
        case True then show ?thesis by simp
      next
        case False
        with i_le_j clo_ij q have |real i / 2 ^ m − real (4 ∗ q + 1) / 2 ^ Suc n′| <
1 / 2 ^ Suc n′
          by (auto simp: field_split_simps)
        then have c(i / 2^m) ≤ b(real(4 ∗ q + 1) / 2 ^ (Suc n′))
```

by (*meson ci_le_bj even_mult_iff even_numeral even_plus_one_iff*)
then show *?thesis*
using *alec* [*of 2∗q+1 n′*] *cleb* [*of 2∗q+1 n′*] *a_ge_0* [*of 2∗q+1 n′*] *b_le_1* [*of 2∗q+1 n′*] *b41* [*of n′ q*] ‹*n′ ≠ 0*›
using *left_right_m* [*of c*((*2∗q* + *1*) / *2^n′*) *a*((*2∗q* + *1*) / *2^n′*) *b*((*2∗q* + *1*) / *2^n′*)]
by (*auto simp*: *algebra_simps*)
qed
also have ... ≤ *c*(*real j* / *2^n*)
proof (*cases j* / *2^n* = (*2∗q* + *1*) / *2^n′*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
with *j_le_j q* **have** *less*: (*2∗q* + *1*) / *2^n′* < *j* / *2^n*
by *auto*
have ∗: ⟦*q* < *i*; *abs*(*i* − *q*) < *s∗2*; *r* = *q* + *s*⟧ ⟹ *abs*(*i* − *r*) < *s* **for** *i q s r*::*real*
by *auto*
have |*real j* / *2* ^ *n* − *real* (*4* ∗ *q* + *3*) / *2* ^ *Suc n′*| < *1* / *2* ^ *Suc n′*
by (*rule* ∗ [*OF less*]) (*use j_le_j clo_jj q* **in** ‹*auto simp*: *field_split_simps*›)
then have *a*(*real*(*4∗q* + *3*) / *2* ^ (*Suc n′*)) ≤ *c*(*j* / *2^n*)
by (*metis Suc3_eq_add_3 add.commute aj_le_ci even_Suc even_mult_iff even_numeral*)
then show *?thesis*
using *alec* [*of 2∗q+1 n′*] *cleb* [*of 2∗q+1 n′*] *a_ge_0* [*of 2∗q+1 n′*] *b_le_1* [*of 2∗q+1 n′*] *a43* [*of n′ q*] ‹*n′ ≠ 0*›
using *left_right_m* [*of c*((*2∗q* + *1*) / *2^n′*) *a*((*2∗q* + *1*) / *2^n′*) *b*((*2∗q* + *1*) / *2^n′*)]
by (*auto simp*: *algebra_simps*)
qed
finally show *?thesis* **.**
qed
have *x* = *y* **if** *0* ≤ *x x* ≤ *1 0* ≤ *y y* ≤ *1 h x* = *h y* **for** *x y*
using *that*
proof (*induction x y rule*: *linorder_class.linorder_less_wlog*)
case (*less x1 x2*)
obtain *m n* **where** *m*: *0* < *m m* < *2* ^ *n*
and *x12*: *x1* < *m* / *2^n m* / *2^n* < *x2*
and *neq*: *h x1* ≠ *h* (*real m* / *2^n*)
proof −
have (*x1* + *x2*) / *2* ∈ *closure D01*
using *cloD01 less.hyps less.prems* **by** *auto*
with *less* **obtain** *y* **where** *y* ∈ *D01* **and** *dist_y*: *dist y* ((*x1* + *x2*) / *2*) < (*x2* − *x1*) / *64*
unfolding *closure_approachable*
by (*metis diff_gt_0_iff_gt less_divide_eq_numeral1*(*1*) *mult_zero_left*)
obtain *m n* **where** *m*: *0* < *m m* < *2* ^ *n*
and *clo*: |*real m* / *2* ^ *n* − (*x1* + *x2*) / *2*| < (*x2* − *x1*) / *64*

```
            and n: 1/2^n < (x2 − x1) / 128
      proof −
        have min 1 ((x2 − x1) / 128) > 0 1/2 < (1::real)
          using less by auto
        then obtain N where N: 1/2^N < min 1 ((x2 − x1) / 128)
          by (metis power_one_over real_arch_pow_inv)
        then have N > 0
          using less_divide_eq_1 by force
        obtain p q where p: p < 2 ^ q p ≠ 0 and yeq: y = real p / 2 ^ q
          using ‹y ∈ D01› by (auto simp: zero_less_divide_iff D01_def)
        show ?thesis
        proof
          show 0 < 2^N * p
            using p by auto
          show 2 ^ N * p < 2 ^ (N+q)
            by (simp add: p power_add)
          have |real (2 ^ N * p) / 2 ^ (N + q) − (x1 + x2) / 2| = |real p / 2 ^
q − (x1 + x2) / 2|
            by (simp add: power_add)
          also have ... = |y − (x1 + x2) / 2|
            by (simp add: yeq)
          also have ... < (x2 − x1) / 64
            using dist_y by (simp add: dist_norm)
          finally show |real (2 ^ N * p) / 2 ^ (N + q) − (x1 + x2) / 2| < (x2
− x1) / 64 .
          have (1::real) / 2 ^ (N + q) ≤ 1/2^N
            by (simp add: field_simps)
          also have ... < (x2 − x1) / 128
            using N by force
          finally show 1/2 ^ (N + q) < (x2 − x1) / 128 .
        qed
      qed
     obtain m′ n′ m′′ n′′ where 0 < m′ m′ < 2 ^ n′ x1 < m′ / 2^n′ m′ / 2^n′
< x2
        and 0 < m′′ m′′ < 2 ^ n′′ x1 < m′′ / 2^n′′ m′′ / 2^n′′ < x2
        and neq: h (real m′′ / 2^n′′) ≠ h (real m′ / 2^n′)
      proof
        show 0 < Suc (2*m)
          by simp
        show m21: Suc (2*m) < 2 ^ Suc n
          using m by auto
        show x1 < real (Suc (2 * m)) / 2 ^ Suc n
          using clo by (simp add: field_simps abs_if split: if_split_asm)
        show real (Suc (2 * m)) / 2 ^ Suc n < x2
          using n clo by (simp add: field_simps abs_if split: if_split_asm)
        show 0 < 4*m + 3
          by simp
        have m+1 ≤ 2 ^ n
          using m by simp
```

```
          then have 4 ∗ (m+1) ≤ 4 ∗ (2 ˆ n)
            by simp
          then show m43: 4∗m + 3 < 2 ˆ (n+2)
            by (simp add: algebra_simps)
          show x1 < real (4 ∗ m + 3) / 2 ˆ (n + 2)
            using clo by (simp add: field_simps abs_if split: if_split_asm)
          show real (4 ∗ m + 3) / 2 ˆ (n + 2) < x2
            using n clo by (simp add: field_simps abs_if split: if_split_asm)
          have c_fold: midpoint (a ((2 ∗ real m + 1) / 2 ˆ Suc n)) (b ((2 ∗ real m
+ 1) / 2 ˆ Suc n)) = c ((2 ∗ real m + 1) / 2 ˆ Suc n)
            by (simp add: c_def)
          define R where R ≡ rightcut (a ((2 ∗ real m + 1) / 2 ˆ Suc n)) (b ((2
∗ real m + 1) / 2 ˆ Suc n)) (c ((2 ∗ real m + 1) / 2 ˆ Suc n))
          have R < b ((2 ∗ real m + 1) / 2 ˆ Suc n)
             unfolding R_def using a_less_b [of 4∗m + 3 n+2] a43 [of Suc n m]
b43 [of Suc n m]
            by simp
          then have Rless: R < midpoint R (b ((2 ∗ real m + 1) / 2 ˆ Suc n))
            by (simp add: midpoint_def)
          have midR_le: midpoint R (b ((2 ∗ real m + 1) / 2 ˆ Suc n)) ≤ b ((2 ∗
real m + 1) / (2 ∗ 2 ˆ n))
            using ‹R < b ((2 ∗ real m + 1) / 2 ˆ Suc n)›
            by (simp add: midpoint_def)
          have (real (Suc (2 ∗ m)) / 2 ˆ Suc n) ∈ D01 real (4 ∗ m + 3) / 2 ˆ (n
+ 2) ∈ D01
           by (simp_all add: D01_def m21 m43 del: power_Suc of_nat_Suc of_nat_add
add_2_eq_Suc') blast+
          then show h (real (4 ∗ m + 3) / 2 ˆ (n + 2)) ≠ h (real (Suc (2 ∗ m))
/ 2 ˆ Suc n)
            using a_less_b [of 4∗m + 3 n+2, THEN conjunct1]
            using a43 [of Suc n m] b43 [of Suc n m]
            using alec [of 2∗m+1 Suc n] cleb [of 2∗m+1 Suc n] a_ge_0 [of 2∗m+1
Suc n] b_le_1 [of 2∗m+1 Suc n]
            apply (simp add: fc_eq [symmetric] c_def del: power_Suc)
            apply (simp only: add.commute [of 1] c_fold R_def [symmetric])
            apply (rule right_neq)
            using Rless apply (simp add: R_def)
              apply (rule midR_le, auto)
            done
        qed
        then show ?thesis by (metis that)
      qed
      have m_div: 0 < m / 2ˆn m / 2ˆn < 1
        using m by (auto simp: field_split_simps)
      have closure0m: {0..m / 2ˆn} = closure ({0<..< m / 2ˆn} ∩ (⋃k m. {real
m / 2 ˆ k}))
        by (subst closure_dyadic_rationals_in_convex_set_pos_1, simp_all add: not_le
m)
      have 2ˆn > m
```

by (*simp add*: *m*(*2*) *not_le*)
then have *closurem1*: {*m* / *2*ˆ*n* .. *1*} = *closure* ({*m* / *2*ˆ*n* <..< *1*} ∩ (⋃ *k* *m*. {*real m* / *2* ˆ *k*}))
using *closure_dyadic_rationals_in_convex_set_pos_1 m_div*(*1*) **by** *fastforce*
have *cont_h'*: *continuous_on* (*closure* ({*u*<..<*v*} ∩ (⋃ *k m*. {*real m* / *2* ˆ *k*}))) *h*
if *0* ≤ *u v* ≤ *1* **for** *u v*
using *that* **by** (*intro continuous_on_subset* [*OF cont_h*] *closure_minimal* [*OF subsetI*]) *auto*
have *closed_f'*: *closed* (*f* ‘ {*u*..*v*}) **if** *0* ≤ *u v* ≤ *1* **for** *u v*
by (*metis compact_continuous_image cont_f compact_interval atLeastatMost_subset_iff*
compact_imp_closed continuous_on_subset that)
have *less_2I*: ⋀*k i*. *real i* / *2* ˆ *k* < *1* ⟹ *i* < *2* ˆ *k*
by *simp*
have *h* ‘ ({*0*<..<*m* / *2* ˆ *n*} ∩ (⋃ *q p*. {*real p* / *2* ˆ *q*})) ⊆ *f* ‘ {*0*..*c* (*m* / *2* ˆ *n*)}
proof *clarsimp*
fix *p q*
assume *p*: *0* < *real p* / *2* ˆ *q real p* / *2* ˆ *q* < *real m* / *2* ˆ *n*
then have [*simp*]: *0* < *p*
by (*simp add*: *field_split_simps*)
have [*simp*]: *p* < *2* ˆ *q*
by (*blast intro*: *p less_2I m_div less_trans*)
have *f* (*c* (*real p* / *2* ˆ *q*)) ∈ *f* ‘ {*0*..*c* (*real m* / *2* ˆ *n*)}
by (*auto simp*: *clec p m*)
then show *h* (*real p* / *2* ˆ *q*) ∈ *f* ‘ {*0*..*c* (*real m* / *2* ˆ *n*)}
by (*simp add*: *h_eq*)
qed
with *m_div* **have** *h* ‘ {*0* .. *m* / *2*ˆ*n*} ⊆ *f* ‘ {*0* .. *c*(*m* / *2*ˆ*n*)}
apply (*subst closure0m*)
by (*rule image_closure_subset* [*OF cont_h' closed_f'*]) *auto*
then have *hx1*: *h x1* ∈ *f* ‘ {*0* .. *c*(*m* / *2*ˆ*n*)}
using *x12 less.prems*(*1*) **by** *auto*
then obtain *t1* **where** *t1*: *h x1* = *f t1 0* ≤ *t1 t1* ≤ *c* (*m* / *2* ˆ *n*)
by *auto*
have *h* ‘ ({*m* / *2* ˆ *n*<..<*1*} ∩ (⋃ *q p*. {*real p* / *2* ˆ *q*})) ⊆ *f* ‘ {*c* (*m* / *2* ˆ *n*)..*1*}
proof *clarsimp*
fix *p q*
assume *p*: *real m* / *2* ˆ *n* < *real p* / *2* ˆ *q* **and** [*simp*]: *p* < *2* ˆ *q*
then have [*simp*]: *0* < *p*
using *gr_zeroI m_div* **by** *fastforce*
have *f* (*c* (*real p* / *2* ˆ *q*)) ∈ *f* ‘ {*c* (*m* / *2* ˆ *n*)..*1*}
by (*auto simp*: *clec p m*)
then show *h* (*real p* / *2* ˆ *q*) ∈ *f* ‘ {*c* (*real m* / *2* ˆ *n*)..*1*}
by (*simp add*: *h_eq*)
qed
with *m* **have** *h* ‘ {*m* / *2*ˆ*n* .. *1*} ⊆ *f* ‘ {*c*(*m* / *2*ˆ*n*) .. *1*}

```
        apply (subst closurem1)
        by (rule image_closure_subset [OF cont_h' closed_f']) auto
      then have hx2: h x2 ∈ f ‘ {c(m / 2^n)..1}
        using x12 less.prems by auto
      then obtain t2 where t2: h x2 = f t2 c (m / 2 ^ n) ≤ t2 t2 ≤ 1
        by auto
      with t1 less neq have False
      using conn [of h x2, unfolded is_interval_connected_1 [symmetric] is_interval_1,
rule_format, of t1 t2 c(m / 2^n)]
        by (simp add: h_eq m)
      then show ?case by blast
    qed auto
    then show ?thesis
      by (auto simp: inj_on_def)
  qed
  ultimately have {0..1::real} homeomorphic f ‘ {0..1}
    using homeomorphic_compact [OF _ cont_h] by blast
  then show ?thesis
    using homeomorphic_sym by blast
qed


theorem path_contains_arc:
  fixes p :: real ⇒ 'a::{complete_space,real_normed_vector}
  assumes path p and a: pathstart p = a and b: pathfinish p = b and a ≠ b
  obtains q where arc q path_image q ⊆ path_image p pathstart q = a pathfinish
q = b
proof −
  have ucont_p: uniformly_continuous_on {0..1} p
    using ‹path p› unfolding path_def
    by (metis compact_Icc compact_uniformly_continuous)
  define φ where φ ≡ λS. S ⊆ {0..1} ∧ 0 ∈ S ∧ 1 ∈ S ∧
                        (∀ x ∈ S. ∀ y ∈ S. open_segment x y ∩ S = {} ⟶ p x = p y)
  obtain T where closed T φ T and T: ⋀U. ⟦closed U; φ U⟧ ⟹ ¬ (U ⊂ T)
  proof (rule Brouwer_reduction_theorem_gen [of {0..1} φ])
    have *: {x<..<y} ∩ {0..1} = {x<..<y} if 0 ≤ x y ≤ 1 x ≤ y for x y::real
      using that by auto
    show φ {0..1}
      by (auto simp: φ_def open_segment_eq_real_ivl *)
    show φ (⋂(F ‘ UNIV))
     if ⋀n. closed (F n) and φ: ⋀n. φ (F n) and Fsub: ⋀n. F (Suc n) ⊆ F n
for F
    proof −
      have F01: ⋀n. F n ⊆ {0..1} ∧ 0 ∈ F n ∧ 1 ∈ F n
        and peq: ⋀n x y. ⟦x ∈ F n; y ∈ F n; open_segment x y ∩ F n = {}⟧ ⟹ p
x = p y
        by (metis φ φ_def)+
      have pqF: False if ∀ u. x ∈ F u ∀ x. y ∈ F x open_segment x y ∩ (⋂x. F x)
= {} and neg: p x ≠ p y
```

for *x y*
using *that*
proof (*induction x y rule*: *linorder_class.linorder_less_wlog*)
case (*less x y*)
have *xy*: $x \in \{0..1\}$ $y \in \{0..1\}$
by (*metis less.prems subsetCE F01*)+
have $norm(p\ x - p\ y) / 2 > 0$
using *less* **by** *auto*
then obtain *e* **where** $e > 0$
and *e*: $\bigwedge u\ v.\ [\![u \in \{0..1\};\ v \in \{0..1\};\ dist\ v\ u < e]\!] \Longrightarrow dist\ (p\ v)\ (p\ u) < norm(p\ x - p\ y) / 2$
by (*metis uniformly_continuous_onE* [*OF ucont_p*])
have *minxy*: $min\ e\ (y - x) < (y - x) * (3 / 2)$
by (*subst min_less_iff_disj*) (*simp add*: *less*)
define *w* **where** $w \equiv x + (min\ e\ (y - x) / 3)$
define *z* **where** $z \equiv y - (min\ e\ (y - x) / 3)$
have $w < z$ **and** *w*: $w \in \{x<..<y\}$ **and** *z*: $z \in \{x<..<y\}$
and *wxe*: $norm(w - x) < e$ **and** *zye*: $norm(z - y) < e$
using *minxy* ‹$0 < e$› *less* **unfolding** *w_def z_def* **by** *auto*
have *Fclo*: $\bigwedge T.\ T \in range\ F \Longrightarrow closed\ T$
by (*metis* ‹$\bigwedge n.\ closed\ (F\ n)$› *image_iff*)
have *eq*: $\{w..z\} \cap \bigcap(F\ `\ UNIV) = \{\}$
using *less w z* **by** (*simp add*: *open_segment_eq_real_ivl disjoint_iff*)
then obtain *K* **where** *finite K* **and** *K*: $\{w..z\} \cap (\bigcap\ (F\ `\ K)) = \{\}$
by (*metis finite_subset_image compact_imp_fip* [*OF compact_interval Fclo*])
then have $K \neq \{\}$
using ‹$w < z$› ‹$\{w..z\} \cap \bigcap(F\ `\ K) = \{\}$› **by** *auto*
define *n* **where** $n \equiv Max\ K$
have $n \in K$ **unfolding** *n_def* **by** (*metis* ‹$K \neq \{\}$› ‹*finite K*› *Max_in*)
have $F\ n \subseteq \bigcap\ (F\ `\ K)$
unfolding *n_def* **by** (*metis Fsub Max_ge* ‹$K \neq \{\}$› ‹*finite K*› *cINF_greatest lift_Suc_antimono_le*)
with *K* **have** *wzF_null*: $\{w..z\} \cap F\ n = \{\}$
by (*metis disjoint_iff_not_equal subset_eq*)
obtain *u* **where** *u*: $u \in F\ n$ $u \in \{x..w\}$ $(\{u..w\} - \{u\}) \cap F\ n = \{\}$
proof (*cases* $w \in F\ n$)
case *True*
then show *?thesis*
by (*metis wzF_null* ‹$w < z$› *atLeastAtMost_iff disjoint_iff_not_equal less_eq_real_def*)
next
case *False*
obtain *u* **where** $u \in F\ n$ $u \in \{x..w\}$ $\{u<..<w\} \cap F\ n = \{\}$
proof (*rule segment_to_point_exists* [*of* $F\ n \cap \{x..w\}$ *w*])
show *closed* $(F\ n \cap \{x..w\})$
by (*metis* ‹$\bigwedge n.\ closed\ (F\ n)$› *closed_Int closed_real_atLeastAtMost*)
show $F\ n \cap \{x..w\} \neq \{\}$
by (*metis atLeastAtMost_iff disjoint_iff_not_equal greaterThanLessThan_iff less.prems*(*1*) *less_eq_real_def w*)

```
        qed (auto simp: open_segment_eq_real_ivl intro!: that)
        with False show thesis
          by (auto simp add: disjoint_iff less_eq_real_def intro!: that)
      qed
      obtain v where v: v ∈ F n v ∈ {z..y} ({z..v} − {v}) ∩ F n = {}
      proof (cases z ∈ F n)
        case True
        have z ∈ {w..z}
          using ‹w < z› by auto
        then show ?thesis
          by (metis wzF_null Int_iff True empty_iff)
      next
        case False
        show ?thesis
        proof (rule segment_to_point_exists [of F n ∩ {z..y} z])
          show closed (F n ∩ {z..y})
            by (metis ‹⋀n. closed (F n)› closed_Int closed_atLeastAtMost)
          show F n ∩ {z..y} ≠ {}
         by (metis atLeastAtMost_iff disjoint_iff_not_equal greaterThanLessThan_iff
less.prems(2) less_eq_real_def z)
          show ⋀b. ⟦b ∈ F n ∩ {z..y}; open_segment z b ∩ (F n ∩ {z..y}) = {}⟧
⟹ thesis
          proof
           show ⋀b. ⟦b ∈ F n ∩ {z..y}; open_segment z b ∩ (F n ∩ {z..y}) = {}⟧
⟹ ({z..b} − {b}) ∩ F n = {}
              using False by (auto simp: open_segment_eq_real_ivl less_eq_real_def)
          qed auto
        qed
      qed
      obtain u v where u ∈ {0..1} v ∈ {0..1} norm(u − x) < e norm(v − y)
< e p u = p v
      proof
        show u ∈ {0..1} v ∈ {0..1}
          by (metis F01 ‹u ∈ F n› ‹v ∈ F n› subsetD)+
        show norm(u − x) < e norm (v − y) < e
          using ‹u ∈ {x..w}› ‹v ∈ {z..y}› atLeastAtMost_iff real_norm_def wxe zye
by auto
        show p u = p v
        proof (rule peq)
          show u ∈ F n v ∈ F n
            by (auto simp: u v)
          have False if ξ ∈ F n u < ξ ξ < v for ξ
          proof −
            have ξ ∉ {z..v}
              by (metis DiffI disjoint_iff_not_equal less_irrefl singletonD that(1,3)
v(3))
            moreover have ξ ∉ {w..z} ∩ F n
              by (metis equals0D wzF_null)
            ultimately have ξ ∈ {u..w}
```

using *that* **by** *auto*
then show *?thesis*
by (*metis DiffI disjoint_iff_not_equal less_eq_real_def not_le singletonD that*(*1*,*2*) *u*(*3*))
qed
moreover
have ⟦ξ ∈ F n; v < ξ; ξ < u⟧ ⟹ *False* **for** ξ
using ‹u ∈ {x..w}› ‹v ∈ {z..y}› ‹w < z› **by** *simp*
ultimately
show *open_segment u v* ∩ *F n* = {}
by (*force simp*: *open_segment_eq_real_ivl*)
qed
qed
then show *?case*
using *e* [*of x u*] *e* [*of y v*] *xy*
by (*metis dist_norm dist_triangle_half_r order_less_irrefl*)
qed (*auto simp*: *open_segment_commute*)
show *?thesis*
unfolding *φ_def* **by** (*metis* (*no_types*, *hide_lams*) *INT_I Inf_lower2 rangeI that*(*3*) *F01 subsetCE pqF*)
qed
show *closed* {*0*..*1*::*real*} **by** *auto*
qed (*meson φ_def*)
then have *T* ⊆ {*0*..*1*} *0* ∈ *T 1* ∈ *T*
and *peq*: ⋀*x y*. ⟦*x* ∈ *T*; *y* ∈ *T*; *open_segment x y* ∩ *T* = {}⟧ ⟹ *p x* = *p y*
unfolding *φ_def* **by** *metis+*
then have *T* ≠ {} **by** *auto*
define *h* **where** *h* ≡ λ*x*. *p*(*SOME y*. *y* ∈ *T* ∧ *open_segment x y* ∩ *T* = {})
have *p y* = *p z* **if** *y* ∈ *T z* ∈ *T* **and** *xyT*: *open_segment x y* ∩ *T* = {} **and** *xzT*: *open_segment x z* ∩ *T* = {}
for *x y z*
proof (*cases x* ∈ *T*)
case *True*
with *that* **show** *?thesis* **by** (*metis* ‹φ *T*› *φ_def*)
next
case *False*
have *insert x* (*open_segment x y* ∪ *open_segment x z*) ∩ *T* = {}
by (*metis False Int_Un_distrib2 Int_insert_left Un_empty_right xyT xzT*)
moreover have *open_segment y z* ∩ *T* ⊆ *insert x* (*open_segment x y* ∪ *open_segment x z*) ∩ *T*
by (*auto simp*: *open_segment_eq_real_ivl*)
ultimately have *open_segment y z* ∩ *T* = {}
by *blast*
with *that peq* **show** *?thesis* **by** *metis*
qed
then have *h_eq_p_gen*: *h x* = *p y* **if** *y* ∈ *T open_segment x y* ∩ *T* = {} **for** *x y*
using *that* **unfolding** *h_def*
by (*metis* (*mono_tags*, *lifting*) *some_eq_ex*)
then have *h_eq_p*: ⋀*x*. *x* ∈ *T* ⟹ *h x* = *p x*

```
    by simp
  have disjoint: ⋀x. ∃y. y ∈ T ∧ open_segment x y ∩ T = {}
    by (meson ‹T ≠ {}› ‹closed T› segment_to_point_exists)
  have heq: h x = h x′ if open_segment x x′ ∩ T = {} for x x′
  proof (cases x ∈ T ∨ x′ ∈ T)
    case True
    then show ?thesis
      by (metis h_eq_p h_eq_p_gen open_segment_commute that)
  next
    case False
    obtain y y′ where y ∈ T open_segment x y ∩ T = {} h x = p y
      y′ ∈ T open_segment x′ y′ ∩ T = {} h x′ = p y′
      by (meson disjoint h_eq_p_gen)
    moreover have open_segment y y′ ⊆ (insert x (insert x′ (open_segment x y ∪
open_segment x′ y′ ∪ open_segment x x′)))
      by (auto simp: open_segment_eq_real_ivl)
    ultimately show ?thesis
      using False that by (fastforce simp add: h_eq_p intro!: peq)
  qed
  have h ‘ {0..1} homeomorphic {0..1::real}
  proof (rule homeomorphic_monotone_image_interval)
    show continuous_on {0..1} h
    proof (clarsimp simp add: continuous_on_iff)
      fix u ε::real
      assume 0 < ε 0 ≤ u u ≤ 1
      then obtain δ where δ > 0 and δ: ⋀v. v ∈ {0..1} ⟹ dist v u < δ ⟶
dist (p v) (p u) < ε / 2
        using ucont_p [unfolded uniformly_continuous_on_def]
        by (metis atLeastAtMost_iff half_gt_zero_iff)
      then have dist (h v) (h u) < ε if v ∈ {0..1} dist v u < δ for v
      proof (cases open_segment u v ∩ T = {})
        case True
        then show ?thesis
          using ‹0 < ε› heq by auto
      next
        case False
        have uvT: closed (closed_segment u v ∩ T) closed_segment u v ∩ T ≠ {}
          using False open_closed_segment by (auto simp: ‹closed T› closed_Int)
        obtain w where w ∈ T and w: w ∈ closed_segment u v open_segment u w
∩ T = {}
        proof (rule segment_to_point_exists [OF uvT])
          fix b
          assume b ∈ closed_segment u v ∩ T open_segment u b ∩ (closed_segment
u v ∩ T) = {}
          then show thesis
              by (metis IntD1 IntD2 ends_in_segment(1) inf.orderE inf_assoc sub-
set_oc_segment that)
        qed
        then have puw: dist (p u) (p w) < ε / 2
```

by (*metis* (*no_types*) ‹$T \subseteq \{0..1\}$› ‹*dist v u* < δ› δ *dist_commute dist_in_closed_segment le_less_trans subsetCE*)
obtain *z* **where** *z* ∈ *T* **and** *z*: *z* ∈ *closed_segment u v open_segment v z* ∩ *T* = {}
proof (*rule segment_to_point_exists* [*OF uvT*])
fix *b*
assume *b* ∈ *closed_segment u v* ∩ *T open_segment v b* ∩ (*closed_segment u v* ∩ *T*) = {}
then show *thesis*
by (*metis IntD1 IntD2 ends_in_segment*(*2*) *inf.orderE inf_assoc subset_oc_segment that*)
qed
then have *dist* (*p u*) (*p z*) < ε / *2*
by (*metis* ‹$T \subseteq \{0..1\}$› ‹*dist v u* < δ› δ *dist_commute dist_in_closed_segment le_less_trans subsetCE*)
then show *?thesis*
using *puw* **by** (*metis* (*no_types*) ‹*w* ∈ *T*› ‹*z* ∈ *T*› *dist_commute dist_triangle_half_l h_eq_p_gen w*(*2*) *z*(*2*))
qed
with ‹*0* < δ› **show** ∃δ>*0*. ∀*v*∈{*0..1*}. *dist v u* < δ ⟶ *dist* (*h v*) (*h u*) < ε **by** *blast*
qed
show *connected* ({*0..1*} ∩ *h* −‘ {*z*}) **for** *z*
proof (*clarsimp simp add*: *connected_iff_connected_component*)
fix *u v*
assume *huv_eq*: *h v* = *h u* **and** *uv*: *0* ≤ *u u* ≤ *1 0* ≤ *v v* ≤ *1*
have ∃ *T*. *connected T* ∧ *T* ⊆ {*0..1*} ∧ *T* ⊆ *h* −‘ {*h u*} ∧ *u* ∈ *T* ∧ *v* ∈ *T*
proof (*intro exI conjI*)
show *connected* (*closed_segment u v*)
by *simp*
show *closed_segment u v* ⊆ {*0..1*}
by (*simp add*: *uv closed_segment_eq_real_ivl*)
have *pxy*: *p x* = *p y*
if *T* ⊆ {*0..1*} *0* ∈ *T 1* ∈ *T x* ∈ *T y* ∈ *T*
and *disjT*: *open_segment x y* ∩ (*T* − *open_segment u v*) = {}
and *xynot*: *x* ∉ *open_segment u v y* ∉ *open_segment u v*
for *x y*
proof (*cases open_segment x y* ∩ *open_segment u v* = {})
case *True*
then show *?thesis*
by (*metis Diff_Int_distrib Diff_empty peq disjT* ‹*x* ∈ *T*› ‹*y* ∈ *T*›)
next
case *False*
then have *open_segment x u* ∪ *open_segment y v* ⊆ *open_segment x y* − *open_segment u v* ∨
open_segment y u ∪ *open_segment x v* ⊆ *open_segment x y* − *open_segment u v* (**is** *?xuyv* ∨ *?yuxv*)
using *xynot* **by** (*fastforce simp add*: *open_segment_eq_real_ivl not_le not_less split*: *if_split_asm*)

```
          then show p x = p y
          proof
            assume ?xuyv
            then have open_segment x u ∩ T = {} open_segment y v ∩ T = {}
              using disjT by auto
            then have h x = h y
              using heq huv_eq by auto
            then show ?thesis
              using h_eq_p ‹x ∈ T› ‹y ∈ T› by auto
          next
            assume ?yuxv
            then have open_segment y u ∩ T = {} open_segment x v ∩ T = {}
              using disjT by auto
            then have h x = h y
              using heq [of y u]  heq [of x v] huv_eq by auto
            then show ?thesis
              using h_eq_p ‹x ∈ T› ‹y ∈ T› by auto
          qed
        qed
        have ¬ T − open_segment u v ⊂ T
        proof (rule T)
          show closed (T − open_segment u v)
            by (simp add: closed_Diff [OF ‹closed T›] open_segment_eq_real_ivl)
          have 0 ∉ open_segment u v 1 ∉ open_segment u v
            using open_segment_eq_real_ivl uv by auto
          then show φ (T − open_segment u v)
            using ‹T ⊆ {0..1}› ‹0 ∈ T› ‹1 ∈ T›
            by (auto simp: φ_def) (meson peq pxy)
        qed
        then have open_segment u v ∩ T = {}
          by blast
        then show closed_segment u v ⊆ h −‘ {h u}
         by (force intro: heq simp: open_segment_eq_real_ivl closed_segment_eq_real_ivl
split: if_split_asm)+
      qed auto
      then show connected_component ({0..1} ∩ h −‘ {h u}) u v
        by (simp add: connected_component_def)
    qed
    show h 1 ≠ h 0
      by (metis ‹φ T› φ_def a ‹a ≠ b› b h_eq_p pathfinish_def pathstart_def)
  qed
  then obtain f and g :: real ⇒ 'a
    where gfeq: (∀x∈h ‘ {0..1}. (g(f x) = x)) and fhim: f ‘ h ‘ {0..1} = {0..1}
and contf: continuous_on (h ‘ {0..1}) f
     and fgeq: (∀y∈{0..1}. (f(g y) = y)) and pag: path_image g = h ‘ {0..1} and
contg: continuous_on {0..1} g
    by (auto simp: homeomorphic_def homeomorphism_def path_image_def)
  then have arc g
    by (metis arc_def path_def inj_on_def)
```

obtain *u v* **where** *u* ∈ {*0..1*} *a* = *g u v* ∈ {*0..1*} *b* = *g v*
by (*metis* (*mono_tags*, *hide_lams*) ‹φ *T*› *φ_def a b fhim gfeq h_eq_p imageI path_image_def pathfinish_def pathfinish_in_path_image pathstart_def pathstart_in_path_image*)
then have *a* ∈ *path_image g b* ∈ *path_image g*
using *path_image_def* **by** *blast+*
have *ph*: *path_image h* ⊆ *path_image p*
by (*metis image_mono image_subset_iff path_image_def disjoint h_eq_p_gen* ‹*T* ⊆ {*0..1*}›)
show *?thesis*
proof
show *pathstart* (*subpath u v g*) = *a pathfinish* (*subpath u v g*) = *b*
by (*simp_all add*: ‹*a* = *g u*› ‹*b* = *g v*›)
show *path_image* (*subpath u v g*) ⊆ *path_image p*
by (*metis* ‹*u* ∈ {*0..1*}› ‹*v* ∈ {*0..1*}› *order_trans pag path_image_def path_image_subpath_subset ph*)
show *arc* (*subpath u v g*)
using ‹*arc g*› ‹*a* = *g u*› ‹*b* = *g v*› ‹*u* ∈ {*0..1*}› ‹*v* ∈ {*0..1*}› *arc_subpath_arc* ‹*a* ≠ *b*› **by** *blast*
qed
qed

corollary *path_connected_arcwise*:
fixes *S* :: *'a*::{*complete_space*,*real_normed_vector*} *set*
shows *path_connected S* ⟷
(∀ *x* ∈ *S*. ∀ *y* ∈ *S*. *x* ≠ *y* ⟶ (∃ *g*. *arc g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*))
(**is** *?lhs* = *?rhs*)
proof (*intro iffI impI ballI*)
fix *x y*
assume *path_connected S x* ∈ *S y* ∈ *S x* ≠ *y*
then obtain *p* **where** *p*: *path p path_image p* ⊆ *S pathstart p* = *x pathfinish p* = *y*
by (*force simp*: *path_connected_def*)
then show ∃ *g*. *arc g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*
by (*metis* ‹*x* ≠ *y*› *order_trans path_contains_arc*)
next
assume *R* [*rule_format*]: *?rhs*
show *?lhs*
unfolding *path_connected_def*
proof (*intro ballI*)
fix *x y*
assume *x* ∈ *S y* ∈ *S*
show ∃ *g*. *path g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*
proof (*cases x* = *y*)
case *True* **with** ‹*x* ∈ *S*› *path_component_def path_component_refl* **show** *?thesis*
by *blast*
next
case *False* **with** *R* [*OF* ‹*x* ∈ *S*› ‹*y* ∈ *S*›] **show** *?thesis*

```
      by (auto intro: arc_imp_path)
    qed
  qed
qed

corollary arc_connected_trans:
  fixes g :: real ⇒ 'a::{complete_space,real_normed_vector}
  assumes arc g arc h pathfinish g = pathstart h pathstart g ≠ pathfinish h
  obtains i where arc i path_image i ⊆ path_image g ∪ path_image h
                  pathstart i = pathstart g pathfinish i = pathfinish h
  by (metis (no_types, hide_lams) arc_imp_path assms path_contains_arc path_image_join
path_join pathfinish_join pathstart_join)
```

6.39.4 Accessibility of frontier points

```
lemma dense_accessible_frontier_points:
  fixes S :: 'a::{complete_space,real_normed_vector} set
  assumes open S and opeSV: openin (top_of_set (frontier S)) V and V ≠ {}
  obtains g where arc g g ` {0..<1} ⊆ S pathstart g ∈ S pathfinish g ∈ V
proof -
  obtain z where z ∈ V
    using ‹V ≠ {}› by auto
  then obtain r where r > 0 and r: ball z r ∩ frontier S ⊆ V
    by (metis openin_contains_ball opeSV)
  then have z ∈ frontier S
    using ‹z ∈ V› opeSV openin_contains_ball by blast
  then have z ∈ closure S z ∉ S
    by (simp_all add: frontier_def assms interior_open)
  with ‹r > 0› have infinite (S ∩ ball z r)
    by (auto simp: closure_def islimpt_eq_infinite_ball)
  then obtain y where y ∈ S and y: y ∈ ball z r
    using infinite_imp_nonempty by force
  then have y ∉ frontier S
    by (meson ‹open S› disjoint_iff_not_equal frontier_disjoint_eq)
  have y ≠ z
    using ‹y ∈ S› ‹z ∉ S› by blast
  have path_connected(ball z r)
    by (simp add: convex_imp_path_connected)
  with y ‹r > 0›  obtain g where arc g and pig: path_image g ⊆ ball z r
                               and g: pathstart g = y pathfinish g = z
    using ‹y ≠ z› by (force simp: path_connected_arcwise)
  have continuous_on {0..1} g
    using ‹arc g› arc_imp_path path_def by blast
  then have compact (g -` frontier S ∩ {0..1})
   by (simp add: bounded_Int closed_Diff closed_vimage_Int compact_eq_bounded_closed)
  moreover have g -` frontier S ∩ {0..1} ≠ {}
  proof -
    have ∃r. r ∈ g -` frontier S ∧ r ∈ {0..1}
```

```
    by (metis ‹z ∈ frontier S› g(2) imageE path_image_def pathfinish_in_path_image
vimageI2)
   then show ?thesis
     by blast
 qed
 ultimately obtain t where gt: g t ∈ frontier S and 0 ≤ t t ≤ 1
              and t: ⋀u. ⟦g u ∈ frontier S; 0 ≤ u; u ≤ 1⟧ ⟹ t ≤ u
   by (force simp: dest!: compact_attains_inf)
 moreover have t ≠ 0
   by (metis ‹y ∉ frontier S› g(1) gt pathstart_def)
 ultimately have t01: 0 < t t ≤ 1
   by auto
 have V ⊆ frontier S
   using opeSV openin_contains_ball by blast
 show ?thesis
 proof
   show arc (subpath 0 t g)
     by (simp add: ‹0 ≤ t› ‹t ≤ 1› ‹arc g› ‹t ≠ 0› arc_subpath_arc)
   have g 0 ∈ S
     by (metis ‹y ∈ S› g(1) pathstart_def)
   then show pathstart (subpath 0 t g) ∈ S
     by auto
   have g t ∈ V
     by (metis IntI atLeastAtMost_iff gt image_eqI path_image_def pig r subsetCE
‹0 ≤ t› ‹t ≤ 1›)
   then show pathfinish (subpath 0 t g) ∈ V
     by auto
   then have inj_on (subpath 0 t g) {0..1}
     using t01 ‹arc (subpath 0 t g)› arc_imp_inj_on by blast
   then have subpath 0 t g ` {0..<1} ⊆ subpath 0 t g ` {0..1} − {subpath 0 t g
1}
     by (force simp: dest: inj_onD)
   moreover have False if subpath 0 t g ` ({0..<1}) − S ≠ {}
   proof −
     have contg: continuous_on {0..1} g
       using ‹arc g› by (auto simp: arc_def path_def)
     have subpath 0 t g ` {0..<1} ∩ frontier S ≠ {}
     proof (rule connected_Int_frontier [OF _ _ that])
       show connected (subpath 0 t g ` {0..<1})
       proof (rule connected_continuous_image)
         show continuous_on {0..<1} (subpath 0 t g)
         by (meson ‹arc (subpath 0 t g)› arc_def atLeastLessThan_subseteq_atLeastAtMost_iff
continuous_on_subset order_refl path_def)
       qed auto
       show subpath 0 t g ` {0..<1} ∩ S ≠ {}
         using ‹y ∈ S› g(1) by (force simp: subpath_def image_def pathstart_def)
     qed
     then obtain x where x ∈ subpath 0 t g ` {0..<1} x ∈ frontier S
       by blast
```

```
      with t01 ‹0 ≤ t› mult_le_one t show False
        by (fastforce simp: subpath_def)
    qed
    then have subpath 0 t g ` {0..1} − {subpath 0 t g 1} ⊆ S
      using subsetD by fastforce
    ultimately  show subpath 0 t g ` {0..<1} ⊆ S
      by auto
  qed
qed


lemma dense_accessible_frontier_points_connected:
  fixes S :: 'a::{complete_space,real_normed_vector} set
  assumes open S connected S x ∈ S V ≠ {}
      and ope: openin (top_of_set (frontier S)) V
  obtains g where arc g g ` {0..<1} ⊆ S pathstart g = x pathfinish g ∈ V
proof −
  have V ⊆ frontier S
    using ope openin_imp_subset by blast
  with ‹open S› ‹x ∈ S› have x ∉ V
    using interior_open by (auto simp: frontier_def)
  obtain g where arc g and g: g ` {0..<1} ⊆ S pathstart g ∈ S pathfinish g ∈ V
    by (metis dense_accessible_frontier_points [OF ‹open S› ope ‹V ≠ {}›])
  then have path_connected S
    by (simp add: assms connected_open_path_connected)
  with ‹pathstart g ∈ S› ‹x ∈ S› have path_component S x (pathstart g)
    by (simp add: path_connected_component)
  then obtain f where path f and f: path_image f ⊆ S pathstart f = x pathfinish
f = pathstart g
    by (auto simp: path_component_def)
  then have path (f +++ g)
    by (simp add: ‹arc g› arc_imp_path)
  then obtain h where arc h
                    and h: path_image h ⊆ path_image (f +++ g) pathstart h = x
pathfinish h = pathfinish g
    using path_contains_arc [of f +++ g x pathfinish g] ‹x ∉ V› ‹pathfinish g ∈ V›
f
    by (metis pathfinish_join pathstart_join)
  have path_image h ⊆ path_image f ∪ path_image g
    using h(1) path_image_join_subset by auto
  then have h ` {0..1} − {h 1} ⊆ S
    using f g h
    apply (simp add: path_image_def pathfinish_def subset_iff image_def Bex_def)
    by (metis le_less)
  then have h ` {0..<1} ⊆ S
    using ‹arc h› by (force simp: arc_def dest: inj_onD)
  then show thesis
    using ‹arc h› g(3) h that by presburger
qed
```

lemma *dense_access_fp_aux*:
fixes *S* :: *'a*::{*complete_space*,*real_normed_vector*} *set*
assumes *S*: *open S connected S*
and *opeSU*: *openin* (*top_of_set* (*frontier S*)) *U*
and *opeSV*: *openin* (*top_of_set* (*frontier S*)) *V*
and *V* ≠ {} ¬ *U* ⊆ *V*
obtains *g* **where** *arc g pathstart g* ∈ *U pathfinish g* ∈ *V g* ' {*0*<..<*1*} ⊆ *S*
proof −
have *S* ≠ {}
using *opeSV* ‹*V* ≠ {}› **by** (*metis frontier_empty openin_subtopology_empty*)
then obtain *x* **where** *x* ∈ *S* **by** *auto*
obtain *g* **where** *arc g* **and** *g*: *g* ' {*0*..<*1*} ⊆ *S pathstart g* = *x pathfinish g* ∈ *V*
using *dense_accessible_frontier_points_connected* [*OF S* ‹*x* ∈ *S*› ‹*V* ≠ {}› *opeSV*]
by *blast*
obtain *h* **where** *arc h* **and** *h*: *h* ' {*0*..<*1*} ⊆ *S pathstart h* = *x pathfinish h* ∈ *U* − {*pathfinish g*}
proof (*rule dense_accessible_frontier_points_connected* [*OF S* ‹*x* ∈ *S*›])
show *U* − {*pathfinish g*} ≠ {}
using ‹*pathfinish g* ∈ *V*› ‹¬ *U* ⊆ *V*› **by** *blast*
show *openin* (*top_of_set* (*frontier S*)) (*U* − {*pathfinish g*})
by (*simp add*: *opeSU openin_delete*)
qed *auto*
obtain γ **where** *arc* γ
and γ: *path_image* γ ⊆ *path_image* (*reversepath h* +++ *g*)
pathstart γ = *pathfinish h pathfinish* γ = *pathfinish g*
proof (*rule path_contains_arc* [*of* (*reversepath h* +++ *g*) *pathfinish h pathfinish g*])
show *path* (*reversepath h* +++ *g*)
by (*simp add*: ‹*arc g*› ‹*arc h*› ‹*pathstart g* = *x*› ‹*pathstart h* = *x*› *arc_imp_path*)
show *pathstart* (*reversepath h* +++ *g*) = *pathfinish h*
pathfinish (*reversepath h* +++ *g*) = *pathfinish g*
by *auto*
show *pathfinish h* ≠ *pathfinish g*
using ‹*pathfinish h* ∈ *U* − {*pathfinish g*}› **by** *auto*
qed *auto*
show *?thesis*
proof
show *arc* γ *pathstart* γ ∈ *U pathfinish* γ ∈ *V*
using γ ‹*arc* γ› ‹*pathfinish h* ∈ *U* − {*pathfinish g*}› ‹*pathfinish g* ∈ *V*› **by** *auto*
have *path_image* γ ⊆ *path_image h* ∪ *path_image g*
by (*metis* γ(*1*) *g*(*2*) *h*(*2*) *path_image_join path_image_reversepath pathfinish_reversepath*)
then have γ ' {*0*..*1*} − {γ *0*, γ *1*} ⊆ *S*
using γ *g h*
apply (*simp add*: *path_image_def pathstart_def pathfinish_def subset_iff image_def Bex_def*)
by (*metis linorder_neqE_linordered_idom not_less*)

```
    then show γ ` {0<..<1} ⊆ S
      using ‹arc h› ‹arc γ›
      by (metis arc_imp_simple_path path_image_def pathfinish_def pathstart_def sim-
ple_path_endless)
  qed
qed

lemma dense_accessible_frontier_point_pairs:
  fixes S :: 'a::{complete_space,real_normed_vector} set
  assumes S: open S connected S
      and opeSU: openin (top_of_set (frontier S)) U
      and opeSV: openin (top_of_set (frontier S)) V
      and U ≠ {} V ≠ {} U ≠ V
    obtains g where arc g pathstart g ∈ U pathfinish g ∈ V g ` {0<..<1} ⊆ S
proof -
  consider ¬ U ⊆ V | ¬ V ⊆ U
    using ‹U ≠ V› by blast
  then show ?thesis
  proof cases
    case 1 then show ?thesis
      using assms dense_access_fp_aux [OF S opeSU opeSV] that by blast
  next
    case 2
    obtain g where arc g and g: pathstart g ∈ V pathfinish g ∈ U g ` {0<..<1}
⊆ S
      using assms dense_access_fp_aux [OF S opeSV opeSU] 2 by blast
    show ?thesis
    proof
      show arc (reversepath g)
        by (simp add: ‹arc g› arc_reversepath)
      show pathstart (reversepath g) ∈ U pathfinish (reversepath g) ∈ V
        using g by auto
      show reversepath g ` {0<..<1} ⊆ S
        using g by (auto simp: reversepath_def)
    qed
  qed
qed

end
```

6.40 Absolute Retracts, Absolute Neighbourhood Retracts and Euclidean Neighbourhood Retracts

```
theory Retracts
imports
  Brouwer_Fixpoint
  Continuous_Extension
```

begin

Absolute retracts (AR), absolute neighbourhood retracts (ANR) and also Euclidean neighbourhood retracts (ENR). We define AR and ANR by specializing the standard definitions for a set to embedding in spaces of higher dimension.

John Harrison writes: "This turns out to be sufficient (since any set in $\mathbb{R}^n$ can be embedded as a closed subset of a convex subset of $\mathbb{R}^{n+1}$) to derive the usual definitions, but we need to split them into two implications because of the lack of type quantifiers. Then ENR turns out to be equivalent to ANR plus local compactness."

```
definition AR :: 'a::topological_space set ⇒ bool where
AR S ≡ ∀ U. ∀ S'::('a ∗ real) set.
  S homeomorphic S' ∧ closedin (top_of_set U) S' ⟶ S' retract_of U

definition ANR :: 'a::topological_space set ⇒ bool where
ANR S ≡ ∀ U. ∀ S'::('a ∗ real) set.
  S homeomorphic S' ∧ closedin (top_of_set U) S'
  ⟶ (∃ T. openin (top_of_set U) T ∧ S' retract_of T)

definition ENR :: 'a::topological_space set ⇒ bool where
ENR S ≡ ∃ U. open U ∧ S retract_of U
```

First, show that we do indeed get the "usual" properties of ARs and ANRs.

```
lemma AR_imp_absolute_extensor:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes AR S and contf: continuous_on T f and f ` T ⊆ S
      and cloUT: closedin (top_of_set U) T
  obtains g where continuous_on U g g ` U ⊆ S ⋀x. x ∈ T ⟹ g x = f x
proof −
  have aff_dim S < int (DIM('b × real))
    using aff_dim_le_DIM [of S] by simp
  then obtain C and S' :: ('b ∗ real) set
          where C: convex C C ≠ {}
            and cloCS: closedin (top_of_set C) S'
            and hom: S homeomorphic S'
    by (metis that homeomorphic_closedin_convex)
  then have S' retract_of C
    using ‹AR S› by (simp add: AR_def)
  then obtain r where S' ⊆ C and contr: continuous_on C r
                and r ` C ⊆ S' and rid: ⋀x. x∈S' ⟹ r x = x
    by (auto simp: retraction_def retract_of_def)
  obtain g h where homeomorphism S S' g h
    using hom by (force simp: homeomorphic_def)
  then have continuous_on (f ` T) g
    by (meson ‹f ` T ⊆ S› continuous_on_subset homeomorphism_def)
  then have contgf: continuous_on T (g ∘ f)
    by (metis continuous_on_compose contf)
```

```
  have gfTC: (g ∘ f) ` T ⊆ C
  proof -
    have g ` S = S'
      by (metis (no_types) ‹homeomorphism S S' g h› homeomorphism_def)
    with ‹S' ⊆ C› ‹f ` T ⊆ S› show ?thesis by force
  qed
  obtain f' where f': continuous_on U f'  f' ` U ⊆ C
                       ⋀x. x ∈ T ⟹ f' x = (g ∘ f) x
    by (metis Dugundji [OF C cloUT contgf gfTC])
  show ?thesis
  proof (rule_tac g = h ∘ r ∘ f' in that)
    show continuous_on U (h ∘ r ∘ f')
    proof (intro continuous_on_compose f')
      show continuous_on (f' ` U) r
        using continuous_on_subset contr f' by blast
      show continuous_on (r ` f' ` U) h
        using ‹homeomorphism S S' g h› ‹f' ` U ⊆ C›
        unfolding homeomorphism_def
        by (metis ‹r ` C ⊆ S'› continuous_on_subset image_mono)
    qed
    show (h ∘ r ∘ f') ` U ⊆ S
      using ‹homeomorphism S S' g h› ‹r ` C ⊆ S'› ‹f' ` U ⊆ C›
      by (fastforce simp: homeomorphism_def)
    show ⋀x. x ∈ T ⟹ (h ∘ r ∘ f') x = f x
      using ‹homeomorphism S S' g h› ‹f ` T ⊆ S› f'
      by (auto simp: rid homeomorphism_def)
  qed
qed

lemma AR_imp_absolute_retract:
  fixes S :: 'a::euclidean_space set and S' :: 'b::euclidean_space set
  assumes AR S S homeomorphic S'
    and clo: closedin (top_of_set U) S'
  shows S' retract_of U
proof -
  obtain g h where hom: homeomorphism S S' g h
    using assms by (force simp: homeomorphic_def)
  obtain h: continuous_on S' h  h ` S' ⊆ S
    using hom homeomorphism_def by blast
  obtain h' where h': continuous_on U h' h' ` U ⊆ S
            and h'h: ⋀x. x ∈ S' ⟹ h' x = h x
    by (blast intro: AR_imp_absolute_extensor [OF ‹AR S› h clo])
  have [simp]: S' ⊆ U using clo closedin_limpt by blast
  show ?thesis
  proof (simp add: retraction_def retract_of_def, intro exI conjI)
    show continuous_on U (g ∘ h')
      by (meson continuous_on_compose continuous_on_subset h' hom homeomor-
phism_cont1)
    show (g ∘ h') ` U ⊆ S'
```

using *h′* **by** *clarsimp* (*metis hom subsetD homeomorphism_def imageI*)
show ∀ *x*∈*S′*. (*g* ∘ *h′*) *x* = *x*
by *clarsimp* (*metis h′h hom homeomorphism_def*)
qed
qed

lemma *AR_imp_absolute_retract_UNIV*:
fixes *S* :: *′a*::*euclidean_space set* **and** *S′* :: *′b*::*euclidean_space set*
assumes *AR S S homeomorphic S′ closed S′*
shows *S′ retract_of UNIV*
using *AR_imp_absolute_retract assms* **by** *fastforce*

lemma *absolute_extensor_imp_AR*:
fixes *S* :: *′a*::*euclidean_space set*
assumes ⋀*f* :: *′a* ∗ *real* ⇒ *′a*.
⋀*U T*. ⟦*continuous_on T f*; *f* ‘ *T* ⊆ *S*;
closedin (*top_of_set U*) *T*⟧
⟹ ∃ *g*. *continuous_on U g* ∧ *g* ‘ *U* ⊆ *S* ∧ (∀ *x* ∈ *T*. *g x* = *f x*)
shows *AR S*
proof (*clarsimp simp*: *AR_def*)
fix *U* **and** *T* :: (*′a* ∗ *real*) *set*
assume *S homeomorphic T* **and** *clo*: *closedin* (*top_of_set U*) *T*
then obtain *g h* **where** *hom*: *homeomorphism S T g h*
by (*force simp*: *homeomorphic_def*)
obtain *h*: *continuous_on T h* *h* ‘ *T* ⊆ *S*
using *hom homeomorphism_def* **by** *blast*
obtain *h′* **where** *h′*: *continuous_on U h′ h′* ‘ *U* ⊆ *S*
and *h′h*: ∀ *x*∈*T*. *h′ x* = *h x*
using *assms* [*OF h clo*] **by** *blast*
have [*simp*]: *T* ⊆ *U*
using *clo closedin_imp_subset* **by** *auto*
show *T retract_of U*
proof (*simp add*: *retraction_def retract_of_def*, *intro exI conjI*)
show *continuous_on U* (*g* ∘ *h′*)
by (*meson continuous_on_compose continuous_on_subset h′ hom homeomorphism_cont1*)
show (*g* ∘ *h′*) ‘ *U* ⊆ *T*
using *h′* **by** *clarsimp* (*metis hom subsetD homeomorphism_def imageI*)
show ∀ *x*∈*T*. (*g* ∘ *h′*) *x* = *x*
by *clarsimp* (*metis h′h hom homeomorphism_def*)
qed
qed

lemma *AR_eq_absolute_extensor*:
fixes *S* :: *′a*::*euclidean_space set*
shows *AR S* ⟷
(∀ *f* :: *′a* ∗ *real* ⇒ *′a*.
∀ *U T*. *continuous_on T f* ⟶ *f* ‘ *T* ⊆ *S* ⟶
closedin (*top_of_set U*) *T* ⟶

(∃ g. continuous_on U g ∧ g ‘ U ⊆ S ∧ (∀ x ∈ T. g x = f x)))
by (*metis* (*mono_tags*, *hide_lams*) *AR_imp_absolute_extensor absolute_extensor_imp_AR*)

lemma *AR_imp_retract*:
fixes S :: 'a::euclidean_space set
assumes AR S ∧ closedin (top_of_set U) S
shows S retract_of U
using *AR_imp_absolute_retract assms homeomorphic_refl* **by** *blast*

lemma *AR_homeomorphic_AR*:
fixes S :: 'a::euclidean_space set **and** T :: 'b::euclidean_space set
assumes AR T S homeomorphic T
shows AR S
unfolding *AR_def*
by (*metis assms AR_imp_absolute_retract homeomorphic_trans* [*of* _ S] *homeomorphic_sym*)

lemma *homeomorphic_AR_iff_AR*:
fixes S :: 'a::euclidean_space set **and** T :: 'b::euclidean_space set
shows S homeomorphic T ⟹ AR S ⟷ AR T
by (*metis AR_homeomorphic_AR homeomorphic_sym*)

lemma *ANR_imp_absolute_neighbourhood_extensor*:
fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
assumes ANR S **and** contf: continuous_on T f **and** f ‘ T ⊆ S
and cloUT: closedin (top_of_set U) T
obtains V g **where** T ⊆ V openin (top_of_set U) V
continuous_on V g
g ‘ V ⊆ S ⋀x. x ∈ T ⟹ g x = f x
proof −
have aff_dim S < int (DIM('b × real))
using *aff_dim_le_DIM* [*of* S] **by** *simp*
then obtain C **and** S' :: ('b ∗ real) set
where C: convex C C ≠ {}
and cloCS: closedin (top_of_set C) S'
and hom: S homeomorphic S'
by (*metis that homeomorphic_closedin_convex*)
then obtain D **where** opD: openin (top_of_set C) D **and** S' retract_of D
using ‹ANR S› **by** (*auto simp*: *ANR_def*)
then obtain r **where** S' ⊆ D **and** contr: continuous_on D r
and r ‘ D ⊆ S' **and** rid: ⋀x. x ∈ S' ⟹ r x = x
by (*auto simp*: *retraction_def retract_of_def*)
obtain g h **where** homgh: homeomorphism S S' g h
using hom **by** (*force simp*: *homeomorphic_def*)
have continuous_on (f ‘ T) g
by (*meson* ‹f ‘ T ⊆ S› *continuous_on_subset homeomorphism_def homgh*)
then have contgf: continuous_on T (g ∘ f)
by (*intro continuous_on_compose contf*)

have *gfTC*: $(g \circ f)$ ‘ $T \subseteq C$
proof −
have g ‘ $S = S'$
by (*metis* (*no_types*) *homeomorphism_def homgh*)
then show *?thesis*
by (*metis* (*no_types*) *assms*(*3*) *cloCS closedin_def image_comp image_mono order.trans topspace_euclidean_subtopology*)
qed
obtain f' **where** *contf'*: *continuous_on* U f'
and f' ‘ $U \subseteq C$
and *eq*: $\bigwedge x.\ x \in T \Longrightarrow f'\ x = (g \circ f)\ x$
by (*metis Dugundji* [*OF C cloUT contgf gfTC*])
show *?thesis*
proof (*rule_tac* $V = U \cap f' -$‘ D **and** $g = h \circ r \circ f'$ **in** *that*)
show $T \subseteq U \cap f' -$‘ D
using *cloUT closedin_imp_subset* ‹$S' \subseteq D$› ‹f ‘ $T \subseteq S$› *eq homeomorphism_image1 homgh*
by *fastforce*
show *ope*: *openin* (*top_of_set* U) ($U \cap f' -$‘ D)
using ‹f' ‘ $U \subseteq C$› **by** (*auto simp*: *opD contf' continuous_openin_preimage*)
have *conth*: *continuous_on* (r ‘ f' ‘ ($U \cap f' -$‘ D)) h
proof (*rule continuous_on_subset* [*of S'*])
show *continuous_on* S' h
using *homeomorphism_def homgh* **by** *blast*
qed (*use* ‹r ‘ $D \subseteq S'$› **in** *blast*)
show *continuous_on* ($U \cap f' -$‘ D) ($h \circ r \circ f'$)
by (*blast intro*: *continuous_on_compose conth continuous_on_subset* [*OF contr*] *continuous_on_subset* [*OF contf'*])
show ($h \circ r \circ f'$) ‘ ($U \cap f' -$‘ D) $\subseteq S$
using ‹*homeomorphism* S S' g h› ‹f' ‘ $U \subseteq C$› ‹r ‘ $D \subseteq S'$›
by (*auto simp*: *homeomorphism_def*)
show $\bigwedge x.\ x \in T \Longrightarrow (h \circ r \circ f')\ x = f\ x$
using ‹*homeomorphism* S S' g h› ‹f ‘ $T \subseteq S$› *eq*
by (*auto simp*: *rid homeomorphism_def*)
qed
qed

corollary *ANR_imp_absolute_neighbourhood_retract*:
fixes S :: *'a::euclidean_space set* **and** S' :: *'b::euclidean_space set*
assumes *ANR S S homeomorphic S'*
and *clo*: *closedin* (*top_of_set* U) S'
obtains V **where** *openin* (*top_of_set* U) V S' *retract_of* V
proof −
obtain g h **where** *hom*: *homeomorphism* S S' g h
using *assms* **by** (*force simp*: *homeomorphic_def*)
obtain h: *continuous_on* S' h h ‘ $S' \subseteq S$
using *hom homeomorphism_def* **by** *blast*
from *ANR_imp_absolute_neighbourhood_extensor* [*OF* ‹*ANR S*› *h clo*]

```
  obtain V h' where S' ⊆ V and opUV: openin (top_of_set U) V
               and h': continuous_on V h' h' ` V ⊆ S
               and h'h:⋀x. x ∈ S' ⟹ h' x = h x
    by (blast intro: ANR_imp_absolute_neighbourhood_extensor [OF ‹ANR S› h clo])
  have S' retract_of V
  proof (simp add: retraction_def retract_of_def, intro exI conjI ‹S' ⊆ V›)
    show continuous_on V (g ∘ h')
        by (meson continuous_on_compose continuous_on_subset h'(1) h'(2) hom
homeomorphism_cont1)
    show (g ∘ h') ` V ⊆ S'
      using h' by clarsimp (metis hom subsetD homeomorphism_def imageI)
    show ∀x∈S'. (g ∘ h') x = x
      by clarsimp (metis h'h hom homeomorphism_def)
  qed
  then show ?thesis
    by (rule that [OF opUV])
qed

corollary ANR_imp_absolute_neighbourhood_retract_UNIV:
  fixes S :: 'a::euclidean_space set and S' :: 'b::euclidean_space set
  assumes ANR S and hom: S homeomorphic S' and clo: closed S'
  obtains V where open V S' retract_of V
  using ANR_imp_absolute_neighbourhood_retract [OF ‹ANR S› hom]
by (metis clo closed_closedin open_openin subtopology_UNIV)

corollary neighbourhood_extension_into_ANR:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes contf: continuous_on S f and fim: f ` S ⊆ T and ANR T closed S
  obtains V g where S ⊆ V open V continuous_on V g
                    g ` V ⊆ T ⋀x. x ∈ S ⟹ g x = f x
  using ANR_imp_absolute_neighbourhood_extensor [OF ‹ANR T› contf fim]
  by (metis ‹closed S› closed_closedin open_openin subtopology_UNIV)

lemma absolute_neighbourhood_extensor_imp_ANR:
  fixes S :: 'a::euclidean_space set
  assumes ⋀f :: 'a ∗ real ⇒ 'a.
          ⋀U T. ⟦continuous_on T f;  f ` T ⊆ S;
                 closedin (top_of_set U) T⟧
                ⟹ ∃V g. T ⊆ V ∧ openin (top_of_set U) V ∧
                     continuous_on V g ∧ g ` V ⊆ S ∧ (∀x ∈ T. g x = f x)
  shows ANR S
proof (clarsimp simp: ANR_def)
  fix U and T :: ('a ∗ real) set
  assume S homeomorphic T and clo: closedin (top_of_set U) T
  then obtain g h where hom: homeomorphism S T g h
    by (force simp: homeomorphic_def)
  obtain h: continuous_on T h  h ` T ⊆ S
    using hom homeomorphism_def by blast
  obtain V h' where T ⊆ V and opV: openin (top_of_set U) V
```

```
        and h': continuous_on V h' h' ` V ⊆ S
       and h'h: ∀x∈T. h' x = h x
    using assms [OF h clo] by blast
  have [simp]: T ⊆ U
    using clo closedin_imp_subset by auto
  have T retract_of V
  proof (simp add: retraction_def retract_of_def, intro exI conjI ‹T ⊆ V›)
    show continuous_on V (g ∘ h')
      by (meson continuous_on_compose continuous_on_subset h' hom homeomor-
phism_cont1)
    show (g ∘ h') ` V ⊆ T
      using h'  by clarsimp (metis hom subsetD homeomorphism_def imageI)
    show ∀x∈T. (g ∘ h') x = x
      by clarsimp (metis h'h hom homeomorphism_def)
  qed
  then show ∃V. openin (top_of_set U) V ∧ T retract_of V
    using opV by blast
qed

lemma ANR_eq_absolute_neighbourhood_extensor:
  fixes S :: 'a::euclidean_space set
  shows ANR S ⟷
        (∀f :: 'a * real ⇒ 'a.
         ∀U T. continuous_on T f ⟶ f ` T ⊆ S ⟶
               closedin (top_of_set U) T ⟶
              (∃V g. T ⊆ V ∧ openin (top_of_set U) V ∧
                     continuous_on V g ∧ g ` V ⊆ S ∧ (∀x ∈ T. g x = f x))) (is _
= ?rhs)
proof
  assume ANR S then show ?rhs
    by (metis ANR_imp_absolute_neighbourhood_extensor)
qed (simp add: absolute_neighbourhood_extensor_imp_ANR)

lemma ANR_imp_neighbourhood_retract:
  fixes S :: 'a::euclidean_space set
  assumes ANR S closedin (top_of_set U) S
  obtains V where openin (top_of_set U) V S retract_of V
using ANR_imp_absolute_neighbourhood_retract assms homeomorphic_refl by blast

lemma ANR_imp_absolute_closed_neighbourhood_retract:
  fixes S :: 'a::euclidean_space set and S' :: 'b::euclidean_space set
  assumes ANR S S homeomorphic S' and US': closedin (top_of_set U) S'
  obtains V W
    where openin (top_of_set U) V
          closedin (top_of_set U) W
          S' ⊆ V V ⊆ W S' retract_of W
proof -
  obtain Z where openin (top_of_set U) Z and S'Z: S' retract_of Z
    by (blast intro: assms ANR_imp_absolute_neighbourhood_retract)
```

then have *UUZ*: *closedin* (*top_of_set U*) (*U* − *Z*)
  by *auto*
have *S′* ∩ (*U* − *Z*) = {}
  using ‹*S′ retract_of Z*› *closedin_retract closedin_subtopology* **by** *fastforce*
then obtain *V W*
    where *openin* (*top_of_set U*) *V*
      and *openin* (*top_of_set U*) *W*
      and *S′* ⊆ *V U* − *Z* ⊆ *W V* ∩ *W* = {}
    using *separation_normal_local* [*OF US′ UUZ*] **by** *auto*
moreover have *S′ retract_of U* − *W*
proof (*rule retract_of_subset* [*OF S′Z*])
  show *S′* ⊆ *U* − *W*
    using *US′* ‹*S′* ⊆ *V*› ‹*V* ∩ *W* = {}› *closedin_subset* **by** *fastforce*
  show *U* − *W* ⊆ *Z*
    using *Diff_subset_conv* ‹*U* − *Z* ⊆ *W*› **by** *blast*
qed
ultimately show *?thesis*
  by (*metis Diff_subset_conv Diff_triv Int_Diff_Un Int_absorb1 openin_closedin_eq that topspace_euclidean_subtopology*)
qed

lemma *ANR_imp_closed_neighbourhood_retract*:
  fixes *S* :: *′a::euclidean_space set*
  assumes *ANR S closedin* (*top_of_set U*) *S*
  obtains *V W* **where** *openin* (*top_of_set U*) *V*
          closedin (*top_of_set U*) *W*
          S ⊆ *V V* ⊆ *W S retract_of W*
by (*meson ANR_imp_absolute_closed_neighbourhood_retract assms homeomorphic_refl*)

lemma *ANR_homeomorphic_ANR*:
  fixes *S* :: *′a::euclidean_space set* **and** *T* :: *′b::euclidean_space set*
  assumes *ANR T S homeomorphic T*
    shows *ANR S*
unfolding *ANR_def*
by (*metis assms ANR_imp_absolute_neighbourhood_retract homeomorphic_trans* [*of _ S*] *homeomorphic_sym*)

lemma *homeomorphic_ANR_iff_ANR*:
  fixes *S* :: *′a::euclidean_space set* **and** *T* :: *′b::euclidean_space set*
  shows *S homeomorphic T* ⟹ *ANR S* ⟷ *ANR T*
by (*metis ANR_homeomorphic_ANR homeomorphic_sym*)

6.40.1 Analogous properties of ENRs

lemma *ENR_imp_absolute_neighbourhood_retract*:
  fixes *S* :: *′a::euclidean_space set* **and** *S′* :: *′b::euclidean_space set*
  assumes *ENR S* **and** *hom*: *S homeomorphic S′*
    and *S′* ⊆ *U*
  obtains *V* **where** *openin* (*top_of_set U*) *V S′ retract_of V*

proof −
  obtain *X* **where** *open X S retract_of X*
    using ‹*ENR S*› **by** (*auto simp*: *ENR_def*)
  then obtain *r* **where** *retraction X S r*
    by (*auto simp*: *retract_of_def*)
  have *locally compact S′*
    using *retract_of_locally_compact open_imp_locally_compact homeomorphic_local_compactness* ‹*S retract_of X*› ‹*open X*› *hom* **by** *blast*
  then obtain *W* **where** *UW*: *openin* (*top_of_set U*) *W*
      and *WS′*: *closedin* (*top_of_set W*) *S′*
    apply (*rule locally_compact_closedin_open*)
    by (*meson Int_lower2 assms*(*3*) *closedin_imp_subset closedin_subset_trans le_inf_iff openin_open*)
  obtain *f g* **where** *hom*: *homeomorphism S S′ f g*
    using *assms* **by** (*force simp*: *homeomorphic_def*)
  have *contg*: *continuous_on S′ g*
    using *hom homeomorphism_def* **by** *blast*
  moreover have *g ‘ S′ ⊆ S* **by** (*metis hom equalityE homeomorphism_def*)
  ultimately obtain *h* **where** *conth*: *continuous_on W h* **and** *hg*: ⋀*x*. *x* ∈ *S′* ⟹ *h x* = *g x*
    using *Tietze_unbounded* [*of S′ g W*] *WS′* **by** *blast*
  have *W ⊆ U* **using** *UW openin_open* **by** *auto*
  have *S′ ⊆ W* **using** *WS′ closedin_closed* **by** *auto*
  have *him*: ⋀*x*. *x* ∈ *S′* ⟹ *h x* ∈ *X*
    by (*metis* (*no_types*) ‹*S retract_of X*› *hg hom homeomorphism_def image_insert insert_absorb insert_iff retract_of_imp_subset subset_eq*)
  have *S′ retract_of* (*W* ∩ *h* −‘ *X*)
  proof (*simp add*: *retraction_def retract_of_def*, *intro exI conjI*)
    show *S′ ⊆ W S′ ⊆ h* −‘ *X*
      using *him WS′ closedin_imp_subset* **by** *blast+*
    show *continuous_on* (*W* ∩ *h* −‘ *X*) (*f* ∘ *r* ∘ *h*)
    proof (*intro continuous_on_compose*)
      show *continuous_on* (*W* ∩ *h* −‘ *X*) *h*
        by (*meson conth continuous_on_subset inf_le1*)
      show *continuous_on* (*h* ‘ (*W* ∩ *h* −‘ *X*)) *r*
      proof −
        have *h* ‘ (*W* ∩ *h* −‘ *X*) ⊆ *X*
          by *blast*
        then show *continuous_on* (*h* ‘ (*W* ∩ *h* −‘ *X*)) *r*
          by (*meson* ‹*retraction X S r*› *continuous_on_subset retraction*)
      qed
      show *continuous_on* (*r* ‘ *h* ‘ (*W* ∩ *h* −‘ *X*)) *f*
      proof (*rule continuous_on_subset* [*of S*])
        show *continuous_on S f*
          using *hom homeomorphism_def* **by** *blast*
        show *r* ‘ *h* ‘ (*W* ∩ *h* −‘ *X*) ⊆ *S*
          by (*metis* ‹*retraction X S r*› *image_mono image_subset_iff_subset_vimage inf_le2 retraction*)
      qed

qed
show (*f* ∘ *r* ∘ *h*) ' (*W* ∩ *h* −' *X*) ⊆ *S'*
using ‹*retraction X S r*› *hom*
by (*auto simp*: *retraction_def homeomorphism_def*)
show ∀ *x*∈*S'*. (*f* ∘ *r* ∘ *h*) *x* = *x*
using ‹*retraction X S r*› *hom* **by** (*auto simp*: *retraction_def homeomorphism_def hg*)
qed
then show *?thesis*
using *UW* ‹*open X*› *conth continuous_openin_preimage_eq openin_trans that* **by** *blast*
qed

corollary *ENR_imp_absolute_neighbourhood_retract_UNIV*:
fixes *S* :: *'a::euclidean_space set* **and** *S'* :: *'b::euclidean_space set*
assumes *ENR S S homeomorphic S'*
obtains *T'* **where** *open T' S' retract_of T'*
by (*metis ENR_imp_absolute_neighbourhood_retract UNIV_I assms*(*1*) *assms*(*2*) *open_openin subsetI subtopology_UNIV*)

lemma *ENR_homeomorphic_ENR*:
fixes *S* :: *'a::euclidean_space set* **and** *T* :: *'b::euclidean_space set*
assumes *ENR T S homeomorphic T*
shows *ENR S*
unfolding *ENR_def*
by (*meson ENR_imp_absolute_neighbourhood_retract_UNIV assms homeomorphic_sym*)

lemma *homeomorphic_ENR_iff_ENR*:
fixes *S* :: *'a::euclidean_space set* **and** *T* :: *'b::euclidean_space set*
assumes *S homeomorphic T*
shows *ENR S* ⟷ *ENR T*
by (*meson ENR_homeomorphic_ENR assms homeomorphic_sym*)

lemma *ENR_translation*:
fixes *S* :: *'a::euclidean_space set*
shows *ENR*(*image* (λ*x*. *a* + *x*) *S*) ⟷ *ENR S*
by (*meson homeomorphic_sym homeomorphic_translation homeomorphic_ENR_iff_ENR*)

lemma *ENR_linear_image_eq*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *linear f inj f*
shows *ENR* (*image f S*) ⟷ *ENR S*
by (*meson assms homeomorphic_ENR_iff_ENR linear_homeomorphic_image*)

Some relations among the concepts. We also relate AR to being a retract of UNIV, which is often a more convenient proxy in the closed case.

lemma *AR_imp_ANR*: *AR S* ⟹ *ANR S*
using *ANR_def AR_def* **by** *fastforce*

lemma *ENR_imp_ANR*:
fixes *S* :: *'a::euclidean_space set*
shows *ENR S* ⟹ *ANR S*
by (*meson ANR_def ENR_imp_absolute_neighbourhood_retract closedin_imp_subset*)

lemma *ENR_ANR*:
fixes *S* :: *'a::euclidean_space set*
shows *ENR S* ⟷ *ANR S* ∧ *locally compact S*
proof
assume *ENR S*
then have *locally compact S*
using *ENR_def open_imp_locally_compact retract_of_locally_compact* **by** *auto*
then show *ANR S* ∧ *locally compact S*
using *ENR_imp_ANR* ‹*ENR S*› **by** *blast*
next
assume *ANR S* ∧ *locally compact S*
then have *ANR S locally compact S* **by** *auto*
then obtain *T* :: (*'a* ∗ *real*) *set* **where** *closed T S homeomorphic T*
using *locally_compact_homeomorphic_closed*
by (*metis DIM_prod DIM_real Suc_eq_plus1 lessI*)
then show *ENR S*
using ‹*ANR S*›
by (*meson ANR_imp_absolute_neighbourhood_retract_UNIV ENR_def ENR_homeomorphic_ENR*)
qed

lemma *AR_ANR*:
fixes *S* :: *'a::euclidean_space set*
shows *AR S* ⟷ *ANR S* ∧ *contractible S* ∧ *S* ≠ {}
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
have *aff_dim S* < *int DIM*(*'a* × *real*)
using *aff_dim_le_DIM* [*of S*] **by** *auto*
then obtain *C* **and** *S'* :: (*'a* ∗ *real*) *set*
where *convex C C* ≠ {} *closedin* (*top_of_set C*) *S' S homeomorphic S'*
using *homeomorphic_closedin_convex* **by** *blast*
with ‹*AR S*› **have** *contractible S*
by (*meson AR_def convex_imp_contractible homeomorphic_contractible_eq retract_of_contractible*)
with ‹*AR S*› **show** *?rhs*
using *AR_imp_ANR AR_imp_retract* **by** *fastforce*
next
assume *?rhs*
then obtain *a* **and** *h*:: *real* × *'a* ⇒ *'a*
where *conth*: *continuous_on* ({*0..1*} × *S*) *h*
and *hS*: *h* ‘ ({*0..1*} × *S*) ⊆ *S*
and [*simp*]: ⋀*x*. *h*(*0*, *x*) = *x*
and [*simp*]: ⋀*x*. *h*(*1*, *x*) = *a*

```
      and ANR S S ≠ {}
    by (auto simp: contractible_def homotopic_with_def)
  then have a ∈ S
    by (metis all_not_in_conv atLeastAtMost_iff image_subset_iff mem_Sigma_iff or-
der_refl zero_le_one)
  have ∃g. continuous_on W g ∧ g ` W ⊆ S ∧ (∀x∈T. g x = f x)
        if      f: continuous_on T f f ` T ⊆ S
          and WT: closedin (top_of_set W) T
        for W T and f :: 'a × real ⇒ 'a
  proof −
    obtain U g
      where T ⊆ U and WU: openin (top_of_set W) U
        and contg: continuous_on U g
        and g ` U ⊆ S and gf: ⋀x. x ∈ T ⟹ g x = f x
     using iffD1 [OF ANR_eq_absolute_neighbourhood_extensor ‹ANR S›, rule_format,
OF f WT]
      by auto
    have WWU: closedin (top_of_set W) (W − U)
      using WU closedin_diff by fastforce
    moreover have (W − U) ∩ T = {}
      using ‹T ⊆ U› by auto
    ultimately obtain V V'
      where WV': openin (top_of_set W) V'
        and WV: openin (top_of_set W) V
        and W − U ⊆ V' T ⊆ V V' ∩ V = {}
      using separation_normal_local [of W W−U T] WT by blast
    then have WVT: T ∩ (W − V) = {}
      by auto
    have WWV: closedin (top_of_set W) (W − V)
      using WV closedin_diff by fastforce
    obtain j :: 'a × real ⇒ real
      where contj: continuous_on W j
        and j: ⋀x. x ∈ W ⟹ j x ∈ {0..1}
        and j0: ⋀x. x ∈ W − V ⟹ j x = 1
        and j1: ⋀x. x ∈ T ⟹ j x = 0
        by (rule Urysohn_local [OF WT WWV WVT, of 0 1::real]) (auto simp:
in_segment)
    have Weq: W = (W − V) ∪ (W − V')
      using ‹V' ∩ V = {}› by force
    show ?thesis
    proof (intro conjI exI)
      have *: continuous_on (W − V') (λx. h (j x, g x))
      proof (rule continuous_on_compose2 [OF conth continuous_on_Pair])
        show continuous_on (W − V') j
          by (rule continuous_on_subset [OF contj Diff_subset])
        show continuous_on (W − V') g
            by (metis Diff_subset_conv ‹W − U ⊆ V'› contg continuous_on_subset
Un_commute)
        show (λx. (j x, g x)) ` (W − V') ⊆ {0..1} × S
```

using *j* ‹*g* ‘ *U* ⊆ *S*› ‹*W* − *U* ⊆ *V'*› **by** *fastforce*
qed
show *continuous_on W* (λ*x*. *if x* ∈ *W* − *V then a else h* (*j x*, *g x*))
proof (*subst Weq*, *rule continuous_on_cases_local*)
show *continuous_on* (*W* − *V'*) (λ*x*. *h* (*j x*, *g x*))
using ∗ **by** *blast*
qed (*use WWV WV' Weq j0 j1* **in** *auto*)
next
have *h* (*j* (*x*, *y*), *g* (*x*, *y*)) ∈ *S* **if** (*x*, *y*) ∈ *W* (*x*, *y*) ∈ *V* **for** *x y*
proof −
have *j*(*x*, *y*) ∈ {*0..1*}
using *j that* **by** *blast*
moreover have *g*(*x*, *y*) ∈ *S*
using ‹*V'* ∩ *V* = {}› ‹*W* − *U* ⊆ *V'*› ‹*g* ‘ *U* ⊆ *S*› *that* **by** *fastforce*
ultimately show *?thesis*
using *hS* **by** *blast*
qed
with ‹*a* ∈ *S*› ‹*g* ‘ *U* ⊆ *S*›
show (λ*x*. *if x* ∈ *W* − *V then a else h* (*j x*, *g x*)) ‘ *W* ⊆ *S*
by *auto*
next
show ∀ *x*∈*T*. (*if x* ∈ *W* − *V then a else h* (*j x*, *g x*)) = *f x*
using ‹*T* ⊆ *V*› **by** (*auto simp*: *j0 j1 gf*)
qed
qed
then show *?lhs*
by (*simp add*: *AR_eq_absolute_extensor*)
qed

lemma *ANR_retract_of_ANR*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *ANR T* **and** *ST*: *S retract_of T*
shows *ANR S*
proof (*clarsimp simp add*: *ANR_eq_absolute_neighbourhood_extensor*)
fix *f*::*'a* × *real* ⇒ *'a* **and** *U W*
assume *W*: *continuous_on W f f* ‘ *W* ⊆ *S closedin* (*top_of_set U*) *W*
then obtain *r* **where** *S* ⊆ *T* **and** *r*: *continuous_on T r r* ‘ *T* ⊆ *S* ∀ *x*∈*S*. *r x* = *x continuous_on W f f* ‘ *W* ⊆ *S*
closedin (*top_of_set U*) *W*
by (*meson ST retract_of_def retraction_def*)
then have *f* ‘ *W* ⊆ *T*
by *blast*
with *W* **obtain** *V g* **where** *V*: *W* ⊆ *V openin* (*top_of_set U*) *V continuous_on V g g* ‘ *V* ⊆ *T* ∀ *x*∈*W*. *g x* = *f x*
by (*metis ANR_imp_absolute_neighbourhood_extensor* ‹*ANR T*›)
with *r* **have** *continuous_on V* (*r* ∘ *g*) ∧ (*r* ∘ *g*) ‘ *V* ⊆ *S* ∧ (∀ *x*∈*W*. (*r* ∘ *g*) *x* = *f x*)
by (*metis* (*no_types*, *lifting*) *comp_apply continuous_on_compose continuous_on_subset*

image_subset_iff)
then show $\exists V.\ W \subseteq V \wedge openin\ (top\_of\_set\ U)\ V \wedge (\exists g.\ continuous\_on\ V\ g \wedge g\ `\ V \subseteq S \wedge (\forall x \in W.\ g\ x = f\ x))$
by (*meson V*)
qed

lemma *AR_retract_of_AR*:
fixes S :: *'a::euclidean_space set*
shows $[\![AR\ T;\ S\ retract\_of\ T]\!] \Longrightarrow AR\ S$
using *ANR_retract_of_ANR AR_ANR retract_of_contractible* **by** *fastforce*

lemma *ENR_retract_of_ENR*:
$[\![ENR\ T;\ S\ retract\_of\ T]\!] \Longrightarrow ENR\ S$
by (*meson ENR_def retract_of_trans*)

lemma *retract_of_UNIV*:
fixes S :: *'a::euclidean_space set*
shows $S\ retract\_of\ UNIV \longleftrightarrow AR\ S \wedge closed\ S$
by (*metis AR_ANR AR_imp_retract ENR_def ENR_imp_ANR closed_UNIV closed_closedin contractible_UNIV empty_not_UNIV open_UNIV retract_of_closed retract_of_contractible retract_of_empty*(*1*) *subtopology_UNIV*)

lemma *compact_AR*:
fixes S :: *'a::euclidean_space set*
shows $compact\ S \wedge AR\ S \longleftrightarrow compact\ S \wedge S\ retract\_of\ UNIV$
using *compact_imp_closed retract_of_UNIV* **by** *blast*

More properties of ARs, ANRs and ENRs

lemma *not_AR_empty* [*simp*]: $\neg\ AR(\{\})$
by (*auto simp*: *AR_def*)

lemma *ENR_empty* [*simp*]: $ENR\ \{\}$
by (*simp add*: *ENR_def*)

lemma *ANR_empty* [*simp*]: $ANR\ (\{\}$ :: *'a::euclidean_space set*)
by (*simp add*: *ENR_imp_ANR*)

lemma *convex_imp_AR*:
fixes S :: *'a::euclidean_space set*
shows $[\![convex\ S;\ S \neq \{\}]\!] \Longrightarrow AR\ S$
by (*metis* (*mono_tags*, *lifting*) *Dugundji absolute_extensor_imp_AR*)

lemma *convex_imp_ANR*:
fixes S :: *'a::euclidean_space set*
shows $convex\ S \Longrightarrow ANR\ S$
using *ANR_empty AR_imp_ANR convex_imp_AR* **by** *blast*

lemma *ENR_convex_closed*:
fixes S :: *'a::euclidean_space set*

shows ⟦*closed S*; *convex S*⟧ ⟹ *ENR S*
using *ENR_def ENR_empty convex_imp_AR retract_of_UNIV* **by** *blast*

lemma *AR_UNIV* [*simp*]: *AR* (*UNIV* :: ′*a*::*euclidean_space set*)
using *retract_of_UNIV* **by** *auto*

lemma *ANR_UNIV* [*simp*]: *ANR* (*UNIV* :: ′*a*::*euclidean_space set*)
by (*simp add*: *AR_imp_ANR*)

lemma *ENR_UNIV* [*simp*]:*ENR UNIV*
using *ENR_def* **by** *blast*

lemma *AR_singleton*:
fixes *a* :: ′*a*::*euclidean_space*
shows *AR* {*a*}
using *retract_of_UNIV* **by** *blast*

lemma *ANR_singleton*:
fixes *a* :: ′*a*::*euclidean_space*
shows *ANR* {*a*}
by (*simp add*: *AR_imp_ANR AR_singleton*)

lemma *ENR_singleton*: *ENR* {*a*}
using *ENR_def* **by** *blast*

ARs closed under union

lemma *AR_closed_Un_local_aux*:
fixes *U* :: ′*a*::*euclidean_space set*
assumes *closedin* (*top_of_set U*) *S*
closedin (*top_of_set U*) *T*
AR S AR T AR(*S* ∩ *T*)
shows (*S* ∪ *T*) *retract_of U*
proof −
have *S* ∩ *T* ≠ {}
using *assms AR_def* **by** *fastforce*
have *S* ⊆ *U T* ⊆ *U*
using *assms* **by** (*auto simp*: *closedin_imp_subset*)
define *S*′ **where** *S*′ ≡ {*x* ∈ *U*. *setdist* {*x*} *S* ≤ *setdist* {*x*} *T*}
define *T*′ **where** *T*′ ≡ {*x* ∈ *U*. *setdist* {*x*} *T* ≤ *setdist* {*x*} *S*}
define *W* **where** *W* ≡ {*x* ∈ *U*. *setdist* {*x*} *S* = *setdist* {*x*} *T*}
have *US*′: *closedin* (*top_of_set U*) *S*′
using *continuous_closedin_preimage* [*of U* λ*x*. *setdist* {*x*} *S* − *setdist* {*x*} *T* {..*0*}]
by (*simp add*: *S*′*_def vimage_def Collect_conj_eq continuous_on_diff continuous_on_setdist*)
have *UT*′: *closedin* (*top_of_set U*) *T*′
using *continuous_closedin_preimage* [*of U* λ*x*. *setdist* {*x*} *T* − *setdist* {*x*} *S* {..*0*}]
by (*simp add*: *T*′*_def vimage_def Collect_conj_eq continuous_on_diff continu-*

```
ous_on_setdist)
  have S ⊆ S'
    using S'_def ‹S ⊆ U› setdist_sing_in_set by fastforce
  have T ⊆ T'
    using T'_def ‹T ⊆ U› setdist_sing_in_set by fastforce
  have S ∩ T ⊆ W W ⊆ U
    using ‹S ⊆ U› by (auto simp: W_def setdist_sing_in_set)
  have (S ∩ T) retract_of W
  proof (rule AR_imp_absolute_retract [OF ‹AR(S ∩ T)›])
    show S ∩ T homeomorphic S ∩ T
      by (simp add: homeomorphic_refl)
    show closedin (top_of_set W) (S ∩ T)
      by (meson ‹S ∩ T ⊆ W› ‹W ⊆ U› assms closedin_Int closedin_subset_trans)
  qed
  then obtain r0
    where S ∩ T ⊆ W and contr0: continuous_on W r0
      and r0 ` W ⊆ S ∩ T
      and r0 [simp]: ⋀x. x ∈ S ∩ T ⟹ r0 x = x
      by (auto simp: retract_of_def retraction_def)
  have ST: x ∈ W ⟹ x ∈ S ⟷ x ∈ T for x
    using setdist_eq_0_closedin ‹S ∩ T ≠ {}› assms
    by (force simp: W_def setdist_sing_in_set)
  have S' ∩ T' = W
    by (auto simp: S'_def T'_def W_def)
  then have cloUW: closedin (top_of_set U) W
    using closedin_Int US' UT' by blast
  define r where r ≡ λx. if x ∈ W then r0 x else x
  have contr: continuous_on (W ∪ (S ∪ T)) r
  unfolding r_def
  proof (rule continuous_on_cases_local [OF _ _ contr0 continuous_on_id])
    show closedin (top_of_set (W ∪ (S ∪ T))) W
     using ‹S ⊆ U› ‹T ⊆ U› ‹W ⊆ U› ‹closedin (top_of_set U) W› closedin_subset_trans
by fastforce
    show closedin (top_of_set (W ∪ (S ∪ T))) (S ∪ T)
     by (meson ‹S ⊆ U› ‹T ⊆ U› ‹W ⊆ U› assms closedin_Un closedin_subset_trans
sup.bounded_iff sup.cobounded2)
    show ⋀x. x ∈ W ∧ x ∉ W ∨ x ∈ S ∪ T ∧ x ∈ W ⟹ r0 x = x
      by (auto simp: ST)
  qed
  have rim: r ` (W ∪ S) ⊆ S r ` (W ∪ T) ⊆ T
    using ‹r0 ` W ⊆ S ∩ T› r_def by auto
  have cloUWS: closedin (top_of_set U) (W ∪ S)
    by (simp add: cloUW assms closedin_Un)
  obtain g where contg: continuous_on U g
    and g ` U ⊆ S and geqr: ⋀x. x ∈ W ∪ S ⟹ g x = r x
  proof (rule AR_imp_absolute_extensor [OF ‹AR S› _ _ cloUWS])
    show continuous_on (W ∪ S) r
      using continuous_on_subset contr sup_assoc by blast
  qed (use rim in auto)
```

have *cloUWT*: *closedin* (*top_of_set U*) (*W* ∪ *T*)
  by (*simp add*: *cloUW assms closedin_Un*)
obtain *h* **where** *conth*: *continuous_on U h*
    and *h* ‘ *U* ⊆ *T* **and** *heqr*: ⋀*x*. *x* ∈ *W* ∪ *T* ⟹ *h x* = *r x*
proof (*rule AR_imp_absolute_extensor* [*OF* ‹*AR T*› _ _ *cloUWT*])
  show *continuous_on* (*W* ∪ *T*) *r*
    using *continuous_on_subset contr sup_assoc* **by** *blast*
qed (*use rim* **in** *auto*)
have *U*: *U* = *S'* ∪ *T'*
  by (*force simp*: *S'_def T'_def*)
have *cont*: *continuous_on U* (λ*x*. *if x* ∈ *S' then g x else h x*)
  unfolding *U*
  apply (*rule continuous_on_cases_local*)
  using *US' UT'* ‹*S'* ∩ *T'* = *W*› ‹*U* = *S'* ∪ *T'*›
      contg conth continuous_on_subset geqr heqr **by** *auto*
have *UST*: (λ*x*. *if x* ∈ *S' then g x else h x*) ‘ *U* ⊆ *S* ∪ *T*
  using ‹*g* ‘ *U* ⊆ *S*› ‹*h* ‘ *U* ⊆ *T*› **by** *auto*
show *?thesis*
  apply (*simp add*: *retract_of_def retraction_def* ‹*S* ⊆ *U*› ‹*T* ⊆ *U*›)
  apply (*rule_tac x*=λ*x*. *if x* ∈ *S' then g x else h x* **in** *exI*)
  using *ST UST* ‹*S* ⊆ *S'*› ‹*S'* ∩ *T'* = *W*› ‹*T* ⊆ *T'*› *cont geqr heqr r_def* **by** *auto*
qed

lemma *AR_closed_Un_local*:
  fixes *S* :: *'a*::*euclidean_space set*
  assumes *STS*: *closedin* (*top_of_set* (*S* ∪ *T*)) *S*
    and *STT*: *closedin* (*top_of_set* (*S* ∪ *T*)) *T*
    and *AR S AR T AR*(*S* ∩ *T*)
  shows *AR*(*S* ∪ *T*)
proof −
  have *C retract_of U*
    if *hom*: *S* ∪ *T homeomorphic C* **and** *UC*: *closedin* (*top_of_set U*) *C*
    for *U* **and** *C* :: (*'a* ∗ *real*) *set*
  proof −
    obtain *f g* **where** *hom*: *homeomorphism* (*S* ∪ *T*) *C f g*
      using *hom* **by** (*force simp*: *homeomorphic_def*)
    have *US*: *closedin* (*top_of_set U*) (*C* ∩ *g* −‘ *S*)
      by (*metis STS continuous_on_imp_closedin hom homeomorphism_def closedin_trans* [*OF* _ *UC*])
    have *UT*: *closedin* (*top_of_set U*) (*C* ∩ *g* −‘ *T*)
      by (*metis STT continuous_on_closed hom homeomorphism_def closedin_trans* [*OF* _ *UC*])
    have *homeomorphism* (*C* ∩ *g* −‘ *S*) *S g f*
      using *hom*
      apply (*auto simp*: *homeomorphism_def elim*!: *continuous_on_subset*)
      apply (*rule_tac x*=*f x* **in** *image_eqI*, *auto*)
      done
    then have *ARS*: *AR* (*C* ∩ *g* −‘ *S*)

```
      using ‹AR S› homeomorphic_AR_iff_AR homeomorphic_def by blast
    have homeomorphism (C ∩ g -` T) T g f
      using hom
      apply (auto simp: homeomorphism_def elim!: continuous_on_subset)
      apply (rule_tac x=f x in image_eqI, auto)
      done
    then have ART: AR (C ∩ g -` T)
      using ‹AR T› homeomorphic_AR_iff_AR homeomorphic_def by blast
    have homeomorphism (C ∩ g -` S ∩ (C ∩ g -` T)) (S ∩ T) g f
      using hom
      apply (auto simp: homeomorphism_def elim!: continuous_on_subset)
      apply (rule_tac x=f x in image_eqI, auto)
      done
    then have ARI: AR ((C ∩ g -` S) ∩ (C ∩ g -` T))
      using ‹AR (S ∩ T)› homeomorphic_AR_iff_AR homeomorphic_def by blast
    have C = (C ∩ g -` S) ∪ (C ∩ g -` T)
      using hom by (auto simp: homeomorphism_def)
    then show ?thesis
      by (metis AR_closed_Un_local_aux [OF US UT ARS ART ARI])
  qed
  then show ?thesis
    by (force simp: AR_def)
qed

corollary AR_closed_Un:
  fixes S :: 'a::euclidean_space set
  shows ⟦closed S; closed T; AR S; AR T; AR (S ∩ T)⟧ ⟹ AR (S ∪ T)
by (metis AR_closed_Un_local_aux closed_closedin retract_of_UNIV subtopology_UNIV)
```

ANRs closed under union

```
lemma ANR_closed_Un_local_aux:
  fixes U :: 'a::euclidean_space set
  assumes US: closedin (top_of_set U) S
      and UT: closedin (top_of_set U) T
      and ANR S ANR T ANR(S ∩ T)
  obtains V where openin (top_of_set U) V (S ∪ T) retract_of V
proof (cases S = {} ∨ T = {})
  case True with assms that show ?thesis
   by (metis ANR_imp_neighbourhood_retract Un_commute inf_bot_right sup_inf_absorb)
next
  case False
  then have [simp]: S ≠ {} T ≠ {} by auto
  have S ⊆ U T ⊆ U
    using assms by (auto simp: closedin_imp_subset)
  define S' where S' ≡ {x ∈ U. setdist {x} S ≤ setdist {x} T}
  define T' where T' ≡ {x ∈ U. setdist {x} T ≤ setdist {x} S}
  define W  where W ≡ {x ∈ U. setdist {x} S = setdist {x} T}
  have cloUS': closedin (top_of_set U) S'
    using continuous_closedin_preimage [of U λx. setdist {x} S - setdist {x} T
```

{..*0*}]
  by (*simp add*: *S′_def vimage_def Collect_conj_eq continuous_on_diff continuous_on_setdist*)
have *cloUT′*: *closedin* (*top_of_set U*) *T′*
  using *continuous_closedin_preimage* [*of U* λ*x*. *setdist* {*x*} *T* − *setdist* {*x*} *S* {..*0*}]
  by (*simp add*: *T′_def vimage_def Collect_conj_eq continuous_on_diff continuous_on_setdist*)
have *S* ⊆ *S′*
  using *S′_def* ‹*S* ⊆ *U*› *setdist_sing_in_set* **by** *fastforce*
have *T* ⊆ *T′*
  using *T′_def* ‹*T* ⊆ *U*› *setdist_sing_in_set* **by** *fastforce*
have *S′* ∪ *T′* = *U*
  by (*auto simp*: *S′_def T′_def*)
have *W* ⊆ *S′*
  by (*simp add*: *Collect_mono S′_def W_def*)
have *W* ⊆ *T′*
  by (*simp add*: *Collect_mono T′_def W_def*)
have *ST_W*: *S* ∩ *T* ⊆ *W* **and** *W* ⊆ *U*
  using ‹*S* ⊆ *U*› **by** (*force simp*: *W_def setdist_sing_in_set*)+
have *S′* ∩ *T′* = *W*
  by (*auto simp*: *S′_def T′_def W_def*)
then have *cloUW*: *closedin* (*top_of_set U*) *W*
  using *closedin_Int cloUS′ cloUT′* **by** *blast*
obtain *W′ W0* **where** *openin* (*top_of_set W*) *W′*
    and *cloWW0*: *closedin* (*top_of_set W*) *W0*
    and *S* ∩ *T* ⊆ *W′ W′* ⊆ *W0*
    and *ret*: (*S* ∩ *T*) *retract_of W0*
  by (*meson ANR_imp_closed_neighbourhood_retract ST_W US UT* ‹*W* ⊆ *U*› ‹*ANR*(*S* ∩ *T*)› *closedin_Int closedin_subset_trans*)
then obtain *U0* **where** *opeUU0*: *openin* (*top_of_set U*) *U0*
    and *U0*: *S* ∩ *T* ⊆ *U0 U0* ∩ *W* ⊆ *W0*
  unfolding *openin_open* **using** ‹*W* ⊆ *U*› **by** *blast*
have *W0* ⊆ *U*
  using ‹*W* ⊆ *U*› *cloWW0 closedin_subset* **by** *fastforce*
obtain *r0*
  where *S* ∩ *T* ⊆ *W0* **and** *contr0*: *continuous_on W0 r0* **and** *r0* ‘ *W0* ⊆ *S* ∩ *T*
    and *r0* [*simp*]: ⋀*x*. *x* ∈ *S* ∩ *T* ⟹ *r0 x* = *x*
  using *ret* **by** (*force simp*: *retract_of_def retraction_def*)
have *ST*: *x* ∈ *W* ⟹ *x* ∈ *S* ⟷ *x* ∈ *T* **for** *x*
  using *assms* **by** (*auto simp*: *W_def setdist_sing_in_set dest*!: *setdist_eq_0_closedin*)
define *r* **where** *r* ≡ λ*x*. *if x* ∈ *W0 then r0 x else x*
have *r* ‘ (*W0* ∪ *S*) ⊆ *S r* ‘ (*W0* ∪ *T*) ⊆ *T*
  using ‹*r0* ‘ *W0* ⊆ *S* ∩ *T*› *r_def* **by** *auto*
have *contr*: *continuous_on* (*W0* ∪ (*S* ∪ *T*)) *r*
unfolding *r_def*
proof (*rule continuous_on_cases_local* [*OF* _ _ *contr0 continuous_on_id*])
  show *closedin* (*top_of_set* (*W0* ∪ (*S* ∪ *T*))) *W0*
    using *closedin_subset_trans* [*of U*]

by (*metis le_sup_iff order_refl cloWW0 cloUW closedin_trans* ‹*W0* ⊆ *U*› ‹*S* ⊆ *U*› ‹*T* ⊆ *U*›)
show *closedin* (*top_of_set* (*W0* ∪ (*S* ∪ *T*))) (*S* ∪ *T*)
by (*meson* ‹*S* ⊆ *U*› ‹*T* ⊆ *U*› ‹*W0* ⊆ *U*› *assms closedin_Un closedin_subset_trans sup.bounded_iff sup.cobounded2*)
show ⋀*x*. *x* ∈ *W0* ∧ *x* ∉ *W0* ∨ *x* ∈ *S* ∪ *T* ∧ *x* ∈ *W0* ⟹ *r0 x* = *x*
using *ST cloWW0 closedin_subset* **by** *fastforce*
qed
have *cloS'WS*: *closedin* (*top_of_set S'*) (*W0* ∪ *S*)
by (*meson closedin_subset_trans US cloUS'* ‹*S* ⊆ *S'*› ‹*W* ⊆ *S'*› *cloUW cloWW0*

closedin_Un closedin_imp_subset closedin_trans)
obtain *W1 g* **where** *W0* ∪ *S* ⊆ *W1* **and** *contg*: *continuous_on W1 g*
and *opeSW1*: *openin* (*top_of_set S'*) *W1*
and *g* ` *W1* ⊆ *S* **and** *geqr*: ⋀*x*. *x* ∈ *W0* ∪ *S* ⟹ *g x* = *r x*
proof (*rule ANR_imp_absolute_neighbourhood_extensor* [*OF* ‹*ANR S*› _ ‹*r* ` (*W0* ∪ *S*) ⊆ *S*› *cloS'WS*])
show *continuous_on* (*W0* ∪ *S*) *r*
using *continuous_on_subset contr sup_assoc* **by** *blast*
qed *auto*
have *cloT'WT*: *closedin* (*top_of_set T'*) (*W0* ∪ *T*)
by (*meson closedin_subset_trans UT cloUT'* ‹*T* ⊆ *T'*› ‹*W* ⊆ *T'*› *cloUW cloWW0*

closedin_Un closedin_imp_subset closedin_trans)
obtain *W2 h* **where** *W0* ∪ *T* ⊆ *W2* **and** *conth*: *continuous_on W2 h*
and *opeSW2*: *openin* (*top_of_set T'*) *W2*
and *h* ` *W2* ⊆ *T* **and** *heqr*: ⋀*x*. *x* ∈ *W0* ∪ *T* ⟹ *h x* = *r x*
proof (*rule ANR_imp_absolute_neighbourhood_extensor* [*OF* ‹*ANR T*› _ ‹*r* ` (*W0* ∪ *T*) ⊆ *T*› *cloT'WT*])
show *continuous_on* (*W0* ∪ *T*) *r*
using *continuous_on_subset contr sup_assoc* **by** *blast*
qed *auto*
have *S'* ∩ *T'* = *W*
by (*force simp*: *S'_def T'_def W_def*)
obtain *O1 O2* **where** *O12*: *open O1 W1* = *S'* ∩ *O1 open O2 W2* = *T'* ∩ *O2*
using *opeSW1 opeSW2* **by** (*force simp*: *openin_open*)
show *?thesis*
proof
have *eq*: *W1* − (*W* − *U0*) ∪ (*W2* − (*W* − *U0*))
= ((*U* − *T'*) ∩ *O1* ∪ (*U* − *S'*) ∩ *O2* ∪ *U* ∩ *O1* ∩ *O2*) − (*W* − *U0*) (**is** *?WW1* ∪ *?WW2* = *?rhs*)
using ‹*U0* ∩ *W* ⊆ *W0*› ‹*W0* ∪ *S* ⊆ *W1*› ‹*W0* ∪ *T* ⊆ *W2*›
by (*auto simp*: ‹*S'* ∪ *T'* = *U*› [*symmetric*] ‹*S'* ∩ *T'* = *W*› [*symmetric*] ‹*W1* = *S'* ∩ *O1*› ‹*W2* = *T'* ∩ *O2*›)
show *openin* (*top_of_set U*) (*?WW1* ∪ *?WW2*)
by (*simp add*: *eq* ‹*open O1*› ‹*open O2*› *cloUS' cloUT' cloUW closedin_diff opeUU0 openin_Int_open openin_Un openin_diff*)
obtain *SU'* **where** *closed SU' S'* = *U* ∩ *SU'*
using *cloUS'* **by** (*auto simp add*: *closedin_closed*)

moreover have *?WW1* = (*?WW1* ∪ *?WW2*) ∩ *SU′*
using ‹*S′* = *U* ∩ *SU′*› ‹*W1* = *S′* ∩ *O1*› ‹*S′* ∪ *T′* = *U*› ‹*W2* = *T′* ∩ *O2*› ‹*S′* ∩ *T′* = *W*› ‹*W0* ∪ *S* ⊆ *W1*› *U0*
by *auto*
ultimately have *cloW1*: *closedin* (*top_of_set* (*W1* − (*W* − *U0*) ∪ (*W2* − (*W* − *U0*)))) (*W1* − (*W* − *U0*))
by (*metis closedin_closed_Int*)
obtain *TU′* **where** *closed TU′ T′* = *U* ∩ *TU′*
using *cloUT′* **by** (*auto simp add*: *closedin_closed*)
moreover have *?WW2* = (*?WW1* ∪ *?WW2*) ∩ *TU′*
using ‹*T′* = *U* ∩ *TU′*› ‹*W1* = *S′* ∩ *O1*› ‹*S′* ∪ *T′* = *U*› ‹*W2* = *T′* ∩ *O2*› ‹*S′* ∩ *T′* = *W*› ‹*W0* ∪ *T* ⊆ *W2*› *U0*
by *auto*
ultimately have *cloW2*: *closedin* (*top_of_set* (*?WW1* ∪ *?WW2*)) *?WW2*
by (*metis closedin_closed_Int*)
let *?gh* = λ*x*. *if x* ∈ *S′ then g x else h x*
have ∃ *r*. *continuous_on* (*?WW1* ∪ *?WW2*) *r* ∧ *r* ‘ (*?WW1* ∪ *?WW2*) ⊆ *S* ∪ *T* ∧ (∀ *x*∈*S* ∪ *T*. *r x* = *x*)
proof (*intro exI conjI*)
show ∀ *x*∈*S* ∪ *T*. *?gh x* = *x*
using *ST* ‹*S′* ∩ *T′* = *W*› *geqr heqr O12*
by (*metis Int_iff Un_iff* ‹*W0* ∪ *S* ⊆ *W1*› ‹*W0* ∪ *T* ⊆ *W2*› *r0 r_def sup.order_iff*)
have ⋀*x*. *x* ∈ *?WW1* ∧ *x* ∉ *S′* ∨ *x* ∈ *?WW2* ∧ *x* ∈ *S′* ⟹ *g x* = *h x*
using *O12*
by (*metis* (*full_types*) *DiffD1 DiffD2 DiffI IntE IntI U0*(*2*) *UnCI* ‹*S′* ∩ *T′* = *W*› *geqr heqr in_mono*)
then show *continuous_on* (*?WW1* ∪ *?WW2*) *?gh*
using *continuous_on_cases_local* [*OF cloW1 cloW2 continuous_on_subset* [*OF contg*] *continuous_on_subset* [*OF conth*]]
by *simp*
show *?gh* ‘ (*?WW1* ∪ *?WW2*) ⊆ *S* ∪ *T*
using ‹*W1* = *S′* ∩ *O1*› ‹*W2* = *T′* ∩ *O2*› ‹*S′* ∩ *T′* = *W*› ‹*g* ‘ *W1* ⊆ *S*› ‹*h* ‘ *W2* ⊆ *T*› ‹*U0* ∩ *W* ⊆ *W0*› ‹*W0* ∪ *S* ⊆ *W1*›
by (*auto simp add*: *image_subset_iff*)
qed
then show *S* ∪ *T retract_of ?WW1* ∪ *?WW2*
using ‹*W0* ∪ *S* ⊆ *W1*› ‹*W0* ∪ *T* ⊆ *W2*› *ST opeUU0 U0*
by (*auto simp*: *retract_of_def retraction_def*)
qed
qed

lemma *ANR_closed_Un_local*:
fixes *S* :: *′a*::*euclidean_space set*
assumes *STS*: *closedin* (*top_of_set* (*S* ∪ *T*)) *S*
and *STT*: *closedin* (*top_of_set* (*S* ∪ *T*)) *T*
and *ANR S ANR T ANR*(*S* ∩ *T*)
shows *ANR*(*S* ∪ *T*)

```
proof -
  have ∃ T. openin (top_of_set U) T ∧ C retract_of T
      if hom: S ∪ T homeomorphic C and UC: closedin (top_of_set U) C
      for U and C :: ('a * real) set
  proof -
    obtain f g where hom: homeomorphism (S ∪ T) C f g
      using hom by (force simp: homeomorphic_def)
    have US: closedin (top_of_set U) (C ∩ g -` S)
     by (metis STS UC closedin_trans continuous_on_imp_closedin hom homeomor-
phism_def)
    have UT: closedin (top_of_set U) (C ∩ g -` T)
     by (metis STT UC closedin_trans continuous_on_imp_closedin hom homeomor-
phism_def)
    have homeomorphism (C ∩ g -` S) S g f
      using hom
      apply (auto simp: homeomorphism_def elim!: continuous_on_subset)
      by (rule_tac x=f x in image_eqI, auto)
    then have ANRS: ANR (C ∩ g -` S)
      using ‹ANR S› homeomorphic_ANR_iff_ANR homeomorphic_def by blast
    have homeomorphism (C ∩ g -` T) T g f
     using hom apply (auto simp: homeomorphism_def elim!: continuous_on_subset)
      by (rule_tac x=f x in image_eqI, auto)
    then have ANRT: ANR (C ∩ g -` T)
      using ‹ANR T› homeomorphic_ANR_iff_ANR homeomorphic_def by blast
    have homeomorphism (C ∩ g -` S ∩ (C ∩ g -` T)) (S ∩ T) g f
      using hom
      apply (auto simp: homeomorphism_def elim!: continuous_on_subset)
      by (rule_tac x=f x in image_eqI, auto)
    then have ANRI: ANR ((C ∩ g -` S) ∩ (C ∩ g -` T))
       using ‹ANR (S ∩ T)› homeomorphic_ANR_iff_ANR homeomorphic_def by
blast
    have C = (C ∩ g -` S) ∪ (C ∩ g -` T)
      using hom by (auto simp: homeomorphism_def)
    then show ?thesis
      by (metis ANR_closed_Un_local_aux [OF US UT ANRS ANRT ANRI])
  qed
  then show ?thesis
    by (auto simp: ANR_def)
qed

corollary ANR_closed_Un:
  fixes S :: 'a::euclidean_space set
  shows ⟦closed S; closed T; ANR S; ANR T; ANR (S ∩ T)⟧ ⟹ ANR (S ∪ T)
by (simp add: ANR_closed_Un_local closedin_def diff_eq open_Compl openin_open_Int)

lemma ANR_openin:
  fixes S :: 'a::euclidean_space set
  assumes ANR T and opeTS: openin (top_of_set T) S
  shows ANR S
```

proof (*clarsimp simp only*: *ANR_eq_absolute_neighbourhood_extensor*)
  fix *f* :: *'a* × *real* ⇒ *'a* **and** *U C*
  assume *contf*: *continuous_on C f* **and** *fim*: *f* ' *C* ⊆ *S*
    and *cloUC*: *closedin* (*top_of_set U*) *C*
  have *f* ' *C* ⊆ *T*
    using *fim opeTS openin_imp_subset* **by** *blast*
  obtain *W g* **where** *C* ⊆ *W*
        and *UW*: *openin* (*top_of_set U*) *W*
        and *contg*: *continuous_on W g*
        and *gim*: *g* ' *W* ⊆ *T*
        and *geq*: ⋀*x*. *x* ∈ *C* ⟹ *g x* = *f x*
    using *ANR_imp_absolute_neighbourhood_extensor* [*OF* ‹*ANR T*› *contf* ‹*f* ' *C* ⊆ *T*› *cloUC*] *fim* **by** *auto*
  show ∃ *V g*. *C* ⊆ *V* ∧ *openin* (*top_of_set U*) *V* ∧ *continuous_on V g* ∧ *g* ' *V* ⊆ *S* ∧ (∀ *x*∈*C*. *g x* = *f x*)
  proof (*intro exI conjI*)
    show *C* ⊆ *W* ∩ *g* −' *S*
      using ‹*C* ⊆ *W*› *fim geq* **by** *blast*
    show *openin* (*top_of_set U*) (*W* ∩ *g* −' *S*)
      by (*metis* (*mono_tags*, *lifting*) *UW contg continuous_openin_preimage gim opeTS openin_trans*)
    show *continuous_on* (*W* ∩ *g* −' *S*) *g*
      by (*blast intro*: *continuous_on_subset* [*OF contg*])
    show *g* ' (*W* ∩ *g* −' *S*) ⊆ *S*
      using *gim* **by** *blast*
    show ∀ *x*∈*C*. *g x* = *f x*
      using *geq* **by** *blast*
  qed
qed

lemma *ENR_openin*:
  fixes *S* :: *'a*::*euclidean_space set*
  assumes *ENR T openin* (*top_of_set T*) *S*
  shows *ENR S*
  by (*meson ANR_openin ENR_ANR assms locally_open_subset*)

lemma *ANR_neighborhood_retract*:
  fixes *S* :: *'a*::*euclidean_space set*
  assumes *ANR U S retract_of T openin* (*top_of_set U*) *T*
  shows *ANR S*
  using *ANR_openin ANR_retract_of_ANR assms* **by** *blast*

lemma *ENR_neighborhood_retract*:
  fixes *S* :: *'a*::*euclidean_space set*
  assumes *ENR U S retract_of T openin* (*top_of_set U*) *T*
  shows *ENR S*
  using *ENR_openin ENR_retract_of_ENR assms* **by** *blast*

lemma *ANR_rel_interior*:

fixes *S* :: *'a::euclidean_space set*
shows *ANR S* ⟹ *ANR(rel_interior S)*
by (*blast intro*: *ANR_openin openin_set_rel_interior*)

lemma *ANR_delete*:
fixes *S* :: *'a::euclidean_space set*
shows *ANR S* ⟹ *ANR(S* − {*a*})
by (*blast intro*: *ANR_openin openin_delete openin_subtopology_self*)

lemma *ENR_rel_interior*:
fixes *S* :: *'a::euclidean_space set*
shows *ENR S* ⟹ *ENR(rel_interior S)*
by (*blast intro*: *ENR_openin openin_set_rel_interior*)

lemma *ENR_delete*:
fixes *S* :: *'a::euclidean_space set*
shows *ENR S* ⟹ *ENR(S* − {*a*})
by (*blast intro*: *ENR_openin openin_delete openin_subtopology_self*)

lemma *open_imp_ENR*: *open S* ⟹ *ENR S*
using *ENR_def* **by** *blast*

lemma *open_imp_ANR*:
fixes *S* :: *'a::euclidean_space set*
shows *open S* ⟹ *ANR S*
by (*simp add*: *ENR_imp_ANR open_imp_ENR*)

lemma *ANR_ball* [*iff*]:
fixes *a* :: *'a::euclidean_space*
shows *ANR(ball a r)*
by (*simp add*: *convex_imp_ANR*)

lemma *ENR_ball* [*iff*]: *ENR(ball a r)*
by (*simp add*: *open_imp_ENR*)

lemma *AR_ball* [*simp*]:
fixes *a* :: *'a::euclidean_space*
shows *AR(ball a r)* ⟷ *0* < *r*
by (*auto simp*: *AR_ANR convex_imp_contractible*)

lemma *ANR_cball* [*iff*]:
fixes *a* :: *'a::euclidean_space*
shows *ANR(cball a r)*
by (*simp add*: *convex_imp_ANR*)

lemma *ENR_cball*:
fixes *a* :: *'a::euclidean_space*
shows *ENR(cball a r)*
using *ENR_convex_closed* **by** *blast*

lemma *AR_cball* [*simp*]:
  fixes *a* :: *'a::euclidean_space*
  shows *AR*(*cball a r*) ⟷ *0* ≤ *r*
 by (*auto simp*: *AR_ANR convex_imp_contractible*)

lemma *ANR_box* [*iff*]:
  fixes *a* :: *'a::euclidean_space*
  shows *ANR*(*cbox a b*) *ANR*(*box a b*)
 by (*auto simp*: *convex_imp_ANR open_imp_ANR*)

lemma *ENR_box* [*iff*]:
  fixes *a* :: *'a::euclidean_space*
  shows *ENR*(*cbox a b*) *ENR*(*box a b*)
 by (*simp_all add*: *ENR_convex_closed closed_cbox open_box open_imp_ENR*)

lemma *AR_box* [*simp*]:
  AR(*cbox a b*) ⟷ *cbox a b* ≠ {} *AR*(*box a b*) ⟷ *box a b* ≠ {}
 by (*auto simp*: *AR_ANR convex_imp_contractible*)

lemma *ANR_interior*:
   fixes *S* :: *'a::euclidean_space set*
   shows *ANR*(*interior S*)
 by (*simp add*: *open_imp_ANR*)

lemma *ENR_interior*:
   fixes *S* :: *'a::euclidean_space set*
   shows *ENR*(*interior S*)
 by (*simp add*: *open_imp_ENR*)

lemma *AR_imp_contractible*:
  fixes *S* :: *'a::euclidean_space set*
  shows *AR S* ⟹ *contractible S*
 by (*simp add*: *AR_ANR*)

lemma *ENR_imp_locally_compact*:
  fixes *S* :: *'a::euclidean_space set*
  shows *ENR S* ⟹ *locally compact S*
 by (*simp add*: *ENR_ANR*)

lemma *ANR_imp_locally_path_connected*:
 fixes *S* :: *'a::euclidean_space set*
 assumes *ANR S*
  shows *locally path_connected S*
proof −
 obtain *U* **and** *T* :: (*'a* × *real*) *set*
   where *convex U U* ≠ {}
    and *UT*: *closedin* (*top_of_set U*) *T* **and** *S homeomorphic T*
 proof (*rule homeomorphic_closedin_convex*)

show *aff_dim S* < *int DIM*('*a* × *real*)
  using *aff_dim_le_DIM* [*of S*] **by** *auto*
qed *auto*
then have *locally path_connected T*
  by (*meson ANR_imp_absolute_neighbourhood_retract*
    assms convex_imp_locally_path_connected locally_open_subset retract_of_locally_path_connected)
then have *S*: *locally path_connected S*
    if *openin* (*top_of_set U*) *V T retract_of V U* ≠ {} **for** *V*
  using ‹*S homeomorphic T*› *homeomorphic_locally homeomorphic_path_connectedness*
by *blast*
obtain *Ta* **where** (*openin* (*top_of_set U*) *Ta* ∧ *T retract_of Ta*)
  using *ANR_def UT* ‹*S homeomorphic T*› *assms* **by** *moura*
then show *?thesis*
  using *S* ‹*U* ≠ {}› **by** *blast*
qed

lemma *ANR_imp_locally_connected*:
  fixes *S* :: '*a*::*euclidean_space set*
  assumes *ANR S*
    shows *locally connected S*
using *locally_path_connected_imp_locally_connected ANR_imp_locally_path_connected*
assms **by** *auto*

lemma *AR_imp_locally_path_connected*:
  fixes *S* :: '*a*::*euclidean_space set*
  assumes *AR S*
    shows *locally path_connected S*
by (*simp add*: *ANR_imp_locally_path_connected AR_imp_ANR assms*)

lemma *AR_imp_locally_connected*:
  fixes *S* :: '*a*::*euclidean_space set*
  assumes *AR S*
    shows *locally connected S*
using *ANR_imp_locally_connected AR_ANR assms* **by** *blast*

lemma *ENR_imp_locally_path_connected*:
  fixes *S* :: '*a*::*euclidean_space set*
  assumes *ENR S*
    shows *locally path_connected S*
by (*simp add*: *ANR_imp_locally_path_connected ENR_imp_ANR assms*)

lemma *ENR_imp_locally_connected*:
  fixes *S* :: '*a*::*euclidean_space set*
  assumes *ENR S*
    shows *locally connected S*
using *ANR_imp_locally_connected ENR_ANR assms* **by** *blast*

lemma *ANR_Times*:
  fixes *S* :: '*a*::*euclidean_space set* **and** *T* :: '*b*::*euclidean_space set*

assumes *ANR S ANR T* **shows** *ANR*(*S* × *T*)
proof (*clarsimp simp only*: *ANR_eq_absolute_neighbourhood_extensor*)
fix *f* :: (′*a* × ′*b*) × *real* ⇒ ′*a* × ′*b* **and** *U C*
assume *continuous_on C f* **and** *fim*: *f* ‘ *C* ⊆ *S* × *T*
and *cloUC*: *closedin* (*top_of_set U*) *C*
have *contf1*: *continuous_on C* (*fst* ∘ *f*)
by (*simp add*: ‹*continuous_on C f*› *continuous_on_fst*)
obtain *W1 g* **where** *C* ⊆ *W1*
and *UW1*: *openin* (*top_of_set U*) *W1*
and *contg*: *continuous_on W1 g*
and *gim*: *g* ‘ *W1* ⊆ *S*
and *geq*: ⋀*x*. *x* ∈ *C* ⟹ *g x* = (*fst* ∘ *f*) *x*
proof (*rule ANR_imp_absolute_neighbourhood_extensor* [*OF* ‹*ANR S*› *contf1* _ *cloUC*])
show (*fst* ∘ *f*) ‘ *C* ⊆ *S*
using *fim* **by** *auto*
qed *auto*
have *contf2*: *continuous_on C* (*snd* ∘ *f*)
by (*simp add*: ‹*continuous_on C f*› *continuous_on_snd*)
obtain *W2 h* **where** *C* ⊆ *W2*
and *UW2*: *openin* (*top_of_set U*) *W2*
and *conth*: *continuous_on W2 h*
and *him*: *h* ‘ *W2* ⊆ *T*
and *heq*: ⋀*x*. *x* ∈ *C* ⟹ *h x* = (*snd* ∘ *f*) *x*
proof (*rule ANR_imp_absolute_neighbourhood_extensor* [*OF* ‹*ANR T*› *contf2* _ *cloUC*])
show (*snd* ∘ *f*) ‘ *C* ⊆ *T*
using *fim* **by** *auto*
qed *auto*
show ∃ *V g*. *C* ⊆ *V* ∧
openin (*top_of_set U*) *V* ∧
continuous_on V g ∧ *g* ‘ *V* ⊆ *S* × *T* ∧ (∀ *x*∈*C*. *g x* = *f x*)
proof (*intro exI conjI*)
show *C* ⊆ *W1* ∩ *W2*
by (*simp add*: ‹*C* ⊆ *W1*› ‹*C* ⊆ *W2*›)
show *openin* (*top_of_set U*) (*W1* ∩ *W2*)
by (*simp add*: *UW1 UW2 openin_Int*)
show *continuous_on* (*W1* ∩ *W2*) (λ*x*. (*g x*, *h x*))
by (*metis* (*no_types*) *contg conth continuous_on_Pair continuous_on_subset inf_commute inf_le1*)
show (λ*x*. (*g x*, *h x*)) ‘ (*W1* ∩ *W2*) ⊆ *S* × *T*
using *gim him* **by** *blast*
show (∀ *x*∈*C*. (*g x*, *h x*) = *f x*)
using *geq heq* **by** *auto*
qed
qed

lemma *AR_Times*:
fixes *S* :: ′*a*::*euclidean_space set* **and** *T* :: ′*b*::*euclidean_space set*

assumes *AR S AR T* **shows** *AR*(*S* × *T*)
using *assms* **by** (*simp add*: *AR_ANR ANR_Times contractible_Times*)

6.40.2 More advanced properties of ANRs and ENRs

lemma *ENR_rel_frontier_convex*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *bounded S convex S*
shows *ENR*(*rel_frontier S*)
proof (*cases S* = {})
case *True* **then show** *?thesis*
by *simp*
next
case *False*
with *assms* **have** *rel_interior S* ≠ {}
by (*simp add*: *rel_interior_eq_empty*)
then obtain *a* **where** *a*: *a* ∈ *rel_interior S*
by *auto*
have *ahS*: *affine hull S* − {*a*} ⊆ {*x*. *closest_point* (*affine hull S*) *x* ≠ *a*}
by (*auto simp*: *closest_point_self*)
have *rel_frontier S retract_of affine hull S* − {*a*}
by (*simp add*: *assms a rel_frontier_retract_of_punctured_affine_hull*)
also have . . . *retract_of* {*x*. *closest_point* (*affine hull S*) *x* ≠ *a*}
unfolding *retract_of_def retraction_def ahS*
apply (*rule_tac x*=*closest_point* (*affine hull S*) **in** *exI*)
apply (*auto simp*: *False closest_point_self affine_imp_convex closest_point_in_set continuous_on_closest_point*)
done
finally have *rel_frontier S retract_of* {*x*. *closest_point* (*affine hull S*) *x* ≠ *a*} **.**
moreover have *openin* (*top_of_set UNIV*) (*UNIV* ∩ *closest_point* (*affine hull S*) −‘ (− {*a*}))
by (*intro continuous_openin_preimage_gen*) (*auto simp*: *False affine_imp_convex continuous_on_closest_point*)
ultimately show *?thesis*
by (*meson ENR_convex_closed ENR_delete ENR_retract_of_ENR* ‹*rel_frontier S retract_of affine hull S* − {*a*}›
closed_affine_hull convex_affine_hull)
qed

lemma *ANR_rel_frontier_convex*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *bounded S convex S*
shows *ANR*(*rel_frontier S*)
by (*simp add*: *ENR_imp_ANR ENR_rel_frontier_convex assms*)

lemma *ENR_closedin_Un_local*:
fixes *S* :: ′*a*::*euclidean_space set*
shows ⟦*ENR S*; *ENR T*; *ENR*(*S* ∩ *T*);
closedin (*top_of_set* (*S* ∪ *T*)) *S*; *closedin* (*top_of_set* (*S* ∪ *T*)) *T*⟧

⟹ *ENR*(*S* ∪ *T*)
by (*simp add*: *ENR_ANR ANR_closed_Un_local locally_compact_closedin_Un*)

lemma *ENR_closed_Un*:
fixes *S* :: ′*a*::*euclidean_space set*
shows ⟦*closed S*; *closed T*; *ENR S*; *ENR T*; *ENR*(*S* ∩ *T*)⟧ ⟹ *ENR*(*S* ∪ *T*)
by (*auto simp*: *closed_subset ENR_closedin_Un_local*)

lemma *absolute_retract_Un*:
fixes *S* :: ′*a*::*euclidean_space set*
shows ⟦*S retract_of UNIV*; *T retract_of UNIV*; (*S* ∩ *T*) *retract_of UNIV*⟧
⟹ (*S* ∪ *T*) *retract_of UNIV*
by (*meson AR_closed_Un_local_aux closed_subset retract_of_UNIV retract_of_imp_subset*)

lemma *retract_from_Un_Int*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *clS*: *closedin* (*top_of_set* (*S* ∪ *T*)) *S*
and *clT*: *closedin* (*top_of_set* (*S* ∪ *T*)) *T*
and *Un*: (*S* ∪ *T*) *retract_of U* **and** *Int*: (*S* ∩ *T*) *retract_of T*
shows *S retract_of U*
proof −
obtain *r* **where** *r*: *continuous_on T r r* ‘ *T* ⊆ *S* ∩ *T* ∀ *x*∈*S* ∩ *T*. *r x* = *x*
using *Int* **by** (*auto simp*: *retraction_def retract_of_def*)
have *S retract_of S* ∪ *T*
unfolding *retraction_def retract_of_def*
proof (*intro exI conjI*)
show *continuous_on* (*S* ∪ *T*) (λ*x*. *if x* ∈ *S then x else r x*)
using *r* **by** (*intro continuous_on_cases_local* [*OF clS clT*]) *auto*
qed (*use r* **in** *auto*)
also have . . . *retract_of U*
by (*rule Un*)
finally show *?thesis* **.**
qed

lemma *AR_from_Un_Int_local*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *clS*: *closedin* (*top_of_set* (*S* ∪ *T*)) *S*
and *clT*: *closedin* (*top_of_set* (*S* ∪ *T*)) *T*
and *Un*: *AR*(*S* ∪ *T*) **and** *Int*: *AR*(*S* ∩ *T*)
shows *AR S*
by (*meson AR_imp_retract AR_retract_of_AR Un assms closedin_closed_subset local.Int*
retract_from_Un_Int retract_of_refl sup_ge2)

lemma *AR_from_Un_Int_local′*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *closedin* (*top_of_set* (*S* ∪ *T*)) *S*
and *closedin* (*top_of_set* (*S* ∪ *T*)) *T*
and *AR*(*S* ∪ *T*) *AR*(*S* ∩ *T*)

shows *AR T*
using *AR_from_Un_Int_local* [*of T S*] *assms* **by** (*simp add*: *Un_commute Int_commute*)

lemma *AR_from_Un_Int*:
fixes *S* :: *'a::euclidean_space set*
assumes *clo*: *closed S closed T* **and** *Un*: *AR*(*S* ∪ *T*) **and** *Int*: *AR*(*S* ∩ *T*)
shows *AR S*
by (*metis AR_from_Un_Int_local* [*OF _ _ Un Int*] *Un_commute clo closed_closedin closedin_closed_subset inf_sup_absorb subtopology_UNIV top_greatest*)

lemma *ANR_from_Un_Int_local*:
fixes *S* :: *'a::euclidean_space set*
assumes *clS*: *closedin* (*top_of_set* (*S* ∪ *T*)) *S*
and *clT*: *closedin* (*top_of_set* (*S* ∪ *T*)) *T*
and *Un*: *ANR*(*S* ∪ *T*) **and** *Int*: *ANR*(*S* ∩ *T*)
shows *ANR S*
proof −
obtain *V* **where** *clo*: *closedin* (*top_of_set* (*S* ∪ *T*)) (*S* ∩ *T*)
and *ope*: *openin* (*top_of_set* (*S* ∪ *T*)) *V*
and *ret*: *S* ∩ *T retract_of V*
using *ANR_imp_neighbourhood_retract* [*OF Int*] **by** (*metis clS clT closedin_Int*)
then obtain *r* **where** *r*: *continuous_on V r* **and** *rim*: *r* ‘ *V* ⊆ *S* ∩ *T* **and** *req*: ∀ *x*∈*S* ∩ *T*. *r x* = *x*
by (*auto simp*: *retraction_def retract_of_def*)
have *Vsub*: *V* ⊆ *S* ∪ *T*
by (*meson ope openin_contains_cball*)
have *Vsup*: *S* ∩ *T* ⊆ *V*
by (*simp add*: *retract_of_imp_subset ret*)
then have *eq*: *S* ∪ *V* = ((*S* ∪ *T*) − *T*) ∪ *V*
by *auto*
have *eq'*: *S* ∪ *V* = *S* ∪ (*V* ∩ *T*)
using *Vsub* **by** *blast*
have *continuous_on* (*S* ∪ *V* ∩ *T*) (λ*x*. *if x* ∈ *S then x else r x*)
proof (*rule continuous_on_cases_local*)
show *closedin* (*top_of_set* (*S* ∪ *V* ∩ *T*)) *S*
using *clS closedin_subset_trans inf.boundedE* **by** *blast*
show *closedin* (*top_of_set* (*S* ∪ *V* ∩ *T*)) (*V* ∩ *T*)
using *clT Vsup* **by** (*auto simp*: *closedin_closed*)
show *continuous_on* (*V* ∩ *T*) *r*
by (*meson Int_lower1 continuous_on_subset r*)
qed (*use req continuous_on_id* **in** *auto*)
with *rim* **have** *S retract_of S* ∪ *V*
unfolding *retraction_def retract_of_def* **using** *eq'* **by** *fastforce*
then show *?thesis*
using *ANR_neighborhood_retract* [*OF Un*]
using ‹*S* ∪ *V* = *S* ∪ *T* − *T* ∪ *V*› *clT ope* **by** *fastforce*
qed

lemma *ANR_from_Un_Int*:

fixes *S* :: *'a::euclidean_space set*
assumes *clo*: *closed S closed T* **and** *Un*: *ANR*(*S* ∪ *T*) **and** *Int*: *ANR*(*S* ∩ *T*)
shows *ANR S*
by (*metis ANR_from_Un_Int_local* [*OF* _ _ *Un Int*] *Un_commute clo closed_closedin closedin_closed_subset inf_sup_absorb subtopology_UNIV top_greatest*)

lemma *ANR_finite_Union_convex_closed*:
fixes $\mathcal{T}$ :: *'a::euclidean_space set set*
assumes $\mathcal{T}$: *finite* $\mathcal{T}$ **and** *clo*: $\bigwedge C.\ C \in \mathcal{T} \Longrightarrow$ *closed C* **and** *con*: $\bigwedge C.\ C \in \mathcal{T} \Longrightarrow$ *convex C*
shows *ANR*($\bigcup\mathcal{T}$)
proof –
have *ANR*($\bigcup\mathcal{T}$) **if** *card* $\mathcal{T} < n$ **for** *n*
using *assms that*
proof (*induction n arbitrary*: $\mathcal{T}$)
case *0* **then show** *?case* **by** *simp*
next
case (*Suc n*)
have *ANR*($\bigcup\mathcal{U}$) **if** *finite* $\mathcal{U}$ $\mathcal{U} \subseteq \mathcal{T}$ **for** $\mathcal{U}$
using *that*
proof (*induction* $\mathcal{U}$)
case *empty*
then show *?case* **by** *simp*
next
case (*insert C* $\mathcal{U}$)
have *ANR* (*C* ∪ $\bigcup\mathcal{U}$)
proof (*rule ANR_closed_Un*)
show *ANR* (*C* ∩ $\bigcup\mathcal{U}$)
unfolding *Int_Union*
proof (*rule Suc*)
show *finite* ((∩) *C* ‘ $\mathcal{U}$)
by (*simp add*: *insert.hyps*(*1*))
show $\bigwedge Ca.\ Ca \in (\cap)\ C$ ‘ $\mathcal{U} \Longrightarrow$ *closed Ca*
by (*metis* (*no_types*, *hide_lams*) *Suc.prems*(*2*) *closed_Int subsetD imageE insert.prems insertI1 insertI2*)
show $\bigwedge Ca.\ Ca \in (\cap)\ C$ ‘ $\mathcal{U} \Longrightarrow$ *convex Ca*
by (*metis* (*mono_tags*, *lifting*) *Suc.prems*(*3*) *convex_Int imageE insert.prems insert_subset subsetCE*)
show *card* ((∩) *C* ‘ $\mathcal{U}$) < *n*
proof –
have *card* $\mathcal{T} \leq n$
by (*meson Suc.prems*(*4*) *not_less not_less_eq*)
then show *?thesis*
by (*metis Suc.prems*(*1*) *card_image_le card_seteq insert.hyps insert.prems insert_subset le_trans not_less*)
qed
qed
show *closed* ($\bigcup\mathcal{U}$)
using *Suc.prems*(*2*) *insert.hyps*(*1*) *insert.prems* **by** *blast*

```
      qed (use Suc.prems convex_imp_ANR insert.prems insert.IH in auto)
      then show ?case
        by simp
    qed
    then show ?case
      using Suc.prems(1) by blast
  qed
  then show ?thesis
    by blast
qed


lemma finite_imp_ANR:
  fixes S :: 'a::euclidean_space set
  assumes finite S
  shows ANR S
proof −
  have ANR(⋃x ∈ S. {x})
    by (blast intro: ANR_finite_Union_convex_closed assms)
  then show ?thesis
    by simp
qed

lemma ANR_insert:
  fixes S :: 'a::euclidean_space set
  assumes ANR S closed S
  shows ANR(insert a S)
 by (metis ANR_closed_Un ANR_empty ANR_singleton Diff_disjoint Diff_insert_absorb
assms closed_singleton insert_absorb insert_is_Un)

lemma ANR_path_component_ANR:
  fixes S :: 'a::euclidean_space set
  shows ANR S ⟹ ANR(path_component_set S x)
 using ANR_imp_locally_path_connected ANR_openin openin_path_component_locally_path_connected
by blast

lemma ANR_connected_component_ANR:
  fixes S :: 'a::euclidean_space set
  shows ANR S ⟹ ANR(connected_component_set S x)
 by (metis ANR_openin openin_connected_component_locally_connected ANR_imp_locally_connected)

lemma ANR_component_ANR:
  fixes S :: 'a::euclidean_space set
  assumes ANR S c ∈ components S
  shows ANR c
  by (metis ANR_connected_component_ANR assms componentsE)
```

6.40.3 Original ANR material, now for ENRs

lemma *ENR_bounded*:
  fixes *S* :: *'a::euclidean_space set*
  assumes *bounded S*
  shows *ENR S* ⟷ (∃ *U*. *open U* ∧ *bounded U* ∧ *S retract_of U*)
    (**is** *?lhs* = *?rhs*)
proof
  obtain *r* **where** *0* < *r* **and** *r*: *S* ⊆ *ball 0 r*
    using *bounded_subset_ballD assms* **by** *blast*
  assume *?lhs*
  then show *?rhs*
   by (*meson ENR_def Elementary_Metric_Spaces.open_ball bounded_Int bounded_ball inf_le2 le_inf_iff*
      open_Int r retract_of_imp_subset retract_of_subset)
next
  assume *?rhs*
  then show *?lhs*
    using *ENR_def* **by** *blast*
qed

lemma *absolute_retract_imp_AR_gen*:
  fixes *S* :: *'a::euclidean_space set* **and** *S'* :: *'b::euclidean_space set*
  assumes *S retract_of T convex T T* ≠ {} *S homeomorphic S' closedin* (*top_of_set U*) *S'*
  shows *S' retract_of U*
proof −
  have *AR T*
    by (*simp add*: *assms convex_imp_AR*)
  then have *AR S*
    using *AR_retract_of_AR assms* **by** *auto*
  then show *?thesis*
    using *assms AR_imp_absolute_retract* **by** *metis*
qed

lemma *absolute_retract_imp_AR*:
  fixes *S* :: *'a::euclidean_space set* **and** *S'* :: *'b::euclidean_space set*
  assumes *S retract_of UNIV S homeomorphic S' closed S'*
  shows *S' retract_of UNIV*
  using *AR_imp_absolute_retract_UNIV assms retract_of_UNIV* **by** *blast*

lemma *homeomorphic_compact_arness*:
  fixes *S* :: *'a::euclidean_space set* **and** *S'* :: *'b::euclidean_space set*
  assumes *S homeomorphic S'*
  shows *compact S* ∧ *S retract_of UNIV* ⟷ *compact S'* ∧ *S' retract_of UNIV*
  using *assms homeomorphic_compactness*
  by (*metis compact_AR homeomorphic_AR_iff_AR*)

lemma *absolute_retract_from_Un_Int*:
  fixes *S* :: *'a::euclidean_space set*

assumes $(S \cup T)$ *retract_of UNIV* $(S \cap T)$ *retract_of UNIV closed S closed T*
shows *S retract_of UNIV*
using *AR_from_Un_Int assms retract_of_UNIV* **by** *auto*

lemma *ENR_from_Un_Int_gen*:
fixes *S* :: *'a::euclidean_space set*
assumes *closedin* (*top_of_set* $(S \cup T)$) *S closedin* (*top_of_set* $(S \cup T)$) *T ENR*$(S \cup T)$ *ENR*$(S \cap T)$
shows *ENR S*
by (*meson ANR_from_Un_Int_local ANR_imp_neighbourhood_retract ENR_ANR ENR_neighborhood_retract assms*)

lemma *ENR_from_Un_Int*:
fixes *S* :: *'a::euclidean_space set*
assumes *closed S closed T ENR*$(S \cup T)$ *ENR*$(S \cap T)$
shows *ENR S*
by (*meson ENR_from_Un_Int_gen assms closed_subset sup_ge1 sup_ge2*)

lemma *ENR_finite_Union_convex_closed*:
fixes $\mathcal{T}$ :: *'a::euclidean_space set set*
assumes $\mathcal{T}$: *finite* $\mathcal{T}$ **and** *clo*: $\bigwedge C.\ C \in \mathcal{T} \Longrightarrow$ *closed C* **and** *con*: $\bigwedge C.\ C \in \mathcal{T} \Longrightarrow$ *convex C*
shows *ENR*$(\bigcup \mathcal{T})$
by (*simp add*: *ENR_ANR ANR_finite_Union_convex_closed* $\mathcal{T}$ *clo closed_Union closed_imp_locally_compact con*)

lemma *finite_imp_ENR*:
fixes *S* :: *'a::euclidean_space set*
shows *finite S* $\Longrightarrow$ *ENR S*
by (*simp add*: *ENR_ANR finite_imp_ANR finite_imp_closed closed_imp_locally_compact*)

lemma *ENR_insert*:
fixes *S* :: *'a::euclidean_space set*
assumes *closed S ENR S*
shows *ENR*(*insert a S*)
proof −
have *ENR* $(\{a\} \cup S)$
by (*metis ANR_insert ENR_ANR Un_commute Un_insert_right assms closed_imp_locally_compact closed_insert sup_bot_right*)
then show *?thesis*
by *auto*
qed

lemma *ENR_path_component_ENR*:
fixes *S* :: *'a::euclidean_space set*
assumes *ENR S*
shows *ENR*(*path_component_set S x*)

by (*metis ANR_imp_locally_path_connected ENR_empty ENR_imp_ANR ENR_openin assms*
locally_path_connected_2 openin_subtopology_self path_component_eq_empty)

6.40.4 Finally, spheres are ANRs and ENRs

lemma *absolute_retract_homeomorphic_convex_compact*:
fixes *S* :: *'a::euclidean_space set* **and** *U* :: *'b::euclidean_space set*
assumes *S homeomorphic U S* ≠ {} *S* ⊆ *T convex U compact U*
shows *S retract_of T*
by (*metis UNIV_I assms compact_AR convex_imp_AR homeomorphic_AR_iff_AR homeomorphic_compactness homeomorphic_empty*(*1*) *retract_of_subset subsetI*)

lemma *frontier_retract_of_punctured_universe*:
fixes *S* :: *'a::euclidean_space set*
assumes *convex S bounded S a* ∈ *interior S*
shows (*frontier S*) *retract_of* (− {*a*})
using *rel_frontier_retract_of_punctured_affine_hull*
by (*metis Compl_eq_Diff_UNIV affine_hull_nonempty_interior assms empty_iff rel_frontier_frontier rel_interior_nonempty_interior*)

lemma *sphere_retract_of_punctured_universe_gen*:
fixes *a* :: *'a::euclidean_space*
assumes *b* ∈ *ball a r*
shows *sphere a r retract_of* (− {*b*})
proof −
have *frontier* (*cball a r*) *retract_of* (− {*b*})
using *assms frontier_retract_of_punctured_universe interior_cball* **by** *blast*
then show *?thesis*
by *simp*
qed

lemma *sphere_retract_of_punctured_universe*:
fixes *a* :: *'a::euclidean_space*
assumes *0* < *r*
shows *sphere a r retract_of* (− {*a*})
by (*simp add*: *assms sphere_retract_of_punctured_universe_gen*)

lemma *ENR_sphere*:
fixes *a* :: *'a::euclidean_space*
shows *ENR*(*sphere a r*)
proof (*cases 0* < *r*)
case *True*
then have *sphere a r retract_of* −{*a*}
by (*simp add*: *sphere_retract_of_punctured_universe*)
with *open_delete* **show** *?thesis*
by (*auto simp*: *ENR_def*)
next
case *False*

then show *?thesis*
using *finite_imp_ENR*
by (*metis finite_insert infinite_imp_nonempty less_linear sphere_eq_empty sphere_trivial*)
qed

corollary *ANR_sphere*:
fixes *a* :: *'a::euclidean_space*
shows *ANR*(*sphere a r*)
by (*simp add*: *ENR_imp_ANR ENR_sphere*)

6.40.5 Spheres are connected, etc

lemma *locally_path_connected_sphere_gen*:
fixes *S* :: *'a::euclidean_space set*
assumes *bounded S* **and** *convex S*
shows *locally path_connected* (*rel_frontier S*)
proof (*cases rel_interior S* = {})
case *True*
with *assms* **show** *?thesis*
by (*simp add*: *rel_interior_eq_empty*)
next
case *False*
then obtain *a* **where** *a*: *a* ∈ *rel_interior S*
by *blast*
show *?thesis*
proof (*rule retract_of_locally_path_connected*)
show *locally path_connected* (*affine hull S* − {*a*})
by (*meson convex_affine_hull convex_imp_locally_path_connected locally_open_subset openin_delete openin_subtopology_self*)
show *rel_frontier S retract_of affine hull S* − {*a*}
using *a assms rel_frontier_retract_of_punctured_affine_hull* **by** *blast*
qed
qed

lemma *locally_connected_sphere_gen*:
fixes *S* :: *'a::euclidean_space set*
assumes *bounded S* **and** *convex S*
shows *locally connected* (*rel_frontier S*)
by (*simp add*: *ANR_imp_locally_connected ANR_rel_frontier_convex assms*)

lemma *locally_path_connected_sphere*:
fixes *a* :: *'a::euclidean_space*
shows *locally path_connected* (*sphere a r*)
using *ENR_imp_locally_path_connected ENR_sphere* **by** *blast*

lemma *locally_connected_sphere*:
fixes *a* :: *'a::euclidean_space*
shows *locally connected*(*sphere a r*)
using *ANR_imp_locally_connected ANR_sphere* **by** *blast*

6.40.6 Borsuk homotopy extension theorem

It's only this late so we can use the concept of retraction, saying that the domain sets or range set are ENRs.

theorem *Borsuk_homotopy_extension_homotopic*:
  fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
  assumes *cloTS*: *closedin* (*top_of_set* T) S
    and *anr*: (*ANR* S ∧ *ANR* T) ∨ *ANR* U
    and *contf*: *continuous_on* T f
    and f ‘ $T \subseteq U$
    and *homotopic_with_canon* (λx. *True*) S U f g
  obtains g' **where** *homotopic_with_canon* (λx. *True*) T U f g'
      continuous_on T g' *image* g' $T \subseteq U$
      $\bigwedge x.\ x \in S \Longrightarrow g'\ x = g\ x$
proof −
  have $S \subseteq T$ **using** *assms closedin_imp_subset* **by** *blast*
  obtain h **where** *conth*: *continuous_on* ({*0..1*} × S) h
      and *him*: h ‘ ({*0..1*} × S) ⊆ U
      and [*simp*]: $\bigwedge x$. h(*0*, x) = f x $\bigwedge x$. h(*1::real*, x) = g x
    using *assms* **by** (*auto simp*: *homotopic_with_def*)
  define h' **where** $h' \equiv$ λz. *if snd z* ∈ S *then h z else* (f ∘ *snd*) z
  define B **where** $B \equiv$ {*0::real*} × T ∪ {*0..1*} × S
  have *clo0T*: *closedin* (*top_of_set* ({*0..1*} × T)) ({*0::real*} × T)
    by (*simp add*: *Abstract_Topology.closedin_Times*)
  moreover have *cloT1S*: *closedin* (*top_of_set* ({*0..1*} × T)) ({*0..1*} × S)
    by (*simp add*: *Abstract_Topology.closedin_Times assms*)
  ultimately have *clo0TB*:*closedin* (*top_of_set* ({*0..1*} × T)) B
    by (*auto simp*: *B_def*)
  have *cloBS*: *closedin* (*top_of_set* B) ({*0..1*} × S)
    by (*metis* (*no_types*) *Un_subset_iff B_def closedin_subset_trans* [*OF cloT1S*] *clo0TB closedin_imp_subset closedin_self*)
  moreover have *cloBT*: *closedin* (*top_of_set* B) ({*0*} × T)
    using ‹$S \subseteq T$› *closedin_subset_trans* [*OF clo0T*]
    by (*metis B_def Un_upper1 clo0TB closedin_closed inf_le1*)
  moreover have *continuous_on* ({*0*} × T) (f ∘ *snd*)
  proof (*rule continuous_intros*)+
    show *continuous_on* (*snd* ‘ ({*0*} × T)) f
      by (*simp add*: *contf*)
  qed
  ultimately have *continuous_on* ({*0..1*} × S ∪ {*0*} × T) (λx. *if snd x* ∈ S *then h x else* (f ∘ *snd*) x)
    by (*auto intro*!: *continuous_on_cases_local conth simp*: *B_def Un_commute* [*of* {*0*} × T])
  then have *conth'*: *continuous_on* B h'
    by (*simp add*: *h'_def B_def Un_commute* [*of* {*0*} × T])
  have *image* h' $B \subseteq U$
    using ‹f ‘ $T \subseteq U$› *him* **by** (*auto simp*: *h'_def B_def*)
  obtain V k **where** $B \subseteq V$ **and** *opeTV*: *openin* (*top_of_set* ({*0..1*} × T)) V
      and *contk*: *continuous_on* V k **and** *kim*: k ‘ $V \subseteq U$

```
            and keq: ⋀x. x ∈ B ⟹ k x = h' x
  using anr
  proof
    assume ST: ANR S ∧ ANR T
    have eq: ({0} × T ∩ {0..1} × S) = {0::real} × S
      using ‹S ⊆ T› by auto
    have ANR B
      unfolding B_def
    proof (rule ANR_closed_Un_local)
      show closedin (top_of_set ({0} × T ∪ {0..1} × S)) ({0::real} × T)
        by (metis cloBT B_def)
      show closedin (top_of_set ({0} × T ∪ {0..1} × S)) ({0..1::real} × S)
        by (metis Un_commute cloBS B_def)
    qed (simp_all add: ANR_Times convex_imp_ANR ANR_singleton ST eq)
    note Vk = that
    have *: thesis if openin (top_of_set ({0..1::real} × T)) V
                  retraction V B r for V r
    proof -
      have continuous_on V (h' ∘ r)
        using conth' continuous_on_compose retractionE that(2) by blast
      moreover have (h' ∘ r) ` V ⊆ U
        by (metis ‹h' ` B ⊆ U› image_comp retractionE that(2))
      ultimately show ?thesis
        using Vk [of V h' ∘ r] by (metis comp_apply retraction that)
    qed
    show thesis
      by (meson * ANR_imp_neighbourhood_retract ‹ANR B› clo0TB retract_of_def)
  next
    assume ANR U
    with ANR_imp_absolute_neighbourhood_extensor ‹h' ` B ⊆ U› clo0TB conth'
that
    show ?thesis by blast
  qed
  define S' where S' ≡ {x. ∃u::real. u ∈ {0..1} ∧ (u, x::'a) ∈ {0..1} × T −
V}
  have closedin (top_of_set T) S'
    unfolding S'_def using closedin_self opeTV
    by (blast intro: closedin_compact_projection)
  have S'_def: S' = {x. ∃u::real.  (u, x::'a) ∈ {0..1} × T − V}
    by (auto simp: S'_def)
  have cloTS': closedin (top_of_set T) S'
    using S'_def ‹closedin (top_of_set T) S'› by blast
  have S ∩ S' = {}
    using S'_def B_def ‹B ⊆ V› by force
  obtain a :: 'a ⇒ real where conta: continuous_on T a
      and ⋀x. x ∈ T ⟹ a x ∈ closed_segment 1 0
      and a1: ⋀x. x ∈ S ⟹ a x = 1
      and a0: ⋀x. x ∈ S' ⟹ a x = 0
    by (rule Urysohn_local [OF cloTS cloTS' ‹S ∩ S' = {}›, of 1 0], blast)
```

then have *ain*: ⋀*x*. *x* ∈ *T* ⟹ *a x* ∈ {*0..1*}
  using *closed_segment_eq_real_ivl* **by** *auto*
have *inV*: (*u* ∗ *a t*, *t*) ∈ *V* **if** *t* ∈ *T 0* ≤ *u u* ≤ *1* **for** *t u*
proof (*rule ccontr*)
  assume (*u* ∗ *a t*, *t*) ∉ *V*
  with *ain* [*OF* ‹*t* ∈ *T*›] **have** *a t* = *0*
    apply *simp*
   by (*metis* (*no_types*, *lifting*) *a0 DiffI S'_def SigmaI atLeastAtMost_iff mem_Collect_eq mult_le_one mult_nonneg_nonneg that*)
  show *False*
    using *B_def* ‹(*u* ∗ *a t*, *t*) ∉ *V*› ‹*B* ⊆ *V*› ‹*a t* = *0*› *that* **by** *auto*
qed
show *?thesis*
proof
  show *hom*: *homotopic_with_canon* (λ*x*. *True*) *T U f* (λ*x*. *k* (*a x*, *x*))
  proof (*simp add*: *homotopic_with*, *intro exI conjI*)
    show *continuous_on* ({*0..1*} × *T*) (*k* ∘ (λ*z*. (*fst z* ∗_R (*a* ∘ *snd*) *z*, *snd z*)))
      apply (*intro continuous_on_compose continuous_intros*)
     apply (*force intro*: *inV continuous_on_subset* [*OF contk*] *continuous_on_subset* [*OF conta*])+
      done
    show (*k* ∘ (λ*z*. (*fst z* ∗_R (*a* ∘ *snd*) *z*, *snd z*))) ‘ ({*0..1*} × *T*) ⊆ *U*
      using *inV kim* **by** *auto*
    show ∀ *x*∈*T*. (*k* ∘ (λ*z*. (*fst z* ∗_R (*a* ∘ *snd*) *z*, *snd z*))) (*0*, *x*) = *f x*
      by (*simp add*: *B_def h'_def keq*)
    show ∀ *x*∈*T*. (*k* ∘ (λ*z*. (*fst z* ∗_R (*a* ∘ *snd*) *z*, *snd z*))) (*1*, *x*) = *k* (*a x*, *x*)
      by *auto*
  qed
show *continuous_on T* (λ*x*. *k* (*a x*, *x*))
  using *homotopic_with_imp_continuous_maps* [*OF hom*] **by** *auto*
show (λ*x*. *k* (*a x*, *x*)) ‘ *T* ⊆ *U*
proof *clarify*
  fix *t*
  assume *t* ∈ *T*
  show *k* (*a t*, *t*) ∈ *U*
   by (*metis* ‹*t* ∈ *T*› *image_subset_iff inV kim not_one_le_zero linear mult_cancel_right1*)
qed
show ⋀*x*. *x* ∈ *S* ⟹ *k* (*a x*, *x*) = *g x*
  by (*simp add*: *B_def a1 h'_def keq*)
qed
qed

corollary *nullhomotopic_into_ANR_extension*:
  fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
  assumes *closed S*
    and *contf*: *continuous_on S f*
    and *ANR T*
    and *fim*: *f* ‘ *S* ⊆ *T*

```
    and S ≠ {}
  shows (∃c. homotopic_with_canon (λx. True) S T f (λx. c)) ⟷
        (∃g. continuous_on UNIV g ∧ range g ⊆ T ∧ (∀x ∈ S. g x = f x))
      (is ?lhs = ?rhs)
proof
  assume ?lhs
  then obtain c where c: homotopic_with_canon (λx. True) S T (λx. c) f
    by (blast intro: homotopic_with_symD)
  have closedin (top_of_set UNIV) S
    using ‹closed S› closed_closedin by fastforce
  then obtain g where continuous_on UNIV g range g ⊆ T
                 ⋀x. x ∈ S ⟹ g x = f x
  proof (rule Borsuk_homotopy_extension_homotopic)
    show range (λx. c) ⊆ T
      using ‹S ≠ {}› c homotopic_with_imp_subset1 by fastforce
  qed (use assms c in auto)
  then show ?rhs by blast
next
  assume ?rhs
  then obtain g where continuous_on UNIV g range g ⊆ T ⋀x. x∈S ⟹ g x =
f x
    by blast
  then obtain c where homotopic_with_canon (λh. True) UNIV T g (λx. c)
     using nullhomotopic_from_contractible [of UNIV g T] contractible_UNIV by
blast
  then have homotopic_with_canon (λx. True) S T g (λx. c)
    by (simp add: homotopic_from_subtopology)
  then show ?lhs
    by (force elim: homotopic_with_eq [of _ _ _ g λx. c] simp: ‹⋀x. x ∈ S ⟹ g x
= f x›)
qed

corollary nullhomotopic_into_rel_frontier_extension:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes closed S
    and contf: continuous_on S f
    and convex T bounded T
    and fim: f ` S ⊆ rel_frontier T
    and S ≠ {}
  shows (∃c. homotopic_with_canon (λx. True) S (rel_frontier T) f (λx. c)) ⟷
        (∃g. continuous_on UNIV g ∧ range g ⊆ rel_frontier T ∧ (∀x ∈ S. g x =
f x))
by (simp add: nullhomotopic_into_ANR_extension assms ANR_rel_frontier_convex)

corollary nullhomotopic_into_sphere_extension:
  fixes f :: 'a::euclidean_space ⇒ 'b :: euclidean_space
  assumes closed S and contf: continuous_on S f
    and S ≠ {} and fim: f ` S ⊆ sphere a r
   shows ((∃c. homotopic_with_canon (λx. True) S (sphere a r) f (λx. c)) ⟷
```

(∃ *g*. *continuous_on UNIV g* ∧ *range g* ⊆ *sphere a r* ∧ (∀ *x* ∈ *S*. *g x* = *f x*)))
(**is** *?lhs* = *?rhs*)
proof (*cases r* = *0*)
case *True* **with** *fim* **show** *?thesis*
by (*metis ANR_sphere* ‹*closed S*› ‹*S* ≠ {}› *contf nullhomotopic_into_ANR_extension*)
next
case *False*
then have *eq*: *sphere a r* = *rel_frontier* (*cball a r*) **by** *simp*
show *?thesis*
using *fim nullhomotopic_into_rel_frontier_extension* [*OF* ‹*closed S*› *contf convex_cball bounded_cball*]
by (*simp add*: ‹*S* ≠ {}› *eq*)
qed

proposition *Borsuk_map_essential_bounded_component*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *compact S* **and** *a* ∉ *S*
shows *bounded* (*connected_component_set* (− *S*) *a*) ⟷
¬(∃ *c*. *homotopic_with_canon* (λ*x*. *True*) *S* (*sphere 0 1*)
(λ*x*. *inverse*(*norm*(*x* − *a*)) $*_R$ (*x* − *a*)) (λ*x*. *c*))
(**is** *?lhs* = *?rhs*)
proof (*cases S* = {})
case *True* **then show** *?thesis*
by *simp*
next
case *False*
have *closed S bounded S*
using ‹*compact S*› *compact_eq_bounded_closed* **by** *auto*
have *s01*: (λ*x*. (*x* − *a*) $/_R$ *norm* (*x* − *a*)) ‘ *S* ⊆ *sphere 0 1*
using ‹*a* ∉ *S*› **by** *clarsimp* (*metis dist_eq_0_iff dist_norm mult.commute right_inverse*)
have *aincc*: *a* ∈ *connected_component_set* (− *S*) *a*
by (*simp add*: ‹*a* ∉ *S*›)
obtain *r* **where** *r*>*0* **and** *r*: *S* ⊆ *ball 0 r*
using *bounded_subset_ballD* ‹*bounded S*› **by** *blast*
have ¬ *?rhs* ⟷ ¬ *?lhs*
proof
assume *notr*: ¬ *?rhs*
have *nog*: ∄ *g*. *continuous_on* (*S* ∪ *connected_component_set* (− *S*) *a*) *g* ∧
g ‘ (*S* ∪ *connected_component_set* (− *S*) *a*) ⊆ *sphere 0 1* ∧
(∀ *x*∈*S*. *g x* = (*x* − *a*) $/_R$ *norm* (*x* − *a*))
if *bounded* (*connected_component_set* (− *S*) *a*)
using *non_extensible_Borsuk_map* [*OF* ‹*compact S*› *componentsI _ aincc*] ‹*a* ∉ *S*› *that* **by** *auto*
obtain *g* **where** *range g* ⊆ *sphere 0 1 continuous_on UNIV g*
⋀*x*. *x* ∈ *S* ⟹ *g x* = (*x* − *a*) $/_R$ *norm* (*x* − *a*)
using *notr*
by (*auto simp*: *nullhomotopic_into_sphere_extension*
[*OF* ‹*closed S*› *continuous_on_Borsuk_map* [*OF* ‹*a* ∉ *S*›] *False s01*])

```
    with ‹a ∉ S› show  ¬ ?lhs
      by (metis UNIV_I continuous_on_subset image_subset_iff nog subsetI)
  next
    assume ¬ ?lhs
    then obtain b where b: b ∈ connected_component_set (− S) a and r ≤ norm
b
      using bounded_iff linear by blast
    then have bnot: b ∉ ball 0 r
      by simp
    have homotopic_with_canon (λx. True) S (sphere 0 1) (λx. (x − a) /R norm
(x − a))
                                             (λx. (x − b) /R norm (x − b))
    proof −
      have path_component (− S) a b
      by (metis (full_types) ‹closed S› b mem_Collect_eq open_Compl open_path_connected_component)
      then show ?thesis
        using Borsuk_maps_homotopic_in_path_component by blast
    qed
    moreover
    obtain c where homotopic_with_canon (λx. True) (ball 0 r) (sphere 0 1)
                              (λx. inverse (norm (x − b)) *R (x − b)) (λx. c)
    proof (rule nullhomotopic_from_contractible)
      show contractible (ball (0::'a) r)
        by (metis convex_imp_contractible convex_ball)
      show continuous_on (ball 0 r) (λx. inverse(norm (x − b)) *R (x − b))
        by (rule continuous_on_Borsuk_map [OF bnot])
      show (λx. (x − b) /R norm (x − b)) ` ball 0 r ⊆ sphere 0 1
        using bnot Borsuk_map_into_sphere by blast
    qed blast
    ultimately have homotopic_with_canon (λx. True) S (sphere 0 1) (λx. (x −
a) /R norm (x − a)) (λx. c)
      by (meson homotopic_with_subset_left homotopic_with_trans r)
    then show ¬ ?rhs
      by blast
  qed
  then show ?thesis by blast
qed

lemma homotopic_Borsuk_maps_in_bounded_component:
  fixes a :: 'a :: euclidean_space
  assumes compact S and a ∉ Sand b ∉ S
      and boc: bounded (connected_component_set (− S) a)
      and hom: homotopic_with_canon (λx. True) S (sphere 0 1)
                              (λx. (x − a) /R norm (x − a))
                              (λx. (x − b) /R norm (x − b))
   shows connected_component (− S) a b
proof (rule ccontr)
  assume notcc: ¬ connected_component (− S) a b
  let ?T = S ∪ connected_component_set (− S) a
```

have ∄ *g*. *continuous_on* (*S* ∪ *connected_component_set* (− *S*) *a*) *g* ∧
g ' (*S* ∪ *connected_component_set* (− *S*) *a*) ⊆ *sphere 0 1* ∧
(∀ *x*∈*S*. *g x* = (*x* − *a*) $/_R$ *norm* (*x* − *a*))
by (*simp add*: ‹*a* ∉ *S*› *componentsI non_extensible_Borsuk_map* [*OF* ‹*compact S*› _ *boc*])
moreover obtain *g* **where** *continuous_on* (*S* ∪ *connected_component_set* (− *S*) *a*) *g*
g ' (*S* ∪ *connected_component_set* (− *S*) *a*) ⊆ *sphere 0 1*
⋀*x*. *x* ∈ *S* ⟹ *g x* = (*x* − *a*) $/_R$ *norm* (*x* − *a*)
proof (*rule Borsuk_homotopy_extension_homotopic*)
show *closedin* (*top_of_set ?T*) *S*
by (*simp add*: ‹*compact S*› *closed_subset compact_imp_closed*)
show *continuous_on ?T* (λ*x*. (*x* − *b*) $/_R$ *norm* (*x* − *b*))
by (*simp add*: ‹*b* ∉ *S*› *notcc continuous_on_Borsuk_map*)
show (λ*x*. (*x* − *b*) $/_R$ *norm* (*x* − *b*)) ' *?T* ⊆ *sphere 0 1*
by (*simp add*: ‹*b* ∉ *S*› *notcc Borsuk_map_into_sphere*)
show *homotopic_with_canon* (λ*x*. *True*) *S* (*sphere 0 1*)
(λ*x*. (*x* − *b*) $/_R$ *norm* (*x* − *b*)) (λ*x*. (*x* − *a*) $/_R$ *norm* (*x* − *a*))
by (*simp add*: *hom homotopic_with_symD*)
qed (*auto simp*: *ANR_sphere intro*: *that*)
ultimately show *False* **by** *blast*
qed

lemma *Borsuk_maps_homotopic_in_connected_component_eq*:
fixes *a* :: '*a* :: *euclidean_space*
assumes *S*: *compact S a* ∉ *S b* ∉ *S* **and** *2*: *2* ≤ *DIM*('*a*)
shows (*homotopic_with_canon* (λ*x*. *True*) *S* (*sphere 0 1*)
(λ*x*. (*x* − *a*) $/_R$ *norm* (*x* − *a*))
(λ*x*. (*x* − *b*) $/_R$ *norm* (*x* − *b*)) ⟷
connected_component (− *S*) *a b*)
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
show *?rhs*
proof (*cases bounded*(*connected_component_set* (− *S*) *a*))
case *True*
show *?thesis*
by (*rule homotopic_Borsuk_maps_in_bounded_component* [*OF S True L*])
next
case *not_bo_a*: *False*
show *?thesis*
proof (*cases bounded*(*connected_component_set* (− *S*) *b*))
case *True*
show *?thesis*
using *homotopic_Borsuk_maps_in_bounded_component* [*OF S*]
by (*simp add*: *L True assms connected_component_sym homotopic_Borsuk_maps_in_bounded_compone*
homotopic_with_sym)
next

```
      case False
      then show ?thesis
          using cobounded_unique_unbounded_component [of −S a b] ‹compact S›
not_bo_a
        by (auto simp: compact_eq_bounded_closed assms connected_component_eq_eq)
    qed
  qed
next
  assume R: ?rhs
  then have path_component (− S) a b
   using assms(1) compact_eq_bounded_closed open_Compl open_path_connected_component_set
by fastforce
  then show ?lhs
    by (simp add: Borsuk_maps_homotopic_in_path_component)
qed
```

6.40.7 More extension theorems

```
lemma extension_from_clopen:
  assumes ope: openin (top_of_set S) T
      and clo: closedin (top_of_set S) T
      and contf: continuous_on T f and fim: f ‘ T ⊆ U and null: U = {} ⟹ S
= {}
 obtains g where continuous_on S g g ‘ S ⊆ U ⋀x. x ∈ T ⟹ g x = f x
proof (cases U = {})
  case True
  then show ?thesis
    by (simp add: null that)
next
  case False
  then obtain a where a ∈ U
    by auto
  let ?g = λx. if x ∈ T then f x else a
  have Seq: S = T ∪ (S − T)
    using clo closedin_imp_subset by fastforce
  show ?thesis
  proof
    have continuous_on (T ∪ (S − T)) ?g
      using Seq clo ope  by (intro continuous_on_cases_local) (auto simp: contf)
    with Seq show continuous_on S ?g
      by metis
    show ?g ‘ S ⊆ U
      using ‹a ∈ U› fim by auto
    show ⋀x. x ∈ T ⟹ ?g x = f x
      by auto
  qed
qed
```

lemma *extension_from_component*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *euclidean_space*
assumes *S*: *locally connected S* ∨ *compact S* **and** *ANR U*
and *C*: *C* ∈ *components S* **and** *contf*: *continuous_on C f* **and** *fim*: *f* ' *C* ⊆ *U*
obtains *g* **where** *continuous_on S g g* ' *S* ⊆ *U* ⋀*x*. *x* ∈ *C* ⟹ *g x* = *f x*
proof −
obtain *T g* **where** *ope*: *openin* (*top_of_set S*) *T*
and *clo*: *closedin* (*top_of_set S*) *T*
and *C* ⊆ *T* **and** *contg*: *continuous_on T g* **and** *gim*: *g* ' *T* ⊆ *U*
and *gf*: ⋀*x*. *x* ∈ *C* ⟹ *g x* = *f x*
using *S*
proof
assume *locally connected S*
show *?thesis*
by (*metis C* ‹*locally connected S*› *openin_components_locally_connected closedin_component contf fim order_refl that*)
next
assume *compact S*
then obtain *W g* **where** *C* ⊆ *W* **and** *opeW*: *openin* (*top_of_set S*) *W*
and *contg*: *continuous_on W g*
and *gim*: *g* ' *W* ⊆ *U* **and** *gf*: ⋀*x*. *x* ∈ *C* ⟹ *g x* = *f x*
using *ANR_imp_absolute_neighbourhood_extensor* [*of U C f S*] *C* ‹*ANR U*› *closedin_component contf fim* **by** *blast*
then obtain *V* **where** *open V* **and** *V*: *W* = *S* ∩ *V*
by (*auto simp*: *openin_open*)
moreover have *locally compact S*
by (*simp add*: ‹*compact S*› *closed_imp_locally_compact compact_imp_closed*)
ultimately obtain *K* **where** *opeK*: *openin* (*top_of_set S*) *K* **and** *compact K* *C* ⊆ *K K* ⊆ *V*
by (*metis C Int_subset_iff* ‹*C* ⊆ *W*› ‹*compact S*› *compact_components Sura_Bura_clopen_subset*)
show *?thesis*
proof
show *closedin* (*top_of_set S*) *K*
by (*meson* ‹*compact K*› ‹*compact S*› *closedin_compact_eq opeK openin_imp_subset*)
show *continuous_on K g*
by (*metis Int_subset_iff V* ‹*K* ⊆ *V*› *contg continuous_on_subset opeK openin_subtopology subset_eq*)
show *g* ' *K* ⊆ *U*
using *V* ‹*K* ⊆ *V*› *gim opeK openin_imp_subset* **by** *fastforce*
qed (*use opeK gf* ‹*C* ⊆ *K*› **in** *auto*)
qed
obtain *h* **where** *continuous_on S h h* ' *S* ⊆ *U* ⋀*x*. *x* ∈ *T* ⟹ *h x* = *g x*
using *extension_from_clopen*
by (*metis C bot.extremum_uniqueI clo contg gim fim image_is_empty in_components_nonempty ope*)
then show *?thesis*
by (*metis* ‹*C* ⊆ *T*› *gf subset_eq that*)
qed

lemma *tube_lemma*:
  fixes S :: $'a$::*euclidean_space set* **and** T :: $'b$::*euclidean_space set*
  assumes *compact* S **and** S: $S \neq \{\}$ $(\lambda x.\ (x,a))$ ` $S \subseteq U$
    and *ope*: *openin* (*top_of_set* $(S \times T)$) U
  obtains V **where** *openin* (*top_of_set* T) V $a \in V$ $S \times V \subseteq U$
proof –
  let $?W = \{y.\ \exists x.\ x \in S \wedge (x, y) \in (S \times T - U)\}$
  have $U \subseteq S \times T$ *closedin* (*top_of_set* $(S \times T)$) $(S \times T - U)$
    using *ope* **by** (*auto simp*: *openin_closedin_eq*)
  then have *closedin* (*top_of_set* T) $?W$
    using ‹*compact* S› *closedin_compact_projection* **by** *blast*
  moreover have $a \in T - ?W$
    using ‹$U \subseteq S \times T$› S **by** *auto*
  moreover have $S \times (T - ?W) \subseteq U$
    by *auto*
  ultimately show *?thesis*
   by (*metis* (*no_types*, *lifting*) *Sigma_cong closedin_def that topspace_euclidean_subtopology*)
qed

lemma *tube_lemma_gen*:
  fixes S :: $'a$::*euclidean_space set* **and** T :: $'b$::*euclidean_space set*
  assumes *compact* S $S \neq \{\}$ $T \subseteq T'$ $S \times T \subseteq U$
    and *ope*: *openin* (*top_of_set* $(S \times T')$) U
  obtains V **where** *openin* (*top_of_set* T') V $T \subseteq V$ $S \times V \subseteq U$
proof –
  have $\bigwedge x.\ x \in T \Longrightarrow \exists V.$ *openin* (*top_of_set* T') $V \wedge x \in V \wedge S \times V \subseteq U$
    using *assms* **by** (*auto intro*: *tube_lemma* [*OF* ‹*compact* S›])
  then obtain F **where** F: $\bigwedge x.\ x \in T \Longrightarrow$ *openin* (*top_of_set* T') $(F\ x) \wedge x \in F$
$x \wedge S \times F\ x \subseteq U$
    by *metis*
  show *?thesis*
  proof
    show *openin* (*top_of_set* T') $(\bigcup(F$ ` $T))$
      using F **by** *blast*
    show $T \subseteq \bigcup(F$ ` $T)$
      using F **by** *blast*
    show $S \times \bigcup(F$ ` $T) \subseteq U$
      using F **by** *auto*
  qed
qed

proposition *homotopic_neighbourhood_extension*:
  fixes f :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
  assumes *contf*: *continuous_on* S f **and** *fim*: f ` $S \subseteq U$
    and *contg*: *continuous_on* S g **and** *gim*: g ` $S \subseteq U$
    and *clo*: *closedin* (*top_of_set* S) T
    and *ANR* U **and** *hom*: *homotopic_with_canon* $(\lambda x.\ True)$ T U f g
   obtains V **where** $T \subseteq V$ *openin* (*top_of_set* S) V

homotopic_with_canon (*λx. True*) *V U f g*
proof −
have *T* ⊆ *S*
using *clo closedin_imp_subset* **by** *blast*
obtain *h* **where** *conth*: *continuous_on* ({*0..1*::*real*} × *T*) *h*
and *him*: *h* ‘ ({*0..1*} × *T*) ⊆ *U*
and *h0*: ⋀*x*. *h*(*0*, *x*) = *f x* **and** *h1*: ⋀*x*. *h*(*1*, *x*) = *g x*
using *hom* **by** (*auto simp*: *homotopic_with_def*)
define *h′* **where** *h′* ≡ *λz*. *if fst z* ∈ {*0*} *then f*(*snd z*)
else if fst z ∈ {*1*} *then g*(*snd z*)
else h z
let *?S0* = {*0*::*real*} × *S* **and** *?S1* = {*1*::*real*} × *S*
have *continuous_on*(*?S0* ∪ (*?S1* ∪ {*0..1*} × *T*)) *h′*
unfolding *h′_def*
proof (*intro continuous_on_cases_local*)
show *closedin* (*top_of_set* (*?S0* ∪ (*?S1* ∪ {*0..1*} × *T*))) *?S0*
closedin (*top_of_set* (*?S1* ∪ {*0..1*} × *T*)) *?S1*
using ‹*T* ⊆ *S*› **by** (*force intro*: *closedin_Times closedin_subset_trans* [*of* {*0..1*} × *S*])+
show *closedin* (*top_of_set* (*?S0* ∪ (*?S1* ∪ {*0..1*} × *T*))) (*?S1* ∪ {*0..1*} × *T*)
closedin (*top_of_set* (*?S1* ∪ {*0..1*} × *T*)) ({*0..1*} × *T*)
using ‹*T* ⊆ *S*› **by** (*force intro*: *clo closedin_Times closedin_subset_trans* [*of* {*0..1*} × *S*])+
show *continuous_on* (*?S0*) (*λx*. *f* (*snd x*))
by (*intro continuous_intros continuous_on_compose2* [*OF contf*]) *auto*
show *continuous_on* (*?S1*) (*λx*. *g* (*snd x*))
by (*intro continuous_intros continuous_on_compose2* [*OF contg*]) *auto*
qed (*use h0 h1 conth* **in** *auto*)
then have *continuous_on* ({*0,1*} × *S* ∪ ({*0..1*} × *T*)) *h′*
by (*metis Sigma_Un_distrib1 Un_assoc insert_is_Un*)
moreover have *h′* ‘ ({*0,1*} × *S* ∪ {*0..1*} × *T*) ⊆ *U*
using *fim gim him* ‹*T* ⊆ *S*› **unfolding** *h′_def* **by** *force*
moreover have *closedin* (*top_of_set* ({*0..1*::*real*} × *S*)) ({*0,1*} × *S* ∪ {*0..1*::*real*} × *T*)
by (*intro closedin_Times closedin_Un clo*) (*simp_all add*: *closed_subset*)
ultimately
obtain *W k* **where** *W*: ({*0,1*} × *S*) ∪ ({*0..1*} × *T*) ⊆ *W*
and *opeW*: *openin* (*top_of_set* ({*0..1*} × *S*)) *W*
and *contk*: *continuous_on W k*
and *kim*: *k* ‘ *W* ⊆ *U*
and *kh′*: ⋀*x*. *x* ∈ ({*0,1*} × *S*) ∪ ({*0..1*} × *T*) ⟹ *k x* = *h′ x*
by (*metis ANR_imp_absolute_neighbourhood_extensor* [*OF* ‹*ANR U*›, *of* ({*0,1*} × *S*) ∪ ({*0..1*} × *T*) *h′* {*0..1*} × *S*])
obtain *T′* **where** *opeT′*: *openin* (*top_of_set S*) *T′*
and *T* ⊆ *T′* **and** *TW*: {*0..1*} × *T′* ⊆ *W*
using *tube_lemma_gen* [*of* {*0..1*::*real*} *T S W*] *W* ‹*T* ⊆ *S*› *opeW* **by** *auto*
moreover have *homotopic_with_canon* (*λx. True*) *T′ U f g*
proof (*simp add*: *homotopic_with*, *intro exI conjI*)
show *continuous_on* ({*0..1*} × *T′*) *k*

using *TW continuous_on_subset contk* **by** *auto*
show k ‘ $(\{0..1\} \times T') \subseteq U$
using *TW kim* **by** *fastforce*
have $T' \subseteq S$
by (*meson opeT′ subsetD openin_imp_subset*)
then show $\forall x{\in}T'.\ k\ (0,\ x) = f\ x$ $\forall x{\in}T'.\ k\ (1,\ x) = g\ x$
by (*auto simp*: *kh′ h′_def*)
qed
ultimately show *?thesis*
by (*blast intro*: *that*)
qed

Homotopy on a union of closed-open sets.

proposition *homotopic_on_clopen_Union*:
fixes $\mathcal{F}$ :: *′a::euclidean_space set set*
assumes $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *closedin* (*top_of_set* $(\bigcup\mathcal{F})$) S
and $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *openin* (*top_of_set* $(\bigcup\mathcal{F})$) S
and $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *homotopic_with_canon* ($\lambda x.$ *True*) $S\ T\ f\ g$
shows *homotopic_with_canon* ($\lambda x.$ *True*) $(\bigcup\mathcal{F})$ $T\ f\ g$
proof −
obtain $\mathcal{V}$ **where** $\mathcal{V} \subseteq \mathcal{F}$ *countable* $\mathcal{V}$ **and** *eqU*: $\bigcup\mathcal{V} = \bigcup\mathcal{F}$
using *Lindelof_openin assms* **by** *blast*
show *?thesis*
proof (*cases* $\mathcal{V} = \{\}$)
case *True*
then show *?thesis*
by (*metis Union_empty eqU homotopic_with_canon_on_empty*)
next
case *False*
then obtain V :: *nat* ⇒ *′a set* **where** V: *range* $V = \mathcal{V}$
using *range_from_nat_into* ‹*countable* $\mathcal{V}$› **by** *metis*
with ‹$\mathcal{V} \subseteq \mathcal{F}$› **have** *clo*: $\bigwedge n.$ *closedin* (*top_of_set* $(\bigcup\mathcal{F})$) $(V\ n)$
and *ope*: $\bigwedge n.$ *openin* (*top_of_set* $(\bigcup\mathcal{F})$) $(V\ n)$
and *hom*: $\bigwedge n.$ *homotopic_with_canon* ($\lambda x.$ *True*) $(V\ n)\ T\ f\ g$
using *assms* **by** *auto*
then obtain h **where** *conth*: $\bigwedge n.$ *continuous_on* $(\{0..1{::}real\} \times V\ n)\ (h\ n)$
and *him*: $\bigwedge n.\ h\ n$ ‘ $(\{0..1\} \times V\ n) \subseteq T$
and *h0*: $\bigwedge n.\ \bigwedge x.\ x \in V\ n \Longrightarrow h\ n\ (0,\ x) = f\ x$
and *h1*: $\bigwedge n.\ \bigwedge x.\ x \in V\ n \Longrightarrow h\ n\ (1,\ x) = g\ x$
by (*simp add*: *homotopic_with*) *metis*
have *wop*: $b \in V\ x \Longrightarrow \exists k.\ b \in V\ k \wedge (\forall j{<}k.\ b \notin V\ j)$ **for** $b\ x$
using *nat_less_induct* [**where** $P = \lambda i.\ b \notin V\ i$] **by** *meson*
obtain ζ **where** *cont*: *continuous_on* $(\{0..1\} \times \bigcup(V$ ‘ $UNIV))\ \zeta$
and *eq*: $\bigwedge x\ i.\ [\![x \in \{0..1\} \times \bigcup(V$ ‘ $UNIV) \cap$
$\{0..1\} \times (V\ i - (\bigcup m{<}i.\ V\ m))]\!] \Longrightarrow \zeta\ x = h\ i\ x$
proof (*rule pasting_lemma_exists*)
let *?X* = *top_of_set* $(\{0..1{::}real\} \times \bigcup(range\ V))$
show *topspace ?X* $\subseteq (\bigcup i.\ \{0..1{::}real\} \times (V\ i - (\bigcup m{<}i.\ V\ m)))$
by (*force simp*: *Ball_def dest*: *wop*)

show *openin* (*top_of_set* ({*0..1*} × ⋃(*V* ' *UNIV*)))
({*0..1*::*real*} × (*V i* − (⋃*m*<*i*. *V m*))) **for** *i*
proof (*intro openin_Times openin_subtopology_self openin_diff*)
show *openin* (*top_of_set* (⋃(*V* ' *UNIV*))) (*V i*)
using *ope V eqU* **by** *auto*
show *closedin* (*top_of_set* (⋃(*V* ' *UNIV*))) (⋃*m*<*i*. *V m*)
using *V clo eqU* **by** (*force intro*: *closedin_Union*)
qed
show *continuous_map* (*subtopology ?X* ({*0..1*} × (*V i* − ⋃ (*V* ' {..<*i*})))) *euclidean* (*h i*) **for** *i*
by (*auto simp add*: *subtopology_subtopology intro*!: *continuous_on_subset* [*OF conth*])
show ⋀*i j x*. *x* ∈ *topspace ?X* ∩ {*0..1*} × (*V i* − (⋃*m*<*i*. *V m*)) ∩ {*0..1*} × (*V j* − (⋃*m*<*j*. *V m*))
⟹ *h i x* = *h j x*
by *clarsimp* (*metis lessThan_iff linorder_neqE_nat*)
qed *auto*
show *?thesis*
proof (*simp add*: *homotopic_with eqU* [*symmetric*], *intro exI conjI ballI*)
show *continuous_on* ({*0..1*} × ⋃𝒱) ζ
using *V eqU* **by** (*blast intro*!: *continuous_on_subset* [*OF cont*])
show ζ' ({*0..1*} × ⋃𝒱) ⊆ *T*
proof *clarsimp*
fix *t* :: *real* **and** *y* :: *'a* **and** *X* :: *'a set*
assume *y* ∈ *X X* ∈ 𝒱 **and** *t*: *0* ≤ *t t* ≤ *1*
then obtain *k* **where** *y* ∈ *V k* **and** *j*: ∀*j*<*k*. *y* ∉ *V j*
by (*metis image_iff V wop*)
with *him t* **show** ζ(*t*, *y*) ∈ *T*
by (*subst eq*) *force*+
qed
fix *X y*
assume *X* ∈ 𝒱 *y* ∈ *X*
then obtain *k* **where** *y* ∈ *V k* **and** *j*: ∀*j*<*k*. *y* ∉ *V j*
by (*metis image_iff V wop*)
then show ζ(*0*, *y*) = *f y* **and** ζ(*1*, *y*) = *g y*
by (*subst eq* [**where** *i*=*k*]; *force simp*: *h0 h1*)+
qed
qed
qed

lemma *homotopic_on_components_eq*:
fixes *S* :: *'a* :: *euclidean_space set* **and** *T* :: *'b* :: *euclidean_space set*
assumes *S*: *locally connected S* ∨ *compact S* **and** *ANR T*
shows *homotopic_with_canon* (λ*x*. *True*) *S T f g* ⟷
(*continuous_on S f* ∧ *f* ' *S* ⊆ *T* ∧ *continuous_on S g* ∧ *g* ' *S* ⊆ *T*) ∧
(∀ *C* ∈ *components S*. *homotopic_with_canon* (λ*x*. *True*) *C T f g*)
(**is** *?lhs* ⟷ *?C* ∧ *?rhs*)
proof −
have *continuous_on S f f* ' *S* ⊆ *T continuous_on S g g* ' *S* ⊆ *T* **if** *?lhs*

```
  using homotopic_with_imp_continuous homotopic_with_imp_subset1 homotopic_with_imp_subset2
that by blast+
 moreover have ?lhs ⟷ ?rhs
   if contf: continuous_on S f and fim: f ` S ⊆ T and contg: continuous_on S g
and gim: g ` S ⊆ T
 proof
   assume ?lhs
   with that show ?rhs
     by (simp add: homotopic_with_subset_left in_components_subset)
 next
   assume R: ?rhs
   have ∃ U. C ⊆ U ∧ closedin (top_of_set S) U ∧
             openin (top_of_set S) U ∧
            homotopic_with_canon (λx. True) U T f g if C: C ∈ components S for
C
   proof -
     have C ⊆ S
       by (simp add: in_components_subset that)
     show ?thesis
       using S
     proof
       assume locally connected S
       show ?thesis
       proof (intro exI conjI)
         show closedin (top_of_set S) C
           by (simp add: closedin_component that)
         show openin (top_of_set S) C
           by (simp add: ‹locally connected S› openin_components_locally_connected
that)
         show homotopic_with_canon (λx. True) C T f g
           by (simp add: R that)
       qed auto
     next
       assume compact S
       have hom: homotopic_with_canon (λx. True) C T f g
         using R that by blast
       obtain U where C ⊆ U and opeU: openin (top_of_set S) U
                 and hom: homotopic_with_canon (λx. True) U T f g
        using homotopic_neighbourhood_extension [OF contf fim contg gim _ ‹ANR
T› hom]
          ‹C ∈ components S› closedin_component by blast
       then obtain V where open V and V: U = S ∩ V
         by (auto simp: openin_open)
       moreover have locally compact S
        by (simp add: ‹compact S› closed_imp_locally_compact compact_imp_closed)
       ultimately obtain K where opeK: openin (top_of_set S) K and compact
K C ⊆ K K ⊆ V
          by (metis C Int_subset_iff Sura_Bura_clopen_subset ‹C ⊆ U› ‹compact S›
compact_components)
```

show *?thesis*
proof (*intro exI conjI*)
show *closedin* (*top_of_set S*) *K*
by (*meson* ‹*compact K*› ‹*compact S*› *closedin_compact_eq opeK openin_imp_subset*)
show *homotopic_with_canon* (*λx. True*) *K T f g*
using *V* ‹*K* ⊆ *V*› *hom homotopic_with_subset_left opeK openin_imp_subset*
by *fastforce*
qed (*use opeK* ‹*C* ⊆ *K*› **in** *auto*)
qed
qed
then obtain *φ* **where** *φ*: ⋀*C*. *C* ∈ *components S* ⟹ *C* ⊆ *φ C*
and *cloφ*: ⋀*C*. *C* ∈ *components S* ⟹ *closedin* (*top_of_set S*) (*φ C*)
and *opeφ*: ⋀*C*. *C* ∈ *components S* ⟹ *openin* (*top_of_set S*) (*φ C*)
and *homφ*: ⋀*C*. *C* ∈ *components S* ⟹ *homotopic_with_canon* (*λx.*
True) (*φ C*) *T f g*
by *metis*
have *Seq*: *S* = ⋃ (*φ* ‘ *components S*)
proof
show *S* ⊆ ⋃ (*φ* ‘ *components S*)
by (*metis Sup_mono Union_components φ imageI*)
show ⋃ (*φ* ‘ *components S*) ⊆ *S*
using *opeφ openin_imp_subset* **by** *fastforce*
qed
show *?lhs*
apply (*subst Seq*)
using *Seq cloφ opeφ homφ* **by** (*intro homotopic_on_clopen_Union*) *auto*
qed
ultimately show *?thesis* **by** *blast*
qed

lemma *cohomotopically_trivial_on_components*:
fixes *S* :: *'a* :: *euclidean_space set* **and** *T* :: *'b* :: *euclidean_space set*
assumes *S*: *locally connected S* ∨ *compact S* **and** *ANR T*
shows
(∀ *f g*. *continuous_on S f* ⟶ *f* ‘ *S* ⊆ *T* ⟶ *continuous_on S g* ⟶ *g* ‘ *S* ⊆ *T*
⟶
homotopic_with_canon (*λx. True*) *S T f g*)
⟷
(∀ *C*∈*components S*.
∀ *f g*. *continuous_on C f* ⟶ *f* ‘ *C* ⊆ *T* ⟶ *continuous_on C g* ⟶ *g* ‘ *C* ⊆
T ⟶
homotopic_with_canon (*λx. True*) *C T f g*)
(**is** *?lhs* = *?rhs*)
proof
assume *L*[*rule_format*]: *?lhs*
show *?rhs*
proof *clarify*
fix *C f g*

assume *contf*: *continuous_on C f* **and** *fim*: *f ‘ C ⊆ T*
and *contg*: *continuous_on C g* **and** *gim*: *g ‘ C ⊆ T* **and** *C*: *C ∈ components S*
obtain *f′* **where** *contf′*: *continuous_on S f′* **and** *f′im*: *f′ ‘ S ⊆ T* **and** *f′f*: $\bigwedge x.\ x \in C \Longrightarrow f'\ x = f\ x$
using *extension_from_component* [*OF S* ‹*ANR T*› *C contf fim*] **by** *metis*
obtain *g′* **where** *contg′*: *continuous_on S g′* **and** *g′im*: *g′ ‘ S ⊆ T* **and** *g′g*: $\bigwedge x.\ x \in C \Longrightarrow g'\ x = g\ x$
using *extension_from_component* [*OF S* ‹*ANR T*› *C contg gim*] **by** *metis*
have *homotopic_with_canon* (*λx. True*) *C T f′ g′*
using *L* [*OF contf′ f′im contg′ g′im*] *homotopic_with_subset_left C in_components_subset* **by** *fastforce*
then show *homotopic_with_canon* (*λx. True*) *C T f g*
using *f′f g′g homotopic_with_eq* **by** *force*
qed
next
assume *R* [*rule_format*]: *?rhs*
show *?lhs*
proof *clarify*
fix *f g*
assume *contf*: *continuous_on S f* **and** *fim*: *f ‘ S ⊆ T*
and *contg*: *continuous_on S g* **and** *gim*: *g ‘ S ⊆ T*
moreover have *homotopic_with_canon* (*λx. True*) *C T f g* **if** *C ∈ components S* **for** *C*
using *R* [*OF that*]
by (*meson contf contg continuous_on_subset fim gim image_mono in_components_subset order.trans that*)
ultimately show *homotopic_with_canon* (*λx. True*) *S T f g*
by (*subst homotopic_on_components_eq* [*OF S* ‹*ANR T*›]) *auto*
qed
qed

6.40.8 The complement of a set and path-connectedness

Complement in dimension N ¿ 1 of set homeomorphic to any interval in any dimension is (path-)connected. This naively generalizes the argument in Ryuji Maehara's paper "The Jordan curve theorem via the Brouwer fixed point theorem", American Mathematical Monthly 1984.

lemma *unbounded_components_complement_absolute_retract*:
fixes *S* :: *′a::euclidean_space set*
assumes *C*: *C ∈ components*(− *S*) **and** *S*: *compact S AR S*
shows ¬ *bounded C*
proof −
obtain *y* **where** *y*: *C* = *connected_component_set* (− *S*) *y* **and** *y ∉ S*
using *C* **by** (*auto simp*: *components_def*)
have *open*(− *S*)
using *S* **by** (*simp add*: *closed_open compact_eq_bounded_closed*)
have *S retract_of UNIV*

using *S compact_AR* **by** *blast*
then obtain *r* **where** *contr*: *continuous_on UNIV r* **and** *ontor*: *range r* ⊆ *S*
and *r*: ⋀*x*. *x* ∈ *S* ⟹ *r x* = *x*
by (*auto simp*: *retract_of_def retraction_def*)
show *?thesis*
proof
assume *bounded C*
have *connected_component_set* (− *S*) *y* ⊆ *S*
proof (*rule frontier_subset_retraction*)
show *bounded* (*connected_component_set* (− *S*) *y*)
using ‹*bounded C*› *y* **by** *blast*
show *frontier* (*connected_component_set* (− *S*) *y*) ⊆ *S*
using *C* ‹*compact S*› *compact_eq_bounded_closed frontier_of_components_closed_complement*
y **by** *blast*
show *continuous_on* (*closure* (*connected_component_set* (− *S*) *y*)) *r*
by (*blast intro*: *continuous_on_subset* [*OF contr*])
qed (*use ontor r* **in** *auto*)
with ‹*y* ∉ *S*› **show** *False* **by** *force*
qed
qed

lemma *connected_complement_absolute_retract*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *S*: *compact S AR S* **and** *2*: *2* ≤ *DIM*(′*a*)
shows *connected*(− *S*)
proof −
have *S retract_of UNIV*
using *S compact_AR* **by** *blast*
show *?thesis*
proof (*clarsimp simp*: *connected_iff_connected_component_eq*)
have ¬ *bounded* (*connected_component_set* (− *S*) *x*) **if** *x* ∉ *S* **for** *x*
by (*meson Compl_iff assms componentsI that unbounded_components_complement_absolute_retract*)
then show *connected_component_set* (− *S*) *x* = *connected_component_set* (−
S) *y*
if *x* ∉ *S y* ∉ *S* **for** *x y*
using *cobounded_unique_unbounded_component* [*OF _ 2*]
by (*metis* ‹*compact S*› *compact_imp_bounded double_compl that*)
qed
qed

lemma *path_connected_complement_absolute_retract*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *compact S AR S 2* ≤ *DIM*(′*a*)
shows *path_connected*(− *S*)
using *connected_complement_absolute_retract* [*OF assms*]
using ‹*compact S*› *compact_eq_bounded_closed connected_open_path_connected* **by**
blast

theorem *connected_complement_homeomorphic_convex_compact*:

```
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes hom: S homeomorphic T and T: convex T compact T and 2: 2 ≤
DIM('a)
    shows connected(− S)
proof (cases S = {})
  case True
  then show ?thesis
    by (simp add: connected_UNIV)
next
  case False
  show ?thesis
  proof (rule connected_complement_absolute_retract)
    show compact S
      using ‹compact T› hom homeomorphic_compactness by auto
    show AR S
     by (meson AR_ANR False ‹convex T› convex_imp_ANR convex_imp_contractible
hom homeomorphic_ANR_iff_ANR homeomorphic_contractible_eq)
  qed (rule 2)
qed

corollary path_connected_complement_homeomorphic_convex_compact:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes hom: S homeomorphic T convex T compact T 2 ≤ DIM('a)
    shows path_connected(− S)
  using connected_complement_homeomorphic_convex_compact [OF assms]
 using ‹compact T› compact_eq_bounded_closed connected_open_path_connected hom
homeomorphic_compactness by blast

lemma path_connected_complement_homeomorphic_interval:
  fixes S :: 'a::euclidean_space set
  assumes S homeomorphic cbox a b 2 ≤ DIM('a)
  shows path_connected(−S)
 using assms compact_cbox convex_box(1) path_connected_complement_homeomorphic_convex_compact
by blast

lemma connected_complement_homeomorphic_interval:
  fixes S :: 'a::euclidean_space set
  assumes S homeomorphic cbox a b 2 ≤ DIM('a)
  shows connected(−S)
 using assms path_connected_complement_homeomorphic_interval path_connected_imp_connected
by blast

end
```

6.41 Extending Continous Maps, Invariance of Domain, etc

Ported from HOL Light (moretop.ml) by L C Paulson

theory *Further_Topology*
imports *Weierstrass_Theorems Polytope Complex_Transcendental Equivalence_Lebesgue_Henstock_Integ*
Retracts
begin

6.41.1 A map from a sphere to a higher dimensional sphere is nullhomotopic

```
lemma spheremap_lemma1:
  fixes f :: 'a::euclidean_space ⇒ 'a::euclidean_space
  assumes subspace S subspace T and dimST: dim S < dim T
      and S ⊆ T
      and diff_f: f differentiable_on sphere 0 1 ∩ S
    shows f ` (sphere 0 1 ∩ S) ≠ sphere 0 1 ∩ T
proof
  assume fim: f ` (sphere 0 1 ∩ S) = sphere 0 1 ∩ T
  have inS: ⋀x. ⟦x ∈ S; x ≠ 0⟧ ⟹ (x /R norm x) ∈ S
    using subspace_mul ‹subspace S› by blast
  have subS01: (λx. x /R norm x) ` (S − {0}) ⊆ sphere 0 1 ∩ S
    using ‹subspace S› subspace_mul by fastforce
  then have diff_f': f differentiable_on (λx. x /R norm x) ` (S − {0})
    by (rule differentiable_on_subset [OF diff_f])
  define g where g ≡ λx. norm x *R f(inverse(norm x) *R x)
  have gdiff: g differentiable_on S − {0}
    unfolding g_def
   by (rule diff_f' derivative_intros differentiable_on_compose [where f=f] | force)+
  have geq: g ` (S − {0}) = T − {0}
  proof
    have ⋀u. ⟦u ∈ S; norm u *R f (u /R norm u) ∉ T⟧ ⟹ u = 0
      by (metis (mono_tags, lifting) DiffI subS01 subspace_mul [OF ‹subspace T›]
fim image_subset_iff inf_le2 singletonD)
    then have g ` (S − {0}) ⊆ T
      using g_def by blast
    moreover have g ` (S − {0}) ⊆ UNIV − {0}
    proof (clarsimp simp: g_def)
      fix y
      assume y ∈ S and f0: f (y /R norm y) = 0
      then have y ≠ 0 ⟹ y /R norm y ∈ sphere 0 1 ∩ S
        by (auto simp: subspace_mul [OF ‹subspace S›])
      then show y = 0
        by (metis fim f0 Int_iff image_iff mem_sphere_0 norm_eq_zero zero_neq_one)
    qed
    ultimately show g ` (S − {0}) ⊆ T − {0}
      by auto
  next
    have *: sphere 0 1 ∩ T ⊆ f ` (sphere 0 1 ∩ S)
      using fim by (simp add: image_subset_iff)
    have x ∈ (λx. norm x *R f (x /R norm x)) ` (S − {0})
         if x ∈ T x ≠ 0 for x
```

proof −
have *x* /$_R$ *norm x* ∈ *T*
using ‹*subspace T*› *subspace_mul that* **by** *blast*
then obtain *u* **where** *u*: *f u* ∈ *T x* /$_R$ *norm x* = *f u norm u* = *1 u* ∈ *S*
using ∗ [*THEN subsetD*, *of x* /$_R$ *norm x*] ‹*x* ≠ *0*› **by** *auto*
with *that* **have** [*simp*]: *norm x* ∗$_R$ *f u* = *x*
by (*metis divideR_right norm_eq_zero*)
moreover have *norm x* ∗$_R$ *u* ∈ *S* − {*0*}
using ‹*subspace S*› *subspace_scale that*(*2*) *u* **by** *auto*
with *u* **show** *?thesis*
by (*simp add*: *image_eqI* [**where** *x*=*norm x* ∗$_R$ *u*])
qed
then have *T* − {*0*} ⊆ (λ*x*. *norm x* ∗$_R$ *f* (*x* /$_R$ *norm x*)) ‘ (*S* − {*0*})
by *force*
then show *T* − {*0*} ⊆ *g* ‘ (*S* − {*0*})
by (*simp add*: *g_def*)
qed
define *T'* **where** *T'* ≡ {*y*. ∀ *x* ∈ *T*. *orthogonal x y*}
have *subspace T'*
by (*simp add*: *subspace_orthogonal_to_vectors T'_def*)
have *dim_eq*: *dim T'* + *dim T* = *DIM*(*'a*)
using *dim_subspace_orthogonal_to_vectors* [*of T UNIV*] ‹*subspace T*›
by (*simp add*: *T'_def*)
have ∃ *v1 v2*. *v1* ∈ *span T* ∧ (∀ *w* ∈ *span T*. *orthogonal v2 w*) ∧ *x* = *v1* + *v2* **for** *x*
by (*force intro*: *orthogonal_subspace_decomp_exists* [*of T x*])
then obtain *p1 p2* **where** *p1span*: *p1 x* ∈ *span T*
and ⋀*w*. *w* ∈ *span T* ⟹ *orthogonal* (*p2 x*) *w*
and *eq*: *p1 x* + *p2 x* = *x* **for** *x*
by *metis*
then have *p1*: ⋀*z*. *p1 z* ∈ *T* **and** *ortho*: ⋀*w*. *w* ∈ *T* ⟹ *orthogonal* (*p2 x*) *w* **for** *x*
using *span_eq_iff* ‹*subspace T*› **by** *blast+*
then have *p2*: ⋀*z*. *p2 z* ∈ *T'*
by (*simp add*: *T'_def orthogonal_commute*)
have *p12_eq*: ⋀*x y*. ⟦*x* ∈ *T*; *y* ∈ *T'*⟧ ⟹ *p1*(*x* + *y*) = *x* ∧ *p2*(*x* + *y*) = *y*
proof (*rule orthogonal_subspace_decomp_unique* [*OF eq p1span*, **where** *T*=*T'*])
show ⋀*x y*. ⟦*x* ∈ *T*; *y* ∈ *T'*⟧ ⟹ *p2* (*x* + *y*) ∈ *span T'*
using *span_eq_iff p2* ‹*subspace T'*› **by** *blast*
show ⋀*a b*. ⟦*a* ∈ *T*; *b* ∈ *T'*⟧ ⟹ *orthogonal a b*
using *T'_def* **by** *blast*
qed (*auto simp*: *span_base*)
then have ⋀*c x*. *p1* (*c* ∗$_R$ *x*) = *c* ∗$_R$ *p1 x* ∧ *p2* (*c* ∗$_R$ *x*) = *c* ∗$_R$ *p2 x*
proof −
fix *c* :: *real* **and** *x* :: *'a*
have *f1*: *c* ∗$_R$ *x* = *c* ∗$_R$ *p1 x* + *c* ∗$_R$ *p2 x*
by (*metis eq pth_6*)
have *f2*: *c* ∗$_R$ *p2 x* ∈ *T'*
by (*simp add*: ‹*subspace T'*› *p2 subspace_scale*)

have $c *_R p1\ x \in T$
by (*metis* (*full_types*) *assms*(*2*) *p1span span_eq_iff subspace_scale*)
then show $p1\ (c *_R x) = c *_R p1\ x \wedge p2\ (c *_R x) = c *_R p2\ x$
using *f2 f1 p12_eq* **by** *presburger*
qed
moreover have *lin_add*: $\bigwedge x\ y.\ p1\ (x + y) = p1\ x + p1\ y \wedge p2\ (x + y) = p2\ x + p2\ y$
proof (*rule orthogonal_subspace_decomp_unique* [*OF* _ *p1span*, **where** $T=T'$])
show $\bigwedge x\ y.\ p1\ (x + y) + p2\ (x + y) = p1\ x + p1\ y + (p2\ x + p2\ y)$
by (*simp add*: *add.assoc add.left_commute eq*)
show $\bigwedge a\ b.\ [\![a \in T;\ b \in T']\!] \Longrightarrow$ *orthogonal a b*
using *T′_def* **by** *blast*
qed (*auto simp*: *p1span p2 span_base span_add*)
ultimately have *linear p1 linear p2*
by *unfold_locales auto*
have *g differentiable_on p1* ‘ $\{x + y \mid x\ y.\ x \in S - \{0\} \wedge y \in T'\}$
using *p12_eq* ‹$S \subseteq T$› **by** (*force intro*: *differentiable_on_subset* [*OF gdiff*])
then have $(\lambda z.\ g\ (p1\ z))$ *differentiable_on* $\{x + y \mid x\ y.\ x \in S - \{0\} \wedge y \in T'\}$
by (*rule differentiable_on_compose* [*OF linear_imp_differentiable_on* [*OF* ‹*linear p1*›]])
then have *diff*: $(\lambda x.\ g\ (p1\ x) + p2\ x)$ *differentiable_on* $\{x + y \mid x\ y.\ x \in S - \{0\} \wedge y \in T'\}$
by (*intro derivative_intros linear_imp_differentiable_on* [*OF* ‹*linear p2*›])
have $dim\ \{x + y \mid x\ y.\ x \in S - \{0\} \wedge y \in T'\} \le dim\ \{x + y \mid x\ y.\ x \in S \wedge y \in T'\}$
by (*blast intro*: *dim_subset*)
also have ... $= dim\ S + dim\ T' - dim\ (S \cap T')$
using *dim_sums_Int* [*OF* ‹*subspace S*› ‹*subspace T′*›]
by (*simp add*: *algebra_simps*)
also have ... $< DIM('a)$
using *dimST dim_eq* **by** *auto*
finally have *neg*: *negligible* $\{x + y \mid x\ y.\ x \in S - \{0\} \wedge y \in T'\}$
by (*rule negligible_lowdim*)
have *negligible* $((\lambda x.\ g\ (p1\ x) + p2\ x)$ ‘ $\{x + y \mid x\ y.\ x \in S - \{0\} \wedge y \in T'\})$
by (*rule negligible_differentiable_image_negligible* [*OF order_refl neg diff*])
then have *negligible* $\{x + y \mid x\ y.\ x \in g$ ‘ $(S - \{0\}) \wedge y \in T'\}$
proof (*rule negligible_subset*)
have $[\![t' \in T';\ s \in S;\ s \neq 0]\!]$
$\Longrightarrow g\ s + t' \in (\lambda x.\ g\ (p1\ x) + p2\ x)$ ‘
$\{x + t' \mid x\ t'.\ x \in S \wedge x \neq 0 \wedge t' \in T'\}$ **for** $t'\ s$
using ‹$S \subseteq T$› *p12_eq* **by** (*rule_tac* $x=s + t'$ **in** *image_eqI*) *auto*
then show $\{x + y \mid x\ y.\ x \in g$ ‘ $(S - \{0\}) \wedge y \in T'\}$
$\subseteq (\lambda x.\ g\ (p1\ x) + p2\ x)$ ‘ $\{x + y \mid x\ y.\ x \in S - \{0\} \wedge y \in T'\}$
by *auto*
qed
moreover have $- T' \subseteq \{x + y \mid x\ y.\ x \in g$ ‘ $(S - \{0\}) \wedge y \in T'\}$
proof *clarsimp*
fix *z* **assume** $z \notin T'$
show $\exists x\ y.\ z = x + y \wedge x \in g$ ‘ $(S - \{0\}) \wedge y \in T'$

by (*metis Diff_iff* ‹$z \notin T'$› *add.left_neutral eq geq p1 p2 singletonD*)
qed
ultimately have *negligible* $(-T')$
using *negligible_subset* **by** *blast*
moreover have *negligible* T'
using *negligible_lowdim*
by (*metis add.commute assms(3) diff_add_inverse2 diff_self_eq_0 dim_eq le_add1 le_antisym linordered_semidom_class.add_diff_inverse not_less0*)
ultimately have *negligible* $(-T' \cup T')$
by (*metis negligible_Un_eq*)
then show *False*
using *negligible_Un_eq non_negligible_UNIV* **by** *simp*
qed

lemma *spheremap_lemma2*:
fixes *f* :: *'a::euclidean_space* ⇒ *'a::euclidean_space*
assumes *ST*: *subspace S subspace T dim S* < *dim T*
and $S \subseteq T$
and *contf*: *continuous_on* (*sphere 0 1* ∩ *S*) *f*
and *fim*: *f* ‘ (*sphere 0 1* ∩ *S*) ⊆ *sphere 0 1* ∩ *T*
shows ∃ *c. homotopic_with_canon* (λ*x. True*) (*sphere 0 1* ∩ *S*) (*sphere 0 1* ∩ *T*) *f* (λ*x. c*)
proof –
have [*simp*]: ⋀*x*. ⟦*norm x* = *1*; *x* ∈ *S*⟧ ⟹ *norm* (*f x*) = *1*
using *fim* **by** (*simp add*: *image_subset_iff*)
have *compact* (*sphere 0 1* ∩ *S*)
by (*simp add*: ‹*subspace S*› *closed_subspace compact_Int_closed*)
then obtain *g* **where** *pfg*: *polynomial_function g* **and** *gim*: *g* ‘ (*sphere 0 1* ∩ *S*) ⊆ *T*
and *g12*: ⋀*x*. *x* ∈ *sphere 0 1* ∩ *S* ⟹ *norm*(*f x* − *g x*) < *1/2*
apply (*rule Stone_Weierstrass_polynomial_function_subspace* [*OF _ contf _* ‹*subspace T*›, *of 1/2*])
using *fim* **by** *auto*
have *gnz*: *g x* ≠ *0* **if** *x* ∈ *sphere 0 1* ∩ *S* **for** *x*
proof –
have *norm* (*f x*) = *1*
using *fim that* **by** (*simp add*: *image_subset_iff*)
then show *?thesis*
using *g12* [*OF that*] **by** *auto*
qed
have *diffg*: *g differentiable_on sphere 0 1* ∩ *S*
by (*metis pfg differentiable_on_polynomial_function*)
define *h* **where** *h* ≡ λ*x. inverse*(*norm*(*g x*)) $*_R$ *g x*
have *h*: *x* ∈ *sphere 0 1* ∩ *S* ⟹ *h x* ∈ *sphere 0 1* ∩ *T* **for** *x*
unfolding *h_def*
using *gnz* [*of x*]
by (*auto simp*: *subspace_mul* [*OF* ‹*subspace T*›] *subsetD* [*OF gim*])
have *diffh*: *h differentiable_on sphere 0 1* ∩ *S*

unfolding *h_def* **using** *gnz*
by (*fastforce intro*: *derivative_intros diffg differentiable_on_compose* [*OF diffg*])
have *homfg*: *homotopic_with_canon* (*λz. True*) (*sphere 0 1* ∩ *S*) (*T* − {*0*}) *f g*
proof (*rule homotopic_with_linear* [*OF contf*])
show *continuous_on* (*sphere 0 1* ∩ *S*) *g*
using *pfg* **by** (*simp add*: *differentiable_imp_continuous_on diffg*)
next
have *non0fg*: *0* ∉ *closed_segment* (*f x*) (*g x*) **if** *norm x* = *1 x* ∈ *S* **for** *x*
proof −
have *f x* ∈ *sphere 0 1*
using *fim that* **by** (*simp add*: *image_subset_iff*)
moreover have *norm*(*f x* − *g x*) < *1/2*
using *g12 that* **by** *auto*
ultimately show *?thesis*
by (*auto simp*: *norm_minus_commute dest*: *segment_bound*)
qed
show *closed_segment* (*f x*) (*g x*) ⊆ *T* − {*0*} **if** *x* ∈ *sphere 0 1* ∩ *S* **for** *x*
proof −
have *convex T*
by (*simp add*: ‹*subspace T*› *subspace_imp_convex*)
then have *convex hull* {*f x*, *g x*} ⊆ *T*
by (*metis IntD2 closed_segment_subset fim gim image_subset_iff segment_convex_hull that*)
then show *?thesis*
using *that non0fg segment_convex_hull* **by** *fastforce*
qed
qed
obtain *d* **where** *d*: *d* ∈ (*sphere 0 1* ∩ *T*) − *h* ‘ (*sphere 0 1* ∩ *S*)
using *h spheremap_lemma1* [*OF ST* ‹*S* ⊆ *T*› *diffh*] **by** *force*
then have *non0hd*: *0* ∉ *closed_segment* (*h x*) (− *d*) **if** *norm x* = *1 x* ∈ *S* **for** *x*
using *midpoint_between* [*of 0 h x −d*] *that h* [*of x*]
by (*auto simp*: *between_mem_segment midpoint_def*)
have *conth*: *continuous_on* (*sphere 0 1* ∩ *S*) *h*
using *differentiable_imp_continuous_on diffh* **by** *blast*
have *hom_hd*: *homotopic_with_canon* (*λz. True*) (*sphere 0 1* ∩ *S*) (*T* − {*0*}) *h* (*λx.* −*d*)
proof (*rule homotopic_with_linear* [*OF conth continuous_on_const*])
fix *x*
assume *x*: *x* ∈ *sphere 0 1* ∩ *S*
have *convex hull* {*h x*, − *d*} ⊆ *T*
proof (*rule hull_minimal*)
show {*h x*, − *d*} ⊆ *T*
using *h d x* **by** (*force simp*: *subspace_neg* [*OF* ‹*subspace T*›])
qed (*simp add*: *subspace_imp_convex* [*OF* ‹*subspace T*›])
with *x segment_convex_hull* **show** *closed_segment* (*h x*) (− *d*) ⊆ *T* − {*0*}
by (*auto simp add*: *subset_Diff_insert non0hd*)
qed
have *conT0*: *continuous_on* (*T* − {*0*}) (*λy. inverse*(*norm y*) $*_R$ *y*)
by (*intro continuous_intros*) *auto*

```
  have sub0T: (λy. y /R norm y) ` (T − {0}) ⊆ sphere 0 1 ∩ T
    by (fastforce simp: assms(2) subspace_mul)
   obtain c where homhc: homotopic_with_canon (λz. True) (sphere 0 1 ∩ S)
(sphere 0 1 ∩ T) h (λx. c)
  proof
    show homotopic_with_canon (λz. True) (sphere 0 1 ∩ S) (sphere 0 1 ∩ T) h
(λx. − d)
      using d
      by (force simp: h_def
              intro: homotopic_with_eq homotopic_with_compose_continuous_left [OF
hom_hd conT0 sub0T])
  qed
  have homotopic_with_canon (λx. True) (sphere 0 1 ∩ S) (sphere 0 1 ∩ T) f h
    by (force simp: h_def
             intro:  homotopic_with_eq homotopic_with_compose_continuous_left [OF
homfg conT0 sub0T])
  then show ?thesis
    by (metis homotopic_with_trans [OF _ homhc])
qed


lemma spheremap_lemma3:
  assumes bounded S convex S subspace U and affSU: aff_dim S ≤ dim U
  obtains T where subspace T T ⊆ U S ≠ {} ⟹ aff_dim T = aff_dim S
                (rel_frontier S) homeomorphic (sphere 0 1 ∩ T)
proof (cases S = {})
  case True
  with ‹subspace U› subspace_0 show ?thesis
    by (rule_tac T = {0} in that) auto
next
  case False
  then obtain a where a ∈ S
    by auto
  then have affS: aff_dim S = int (dim ((λx. −a+x) ` S))
    by (metis hull_inc aff_dim_eq_dim)
  with affSU have dim ((λx. −a+x) ` S) ≤ dim U
    by linarith
  with choose_subspace_of_subspace
 obtain T where subspace T T ⊆ span U and dimT: dim T = dim ((λx. −a+x)
` S) .
  show ?thesis
  proof (rule that [OF ‹subspace T›])
    show T ⊆ U
      using span_eq_iff ‹subspace U› ‹T ⊆ span U› by blast
    show aff_dim T = aff_dim S
      using dimT ‹subspace T› affS aff_dim_subspace by fastforce
    show rel_frontier S homeomorphic sphere 0 1 ∩ T
    proof −
      have aff_dim (ball 0 1 ∩ T) = aff_dim (T)
```

by (*metis IntI interior_ball ‹subspace T› aff_dim_convex_Int_nonempty_interior centre_in_ball empty_iff inf_commute subspace_0 subspace_imp_convex zero_less_one*)
then have *affS_eq*: *aff_dim S* = *aff_dim* (*ball 0 1* ∩ *T*)
using ‹*aff_dim T* = *aff_dim S*› **by** *simp*
have *rel_frontier S homeomorphic rel_frontier*(*ball 0 1* ∩ *T*)
proof (*rule homeomorphic_rel_frontiers_convex_bounded_sets* [*OF* ‹*convex S*› ‹*bounded S*›])
show *convex* (*ball 0 1* ∩ *T*)
by (*simp add*: ‹*subspace T*› *convex_Int subspace_imp_convex*)
show *bounded* (*ball 0 1* ∩ *T*)
by (*simp add*: *bounded_Int*)
show *aff_dim S* = *aff_dim* (*ball 0 1* ∩ *T*)
by (*rule affS_eq*)
qed
also have ... = *frontier* (*ball 0 1*) ∩ *T*
proof (*rule convex_affine_rel_frontier_Int* [*OF convex_ball*])
show *affine T*
by (*simp add*: ‹*subspace T*› *subspace_imp_affine*)
show *interior* (*ball 0 1*) ∩ *T* ≠ {}
using ‹*subspace T*› *subspace_0* **by** *force*
qed
also have ... = *sphere 0 1* ∩ *T*
by *auto*
finally show *?thesis* **.**
qed
qed
qed

proposition *inessential_spheremap_lowdim_gen*:
fixes *f* :: *'M*::*euclidean_space* ⇒ *'a*::*euclidean_space*
assumes *convex S bounded S convex T bounded T*
and *affST*: *aff_dim S* < *aff_dim T*
and *contf*: *continuous_on* (*rel_frontier S*) *f*
and *fim*: *f* ‘ (*rel_frontier S*) ⊆ *rel_frontier T*
obtains *c* **where** *homotopic_with_canon* (λ*z. True*) (*rel_frontier S*) (*rel_frontier T*) *f* (λ*x. c*)
proof (*cases S* = {})
case *True*
then show *?thesis*
by (*simp add*: *that*)
next
case *False*
then show *?thesis*
proof (*cases T* = {})
case *True*
then show *?thesis*
using *fim that* **by** *auto*
next

```
    case False
    obtain T'':: 'a set
      where subspace T' and affT': aff_dim T' = aff_dim T
        and homT: rel_frontier T homeomorphic sphere 0 1 ∩ T'
      apply (rule spheremap_lemma3 [OF ‹bounded T› ‹convex T› subspace_UNIV,
where 'b='a])
      using ‹T ≠ {}› by (auto simp add: aff_dim_le_DIM)
    with homeomorphic_imp_homotopy_eqv
    have relT: sphere 0 1 ∩ T' homotopy_eqv rel_frontier T
      using homotopy_equivalent_space_sym by blast
    have aff_dim S ≤ int (dim T')
      using affT' ‹subspace T'› affST aff_dim_subspace by force
    with spheremap_lemma3 [OF ‹bounded S› ‹convex S› ‹subspace T'›] ‹S ≠ {}›
    obtain S':: 'a set where subspace S' S' ⊆ T'
       and affS': aff_dim S' = aff_dim S
       and homT: rel_frontier S homeomorphic sphere 0 1 ∩ S'
        by metis
    with homeomorphic_imp_homotopy_eqv
    have relS: sphere 0 1 ∩ S' homotopy_eqv rel_frontier S
      using homotopy_equivalent_space_sym by blast
    have dimST': dim S' < dim T'
      by (metis ‹S' ⊆ T'› ‹subspace S'› ‹subspace T'› affS' affST affT' less_irrefl
not_le subspace_dim_equal)
    have ∃c. homotopic_with_canon (λz. True) (rel_frontier S) (rel_frontier T) f
(λx. c)
      apply (rule homotopy_eqv_homotopic_triviality_null_imp [OF relT contf fim])
      apply (rule homotopy_eqv_cohomotopic_triviality_null[OF relS, THEN iffD1,
rule_format])
      apply (metis dimST' ‹subspace S'› ‹subspace T'› ‹S' ⊆ T'› spheremap_lemma2,
blast)
      done
    with that show ?thesis by blast
  qed
qed

lemma inessential_spheremap_lowdim:
  fixes f :: 'M::euclidean_space ⇒ 'a::euclidean_space
  assumes
   DIM('M) < DIM('a) and f: continuous_on (sphere a r) f f ` (sphere a r) ⊆
(sphere b s)
   obtains c where homotopic_with_canon (λz. True) (sphere a r) (sphere b s) f
(λx. c)
proof (cases s ≤ 0)
  case True then show ?thesis
    by (meson nullhomotopic_into_contractible f contractible_sphere that)
next
  case False
  show ?thesis
  proof (cases r ≤ 0)
```

case *True* **then show** *?thesis*
by (*meson f nullhomotopic_from_contractible contractible_sphere that*)
next
case *False*
with ‹¬ $s \leq 0$› **have** $r > 0$ $s > 0$ **by** *auto*
show *thesis*
apply (*rule inessential_spheremap_lowdim_gen* [*of cball a r cball b s f*])
using ‹$0 < r$› ‹$0 < s$› *assms*(*1*) *that* **by** (*simp_all add*: *f aff_dim_cball*)
qed
qed

6.41.2 Some technical lemmas about extending maps from cell complexes

lemma *extending_maps_Union_aux*:
assumes *fin*: *finite* $\mathcal{F}$
and $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *closed* S
and $\bigwedge S\ T.\ [\![S \in \mathcal{F};\ T \in \mathcal{F};\ S \neq T]\!] \Longrightarrow S \cap T \subseteq K$
and $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow \exists\, g.$ *continuous_on* S $g \wedge g$ ‘ $S \subseteq T \wedge (\forall x \in S \cap K.\ g$ $x = h\ x)$
shows $\exists\, g.$ *continuous_on* $(\bigcup \mathcal{F})$ $g \wedge g$ ‘ $(\bigcup \mathcal{F}) \subseteq T \wedge (\forall x \in \bigcup \mathcal{F} \cap K.\ g\ x = h\ x)$
using *assms*
proof (*induction* $\mathcal{F}$)
case *empty* **show** *?case* **by** *simp*
next
case (*insert* S $\mathcal{F}$)
then obtain *f* **where** *contf*: *continuous_on* (S) *f* **and** *fim*: *f* ‘ $S \subseteq T$ **and** *feq*: $\forall x \in S \cap K.\ f\ x = h\ x$
by (*meson insertI1*)
obtain *g* **where** *contg*: *continuous_on* $(\bigcup \mathcal{F})$ *g* **and** *gim*: *g* ‘ $\bigcup \mathcal{F} \subseteq T$ **and** *geq*: $\forall x \in \bigcup \mathcal{F} \cap K.\ g\ x = h\ x$
using *insert* **by** *auto*
have *fg*: $f\ x = g\ x$ **if** $x \in T$ $T \in \mathcal{F}$ $x \in S$ **for** x T
proof −
have $T \cap S \subseteq K \vee S = T$
using *that* **by** (*metis* (*no_types*) *insert.prems*(*2*) *insertCI*)
then show *?thesis*
using *UnionI feq geq* ‹$S \notin \mathcal{F}$› *subsetD that* **by** *fastforce*
qed
show *?case*
apply (*rule_tac* $x=\lambda x.$ *if* $x \in S$ *then* $f\ x$ *else* $g\ x$ **in** *exI*, *simp*)
apply (*intro conjI continuous_on_cases*)
using *fim gim feq geq*
apply (*force simp*: *insert closed_Union contf contg inf_commute intro*: *fg*)+
done
qed

lemma *extending_maps_Union*:

assumes *fin*: *finite* $\mathcal{F}$
and $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow \exists g.\ continuous\_on\ S\ g \wedge g\ `\ S \subseteq T \wedge (\forall x \in S \cap K.\ g\ x = h\ x)$
and $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow closed\ S$
and K: $\bigwedge X\ Y.\ [\![X \in \mathcal{F};\ Y \in \mathcal{F};\ \neg\ X \subseteq Y;\ \neg\ Y \subseteq X]\!] \Longrightarrow X \cap Y \subseteq K$
shows $\exists g.\ continuous\_on\ (\bigcup \mathcal{F})\ g \wedge g\ `\ (\bigcup \mathcal{F}) \subseteq T \wedge (\forall x \in \bigcup \mathcal{F} \cap K.\ g\ x = h\ x)$
apply (*simp flip*: *Union_maximal_sets* [*OF fin*])
apply (*rule extending_maps_Union_aux*)
apply (*simp_all add*: *Union_maximal_sets* [*OF fin*] *assms*)
by (*metis K psubsetI*)

lemma *extend_map_lemma*:
assumes *finite* $\mathcal{F}$ $\mathcal{G} \subseteq \mathcal{F}$ *convex* T *bounded* T
and *poly*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow polytope\ X$
and *aff*: $\bigwedge X.\ X \in \mathcal{F} - \mathcal{G} \Longrightarrow aff\_dim\ X < aff\_dim\ T$
and *face*: $\bigwedge S\ T.\ [\![S \in \mathcal{F};\ T \in \mathcal{F}]\!] \Longrightarrow (S \cap T)\ face\_of\ S$
and *contf*: $continuous\_on\ (\bigcup \mathcal{G})\ f$ **and** *fim*: $f\ `\ (\bigcup \mathcal{G}) \subseteq rel\_frontier\ T$
obtains g **where** $continuous\_on\ (\bigcup \mathcal{F})\ g\ \ g\ `\ (\bigcup \mathcal{F}) \subseteq rel\_frontier\ T\ \ \bigwedge x.\ x \in \bigcup \mathcal{G} \Longrightarrow g\ x = f\ x$
proof (*cases* $\mathcal{F} - \mathcal{G} = \{\}$)
case *True*
show *?thesis*
proof
show $continuous\_on\ (\bigcup\ \mathcal{F})\ f$
using *True* ‹$\mathcal{G} \subseteq \mathcal{F}$› *contf* **by** *auto*
show $f\ `\ \bigcup\ \mathcal{F} \subseteq rel\_frontier\ T$
using *True fim* **by** *auto*
qed *auto*
next
case *False*
then have $0 \leq aff\_dim\ T$
by (*metis aff aff_dim_empty aff_dim_geq aff_dim_negative_iff all_not_in_conv not_less*)
then obtain i::*nat* **where** i: $int\ i = aff\_dim\ T$
by (*metis nonneg_eq_int*)
have *Union_empty_eq*: $\bigcup \{D.\ D = \{\} \wedge P\ D\} = \{\}$ **for** P :: *'a set* ⇒ *bool*
by *auto*
have *face'*: $\bigwedge S\ T.\ [\![S \in \mathcal{F};\ T \in \mathcal{F}]\!] \Longrightarrow (S \cap T)\ face\_of\ S \wedge (S \cap T)\ face\_of\ T$
by (*metis face inf_commute*)
have *extendf*: $\exists g.\ continuous\_on\ (\bigcup (\mathcal{G} \cup \{D.\ \exists C \in \mathcal{F}.\ D\ face\_of\ C \wedge aff\_dim\ D < i\}))\ g \wedge$
$g\ `\ (\bigcup\ (\mathcal{G} \cup \{D.\ \exists C \in \mathcal{F}.\ D\ face\_of\ C \wedge aff\_dim\ D < i\})) \subseteq rel\_frontier\ T \wedge$
$(\forall x \in \bigcup \mathcal{G}.\ g\ x = f\ x)$
if $i \leq aff\_dim\ T$ **for** i::*nat*
using *that*
proof (*induction i*)

case *0*
show *?case*
using *0 contf fim* **by** (*auto simp add*: *Union_empty_eq*)
next
case (*Suc p*)
with ‹*bounded T*› **have** *rel_frontier T* $\neq$ {}
by (*auto simp*: *rel_frontier_eq_empty affine_bounded_eq_lowdim* [*of T*])
then obtain *t* **where** *t*: *t* $\in$ *rel_frontier T* **by** *auto*
have *ple*: *int p* $\leq$ *aff_dim T* **using** *Suc.prems* **by** *force*
obtain *h* **where** *conth*: *continuous_on* ($\bigcup$($\mathcal{G}$ $\cup$ {*D*. $\exists$ *C* $\in$ $\mathcal{F}$. *D face_of C* $\wedge$ *aff_dim D* < *p*})) *h*
and *him*: *h* ‘ ($\bigcup$ ($\mathcal{G}$ $\cup$ {*D*. $\exists$ *C* $\in$ $\mathcal{F}$. *D face_of C* $\wedge$ *aff_dim D* < *p*}))
$\subseteq$ *rel_frontier T*
and *heq*: $\bigwedge$*x*. *x* $\in$ $\bigcup\mathcal{G}$ $\Longrightarrow$ *h x* = *f x*
using *Suc.IH* [*OF ple*] **by** *auto*
let *?Faces* = {*D*. $\exists$ *C* $\in$ $\mathcal{F}$. *D face_of C* $\wedge$ *aff_dim D* $\leq$ *p*}
have *extendh*: $\exists$ *g*. *continuous_on D g* $\wedge$
g ‘ *D* $\subseteq$ *rel_frontier T* $\wedge$
($\forall$ *x* $\in$ *D* $\cap$ $\bigcup$($\mathcal{G}$ $\cup$ {*D*. $\exists$ *C* $\in$ $\mathcal{F}$. *D face_of C* $\wedge$ *aff_dim D* < *p*}). *g x* = *h x*)
if *D*: *D* $\in$ $\mathcal{G}$ $\cup$ *?Faces* **for** *D*
proof (*cases D* $\subseteq$ $\bigcup$($\mathcal{G}$ $\cup$ {*D*. $\exists$ *C* $\in$ $\mathcal{F}$. *D face_of C* $\wedge$ *aff_dim D* < *p*}))
case *True*
have *continuous_on D h*
using *True conth continuous_on_subset* **by** *blast*
moreover have *h* ‘ *D* $\subseteq$ *rel_frontier T*
using *True him* **by** *blast*
ultimately show *?thesis*
by *blast*
next
case *False*
note *notDsub* = *False*
show *?thesis*
proof (*cases* $\exists$ *a*. *D* = {*a*})
case *True*
then obtain *a* **where** *D* = {*a*} **by** *auto*
with *notDsub t* **show** *?thesis*
by (*rule_tac x*=λ*x*. *t* **in** *exI*) *simp*
next
case *False*
have *D* $\neq$ {} **using** *notDsub* **by** *auto*
have *Dnotin*: *D* $\notin$ $\mathcal{G}$ $\cup$ {*D*. $\exists$ *C* $\in$ $\mathcal{F}$. *D face_of C* $\wedge$ *aff_dim D* < *p*}
using *notDsub* **by** *auto*
then have *D* $\notin$ $\mathcal{G}$ **by** *simp*
have *D* $\in$ *?Faces* − {*D*. $\exists$ *C* $\in$ $\mathcal{F}$. *D face_of C* $\wedge$ *aff_dim D* < *p*}
using *Dnotin that* **by** *auto*
then obtain *C* **where** *C* $\in$ $\mathcal{F}$ *D face_of C* **and** *affD*: *aff_dim D* = *int p*
by *auto*
then have *bounded D*

using *face_of_polytope_polytope poly polytope_imp_bounded* **by** *blast*
then have [*simp*]: ¬ *affine D*
using *affine_bounded_eq_trivial False* ‹*D* ≠ {}› ‹*bounded D*› **by** *blast*
have {*F*. *F facet_of D*} ⊆ {*E*. *E face_of C* ∧ *aff_dim E* < *int p*}
by *clarify* (*metis* ‹*D face_of C*› *affD eq_iff face_of_trans facet_of_def zle_diff1_eq*)
moreover have *polyhedron D*
using ‹*C* ∈ *F*› ‹*D face_of C*› *face_of_polytope_polytope poly polytope_imp_polyhedron* **by** *auto*
ultimately have *relf_sub*: *rel_frontier D* ⊆ ⋃ {*E*. *E face_of C* ∧ *aff_dim E* < *p*}
by (*simp add*: *rel_frontier_of_polyhedron Union_mono*)
then have *him_relf*: *h* ‘ *rel_frontier D* ⊆ *rel_frontier T*
using ‹*C* ∈ *F*› *him* **by** *blast*
have *convex D*
by (*simp add*: ‹*polyhedron D*› *polyhedron_imp_convex*)
have *affD_lessT*: *aff_dim D* < *aff_dim T*
using *Suc.prems affD* **by** *linarith*
have *contDh*: *continuous_on* (*rel_frontier D*) *h*
using ‹*C* ∈ *F*› *relf_sub* **by** (*blast intro*: *continuous_on_subset* [*OF conth*])
then have ∗: (∃ *c*. *homotopic_with_canon* (λ*x*. *True*) (*rel_frontier D*) (*rel_frontier T*) *h* (λ*x*. *c*)) =
(∃ *g*. *continuous_on UNIV g* ∧ *range g* ⊆ *rel_frontier T* ∧
(∀ *x*∈*rel_frontier D*. *g x* = *h x*))
by (*simp add*: *assms rel_frontier_eq_empty him_relf nullhomotopic_into_rel_frontier_extension* [*OF closed_rel_frontier*])
have (∃ *c*. *homotopic_with_canon* (λ*x*. *True*) (*rel_frontier D*) (*rel_frontier T*) *h* (λ*x*. *c*))
by (*metis inessential_spheremap_lowdim_gen*
[*OF* ‹*convex D*› ‹*bounded D*› ‹*convex T*› ‹*bounded T*› *affD_lessT contDh him_relf*])
then obtain *g* **where** *contg*: *continuous_on UNIV g*
and *gim*: *range g* ⊆ *rel_frontier T*
and *gh*: ⋀*x*. *x* ∈ *rel_frontier D* ⟹ *g x* = *h x*
by (*metis* ∗)
have *D* ∩ *E* ⊆ *rel_frontier D*
if *E* ∈ *G* ∪ {*D*. *Bex F* ((*face_of*) *D*) ∧ *aff_dim D* < *int p*} **for** *E*
proof (*rule face_of_subset_rel_frontier*)
show *D* ∩ *E face_of D*
using *that*
proof *safe*
assume *E* ∈ *G*
then show *D* ∩ *E face_of D*
by (*meson* ‹*C* ∈ *F*› ‹*D face_of C*› *assms*(*2*) *face'* *face_of_Int_subface face_of_refl_eq poly polytope_imp_convex subsetD*)
next
fix *x*
assume *aff_dim E* < *int p x* ∈ *F* *E face_of x*
then show *D* ∩ *E face_of D*

by (*meson* ‹$C \in \mathcal{F}$› ‹*D face_of C*› *face′ face_of_Int_subface that*)
qed
show $D \cap E \neq D$
using *that notDsub* **by** *auto*
qed
moreover have *continuous_on D g*
using *contg continuous_on_subset* **by** *blast*
ultimately show *?thesis*
by (*rule_tac x=g* **in** *exI*) (*use gh gim* **in** *fastforce*)
qed
qed
have *intle*: $i < 1 + int\ j \longleftrightarrow i \leq int\ j$ **for** *i j*
by *auto*
have *finite* $\mathcal{G}$
using ‹*finite* $\mathcal{F}$› ‹$\mathcal{G} \subseteq \mathcal{F}$› *rev_finite_subset* **by** *blast*
moreover have *finite* (*?Faces*)
proof −
have §: *finite* ($\bigcup$ {{*D. D face_of C*} |*C*. $C \in \mathcal{F}$})
by (*auto simp*: ‹*finite* $\mathcal{F}$› *finite_polytope_faces poly*)
show *?thesis*
by (*auto intro*: *finite_subset* [*OF* _ §])
qed
ultimately have *fin*: *finite* ($\mathcal{G} \cup$ *?Faces*)
by *simp*
have *clo*: *closed S* **if** $S \in \mathcal{G} \cup$ *?Faces* **for** *S*
using *that* ‹$\mathcal{G} \subseteq \mathcal{F}$› *face_of_polytope_polytope poly polytope_imp_closed* **by** *blast*
have *K*: $X \cap Y \subseteq \bigcup(\mathcal{G} \cup \{D.\ \exists C{\in}\mathcal{F}.\ D$ *face_of* $C \wedge$ *aff_dim* $D <$ *int* $p\})$
if $X \in \mathcal{G} \cup$ *?Faces* $Y \in \mathcal{G} \cup$ *?Faces* $\neg\ Y \subseteq X$ **for** *X Y*
proof −
have *ff*: $X \cap Y$ *face_of* $X \wedge X \cap Y$ *face_of* Y
if *XY*: *X face_of D Y face_of E* **and** *DE*: $D \in \mathcal{F}$ $E \in \mathcal{F}$ **for** *D E*
by (*rule face_of_Int_subface* [*OF* _ _ *XY*]) (*auto simp*: *face′ DE*)
show *?thesis*
using *that*
apply *auto*
apply (*drule_tac x*=$X \cap Y$ **in** *spec*, *safe*)
using *ff face_of_imp_convex* [*of X*] *face_of_imp_convex* [*of Y*]
apply (*fastforce dest*: *face_of_aff_dim_lt*)
by (*meson face_of_trans ff*)
qed
obtain *g* **where** *continuous_on* ($\bigcup(\mathcal{G} \cup$ *?Faces*)) *g*
g ‘ $\bigcup(\mathcal{G} \cup$ *?Faces*) $\subseteq$ *rel_frontier T*
($\forall x \in \bigcup(\mathcal{G} \cup$ *?Faces*) $\cap$
$\bigcup(\mathcal{G} \cup \{D.\ \exists C{\in}\mathcal{F}.\ D$ *face_of* $C \wedge$ *aff_dim* $D < p\})$. $g\ x = h$
x)
by (*rule exE* [*OF extending_maps_Union* [*OF fin extendh clo K*]], *blast+*)
then show *?case*
by (*simp add*: *intle local.heq* [*symmetric*], *blast*)
qed

have *eq*: $\bigcup(\mathcal{G} \cup \{D.\ \exists\, C \in \mathcal{F}.\ D\ \mathit{face\_of}\ C \wedge \mathit{aff\_dim}\ D < i\}) = \bigcup\mathcal{F}$
proof
  show $\bigcup(\mathcal{G} \cup \{D.\ \exists\, C{\in}\mathcal{F}.\ D\ \mathit{face\_of}\ C \wedge \mathit{aff\_dim}\ D < \mathit{int}\ i\}) \subseteq \bigcup\mathcal{F}$
    using ‹$\mathcal{G} \subseteq \mathcal{F}$› *face_of_imp_subset* **by** *fastforce*
  show $\bigcup\mathcal{F} \subseteq \bigcup(\mathcal{G} \cup \{D.\ \exists\, C{\in}\mathcal{F}.\ D\ \mathit{face\_of}\ C \wedge \mathit{aff\_dim}\ D < i\})$
  proof (*rule Union_mono*)
    show $\mathcal{F} \subseteq \mathcal{G} \cup \{D.\ \exists\, C{\in}\mathcal{F}.\ D\ \mathit{face\_of}\ C \wedge \mathit{aff\_dim}\ D < \mathit{int}\ i\}$
      using *face* **by** (*fastforce simp*: *aff i*)
  qed
qed
have *int i* ≤ *aff_dim T* **by** (*simp add*: *i*)
then show *?thesis*
  using *extendf* [*of i*] **unfolding** *eq* **by** (*metis that*)
qed

lemma *extend_map_lemma_cofinite0*:
assumes *finite* $\mathcal{F}$
  and *pairwise* $(\lambda S\ T.\ S \cap T \subseteq K)\ \mathcal{F}$
  and $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow \exists\, a\ g.\ a \notin U \wedge \mathit{continuous\_on}\ (S - \{a\})\ g \wedge g\ `\ (S - \{a\}) \subseteq T \wedge (\forall x \in S \cap K.\ g\ x = h\ x)$
  and $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow \mathit{closed}\ S$
shows $\exists\, C\ g.\ \mathit{finite}\ C \wedge \mathit{disjnt}\ C\ U \wedge \mathit{card}\ C \le \mathit{card}\ \mathcal{F} \wedge$
  $\mathit{continuous\_on}\ (\bigcup\mathcal{F} - C)\ g \wedge g\ `\ (\bigcup\mathcal{F} - C) \subseteq T$
  $\wedge (\forall x \in (\bigcup\mathcal{F} - C) \cap K.\ g\ x = h\ x)$
using *assms*
proof *induction*
  case *empty* **then show** *?case*
    by *force*
next
  case (*insert X* $\mathcal{F}$)
  then have *closed X* **and** *clo*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow \mathit{closed}\ X$
    and $\mathcal{F}$: $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow \exists\, a\ g.\ a \notin U \wedge \mathit{continuous\_on}\ (S - \{a\})\ g \wedge g\ `\ (S - \{a\}) \subseteq T \wedge (\forall x \in S \cap K.\ g\ x = h\ x)$
    and *pwX*: $\bigwedge Y.\ Y \in \mathcal{F} \wedge Y \neq X \longrightarrow X \cap Y \subseteq K \wedge Y \cap X \subseteq K$
    and *pwF*: *pairwise* $(\lambda\ S\ T.\ S \cap T \subseteq K)\ \mathcal{F}$
    by (*simp_all add*: *pairwise_insert*)
  obtain *C g* **where** *C*: *finite C disjnt C U card C* ≤ *card* $\mathcal{F}$
      and *contg*: $\mathit{continuous\_on}\ (\bigcup\mathcal{F} - C)\ g$
      and *gim*: $g\ `\ (\bigcup\mathcal{F} - C) \subseteq T$
      and *gh*: $\bigwedge x.\ x \in (\bigcup\mathcal{F} - C) \cap K \Longrightarrow g\ x = h\ x$
    using *insert.IH* [*OF pwF* $\mathcal{F}$ *clo*] **by** *auto*
  obtain *a f* **where** $a \notin U$
      and *contf*: $\mathit{continuous\_on}\ (X - \{a\})\ f$
      and *fim*: $f\ `\ (X - \{a\}) \subseteq T$
      and *fh*: $(\forall x \in X \cap K.\ f\ x = h\ x)$
    using *insert.prems* **by** (*meson insertI1*)
  show *?case*
  proof (*intro exI conjI*)
    show *finite* (*insert a C*)

by (*simp add*: *C*)
show *disjnt* (*insert a C*) *U*
using *C* ‹*a* ∉ *U*› **by** *simp*
show *card* (*insert a C*) ≤ *card* (*insert X* 𝓕)
by (*simp add*: *C card_insert_if insert.hyps le_SucI*)
have *closed* (⋃𝓕)
using *clo insert.hyps* **by** *blast*
have *continuous_on* (*X* − *insert a C*) *f*
using *contf* **by** (*force simp*: *elim*: *continuous_on_subset*)
moreover have *continuous_on* (⋃ 𝓕 − *insert a C*) *g*
using *contg* **by** (*force simp*: *elim*: *continuous_on_subset*)
ultimately
have *continuous_on* (*X* − *insert a C* ∪ (⋃𝓕 − *insert a C*)) (λ*x*. *if x* ∈ *X then f x else g x*)
apply (*intro continuous_on_cases_local*; *simp add*: *closedin_closed*)
using ‹*closed X*› **apply** *blast*
using ‹*closed* (⋃𝓕)› **apply** *blast*
using *fh gh insert.hyps pwX* **by** *fastforce*
then show *continuous_on* (⋃(*insert X* 𝓕) − *insert a C*) (λ*a*. *if a* ∈ *X then f a else g a*)
by (*blast intro*: *continuous_on_subset*)
show ∀ *x*∈(⋃(*insert X* 𝓕) − *insert a C*) ∩ *K*. (*if x* ∈ *X then f x else g x*) = *h x*
using *gh* **by** (*auto simp*: *fh*)
show (λ*a*. *if a* ∈ *X then f a else g a*) ‘ (⋃(*insert X* 𝓕) − *insert a C*) ⊆ *T*
using *fim gim* **by** *auto force*
qed
qed

lemma *extend_map_lemma_cofinite1*:
assumes *finite* 𝓕
and 𝓕: ⋀*X*. *X* ∈ 𝓕 ⟹ ∃ *a g*. *a* ∉ *U* ∧ *continuous_on* (*X* − {*a*}) *g* ∧ *g* ‘ (*X* − {*a*}) ⊆ *T* ∧ (∀ *x* ∈ *X* ∩ *K*. *g x* = *h x*)
and *clo*: ⋀*X*. *X* ∈ 𝓕 ⟹ *closed X*
and *K*: ⋀*X Y*. ⟦*X* ∈ 𝓕; *Y* ∈ 𝓕; ¬ *X* ⊆ *Y*; ¬ *Y* ⊆ *X*⟧ ⟹ *X* ∩ *Y* ⊆ *K*
obtains *C g* **where** *finite C disjnt C U card C* ≤ *card* 𝓕 *continuous_on* (⋃𝓕 − *C*) *g*
g ‘ (⋃𝓕 − *C*) ⊆ *T*
⋀*x*. *x* ∈ (⋃𝓕 − *C*) ∩ *K* ⟹ *g x* = *h x*
proof −
let *?*𝓕 = {*X* ∈ 𝓕. ∀ *Y*∈𝓕. ¬ *X* ⊂ *Y*}
have [*simp*]: ⋃ *?*𝓕 = ⋃𝓕
by (*simp add*: *Union_maximal_sets assms*)
have *fin*: *finite ?*𝓕
by (*force intro*: *finite_subset* [*OF* _ ‹*finite* 𝓕›])
have *pw*: *pairwise* (λ *S T*. *S* ∩ *T* ⊆ *K*) *?*𝓕
by (*simp add*: *pairwise_def*) (*metis K psubsetI*)
have *card* {*X* ∈ 𝓕. ∀ *Y*∈𝓕. ¬ *X* ⊂ *Y*} ≤ *card* 𝓕

by (*simp add*: ‹*finite* $\mathcal{F}$› *card_mono*)
moreover
obtain *C g* **where** *finite C* ∧ *disjnt C U* ∧ *card C* ≤ *card ?*$\mathcal{F}$ ∧
continuous_on (⋃ ?$\mathcal{F}$ − *C*) *g* ∧ *g* ' (⋃ ?$\mathcal{F}$ − *C*) ⊆ *T*
∧ (∀ *x* ∈ (⋃ ?$\mathcal{F}$ − *C*) ∩ *K*. *g x* = *h x*)
using *extend_map_lemma_cofinite0* [*OF fin pw*, *of U T h*] **by** (*fastforce intro*!: *clo* $\mathcal{F}$)
ultimately show *?thesis*
by (*rule_tac C*=*C* **and** *g*=*g* **in** *that*) *auto*
qed

lemma *extend_map_lemma_cofinite*:
assumes *finite* $\mathcal{F}$ $\mathcal{G}$ ⊆ $\mathcal{F}$ **and** *T*: *convex T bounded T*
and *poly*: ⋀*X*. *X* ∈ $\mathcal{F}$ ⟹ *polytope X*
and *contf*: *continuous_on* (⋃$\mathcal{G}$) *f* **and** *fim*: *f* ' (⋃$\mathcal{G}$) ⊆ *rel_frontier T*
and *face*: ⋀*X Y*. ⟦*X* ∈ $\mathcal{F}$; *Y* ∈ $\mathcal{F}$⟧ ⟹ (*X* ∩ *Y*) *face_of X*
and *aff*: ⋀*X*. *X* ∈ $\mathcal{F}$ − $\mathcal{G}$ ⟹ *aff_dim X* ≤ *aff_dim T*
obtains *C g* **where**
finite C disjnt C (⋃$\mathcal{G}$) *card C* ≤ *card* $\mathcal{F}$ *continuous_on* (⋃$\mathcal{F}$ − *C*) *g*
g ' (⋃ $\mathcal{F}$ − *C*) ⊆ *rel_frontier T* ⋀*x*. *x* ∈ ⋃$\mathcal{G}$ ⟹ *g x* = *f x*
proof −
define $\mathcal{H}$ **where** $\mathcal{H}$ ≡ $\mathcal{G}$ ∪ {*D*. ∃ *C* ∈ $\mathcal{F}$ − $\mathcal{G}$. *D face_of C* ∧ *aff_dim D* < *aff_dim T*}
have *finite* $\mathcal{G}$
using *assms finite_subset* **by** *blast*
have ∗: *finite* (⋃{{*D*. *D face_of C*} |*C*. *C* ∈ $\mathcal{F}$})
using *finite_polytope_faces poly* ‹*finite* $\mathcal{F}$› **by** *force*
then have *finite* $\mathcal{H}$
by (*auto simp*: $\mathcal{H}$*_def* ‹*finite* $\mathcal{G}$› *intro*: *finite_subset* [*OF* _ ∗])
have *face'*: ⋀*S T*. ⟦*S* ∈ $\mathcal{F}$; *T* ∈ $\mathcal{F}$⟧ ⟹ (*S* ∩ *T*) *face_of S* ∧ (*S* ∩ *T*) *face_of T*
by (*metis face inf_commute*)
have ∗: ⋀*X Y*. ⟦*X* ∈ $\mathcal{H}$; *Y* ∈ $\mathcal{H}$⟧ ⟹ *X* ∩ *Y face_of X*
unfolding $\mathcal{H}$*_def*
using *subsetD* [*OF* ‹$\mathcal{G}$ ⊆ $\mathcal{F}$›] **apply** (*auto simp add*: *face*)
apply (*meson face' face_of_Int_subface face_of_refl_eq poly polytope_imp_convex*)+
done
obtain *h* **where** *conth*: *continuous_on* (⋃$\mathcal{H}$) *h* **and** *him*: *h* ' (⋃$\mathcal{H}$) ⊆ *rel_frontier T*
and *hf*: ⋀*x*. *x* ∈ ⋃$\mathcal{G}$ ⟹ *h x* = *f x*
proof (*rule extend_map_lemma* [*OF* ‹*finite* $\mathcal{H}$› [*unfolded* $\mathcal{H}$*_def*] *Un_upper1 T*])
show ⋀*X*. ⟦*X* ∈ $\mathcal{G}$ ∪ {*D*. ∃ *C*∈$\mathcal{F}$ − $\mathcal{G}$. *D face_of C* ∧ *aff_dim D* < *aff_dim T*}⟧ ⟹ *polytope X*
using ‹$\mathcal{G}$ ⊆ $\mathcal{F}$› *face_of_polytope_polytope poly* **by** *fastforce*
qed (*use* ∗ $\mathcal{H}$*_def contf fim* **in** *auto*)
have *bounded* (⋃$\mathcal{G}$)
using ‹*finite* $\mathcal{G}$› ‹$\mathcal{G}$ ⊆ $\mathcal{F}$› *poly polytope_imp_bounded* **by** *blast*
then have ⋃$\mathcal{G}$ ≠ *UNIV*
by *auto*

then obtain *a* **where** *a*: $a \notin \bigcup\mathcal{G}$
  by *blast*
have $\mathcal{F}$: $\exists\, a\ g.\ a \notin \bigcup\mathcal{G} \wedge$ *continuous_on* $(D - \{a\})\ g \wedge$
        $g\ `\ (D - \{a\}) \subseteq$ *rel_frontier* $T \wedge (\forall\, x \in D \cap \bigcup\mathcal{H}.\ g\ x = h\ x)$
    if $D \in \mathcal{F}$ **for** D
proof (*cases* $D \subseteq \bigcup\mathcal{H}$)
  case *True*
  then have $h\ `\ (D - \{a\}) \subseteq$ *rel_frontier* T *continuous_on* $(D - \{a\})\ h$
    using *him* **by** (*blast intro*!: ‹$a \notin \bigcup\mathcal{G}$› *continuous_on_subset* [*OF conth*])+
  then show *?thesis*
    using *a* **by** *blast*
next
  case *False*
  note *D_not_subset* = *False*
  show *?thesis*
  proof (*cases* $D \in \mathcal{G}$)
    case *True*
    with *D_not_subset* **show** *?thesis*
      by (*auto simp*: $\mathcal{H}$*_def*)
  next
    case *False*
    then have *affD*: *aff_dim* $D \leq$ *aff_dim* T
      by (*simp add*: ‹$D \in \mathcal{F}$› *aff*)
    show *?thesis*
    proof (*cases rel_interior* $D = \{\}$)
      case *True*
      with ‹$D \in \mathcal{F}$› *poly a* **show** *?thesis*
        by (*force simp*: *rel_interior_eq_empty polytope_imp_convex*)
    next
      case *False*
      then obtain *b* **where** *brelD*: $b \in$ *rel_interior* D
        by *blast*
      have *polyhedron* D
        by (*simp add*: *poly polytope_imp_polyhedron that*)
      have *rel_frontier* D *retract_of affine hull* $D - \{b\}$
        by (*simp add*: *rel_frontier_retract_of_punctured_affine_hull poly polytope_imp_bounded polytope_imp_convex that brelD*)
      then obtain *r* **where** *relfD*: *rel_frontier* $D \subseteq$ *affine hull* $D - \{b\}$
          and *contr*: *continuous_on* (*affine hull* $D - \{b\}$) r
          and *rim*: $r\ `$ (*affine hull* $D - \{b\}$) $\subseteq$ *rel_frontier* D
          and *rid*: $\bigwedge x.\ x \in$ *rel_frontier* $D \Longrightarrow r\ x = x$
        by (*auto simp*: *retract_of_def retraction_def*)
      show *?thesis*
      proof (*intro exI conjI ballI*)
        show $b \notin \bigcup\mathcal{G}$
        proof *clarify*
          fix E
          assume $b \in E$ $E \in \mathcal{G}$
          then have $E \cap D$ *face_of* $E \wedge E \cap D$ *face_of* D

using ‹$\mathcal{G} \subseteq \mathcal{F}$› *face′ that* **by** *auto*
with *face_of_subset_rel_frontier* ‹$E \in \mathcal{G}$› ‹$b \in E$› *brelD rel_interior_subset* [*of D*]
D_not_subset rel_frontier_def $\mathcal{H}$*_def*
show *False*
by *blast*
qed
have r ‘ $(D - \{b\}) \subseteq r$ ‘ (*affine hull D* $- \{b\}$)
by (*simp add*: *Diff_mono hull_subset image_mono*)
also have ... $\subseteq$ *rel_frontier D*
by (*rule rim*)
also have ... $\subseteq \bigcup\{E.\ E$ *face_of D* $\wedge$ *aff_dim E* $<$ *aff_dim T*$\}$
using *affD*
by (*force simp*: *rel_frontier_of_polyhedron* [*OF* ‹*polyhedron D*›] *facet_of_def*)
also have ... $\subseteq \bigcup(\mathcal{H})$
using *D_not_subset* $\mathcal{H}$*_def that* **by** *fastforce*
finally have *rsub*: r ‘ $(D - \{b\}) \subseteq \bigcup(\mathcal{H})$ **.**
show *continuous_on* $(D - \{b\})$ $(h \circ r)$
proof (*rule continuous_on_compose*)
show *continuous_on* $(D - \{b\})$ r
by (*meson Diff_mono continuous_on_subset contr hull_subset order_refl*)
show *continuous_on* (r ‘ $(D - \{b\})$) h
by (*simp add*: *Diff_mono hull_subset continuous_on_subset* [*OF conth rsub*])
qed
show $(h \circ r)$ ‘ $(D - \{b\}) \subseteq$ *rel_frontier T*
using *brelD him rsub* **by** *fastforce*
show $(h \circ r)\ x = h\ x$ **if** x: $x \in D \cap \bigcup\mathcal{H}$ **for** x
proof −
consider A **where** $x \in D$ $A \in \mathcal{G}$ $x \in A$
| A B **where** $x \in D$ A *face_of* B $B \in \mathcal{F}$ $B \notin \mathcal{G}$ *aff_dim A* $<$ *aff_dim T* $x \in A$
using x **by** (*auto simp*: $\mathcal{H}$*_def*)
then have *xrel*: $x \in$ *rel_frontier D*
proof *cases*
case *1* **show** *?thesis*
proof (*rule face_of_subset_rel_frontier* [*THEN subsetD*])
show $D \cap A$ *face_of D*
using ‹$A \in \mathcal{G}$› ‹$\mathcal{G} \subseteq \mathcal{F}$› *face* ‹$D \in \mathcal{F}$› **by** *blast*
show $D \cap A \neq D$
using ‹$A \in \mathcal{G}$› *D_not_subset* $\mathcal{H}$*_def* **by** *blast*
qed (*auto simp*: *1*)
next
case *2* **show** *?thesis*
proof (*rule face_of_subset_rel_frontier* [*THEN subsetD*])
have *D face_of D*
by (*simp add*: ‹*polyhedron D*› *polyhedron_imp_convex face_of_refl*)
then show $D \cap A$ *face_of D*
by (*meson 2*(*2*) *2*(*3*) ‹$D \in \mathcal{F}$› *face′ face_of_Int_Int face_of_face*)

show $D \cap A \neq D$
using *2 D_not_subset $\mathcal{H}$_def* **by** *blast*
qed (*auto simp*: *2*)
qed
show *?thesis*
by (*simp add*: *rid xrel*)
qed
qed
qed
qed
qed
have *clo*: $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *closed S*
by (*simp add*: *poly polytope_imp_closed*)
obtain *C g* **where** *finite C disjnt C* ($\bigcup\mathcal{G}$) *card C* $\leq$ *card* $\mathcal{F}$ *continuous_on* ($\bigcup\mathcal{F}$ $-$ *C*) *g*
g ‘ ($\bigcup\mathcal{F}$ $-$ *C*) $\subseteq$ *rel_frontier T*
and *gh*: $\bigwedge x.\ x \in (\bigcup\mathcal{F} - C) \cap \bigcup\mathcal{H} \Longrightarrow g\ x = h\ x$
proof (*rule extend_map_lemma_cofinite1* [*OF* ‹*finite* $\mathcal{F}$› $\mathcal{F}$ *clo*])
show $X \cap Y \subseteq \bigcup\mathcal{H}$ **if** *XY*: $X \in \mathcal{F}$ $Y \in \mathcal{F}$ **and** $\neg\ X \subseteq Y$ $\neg\ Y \subseteq X$ **for** *X Y*
proof (*cases* $X \in \mathcal{G}$)
case *True*
then show *?thesis*
by (*auto simp*: $\mathcal{H}$*_def*)
next
case *False*
have $X \cap Y \neq X$
using ‹$\neg\ X \subseteq Y$› **by** *blast*
with *XY*
show *?thesis*
by (*clarsimp simp*: $\mathcal{H}$*_def*)
(*metis Diff_iff Int_iff aff antisym_conv face face_of_aff_dim_lt face_of_refl not_le poly polytope_imp_convex*)
qed
qed (*blast*)+
with ‹$\mathcal{G} \subseteq \mathcal{F}$› **show** *?thesis*
by (*rule_tac C=C* **and** *g=g* **in** *that*) (*auto simp*: *disjnt_def hf* [*symmetric*] $\mathcal{H}$*_def intro*!: *gh*)
qed

The next two proofs are similar

theorem *extend_map_cell_complex_to_sphere*:
assumes *finite* $\mathcal{F}$ **and** *S*: $S \subseteq \bigcup\mathcal{F}$ *closed S* **and** *T*: *convex T bounded T*
and *poly*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *polytope X*
and *aff*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *aff_dim X* $<$ *aff_dim T*
and *face*: $\bigwedge X\ Y.\ [\![X \in \mathcal{F};\ Y \in \mathcal{F}]\!] \Longrightarrow (X \cap Y)$ *face_of X*
and *contf*: *continuous_on S f* **and** *fim*: *f* ‘ *S* $\subseteq$ *rel_frontier T*
obtains *g* **where** *continuous_on* ($\bigcup\mathcal{F}$) *g*
g ‘ ($\bigcup\mathcal{F}$) $\subseteq$ *rel_frontier T* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
proof $-$

obtain *V g* **where** $S \subseteq V$ *open V continuous_on V g* **and** *gim*: *g ‘ V* $\subseteq$ *rel_frontier T* **and** *gf*: $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
using *neighbourhood_extension_into_ANR* [*OF contf fim* _ ‹*closed S*›] *ANR_rel_frontier_convex T* **by** *blast*
have *compact S*
by (*meson assms compact_Union poly polytope_imp_compact seq_compact_closed_subset seq_compact_eq_compact*)
then obtain *d* **where** $d > 0$ **and** *d*: $\bigwedge x\ y.\ [\![x \in S;\ y \in -\ V]\!] \Longrightarrow d \leq dist\ x\ y$
using *separate_compact_closed* [*of S* $-V$] ‹*open V*› ‹$S \subseteq V$› **by** *force*
obtain $\mathcal{G}$ **where** *finite* $\mathcal{G}$ $\bigcup\mathcal{G} = \bigcup\mathcal{F}$
and *diaG*: $\bigwedge X.\ X \in \mathcal{G} \Longrightarrow$ *diameter X* $< d$
and *polyG*: $\bigwedge X.\ X \in \mathcal{G} \Longrightarrow$ *polytope X*
and *affG*: $\bigwedge X.\ X \in \mathcal{G} \Longrightarrow$ *aff_dim X* $\leq$ *aff_dim T* $-$ *1*
and *faceG*: $\bigwedge X\ Y.\ [\![X \in \mathcal{G};\ Y \in \mathcal{G}]\!] \Longrightarrow X \cap Y$ *face_of X*
proof (*rule cell_complex_subdivision_exists* [*OF* ‹*d>0*› ‹*finite* $\mathcal{F}$› *poly* _ *face*])
show $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *aff_dim X* $\leq$ *aff_dim T* $-$ *1*
by (*simp add*: *aff*)
qed *auto*
obtain *h* **where** *conth*: *continuous_on* ($\bigcup\mathcal{G}$) *h* **and** *him*: *h ‘* $\bigcup\mathcal{G} \subseteq$ *rel_frontier T* **and** *hg*: $\bigwedge x.\ x \in \bigcup(\mathcal{G} \cap Pow\ V) \Longrightarrow h\ x = g\ x$
proof (*rule extend_map_lemma* [*of* $\mathcal{G}$ $\mathcal{G} \cap$ *Pow V T g*])
show *continuous_on* ($\bigcup(\mathcal{G} \cap Pow\ V)$) *g*
by (*metis Union_Int_subset Union_Pow_eq* ‹*continuous_on V g*› *continuous_on_subset le_inf_iff*)
qed (*use* ‹*finite* $\mathcal{G}$› *T polyG affG faceG gim* **in** *fastforce*)+
show *?thesis*
proof
show *continuous_on* ($\bigcup\mathcal{F}$) *h*
using ‹$\bigcup\mathcal{G} = \bigcup\mathcal{F}$› *conth* **by** *auto*
show *h ‘* $\bigcup\mathcal{F} \subseteq$ *rel_frontier T*
using ‹$\bigcup\mathcal{G} = \bigcup\mathcal{F}$› *him* **by** *auto*
show $h\ x = f\ x$ **if** $x \in S$ **for** *x*
proof −
have $x \in \bigcup\mathcal{G}$
using ‹$\bigcup\mathcal{G} = \bigcup\mathcal{F}$› ‹$S \subseteq \bigcup\mathcal{F}$› *that* **by** *auto*
then obtain *X* **where** $x \in X$ $X \in \mathcal{G}$ **by** *blast*
then have *diameter X* $< d$ *bounded X*
by (*auto simp*: *diaG* ‹$X \in \mathcal{G}$› *polyG polytope_imp_bounded*)
then have $X \subseteq V$ **using** *d* [*OF* ‹$x \in S$›] *diameter_bounded_bound* [*OF* ‹*bounded X*› ‹$x \in X$›]
by *fastforce*
have $h\ x = g\ x$
using ‹$X \in \mathcal{G}$› ‹$X \subseteq V$› ‹$x \in X$› *hg* **by** *auto*
also have ... $= f\ x$
by (*simp add*: *gf that*)
finally show $h\ x = f\ x$ **.**
qed
qed
qed

theorem *extend_map_cell_complex_to_sphere_cofinite*:
  assumes *finite* $\mathcal{F}$ **and** *S*: $S \subseteq \bigcup\mathcal{F}$ *closed S* **and** *T*: *convex T bounded T*
    and *poly*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *polytope X*
    and *aff*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *aff_dim X* $\leq$ *aff_dim T*
    and *face*: $\bigwedge X\ Y.\ [\![X \in \mathcal{F};\ Y \in \mathcal{F}]\!] \Longrightarrow (X \cap Y)$ *face_of X*
    and *contf*: *continuous_on S f* **and** *fim*: *f ‘ S* $\subseteq$ *rel_frontier T*
  obtains *C g* **where** *finite C disjnt C S continuous_on* $(\bigcup\mathcal{F} - C)$ *g*
   g ‘ $(\bigcup\mathcal{F} - C) \subseteq$ *rel_frontier T* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
proof –
 obtain *V g* **where** $S \subseteq V$ *open V continuous_on V g* **and** *gim*: *g ‘ V* $\subseteq$ *rel_frontier T* **and** *gf*: $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
  using *neighbourhood_extension_into_ANR* [*OF contf fim _* ‹*closed S*›] *ANR_rel_frontier_convex T* **by** *blast*
 have *compact S*
  by (*meson assms compact_Union poly polytope_imp_compact seq_compact_closed_subset seq_compact_eq_compact*)
 then obtain *d* **where** $d > 0$ **and** *d*: $\bigwedge x\ y.\ [\![x \in S;\ y \in -\ V]\!] \Longrightarrow d \leq$ *dist x y*
  using *separate_compact_closed* [*of S* $-V$] ‹*open V*› ‹$S \subseteq V$› **by** *force*
 obtain $\mathcal{G}$ **where** *finite* $\mathcal{G}$ $\bigcup\mathcal{G} = \bigcup\mathcal{F}$
   and *diaG*: $\bigwedge X.\ X \in \mathcal{G} \Longrightarrow$ *diameter X* $< d$
   and *polyG*: $\bigwedge X.\ X \in \mathcal{G} \Longrightarrow$ *polytope X*
   and *affG*: $\bigwedge X.\ X \in \mathcal{G} \Longrightarrow$ *aff_dim X* $\leq$ *aff_dim T*
   and *faceG*: $\bigwedge X\ Y.\ [\![X \in \mathcal{G};\ Y \in \mathcal{G}]\!] \Longrightarrow X \cap Y$ *face_of X*
  by (*rule cell_complex_subdivision_exists* [*OF* ‹$d>0$› ‹*finite* $\mathcal{F}$› *poly aff face*]) *auto*
 obtain *C h* **where** *finite C* **and** *dis*: *disjnt C* $(\bigcup(\mathcal{G} \cap Pow\ V))$
   and *card*: *card C* $\leq$ *card* $\mathcal{G}$ **and** *conth*: *continuous_on* $(\bigcup\mathcal{G} - C)$ *h*
   and *him*: *h ‘* $(\bigcup\mathcal{G} - C) \subseteq$ *rel_frontier T*
   and *hg*: $\bigwedge x.\ x \in \bigcup(\mathcal{G} \cap Pow\ V) \Longrightarrow h\ x = g\ x$
 proof (*rule extend_map_lemma_cofinite* [*of* $\mathcal{G}$ $\mathcal{G} \cap Pow\ V$ *T g*])
  show *continuous_on* $(\bigcup(\mathcal{G} \cap Pow\ V))$ *g*
    by (*metis Union_Int_subset Union_Pow_eq* ‹*continuous_on V g*› *continuous_on_subset le_inf_iff*)
  show *g ‘* $\bigcup(\mathcal{G} \cap Pow\ V) \subseteq$ *rel_frontier T*
   using *gim* **by** *force*
 qed (*auto intro*: ‹*finite* $\mathcal{G}$› *T polyG affG dest*: *faceG*)
 have *Ssub*: $S \subseteq \bigcup(\mathcal{G} \cap Pow\ V)$
 proof
  fix *x*
  assume $x \in S$
  then have $x \in \bigcup\mathcal{G}$
   using ‹$\bigcup\mathcal{G} = \bigcup\mathcal{F}$› ‹$S \subseteq \bigcup\mathcal{F}$› **by** *auto*
  then obtain *X* **where** $x \in X$ $X \in \mathcal{G}$ **by** *blast*
  then have *diameter X* $< d$ *bounded X*
   by (*auto simp*: *diaG* ‹$X \in \mathcal{G}$› *polyG polytope_imp_bounded*)
  then have $X \subseteq V$ **using** *d* [*OF* ‹$x \in S$›] *diameter_bounded_bound* [*OF* ‹*bounded X*› ‹$x \in X$›]
   by *fastforce*

```
    then show x ∈ ⋃(𝒢 ∩ Pow V)
      using ‹X ∈ 𝒢› ‹x ∈ X› by blast
  qed
  show ?thesis
  proof
    show continuous_on (⋃ℱ−C) h
      using ‹⋃𝒢 = ⋃ℱ› conth by auto
    show h ‘ (⋃ℱ − C) ⊆ rel_frontier T
      using ‹⋃𝒢 = ⋃ℱ› him by auto
    show h x = f x if x ∈ S for x
    proof −
      have h x = g x
        using Ssub hg that by blast
      also have ... = f x
        by (simp add: gf that)
      finally show h x = f x .
    qed
    show disjnt C S
      using dis Ssub  by (meson disjnt_iff subset_eq)
  qed (intro ‹finite C›)
qed
```

6.41.3 Special cases and corollaries involving spheres

```
lemma disjnt_Diff1: X ⊆ Y' ⟹ disjnt (X − Y) (X' − Y')
  by (auto simp: disjnt_def)

proposition extend_map_affine_to_sphere_cofinite_simple:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes compact S convex U bounded U
      and aff: aff_dim T ≤ aff_dim U
      and S ⊆ T and contf: continuous_on S f
      and fim: f ‘ S ⊆ rel_frontier U
 obtains K g where finite K K ⊆ T disjnt K S continuous_on (T − K) g
                  g ‘ (T − K) ⊆ rel_frontier U
                  ⋀x. x ∈ S ⟹ g x = f x
proof −
  have ∃K g. finite K ∧ disjnt K S ∧ continuous_on (T − K) g ∧
            g ‘ (T − K) ⊆ rel_frontier U ∧ (∀x ∈ S. g x = f x)
       if affine T S ⊆ T and aff: aff_dim T ≤ aff_dim U  for T
  proof (cases S = {})
    case True
    show ?thesis
    proof (cases rel_frontier U = {})
      case True
      with ‹bounded U› have aff_dim U ≤ 0
        using affine_bounded_eq_lowdim rel_frontier_eq_empty by auto
      with aff have aff_dim T ≤ 0 by auto
      then obtain a where T ⊆ {a}
```

using ‹*affine T*› *affine_bounded_eq_lowdim affine_bounded_eq_trivial* **by** *auto*
then show *?thesis*
using ‹*S* = {}› *fim*
by (*metis Diff_cancel contf disjnt_empty2 finite.emptyI finite_insert finite_subset*)
next
case *False*
then obtain *a* **where** *a* ∈ *rel_frontier U*
by *auto*
then show *?thesis*
using *continuous_on_const* [*of* _ *a*] ‹*S* = {}› **by** *force*
qed
next
case *False*
have *bounded S*
by (*simp add*: ‹*compact S*› *compact_imp_bounded*)
then obtain *b* **where** *b*: *S* ⊆ *cbox* (−*b*) *b*
using *bounded_subset_cbox_symmetric* **by** *blast*
define *bbox* **where** *bbox* ≡ *cbox* (−(*b*+*One*)) (*b*+*One*)
have *cbox* (−*b*) *b* ⊆ *bbox*
by (*auto simp*: *bbox_def algebra_simps intro*!: *subset_box_imp*)
with *b* ‹*S* ⊆ *T*› **have** *S* ⊆ *bbox* ∩ *T*
by *auto*
then have *Ssub*: *S* ⊆ ⋃{*bbox* ∩ *T*}
by *auto*
then have *aff_dim* (*bbox* ∩ *T*) ≤ *aff_dim U*
by (*metis aff aff_dim_subset inf_commute inf_le1 order_trans*)
obtain *K g* **where** *K*: *finite K disjnt K S*
and *contg*: *continuous_on* (⋃{*bbox* ∩ *T*} − *K*) *g*
and *gim*: *g* ‘ (⋃{*bbox* ∩ *T*} − *K*) ⊆ *rel_frontier U*
and *gf*: ⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x*
proof (*rule extend_map_cell_complex_to_sphere_cofinite*
[*OF* _ *Ssub* _ ‹*convex U*› ‹*bounded U*› _ _ _ *contf fim*])
show *closed S*
using ‹*compact S*› *compact_eq_bounded_closed* **by** *auto*
show *poly*: ⋀*X*. *X* ∈ {*bbox* ∩ *T*} ⟹ *polytope X*
by (*simp add*: *polytope_Int_polyhedron bbox_def polytope_interval affine_imp_polyhedron* ‹*affine T*›)
show ⋀*X Y*. ⟦*X* ∈ {*bbox* ∩ *T*}; *Y* ∈ {*bbox* ∩ *T*}⟧ ⟹ *X* ∩ *Y face_of X*
by (*simp add*:*poly face_of_refl polytope_imp_convex*)
show ⋀*X*. *X* ∈ {*bbox* ∩ *T*} ⟹ *aff_dim X* ≤ *aff_dim U*
by (*simp add*: ‹*aff_dim* (*bbox* ∩ *T*) ≤ *aff_dim U*›)
qed *auto*
define *fro* **where** *fro* ≡ λ*d*. *frontier*(*cbox* (−(*b* + *d* $*_R$ *One*)) (*b* + *d* $*_R$ *One*))
obtain *d* **where** *d12*: *1/2* ≤ *d d* ≤ *1* **and** *dd*: *disjnt K* (*fro d*)
proof (*rule disjoint_family_elem_disjnt* [*OF* _ ‹*finite K*›])
show *infinite* {*1/2..1*::*real*}
by (*simp add*: *infinite_Icc*)
have *dis1*: *disjnt* (*fro x*) (*fro y*) **if** *x*<*y* **for** *x y*

by (*auto simp*: *algebra_simps that subset_box_imp disjnt_Diff1 frontier_def fro_def*)
then show *disjoint_family_on fro* {*1*/*2*..*1*}
by (*auto simp*: *disjoint_family_on_def disjnt_def neq_iff*)
qed *auto*
define *c* **where** $c \equiv b + d *_R One$
have *cbsub*: *cbox* (−*b*) *b* ⊆ *box* (−*c*) *c cbox* (−*b*) *b* ⊆ *cbox* (−*c*) *c* *cbox* (−*c*) *c* ⊆ *bbox*
using *d12* **by** (*auto simp*: *algebra_simps subset_box_imp c_def bbox_def*)
have *clo_cbT*: *closed* (*cbox* (− *c*) *c* ∩ *T*)
by (*simp add*: *affine_closed closed_Int closed_cbox* ‹*affine T*›)
have *cpT_ne*: *cbox* (− *c*) *c* ∩ *T* ≠ {}
using ‹*S* ≠ {}› *b cbsub*(*2*) ‹*S* ⊆ *T*› **by** *fastforce*
have *closest_point* (*cbox* (− *c*) *c* ∩ *T*) *x* ∉ *K* **if** *x* ∈ *T x* ∉ *K* **for** *x*
proof (*cases x* ∈ *cbox* (−*c*) *c*)
case *True* **with** *that* **show** *?thesis*
by (*simp add*: *closest_point_self*)
next
case *False*
have *int_ne*: *interior* (*cbox* (−*c*) *c*) ∩ *T* ≠ {}
using ‹*S* ≠ {}› ‹*S* ⊆ *T*› *b* ‹*cbox* (− *b*) *b* ⊆ *box* (− *c*) *c*› **by** *force*
have *convex T*
by (*meson* ‹*affine T*› *affine_imp_convex*)
then have *x* ∈ *affine hull* (*cbox* (− *c*) *c* ∩ *T*)
by (*metis Int_commute Int_iff* ‹*S* ≠ {}› ‹*S* ⊆ *T*› *cbsub*(*1*) ‹*x* ∈ *T*› *affine_hull_convex_Int_nonempty_interior all_not_in_conv b hull_inc inf*.*orderE interior_cbox*)
then have *x* ∈ *affine hull* (*cbox* (− *c*) *c* ∩ *T*) − *rel_interior* (*cbox* (− *c*) *c* ∩ *T*)
by (*meson DiffI False Int_iff rel_interior_subset subsetCE*)
then have *closest_point* (*cbox* (− *c*) *c* ∩ *T*) *x* ∈ *rel_frontier* (*cbox* (− *c*) *c* ∩ *T*)
by (*rule closest_point_in_rel_frontier* [*OF clo_cbT cpT_ne*])
moreover have (*rel_frontier* (*cbox* (− *c*) *c* ∩ *T*)) ⊆ *fro d*
by (*subst convex_affine_rel_frontier_Int* [*OF* _ ‹*affine T*› *int_ne*]) (*auto simp*: *fro_def c_def*)
ultimately show *?thesis*
using *dd* **by** (*force simp*: *disjnt_def*)
qed
then have *cpt_subset*: *closest_point* (*cbox* (− *c*) *c* ∩ *T*) ‘ (*T* − *K*) ⊆ ⋃{*bbox* ∩ *T*} − *K*
using *closest_point_in_set* [*OF clo_cbT cpT_ne*] *cbsub*(*3*) **by** *force*
show *?thesis*
proof (*intro conjI ballI exI*)
have *continuous_on* (*T* − *K*) (*closest_point* (*cbox* (− *c*) *c* ∩ *T*))
proof (*rule continuous_on_closest_point*)
show *convex* (*cbox* (− *c*) *c* ∩ *T*)
by (*simp add*: *affine_imp_convex convex_Int* ‹*affine T*›)
show *closed* (*cbox* (− *c*) *c* ∩ *T*)

using *clo_cbT* **by** *blast*
show *cbox* $(- c)$ $c \cap T \neq \{\}$
using ‹$S \neq \{\}$› *cbsub(2)* *b that* **by** *auto*
qed
then show *continuous_on* $(T - K)$ $(g \circ$ *closest_point* (*cbox* $(- c)$ $c \cap T))$
by (*metis continuous_on_compose continuous_on_subset* [*OF contg cpt_subset*])
have $(g \circ$ *closest_point* (*cbox* $(- c)$ $c \cap T))$ ‘ $(T - K) \subseteq g$ ‘ $(\bigcup\{bbox \cap T\} - K)$
by (*metis image_comp image_mono cpt_subset*)
also have ... $\subseteq$ *rel_frontier U*
by (*rule gim*)
finally show $(g \circ$ *closest_point* (*cbox* $(- c)$ $c \cap T))$ ‘ $(T - K) \subseteq$ *rel_frontier U* **.**
show $(g \circ$ *closest_point* (*cbox* $(- c)$ $c \cap T))$ $x = f\,x$ **if** $x \in S$ **for** x
proof −
have $(g \circ$ *closest_point* (*cbox* $(- c)$ $c \cap T))$ $x = g\,x$
unfolding *o_def*
by (*metis IntI* ‹$S \subseteq T$› *b cbsub(2) closest_point_self subset_eq that*)
also have ... $= f\,x$
by (*simp add*: *that gf*)
finally show *?thesis* **.**
qed
qed (*auto simp*: *K*)
qed
then obtain *K g* **where** *finite K disjnt K S*
and *contg*: *continuous_on* (*affine hull* $T - K$) g
and *gim*: g ‘ (*affine hull* $T - K$) $\subseteq$ *rel_frontier U*
and *gf*: $\bigwedge x.\ x \in S \Longrightarrow g\,x = f\,x$
by (*metis aff affine_affine_hull aff_dim_affine_hull*
order_trans [*OF* ‹$S \subseteq T$› *hull_subset* [*of T affine*]])
then obtain *K g* **where** *finite K disjnt K S*
and *contg*: *continuous_on* $(T - K)$ g
and *gim*: g ‘ $(T - K) \subseteq$ *rel_frontier U*
and *gf*: $\bigwedge x.\ x \in S \Longrightarrow g\,x = f\,x$
by (*rule_tac* $K=K$ **and** $g=g$ **in** *that*) (*auto simp*: *hull_inc elim*: *continuous_on_subset*)
then show *?thesis*
by (*rule_tac* $K=K \cap T$ **and** $g=g$ **in** *that*) (*auto simp*: *disjnt_iff Diff_Int contg*)
qed

6.41.4 Extending maps to spheres

lemma *extend_map_affine_to_sphere1*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::topological_space*
assumes *finite K affine U* **and** *contf*: *continuous_on* $(U - K)$ f
and *fim*: f ‘ $(U - K) \subseteq T$
and *comps*: $\bigwedge C.\ [\![C \in components(U - S);\ C \cap K \neq \{\}]\!] \Longrightarrow C \cap L \neq \{\}$
and *clo*: *closedin* (*top_of_set U*) *S* **and** *K*: *disjnt K S* $K \subseteq U$
obtains *g* **where** *continuous_on* $(U - L)$ g g ‘ $(U - L) \subseteq T$ $\bigwedge x.\ x \in S \Longrightarrow g$

x = *f x*
proof (*cases K* = {})
  case *True*
  then show *?thesis*
    by (*metis Diff_empty Diff_subset contf fim continuous_on_subset image_subsetI rev_image_eqI subset_iff that*)
next
  case *False*
  have *S* ⊆ *U*
    using *clo closedin_limpt* **by** *blast*
  then have (*U* − *S*) ∩ *K* ≠ {}
    by (*metis Diff_triv False Int_Diff K disjnt_def inf.absorb_iff2 inf_commute*)
  then have ⋃(*components* (*U* − *S*)) ∩ *K* ≠ {}
    using *Union_components* **by** *simp*
  then obtain *C0* **where** *C0*: *C0* ∈ *components* (*U* − *S*) *C0* ∩ *K* ≠ {}
    by *blast*
  have *convex U*
    by (*simp add*: *affine_imp_convex* ‹*affine U*›)
  then have *locally connected U*
    by (*rule convex_imp_locally_connected*)
  have ∃ *a g*. *a* ∈ *C* ∧ *a* ∈ *L* ∧ *continuous_on* (*S* ∪ (*C* − {*a*})) *g* ∧
            g ‘ (*S* ∪ (*C* − {*a*})) ⊆ *T* ∧ (∀ *x* ∈ *S*. *g x* = *f x*)
      if *C*: *C* ∈ *components* (*U* − *S*) **and** *CK*: *C* ∩ *K* ≠ {} **for** *C*
  proof −
    have *C* ⊆ *U*−*S C* ∩ *L* ≠ {}
      by (*simp_all add*: *in_components_subset comps that*)
    then obtain *a* **where** *a*: *a* ∈ *C a* ∈ *L* **by** *auto*
    have *opeUC*: *openin* (*top_of_set U*) *C*
    proof (*rule openin_trans*)
      show *openin* (*top_of_set* (*U*−*S*)) *C*
      by (*simp add*: ‹*locally connected U*› *clo locally_diff_closed openin_components_locally_connected* [*OF* _ *C*])
      show *openin* (*top_of_set U*) (*U* − *S*)
        by (*simp add*: *clo openin_diff*)
    qed
    then obtain *d* **where** *C* ⊆ *U 0* < *d* **and** *d*: *cball a d* ∩ *U* ⊆ *C*
      using *openin_contains_cball* **by** (*metis* ‹*a* ∈ *C*›)
    then have *ball a d* ∩ *U* ⊆ *C*
      by *auto*
    obtain *h k* **where** *homhk*: *homeomorphism* (*S* ∪ *C*) (*S* ∪ *C*) *h k*
              and *subC*: {*x*. (¬ (*h x* = *x* ∧ *k x* = *x*))} ⊆ *C*
              and *bou*: *bounded* {*x*. (¬ (*h x* = *x* ∧ *k x* = *x*))}
              and *hin*: ⋀*x*. *x* ∈ *C* ∩ *K* ⟹ *h x* ∈ *ball a d* ∩ *U*
    proof (*rule homeomorphism_grouping_points_exists_gen* [*of C ball a d* ∩ *U C* ∩ *K S* ∪ *C*])
      show *openin* (*top_of_set C*) (*ball a d* ∩ *U*)
        by (*metis open_ball* ‹*C* ⊆ *U*› ‹*ball a d* ∩ *U* ⊆ *C*› *inf.absorb_iff2 inf.orderE inf_assoc open_openin openin_subtopology*)
      show *openin* (*top_of_set* (*affine hull C*)) *C*

by (*metis* ‹*a* ∈ *C*› ‹*openin* (*top_of_set U*) *C*› *affine_hull_eq affine_hull_openin all_not_in_conv* ‹*affine U*›)
show *ball a d* ∩ *U* ≠ {}
using ‹*0* < *d*› ‹*C* ⊆ *U*› ‹*a* ∈ *C*› **by** *force*
show *finite* (*C* ∩ *K*)
by (*simp add*: ‹*finite K*›)
show *S* ∪ *C* ⊆ *affine hull C*
by (*metis* ‹*C* ⊆ *U*› ‹*S* ⊆ *U*› ‹*a* ∈ *C*› *opeUC affine_hull_eq affine_hull_openin all_not_in_conv assms*(*2*) *sup.bounded_iff*)
show *connected C*
by (*metis C in_components_connected*)
qed *auto*
have *a_BU*: *a* ∈ *ball a d* ∩ *U*
using ‹*0* < *d*› ‹*C* ⊆ *U*› ‹*a* ∈ *C*› **by** *auto*
have *rel_frontier* (*cball a d* ∩ *U*) *retract_of* (*affine hull* (*cball a d* ∩ *U*) − {*a*})
proof (*rule rel_frontier_retract_of_punctured_affine_hull*)
show *bounded* (*cball a d* ∩ *U*) *convex* (*cball a d* ∩ *U*)
by (*auto simp*: ‹*convex U*› *convex_Int*)
show *a* ∈ *rel_interior* (*cball a d* ∩ *U*)
by (*metis* ‹*affine U*› *convex_cball empty_iff interior_cball a_BU rel_interior_convex_Int_affine*)
qed
moreover have *rel_frontier* (*cball a d* ∩ *U*) = *frontier* (*cball a d*) ∩ *U*
by (*metis a_BU* ‹*affine U*› *convex_affine_rel_frontier_Int convex_cball equals0D interior_cball*)
moreover have *affine hull* (*cball a d* ∩ *U*) = *U*
by (*metis* ‹*convex U*› *a_BU affine_hull_convex_Int_nonempty_interior affine_hull_eq* ‹*affine U*› *equals0D inf.commute interior_cball*)
ultimately have *frontier* (*cball a d*) ∩ *U retract_of* (*U* − {*a*})
by *metis*
then obtain *r* **where** *contr*: *continuous_on* (*U* − {*a*}) *r*
and *rim*: *r* ‘ (*U* − {*a*}) ⊆ *sphere a d* *r* ‘ (*U* − {*a*}) ⊆ *U*
and *req*: ⋀*x*. *x* ∈ *sphere a d* ∩ *U* ⟹ *r x* = *x*
using ‹*affine U*› **by** (*auto simp*: *retract_of_def retraction_def hull_same*)
define *j* **where** *j* ≡ λ*x*. *if x* ∈ *ball a d then r x else x*
have *kj*: ⋀*x*. *x* ∈ *S* ⟹ *k* (*j x*) = *x*
using ‹*C* ⊆ *U* − *S*› ‹*S* ⊆ *U*› ‹*ball a d* ∩ *U* ⊆ *C*› *j_def subC* **by** *auto*
have *Uaeq*: *U* − {*a*} = (*cball a d* − {*a*}) ∩ *U* ∪ (*U* − *ball a d*)
using ‹*0* < *d*› **by** *auto*
have *jim*: *j* ‘ (*S* ∪ (*C* − {*a*})) ⊆ (*S* ∪ *C*) − *ball a d*
proof *clarify*
fix *y* **assume** *y* ∈ *S* ∪ (*C* − {*a*})
then have *y* ∈ *U* − {*a*}
using ‹*C* ⊆ *U* − *S*› ‹*S* ⊆ *U*› ‹*a* ∈ *C*› **by** *auto*
then have *r y* ∈ *sphere a d*
using *rim* **by** *auto*
then show *j y* ∈ *S* ∪ *C* − *ball a d*
unfolding *j_def*
using ‹*r y* ∈ *sphere a d*› ‹*y* ∈ *U* − {*a*}› ‹*y* ∈ *S* ∪ (*C* − {*a*})› *d rim*
by (*metis Diff_iff Int_iff Un_iff subsetD cball_diff_eq_sphere image_subset_iff*)

qed
have *contj*: *continuous_on* (*U* − {*a*}) *j*
unfolding *j_def Uaeq*
proof (*intro continuous_on_cases_local continuous_on_id*, *simp_all add*: *req closedin_closed Uaeq* [*symmetric*])
show ∃ *T*. *closed T* ∧ (*cball a d* − {*a*}) ∩ *U* = (*U* − {*a*}) ∩ *T*
using *affine_closed* ‹*affine U*› **by** (*rule_tac x*=(*cball a d*) ∩ *U* **in** *exI*) *blast*
show ∃ *T*. *closed T* ∧ *U* − *ball a d* = (*U* − {*a*}) ∩ *T*
using ‹*0* < *d*› ‹*affine U*›
by (*rule_tac x*=*U* − *ball a d* **in** *exI*) (*force simp*: *affine_closed*)
show *continuous_on* ((*cball a d* − {*a*}) ∩ *U*) *r*
by (*force intro*: *continuous_on_subset* [*OF contr*])
qed
have *fT*: *x* ∈ *U* − *K* ⟹ *f x* ∈ *T* **for** *x*
using *fim* **by** *blast*
show *?thesis*
proof (*intro conjI exI*)
show *continuous_on* (*S* ∪ (*C* − {*a*})) (*f* ∘ *k* ∘ *j*)
proof (*intro continuous_on_compose*)
have *S* ∪ (*C* − {*a*}) ⊆ *U* − {*a*}
using ‹*C* ⊆ *U* − *S*› ‹*S* ⊆ *U*› ‹*a* ∈ *C*› **by** *force*
then show *continuous_on* (*S* ∪ (*C* − {*a*})) *j*
by (*rule continuous_on_subset* [*OF contj*])
have *j* ‘ (*S* ∪ (*C* − {*a*})) ⊆ *S* ∪ *C*
using *jim* ‹*C* ⊆ *U* − *S*› ‹*S* ⊆ *U*› ‹*ball a d* ∩ *U* ⊆ *C*› *j_def* **by** *blast*
then show *continuous_on* (*j* ‘ (*S* ∪ (*C* − {*a*}))) *k*
by (*rule continuous_on_subset* [*OF homeomorphism_cont2* [*OF homhk*]])
show *continuous_on* (*k* ‘ *j* ‘ (*S* ∪ (*C* − {*a*}))) *f*
proof (*clarify intro*!: *continuous_on_subset* [*OF contf*])
fix *y* **assume** *y* ∈ *S* ∪ (*C* − {*a*})
have *ky*: *k y* ∈ *S* ∪ *C*
using *homeomorphism_image2* [*OF homhk*] ‹*y* ∈ *S* ∪ (*C* − {*a*})› **by** *blast*
have *jy*: *j y* ∈ *S* ∪ *C* − *ball a d*
using *Un_iff* ‹*y* ∈ *S* ∪ (*C* − {*a*})› *jim* **by** *auto*
have *k* (*j y*) ∈ *U*
using ‹*C* ⊆ *U*› ‹*S* ⊆ *U*› *homeomorphism_image2* [*OF homhk*] *jy* **by** *blast*
moreover have *k* (*j y*) ∉ *K*
using *K* **unfolding** *disjnt_iff*
by (*metis DiffE Int_iff Un_iff hin homeomorphism_def homhk image_eqI jy*)
ultimately show *k* (*j y*) ∈ *U* − *K*
by *blast*
qed
qed
have *ST*: ⋀*x*. *x* ∈ *S* ⟹ (*f* ∘ *k* ∘ *j*) *x* ∈ *T*
proof (*simp add*: *kj*)
show ⋀*x*. *x* ∈ *S* ⟹ *f x* ∈ *T*

using *K* **unfolding** *disjnt_iff* **by** (*metis DiffI* ‹*S* ⊆ *U*› *subsetD fim image_subset_iff*)
qed
moreover have (*f* ∘ *k* ∘ *j*) *x* ∈ *T* **if** *x* ∈ *C* *x* ≠ *a* *x* ∉ *S* **for** *x*
proof −
have *rx*: *r x* ∈ *sphere a d*
using ‹*C* ⊆ *U*› *rim that* **by** *fastforce*
have *jj*: *j x* ∈ *S* ∪ *C* − *ball a d*
using *jim that* **by** *blast*
have *k* (*j x*) = *j x* ⟶ *k* (*j x*) ∈ *C* ∨ *j x* ∈ *C*
by (*metis Diff_iff Int_iff Un_iff* ‹*S* ⊆ *U*› *subsetD d j_def jj rx sphere_cball that*(*1*))
then have *kj*: *k* (*j x*) ∈ *C*
using *homeomorphism_apply2* [*OF homhk*, *of j x*] ‹*C* ⊆ *U*› ‹*S* ⊆ *U*› *a rx*
by (*metis* (*mono_tags*, *lifting*) *Diff_iff subsetD jj mem_Collect_eq subC*)
then show *?thesis*
by (*metis DiffE DiffI IntD1 IntI* ‹*C* ⊆ *U*› *comp_apply fT hin homeomorphism_apply2 homhk jj kj subset_eq*)
qed
ultimately show (*f* ∘ *k* ∘ *j*) ‘ (*S* ∪ (*C* − {*a*})) ⊆ *T*
by *force*
show ∀ *x*∈*S*. (*f* ∘ *k* ∘ *j*) *x* = *f x* **using** *kj* **by** *simp*
qed (*auto simp*: *a*)
qed
then obtain *a h* **where**
ah: ⋀*C*. ⟦*C* ∈ *components* (*U* − *S*); *C* ∩ *K* ≠ {}⟧
⟹ *a C* ∈ *C* ∧ *a C* ∈ *L* ∧ *continuous_on* (*S* ∪ (*C* − {*a C*})) (*h C*) ∧
h C ‘ (*S* ∪ (*C* − {*a C*})) ⊆ *T* ∧ (∀ *x* ∈ *S*. *h C x* = *f x*)
using *that* **by** *metis*
define *F* **where** *F* ≡ {*C* ∈ *components* (*U* − *S*). *C* ∩ *K* ≠ {}}
define *G* **where** *G* ≡ {*C* ∈ *components* (*U* − *S*). *C* ∩ *K* = {}}
define *UF* **where** *UF* ≡ (⋃ *C*∈*F*. *C* − {*a C*})
have *C0* ∈ *F*
by (*auto simp*: *F_def C0*)
have *finite F*
proof (*subst finite_image_iff* [*of* λ*C*. *C* ∩ *K F*, *symmetric*])
show *inj_on* (λ*C*. *C* ∩ *K*) *F*
unfolding *F_def inj_on_def*
using *components_nonoverlap* **by** *blast*
show *finite* ((λ*C*. *C* ∩ *K*) ‘ *F*)
unfolding *F_def*
by (*rule finite_subset* [*of* _ *Pow K*]) (*auto simp*: ‹*finite K*›)
qed
obtain *g* **where** *contg*: *continuous_on* (*S* ∪ *UF*) *g*
and *gh*: ⋀*x i*. ⟦*i* ∈ *F*; *x* ∈ (*S* ∪ *UF*) ∩ (*S* ∪ (*i* − {*a i*}))⟧
⟹ *g x* = *h i x*
proof (*rule pasting_lemma_exists_closed* [*OF* ‹*finite F*›])
let *?X* = *top_of_set* (*S* ∪ *UF*)
show *topspace ?X* ⊆ (⋃ *C*∈*F*. *S* ∪ (*C* − {*a C*}))

```
      using ‹C0 ∈ F› by (force simp: UF_def)
    show closedin (top_of_set (S ∪ UF)) (S ∪ (C − {a C}))
         if C ∈ F for C
    proof (rule closedin_closed_subset [of U S ∪ C])
      have C ∈ components (U − S)
        using F_def that by blast
      then show closedin (top_of_set U) (S ∪ C)
       by (rule closedin_Un_complement_component [OF ‹locally connected U› clo])
    next
      have x = a C' if C' ∈ F  x ∈ C' x ∉ U for x C'
      proof −
        have ∀ A. x ∈ ⋃A ∨ C' ∉ A
          using ‹x ∈ C'› by blast
        with that show x = a C'
          by (metis (lifting) DiffD1 F_def Union_components mem_Collect_eq)
      qed
      then show S ∪ UF ⊆ U
        using ‹S ⊆ U› by (force simp: UF_def)
    next
      show S ∪ (C − {a C}) = (S ∪ C) ∩ (S ∪ UF)
        using F_def UF_def components_nonoverlap that by auto
    qed
    show continuous_map (subtopology ?X (S ∪ (C' − {a C'}))) euclidean (h C')
if C' ∈ F for C'
    proof −
      have C': C' ∈ components (U − S) C' ∩ K ≠ {}
        using F_def that by blast+
      show ?thesis
          using ah [OF C'] by (auto simp: F_def subtopology_subtopology intro:
continuous_on_subset)
    qed
    show ⋀i j x. ⟦i ∈ F; j ∈ F;
                  x ∈ topspace ?X ∩ (S ∪ (i − {a i})) ∩ (S ∪ (j − {a j}))⟧
                 ⟹ h i x = h j x
      using components_eq by (fastforce simp: components_eq F_def ah)
  qed auto
  have SU': S ∪ ⋃G ∪ (S ∪ UF) ⊆ U
    using ‹S ⊆ U› in_components_subset by (auto simp: F_def G_def UF_def)
  have clo1: closedin (top_of_set (S ∪ ⋃G ∪ (S ∪ UF))) (S ∪ ⋃G)
  proof (rule closedin_closed_subset [OF _ SU'])
    have ∗: ⋀C. C ∈ F ⟹ openin (top_of_set U) C
      unfolding F_def
       by clarify (metis (no_types, lifting) ‹locally connected U› clo closedin_def lo-
cally_diff_closed openin_components_locally_connected openin_trans topspace_euclidean_subtopology)
    show closedin (top_of_set U) (U − UF)
      unfolding UF_def
      by (force intro: openin_delete ∗)
    show S ∪ ⋃G = (U − UF) ∩ (S ∪ ⋃G ∪ (S ∪ UF))
      using ‹S ⊆ U› apply (auto simp: F_def G_def UF_def)
```

apply (*metis Diff_iff UnionI Union_components*)
apply (*metis DiffD1 UnionI Union_components*)
by (*metis* (*no_types*, *lifting*) *IntI components_nonoverlap empty_iff*)
qed
have *clo2*: *closedin* (*top_of_set* ($S \cup \bigcup G \cup (S \cup UF)$)) ($S \cup UF$)
proof (*rule closedin_closed_subset* [*OF _ SU'*])
show *closedin* (*top_of_set U*) ($\bigcup C \in F.\ S \cup C$)
proof (*rule closedin_Union*)
show $\bigwedge T.\ T \in (\cup)\ S$ ‘ $F \Longrightarrow$ *closedin* (*top_of_set U*) *T*
using *F_def* ‹*locally connected U*› *clo closedin_Un_complement_component* **by** *blast*
qed (*simp add*: ‹*finite F*›)
show $S \cup UF = (\bigcup C \in F.\ S \cup C) \cap (S \cup \bigcup G \cup (S \cup UF))$
using ‹$S \subseteq U$› **apply** (*auto simp*: *F_def G_def UF_def*)
using *C0* **apply** *blast*
by (*metis components_nonoverlap disjoint_iff*)
qed
have *SUG*: $S \cup \bigcup G \subseteq U - K$
using ‹$S \subseteq U$› *K* **apply** (*auto simp*: *G_def disjnt_iff*)
by (*meson Diff_iff subsetD in_components_subset*)
then have *contf'*: *continuous_on* ($S \cup \bigcup G$) *f*
by (*rule continuous_on_subset* [*OF contf*])
have *contg'*: *continuous_on* ($S \cup UF$) *g*
by (*simp add*: *contg*)
have $\bigwedge x.\ [\![S \subseteq U;\ x \in S]\!] \Longrightarrow f\ x = g\ x$
by (*subst gh*) (*auto simp*: *ah C0 intro*: ‹$C0 \in F$›)
then have *f_eq_g*: $\bigwedge x.\ x \in S \cup UF \wedge x \in S \cup \bigcup G \Longrightarrow f\ x = g\ x$
using ‹$S \subseteq U$› **apply** (*auto simp*: *F_def G_def UF_def dest*: *in_components_subset*)
using *components_eq* **by** *blast*
have *cont*: *continuous_on* ($S \cup \bigcup G \cup (S \cup UF)$) ($\lambda x.$ *if* $x \in S \cup \bigcup G$ *then* $f\ x$ *else* $g\ x$)
by (*blast intro*: *continuous_on_cases_local* [*OF clo1 clo2 contf' contg' f_eq_g*, *of* $\lambda x.\ x \in S \cup \bigcup G$])
show *?thesis*
proof
have *UF*: $\bigcup F - L \subseteq UF$
unfolding *F_def UF_def* **using** *ah* **by** *blast*
have $U - S - L = \bigcup$(*components* ($U - S$)) $- L$
by *simp*
also have ... $= \bigcup F \cup \bigcup G - L$
unfolding *F_def G_def* **by** *blast*
also have ... $\subseteq UF \cup \bigcup G$
using *UF* **by** *blast*
finally have $U - L \subseteq S \cup \bigcup G \cup (S \cup UF)$
by *blast*
then show *continuous_on* ($U - L$) ($\lambda x.$ *if* $x \in S \cup \bigcup G$ *then* $f\ x$ *else* $g\ x$)
by (*rule continuous_on_subset* [*OF cont*])
have $((U - L) \cap \{x.\ x \notin S \wedge (\forall xa \in G.\ x \notin xa)\}) \subseteq ((U - L) \cap (-S \cap UF))$
using ‹$U - L \subseteq S \cup \bigcup G \cup (S \cup UF)$› **by** *auto*

moreover have g ‘ $((U - L) \cap (-S \cap UF)) \subseteq T$
proof −
have $g\ x \in T$ **if** $x \in U\ x \notin L\ x \notin S\ C \in F\ x \in C\ x \neq a\ C$ **for** $x\ C$
proof (*subst gh*)
show $x \in (S \cup UF) \cap (S \cup (C - \{a\ C\}))$
using *that* **by** (*auto simp*: *UF_def*)
show $h\ C\ x \in T$
using *ah that* **by** (*fastforce simp add*: *F_def*)
qed (*rule that*)
then show *?thesis*
by (*force simp*: *UF_def*)
qed
ultimately have g ‘ $((U - L) \cap \{x.\ x \notin S \wedge (\forall xa \in G.\ x \notin xa)\}) \subseteq T$
using *image_mono order_trans* **by** *blast*
moreover have f ‘ $((U - L) \cap (S \cup \bigcup G)) \subseteq T$
using *fim SUG* **by** *blast*
ultimately show $(\lambda x.\ \mathit{if}\ x \in S \cup \bigcup G\ \mathit{then}\ f\ x\ \mathit{else}\ g\ x)$ ‘ $(U - L) \subseteq T$
by *force*
show $\bigwedge x.\ x \in S \Longrightarrow (\mathit{if}\ x \in S \cup \bigcup G\ \mathit{then}\ f\ x\ \mathit{else}\ g\ x) = f\ x$
by (*simp add*: *F_def G_def*)
qed
qed

lemma *extend_map_affine_to_sphere2*:
fixes f :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *compact S convex U bounded U affine T* $S \subseteq T$
and *affTU*: *aff_dim T* ≤ *aff_dim U*
and *contf*: *continuous_on S f*
and *fim*: f ‘ $S \subseteq$ *rel_frontier U*
and *ovlap*: $\bigwedge C.\ C \in \mathit{components}(T - S) \Longrightarrow C \cap L \neq \{\}$
obtains $K\ g$ **where** *finite K* $K \subseteq L$ $K \subseteq T$ *disjnt K S*
continuous_on $(T - K)$ g g ‘ $(T - K) \subseteq$ *rel_frontier U*
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
proof −
obtain $K\ g$ **where** K: *finite K* $K \subseteq T$ *disjnt K S*
and *contg*: *continuous_on* $(T - K)$ g
and *gim*: g ‘ $(T - K) \subseteq$ *rel_frontier U*
and *gf*: $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
using *assms extend_map_affine_to_sphere_cofinite_simple* **by** *metis*
have $(\exists y\ C.\ C \in \mathit{components}\ (T - S) \wedge x \in C \wedge y \in C \wedge y \in L)$ **if** $x \in K$ **for** x
proof −
have $x \in T-S$
using ‹$K \subseteq T$› ‹*disjnt K S*› *disjnt_def that* **by** *fastforce*
then obtain C **where** $C \in \mathit{components}(T - S)$ $x \in C$
by (*metis UnionE Union_components*)
with *ovlap* [*of C*] **show** *?thesis*
by *blast*

qed
then obtain ξ **where** ξ: $\bigwedge x.\ x \in K \Longrightarrow \exists\, C.\ C \in$ *components* $(T - S) \wedge x \in C \wedge \xi\ x \in C \wedge \xi\ x \in L$
by *metis*
obtain h **where** *conth*: *continuous_on* $(T - \xi$ ‘ $K)$ h
and *him*: h ‘ $(T - \xi$ ‘ $K) \subseteq$ *rel_frontier* U
and *hg*: $\bigwedge x.\ x \in S \Longrightarrow h\ x = g\ x$
proof (*rule extend_map_affine_to_sphere1* [*OF* ‹*finite K*› ‹*affine T*› *contg gim*, *of* S ξ ‘ K])
show *cloTS*: *closedin* (*top_of_set* T) S
by (*simp add*: ‹*compact S*› ‹$S \subseteq T$› *closed_subset compact_imp_closed*)
show $\bigwedge C.\ [\![C \in$ *components* $(T - S);\ C \cap K \neq \{\}]\!] \Longrightarrow C \cap \xi$ ‘ $K \neq \{\}$
using ξ *components_eq* **by** *blast*
qed (*use* K **in** *auto*)
show *?thesis*
proof
show $*$: ξ ‘ $K \subseteq L$
using ξ **by** *blast*
show *finite* (ξ ‘ K)
by (*simp add*: K)
show ξ ‘ $K \subseteq T$
by *clarify* (*meson* ξ *Diff_iff contra_subsetD in_components_subset*)
show *continuous_on* $(T - \xi$ ‘ $K)$ h
by (*rule conth*)
show *disjnt* (ξ ‘ K) S
using K ξ *in_components_subset* **by** (*fastforce simp*: *disjnt_def*)
qed (*simp_all add*: *him hg gf*)
qed

proposition *extend_map_affine_to_sphere_cofinite_gen*:
fixes f :: *'a::euclidean_space* $\Rightarrow$ *'b::euclidean_space*
assumes *SUT*: *compact S convex U bounded U affine T* $S \subseteq T$
and *aff*: *aff_dim* $T \leq$ *aff_dim* U
and *contf*: *continuous_on* S f
and *fim*: f ‘ $S \subseteq$ *rel_frontier* U
and *dis*: $\bigwedge C.\ [\![C \in$ *components*$(T - S)$; *bounded* $C]\!] \Longrightarrow C \cap L \neq \{\}$
obtains K g **where** *finite* K $K \subseteq L$ $K \subseteq T$ *disjnt* K S *continuous_on* $(T - K)$ g
g ‘ $(T - K) \subseteq$ *rel_frontier* U
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
proof (*cases* $S = \{\}$)
case *True*
show *?thesis*
proof (*cases rel_frontier* $U = \{\}$)
case *True*
with *aff* **have** *aff_dim* $T \leq 0$
using *affine_bounded_eq_lowdim* ‹*bounded U*› *order_trans*
by (*auto simp add*: *rel_frontier_eq_empty*)
with *aff_dim_geq* [*of T*] **consider** *aff_dim* $T = -1$ | *aff_dim* $T = 0$

```
      by linarith
    then show ?thesis
    proof cases
      assume aff_dim T = -1
      then have T = {}
        by (simp add: aff_dim_empty)
      then show ?thesis
        by (rule_tac K={} in that) auto
    next
      assume aff_dim T = 0
      then obtain a where T = {a}
        using aff_dim_eq_0 by blast
      then have a ∈ L
        using dis [of {a}] ‹S = {}› by (auto simp: in_components_self)
      with ‹S = {}› ‹T = {a}› show ?thesis
        by (rule_tac K={a} and g=f in that) auto
    qed
  next
    case False
    then obtain y where y ∈ rel_frontier U
      by auto
    with ‹S = {}› show ?thesis
      by (rule_tac K={} and g=λx. y in that)  (auto)
  qed
next
  case False
  have bounded S
    by (simp add: assms compact_imp_bounded)
  then obtain b where b: S ⊆ cbox (-b) b
    using bounded_subset_cbox_symmetric by blast
  define LU where LU ≡ L ∪ (⋃ {C ∈ components (T - S). ¬bounded C} -
cbox (-(b+One)) (b+One))
  obtain K g where finite K K ⊆ LU K ⊆ T disjnt K S
             and contg: continuous_on (T - K) g
             and gim: g ` (T - K) ⊆ rel_frontier U
             and gf: ⋀x. x ∈ S ⟹ g x = f x
  proof (rule extend_map_affine_to_sphere2 [OF SUT aff contf fim])
    show C ∩ LU ≠ {} if C ∈ components (T - S) for C
    proof (cases bounded C)
      case True
      with dis that show ?thesis
        unfolding LU_def by fastforce
    next
      case False
      then have ¬ bounded (⋃{C ∈ components (T - S). ¬ bounded C})
       by (metis (no_types, lifting) Sup_upper bounded_subset mem_Collect_eq that)
      then show ?thesis
       apply (clarsimp simp: LU_def Int_Un_distrib Diff_Int_distrib Int_UN_distrib)
        by (metis (no_types, lifting) False Sup_upper bounded_cbox bounded_subset
```

inf.orderE mem_Collect_eq that)
qed
qed *blast*
have ∗: *False* **if** $x \in$ *cbox* (− *b* − *m* $*_R$ *One*) (*b* + *m* $*_R$ *One*)
$x \notin$ *box* (− *b* − *n* $*_R$ *One*) (*b* + *n* $*_R$ *One*)
$0 \le m$ $m < n$ $n \le 1$ **for** *m n x*
using *that* **by** (*auto simp*: *mem_box algebra_simps*)
have *disjoint_family_on* (λ*d*. *frontier* (*cbox* (− *b* − *d* $*_R$ *One*) (*b* + *d* $*_R$ *One*))) {*1* / *2*..*1*}
by (*auto simp*: *disjoint_family_on_def neq_iff frontier_def dest*: ∗)
then obtain *d* **where** *d12*: *1*/*2* ≤ *d* *d* ≤ *1*
and *ddis*: *disjnt K* (*frontier* (*cbox* (−(*b* + *d* $*_R$ *One*)) (*b* + *d* $*_R$ *One*)))
using *disjoint_family_elem_disjnt* [*of* {*1*/*2*..*1*::*real*} *K* λ*d*. *frontier* (*cbox* (−(*b* + *d* $*_R$ *One*)) (*b* + *d* $*_R$ *One*))]
by (*auto simp*: ‹*finite K*›)
define *c* **where** *c* ≡ *b* + *d* $*_R$ *One*
have *cbsub*: *cbox* (−*b*) *b* ⊆ *box* (−*c*) *c*
cbox (−*b*) *b* ⊆ *cbox* (−*c*) *c*
cbox (−*c*) *c* ⊆ *cbox* (−(*b*+*One*)) (*b*+*One*)
using *d12* **by** (*simp_all add*: *subset_box c_def inner_diff_left inner_left_distrib*)
have *clo_cT*: *closed* (*cbox* (− *c*) *c* ∩ *T*)
using *affine_closed* ‹*affine T*› **by** *blast*
have *cT_ne*: *cbox* (− *c*) *c* ∩ *T* ≠ {}
using ‹*S* ≠ {}› ‹*S* ⊆ *T*› *b cbsub* **by** *fastforce*
have *S_sub_cc*: *S* ⊆ *cbox* (− *c*) *c*
using ‹*cbox* (− *b*) *b* ⊆ *cbox* (− *c*) *c*› *b* **by** *auto*
show *?thesis*
proof
show *finite* (*K* ∩ *cbox* (−(*b*+*One*)) (*b*+*One*))
using ‹*finite K*› **by** *blast*
show *K* ∩ *cbox* (− (*b* + *One*)) (*b* + *One*) ⊆ *L*
using ‹*K* ⊆ *LU*› **by** (*auto simp*: *LU_def*)
show *K* ∩ *cbox* (− (*b* + *One*)) (*b* + *One*) ⊆ *T*
using ‹*K* ⊆ *T*› **by** *auto*
show *disjnt* (*K* ∩ *cbox* (− (*b* + *One*)) (*b* + *One*)) *S*
using ‹*disjnt K S*› **by** (*simp add*: *disjnt_def disjoint_eq_subset_Compl inf.coboundedI1*)
have *cloTK*: *closest_point* (*cbox* (− *c*) *c* ∩ *T*) *x* ∈ *T* − *K*
if *x* ∈ *T* **and** *Knot*: *x* ∈ *K* ⟶ *x* ∉ *cbox* (− *b* − *One*) (*b* + *One*)
for *x*
proof (*cases x* ∈ *cbox* (− *c*) *c*)
case *True*
with ‹*x* ∈ *T*› **show** *?thesis*
using *cbsub*(*3*) *Knot* **by** (*force simp*: *closest_point_self*)
next
case *False*
have *clo_in_rf*: *closest_point* (*cbox* (− *c*) *c* ∩ *T*) *x* ∈ *rel_frontier* (*cbox* (− *c*) *c* ∩ *T*)
proof (*intro closest_point_in_rel_frontier* [*OF clo_cT cT_ne*] *DiffI notI*)

have *T* ∩ *interior* (*cbox* (− *c*) *c*) ≠ {}
using ‹*S* ≠ {}› ‹*S* ⊆ *T*› *b cbsub*(*1*) **by** *fastforce*
then show *x* ∈ *affine hull* (*cbox* (− *c*) *c* ∩ *T*)
by (*simp add*: *Int_commute affine_hull_affine_Int_nonempty_interior* ‹*affine T*› *hull_inc that*(*1*))
next
show *False* **if** *x* ∈ *rel_interior* (*cbox* (− *c*) *c* ∩ *T*)
proof −
have *interior* (*cbox* (− *c*) *c*) ∩ *T* ≠ {}
using ‹*S* ≠ {}› ‹*S* ⊆ *T*› *b cbsub*(*1*) **by** *fastforce*
then have *affine hull* (*T* ∩ *cbox* (− *c*) *c*) = *T*
using *affine_hull_convex_Int_nonempty_interior* [*of T cbox* (− *c*) *c*]
by (*simp add*: *affine_imp_convex* ‹*affine T*› *inf_commute*)
then show *?thesis*
by (*meson subsetD le_inf_iff rel_interior_subset that False*)
qed
qed
have *closest_point* (*cbox* (− *c*) *c* ∩ *T*) *x* ∉ *K*
proof
assume *inK*: *closest_point* (*cbox* (− *c*) *c* ∩ *T*) *x* ∈ *K*
have $\bigwedge$*x*. *x* ∈ *K* ⟹ *x* ∉ *frontier* (*cbox* (− (*b* + *d* $*_R$ *One*)) (*b* + *d* $*_R$ *One*))
by (*metis ddis disjnt_iff*)
then show *False*
by (*metis DiffI Int_iff* ‹*affine T*› *cT_ne c_def clo_cT clo_in_rf closest_point_in_set*
convex_affine_rel_frontier_Int convex_box(*1*) *empty_iff frontier_cbox inK interior_cbox*)
qed
then show *?thesis*
using *cT_ne clo_cT closest_point_in_set* **by** *blast*
qed
show *continuous_on* (*T* − *K* ∩ *cbox* (− (*b* + *One*)) (*b* + *One*)) (*g* ∘ *closest_point* (*cbox* (−*c*) *c* ∩ *T*))
using *cloTK*
apply (*intro continuous_on_compose continuous_on_closest_point continuous_on_subset* [*OF contg*])
by (*auto simp add*: *clo_cT affine_imp_convex* ‹*affine T*› *convex_Int cT_ne*)
have *g* (*closest_point* (*cbox* (− *c*) *c* ∩ *T*) *x*) ∈ *rel_frontier U*
if *x* ∈ *T x* ∈ *K* ⟶ *x* ∉ *cbox* (− *b* − *One*) (*b* + *One*) **for** *x*
using *gim* [*THEN subsetD*] *that cloTK* **by** *blast*
then show (*g* ∘ *closest_point* (*cbox* (− *c*) *c* ∩ *T*)) ‘ (*T* − *K* ∩ *cbox* (− (*b* + *One*)) (*b* + *One*))
⊆ *rel_frontier U*
by *force*
show $\bigwedge$*x*. *x* ∈ *S* ⟹ (*g* ∘ *closest_point* (*cbox* (− *c*) *c* ∩ *T*)) *x* = *f x*
by *simp* (*metis* (*mono_tags*, *lifting*) *IntI* ‹*S* ⊆ *T*› *cT_ne clo_cT closest_point_refl gf subsetD S_sub_cc*)
qed

qed

corollary *extend_map_affine_to_sphere_cofinite*:
  fixes f :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
  assumes *SUT*: *compact S affine T S* ⊆ *T*
    and *aff*: *aff_dim T* ≤ *DIM*($'b$) **and** *0* ≤ *r*
    and *contf*: *continuous_on S f*
    and *fim*: *f* ‘ *S* ⊆ *sphere a r*
    and *dis*: ⋀*C*. ⟦*C* ∈ *components*(*T* − *S*); *bounded C*⟧ ⟹ *C* ∩ *L* ≠ {}
  obtains *K g* **where** *finite K K* ⊆ *L K* ⊆ *T disjnt K S continuous_on* (*T* − *K*)
g
    g ‘ (*T* − *K*) ⊆ *sphere a r* ⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x*
proof (*cases r* = *0*)
  case *True*
  with *fim* **show** *?thesis*
    by (*rule_tac K*={} **and** *g* = λ*x*. *a* **in** *that*) (*auto*)
next
  case *False*
  with *assms* **have** *0* < *r* **by** *auto*
  then have *aff_dim T* ≤ *aff_dim* (*cball a r*)
    by (*simp add*: *aff aff_dim_cball*)
  then show *?thesis*
    apply (*rule extend_map_affine_to_sphere_cofinite_gen*
      [*OF* ‹*compact S*› *convex_cball bounded_cball* ‹*affine T*› ‹*S* ⊆ *T*› _ *contf*])
    using *fim* **apply** (*auto simp*: *assms False that dest*: *dis*)
    done
qed

corollary *extend_map_UNIV_to_sphere_cofinite*:
  fixes f :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
  assumes *DIM*($'a$) ≤ *DIM*($'b$) **and** *0* ≤ *r*
    and *compact S*
    and *continuous_on S f*
    and *f* ‘ *S* ⊆ *sphere a r*
    and ⋀*C*. ⟦*C* ∈ *components*(− *S*); *bounded C*⟧ ⟹ *C* ∩ *L* ≠ {}
  obtains *K g* **where** *finite K K* ⊆ *L disjnt K S continuous_on* (− *K*) *g*
    g ‘ (− *K*) ⊆ *sphere a r* ⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x*
  using *extend_map_affine_to_sphere_cofinite*
    [*OF* ‹*compact S*› *affine_UNIV subset_UNIV*] *assms*
  by (*metis Compl_eq_Diff_UNIV aff_dim_UNIV of_nat_le_iff*)

corollary *extend_map_UNIV_to_sphere_no_bounded_component*:
  fixes f :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
  assumes *aff*: *DIM*($'a$) ≤ *DIM*($'b$) **and** *0* ≤ *r*
    and *SUT*: *compact S*
    and *contf*: *continuous_on S f*
    and *fim*: *f* ‘ *S* ⊆ *sphere a r*
    and *dis*: ⋀*C*. *C* ∈ *components*(− *S*) ⟹ ¬ *bounded C*

```
  obtains g where continuous_on UNIV g g ‘ UNIV ⊆ sphere a r ⋀x. x ∈ S ⟹
g x = f x
   apply (rule extend_map_UNIV_to_sphere_cofinite [OF aff ‹0 ≤ r› ‹compact S›
contf fim, of {}])
   apply (auto dest: dis)
done

theorem Borsuk_separation_theorem_gen:
  fixes S :: 'a::euclidean_space set
  assumes compact S
    shows (∀ c ∈ components(− S). ¬bounded c) ⟷
          (∀ f. continuous_on S f ∧ f ‘ S ⊆ sphere (0::'a) 1
              ⟶ (∃ c. homotopic_with_canon (λx. True) S (sphere 0 1) f (λx. c)))
       (is ?lhs = ?rhs)
proof
  assume L [rule_format]: ?lhs
  show ?rhs
  proof clarify
    fix f :: 'a ⇒ 'a
    assume contf: continuous_on S f and fim: f ‘ S ⊆ sphere 0 1
    obtain g where contg: continuous_on UNIV g and gim: range g ⊆ sphere 0 1
                and gf: ⋀x. x ∈ S ⟹ g x = f x
      by (rule extend_map_UNIV_to_sphere_no_bounded_component [OF _ _ ‹compact
S› contf fim L]) auto
    then obtain c where c: homotopic_with_canon (λh. True) UNIV (sphere 0 1)
g (λx. c)
      using contractible_UNIV nullhomotopic_from_contractible by blast
    then show ∃ c. homotopic_with_canon (λx. True) S (sphere 0 1) f (λx. c)
      by (metis assms compact_imp_closed contf contg contractible_empty fim gf gim
nullhomotopic_from_contractible nullhomotopic_into_sphere_extension)
  qed
next
  assume R [rule_format]: ?rhs
  show ?lhs
    unfolding components_def
  proof clarify
    fix a
    assume a ∉ S and a: bounded (connected_component_set (− S) a)
    have ∀ x∈S. norm (x − a) ≠ 0
      using ‹a ∉ S› by auto
    then have cont: continuous_on S (λx. inverse(norm(x − a)) *R (x − a))
      by (intro continuous_intros)
    have im: (λx. inverse(norm(x − a)) *R (x − a)) ‘ S ⊆ sphere 0 1
      by clarsimp (metis ‹a ∉ S› eq_iff_diff_eq_0 left_inverse norm_eq_zero)
    show False
      using R cont im Borsuk_map_essential_bounded_component [OF ‹compact S›
‹a ∉ S›] a by blast
  qed
qed
```

corollary *Borsuk_separation_theorem*:
fixes *S* :: *'a::euclidean_space set*
assumes *compact S* **and** *2*: *2* ≤ *DIM('a)*
shows *connected(− S)* ⟷
(∀ *f*. *continuous_on S f* ∧ *f* ‘ *S* ⊆ *sphere* (*0::'a*) *1*
⟶ (∃ *c*. *homotopic_with_canon* (λ*x*. *True*) *S* (*sphere 0 1*) *f* (λ*x*. *c*)))
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
show *?rhs*
proof (*cases S* = {})
case *True*
then show *?thesis* **by** *auto*
next
case *False*
then have (∀ *c*∈*components* (− *S*). ¬ *bounded c*)
by (*metis L assms(1) bounded_empty cobounded_imp_unbounded compact_imp_bounded in_components_maximal order_refl*)
then show *?thesis*
by (*simp add*: *Borsuk_separation_theorem_gen* [*OF* ‹*compact S*›])
qed
next
assume *R*: *?rhs*
then show *?lhs*
apply (*simp add*: *Borsuk_separation_theorem_gen* [*OF* ‹*compact S*›, *symmetric*])
apply (*auto simp*: *components_def connected_iff_eq_connected_component_set*)
using *connected_component_in* **apply** *fastforce*
using *cobounded_unique_unbounded_component* [*OF* _ *2*, *of* −*S*] ‹*compact S*› *compact_eq_bounded_closed* **by** *fastforce*
qed

lemma *homotopy_eqv_separation*:
fixes *S* :: *'a::euclidean_space set* **and** *T* :: *'a set*
assumes *S homotopy_eqv T* **and** *compact S* **and** *compact T*
shows *connected(− S)* ⟷ *connected(− T)*
proof −
consider *DIM('a)* = *1* | *2* ≤ *DIM('a)*
by (*metis DIM_ge_Suc0 One_nat_def Suc_1 dual_order.antisym not_less_eq_eq*)
then show *?thesis*
proof *cases*
case *1*
then show *?thesis*
using *bounded_connected_Compl_1 compact_imp_bounded homotopy_eqv_empty1 homotopy_eqv_empty2 assms* **by** *metis*
next
case *2*

```
    with assms show ?thesis
     by (simp add: Borsuk_separation_theorem homotopy_eqv_cohomotopic_triviality_null)
  qed
qed

proposition Jordan_Brouwer_separation:
  fixes S :: 'a::euclidean_space set and a::'a
  assumes hom: S homeomorphic sphere a r and 0 < r
    shows ¬ connected(− S)
proof −
  have − sphere a r ∩ ball a r ≠ {}
    using ‹0 < r› by (simp add: Int_absorb1 subset_eq)
  moreover
  have eq: − sphere a r − ball a r = − cball a r
    by auto
  have − cball a r ≠ {}
  proof −
    have frontier (cball a r) ≠ {}
      using ‹0 < r› by auto
    then show ?thesis
      by (metis frontier_complement frontier_empty)
  qed
  with eq have − sphere a r − ball a r ≠ {}
    by auto
  moreover
  have connected (− S) = connected (− sphere a r)
  proof (rule homotopy_eqv_separation)
    show S homotopy_eqv sphere a r
      using hom homeomorphic_imp_homotopy_eqv by blast
    show compact (sphere a r)
      by simp
    then show  compact S
      using hom homeomorphic_compactness by blast
  qed
  ultimately show ?thesis
    using connected_Int_frontier [of − sphere a r ball a r] by (auto simp: ‹0 < r›)
qed


proposition Jordan_Brouwer_frontier:
  fixes S :: 'a::euclidean_space set and a::'a
  assumes S: S homeomorphic sphere a r and T: T ∈ components(− S) and 2:
2 ≤ DIM('a)
    shows frontier T = S
proof (cases r rule: linorder_cases)
  assume r < 0
  with S T show ?thesis by auto
next
  assume r = 0
```

with *S T card_eq_SucD* **obtain** *b* **where** $S = \{b\}$
by (*auto simp*: *homeomorphic_finite* [*of* {*a*} *S*])
have *components* $(- \{b\}) = \{ -\{b\}\}$
using *T* ‹$S = \{b\}$› **by** (*auto simp*: *components_eq_sing_iff connected_punctured_universe 2*)
with *T* **show** *?thesis*
by (*metis* ‹$S = \{b\}$› *cball_trivial frontier_cball frontier_complement singletonD sphere_trivial*)
next
assume $r > 0$
have *compact S*
using *homeomorphic_compactness compact_sphere S* **by** *blast*
show *?thesis*
proof (*rule frontier_minimal_separating_closed*)
show *closed S*
using ‹*compact S*› *compact_eq_bounded_closed* **by** *blast*
show ¬ *connected* (− *S*)
using *Jordan_Brouwer_separation S* ‹$0 < r$› **by** *blast*
obtain *f g* **where** *hom*: *homeomorphism S* (*sphere a r*) *f g*
using *S* **by** (*auto simp*: *homeomorphic_def*)
show *connected* (− *T*) **if** *closed T* $T \subset S$ **for** *T*
proof −
have *f* ‘ $T \subseteq$ *sphere a r*
using ‹$T \subset S$› *hom homeomorphism_image1* **by** *blast*
moreover have *f* ‘ $T \neq$ *sphere a r*
using ‹$T \subset S$› *hom*
by (*metis homeomorphism_image2 homeomorphism_of_subsets order_refl psubsetE*)
ultimately have *f* ‘ $T \subset$ *sphere a r* **by** *blast*
then have *connected* (− *f* ‘ *T*)
by (*rule psubset_sphere_Compl_connected* [*OF* _ ‹$0 < r$› *2*])
moreover have *compact T*
using ‹*compact S*› *bounded_subset compact_eq_bounded_closed that* **by** *blast*
moreover then have *compact* (*f* ‘ *T*)
by (*meson compact_continuous_image continuous_on_subset hom homeomorphism_def psubsetE* ‹$T \subset S$›)
moreover have *T homotopy_eqv f* ‘ *T*
by (*meson* ‹*f* ‘ $T \subseteq$ *sphere a r*› *dual_order.strict_implies_order hom homeomorphic_def homeomorphic_imp_homotopy_eqv homeomorphism_of_subsets* ‹$T \subset S$›)
ultimately show *?thesis*
using *homotopy_eqv_separation* [*of T f*‘*T*] **by** *blast*
qed
qed (*rule T*)
qed

proposition *Jordan_Brouwer_nonseparation*:
fixes *S* :: ′*a*::*euclidean_space set* **and** *a*::′*a*
assumes *S*: *S homeomorphic sphere a r* **and** $T \subset S$ **and** *2*: $2 \leq DIM('a)$
shows *connected*(− *T*)

proof −
have ∗: *connected*$(C \cup (S - T))$ **if** $C \in$ *components*$(- S)$ **for** C
proof (*rule connected_intermediate_closure*)
show *connected* C
using *in_components_connected that* **by** *auto*
have $S =$ *frontier* C
using *2 Jordan_Brouwer_frontier S that* **by** *blast*
with *closure_subset* **show** $C \cup (S - T) \subseteq$ *closure* C
by (*auto simp*: *frontier_def*)
qed *auto*
have *components*$(- S) \neq \{\}$
by (*metis S bounded_empty cobounded_imp_unbounded compact_eq_bounded_closed compact_sphere*
components_eq_empty homeomorphic_compactness)
then have $- T = (\bigcup C \in$ *components*$(- S). C \cup (S - T))$
using *Union_components* [*of* $-S$] ‹$T \subset S$› **by** *auto*
moreover have *connected* ...
using ‹$T \subset S$› **by** (*intro connected_Union*) (*auto simp*: ∗)
ultimately show *?thesis*
by *simp*
qed

6.41.5 Invariance of domain and corollaries

lemma *invariance_of_domain_ball*:
fixes f :: $'a \Rightarrow 'a$::*euclidean_space*
assumes *contf*: *continuous_on* (*cball a r*) f **and** $0 < r$
and *inj*: *inj_on* f (*cball a r*)
shows *open*(f ‘ *ball a r*)
proof (*cases DIM*$('a) = 1$)
case *True*
obtain h::$'a \Rightarrow$*real* **and** k
where *linear h linear k h* ‘ *UNIV* = *UNIV k* ‘ *UNIV* = *UNIV*
$\bigwedge x.$ *norm*$(h\ x) =$ *norm* x $\bigwedge x.$ *norm*$(k\ x) =$ *norm* x
and *kh*: $\bigwedge x.\ k(h\ x) = x$ **and** $\bigwedge x.\ h(k\ x) = x$
proof (*rule isomorphisms_UNIV_UNIV*)
show *DIM*$('a) =$ *DIM*(*real*)
using *True* **by** *force*
qed (*metis UNIV_I UNIV_eq_I imageI*)
have *cont*: *continuous_on S h* *continuous_on T k* **for** S T
by (*simp_all add*: ‹*linear h*› ‹*linear k*› *linear_continuous_on linear_linear*)
have *continuous_on* (h ‘ *cball a r*) $(h \circ f \circ k)$
by (*intro continuous_on_compose cont continuous_on_subset* [*OF contf*]) (*auto simp*: *kh*)
moreover have *is_interval* (h ‘ *cball a r*)
by (*simp add*: *is_interval_connected_1* ‹*linear h*› *linear_continuous_on linear_linear connected_continuous_image*)
moreover have *inj_on* $(h \circ f \circ k)$ (h ‘ *cball a r*)
using *inj* **by** (*simp add*: *inj_on_def*) (*metis* ‹$\bigwedge x.\ k\ (h\ x) = x$›)

ultimately have ∗: ⋀*T*. ⟦*open T*; *T* ⊆ *h* ‘ *cball a r*⟧ ⟹ *open* ((*h* ∘ *f* ∘ *k*) ‘ *T*)
using *injective_eq_1d_open_map_UNIV* **by** *blast*
have *open* ((*h* ∘ *f* ∘ *k*) ‘ (*h* ‘ *ball a r*))
by (*rule* ∗) (*auto simp*: ‹*linear h*› ‹*range h* = *UNIV*› *open_surjective_linear_image*)
then have *open* ((*h* ∘ *f*) ‘ *ball a r*)
by (*simp add*: *image_comp* ‹⋀*x*. *k* (*h x*) = *x*› *cong*: *image_cong*)
then show *?thesis*
unfolding *image_comp* [*symmetric*]
by (*metis open_bijective_linear_image_eq* ‹*linear h*› *kh* ‹*range h* = *UNIV*› *bijI inj_on_def*)
next
case *False*
then have *2*: *DIM*(′*a*) ≥ *2*
by (*metis DIM_ge_Suc0 One_nat_def Suc_1 antisym not_less_eq_eq*)
have *fimsub*: *f* ‘ *ball a r* ⊆ − *f* ‘ *sphere a r*
using *inj* **by** *clarsimp* (*metis inj_onD less_eq_real_def mem_cball order_less_irrefl*)
have *hom*: *f* ‘ *sphere a r homeomorphic sphere a r*
by (*meson compact_sphere contf continuous_on_subset homeomorphic_compact homeomorphic_sym inj inj_on_subset sphere_cball*)
then have *nconn*: ¬ *connected* (− *f* ‘ *sphere a r*)
by (*rule Jordan_Brouwer_separation*) (*auto simp*: ‹*0* < *r*›)
have *bounded* (*f* ‘ *sphere a r*)
by (*meson compact_imp_bounded compact_continuous_image_eq compact_sphere contf inj sphere_cball*)
then obtain *C* **where** *C*: *C* ∈ *components* (− *f* ‘ *sphere a r*) **and** *bounded C*
using *cobounded_has_bounded_component* [*OF* _ *nconn*] *2* **by** *auto*
moreover have *f* ‘ (*ball a r*) = *C*
proof
have *C* ≠ {}
by (*rule in_components_nonempty* [*OF C*])
show *C* ⊆ *f* ‘ *ball a r*
proof (*rule ccontr*)
assume *nonsub*: ¬ *C* ⊆ *f* ‘ *ball a r*
have − *f* ‘ *cball a r* ⊆ *C*
proof (*rule components_maximal* [*OF C*])
have *f* ‘ *cball a r homeomorphic cball a r*
using *compact_cball contf homeomorphic_compact homeomorphic_sym inj* **by** *blast*
then show *connected* (− *f* ‘ *cball a r*)
by (*auto intro*: *connected_complement_homeomorphic_convex_compact 2*)
show − *f* ‘ *cball a r* ⊆ − *f* ‘ *sphere a r*
by *auto*
then show *C* ∩ − *f* ‘ *cball a r* ≠ {}
using ‹*C* ≠ {}› *in_components_subset* [*OF C*] *nonsub*
using *image_iff* **by** *fastforce*
qed
then have *bounded* (− *f* ‘ *cball a r*)
using *bounded_subset* ‹*bounded C*› **by** *auto*

then have ¬ *bounded* (*f* ‘ *cball a r*)
using *cobounded_imp_unbounded* **by** *blast*
then show *False*
using *compact_continuous_image* [*OF contf*] *compact_cball compact_imp_bounded*
by *blast*
qed
with ‹*C* ≠ {}› **have** *C* ∩ *f* ‘ *ball a r* ≠ {}
by (*simp add*: *inf.absorb_iff1*)
then show *f* ‘ *ball a r* ⊆ *C*
by (*metis components_maximal* [*OF C* _ *fimsub*] *connected_continuous_image*
ball_subset_cball connected_ball contf continuous_on_subset)
qed
moreover have *open* (− *f* ‘ *sphere a r*)
using *hom compact_eq_bounded_closed compact_sphere homeomorphic_compactness*
by *blast*
ultimately show *?thesis*
using *open_components* **by** *blast*
qed

Proved by L. E. J. Brouwer (1912)

theorem *invariance_of_domain*:
fixes *f* :: *'a* ⇒ *'a*::*euclidean_space*
assumes *continuous_on S f open S inj_on f S*
shows *open*(*f* ‘ *S*)
unfolding *open_subopen* [*of f*‘*S*]
proof *clarify*
fix *a*
assume *a* ∈ *S*
obtain δ **where** δ > *0* **and** δ: *cball a* δ ⊆ *S*
using ‹*open S*› ‹*a* ∈ *S*› *open_contains_cball_eq* **by** *blast*
show ∃ *T*. *open T* ∧ *f a* ∈ *T* ∧ *T* ⊆ *f* ‘ *S*
proof (*intro exI conjI*)
show *open* (*f* ‘ (*ball a* δ))
by (*meson* δ ‹*0* < δ› *assms continuous_on_subset inj_on_subset invariance_of_domain_ball*)
show *f a* ∈ *f* ‘ *ball a* δ
by (*simp add*: ‹*0* < δ›)
show *f* ‘ *ball a* δ ⊆ *f* ‘ *S*
using δ *ball_subset_cball* **by** *blast*
qed
qed

lemma *inv_of_domain_ss0*:
fixes *f* :: *'a* ⇒ *'a*::*euclidean_space*
assumes *contf*: *continuous_on U f* **and** *injf*: *inj_on f U* **and** *fim*: *f* ‘ *U* ⊆ *S*
and *subspace S* **and** *dimS*: *dim S* = *DIM*(*'b*::*euclidean_space*)
and *ope*: *openin* (*top_of_set S*) *U*
shows *openin* (*top_of_set S*) (*f* ‘ *U*)
proof −
have *U* ⊆ *S*

using *ope openin_imp_subset* **by** *blast*
have (*UNIV*::′*b set*) *homeomorphic S*
by (*simp add*: ‹*subspace S*› *dimS homeomorphic_subspaces*)
then obtain *h k* **where** *homhk*: *homeomorphism* (*UNIV*::′*b set*) *S h k*
using *homeomorphic_def* **by** *blast*
have *homkh*: *homeomorphism S* (*k* ‘ *S*) *k h*
using *homhk homeomorphism_image2 homeomorphism_sym* **by** *fastforce*
have *open* ((*k* ∘ *f* ∘ *h*) ‘ *k* ‘ *U*)
proof (*rule invariance_of_domain*)
show *continuous_on* (*k* ‘ *U*) (*k* ∘ *f* ∘ *h*)
proof (*intro continuous_intros*)
show *continuous_on* (*k* ‘ *U*) *h*
by (*meson continuous_on_subset* [*OF homeomorphism_cont1* [*OF homhk*]] *top_greatest*)
have *h* ‘ *k* ‘ *U* ⊆ *U*
by (*metis* ‹*U* ⊆ *S*› *dual_order.eq_iff homeomorphism_image2 homeomorphism_of_subsets homkh*)
then show *continuous_on* (*h* ‘ *k* ‘ *U*) *f*
by (*rule continuous_on_subset* [*OF contf*])
have *f* ‘ *h* ‘ *k* ‘ *U* ⊆ *S*
using ‹*h* ‘ *k* ‘ *U* ⊆ *U*› *fim* **by** *blast*
then show *continuous_on* (*f* ‘ *h* ‘ *k* ‘ *U*) *k*
by (*rule continuous_on_subset* [*OF homeomorphism_cont2* [*OF homhk*]])
qed
have *ope_iff*: ⋀*T*. *open T* ⟷ *openin* (*top_of_set* (*k* ‘ *S*)) *T*
using *homhk homeomorphism_image2 open_openin* **by** *fastforce*
show *open* (*k* ‘ *U*)
by (*simp add*: *ope_iff homeomorphism_imp_open_map* [*OF homkh ope*])
show *inj_on* (*k* ∘ *f* ∘ *h*) (*k* ‘ *U*)
apply (*clarsimp simp*: *inj_on_def*)
by (*metis* ‹*U* ⊆ *S*› *fim homeomorphism_apply2 homhk image_subset_iff inj_onD injf subsetD*)
qed
moreover
have *eq*: *f* ‘ *U* = *h* ‘ (*k* ∘ *f* ∘ *h* ∘ *k*) ‘ *U*
unfolding *image_comp* [*symmetric*] **using** ‹*U* ⊆ *S*› *fim*
by (*metis homeomorphism_image2 homeomorphism_of_subsets homkh subset_image_iff*)
ultimately show *?thesis*
by (*metis* (*no_types*, *hide_lams*) *homeomorphism_imp_open_map homhk image_comp open_openin subtopology_UNIV*)
qed

lemma *inv_of_domain_ss1*:
fixes *f* :: ′*a* ⇒ ′*a*::*euclidean_space*
assumes *contf*: *continuous_on U f* **and** *injf*: *inj_on f U* **and** *fim*: *f* ‘ *U* ⊆ *S*
and *subspace S*
and *ope*: *openin* (*top_of_set S*) *U*
shows *openin* (*top_of_set S*) (*f* ‘ *U*)
proof −

define S' **where** $S' \equiv \{y.\ \forall x \in S.\ orthogonal\ x\ y\}$
have *subspace* S'
by (*simp add*: S'*_def subspace_orthogonal_to_vectors*)
define g **where** $g \equiv \lambda z::'a*'a.\ ((f \circ fst)z,\ snd\ z)$
have *openin* (*top_of_set* ($S \times S'$)) (g ` ($U \times S'$))
proof (*rule inv_of_domain_ss0*)
show *continuous_on* ($U \times S'$) g
unfolding *g_def*
by (*auto intro*!: *continuous_intros continuous_on_compose2* [*OF contf continuous_on_fst*])
show g ` ($U \times S'$) $\subseteq S \times S'$
using *fim* **by** (*auto simp*: *g_def*)
show *inj_on* g ($U \times S'$)
using *injf* **by** (*auto simp*: *g_def inj_on_def*)
show *subspace* ($S \times S'$)
by (*simp add*: ‹*subspace* S'› ‹*subspace* S› *subspace_Times*)
show *openin* (*top_of_set* ($S \times S'$)) ($U \times S'$)
by (*simp add*: *openin_Times* [*OF ope*])
have *dim* ($S \times S'$) = *dim* S + *dim* S'
by (*simp add*: ‹*subspace* S'› ‹*subspace* S› *dim_Times*)
also have ... = *DIM*('*a*)
using *dim_subspace_orthogonal_to_vectors* [*OF* ‹*subspace* S› *subspace_UNIV*]
by (*simp add*: *add.commute* S'*_def*)
finally show *dim* ($S \times S'$) = *DIM*('*a*) **.**
qed
moreover have g ` ($U \times S'$) = $f$ ` $U \times S'$
by (*auto simp*: *g_def image_iff*)
moreover have $0 \in S'$
using ‹*subspace* S'› *subspace_affine* **by** *blast*
ultimately show *?thesis*
by (*auto simp*: *openin_Times_eq*)
qed

corollary *invariance_of_domain_subspaces*:
fixes f :: '*a*::*euclidean_space* ⇒ '*b*::*euclidean_space*
assumes *ope*: *openin* (*top_of_set* U) S
and *subspace* U *subspace* V **and** *VU*: *dim* $V \le$ *dim* U
and *contf*: *continuous_on* S f **and** *fim*: f ` $S \subseteq V$
and *injf*: *inj_on* f S
shows *openin* (*top_of_set* V) (f ` S)
proof −
obtain V' **where** *subspace* V' $V' \subseteq U$ *dim* V' = *dim* V
using *choose_subspace_of_subspace* [*OF VU*]
by (*metis span_eq_iff* ‹*subspace* U›)
then have V *homeomorphic* V'
by (*simp add*: ‹*subspace* V› *homeomorphic_subspaces*)
then obtain h k **where** *homhk*: *homeomorphism* V V' h k
using *homeomorphic_def* **by** *blast*

have *eq*: *f* ‘ *S* = *k* ‘ (*h* ∘ *f*) ‘ *S*
proof −
have *k* ‘ *h* ‘ *f* ‘ *S* = *f* ‘ *S*
by (*meson fim homeomorphism_def homeomorphism_of_subsets homhk subset_refl*)
then show *?thesis*
by (*simp add*: *image_comp*)
qed
show *?thesis*
unfolding *eq*
proof (*rule homeomorphism_imp_open_map*)
show *homkh*: *homeomorphism V′ V k h*
by (*simp add*: *homeomorphism_symD homhk*)
have *hfV′*: (*h* ∘ *f*) ‘ *S* ⊆ *V′*
using *fim homeomorphism_image1 homhk* **by** *fastforce*
moreover have *openin* (*top_of_set U*) ((*h* ∘ *f*) ‘ *S*)
proof (*rule inv_of_domain_ss1*)
show *continuous_on S* (*h* ∘ *f*)
by (*meson contf continuous_on_compose continuous_on_subset fim homeomorphism_cont1 homhk*)
show *inj_on* (*h* ∘ *f*) *S*
apply (*clarsimp simp*: *inj_on_def*)
by (*metis fim homeomorphism_apply2* [*OF homkh*] *image_subset_iff inj_onD injf*)
show (*h* ∘ *f*) ‘ *S* ⊆ *U*
using ‹*V′* ⊆ *U*› *hfV′* **by** *auto*
qed (*auto simp*: *assms*)
ultimately show *openin* (*top_of_set V′*) ((*h* ∘ *f*) ‘ *S*)
using *openin_subset_trans* ‹*V′* ⊆ *U*› **by** *force*
qed
qed

corollary *invariance_of_dimension_subspaces*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
assumes *ope*: *openin* (*top_of_set U*) *S*
and *subspace U subspace V*
and *contf*: *continuous_on S f* **and** *fim*: *f* ‘ *S* ⊆ *V*
and *injf*: *inj_on f S* **and** *S* ≠ {}
shows *dim U* ≤ *dim V*
proof −
have *False* **if** *dim V* < *dim U*
proof −
obtain *T* **where** *subspace T T* ⊆ *U dim T* = *dim V*
using *choose_subspace_of_subspace* [*of dim V U*]
by (*metis* ‹*dim V* < *dim U*› *assms*(*2*) *order.strict_implies_order span_eq_iff*)
then have *V homeomorphic T*
by (*simp add*: ‹*subspace V*› *homeomorphic_subspaces*)
then obtain *h k* **where** *homhk*: *homeomorphism V T h k*
using *homeomorphic_def* **by** *blast*

have *continuous_on S* (*h* ∘ *f*)
by (*meson contf continuous_on_compose continuous_on_subset fim homeomorphism_cont1 homhk*)
moreover have (*h* ∘ *f*) ' *S* ⊆ *U*
using ‹*T* ⊆ *U*› *fim homeomorphism_image1 homhk* **by** *fastforce*
moreover have *inj_on* (*h* ∘ *f*) *S*
apply (*clarsimp simp*: *inj_on_def*)
by (*metis fim homeomorphism_apply1 homhk image_subset_iff inj_onD injf*)
ultimately have *ope_hf*: *openin* (*top_of_set U*) ((*h* ∘ *f*) ' *S*)
using *invariance_of_domain_subspaces* [*OF ope* ‹*subspace U*› ‹*subspace U*›] **by** *blast*
have (*h* ∘ *f*) ' *S* ⊆ *T*
using *fim homeomorphism_image1 homhk* **by** *fastforce*
then have *dim* ((*h* ∘ *f*) ' *S*) ≤ *dim T*
by (*rule dim_subset*)
also have *dim* ((*h* ∘ *f*) ' *S*) = *dim U*
using ‹*S* ≠ {}› ‹*subspace U*›
by (*blast intro*: *dim_openin ope_hf*)
finally show *False*
using ‹*dim V* < *dim U*› ‹*dim T* = *dim V*› **by** *simp*
qed
then show *?thesis*
using *not_less* **by** *blast*
qed

corollary *invariance_of_domain_affine_sets*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *ope*: *openin* (*top_of_set U*) *S*
and *aff*: *affine U affine V aff_dim V* ≤ *aff_dim U*
and *contf*: *continuous_on S f* **and** *fim*: *f* ' *S* ⊆ *V*
and *injf*: *inj_on f S*
shows *openin* (*top_of_set V*) (*f* ' *S*)
proof (*cases S* = {})
case *True*
then show *?thesis* **by** *auto*
next
case *False*
obtain *a b* **where** *a* ∈ *S a* ∈ *U b* ∈ *V*
using *False fim ope openin_contains_cball* **by** *fastforce*
have *openin* (*top_of_set* ((+) (− *b*) ' *V*)) (((+) (− *b*) ∘ *f* ∘ (+) *a*) ' (+) (− *a*) ' *S*)
proof (*rule invariance_of_domain_subspaces*)
show *openin* (*top_of_set* ((+) (− *a*) ' *U*)) ((+) (− *a*) ' *S*)
by (*metis ope homeomorphism_imp_open_map homeomorphism_translation translation_galois*)
show *subspace* ((+) (− *a*) ' *U*)
by (*simp add*: ‹*a* ∈ *U*› *affine_diffs_subspace_subtract* ‹*affine U*› *cong*: *image_cong_simp*)
show *subspace* ((+) (− *b*) ' *V*)

by (*simp add*: ‹*b* ∈ *V*› *affine_diffs_subspace_subtract* ‹*affine V*› *cong*: *image_cong_simp*)
show *dim* ((+) (− *b*) ‘ *V*) ≤ *dim* ((+) (− *a*) ‘ *U*)
by (*metis* ‹*a* ∈ *U*› ‹*b* ∈ *V*› *aff_dim_eq_dim affine_hull_eq aff of_nat_le_iff*)
show *continuous_on* ((+) (− *a*) ‘ *S*) ((+) (− *b*) ∘ *f* ∘ (+) *a*)
by (*metis contf continuous_on_compose homeomorphism_cont2 homeomorphism_translation translation_galois*)
show ((+) (− *b*) ∘ *f* ∘ (+) *a*) ‘ (+) (− *a*) ‘ *S* ⊆ (+) (− *b*) ‘ *V*
using *fim* **by** *auto*
show *inj_on* ((+) (− *b*) ∘ *f* ∘ (+) *a*) ((+) (− *a*) ‘ *S*)
by (*auto simp*: *inj_on_def*) (*meson inj_onD injf*)
qed
then show *?thesis*
by (*metis* (*no_types*, *lifting*) *homeomorphism_imp_open_map homeomorphism_translation image_comp translation_galois*)
qed

corollary *invariance_of_dimension_affine_sets*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
assumes *ope*: *openin* (*top_of_set U*) *S*
and *aff*: *affine U affine V*
and *contf*: *continuous_on S f* **and** *fim*: *f* ‘ *S* ⊆ *V*
and *injf*: *inj_on f S* **and** *S* ≠ {}
shows *aff_dim U* ≤ *aff_dim V*
proof −
obtain *a b* **where** *a* ∈ *S a* ∈ *U b* ∈ *V*
using ‹*S* ≠ {}› *fim ope openin_contains_cball* **by** *fastforce*
have *dim* ((+) (− *a*) ‘ *U*) ≤ *dim* ((+) (− *b*) ‘ *V*)
proof (*rule invariance_of_dimension_subspaces*)
show *openin* (*top_of_set* ((+) (− *a*) ‘ *U*)) ((+) (− *a*) ‘ *S*)
by (*metis ope homeomorphism_imp_open_map homeomorphism_translation translation_galois*)
show *subspace* ((+) (− *a*) ‘ *U*)
by (*simp add*: ‹*a* ∈ *U*› *affine_diffs_subspace_subtract* ‹*affine U*› *cong*: *image_cong_simp*)
show *subspace* ((+) (− *b*) ‘ *V*)
by (*simp add*: ‹*b* ∈ *V*› *affine_diffs_subspace_subtract* ‹*affine V*› *cong*: *image_cong_simp*)
show *continuous_on* ((+) (− *a*) ‘ *S*) ((+) (− *b*) ∘ *f* ∘ (+) *a*)
by (*metis contf continuous_on_compose homeomorphism_cont2 homeomorphism_translation translation_galois*)
show ((+) (− *b*) ∘ *f* ∘ (+) *a*) ‘ (+) (− *a*) ‘ *S* ⊆ (+) (− *b*) ‘ *V*
using *fim* **by** *auto*
show *inj_on* ((+) (− *b*) ∘ *f* ∘ (+) *a*) ((+) (− *a*) ‘ *S*)
by (*auto simp*: *inj_on_def*) (*meson inj_onD injf*)
qed (*use* ‹*S* ≠ {}› **in** *auto*)
then show *?thesis*
by (*metis* ‹*a* ∈ *U*› ‹*b* ∈ *V*› *aff_dim_eq_dim affine_hull_eq aff of_nat_le_iff*)
qed

corollary *invariance_of_dimension*:
  fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
  assumes *contf*: *continuous_on S f* **and** *open S*
    and *injf*: *inj_on f S* **and** *S* ≠ {}
   shows *DIM('a)* ≤ *DIM('b)*
  using *invariance_of_dimension_subspaces* [*of UNIV S UNIV f*] *assms*
  by *auto*

corollary *continuous_injective_image_subspace_dim_le*:
  fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
  assumes *subspace S subspace T*
    and *contf*: *continuous_on S f* **and** *fim*: *f ' S* ⊆ *T*
    and *injf*: *inj_on f S*
   shows *dim S* ≤ *dim T*
  using *invariance_of_dimension_subspaces* [*of S S _ f*] *assms* **by** (*auto simp*: *subspace_affine*)

lemma *invariance_of_dimension_convex_domain*:
  fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
  assumes *convex S*
    and *contf*: *continuous_on S f* **and** *fim*: *f ' S* ⊆ *affine hull T*
    and *injf*: *inj_on f S*
   shows *aff_dim S* ≤ *aff_dim T*
proof (*cases S* = {})
  case *True*
  then show *?thesis* **by** (*simp add*: *aff_dim_geq*)
next
  case *False*
  have *aff_dim* (*affine hull S*) ≤ *aff_dim* (*affine hull T*)
  proof (*rule invariance_of_dimension_affine_sets*)
   show *openin* (*top_of_set* (*affine hull S*)) (*rel_interior S*)
    by (*simp add*: *openin_rel_interior*)
   show *continuous_on* (*rel_interior S*) *f*
    using *contf continuous_on_subset rel_interior_subset* **by** *blast*
   show *f ' rel_interior S* ⊆ *affine hull T*
    using *fim rel_interior_subset* **by** *blast*
   show *inj_on f* (*rel_interior S*)
    using *inj_on_subset injf rel_interior_subset* **by** *blast*
   show *rel_interior S* ≠ {}
    by (*simp add*: *False* ‹*convex S*› *rel_interior_eq_empty*)
  qed *auto*
  then show *?thesis*
   by *simp*
qed

lemma *homeomorphic_convex_sets_le*:

assumes *convex S S homeomorphic T*
shows *aff_dim S* ≤ *aff_dim T*
proof −
obtain *h k* **where** *homhk*: *homeomorphism S T h k*
using *homeomorphic_def assms* **by** *blast*
show *?thesis*
proof (*rule invariance_of_dimension_convex_domain* [*OF* ‹*convex S*›])
show *continuous_on S h*
using *homeomorphism_def homhk* **by** *blast*
show *h* ‘ *S* ⊆ *affine hull T*
by (*metis homeomorphism_def homhk hull_subset*)
show *inj_on h S*
by (*meson homeomorphism_apply1 homhk inj_on_inverseI*)
qed
qed

lemma *homeomorphic_convex_sets*:
assumes *convex S convex T S homeomorphic T*
shows *aff_dim S* = *aff_dim T*
by (*meson assms dual_order.antisym homeomorphic_convex_sets_le homeomorphic_sym*)

lemma *homeomorphic_convex_compact_sets_eq*:
assumes *convex S compact S convex T compact T*
shows *S homeomorphic T* ⟷ *aff_dim S* = *aff_dim T*
by (*meson assms homeomorphic_convex_compact_sets homeomorphic_convex_sets*)

lemma *invariance_of_domain_gen*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
assumes *open S continuous_on S f inj_on f S DIM*(′*b*) ≤ *DIM*(′*a*)
shows *open*(*f* ‘ *S*)
using *invariance_of_domain_subspaces* [*of UNIV S UNIV f*] *assms* **by** *auto*

lemma *injective_into_1d_imp_open_map_UNIV*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ *real*
assumes *open T continuous_on S f inj_on f S T* ⊆ *S*
shows *open* (*f* ‘ *T*)
apply (*rule invariance_of_domain_gen* [*OF* ‹*open T*›])
using *assms* **by** (*auto simp*: *elim*: *continuous_on_subset subset_inj_on*)

lemma *continuous_on_inverse_open*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
assumes *open S continuous_on S f DIM*(′*b*) ≤ *DIM*(′*a*) **and** *gf*: ⋀*x*. *x* ∈ *S* ⟹ *g*(*f x*) = *x*
shows *continuous_on* (*f* ‘ *S*) *g*
proof (*clarsimp simp add*: *continuous_openin_preimage_eq*)
fix *T* :: ′*a set*
assume *open T*
have *eq*: *f* ‘ *S* ∩ *g* −‘ *T* = *f* ‘ (*S* ∩ *T*)

```
    by (auto simp: gf)
  have open (f ‘ S)
    by (rule invariance_of_domain_gen) (use assms inj_on_inverseI in auto)
  moreover have open (f ‘ (S ∩ T))
    using assms
   by (metis ‹open T› continuous_on_subset inj_onI inj_on_subset invariance_of_domain_gen
openin_open openin_open_eq)
  ultimately show openin (top_of_set (f ‘ S)) (f ‘ S ∩ g −‘ T)
    unfolding eq by (auto intro: open_openin_trans)
qed

lemma invariance_of_domain_homeomorphism:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes open S continuous_on S f DIM('b) ≤ DIM('a) inj_on f S
  obtains g where homeomorphism S (f ‘ S) f g
proof
  show homeomorphism S (f ‘ S) f (inv_into S f)
    by (simp add: assms continuous_on_inverse_open homeomorphism_def)
qed

corollary invariance_of_domain_homeomorphic:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes open S continuous_on S f DIM('b) ≤ DIM('a) inj_on f S
  shows S homeomorphic (f ‘ S)
  using invariance_of_domain_homeomorphism [OF assms]
  by (meson homeomorphic_def)

lemma continuous_image_subset_interior:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes continuous_on S f inj_on f S DIM('b) ≤ DIM('a)
  shows f ‘ (interior S) ⊆ interior(f ‘ S)
proof −
  have open (f ‘ interior S)
    using assms
     by (intro invariance_of_domain_gen) (auto simp: subset_inj_on interior_subset
continuous_on_subset)
  then show ?thesis
    by (simp add: image_mono interior_maximal interior_subset)
qed

lemma homeomorphic_interiors_same_dimension:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T and dimeq: DIM('a) = DIM('b)
  shows (interior S) homeomorphic (interior T)
  using assms [unfolded homeomorphic_minimal]
  unfolding homeomorphic_def
proof (clarify elim!: ex_forward)
  fix f g
  assume S: ∀x∈S. f x ∈ T ∧ g (f x) = x and T: ∀y∈T. g y ∈ S ∧ f (g y) = y
```

and *contf*: *continuous_on S f* **and** *contg*: *continuous_on T g*
then have *fST*: *f ‘ S* = *T* **and** *gTS*: *g ‘ T* = *S* **and** *inj_on f S inj_on g T*
by (*auto simp*: *inj_on_def intro*: *rev_image_eqI*) *metis+*
have *fim*: *f ‘ interior S* ⊆ *interior T*
using *continuous_image_subset_interior* [*OF contf* ‹*inj_on f S*›] *dimeq fST* **by** *simp*
have *gim*: *g ‘ interior T* ⊆ *interior S*
using *continuous_image_subset_interior* [*OF contg* ‹*inj_on g T*›] *dimeq gTS* **by** *simp*
show *homeomorphism* (*interior S*) (*interior T*) *f g*
unfolding *homeomorphism_def*
proof (*intro conjI ballI*)
show ⋀*x*. *x* ∈ *interior S* ⟹ *g* (*f x*) = *x*
by (*meson* ‹∀ *x*∈*S*. *f x* ∈ *T* ∧ *g* (*f x*) = *x*› *subsetD interior_subset*)
have *interior T* ⊆ *f ‘ interior S*
proof
fix *x* **assume** *x* ∈ *interior T*
then have *g x* ∈ *interior S*
using *gim* **by** *blast*
then show *x* ∈ *f ‘ interior S*
by (*metis T* ‹*x* ∈ *interior T*› *image_iff interior_subset subsetCE*)
qed
then show *f ‘ interior S* = *interior T*
using *fim* **by** *blast*
show *continuous_on* (*interior S*) *f*
by (*metis interior_subset continuous_on_subset contf*)
show ⋀*y*. *y* ∈ *interior T* ⟹ *f* (*g y*) = *y*
by (*meson T subsetD interior_subset*)
have *interior S* ⊆ *g ‘ interior T*
proof
fix *x* **assume** *x* ∈ *interior S*
then have *f x* ∈ *interior T*
using *fim* **by** *blast*
then show *x* ∈ *g ‘ interior T*
by (*metis S* ‹*x* ∈ *interior S*› *image_iff interior_subset subsetCE*)
qed
then show *g ‘ interior T* = *interior S*
using *gim* **by** *blast*
show *continuous_on* (*interior T*) *g*
by (*metis interior_subset continuous_on_subset contg*)
qed
qed

lemma *homeomorphic_open_imp_same_dimension*:
fixes *S* :: *'a*::*euclidean_space set* **and** *T* :: *'b*::*euclidean_space set*
assumes *S homeomorphic T open S S* ≠ {} *open T T* ≠ {}
shows *DIM*(*'a*) = *DIM*(*'b*)
using *assms*
apply (*simp add*: *homeomorphic_minimal*)

apply (*rule order_antisym*; *metis inj_onI invariance_of_dimension*)
done

proposition *homeomorphic_interiors*:
fixes *S* :: *'a::euclidean_space set* **and** *T* :: *'b::euclidean_space set*
assumes *S homeomorphic T interior S* = {} ⟷ *interior T* = {}
shows (*interior S*) *homeomorphic* (*interior T*)
proof (*cases interior T* = {})
case *True*
with *assms* **show** *?thesis* **by** *auto*
next
case *False*
then have *DIM('a)* = *DIM('b)*
using *assms*
apply (*simp add*: *homeomorphic_minimal*)
apply (*rule order_antisym*; *metis continuous_on_subset inj_onI inj_on_subset interior_subset invariance_of_dimension open_interior*)
done
then show *?thesis*
by (*rule homeomorphic_interiors_same_dimension* [*OF* ‹*S homeomorphic T*›])
qed

lemma *homeomorphic_frontiers_same_dimension*:
fixes *S* :: *'a::euclidean_space set* **and** *T* :: *'b::euclidean_space set*
assumes *S homeomorphic T closed S closed T* **and** *dimeq*: *DIM('a)* = *DIM('b)*
shows (*frontier S*) *homeomorphic* (*frontier T*)
using *assms* [*unfolded homeomorphic_minimal*]
unfolding *homeomorphic_def*
proof (*clarify elim*!: *ex_forward*)
fix *f g*
assume *S*: ∀ *x*∈*S*. *f x* ∈ *T* ∧ *g* (*f x*) = *x* **and** *T*: ∀ *y*∈*T*. *g y* ∈ *S* ∧ *f* (*g y*) = *y*
and *contf*: *continuous_on S f* **and** *contg*: *continuous_on T g*
then have *fST*: *f* ‘ *S* = *T* **and** *gTS*: *g* ‘ *T* = *S* **and** *inj_on f S inj_on g T*
by (*auto simp*: *inj_on_def intro*: *rev_image_eqI*) *metis+*
have *g* ‘ *interior T* ⊆ *interior S*
using *continuous_image_subset_interior* [*OF contg* ‹*inj_on g T*›] *dimeq gTS* **by** *simp*
then have *fim*: *f* ‘ *frontier S* ⊆ *frontier T*
unfolding *frontier_def*
using *continuous_image_subset_interior assms*(*2*) *assms*(*3*) *S* **by** *auto*
have *f* ‘ *interior S* ⊆ *interior T*
using *continuous_image_subset_interior* [*OF contf* ‹*inj_on f S*›] *dimeq fST* **by** *simp*
then have *gim*: *g* ‘ *frontier T* ⊆ *frontier S*
unfolding *frontier_def*
using *continuous_image_subset_interior T assms*(*2*) *assms*(*3*) **by** *auto*
show *homeomorphism* (*frontier S*) (*frontier T*) *f g*
unfolding *homeomorphism_def*
proof (*intro conjI ballI*)

show *gf*: $\bigwedge x.\ x \in$ *frontier* $S \Longrightarrow g\ (f\ x) = x$
by (*simp add*: *S assms*(*2*) *frontier_def*)
show *fg*: $\bigwedge y.\ y \in$ *frontier* $T \Longrightarrow f\ (g\ y) = y$
by (*simp add*: *T assms*(*3*) *frontier_def*)
have *frontier* $T \subseteq f$ ‘ *frontier* S
proof
fix *x* **assume** $x \in$ *frontier* T
then have $g\ x \in$ *frontier* S
using *gim* **by** *blast*
then show $x \in f$ ‘ *frontier* S
by (*metis fg* ‹$x \in$ *frontier* T› *imageI*)
qed
then show f ‘ *frontier* S = *frontier* T
using *fim* **by** *blast*
show *continuous_on* (*frontier* S) f
by (*metis Diff_subset assms*(*2*) *closure_eq contf continuous_on_subset frontier_def*)
have *frontier* $S \subseteq g$ ‘ *frontier* T
proof
fix *x* **assume** $x \in$ *frontier* S
then have $f\ x \in$ *frontier* T
using *fim* **by** *blast*
then show $x \in g$ ‘ *frontier* T
by (*metis gf* ‹$x \in$ *frontier* S› *imageI*)
qed
then show g ‘ *frontier* T = *frontier* S
using *gim* **by** *blast*
show *continuous_on* (*frontier* T) g
by (*metis Diff_subset assms*(*3*) *closure_closed contg continuous_on_subset frontier_def*)
qed
qed

lemma *homeomorphic_frontiers*:
fixes S :: *'a::euclidean_space set* **and** T :: *'b::euclidean_space set*
assumes *S homeomorphic T closed S closed T*
interior $S = \{\} \longleftrightarrow$ *interior* $T = \{\}$
shows (*frontier* S) *homeomorphic* (*frontier* T)
proof (*cases interior* $T = \{\}$)
case *True*
then show *?thesis*
by (*metis Diff_empty assms closure_eq frontier_def*)
next
case *False*
then have *DIM*(*'a*) = *DIM*(*'b*)
using *assms homeomorphic_interiors homeomorphic_open_imp_same_dimension* **by** *blast*
then show *?thesis*
using *assms homeomorphic_frontiers_same_dimension* **by** *blast*

qed

lemma *continuous_image_subset_rel_interior*:
  fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
  assumes *contf*: *continuous_on S f* **and** *injf*: *inj_on f S* **and** *fim*: *f ' S* ⊆ *T*
    and *TS*: *aff_dim T* ≤ *aff_dim S*
  shows *f ' (rel_interior S)* ⊆ *rel_interior(f ' S)*
proof (*rule rel_interior_maximal*)
  show *f ' rel_interior S* ⊆ *f ' S*
    by(*simp add*: *image_mono rel_interior_subset*)
  show *openin* (*top_of_set* (*affine hull f ' S*)) (*f ' rel_interior S*)
  proof (*rule invariance_of_domain_affine_sets*)
    show *openin* (*top_of_set* (*affine hull S*)) (*rel_interior S*)
      by (*simp add*: *openin_rel_interior*)
    show *aff_dim* (*affine hull f ' S*) ≤ *aff_dim* (*affine hull S*)
      by (*metis aff_dim_affine_hull aff_dim_subset fim TS order_trans*)
    show *f ' rel_interior S* ⊆ *affine hull f ' S*
      by (*meson* ‹*f ' rel_interior S* ⊆ *f ' S*› *hull_subset order_trans*)
    show *continuous_on* (*rel_interior S*) *f*
      using *contf continuous_on_subset rel_interior_subset* **by** *blast*
    show *inj_on f* (*rel_interior S*)
      using *inj_on_subset injf rel_interior_subset* **by** *blast*
  qed *auto*
qed

lemma *homeomorphic_rel_interiors_same_dimension*:
  fixes *S* :: *'a::euclidean_space set* **and** *T* :: *'b::euclidean_space set*
  assumes *S homeomorphic T* **and** *aff*: *aff_dim S* = *aff_dim T*
  shows (*rel_interior S*) *homeomorphic* (*rel_interior T*)
  using *assms* [*unfolded homeomorphic_minimal*]
  unfolding *homeomorphic_def*
proof (*clarify elim*!: *ex_forward*)
  fix *f g*
  assume *S*: ∀ *x*∈*S*. *f x* ∈ *T* ∧ *g* (*f x*) = *x* **and** *T*: ∀ *y*∈*T*. *g y* ∈ *S* ∧ *f* (*g y*) = *y*
    and *contf*: *continuous_on S f* **and** *contg*: *continuous_on T g*
  then have *fST*: *f ' S* = *T* **and** *gTS*: *g ' T* = *S* **and** *inj_on f S inj_on g T*
    by (*auto simp*: *inj_on_def intro*: *rev_image_eqI*) *metis+*
  have *fim*: *f ' rel_interior S* ⊆ *rel_interior T*
    by (*metis* ‹*inj_on f S*› *aff contf continuous_image_subset_rel_interior fST order_refl*)
  have *gim*: *g ' rel_interior T* ⊆ *rel_interior S*
    by (*metis* ‹*inj_on g T*› *aff contg continuous_image_subset_rel_interior gTS order_refl*)
  show *homeomorphism* (*rel_interior S*) (*rel_interior T*) *f g*
    unfolding *homeomorphism_def*
  proof (*intro conjI ballI*)
    show *gf*: ⋀*x*. *x* ∈ *rel_interior S* ⟹ *g* (*f x*) = *x*
      using *S rel_interior_subset* **by** *blast*
    show *fg*: ⋀*y*. *y* ∈ *rel_interior T* ⟹ *f* (*g y*) = *y*

using *T mem_rel_interior_ball* **by** *blast*
have *rel_interior T* ⊆ *f* ' *rel_interior S*
proof
fix *x* **assume** *x* ∈ *rel_interior T*
then have *g x* ∈ *rel_interior S*
using *gim* **by** *blast*
then show *x* ∈ *f* ' *rel_interior S*
by (*metis fg* ‹*x* ∈ *rel_interior T*› *imageI*)
qed
moreover have *f* ' *rel_interior S* ⊆ *rel_interior T*
by (*metis* ‹*inj_on f S*› *aff contf continuous_image_subset_rel_interior fST order_refl*)
ultimately show *f* ' *rel_interior S* = *rel_interior T*
by *blast*
show *continuous_on* (*rel_interior S*) *f*
using *contf continuous_on_subset rel_interior_subset* **by** *blast*
have *rel_interior S* ⊆ *g* ' *rel_interior T*
proof
fix *x* **assume** *x* ∈ *rel_interior S*
then have *f x* ∈ *rel_interior T*
using *fim* **by** *blast*
then show *x* ∈ *g* ' *rel_interior T*
by (*metis gf* ‹*x* ∈ *rel_interior S*› *imageI*)
qed
then show *g* ' *rel_interior T* = *rel_interior S*
using *gim* **by** *blast*
show *continuous_on* (*rel_interior T*) *g*
using *contg continuous_on_subset rel_interior_subset* **by** *blast*
qed
qed

lemma *homeomorphic_aff_dim_le*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *S homeomorphic T rel_interior S* ≠ {}
shows *aff_dim* (*affine hull S*) ≤ *aff_dim* (*affine hull T*)
proof −
obtain *f g*
where *S*: ∀ *x*∈*S*. *f x* ∈ *T* ∧ *g* (*f x*) = *x* **and** *T*: ∀ *y*∈*T*. *g y* ∈ *S* ∧ *f* (*g y*) = *y*
and *contf*: *continuous_on S f* **and** *contg*: *continuous_on T g*
using *assms* [*unfolded homeomorphic_minimal*] **by** *auto*
show *?thesis*
proof (*rule invariance_of_dimension_affine_sets*)
show *continuous_on* (*rel_interior S*) *f*
using *contf continuous_on_subset rel_interior_subset* **by** *blast*
show *f* ' *rel_interior S* ⊆ *affine hull T*
by (*meson S hull_subset image_subsetI rel_interior_subset rev_subsetD*)
show *inj_on f* (*rel_interior S*)
by (*metis S inj_on_inverseI inj_on_subset rel_interior_subset*)

qed (*simp_all add*: *openin_rel_interior assms*)
qed

lemma *homeomorphic_rel_interiors*:
fixes *S* :: *'a*::*euclidean_space set* **and** *T* :: *'b*::*euclidean_space set*
assumes *S homeomorphic T rel_interior S* = {} ⟷ *rel_interior T* = {}
shows (*rel_interior S*) *homeomorphic* (*rel_interior T*)
proof (*cases rel_interior T* = {})
case *True*
with *assms* **show** *?thesis* **by** *auto*
next
case *False*
have *aff_dim* (*affine hull S*) ≤ *aff_dim* (*affine hull T*)
using *False assms homeomorphic_aff_dim_le* **by** *blast*
moreover have *aff_dim* (*affine hull T*) ≤ *aff_dim* (*affine hull S*)
using *False assms*(*1*) *homeomorphic_aff_dim_le homeomorphic_sym* **by** *auto*
ultimately have *aff_dim S* = *aff_dim T* **by** *force*
then show *?thesis*
by (*rule homeomorphic_rel_interiors_same_dimension* [*OF* ‹*S homeomorphic T*›])
qed

lemma *homeomorphic_rel_boundaries_same_dimension*:
fixes *S* :: *'a*::*euclidean_space set* **and** *T* :: *'b*::*euclidean_space set*
assumes *S homeomorphic T* **and** *aff*: *aff_dim S* = *aff_dim T*
shows (*S* − *rel_interior S*) *homeomorphic* (*T* − *rel_interior T*)
using *assms* [*unfolded homeomorphic_minimal*]
unfolding *homeomorphic_def*
proof (*clarify elim*!: *ex_forward*)
fix *f g*
assume *S*: ∀ *x*∈*S*. *f x* ∈ *T* ∧ *g* (*f x*) = *x* **and** *T*: ∀ *y*∈*T*. *g y* ∈ *S* ∧ *f* (*g y*) = *y*
and *contf*: *continuous_on S f* **and** *contg*: *continuous_on T g*
then have *fST*: *f* ‘ *S* = *T* **and** *gTS*: *g* ‘ *T* = *S* **and** *inj_on f S inj_on g T*
by (*auto simp*: *inj_on_def intro*: *rev_image_eqI*) *metis*+
have *fim*: *f* ‘ *rel_interior S* ⊆ *rel_interior T*
by (*metis* ‹*inj_on f S*› *aff contf continuous_image_subset_rel_interior fST order_refl*)
have *gim*: *g* ‘ *rel_interior T* ⊆ *rel_interior S*
by (*metis* ‹*inj_on g T*› *aff contg continuous_image_subset_rel_interior gTS order_refl*)
show *homeomorphism* (*S* − *rel_interior S*) (*T* − *rel_interior T*) *f g*
unfolding *homeomorphism_def*
proof (*intro conjI ballI*)
show *gf*: ⋀*x*. *x* ∈ *S* − *rel_interior S* ⟹ *g* (*f x*) = *x*
using *S rel_interior_subset* **by** *blast*
show *fg*: ⋀*y*. *y* ∈ *T* − *rel_interior T* ⟹ *f* (*g y*) = *y*
using *T mem_rel_interior_ball* **by** *blast*
show *f* ‘ (*S* − *rel_interior S*) = *T* − *rel_interior T*

using *S fST fim gim* **by** *auto*
show *continuous_on* (*S* − *rel_interior S*) *f*
using *contf continuous_on_subset rel_interior_subset* **by** *blast*
show *g* ‘ (*T* − *rel_interior T*) = *S* − *rel_interior S*
using *T gTS gim fim* **by** *auto*
show *continuous_on* (*T* − *rel_interior T*) *g*
using *contg continuous_on_subset rel_interior_subset* **by** *blast*
qed
qed

lemma *homeomorphic_rel_boundaries*:
fixes *S* :: ′*a*::*euclidean_space set* **and** *T* :: ′*b*::*euclidean_space set*
assumes *S homeomorphic T rel_interior S* = {} ⟷ *rel_interior T* = {}
shows (*S* − *rel_interior S*) *homeomorphic* (*T* − *rel_interior T*)
proof (*cases rel_interior T* = {})
case *True*
with *assms* **show** *?thesis* **by** *auto*
next
case *False*
obtain *f g*
where *S*: ∀ *x*∈*S*. *f x* ∈ *T* ∧ *g* (*f x*) = *x* **and** *T*: ∀ *y*∈*T*. *g y* ∈ *S* ∧ *f* (*g y*) = *y*
and *contf*: *continuous_on S f* **and** *contg*: *continuous_on T g*
using *assms* [*unfolded homeomorphic_minimal*] **by** *auto*
have *aff_dim* (*affine hull S*) ≤ *aff_dim* (*affine hull T*)
using *False assms homeomorphic_aff_dim_le* **by** *blast*
moreover have *aff_dim* (*affine hull T*) ≤ *aff_dim* (*affine hull S*)
by (*meson False assms*(*1*) *homeomorphic_aff_dim_le homeomorphic_sym*)
ultimately have *aff_dim S* = *aff_dim T* **by** *force*
then show *?thesis*
by (*rule homeomorphic_rel_boundaries_same_dimension* [*OF* ‹*S homeomorphic T*›])
qed

proposition *uniformly_continuous_homeomorphism_UNIV_trivial*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*a*
assumes *contf*: *uniformly_continuous_on S f* **and** *hom*: *homeomorphism S UNIV f g*
shows *S* = *UNIV*
proof (*cases S* = {})
case *True*
then show *?thesis*
by (*metis UNIV_I hom empty_iff homeomorphism_def image_eqI*)
next
case *False*
have *inj g*
by (*metis UNIV_I hom homeomorphism_apply2 injI*)
then have *open* (*g* ‘ *UNIV*)
by (*blast intro*: *invariance_of_domain hom homeomorphism_cont2*)
then have *open S*

using *hom homeomorphism_image2* **by** *blast*
moreover have *complete S*
unfolding *complete_def*
proof *clarify*
fix σ
assume σ: ∀ *n*. σ *n* ∈ *S* **and** *Cauchy* σ
have *Cauchy* (*f o* σ)
using *uniformly_continuous_imp_Cauchy_continuous* ‹*Cauchy* σ› σ *contf* **by** *blast*
then obtain *l* **where** (*f* ∘ σ) ⟶ *l*
by (*auto simp*: *convergent_eq_Cauchy* [*symmetric*])
show ∃ *l*∈*S*. σ ⟶ *l*
proof
show *g l* ∈ *S*
using *hom homeomorphism_image2* **by** *blast*
have (*g* ∘ (*f* ∘ σ)) ⟶ *g l*
by (*meson UNIV_I* ‹(*f* ∘ σ) ⟶ *l*› *continuous_on_sequentially hom homeomorphism_cont2*)
then show σ ⟶ *g l*
proof −
have ∀ *n*. σ *n* = (*g* ∘ (*f* ∘ σ)) *n*
by (*metis* (*no_types*) σ *comp_eq_dest_lhs hom homeomorphism_apply1*)
then show *?thesis*
by (*metis* (*no_types*) *LIMSEQ_iff* ‹(*g* ∘ (*f* ∘ σ)) ⟶ *g l*›)
qed
qed
qed
then have *closed S*
by (*simp add*: *complete_eq_closed*)
ultimately show *?thesis*
using *clopen* [*of S*] *False* **by** *simp*
qed

6.41.6 Formulation of loop homotopy in terms of maps out of type complex

lemma *homotopic_circlemaps_imp_homotopic_loops*:
assumes *homotopic_with_canon* (λ*h*. *True*) (*sphere 0 1*) *S f g*
shows *homotopic_loops S* (*f* ∘ *exp* ∘ (λ*t*. *2* ∗ *of_real pi* ∗ *of_real t* ∗ i))
(*g* ∘ *exp* ∘ (λ*t*. *2* ∗ *of_real pi* ∗ *of_real t* ∗ i))
proof −
have *homotopic_with_canon* (λ*f*. *True*) {*z*. *cmod z* = *1*} *S f g*
using *assms* **by** (*auto simp*: *sphere_def*)
moreover have *continuous_on* {*0*..*1*} (*exp* ∘ (λ*t*. *2* ∗ *of_real pi* ∗ *of_real t* ∗ i))
by (*intro continuous_intros*)
moreover have (*exp* ∘ (λ*t*. *2* ∗ *of_real pi* ∗ *of_real t* ∗ i)) ‘ {*0*..*1*} ⊆ {*z*. *cmod z* = *1*}
by (*auto simp*: *norm_mult*)
ultimately

show *?thesis*
apply (*simp add*: *homotopic_loops_def comp_assoc*)
apply (*rule homotopic_with_compose_continuous_right*)
apply (*auto simp*: *pathstart_def pathfinish_def*)
done
qed

lemma *homotopic_loops_imp_homotopic_circlemaps*:
assumes *homotopic_loops S p q*
shows *homotopic_with_canon* (*λh. True*) (*sphere 0 1*) *S*
(*p* ∘ (*λz.* (*Arg2pi z* / (*2* ∗ *pi*))))
(*q* ∘ (*λz.* (*Arg2pi z* / (*2* ∗ *pi*))))
proof −
obtain *h* **where** *conth*: *continuous_on* ({*0..1*::*real*} × {*0..1*}) *h*
and *him*: *h* ‘ ({*0..1*} × {*0..1*}) ⊆ *S*
and *h0*: (∀ *x*. *h* (*0*, *x*) = *p x*)
and *h1*: (∀ *x*. *h* (*1*, *x*) = *q x*)
and *h01*: (∀ *t*∈{*0..1*}. *h* (*t*, *1*) = *h* (*t*, *0*))
using *assms*
by (*auto simp*: *homotopic_loops_def sphere_def homotopic_with_def pathstart_def pathfinish_def*)
define *j* **where** *j* ≡ *λz. if 0* ≤ *Im* (*snd z*)
then h (*fst z*, *Arg2pi* (*snd z*) / (*2* ∗ *pi*))
else h (*fst z*, *1* − *Arg2pi* (*cnj* (*snd z*)) / (*2* ∗ *pi*))
have *Arg2pi_eq*: *1* − *Arg2pi* (*cnj y*) / (*2* ∗ *pi*) = *Arg2pi y* / (*2* ∗ *pi*) ∨ *Arg2pi y* = *0* ∧ *Arg2pi* (*cnj y*) = *0* **if** *cmod y* = *1* **for** *y*
using *that Arg2pi_eq_0_pi Arg2pi_eq_pi* **by** (*force simp*: *Arg2pi_cnj field_split_simps*)
show *?thesis*
proof (*simp add*: *homotopic_with*; *intro conjI ballI exI*)
show *continuous_on* ({*0..1*} × *sphere 0 1*) (*λw. h* (*fst w*, *Arg2pi* (*snd w*) / (*2* ∗ *pi*)))
proof (*rule continuous_on_eq*)
show *j*: *j x* = *h* (*fst x*, *Arg2pi* (*snd x*) / (*2* ∗ *pi*)) **if** *x* ∈ {*0..1*} × *sphere 0 1* **for** *x*
using *Arg2pi_eq that h01* **by** (*force simp*: *j_def*)
have *eq*: *S* = *S* ∩ (*UNIV* × {*z. 0* ≤ *Im z*}) ∪ *S* ∩ (*UNIV* × {*z. Im z* ≤ *0*}) **for** *S* :: (*real*∗*complex*)*set*
by *auto*
have *c1*: *continuous_on* ({*0..1*} × *sphere 0 1* ∩ *UNIV* × {*z. 0* ≤ *Im z*}) (*λx. h* (*fst x*, *Arg2pi* (*snd x*) / (*2* ∗ *pi*)))
apply (*intro continuous_intros continuous_on_compose2* [*OF conth*] *continuous_on_compose2* [*OF continuous_on_upperhalf_Arg2pi*])
apply (*auto simp*: *Arg2pi*)
apply (*meson Arg2pi_lt_2pi linear not_le*)
done
have *c2*: *continuous_on* ({*0..1*} × *sphere 0 1* ∩ *UNIV* × {*z. Im z* ≤ *0*}) (*λx. h* (*fst x*, *1* − *Arg2pi* (*cnj* (*snd x*)) / (*2* ∗ *pi*)))
apply (*intro continuous_intros continuous_on_compose2* [*OF conth*] *continuous_on_compose2* [*OF continuous_on_upperhalf_Arg2pi*])

```
apply (auto simp: Arg2pi)
apply (meson Arg2pi_lt_2pi linear not_le)
done
show continuous_on ({0..1} × sphere 0 1) j
apply (simp add: j_def)
apply (subst eq)
apply (rule continuous_on_cases_local)
using Arg2pi_eq h01
by (force simp add: eq [symmetric] closedin_closed_Int closed_Times closed_halfspace_Im_le
closed_halfspace_Im_ge c1 c2)+
qed
have (λw. h (fst w, Arg2pi (snd w) / (2 ∗ pi))) ‘ ({0..1} × sphere 0 1) ⊆ h ‘
({0..1} × {0..1})
by (auto simp: Arg2pi_ge_0 Arg2pi_lt_2pi less_imp_le)
also have ... ⊆ S
using him by blast
finally show (λw. h (fst w, Arg2pi (snd w) / (2 ∗ pi))) ‘ ({0..1} × sphere 0
1) ⊆ S .
qed (auto simp: h0 h1)
qed
```

```
lemma simply_connected_homotopic_loops:
simply_connected S ⟷
(∀ p q. homotopic_loops S p p ∧ homotopic_loops S q q ⟶ homotopic_loops
S p q)
unfolding simply_connected_def using homotopic_loops_refl by metis
```

```
lemma simply_connected_eq_homotopic_circlemaps1:
fixes f :: complex ⇒ 'a::topological_space and g :: complex ⇒ 'a
assumes S: simply_connected S
and contf: continuous_on (sphere 0 1) f and fim: f ‘ (sphere 0 1) ⊆ S
and contg: continuous_on (sphere 0 1) g and gim: g ‘ (sphere 0 1) ⊆ S
shows homotopic_with_canon (λh. True) (sphere 0 1) S f g
proof −
have homotopic_loops S (f ◦ exp ◦ (λt. of_real(2 ∗ pi ∗ t) ∗ i)) (g ◦ exp ◦ (λt.
of_real(2 ∗ pi ∗ t) ∗ i))
apply (rule S [unfolded simply_connected_homotopic_loops, rule_format])
apply (simp add: homotopic_circlemaps_imp_homotopic_loops contf fim contg
gim)
done
then show ?thesis
apply (rule homotopic_with_eq [OF homotopic_loops_imp_homotopic_circlemaps])
apply (auto simp: o_def complex_norm_eq_1_exp mult.commute)
done
qed
```

```
lemma simply_connected_eq_homotopic_circlemaps2a:
fixes h :: complex ⇒ 'a::topological_space
```

assumes *conth*: *continuous_on* (*sphere 0 1*) *h* **and** *him*: *h* ‘ (*sphere 0 1*) ⊆ *S*
and *hom*: ⋀*f g*::*complex* ⇒ ′*a*.
⟦*continuous_on* (*sphere 0 1*) *f*; *f* ‘ (*sphere 0 1*) ⊆ *S*;
continuous_on (*sphere 0 1*) *g*; *g* ‘ (*sphere 0 1*) ⊆ *S*⟧
⟹ *homotopic_with_canon* (λ*h*. *True*) (*sphere 0 1*) *S f g*
shows ∃ *a*. *homotopic_with_canon* (λ*h*. *True*) (*sphere 0 1*) *S h* (λ*x*. *a*)
apply (*rule_tac x*=*h 1* **in** *exI*)
apply (*rule hom*)
using *assms* **by** (*auto*)

lemma *simply_connected_eq_homotopic_circlemaps2b*:
fixes *S* :: ′*a*::*real_normed_vector set*
assumes ⋀*f g*::*complex* ⇒ ′*a*.
⟦*continuous_on* (*sphere 0 1*) *f*; *f* ‘ (*sphere 0 1*) ⊆ *S*;
continuous_on (*sphere 0 1*) *g*; *g* ‘ (*sphere 0 1*) ⊆ *S*⟧
⟹ *homotopic_with_canon* (λ*h*. *True*) (*sphere 0 1*) *S f g*
shows *path_connected S*
proof (*clarsimp simp add*: *path_connected_eq_homotopic_points*)
fix *a b*
assume *a* ∈ *S b* ∈ *S*
then show *homotopic_loops S* (*linepath a a*) (*linepath b b*)
using *homotopic_circlemaps_imp_homotopic_loops* [*OF assms* [*of* λ*x*. *a* λ*x*. *b*]]
by (*auto simp*: *o_def linepath_def*)
qed

lemma *simply_connected_eq_homotopic_circlemaps3*:
fixes *h* :: *complex* ⇒ ′*a*::*real_normed_vector*
assumes *path_connected S*
and *hom*: ⋀*f*::*complex* ⇒ ′*a*.
⟦*continuous_on* (*sphere 0 1*) *f*; *f* ‘(*sphere 0 1*) ⊆ *S*⟧
⟹ ∃ *a*. *homotopic_with_canon* (λ*h*. *True*) (*sphere 0 1*) *S f* (λ*x*. *a*)
shows *simply_connected S*
proof (*clarsimp simp add*: *simply_connected_eq_contractible_loop_some assms*)
fix *p*
assume *p*: *path p path_image p* ⊆ *S pathfinish p* = *pathstart p*
then have *homotopic_loops S p p*
by (*simp add*: *homotopic_loops_refl*)
then obtain *a* **where** *homp*: *homotopic_with_canon* (λ*h*. *True*) (*sphere 0 1*) *S* (*p* ∘ (λ*z*. *Arg2pi z* / (*2* ∗ *pi*))) (λ*x*. *a*)
by (*metis homotopic_with_imp_subset2 homotopic_loops_imp_homotopic_circlemaps homotopic_with_imp_continuous hom*)
show ∃ *a*. *a* ∈ *S* ∧ *homotopic_loops S p* (*linepath a a*)
proof (*intro exI conjI*)
show *a* ∈ *S*
using *homotopic_with_imp_subset2* [*OF homp*]
by (*metis dist_0_norm image_subset_iff mem_sphere norm_one*)
have *teq*: ⋀*t*. ⟦*0* ≤ *t*; *t* ≤ *1*⟧
⟹ *t* = *Arg2pi* (*exp* (*2* ∗ *of_real pi* ∗ *of_real t* ∗ i)) / (*2* ∗ *pi*) ∨ *t*=*1* ∧ *Arg2pi* (*exp* (*2* ∗ *of_real pi* ∗ *of_real t* ∗ i)) = *0*

    using *Arg2pi_of_real* [*of 1*] **by** (*force simp*: *Arg2pi_exp*)
   have *homotopic_loops S p* (*p* ∘ (λ*z*. *Arg2pi z* / (*2* ∗ *pi*)) ∘ *exp* ∘ (λ*t*. *2* ∗ *complex_of_real pi* ∗ *complex_of_real t* ∗ i))
    using *p teq* **by** (*fastforce simp*: *pathfinish_def pathstart_def intro*: *homotopic_loops_eq* [*OF p*])
  then show *homotopic_loops S p* (*linepath a a*)
   by (*simp add*: *linepath_refl homotopic_loops_trans* [*OF _ homotopic_circlemaps_imp_homotopic_loops* [*OF homp*, *simplified K_record_comp*]])
 qed
qed

proposition *simply_connected_eq_homotopic_circlemaps*:
 fixes *S* :: *'a*::*real_normed_vector set*
 shows *simply_connected S* ⟷
   (∀ *f g*::*complex* ⇒ *'a*.
     continuous_on (*sphere 0 1*) *f* ∧ *f* ' (*sphere 0 1*) ⊆ *S* ∧
     continuous_on (*sphere 0 1*) *g* ∧ *g* ' (*sphere 0 1*) ⊆ *S*
     ⟶ *homotopic_with_canon* (λ*h*. *True*) (*sphere 0 1*) *S f g*)
 apply (*rule iffI*)
  apply (*blast dest*: *simply_connected_eq_homotopic_circlemaps1*)
 by (*simp add*: *simply_connected_eq_homotopic_circlemaps2a simply_connected_eq_homotopic_circlemaps2b simply_connected_eq_homotopic_circlemaps3*)

proposition *simply_connected_eq_contractible_circlemap*:
 fixes *S* :: *'a*::*real_normed_vector set*
 shows *simply_connected S* ⟷
   path_connected S ∧
   (∀ *f*::*complex* ⇒ *'a*.
     continuous_on (*sphere 0 1*) *f* ∧ *f* '(*sphere 0 1*) ⊆ *S*
     ⟶ (∃ *a*. *homotopic_with_canon* (λ*h*. *True*) (*sphere 0 1*) *S f* (λ*x*. *a*)))
 apply (*rule iffI*)
 apply (*simp add*: *simply_connected_eq_homotopic_circlemaps1 simply_connected_eq_homotopic_circlemaps2a simply_connected_eq_homotopic_circlemaps2b*)
 using *simply_connected_eq_homotopic_circlemaps3* **by** *blast*

corollary *homotopy_eqv_simple_connectedness*:
 fixes *S* :: *'a*::*real_normed_vector set* **and** *T* :: *'b*::*real_normed_vector set*
 shows *S homotopy_eqv T* ⟹ *simply_connected S* ⟷ *simply_connected T*
 by (*simp add*: *simply_connected_eq_homotopic_circlemaps homotopy_eqv_homotopic_triviality*)

6.41.7 Homeomorphism of simple closed curves to circles

proposition *homeomorphic_simple_path_image_circle*:
 fixes *a* :: *complex* **and** γ :: *real* ⇒ *'a*::*t2_space*
 assumes *simple_path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ **and** *0* < *r*
 shows (*path_image* γ) *homeomorphic sphere a r*
proof −
 have *homotopic_loops* (*path_image* γ) γ γ

by (*simp add*: *assms homotopic_loops_refl simple_path_imp_path*)
then have *hom*: *homotopic_with_canon* (λ*h*. *True*) (*sphere 0 1*) (*path_image* γ)
(γ ∘ (λ*z*. *Arg2pi z* / (*2*∗*pi*))) (γ ∘ (λ*z*. *Arg2pi z* / (*2*∗*pi*)))
by (*rule homotopic_loops_imp_homotopic_circlemaps*)
have ∃*g*. *homeomorphism* (*sphere 0 1*) (*path_image* γ) (γ ∘ (λ*z*. *Arg2pi z* / (*2*∗*pi*))) *g*
proof (*rule homeomorphism_compact*)
show *continuous_on* (*sphere 0 1*) (γ ∘ (λ*z*. *Arg2pi z* / (*2*∗*pi*)))
using *hom homotopic_with_imp_continuous* **by** *blast*
show *inj_on* (γ ∘ (λ*z*. *Arg2pi z* / (*2*∗*pi*))) (*sphere 0 1*)
proof
fix *x y*
assume *xy*: *x* ∈ *sphere 0 1 y* ∈ *sphere 0 1*
and *eq*: (γ ∘ (λ*z*. *Arg2pi z* / (*2*∗*pi*))) *x* = (γ ∘ (λ*z*. *Arg2pi z* / (*2*∗*pi*))) *y*
then have (*Arg2pi x* / (*2*∗*pi*)) = (*Arg2pi y* / (*2*∗*pi*))
proof −
have (*Arg2pi x* / (*2*∗*pi*)) ∈ {*0..1*} (*Arg2pi y* / (*2*∗*pi*)) ∈ {*0..1*}
using *Arg2pi_ge_0 Arg2pi_lt_2pi dual_order.strict_iff_order* **by** *fastforce+*
with *eq* **show** *?thesis*
using ‹*simple_path* γ› *Arg2pi_lt_2pi* **unfolding** *simple_path_def o_def*
by (*metis eq_divide_eq_1 not_less_iff_gr_or_eq*)
qed
with *xy* **show** *x* = *y*
by (*metis is_Arg_def Arg2pi Arg2pi_0 dist_0_norm divide_cancel_right dual_order.strict_iff_order mem_sphere*)
qed
have ⋀*z*. *cmod z* = *1* ⟹ ∃*x*∈{*0..1*}. γ (*Arg2pi z* / (*2*∗*pi*)) = γ *x*
by (*metis Arg2pi_ge_0 Arg2pi_lt_2pi atLeastAtMost_iff divide_less_eq_1 less_eq_real_def zero_less_mult_iff pi_gt_zero zero_le_divide_iff zero_less_numeral*)
moreover have ∃*z*∈*sphere 0 1*. γ *x* = γ (*Arg2pi z* / (*2*∗*pi*)) **if** *0* ≤ *x x* ≤ *1* **for** *x*
proof (*cases x*=*1*)
case *True*
with *Arg2pi_of_real* [*of 1*] *loop* **show** *?thesis*
by (*rule_tac x*=*1* **in** *bexI*) (*auto simp*: *pathfinish_def pathstart_def* ‹*0* ≤ *x*›)
next
case *False*
then have ∗: (*Arg2pi* (*exp* (i∗(*2*∗ *of_real pi*∗ *of_real x*))) / (*2*∗*pi*)) = *x*
using *that* **by** (*auto simp*: *Arg2pi_exp field_split_simps*)
show *?thesis*
by (*rule_tac x*=*exp*(i ∗ *of_real*(*2*∗*pi*∗*x*)) **in** *bexI*) (*auto simp*: ∗)
qed
ultimately show (γ ∘ (λ*z*. *Arg2pi z* / (*2*∗*pi*))) ‘ *sphere 0 1* = *path_image* γ
by (*auto simp*: *path_image_def image_iff*)
qed *auto*
then have *path_image* γ *homeomorphic sphere* (*0*::*complex*) *1*
using *homeomorphic_def homeomorphic_sym* **by** *blast*
also have ... *homeomorphic sphere a r*
by (*simp add*: *assms homeomorphic_spheres*)

finally show *?thesis* **.**
qed

lemma *homeomorphic_simple_path_images*:
fixes *γ1* :: *real* ⇒ *'a*::*t2_space* **and** *γ2* :: *real* ⇒ *'b*::*t2_space*
assumes *simple_path γ1* **and** *loop*: *pathfinish γ1* = *pathstart γ1*
assumes *simple_path γ2* **and** *loop*: *pathfinish γ2* = *pathstart γ2*
shows (*path_image γ1*) *homeomorphic* (*path_image γ2*)
by (*meson assms homeomorphic_simple_path_image_circle homeomorphic_sym homeomorphic_trans loop pi_gt_zero*)

6.41.8 Dimension-based conditions for various homeomorphisms

lemma *homeomorphic_subspaces_eq*:
fixes *S* :: *'a*::*euclidean_space set* **and** *T* :: *'b*::*euclidean_space set*
assumes *subspace S subspace T*
shows *S homeomorphic T* ⟷ *dim S* = *dim T*
proof
assume *S homeomorphic T*
then obtain *f g* **where** *hom*: *homeomorphism S T f g*
using *homeomorphic_def* **by** *blast*
show *dim S* = *dim T*
proof (*rule order_antisym*)
show *dim S* ≤ *dim T*
by (*metis assms dual_order.refl inj_onI homeomorphism_cont1* [*OF hom*] *homeomorphism_apply1* [*OF hom*] *homeomorphism_image1* [*OF hom*] *continuous_injective_image_subspace_dim_le*)
show *dim T* ≤ *dim S*
by (*metis assms dual_order.refl inj_onI homeomorphism_cont2* [*OF hom*] *homeomorphism_apply2* [*OF hom*] *homeomorphism_image2* [*OF hom*] *continuous_injective_image_subspace_dim_le*)
qed
next
assume *dim S* = *dim T*
then show *S homeomorphic T*
by (*simp add*: *assms homeomorphic_subspaces*)
qed

lemma *homeomorphic_affine_sets_eq*:
fixes *S* :: *'a*::*euclidean_space set* **and** *T* :: *'b*::*euclidean_space set*
assumes *affine S affine T*
shows *S homeomorphic T* ⟷ *aff_dim S* = *aff_dim T*
proof (*cases S* = {} ∨ *T* = {})
case *True*
then show *?thesis*
using *assms homeomorphic_affine_sets* **by** *force*
next
case *False*
then obtain *a b* **where** *a* ∈ *S b* ∈ *T*
by *blast*

then have *subspace* ((+) (− *a*) ‘ *S*) *subspace* ((+) (− *b*) ‘ *T*)
using *affine_diffs_subspace assms* **by** *blast+*
then show *?thesis*
by (*metis affine_imp_convex assms homeomorphic_affine_sets homeomorphic_convex_sets*)
qed

lemma *homeomorphic_hyperplanes_eq*:
fixes *a* :: *'a::euclidean_space* **and** *c* :: *'b::euclidean_space*
assumes $a \neq 0$ $c \neq 0$
shows ({*x*. *a* · *x* = *b*} *homeomorphic* {*x*. *c* · *x* = *d*} ⟷ *DIM*(*'a*) = *DIM*(*'b*))
apply (*auto simp*: *homeomorphic_affine_sets_eq affine_hyperplane assms*)
by (*metis DIM_positive Suc_pred*)

lemma *homeomorphic_UNIV_UNIV*:
shows (*UNIV::'a set*) *homeomorphic* (*UNIV::'b set*) ⟷
DIM(*'a::euclidean_space*) = *DIM*(*'b::euclidean_space*)
by (*simp add*: *homeomorphic_subspaces_eq*)

lemma *simply_connected_sphere_gen*:
assumes *convex S bounded S* **and** *3*: *3* ≤ *aff_dim S*
shows *simply_connected*(*rel_frontier S*)
proof −
have *pa*: *path_connected* (*rel_frontier S*)
using *assms* **by** (*simp add*: *path_connected_sphere_gen*)
show *?thesis*
proof (*clarsimp simp add*: *simply_connected_eq_contractible_circlemap pa*)
fix *f*
assume *f*: *continuous_on* (*sphere* (*0::complex*) *1*) *f f* ‘ *sphere 0 1* ⊆ *rel_frontier S*
have *eq*: *sphere* (*0::complex*) *1* = *rel_frontier*(*cball 0 1*)
by *simp*
have *convex* (*cball* (*0::complex*) *1*)
by (*rule convex_cball*)
then obtain *c* **where** *homotopic_with_canon* (λ*z*. *True*) (*sphere* (*0::complex*) *1*) (*rel_frontier S*) *f* (λ*x*. *c*)
apply (*rule inessential_spheremap_lowdim_gen* [*OF _ bounded_cball* ‹*convex S*› ‹*bounded S*›, **where** *f=f*])
using *f 3*
apply (*auto simp*: *aff_dim_cball*)
done
then show ∃ *a*. *homotopic_with_canon* (λ*h*. *True*) (*sphere 0 1*) (*rel_frontier S*) *f* (λ*x*. *a*)
by *blast*
qed
qed

6.41.9 more invariance of domain

proposition *invariance_of_domain_sphere_affine_set_gen*:

fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *contf*: *continuous_on S f* **and** *injf*: *inj_on f S* **and** *fim*: *f ' S* ⊆ *T*
and *U*: *bounded U convex U*
and *affine T* **and** *affTU*: *aff_dim T* < *aff_dim U*
and *ope*: *openin* (*top_of_set* (*rel_frontier U*)) *S*
shows *openin* (*top_of_set T*) (*f ' S*)
proof (*cases rel_frontier U* = {})
case *True*
then show *?thesis*
using *ope openin_subset* **by** *force*
next
case *False*
obtain *b c* **where** *b*: *b* ∈ *rel_frontier U* **and** *c*: *c* ∈ *rel_frontier U* **and** *b* ≠ *c*
using ‹*bounded U*› *rel_frontier_not_sing* [*of U*] *subset_singletonD False* **by** *fastforce*
obtain *V* :: *'a set* **where** *affine V* **and** *affV*: *aff_dim V* = *aff_dim U* − *1*
proof (*rule choose_affine_subset* [*OF affine_UNIV*])
show − *1* ≤ *aff_dim U* − *1*
by (*metis aff_dim_empty aff_dim_geq aff_dim_negative_iff affTU diff_0 diff_right_mono not_le*)
show *aff_dim U* − *1* ≤ *aff_dim* (*UNIV::'a set*)
by (*metis aff_dim_UNIV aff_dim_le_DIM le_cases not_le zle_diff1_eq*)
qed *auto*
have *SU*: *S* ⊆ *rel_frontier U*
using *ope openin_imp_subset* **by** *auto*
have *homb*: *rel_frontier U* − {*b*} *homeomorphic V*
and *homc*: *rel_frontier U* − {*c*} *homeomorphic V*
using *homeomorphic_punctured_sphere_affine_gen* [*of U _ V*]
by (*simp_all add*: ‹*affine V*› *affV U b c*)
then obtain *g h j k*
where *gh*: *homeomorphism* (*rel_frontier U* − {*b*}) *V g h*
and *jk*: *homeomorphism* (*rel_frontier U* − {*c*}) *V j k*
by (*auto simp*: *homeomorphic_def*)
with *SU* **have** *hgsub*: (*h ' g '* (*S* − {*b*})) ⊆ *S* **and** *kjsub*: (*k ' j '* (*S* − {*c*})) ⊆ *S*
by (*simp_all add*: *homeomorphism_def subset_eq*)
have [*simp*]: *aff_dim T* ≤ *aff_dim V*
by (*simp add*: *affTU affV*)
have *openin* (*top_of_set T*) ((*f* ∘ *h*) *' g '* (*S* − {*b*}))
proof (*rule invariance_of_domain_affine_sets* [*OF _* ‹*affine V*›])
have *openin* (*top_of_set* (*rel_frontier U* − {*b*})) (*S* − {*b*})
by (*meson Diff_mono Diff_subset SU ope openin_delete openin_subset_trans order_refl*)
then show *openin* (*top_of_set V*) (*g '* (*S* − {*b*}))
by (*rule homeomorphism_imp_open_map* [*OF gh*])
show *continuous_on* (*g '* (*S* − {*b*})) (*f* ∘ *h*)
proof (*rule continuous_on_compose*)
show *continuous_on* (*g '* (*S* − {*b*})) *h*
by (*meson Diff_mono SU homeomorphism_def homeomorphism_of_subsets gh set_eq_subset*)

qed (*use contf continuous_on_subset hgsub* **in** *blast*)
show *inj_on* $(f \circ h)$ $(g$ ‘ $(S - \{b\}))$
using *kjsub*
apply (*clarsimp simp add*: *inj_on_def*)
by (*metis SU b homeomorphism_def inj_onD injf insert_Diff insert_iff gh rev_subsetD*)
show $(f \circ h)$ ‘ g ‘ $(S - \{b\}) \subseteq T$
by (*metis fim image_comp image_mono hgsub subset_trans*)
qed (*auto simp*: *assms*)
moreover
have *openin* (*top_of_set* T) $((f \circ k)$ ‘ j ‘ $(S - \{c\}))$
proof (*rule invariance_of_domain_affine_sets* [*OF* _ ‹*affine V*›])
show *openin* (*top_of_set* V) $(j$ ‘ $(S - \{c\}))$
by (*meson Diff_mono Diff_subset SU ope openin_delete openin_subset_trans order_refl homeomorphism_imp_open_map* [*OF jk*])
show *continuous_on* $(j$ ‘ $(S - \{c\}))$ $(f \circ k)$
proof (*rule continuous_on_compose*)
show *continuous_on* $(j$ ‘ $(S - \{c\}))$ k
by (*meson Diff_mono SU homeomorphism_def homeomorphism_of_subsets jk set_eq_subset*)
qed (*use contf continuous_on_subset kjsub* **in** *blast*)
show *inj_on* $(f \circ k)$ $(j$ ‘ $(S - \{c\}))$
using *kjsub*
apply (*clarsimp simp add*: *inj_on_def*)
by (*metis SU c homeomorphism_def inj_onD injf insert_Diff insert_iff jk rev_subsetD*)
show $(f \circ k)$ ‘ j ‘ $(S - \{c\}) \subseteq T$
by (*metis fim image_comp image_mono kjsub subset_trans*)
qed (*auto simp*: *assms*)
ultimately have *openin* (*top_of_set* T) $((f \circ h)$ ‘ g ‘ $(S - \{b\}) \cup ((f \circ k)$ ‘ j ‘ $(S - \{c\})))$
by (*rule openin_Un*)
moreover have $(f \circ h)$ ‘ g ‘ $(S - \{b\}) = f$ ‘ $(S - \{b\})$
proof −
have h ‘ g ‘ $(S - \{b\}) = (S - \{b\})$
proof
show h ‘ g ‘ $(S - \{b\}) \subseteq S - \{b\}$
using *homeomorphism_apply1* [*OF gh*] *SU*
by (*fastforce simp add*: *image_iff image_subset_iff*)
show $S - \{b\} \subseteq h$ ‘ g ‘ $(S - \{b\})$
apply *clarify*
by (*metis SU subsetD homeomorphism_apply1* [*OF gh*] *image_iff member_remove remove_def*)
qed
then show *?thesis*
by (*metis image_comp*)
qed
moreover have $(f \circ k)$ ‘ j ‘ $(S - \{c\}) = f$ ‘ $(S - \{c\})$
proof −

```
    have k ` j ` (S − {c}) = (S − {c})
    proof
      show k ` j ` (S − {c}) ⊆ S − {c}
        using homeomorphism_apply1 [OF jk] SU
        by (fastforce simp add: image_iff image_subset_iff)
      show S − {c} ⊆ k ` j ` (S − {c})
        apply clarify
          by (metis SU subsetD homeomorphism_apply1 [OF jk] image_iff member_remove remove_def)
    qed
    then show ?thesis
      by (metis image_comp)
  qed
  moreover have f ` (S − {b}) ∪ f ` (S − {c}) = f ` (S)
    using ‹b ≠ c› by blast
  ultimately show ?thesis
    by simp
qed


lemma invariance_of_domain_sphere_affine_set:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes contf: continuous_on S f and injf: inj_on f S and fim: f ` S ⊆ T
      and r ≠ 0 affine T and affTU: aff_dim T < DIM('a)
      and ope: openin (top_of_set (sphere a r)) S
   shows openin (top_of_set T) (f ` S)
proof (cases sphere a r = {})
  case True
  then show ?thesis
    using ope openin_subset by force
next
  case False
  show ?thesis
 proof (rule invariance_of_domain_sphere_affine_set_gen [OF contf injf fim bounded_cball convex_cball ‹affine T›])
    show aff_dim T < aff_dim (cball a r)
      by (metis False affTU aff_dim_cball assms(4) linorder_cases sphere_empty)
    show openin (top_of_set (rel_frontier (cball a r))) S
      by (simp add: ‹r ≠ 0› ope)
  qed
qed

lemma no_embedding_sphere_lowdim:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
   assumes contf: continuous_on (sphere a r) f and injf: inj_on f (sphere a r) and r > 0
   shows DIM('a) ≤ DIM('b)
proof −
  have False if DIM('a) > DIM('b)
```

```
  proof -
    have compact (f ` sphere a r)
      using compact_continuous_image
      by (simp add: compact_continuous_image contf)
    then have ¬ open (f ` sphere a r)
      using compact_open
      by (metis assms(3) image_is_empty not_less_iff_gr_or_eq sphere_eq_empty)
    then show False
      using invariance_of_domain_sphere_affine_set [OF contf injf subset_UNIV] ‹r
> 0›
       by (metis aff_dim_UNIV affine_UNIV less_irrefl of_nat_less_iff open_openin
openin_subtopology_self subtopology_UNIV that)
  qed
  then show ?thesis
    using not_less by blast
qed

lemma simply_connected_sphere:
  fixes a :: 'a::euclidean_space
  assumes 3 ≤ DIM('a)
    shows simply_connected(sphere a r)
proof (cases rule: linorder_cases [of r 0])
  case less
  then show ?thesis by simp
next
  case equal
  then show ?thesis  by (auto simp: convex_imp_simply_connected)
next
  case greater
  then show ?thesis
    using simply_connected_sphere_gen [of cball a r] assms
    by (simp add: aff_dim_cball)
qed

lemma simply_connected_sphere_eq:
  fixes a :: 'a::euclidean_space
  shows simply_connected(sphere a r) ⟷ 3 ≤ DIM('a) ∨ r ≤ 0  (is ?lhs = ?rhs)
proof (cases r ≤ 0)
  case True
  have simply_connected (sphere a r)
    using True less_eq_real_def by (auto intro: convex_imp_simply_connected)
  with True show ?thesis by auto
next
  case False
  show ?thesis
  proof
    assume L: ?lhs
    have False if DIM('a) = 1 ∨ DIM('a) = 2
      using that
```

```
    proof
      assume DIM('a) = 1
      with L show False
        using connected_sphere_eq simply_connected_imp_connected
        by (metis False Suc_1 not_less_eq_eq order_refl)
    next
      assume DIM('a) = 2
      then have sphere a r homeomorphic sphere (0::complex) 1
      by (metis DIM_complex False homeomorphic_spheres_gen not_less zero_less_one)
      then have simply_connected(sphere (0::complex) 1)
        using L homeomorphic_simply_connected_eq by blast
      then obtain a::complex where homotopic_with_canon (λh. True) (sphere 0
1) (sphere 0 1) id (λx. a)
      by (metis continuous_on_id' id_apply image_id subset_refl simply_connected_eq_contractible_circlemap)
      then show False
        using contractible_sphere contractible_def not_one_le_zero by blast
    qed
    with False show ?rhs
      apply simp
     by (metis DIM_ge_Suc0 le_antisym not_less_eq_eq numeral_2_eq_2 numeral_3_eq_3)
  next
    assume ?rhs
    with False show ?lhs by (simp add: simply_connected_sphere)
  qed
qed


lemma simply_connected_punctured_universe_eq:
  fixes a :: 'a::euclidean_space
  shows simply_connected(− {a}) ⟷ 3 ≤ DIM('a)
proof −
  have [simp]: a ∈ rel_interior (cball a 1)
    by (simp add: rel_interior_nonempty_interior)
  have [simp]: affine hull cball a 1 − {a} = −{a}
    by (metis Compl_eq_Diff_UNIV aff_dim_cball aff_dim_lt_full not_less_iff_gr_or_eq
zero_less_one)
  have sphere a 1 homotopy_eqv − {a}
    using homotopy_eqv_rel_frontier_punctured_affine_hull [of cball a 1 a] by auto
  then have simply_connected(− {a}) ⟷ simply_connected(sphere a 1)
    using homotopy_eqv_simple_connectedness by blast
  also have ... ⟷ 3 ≤ DIM('a)
    by (simp add: simply_connected_sphere_eq)
  finally show ?thesis .
qed

lemma not_simply_connected_circle:
  fixes a :: complex
  shows 0 < r ⟹ ¬ simply_connected(sphere a r)
by (simp add: simply_connected_sphere_eq)
```

proposition *simply_connected_punctured_convex*:
fixes *a* :: *'a::euclidean_space*
assumes *convex S* **and** *3*: *3* ≤ *aff_dim S*
shows *simply_connected*(*S* − {*a*})
proof (*cases a* ∈ *rel_interior S*)
case *True*
then obtain *e* **where** *a* ∈ *S 0* < *e* **and** *e*: *cball a e* ∩ *affine hull S* ⊆ *S*
by (*auto simp*: *rel_interior_cball*)
have *con*: *convex* (*cball a e* ∩ *affine hull S*)
by (*simp add*: *convex_Int*)
have *bo*: *bounded* (*cball a e* ∩ *affine hull S*)
by (*simp add*: *bounded_Int*)
have *affine hull S* ∩ *interior* (*cball a e*) ≠ {}
using ‹*0* < *e*› ‹*a* ∈ *S*› *hull_subset* **by** *fastforce*
then have *3* ≤ *aff_dim* (*affine hull S* ∩ *cball a e*)
by (*simp add*: *3 aff_dim_convex_Int_nonempty_interior* [*OF convex_affine_hull*])
also have ... = *aff_dim* (*cball a e* ∩ *affine hull S*)
by (*simp add*: *Int_commute*)
finally have *3* ≤ *aff_dim* (*cball a e* ∩ *affine hull S*) **.**
moreover have *rel_frontier* (*cball a e* ∩ *affine hull S*) *homotopy_eqv S* − {*a*}
proof (*rule homotopy_eqv_rel_frontier_punctured_convex*)
show *a* ∈ *rel_interior* (*cball a e* ∩ *affine hull S*)
by (*meson IntI Int_mono* ‹*a* ∈ *S*› ‹*0* < *e*› *e* ‹*cball a e* ∩ *affine hull S* ⊆ *S*›
ball_subset_cball centre_in_cball dual_order.strict_implies_order hull_inc hull_mono mem_rel_interior_ball)
have *closed* (*cball a e* ∩ *affine hull S*)
by *blast*
then show *rel_frontier* (*cball a e* ∩ *affine hull S*) ⊆ *S*
by (*metis Diff_subset closure_closed dual_order.trans e rel_frontier_def*)
show *S* ⊆ *affine hull* (*cball a e* ∩ *affine hull S*)
by (*metis* (*no_types*, *lifting*) *IntI* ‹*a* ∈ *S*› ‹*0* < *e*› *affine_hull_convex_Int_nonempty_interior*
centre_in_ball convex_affine_hull empty_iff hull_subset inf_commute interior_cball sub-
setCE subsetI)
qed (*auto simp*: *assms con bo*)
ultimately show *?thesis*
using *homotopy_eqv_simple_connectedness simply_connected_sphere_gen* [*OF con*
bo]
by *blast*
next
case *False*
then have *rel_interior S* ⊆ *S* − {*a*}
by (*simp add*: *False rel_interior_subset subset_Diff_insert*)
moreover have *S* − {*a*} ⊆ *closure S*
by (*meson Diff_subset closure_subset subset_trans*)
ultimately show *?thesis*
by (*metis contractible_imp_simply_connected contractible_convex_tweak_boundary_points*
[*OF* ‹*convex S*›])
qed

corollary *simply_connected_punctured_universe*:
fixes *a* :: *'a::euclidean_space*
assumes *3* ≤ *DIM('a)*
shows *simply_connected*(− {*a*})
proof −
have [*simp*]: *affine hull cball a 1* = *UNIV*
by (*simp add*: *aff_dim_cball affine_hull_UNIV*)
have *a* ∈ *rel_interior* (*cball a 1*)
by (*simp add*: *rel_interior_interior*)
then
have *simply_connected* (*rel_frontier* (*cball a 1*)) = *simply_connected* (*affine hull cball a 1* − {*a*})
using *homotopy_eqv_rel_frontier_punctured_affine_hull homotopy_eqv_simple_connectedness*
by *blast*
then show *?thesis*
using *simply_connected_sphere* [*of a 1*, *OF assms*] **by** (*auto simp*: *Compl_eq_Diff_UNIV*)
qed

6.41.10 The power, squaring and exponential functions as covering maps

proposition *covering_space_power_punctured_plane*:
assumes *0* < *n*
shows *covering_space* (− {*0*}) (λ*z::complex*. *z*ˆ*n*) (− {*0*})
proof −
consider *n* = *1* | *2* ≤ *n* **using** *assms* **by** *linarith*
then obtain *e* **where** *0* < *e*
and *e*: ⋀*w z*. *cmod*(*w* − *z*) < *e* ∗ *cmod z* ⟹ (*w*ˆ*n* = *z*ˆ*n* ⟷ *w* = *z*)
proof *cases*
assume *n* = *1* **then show** *?thesis*
by (*rule_tac e=1* **in** *that*) *auto*
next
assume *2* ≤ *n*
have *eq_if_pow_eq*:
w = *z* **if** *lt*: *cmod* (*w* − *z*) < *2* ∗ *sin* (*pi* / *real n*) ∗ *cmod z*
and *eq*: *w*ˆ*n* = *z*ˆ*n* **for** *w z*
proof (*cases z* = *0*)
case *True* **with** *eq assms* **show** *?thesis* **by** (*auto simp*: *power_0_left*)
next
case *False*
then have *z* ≠ *0* **by** *auto*
have (*w*/*z*)ˆ*n* = *1*
by (*metis False divide_self_if eq power_divide power_one*)
then obtain *j* **where** *j*: *w* / *z* = *exp* (*2* ∗ *of_real pi* ∗ i ∗ *j* / *n*) **and** *j* < *n*
using *Suc_leI assms* ‹*2* ≤ *n*› *complex_roots_unity* [*THEN eqset_imp_iff*, *of n w*/*z*]
by *force*

```
      have cmod (w/z − 1) < 2 ∗ sin (pi / real n)
        using lt assms ‹z ≠ 0› by (simp add: field_split_simps norm_divide)
      then have cmod (exp (i ∗ of_real (2 ∗ pi ∗ j / n)) − 1) < 2 ∗ sin (pi / real
n)
        by (simp add: j field_simps)
      then have 2 ∗ |sin((2 ∗ pi ∗ j / n) / 2)| < 2 ∗ sin (pi / real n)
        by (simp only: dist_exp_i_1)
      then have sin_less: sin((pi ∗ j / n)) < sin (pi / real n)
        by (simp add: field_simps)
      then have w / z = 1
      proof (cases j = 0)
        case True then show ?thesis by (auto simp: j)
      next
        case False
        then have sin (pi / real n) ≤ sin((pi ∗ j / n))
        proof (cases j / n ≤ 1/2)
          case True
          show ?thesis
            using ‹j ≠ 0 › ‹j < n› True
            by (intro sin_monotone_2pi_le) (auto simp: field_simps intro: order_trans
[of _ 0])
        next
          case False
          then have seq: sin(pi ∗ j / n) = sin(pi ∗ (n − j) / n)
           using ‹j < n› by (simp add: algebra_simps diff_divide_distrib of_nat_diff)
          show ?thesis
            unfolding seq
            using ‹j < n› False
            by (intro sin_monotone_2pi_le) (auto simp: field_simps intro: order_trans
[of _ 0])
        qed
        with sin_less show ?thesis by force
      qed
      then show ?thesis by simp
    qed
    show ?thesis
    proof
      show 0 < 2 ∗ sin (pi / real n)
        by (force simp: ‹2 ≤ n› sin_pi_divide_n_gt_0)
    qed (meson eq_if_pow_eq)
  qed
  have zn1: continuous_on (− {0}) (λz::complex. zˆn)
    by (rule continuous_intros)+
  have zn2: (λz::complex. zˆn) ‘ (− {0}) = − {0}
   using assms by (auto simp: image_def elim: exists_complex_root_nonzero [where
n = n])
  have zn3: ∃ T. zˆn ∈ T ∧ open T ∧ 0 ∉ T ∧
             (∃ v. ⋃v = −{0} ∩ (λz. z ˆ n) −‘ T ∧
                  (∀ u∈v. open u ∧ 0 ∉ u) ∧
```

```
              pairwise disjnt v ∧
              (∀ u∈v. Ex (homeomorphism u T (λz. z^n))))
        if z ≠ 0 for z::complex
  proof –
    define d where d ≡ min (1/2) (e/4) * norm z
    have 0 < d
      by (simp add: d_def ‹0 < e› ‹z ≠ 0›)
    have iff_x_eq_y: x^n = y^n ⟷ x = y
        if eq: w^n = z^n and x: x ∈ ball w d and y: y ∈ ball w d for w x y
    proof –
      have [simp]: norm z = norm w using that
        by (simp add: assms power_eq_imp_eq_norm)
      show ?thesis
      proof (cases w = 0)
        case True with ‹z ≠ 0› assms eq
        show ?thesis by (auto simp: power_0_left)
      next
        case False
        have cmod (x - y) < 2*d
          using x y
            by (simp add: dist_norm [symmetric]) (metis dist_commute mult_2
dist_triangle_less_add)
        also have ... ≤ 2 * e / 4 * norm w
          using ‹e > 0› by (simp add: d_def min_mult_distrib_right)
        also have ... = e * (cmod w / 2)
          by simp
        also have ... ≤ e * cmod y
        proof (rule mult_left_mono)
          have cmod (w - y) < cmod w / 2 ⟹ cmod w / 2 ≤ cmod y
            by (metis (no_types) dist_0_norm dist_norm norm_triangle_half_l not_le
order_less_irrefl)
          then show cmod w / 2 ≤ cmod y
            using y by (simp add: dist_norm d_def min_mult_distrib_right)
        qed (use ‹e > 0› in auto)
        finally have cmod (x - y) < e * cmod y .
        then show ?thesis by (rule e)
      qed
    qed
    then have inj: inj_on (λw. w^n) (ball z d)
      by (simp add: inj_on_def)
    have cont: continuous_on (ball z d) (λw. w ^ n)
      by (intro continuous_intros)
    have noncon: ¬ (λw::complex. w^n) constant_on UNIV
      by (metis UNIV_I assms constant_on_def power_one zero_neq_one zero_power)
    have im_eq: (λw. w^n) ` ball z' d = (λw. w^n) ` ball z d
            if z': z'^n = z^n for z'
    proof –
      have nz': norm z' = norm z using that assms power_eq_imp_eq_norm by blast
      have (w ∈ (λw. w^n) ` ball z' d) = (w ∈ (λw. w^n) ` ball z d) for w
```

proof (*cases w=0*)
case *True* **with** *assms* **show** *?thesis*
by (*simp add*: *image_def ball_def nz′*)
next
case *False*
have $z' \neq 0$ **using** ‹$z \neq 0$› *nz′* **by** *force*
have *1*: $(z{*}x\ /\ z')\hat{}n = x\hat{}n$ **if** $x \neq 0$ **for** *x*
using *z′ that* **by** (*simp add*: *field_simps* ‹$z \neq 0$›)
have *2*: *cmod* $(z - z * x\ /\ z')$ = *cmod* $(z' - x)$ **if** $x \neq 0$ **for** *x*
proof −
have *cmod* $(z - z * x\ /\ z')$ = *cmod z* ∗ *cmod* $(1 - x\ /\ z')$
by (*metis* (*no_types*) *ab_semigroup_mult_class.mult_ac*(*1*) *divide_complex_def mult.right_neutral norm_mult right_diff_distrib′*)
also have ... = *cmod z′* ∗ *cmod* $(1 - x\ /\ z')$
by (*simp add*: *nz′*)
also have ... = *cmod* $(z' - x)$
by (*simp add*: ‹$z' \neq 0$› *diff_divide_eq_iff norm_divide*)
finally show *?thesis* **.**
qed
have *3*: $(z'{*}x\ /\ z)\hat{}n = x\hat{}n$ **if** $x \neq 0$ **for** *x*
using *z′ that* **by** (*simp add*: *field_simps* ‹$z \neq 0$›)
have *4*: *cmod* $(z' - z' * x\ /\ z)$ = *cmod* $(z - x)$ **if** $x \neq 0$ **for** *x*
proof −
have *cmod* (*z* ∗ (*1* − *x* ∗ *inverse z*)) = *cmod* $(z - x)$
by (*metis* ‹$z \neq 0$› *diff_divide_distrib divide_complex_def divide_self_if nonzero_eq_divide_eq semiring_normalization_rules*(*7*))
then show *?thesis*
by (*metis* (*no_types*) *mult.assoc divide_complex_def mult.right_neutral norm_mult nz′ right_diff_distrib′*)
qed
show *?thesis*
by (*simp add*: *set_eq_iff image_def ball_def*) (*metis 1 2 3 4 diff_zero dist_norm nz′*)
qed
then show *?thesis* **by** *blast*
qed

have *ex_ball*: ∃ *B*. (∃ *z′*. *B* = *ball z′ d* ∧ $z'\hat{}n = z\hat{}n$) ∧ *x* ∈ *B*
if $x \neq 0$ **and** *eq*: $x\hat{}n = w\hat{}n$ **and** *dzw*: *dist z w* < *d* **for** *x w*
proof −
have $w \neq 0$ **by** (*metis assms power_eq_0_iff that*(*1*) *that*(*2*))
have [*simp*]: *cmod x* = *cmod w*
using *assms power_eq_imp_eq_norm eq* **by** *blast*
have [*simp*]: *cmod* $(x * z\ /\ w - x)$ = *cmod* $(z - w)$
proof −
have *cmod* $(x * z\ /\ w - x)$ = *cmod x* ∗ *cmod* $(z\ /\ w - 1)$
by (*metis* (*no_types*) *mult.right_neutral norm_mult right_diff_distrib′ times_divide_eq_right*)
also have ... = *cmod w* ∗ *cmod* $(z\ /\ w - 1)$

```
        by simp
      also have ... = cmod (z − w)
        by (simp add: ‹w ≠ 0› divide_diff_eq_iff nonzero_norm_divide)
      finally show ?thesis .
    qed
    show ?thesis
    proof (intro exI conjI)
      show (z / w * x) ^ n = z ^ n
        by (metis ‹w ≠ 0› eq nonzero_eq_divide_eq power_mult_distrib)
      show x ∈ ball (z / w * x) d
        using ‹d > 0› that
          by (simp add: ball_eq_ball_iff ‹z ≠ 0› ‹w ≠ 0› field_simps) (simp add:
dist_norm)
    qed auto
  qed

  show ?thesis
  proof (rule exI, intro conjI)
    show z ^ n ∈ (λw. w ^ n) ` ball z d
      using ‹d > 0› by simp
    show open ((λw. w ^ n) ` ball z d)
      by (rule invariance_of_domain [OF cont open_ball inj])
    show 0 ∉ (λw. w ^ n) ` ball z d
      using ‹z ≠ 0› assms by (force simp: d_def)
    show ∃ v. ⋃ v = − {0} ∩ (λz. z ^ n) −` (λw. w ^ n) ` ball z d ∧
            (∀ u∈v. open u ∧ 0 ∉ u) ∧
            disjoint v ∧
            (∀ u∈v. Ex (homeomorphism u ((λw. w ^ n) ` ball z d) (λz. z ^ n)))
    proof (rule exI, intro ballI conjI)
      show ⋃ {ball z′ d |z′. z′^n = z^n} = − {0} ∩ (λz. z ^ n) −` (λw. w ^ n)
` ball z d (is ?l = ?r)
      proof
        have ⋀z′. cmod z′ < d ⟹ z′ ^ n ≠ z ^ n
          by (auto simp add: assms d_def power_eq_imp_eq_norm that)
        then show ?l ⊆ ?r
          by auto (metis im_eq image_eqI mem_ball)
        show ?r ⊆ ?l
          by auto (meson ex_ball)
      qed
      show ⋀u. u ∈ {ball z′ d |z′. z′ ^ n = z ^ n} ⟹ 0 ∉ u
        by (force simp add: assms d_def power_eq_imp_eq_norm that)

      show disjoint {ball z′ d |z′. z′ ^ n = z ^ n}
      proof (clarsimp simp add: pairwise_def disjnt_iff)
        fix ξ ζ x
        assume ξ^n = z^n ζ^n = z^n ball ξ d ≠ ball ζ d
          and dist ξ x < d dist ζ x < d
        then have dist ξ ζ < d+d
          using dist_triangle_less_add by blast
```

then have *cmod* (ξ − ζ) < *2*∗*d*
by (*simp add*: *dist_norm*)
also have ... ≤ *e* ∗ *cmod z*
using *mult_right_mono* ‹*0* < *e*› *that* **by** (*auto simp*: *d_def*)
finally have *cmod* (ξ − ζ) < *e* ∗ *cmod z* **.**
with *e* **have** ξ = ζ
by (*metis* ‹ξ ˆ*n* = *z*ˆ*n*› ‹ζ ˆ*n* = *z*ˆ*n*› *assms power_eq_imp_eq_norm*)
then show *False*
using ‹*ball* ξ *d* ≠ *ball* ζ *d*› **by** *blast*
qed
show *Ex* (*homeomorphism u* ((λ*w*. *w* ˆ *n*) ‘ *ball z d*) (λ*z*. *z* ˆ *n*))
if *u* ∈ {*ball z*′ *d* |*z*′. *z*′ ˆ *n* = *z* ˆ *n*} **for** *u*
proof (*rule invariance_of_domain_homeomorphism* [*of u* λ*z*. *z*ˆ*n*])
show *open u*
using *that* **by** *auto*
show *continuous_on u* (λ*z*. *z* ˆ *n*)
by (*intro continuous_intros*)
show *inj_on* (λ*z*. *z* ˆ *n*) *u*
using *that* **by** (*auto simp*: *iff_x_eq_y inj_on_def*)
show ⋀*g*. *homeomorphism u* ((λ*z*. *z* ˆ *n*) ‘ *u*) (λ*z*. *z* ˆ *n*) *g* ⟹ *Ex* (*homeomorphism u* ((λ*w*. *w* ˆ *n*) ‘ *ball z d*) (λ*z*. *z* ˆ *n*))
using *im_eq that* **by** *clarify metis*
qed *auto*
qed *auto*
qed
qed
show *?thesis*
using *assms*
apply (*simp add*: *covering_space_def zn1 zn2*)
apply (*subst zn2* [*symmetric*])
apply (*simp add*: *openin_open_eq open_Compl zn3*)
done
qed

corollary *covering_space_square_punctured_plane*:
covering_space (− {*0*}) (λ*z*::*complex*. *z*ˆ*2*) (− {*0*})
by (*simp add*: *covering_space_power_punctured_plane*)

proposition *covering_space_exp_punctured_plane*:
covering_space UNIV (λ*z*::*complex*. *exp z*) (− {*0*})
proof (*simp add*: *covering_space_def*, *intro conjI ballI*)
show *continuous_on UNIV* (λ*z*::*complex*. *exp z*)
by (*rule continuous_on_exp* [*OF continuous_on_id*])
show *range exp* = − {*0*::*complex*}
by *auto* (*metis exp_Ln range_eqI*)
show ∃ *T*. *z* ∈ *T* ∧ *openin* (*top_of_set* (− {*0*})) *T* ∧
(∃ *v*. ⋃*v* = *exp* −‘ *T* ∧ (∀ *u*∈*v*. *open u*) ∧ *disjoint v* ∧
(∀ *u*∈*v*. ∃ *q*. *homeomorphism u T exp q*))

```
      if z ∈ − {0::complex} for z
  proof −
    have z ≠ 0
      using that by auto
    have ball (Ln z) 1 ⊆ ball (Ln z) pi
      using pi_ge_two by (simp add: ball_subset_ball_iff)
    then have inj_exp: inj_on exp (ball (Ln z) 1)
      using inj_on_exp_pi inj_on_subset by blast
     define 𝒱 where 𝒱 ≡ range (λn. (λx. x + of_real (2 ∗ of_int n ∗ pi) ∗ i) ‘
(ball(Ln z) 1))
    show ?thesis
    proof (intro exI conjI)
      show z ∈ exp ‘ (ball(Ln z) 1)
        by (metis ‹z ≠ 0› centre_in_ball exp_Ln rev_image_eqI zero_less_one)
      have open (− {0::complex})
        by blast
      with inj_exp show openin (top_of_set (− {0})) (exp ‘ ball (Ln z) 1)
        by (auto simp: openin_open_eq invariance_of_domain continuous_on_exp [OF
continuous_on_id])
      show ⋃𝒱 = exp −‘ exp ‘ ball (Ln z) 1
        by (force simp: 𝒱_def Complex_Transcendental.exp_eq image_iff)
      show ∀ V∈𝒱. open V
        by (auto simp: 𝒱_def inj_on_def continuous_intros invariance_of_domain)
      have xy: 2 ≤ cmod (2 ∗ of_int x ∗ of_real pi ∗ i − 2 ∗ of_int y ∗ of_real pi ∗
i)
               if x < y for x y
      proof −
        have 1 ≤ abs (x − y)
          using that by linarith
        then have 1 ≤ cmod (of_int x − of_int y) ∗ 1
        by (metis mult.right_neutral norm_of_int of_int_1_le_iff of_int_abs of_int_diff)
        also have ... ≤ cmod (of_int x − of_int y) ∗ of_real pi
          using pi_ge_two
           by (intro mult_left_mono) auto
        also have ... ≤ cmod ((of_int x − of_int y) ∗ of_real pi ∗ i)
          by (simp add: norm_mult)
        also have ... ≤ cmod (of_int x ∗ of_real pi ∗ i − of_int y ∗ of_real pi ∗ i)
          by (simp add: algebra_simps)
       finally have 1 ≤ cmod (of_int x ∗ of_real pi ∗ i − of_int y ∗ of_real pi ∗ i) .
        then have 2 ∗ 1 ≤ cmod (2 ∗ (of_int x ∗ of_real pi ∗ i − of_int y ∗ of_real
pi ∗ i))
          by (metis mult_le_cancel_left_pos norm_mult_numeral1 zero_less_numeral)
        then show ?thesis
          by (simp add: algebra_simps)
      qed
      show disjoint 𝒱
        apply (clarsimp simp add: 𝒱_def pairwise_def disjnt_def add.commute [of _
x∗y for x y]
                        ball_eq_ball_iff intro!: disjoint_ballI)
```

```
      apply (auto simp: dist_norm neq_iff)
      by (metis norm_minus_commute xy)+
    show ∀ u∈𝒱. ∃ q. homeomorphism u (exp ` ball (Ln z) 1) exp q
    proof
      fix u
      assume u ∈ 𝒱
       then obtain n where n: u = (λx. x + of_real (2 * of_int n * pi) * i) `
(ball(Ln z) 1)
        by (auto simp: 𝒱_def)
      have compact (cball (Ln z) 1)
        by simp
      moreover have continuous_on (cball (Ln z) 1) exp
        by (rule continuous_on_exp [OF continuous_on_id])
      moreover have inj_on exp (cball (Ln z) 1)
        apply (rule inj_on_subset [OF inj_on_exp_pi [of Ln z]])
        using pi_ge_two by (simp add: cball_subset_ball_iff)
      ultimately obtain γ where hom: homeomorphism (cball (Ln z) 1) (exp `
cball (Ln z) 1) exp γ
        using homeomorphism_compact by blast
      have eq1: exp ` u = exp ` ball (Ln z) 1
        apply (auto simp: algebra_simps n)
        apply (rule_tac x = _ + i * (of_int n * (of_real pi * 2)) in image_eqI)
        apply (auto simp: image_iff)
        done
      have γexp: γ (exp x) + 2 * of_int n * of_real pi * i = x if x ∈ u for x
      proof −
        have exp x = exp (x − 2 * of_int n * of_real pi * i)
          by (simp add: exp_eq)
        then have γ (exp x) = γ (exp (x − 2 * of_int n * of_real pi * i))
          by simp
        also have ... = x − 2 * of_int n * of_real pi * i
         using ‹x ∈ u› by (auto simp: n intro: homeomorphism_apply1 [OF hom])
        finally show ?thesis
          by simp
      qed
     have exp2n: exp (γ (exp x) + 2 * of_int n * complex_of_real pi * i) = exp x
            if dist (Ln z) x < 1 for x
        using that by (auto simp: exp_eq homeomorphism_apply1 [OF hom])
      have continuous_on (exp ` ball (Ln z) 1) γ
      by (meson ball_subset_cball continuous_on_subset hom homeomorphism_cont2
image_mono)
      then have cont: continuous_on (exp ` ball (Ln z) 1) (λx. γ x + 2 * of_int
n * complex_of_real pi * i)
        by (intro continuous_intros)
      show ∃ q. homeomorphism u (exp ` ball (Ln z) 1) exp q
        apply (rule_tac x=(λx. x + of_real(2 * n * pi) * i) ∘ γ in exI)
        unfolding homeomorphism_def
        apply (intro conjI ballI eq1 continuous_on_exp [OF continuous_on_id])
          apply (auto simp: γexp exp2n cont n)
```

```
          apply (force simp: image_iff homeomorphism_apply1 [OF hom])+
         done
     qed
   qed
  qed
qed
```

6.41.11 Hence the Borsukian results about mappings into circles

```
lemma inessential_eq_continuous_logarithm:
  fixes f :: 'a::real_normed_vector ⇒ complex
  shows (∃ a. homotopic_with_canon (λh. True) S (−{0}) f (λt. a)) ⟷
         (∃ g. continuous_on S g ∧ (∀ x ∈ S. f x = exp(g x)))
  (is ?lhs ⟷ ?rhs)
proof
  assume ?lhs thus ?rhs
   by (metis covering_space_lift_inessential_function covering_space_exp_punctured_plane)
next
  assume ?rhs
  then obtain g where contg: continuous_on S g and f: ⋀x. x ∈ S ⟹ f x =
exp(g x)
    by metis
  obtain a where homotopic_with_canon (λh. True) S (− {of_real 0}) (exp ∘ g)
(λx. a)
  proof (rule nullhomotopic_through_contractible [OF contg subset_UNIV _ _ con-
tractible_UNIV])
    show continuous_on (UNIV::complex set) exp
      by (intro continuous_intros)
    show range exp ⊆ − {0}
      by auto
  qed force
  then have homotopic_with_canon (λh. True) S (− {0}) f (λt. a)
    using f homotopic_with_eq by fastforce
  then show ?lhs ..
qed

corollary inessential_imp_continuous_logarithm_circle:
  fixes f :: 'a::real_normed_vector ⇒ complex
  assumes homotopic_with_canon (λh. True) S (sphere 0 1) f (λt. a)
  obtains g where continuous_on S g and ⋀x. x ∈ S ⟹ f x = exp(g x)
proof −
  have homotopic_with_canon (λh. True) S (− {0}) f (λt. a)
    using assms homotopic_with_subset_right by fastforce
  then show ?thesis
    by (metis inessential_eq_continuous_logarithm that)
qed
```

lemma *inessential_eq_continuous_logarithm_circle*:
fixes f :: *'a::real_normed_vector* ⇒ *complex*
shows (∃ *a*. *homotopic_with_canon* (λ*h*. *True*) *S* (*sphere 0 1*) *f* (λ*t*. *a*)) ⟷
(∃ *g*. *continuous_on S g* ∧ (∀ *x* ∈ *S*. *f x* = *exp*(i ∗ *of_real*(*g x*))))
(**is** *?lhs* ⟷ *?rhs*)
proof
assume *L*: *?lhs*
then obtain *g* **where** *contg*: *continuous_on S g* **and** *g*: ⋀*x*. *x* ∈ *S* ⟹ *f x* = *exp*(*g x*)
using *inessential_imp_continuous_logarithm_circle* **by** *blast*
have *f* ' *S* ⊆ *sphere 0 1*
by (*metis L homotopic_with_imp_subset1*)
then have ⋀*x*. *x* ∈ *S* ⟹ *Re* (*g x*) = *0*
using *g* **by** *auto*
then show *?rhs*
by (*rule_tac x=Im* ∘ *g* **in** *exI*) (*auto simp*: *Euler g intro*: *contg continuous_intros*)
next
assume *?rhs*
then obtain *g* **where** *contg*: *continuous_on S g* **and** *g*: ⋀*x*. *x* ∈ *S* ⟹ *f x* = *exp*(i∗ *of_real*(*g x*))
by *metis*
obtain *a* **where** *homotopic_with_canon* (λ*h*. *True*) *S* (*sphere 0 1*) ((*exp* ∘ (λ*z*. i∗*z*)) ∘ (*of_real* ∘ *g*)) (λ*x*. *a*)
proof (*rule nullhomotopic_through_contractible*)
show *continuous_on S* (*complex_of_real* ∘ *g*)
by (*intro conjI contg continuous_intros*)
show (*complex_of_real* ∘ *g*) ' *S* ⊆ ℝ
by *auto*
show *continuous_on* ℝ (*exp* ∘ (∗)i)
by (*intro continuous_intros*)
show (*exp* ∘ (∗)i) ' ℝ ⊆ *sphere 0 1*
by (*auto simp*: *complex_is_Real_iff*)
qed (*auto simp*: *convex_Reals convex_imp_contractible*)
moreover have ⋀*x*. *x* ∈ *S* ⟹ (*exp* ∘ (∗)i ∘ (*complex_of_real* ∘ *g*)) *x* = *f x*
by (*simp add*: *g*)
ultimately have *homotopic_with_canon* (λ*h*. *True*) *S* (*sphere 0 1*) *f* (λ*t*. *a*)
using *homotopic_with_eq* **by** *force*
then show *?lhs* **..**
qed

proposition *homotopic_with_sphere_times*:
fixes *f* :: *'a::real_normed_vector* ⇒ *complex*
assumes *hom*: *homotopic_with_canon* (λ*x*. *True*) *S* (*sphere 0 1*) *f g* **and** *conth*: *continuous_on S h*
and *hin*: ⋀*x*. *x* ∈ *S* ⟹ *h x* ∈ *sphere 0 1*
shows *homotopic_with_canon* (λ*x*. *True*) *S* (*sphere 0 1*) (λ*x*. *f x* ∗ *h x*) (λ*x*. *g x* ∗ *h x*)
proof −
obtain *k* **where** *contk*: *continuous_on* ({*0..1*::*real*} × *S*) *k*

```
          and kim: k ‘ ({0..1} × S) ⊆ sphere 0 1
          and k0:  ⋀x. k(0, x) = f x
          and k1: ⋀x. k(1, x) = g x
    using hom by (auto simp: homotopic_with_def)
  show ?thesis
    apply (simp add: homotopic_with)
    apply (rule_tac x=λz. k z∗(h ∘ snd)z in exI)
    using kim hin by (fastforce simp: conth norm_mult k0 k1 intro!: contk contin-
uous_intros)+
qed

proposition homotopic_circlemaps_divide:
  fixes f :: 'a::real_normed_vector ⇒ complex
    shows homotopic_with_canon (λx. True) S (sphere 0 1) f g ⟷
           continuous_on S f ∧ f ‘ S ⊆ sphere 0 1 ∧
           continuous_on S g ∧ g ‘ S ⊆ sphere 0 1 ∧
          (∃ c. homotopic_with_canon (λx. True) S (sphere 0 1) (λx. f x / g x) (λx.
c))
proof −
  have homotopic_with_canon (λx. True) S (sphere 0 1) (λx. f x / g x) (λx. 1)
       if homotopic_with_canon (λx. True) S (sphere 0 1) (λx. f x / g x) (λx. c)
for c
  proof −
    have S = {} ∨ path_component (sphere 0 1) 1 c
    using homotopic_with_imp_subset2 [OF that] path_connected_sphere [of 0::complex
1]
      by (auto simp: path_connected_component)
    then have homotopic_with_canon (λx. True) S (sphere 0 1) (λx. 1) (λx. c)
      by (simp add: homotopic_constant_maps)
    then show ?thesis
      using homotopic_with_symD homotopic_with_trans that by blast
  qed
  then have ∗: (∃ c. homotopic_with_canon (λx. True) S (sphere 0 1) (λx. f x / g
x) (λx. c)) ⟷
              homotopic_with_canon (λx. True) S (sphere 0 1) (λx. f x / g x) (λx.
1)
    by auto
  have homotopic_with_canon (λx. True) S (sphere 0 1) f g ⟷
           continuous_on S f ∧ f ‘ S ⊆ sphere 0 1 ∧
           continuous_on S g ∧ g ‘ S ⊆ sphere 0 1 ∧
           homotopic_with_canon (λx. True) S (sphere 0 1) (λx. f x / g x) (λx. 1)
        (is ?lhs ⟷ ?rhs)
  proof
    assume L: ?lhs
    have geq1 [simp]: ⋀x. x ∈ S ⟹ cmod (g x) = 1
      using homotopic_with_imp_subset2 [OF L]
      by (simp add: image_subset_iff)
    have cont: continuous_on S (inverse ∘ g)
    proof (rule continuous_intros)
```

show *continuous_on S g*
  using *homotopic_with_imp_continuous* [*OF L*] **by** *blast*
show *continuous_on* (*g ‘ S*) *inverse*
  by (*rule continuous_on_subset* [*of sphere 0 1*, *OF continuous_on_inverse*]) *auto*
qed
have [*simp*]: $\bigwedge x.\ x \in S \Longrightarrow g\ x \neq 0$
  using *geq1* **by** *fastforce*
have *homotopic_with_canon* ($\lambda x.$ *True*) *S* (*sphere 0 1*) ($\lambda x.$ *f x* / *g x*) ($\lambda x.$ *1*)
  apply (*rule homotopic_with_eq* [*OF homotopic_with_sphere_times* [*OF L cont*]])
  by (*auto simp*: *divide_inverse norm_inverse*)
with *L* **show** *?rhs*
  by (*auto simp*: *homotopic_with_imp_continuous dest*: *homotopic_with_imp_subset1 homotopic_with_imp_subset2*)
next
  assume *?rhs* **then show** *?lhs*
    by (*elim conjE homotopic_with_eq* [*OF homotopic_with_sphere_times*]; *force*)
qed
then show *?thesis*
  by (*simp add*: ∗)
qed

6.41.12 Upper and lower hemicontinuous functions

And relation in the case of preimage map to open and closed maps, and fact that upper and lower hemicontinuity together imply continuity in the sense of the Hausdorff metric (at points where the function gives a bounded and nonempty set).

Many similar proofs below.

lemma *upper_hemicontinuous*:
  assumes $\bigwedge x.\ x \in S \Longrightarrow f\ x \subseteq T$
  shows (($\forall U$. *openin* (*top_of_set T*) *U*
      $\longrightarrow$ *openin* (*top_of_set S*) $\{x \in S.\ f\ x \subseteq U\}$) $\longleftrightarrow$
    ($\forall U$. *closedin* (*top_of_set T*) *U*
      $\longrightarrow$ *closedin* (*top_of_set S*) $\{x \in S.\ f\ x \cap U \neq \{\}\}$))
    (**is** *?lhs* = *?rhs*)
proof (*intro iffI allI impI*)
  fix *U*
  assume ∗ [*rule_format*]: *?lhs* **and** *closedin* (*top_of_set T*) *U*
  then have *openin* (*top_of_set T*) (*T* − *U*)
    by (*simp add*: *openin_diff*)
  then have *openin* (*top_of_set S*) $\{x \in S.\ f\ x \subseteq T - U\}$
    using ∗ [*of T*−*U*] **by** *blast*
  moreover have $S - \{x \in S.\ f\ x \subseteq T - U\} = \{x \in S.\ f\ x \cap U \neq \{\}\}$
    using *assms* **by** *blast*
  ultimately show *closedin* (*top_of_set S*) $\{x \in S.\ f\ x \cap U \neq \{\}\}$
    by (*simp add*: *openin_closedin_eq*)
next

fix *U*
assume ∗ [*rule_format*]: *?rhs* **and** *openin* (*top_of_set T*) *U*
then have *closedin* (*top_of_set T*) (*T* − *U*)
by (*simp add*: *closedin_diff*)
then have *closedin* (*top_of_set S*) {*x* ∈ *S*. *f x* ∩ (*T* − *U*) ≠ {}}
using ∗ [*of T*−*U*] **by** *blast*
moreover have {*x* ∈ *S*. *f x* ∩ (*T* − *U*) ≠ {}} = *S* − {*x* ∈ *S*. *f x* ⊆ *U*}
using *assms* **by** *auto*
ultimately show *openin* (*top_of_set S*) {*x* ∈ *S*. *f x* ⊆ *U*}
by (*simp add*: *openin_closedin_eq*)
qed

lemma *lower_hemicontinuous*:
assumes ⋀*x*. *x* ∈ *S* ⟹ *f x* ⊆ *T*
shows ((∀ *U*. *closedin* (*top_of_set T*) *U*
⟶ *closedin* (*top_of_set S*) {*x* ∈ *S*. *f x* ⊆ *U*}) ⟷
(∀ *U*. *openin* (*top_of_set T*) *U*
⟶ *openin* (*top_of_set S*) {*x* ∈ *S*. *f x* ∩ *U* ≠ {}}))
(**is** *?lhs* = *?rhs*)
proof (*intro iffI allI impI*)
fix *U*
assume ∗ [*rule_format*]: *?lhs* **and** *openin* (*top_of_set T*) *U*
then have *closedin* (*top_of_set T*) (*T* − *U*)
by (*simp add*: *closedin_diff*)
then have *closedin* (*top_of_set S*) {*x* ∈ *S*. *f x* ⊆ *T*−*U*}
using ∗ [*of T*−*U*] **by** *blast*
moreover have {*x* ∈ *S*. *f x* ⊆ *T*−*U*} = *S* − {*x* ∈ *S*. *f x* ∩ *U* ≠ {}}
using *assms* **by** *auto*
ultimately show *openin* (*top_of_set S*) {*x* ∈ *S*. *f x* ∩ *U* ≠ {}}
by (*simp add*: *openin_closedin_eq*)
next
fix *U*
assume ∗ [*rule_format*]: *?rhs* **and** *closedin* (*top_of_set T*) *U*
then have *openin* (*top_of_set T*) (*T* − *U*)
by (*simp add*: *openin_diff*)
then have *openin* (*top_of_set S*) {*x* ∈ *S*. *f x* ∩ (*T* − *U*) ≠ {}}
using ∗ [*of T*−*U*] **by** *blast*
moreover have *S* − {*x* ∈ *S*. *f x* ∩ (*T* − *U*) ≠ {}} = {*x* ∈ *S*. *f x* ⊆ *U*}
using *assms* **by** *blast*
ultimately show *closedin* (*top_of_set S*) {*x* ∈ *S*. *f x* ⊆ *U*}
by (*simp add*: *openin_closedin_eq*)
qed

lemma *open_map_iff_lower_hemicontinuous_preimage*:
assumes *f* ‘ *S* ⊆ *T*
shows ((∀ *U*. *openin* (*top_of_set S*) *U*
⟶ *openin* (*top_of_set T*) (*f* ‘ *U*)) ⟷
(∀ *U*. *closedin* (*top_of_set S*) *U*
⟶ *closedin* (*top_of_set T*) {*y* ∈ *T*. {*x*. *x* ∈ *S* ∧ *f x* = *y*} ⊆ *U*}))

(**is** *?lhs* = *?rhs*)
proof (*intro iffI allI impI*)
fix *U*
assume ∗ [*rule_format*]: *?lhs* **and** *closedin* (*top_of_set S*) *U*
then have *openin* (*top_of_set S*) (*S* − *U*)
by (*simp add*: *openin_diff*)
then have *openin* (*top_of_set T*) (*f* ‘ (*S* − *U*))
using ∗ [*of S*−*U*] **by** *blast*
moreover have *T* − (*f* ‘ (*S* − *U*)) = {*y* ∈ *T*. {*x* ∈ *S*. *f x* = *y*} ⊆ *U*}
using *assms* **by** *blast*
ultimately show *closedin* (*top_of_set T*) {*y* ∈ *T*. {*x* ∈ *S*. *f x* = *y*} ⊆ *U*}
by (*simp add*: *openin_closedin_eq*)
next
fix *U*
assume ∗ [*rule_format*]: *?rhs* **and** *opeSU*: *openin* (*top_of_set S*) *U*
then have *closedin* (*top_of_set S*) (*S* − *U*)
by (*simp add*: *closedin_diff*)
then have *closedin* (*top_of_set T*) {*y* ∈ *T*. {*x* ∈ *S*. *f x* = *y*} ⊆ *S* − *U*}
using ∗ [*of S*−*U*] **by** *blast*
moreover have {*y* ∈ *T*. {*x* ∈ *S*. *f x* = *y*} ⊆ *S* − *U*} = *T* − (*f* ‘ *U*)
using *assms openin_imp_subset* [*OF opeSU*] **by** *auto*
ultimately show *openin* (*top_of_set T*) (*f* ‘ *U*)
using *assms openin_imp_subset* [*OF opeSU*] **by** (*force simp*: *openin_closedin_eq*)
qed

lemma *closed_map_iff_upper_hemicontinuous_preimage*:
assumes *f* ‘ *S* ⊆ *T*
shows ((∀ *U*. *closedin* (*top_of_set S*) *U*
⟶ *closedin* (*top_of_set T*) (*f* ‘ *U*)) ⟷
(∀ *U*. *openin* (*top_of_set S*) *U*
⟶ *openin* (*top_of_set T*) {*y* ∈ *T*. {*x*. *x* ∈ *S* ∧ *f x* = *y*} ⊆ *U*}))
(**is** *?lhs* = *?rhs*)
proof (*intro iffI allI impI*)
fix *U*
assume ∗ [*rule_format*]: *?lhs* **and** *opeSU*: *openin* (*top_of_set S*) *U*
then have *closedin* (*top_of_set S*) (*S* − *U*)
by (*simp add*: *closedin_diff*)
then have *closedin* (*top_of_set T*) (*f* ‘ (*S* − *U*))
using ∗ [*of S*−*U*] **by** *blast*
moreover have *f* ‘ (*S* − *U*) = *T* − {*y* ∈ *T*. {*x*. *x* ∈ *S* ∧ *f x* = *y*} ⊆ *U*}
using *assms openin_imp_subset* [*OF opeSU*] **by** *auto*
ultimately show *openin* (*top_of_set T*) {*y* ∈ *T*. {*x*. *x* ∈ *S* ∧ *f x* = *y*} ⊆ *U*}
using *assms openin_imp_subset* [*OF opeSU*] **by** (*force simp*: *openin_closedin_eq*)
next
fix *U*
assume ∗ [*rule_format*]: *?rhs* **and** *cloSU*: *closedin* (*top_of_set S*) *U*
then have *openin* (*top_of_set S*) (*S* − *U*)
by (*simp add*: *openin_diff*)
then have *openin* (*top_of_set T*) {*y* ∈ *T*. {*x* ∈ *S*. *f x* = *y*} ⊆ *S* − *U*}

```
    using ∗ [of S−U] by blast
  moreover have (f ‘ U) = T − {y ∈ T. {x ∈ S. f x = y} ⊆ S − U}
    using assms closedin_imp_subset [OF cloSU] by auto
  ultimately show closedin (top_of_set T) (f ‘ U)
    by (simp add: openin_closedin_eq)
qed

proposition upper_lower_hemicontinuous_explicit:
  fixes T :: ('b::{real_normed_vector,heine_borel}) set
  assumes fST: ⋀x. x ∈ S ⟹ f x ⊆ T
      and ope: ⋀U. openin (top_of_set T) U
                    ⟹ openin (top_of_set S) {x ∈ S. f x ⊆ U}
      and clo: ⋀U. closedin (top_of_set T) U
                    ⟹ closedin (top_of_set S) {x ∈ S. f x ⊆ U}
      and x ∈ S 0 < e and bofx: bounded(f x) and fx_ne: f x ≠ {}
  obtains d where 0 < d
           ⋀x'. ⟦x' ∈ S; dist x x' < d⟧
                    ⟹ (∀ y ∈ f x. ∃ y'. y' ∈ f x' ∧ dist y y' < e) ∧
                       (∀ y' ∈ f x'. ∃ y. y ∈ f x ∧ dist y' y < e)
proof −
  have openin (top_of_set T) (T ∩ (⋃a∈f x. ⋃b∈ball 0 e. {a + b}))
    by (auto simp: open_sums openin_open_Int)
  with ope have openin (top_of_set S)
                  {u ∈ S. f u ⊆ T ∩ (⋃a∈f x. ⋃b∈ball 0 e. {a + b})} by blast
  with ‹0 < e› ‹x ∈ S› obtain d1 where d1 > 0 and
        d1: ⋀x'. ⟦x' ∈ S; dist x' x < d1⟧ ⟹ f x' ⊆ T ∧ f x' ⊆ (⋃a ∈ f x. ⋃b ∈
ball 0 e. {a + b})
    by (force simp: openin_euclidean_subtopology_iff dest: fST)
  have oo: ⋀U. openin (top_of_set T) U ⟹
                openin (top_of_set S) {x ∈ S. f x ∩ U ≠ {}}
    apply (rule lower_hemicontinuous [THEN iffD1, rule_format])
    using fST clo by auto
  have compact (closure(f x))
    by (simp add: bofx)
  moreover have closure(f x) ⊆ (⋃a ∈ f x. ball a (e/2))
   using ‹0 < e› by (force simp: closure_approachable simp del: divide_const_simps)
  ultimately obtain C where C ⊆ f x finite C closure(f x) ⊆ (⋃a ∈ C. ball a
(e/2))
    apply (rule compactE, force)
    by (metis finite_subset_image)
  then have fx_cover: f x ⊆ (⋃a ∈ C. ball a (e/2))
    by (meson closure_subset order_trans)
  with fx_ne have C ≠ {}
    by blast
  have xin: x ∈ (⋂a ∈ C. {x ∈ S. f x ∩ T ∩ ball a (e/2) ≠ {}})
    using ‹x ∈ S› ‹0 < e› fST ‹C ⊆ f x› by force
  have openin (top_of_set S) {x ∈ S. f x ∩ (T ∩ ball a (e/2)) ≠ {}} for a
    by (simp add: openin_open_Int oo)
  then have openin (top_of_set S) (⋂a ∈ C. {x ∈ S. f x ∩ T ∩ ball a (e/2) ≠
```

{}})
    by (*simp add*: *Int_assoc openin_INT2* [*OF* ‹*finite C*› ‹*C* ≠ {}›])
  with *xin* **obtain** *d2* **where** *d2*>*0*
             and *d2*: ⋀*u v*. ⟦*u* ∈ *S*; *dist u x* < *d2*; *v* ∈ *C*⟧ ⟹ *f u* ∩ *T* ∩ *ball v* (*e*/*2*) ≠ {}
    unfolding *openin_euclidean_subtopology_iff* **using** *xin* **by** *fastforce*
  show *?thesis*
  proof (*intro that conjI ballI*)
    show *0* < *min d1 d2*
      using ‹*0* < *d1*› ‹*0* < *d2*› **by** *linarith*
  next
    fix *x*′ *y*
    assume *x*′ ∈ *S dist x x*′ < *min d1 d2 y* ∈ *f x*
    then have *dd2*: *dist x*′ *x* < *d2*
      by (*auto simp*: *dist_commute*)
    obtain *a* **where** *a* ∈ *C y* ∈ *ball a* (*e*/*2*)
      using *fx_cover* ‹*y* ∈ *f x*› **by** *auto*
    then show ∃ *y*′. *y*′ ∈ *f x*′ ∧ *dist y y*′ < *e*
      using *d2* [*OF* ‹*x*′ ∈ *S*› *dd2*] *dist_triangle_half_r* **by** *fastforce*
  next
    fix *x*′ *y*′
    assume *x*′ ∈ *S dist x x*′ < *min d1 d2 y*′ ∈ *f x*′
    then have *dist x*′ *x* < *d1*
      by (*auto simp*: *dist_commute*)
    then have *y*′ ∈ (⋃ *a*∈*f x*. ⋃ *b*∈*ball 0 e*. {*a* + *b*})
      using *d1* [*OF* ‹*x*′ ∈ *S*›] ‹*y*′ ∈ *f x*′› **by** *force*
    then show ∃ *y*. *y* ∈ *f x* ∧ *dist y*′ *y* < *e*
      by *clarsimp* (*metis add_diff_cancel_left*′ *dist_norm*)
  qed
qed

6.41.13 Complex logs exist on various "well-behaved" sets

lemma *continuous_logarithm_on_contractible*:
  fixes *f* :: ′*a*::*real_normed_vector* ⇒ *complex*
  assumes *continuous_on S f contractible S* ⋀*z*. *z* ∈ *S* ⟹ *f z* ≠ *0*
  obtains *g* **where** *continuous_on S g* ⋀*x*. *x* ∈ *S* ⟹ *f x* = *exp*(*g x*)
proof −
  obtain *c* **where** *hom*: *homotopic_with_canon* (λ*h*. *True*) *S* (−{*0*}) *f* (λ*x*. *c*)
    using *nullhomotopic_from_contractible assms*
    by (*metis imageE subset_Compl_singleton*)
  then show *?thesis*
    by (*metis inessential_eq_continuous_logarithm that*)
qed

lemma *continuous_logarithm_on_simply_connected*:
  fixes *f* :: ′*a*::*real_normed_vector* ⇒ *complex*
  assumes *contf*: *continuous_on S f* **and** *S*: *simply_connected S locally path_connected S*

and *f*: ⋀*z*. *z* ∈ *S* ⟹ *f z* ≠ *0*
obtains *g* **where** *continuous_on S g* ⋀*x*. *x* ∈ *S* ⟹ *f x* = *exp*(*g x*)
using *covering_space_lift* [*OF covering_space_exp_punctured_plane S contf*]
by (*metis* (*full_types*) *f imageE subset_Compl_singleton*)

lemma *continuous_logarithm_on_cball*:
fixes *f* :: *'a*::*real_normed_vector* ⇒ *complex*
assumes *continuous_on* (*cball a r*) *f* **and** ⋀*z*. *z* ∈ *cball a r* ⟹ *f z* ≠ *0*
obtains *h* **where** *continuous_on* (*cball a r*) *h* ⋀*z*. *z* ∈ *cball a r* ⟹ *f z* = *exp*(*h z*)
using *assms continuous_logarithm_on_contractible convex_imp_contractible* **by** *blast*

lemma *continuous_logarithm_on_ball*:
fixes *f* :: *'a*::*real_normed_vector* ⇒ *complex*
assumes *continuous_on* (*ball a r*) *f* **and** ⋀*z*. *z* ∈ *ball a r* ⟹ *f z* ≠ *0*
obtains *h* **where** *continuous_on* (*ball a r*) *h* ⋀*z*. *z* ∈ *ball a r* ⟹ *f z* = *exp*(*h z*)
using *assms continuous_logarithm_on_contractible convex_imp_contractible* **by** *blast*

lemma *continuous_sqrt_on_contractible*:
fixes *f* :: *'a*::*real_normed_vector* ⇒ *complex*
assumes *continuous_on S f contractible S*
and ⋀*z*. *z* ∈ *S* ⟹ *f z* ≠ *0*
obtains *g* **where** *continuous_on S g* ⋀*x*. *x* ∈ *S* ⟹ *f x* = (*g x*) ˆ *2*
proof −
obtain *g* **where** *contg*: *continuous_on S g* **and** *feq*: ⋀*x*. *x* ∈ *S* ⟹ *f x* = *exp*(*g x*)
using *continuous_logarithm_on_contractible* [*OF assms*] **by** *blast*
show *?thesis*
proof
show *continuous_on S* (λ*z*. *exp* (*g z* / *2*))
by (*rule continuous_on_compose2* [*of UNIV exp*]; *intro continuous_intros contg subset_UNIV*) *auto*
show ⋀*x*. *x* ∈ *S* ⟹ *f x* = (*exp* (*g x* / *2*))^2
by (*metis exp_double feq nonzero_mult_div_cancel_left times_divide_eq_right zero_neq_numeral*)
qed
qed

lemma *continuous_sqrt_on_simply_connected*:
fixes *f* :: *'a*::*real_normed_vector* ⇒ *complex*
assumes *contf*: *continuous_on S f* **and** *S*: *simply_connected S locally path_connected S*
and *f*: ⋀*z*. *z* ∈ *S* ⟹ *f z* ≠ *0*
obtains *g* **where** *continuous_on S g* ⋀*x*. *x* ∈ *S* ⟹ *f x* = (*g x*) ˆ *2*
proof −
obtain *g* **where** *contg*: *continuous_on S g* **and** *feq*: ⋀*x*. *x* ∈ *S* ⟹ *f x* = *exp*(*g x*)
using *continuous_logarithm_on_simply_connected* [*OF assms*] **by** *blast*
show *?thesis*

proof
  show *continuous_on S* (*λz. exp* (*g z / 2*))
   by (*rule continuous_on_compose2* [*of UNIV exp*]; *intro continuous_intros contg subset_UNIV*) *auto*
  show ⋀*x. x* ∈ *S* ⟹ *f x* = (*exp* (*g x / 2*))^2
    by (*metis exp_double feq nonzero_mult_div_cancel_left times_divide_eq_right zero_neq_numeral*)
 qed
qed

6.41.14 Another simple case where sphere maps are nullhomotopic

lemma *inessential_spheremap_2_aux*:
 fixes *f* :: ′*a*::*euclidean_space* ⇒ *complex*
 assumes *2*: *2* < *DIM*(′*a*) **and** *contf*: *continuous_on* (*sphere a r*) *f*
   and *fim*: *f* ‘(*sphere a r*) ⊆ (*sphere 0 1*)
 obtains *c* **where** *homotopic_with_canon* (*λz. True*) (*sphere a r*) (*sphere 0 1*) *f* (*λx. c*)
proof −
 obtain *g* **where** *contg*: *continuous_on* (*sphere a r*) *g*
      and *feq*: ⋀*x. x* ∈ *sphere a r* ⟹ *f x* = *exp*(*g x*)
 proof (*rule continuous_logarithm_on_simply_connected* [*OF contf*])
  show *simply_connected* (*sphere a r*)
   using *2* **by** (*simp add*: *simply_connected_sphere_eq*)
  show *locally path_connected* (*sphere a r*)
   by (*simp add*: *locally_path_connected_sphere*)
  show ⋀*z*. *z* ∈ *sphere a r* ⟹ *f z* ≠ *0*
   using *fim* **by** *force*
 qed *auto*
 have ∃ *g*. *continuous_on* (*sphere a r*) *g* ∧ (∀ *x*∈*sphere a r*. *f x* = *exp* (i ∗ *complex_of_real* (*g x*)))
 proof (*intro exI conjI*)
  show *continuous_on* (*sphere a r*) (*Im* ∘ *g*)
   by (*intro contg continuous_intros continuous_on_compose*)
  show ∀ *x*∈*sphere a r*. *f x* = *exp* (i ∗ *complex_of_real* ((*Im* ∘ *g*) *x*))
   using *exp_eq_polar feq fim norm_exp_eq_Re* **by** *auto*
 qed
 with *inessential_eq_continuous_logarithm_circle that* **show** *?thesis*
  by *metis*
qed

lemma *inessential_spheremap_2*:
 fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
 assumes *a2*: *2* < *DIM*(′*a*) **and** *b2*: *DIM*(′*b*) = *2*
   and *contf*: *continuous_on* (*sphere a r*) *f* **and** *fim*: *f* ‘(*sphere a r*) ⊆ (*sphere b s*)
 obtains *c* **where** *homotopic_with_canon* (*λz. True*) (*sphere a r*) (*sphere b s*) *f* (*λx. c*)

proof (*cases s ≤ 0*)
  case *True*
  then show *?thesis*
    using *contf contractible_sphere fim nullhomotopic_into_contractible that* **by** *blast*
next
  case *False*
  then have *sphere b s homeomorphic sphere (0::complex) 1*
    using *assms* **by** (*simp add*: *homeomorphic_spheres_gen*)
  then obtain *h k* **where** *hk*: *homeomorphism* (*sphere b s*) (*sphere* (*0*::*complex*) *1*) *h k*
    by (*auto simp*: *homeomorphic_def*)
  then have *conth*: *continuous_on* (*sphere b s*) *h*
      and *contk*: *continuous_on* (*sphere 0 1*) *k*
      and *him*: *h ‘ sphere b s ⊆ sphere 0 1*
      and *kim*: *k ‘ sphere 0 1 ⊆ sphere b s*
    by (*simp_all add*: *homeomorphism_def*)
  obtain *c* **where** *homotopic_with_canon* (*λz. True*) (*sphere a r*) (*sphere 0 1*) (*h ∘ f*) (*λx. c*)
  proof (*rule inessential_spheremap_2_aux* [*OF a2*])
    show *continuous_on* (*sphere a r*) (*h ∘ f*)
     by (*meson continuous_on_compose* [*OF contf*] *conth continuous_on_subset fim*)
    show (*h ∘ f*) *‘ sphere a r ⊆ sphere 0 1*
      using *fim him* **by** *force*
  qed *auto*
  then have *homotopic_with_canon* (*λf. True*) (*sphere a r*) (*sphere b s*) (*k ∘ (h ∘ f)*) (*k ∘ (λx. c)*)
    by (*rule homotopic_with_compose_continuous_left* [*OF _ contk kim*])
  then have *homotopic_with_canon* (*λz. True*) (*sphere a r*) (*sphere b s*) *f* (*λx. k c*)
    apply (*rule homotopic_with_eq*, *auto*)
    by (*metis fim hk homeomorphism_def image_subset_iff mem_sphere*)
  then show *?thesis*
    by (*metis that*)
qed

6.41.15 Holomorphic logarithms and square roots

lemma *g_imp_holomorphic_log*:
  assumes *holf*: *f holomorphic_on S*
    and *contg*: *continuous_on S g* **and** *feq*: $\bigwedge x.\ x \in S \Longrightarrow f\ x = exp\ (g\ x)$
    and *fnz*: $\bigwedge z.\ z \in S \Longrightarrow f\ z \neq 0$
  obtains *g* **where** *g holomorphic_on S* $\bigwedge z.\ z \in S \Longrightarrow f\ z = exp(g\ z)$
proof –
  have *contf*: *continuous_on S f*
    by (*simp add*: *holf holomorphic_on_imp_continuous_on*)
  have *g field_differentiable at z within S* **if** *f field_differentiable at z within S z ∈ S* **for** *z*
  proof –
    obtain *f′* **where** *f′*: $((\lambda y.\ (f\ y - f\ z) / (y - z)) \longrightarrow f')$ (*at z within S*)

```
    using ‹f field_differentiable at z within S› by (auto simp: field_differentiable_def
has_field_derivative_iff)
   then have ee: ((λx. (exp(g x) − exp(g z)) / (x − z)) ⟶ f') (at z within S)
     by (simp add: feq ‹z ∈ S› Lim_transform_within [OF _ zero_less_one])
   have (((λy. if y = g z then exp (g z) else (exp y − exp (g z)) / (y − g z)) ∘
g) ⟶ exp (g z))
        (at z within S)
   proof (rule tendsto_compose_at)
     show (g ⟶ g z) (at z within S)
       using contg continuous_on ‹z ∈ S› by blast
     show (λy. if y = g z then exp (g z) else (exp y − exp (g z)) / (y − g z)) −g
z→ exp (g z)
       by (simp add: LIM_offset_zero_iff DERIV_D cong: if_cong Lim_cong_within)
     qed auto
   then have dd: ((λx. if g x = g z then exp(g z) else (exp(g x) − exp(g z)) / (g
x − g z)) ⟶ exp(g z)) (at z within S)
     by (simp add: o_def)
   have continuous (at z within S) g
     using contg continuous_on_eq_continuous_within ‹z ∈ S› by blast
   then have (∀F x in at z within S. dist (g x) (g z) < 2*pi)
     by (simp add: continuous_within tendsto_iff)
   then have ∀F x in at z within S. exp (g x) = exp (g z) ⟶ g x ≠ g z ⟶ x
= z
     by (rule eventually_mono) (auto simp: exp_eq dist_norm norm_mult)
   then have ((λy. (g y − g z) / (y − z)) ⟶ f' / exp (g z)) (at z within S)
     by (auto intro!: Lim_transform_eventually [OF tendsto_divide [OF ee dd]])
   then show ?thesis
     by (auto simp: field_differentiable_def has_field_derivative_iff)
  qed
  then have g holomorphic_on S
   using holf holomorphic_on_def by auto
  then show ?thesis
   using feq that by auto
qed

lemma contractible_imp_holomorphic_log:
  assumes holf: f holomorphic_on S
     and S: contractible S
     and fnz: ⋀z. z ∈ S ⟹ f z ≠ 0
  obtains g where g holomorphic_on S ⋀z. z ∈ S ⟹ f z = exp(g z)
proof −
  have contf: continuous_on S f
   by (simp add: holf holomorphic_on_imp_continuous_on)
  obtain g where contg: continuous_on S g and feq: ⋀x. x ∈ S ⟹ f x = exp (g
x)
   by (metis continuous_logarithm_on_contractible [OF contf S fnz])
  then show thesis
   using fnz g_imp_holomorphic_log holf that by blast
qed
```

lemma *simply_connected_imp_holomorphic_log*:
assumes *holf*: *f holomorphic_on S*
and *S*: *simply_connected S locally path_connected S*
and *fnz*: $\bigwedge z.\ z \in S \Longrightarrow f\ z \neq 0$
obtains *g* **where** *g holomorphic_on S* $\bigwedge z.\ z \in S \Longrightarrow f\ z = exp(g\ z)$
proof −
have *contf*: *continuous_on S f*
by (*simp add*: *holf holomorphic_on_imp_continuous_on*)
obtain *g* **where** *contg*: *continuous_on S g* **and** *feq*: $\bigwedge x.\ x \in S \Longrightarrow f\ x = exp\ (g\ x)$
by (*metis continuous_logarithm_on_simply_connected* [*OF contf S fnz*])
then show *thesis*
using *fnz g_imp_holomorphic_log holf that* **by** *blast*
qed

lemma *contractible_imp_holomorphic_sqrt*:
assumes *holf*: *f holomorphic_on S*
and *S*: *contractible S*
and *fnz*: $\bigwedge z.\ z \in S \Longrightarrow f\ z \neq 0$
obtains *g* **where** *g holomorphic_on S* $\bigwedge z.\ z \in S \Longrightarrow f\ z = g\ z\ \hat{}\ 2$
proof −
obtain *g* **where** *holg*: *g holomorphic_on S* **and** *feq*: $\bigwedge z.\ z \in S \Longrightarrow f\ z = exp(g\ z)$
using *contractible_imp_holomorphic_log* [*OF assms*] **by** *blast*
show *?thesis*
proof
show $exp \circ (\lambda z.\ z\ /\ 2) \circ g$ *holomorphic_on S*
by (*intro holomorphic_on_compose holg holomorphic_intros*) *auto*
show $\bigwedge z.\ z \in S \Longrightarrow f\ z = ((exp \circ (\lambda z.\ z\ /\ 2) \circ g)\ z)^2$
by (*simp add*: *feq flip*: *exp_double*)
qed
qed

lemma *simply_connected_imp_holomorphic_sqrt*:
assumes *holf*: *f holomorphic_on S*
and *S*: *simply_connected S locally path_connected S*
and *fnz*: $\bigwedge z.\ z \in S \Longrightarrow f\ z \neq 0$
obtains *g* **where** *g holomorphic_on S* $\bigwedge z.\ z \in S \Longrightarrow f\ z = g\ z\ \hat{}\ 2$
proof −
obtain *g* **where** *holg*: *g holomorphic_on S* **and** *feq*: $\bigwedge z.\ z \in S \Longrightarrow f\ z = exp(g\ z)$
using *simply_connected_imp_holomorphic_log* [*OF assms*] **by** *blast*
show *?thesis*
proof
show $exp \circ (\lambda z.\ z\ /\ 2) \circ g$ *holomorphic_on S*
by (*intro holomorphic_on_compose holg holomorphic_intros*) *auto*
show $\bigwedge z.\ z \in S \Longrightarrow f\ z = ((exp \circ (\lambda z.\ z\ /\ 2) \circ g)\ z)^2$
by (*simp add*: *feq flip*: *exp_double*)

qed
qed

Related theorems about holomorphic inverse cosines.

lemma *contractible_imp_holomorphic_arccos*:
assumes *holf*: *f holomorphic_on S* **and** *S*: *contractible S*
and *non1*: $\bigwedge z.\ z \in S \Longrightarrow f\ z \neq 1 \wedge f\ z \neq -1$
obtains *g* **where** *g holomorphic_on S* $\bigwedge z.\ z \in S \Longrightarrow f\ z = cos(g\ z)$
proof −
have *hol1f*: ($\lambda z.\ 1 - f\ z\ \hat{}\ 2$) *holomorphic_on S*
by (*intro holomorphic_intros holf*)
obtain *g* **where** *holg*: *g holomorphic_on S* **and** *eq*: $\bigwedge z.\ z \in S \Longrightarrow 1 - (f\ z)^2 = (g\ z)^2$
using *contractible_imp_holomorphic_sqrt* [*OF hol1f S*]
by (*metis eq_iff_diff_eq_0 non1 power2_eq_1_iff*)
have *holfg*: ($\lambda z.\ f\ z + \mathrm{i}*g\ z$) *holomorphic_on S*
by (*intro holf holg holomorphic_intros*)
have $\bigwedge z.\ z \in S \Longrightarrow f\ z + \mathrm{i}*g\ z \neq 0$
by (*metis Arccos_body_lemma eq add.commute add.inverse_unique complex_i_mult_minus power2_csqrt power2_eq_iff*)
then obtain *h* **where** *holh*: *h holomorphic_on S* **and** *fgeq*: $\bigwedge z.\ z \in S \Longrightarrow f\ z + \mathrm{i}*g\ z = exp\ (h\ z)$
using *contractible_imp_holomorphic_log* [*OF holfg S*] **by** *metis*
show *?thesis*
proof
show ($\lambda z.\ -\mathrm{i}*h\ z$) *holomorphic_on S*
by (*intro holh holomorphic_intros*)
show $f\ z = cos\ (-\ \mathrm{i}*h\ z)$ **if** $z \in S$ **for** *z*
proof −
have $(f\ z + \mathrm{i}*g\ z)*(f\ z - \mathrm{i}*g\ z) = 1$
using *that eq* **by** (*auto simp*: *algebra_simps power2_eq_square*)
then have $f\ z - \mathrm{i}*g\ z = inverse\ (f\ z + \mathrm{i}*g\ z)$
using *inverse_unique* **by** *force*
also have ... $= exp\ (-\ h\ z)$
by (*simp add*: *exp_minus fgeq that*)
finally have $f\ z = exp\ (-\ h\ z) + \mathrm{i}*g\ z$
by (*simp add*: *diff_eq_eq*)
then show *?thesis*
apply (*simp add*: *cos_exp_eq*)
by (*metis fgeq add.assoc mult_2_right that*)
qed
qed
qed

lemma *contractible_imp_holomorphic_arccos_bounded*:
assumes *holf*: *f holomorphic_on S* **and** *S*: *contractible S* **and** $a \in S$
and *non1*: $\bigwedge z.\ z \in S \Longrightarrow f\ z \neq 1 \wedge f\ z \neq -1$
obtains *g* **where** *g holomorphic_on S* $norm(g\ a) \le pi + norm(f\ a)$ $\bigwedge z.\ z \in S$

⟹ *f z* = *cos*(*g z*)
proof −
  obtain *g* **where** *holg*: *g holomorphic_on S* **and** *feq*: ⋀*z*. *z* ∈ *S* ⟹ *f z* = *cos* (*g z*)
    using *contractible_imp_holomorphic_arccos* [*OF holf S non1*] **by** *blast*
  obtain *b* **where** *cos b* = *f a norm b* ≤ *pi* + *norm* (*f a*)
    using *cos_Arccos norm_Arccos_bounded* **by** *blast*
  then have *cos b* = *cos* (*g a*)
    by (*simp add*: ‹*a* ∈ *S*› *feq*)
  then consider *n* **where** *n* ∈ ℤ *b* = *g a* + *of_real*(*2*∗*n*∗*pi*) | *n* **where** *n* ∈ ℤ *b* = −*g a* + *of_real*(*2*∗*n*∗*pi*)
    by (*auto simp*: *complex_cos_eq*)
  then show *?thesis*
  proof *cases*
    case *1*
    show *?thesis*
    proof
      show (λ*z*. *g z* + *of_real*(*2*∗*n*∗*pi*)) *holomorphic_on S*
        by (*intro holomorphic_intros holg*)
      show *cmod* (*g a* + *of_real*(*2*∗*n*∗*pi*)) ≤ *pi* + *cmod* (*f a*)
        using *1* ‹*cmod b* ≤ *pi* + *cmod* (*f a*)› **by** *blast*
      show ⋀*z*. *z* ∈ *S* ⟹ *f z* = *cos* (*g z* + *complex_of_real* (*2*∗*n*∗*pi*))
        by (*metis* ‹*n* ∈ ℤ› *complex_cos_eq feq*)
    qed
  next
    case *2*
    show *?thesis*
    proof
      show (λ*z*. −*g z* + *of_real*(*2*∗*n*∗*pi*)) *holomorphic_on S*
        by (*intro holomorphic_intros holg*)
      show *cmod* (−*g a* + *of_real*(*2*∗*n*∗*pi*)) ≤ *pi* + *cmod* (*f a*)
        using *2* ‹*cmod b* ≤ *pi* + *cmod* (*f a*)› **by** *blast*
      show ⋀*z*. *z* ∈ *S* ⟹ *f z* = *cos* (−*g z* + *complex_of_real* (*2*∗*n*∗*pi*))
        by (*metis* ‹*n* ∈ ℤ› *complex_cos_eq feq*)
    qed
  qed
qed

6.41.16 The "Borsukian" property of sets

This doesn't have a standard name. Kuratowski uses "contractible with respect to $[S^1]$" while Whyburn uses "property b". It's closely related to unicoherence.

definition *Borsukian* **where**
  Borsukian S ≡
    ∀*f*. *continuous_on S f* ∧ *f* ‘ *S* ⊆ (− {*0*::*complex*})
      ⟶ (∃*a*. *homotopic_with_canon* (λ*h*. *True*) *S* (− {*0*}) *f* (λ*x*. *a*))

lemma *Borsukian_retraction_gen*:

```
  assumes Borsukian S continuous_on S h h ` S = T
          continuous_on T k  k ` T ⊆ S  ⋀y. y ∈ T ⟹ h(k y) = y
    shows Borsukian T
proof -
  interpret R: Retracts S h T k
    using assms by (simp add: Retracts.intro)
  show ?thesis
    using assms
    apply (clarsimp simp add: Borsukian_def)
    apply (rule R.cohomotopically_trivial_retraction_null_gen [OF TrueI TrueI refl,
of -{0}], auto)
    done
qed

lemma retract_of_Borsukian: ⟦Borsukian T; S retract_of T⟧ ⟹ Borsukian S
  apply (auto simp: retract_of_def retraction_def)
  apply (erule (1) Borsukian_retraction_gen)
  apply (meson retraction retraction_def)
    apply (auto)
    done

lemma homeomorphic_Borsukian: ⟦Borsukian S; S homeomorphic T⟧ ⟹ Bor-
sukian T
  using Borsukian_retraction_gen order_refl
  by (fastforce simp add: homeomorphism_def homeomorphic_def)

lemma homeomorphic_Borsukian_eq:
   S homeomorphic T ⟹ Borsukian S ⟷ Borsukian T
  by (meson homeomorphic_Borsukian homeomorphic_sym)

lemma Borsukian_translation:
  fixes S :: 'a::real_normed_vector set
  shows Borsukian (image (λx. a + x) S) ⟷ Borsukian S
  using homeomorphic_Borsukian_eq homeomorphic_translation by blast

lemma Borsukian_injective_linear_image:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes linear f inj f
    shows Borsukian(f ` S) ⟷ Borsukian S
  using assms homeomorphic_Borsukian_eq linear_homeomorphic_image by blast

lemma homotopy_eqv_Borsukianness:
  fixes S :: 'a::real_normed_vector set
    and T :: 'b::real_normed_vector set
   assumes S homotopy_eqv T
     shows (Borsukian S ⟷ Borsukian T)
  by (meson Borsukian_def assms homotopy_eqv_cohomotopic_triviality_null)

lemma Borsukian_alt:
```

```
  fixes S :: 'a::real_normed_vector set
  shows
   Borsukian S ⟷
      (∀f g. continuous_on S f ∧ f ` S ⊆ −{0} ∧
             continuous_on S g ∧ g ` S ⊆ −{0}
             ⟶ homotopic_with_canon (λh. True) S (− {0::complex}) f g)
  unfolding Borsukian_def homotopic_triviality
  by (simp add: path_connected_punctured_universe)

lemma Borsukian_continuous_logarithm:
  fixes S :: 'a::real_normed_vector set
  shows Borsukian S ⟷
          (∀f. continuous_on S f ∧ f ` S ⊆ (− {0::complex})
               ⟶ (∃g. continuous_on S g ∧ (∀x ∈ S. f x = exp(g x))))
  by (simp add: Borsukian_def inessential_eq_continuous_logarithm)

lemma Borsukian_continuous_logarithm_circle:
  fixes S :: 'a::real_normed_vector set
  shows Borsukian S ⟷
           (∀f. continuous_on S f ∧ f ` S ⊆ sphere (0::complex) 1
                ⟶ (∃g. continuous_on S g ∧ (∀x ∈ S. f x = exp(g x))))
   (is ?lhs = ?rhs)
proof
  assume ?lhs then show ?rhs
    by (force simp: Borsukian_continuous_logarithm)
next
  assume RHS [rule_format]: ?rhs
  show ?lhs
  proof (clarsimp simp: Borsukian_continuous_logarithm)
    fix f :: 'a ⇒ complex
    assume contf: continuous_on S f and 0: 0 ∉ f ` S
    then have continuous_on S (λx. f x / complex_of_real (cmod (f x)))
      by (intro continuous_intros) auto
    moreover have (λx. f x / complex_of_real (cmod (f x))) ` S ⊆ sphere 0 1
      using 0 by (auto simp: norm_divide)
    ultimately obtain g where contg: continuous_on S g
                  and fg: ∀x ∈ S. f x / complex_of_real (cmod (f x)) = exp(g x)
      using RHS [of λx. f x / of_real(norm(f x))] by auto
    show ∃g. continuous_on S g ∧ (∀x∈S. f x = exp (g x))
    proof (intro exI ballI conjI)
      show continuous_on S (λx. (Ln ∘ of_real ∘ norm ∘ f)x + g x)
        by (intro continuous_intros contf contg conjI) (use 0 in auto)
      show f x = exp ((Ln ∘ complex_of_real ∘ cmod ∘ f) x + g x) if x ∈ S for x
        using 0 that
        apply (simp add: exp_add)
      by (metis div_by_0 exp_Ln exp_not_eq_zero fg mult.commute nonzero_eq_divide_eq)
    qed
  qed
qed
```

lemma *Borsukian_continuous_logarithm_circle_real*:
fixes *S* :: *'a::real_normed_vector set*
shows *Borsukian S* ⟷
(∀ *f*. *continuous_on S f* ∧ *f* ‘ *S* ⊆ *sphere* (*0::complex*) *1*
⟶ (∃ *g*. *continuous_on S* (*complex_of_real* ∘ *g*) ∧ (∀ *x* ∈ *S*. *f x* = *exp*(i
∗ *of_real*(*g x*)))))
(**is** *?lhs* = *?rhs*)
proof
assume *LHS*: *?lhs*
show *?rhs*
proof (*clarify*)
fix *f* :: *'a* ⇒ *complex*
assume *continuous_on S f* **and** *f01*: *f* ‘ *S* ⊆ *sphere 0 1*
then obtain *g* **where** *contg*: *continuous_on S g* **and** ⋀*x*. *x* ∈ *S* ⟹ *f x* =
exp(*g x*)
using *LHS* **by** (*auto simp*: *Borsukian_continuous_logarithm_circle*)
then have ∀ *x*∈*S*. *f x* = *exp* (i ∗ *complex_of_real* ((*Im* ∘ *g*) *x*))
using *f01 exp_eq_polar norm_exp_eq_Re* **by** *auto*
then show ∃ *g*. *continuous_on S* (*complex_of_real* ∘ *g*) ∧ (∀ *x*∈*S*. *f x* = *exp* (i
∗ *complex_of_real* (*g x*)))
by (*rule_tac x*=*Im* ∘ *g* **in** *exI*) (*force intro*: *continuous_intros contg*)
qed
next
assume *RHS* [*rule_format*]: *?rhs*
show *?lhs*
proof (*clarsimp simp*: *Borsukian_continuous_logarithm_circle*)
fix *f* :: *'a* ⇒ *complex*
assume *continuous_on S f* **and** *f01*: *f* ‘ *S* ⊆ *sphere 0 1*
then obtain *g* **where** *contg*: *continuous_on S* (*complex_of_real* ∘ *g*) **and** ⋀*x*.
x ∈ *S* ⟹ *f x* = *exp*(i ∗ *of_real*(*g x*))
by (*metis RHS*)
then show ∃ *g*. *continuous_on S g* ∧ (∀ *x*∈*S*. *f x* = *exp* (*g x*))
by (*rule_tac x*=λ*x*. i∗ *of_real*(*g x*) **in** *exI*) (*auto simp*: *continuous_intros contg*)
qed
qed

lemma *Borsukian_circle*:
fixes *S* :: *'a::real_normed_vector set*
shows *Borsukian S* ⟷
(∀ *f*. *continuous_on S f* ∧ *f* ‘ *S* ⊆ *sphere* (*0::complex*) *1*
⟶ (∃ *a*. *homotopic_with_canon* (λ*h*. *True*) *S* (*sphere* (*0::complex*) *1*)
f (λ*x*. *a*)))
by (*simp add*: *inessential_eq_continuous_logarithm_circle Borsukian_continuous_logarithm_circle_real*)

lemma *contractible_imp_Borsukian*: *contractible S* ⟹ *Borsukian S*
by (*meson Borsukian_def nullhomotopic_from_contractible*)

lemma *simply_connected_imp_Borsukian*:
fixes *S* :: *'a::real_normed_vector set*
shows ⟦*simply_connected S*; *locally path_connected S*⟧ ⟹ *Borsukian S*
by (*metis* (*no_types*, *lifting*) *Borsukian_continuous_logarithm continuous_logarithm_on_simply_connected image_eqI subset_Compl_singleton*)

lemma *starlike_imp_Borsukian*:
fixes *S* :: *'a::real_normed_vector set*
shows *starlike S* ⟹ *Borsukian S*
by (*simp add*: *contractible_imp_Borsukian starlike_imp_contractible*)

lemma *Borsukian_empty*: *Borsukian* {}
by (*auto simp*: *contractible_imp_Borsukian*)

lemma *Borsukian_UNIV*: *Borsukian* (*UNIV* :: *'a::real_normed_vector set*)
by (*auto simp*: *contractible_imp_Borsukian*)

lemma *convex_imp_Borsukian*:
fixes *S* :: *'a::real_normed_vector set*
shows *convex S* ⟹ *Borsukian S*
by (*meson Borsukian_def convex_imp_contractible nullhomotopic_from_contractible*)

proposition *Borsukian_sphere*:
fixes *a* :: *'a::euclidean_space*
shows *3* ≤ *DIM*(*'a*) ⟹ *Borsukian* (*sphere a r*)
using *ENR_sphere*
by (*blast intro*: *simply_connected_imp_Borsukian ENR_imp_locally_path_connected simply_connected_sphere*)

lemma *Borsukian_Un_lemma*:
fixes *S* :: *'a::real_normed_vector set*
assumes *BS*: *Borsukian S* **and** *BT*: *Borsukian T* **and** *ST*: *connected*(*S* ∩ *T*)
and ∗: ⋀*f g*::*'a* ⇒ *complex*.
⟦*continuous_on S f*; *continuous_on T g*; ⋀*x*. *x* ∈ *S* ∧ *x* ∈ *T* ⟹ *f x* = *g x*⟧
⟹ *continuous_on* (*S* ∪ *T*) (λ*x*. *if x* ∈ *S then f x else g x*)
shows *Borsukian*(*S* ∪ *T*)
proof (*clarsimp simp add*: *Borsukian_continuous_logarithm*)
fix *f* :: *'a* ⇒ *complex*
assume *contf*: *continuous_on* (*S* ∪ *T*) *f* **and** *0*: *0* ∉ *f* ‘ (*S* ∪ *T*)
then have *contfS*: *continuous_on S f* **and** *contfT*: *continuous_on T f*
using *continuous_on_subset* **by** *auto*
have ⟦*continuous_on S f*; *f* ‘ *S* ⊆ −{*0*}⟧ ⟹ ∃ *g*. *continuous_on S g* ∧ (∀ *x* ∈ *S*. *f x* = *exp*(*g x*))
using *BS* **by** (*auto simp*: *Borsukian_continuous_logarithm*)
then obtain *g* **where** *contg*: *continuous_on S g* **and** *fg*: ⋀*x*. *x* ∈ *S* ⟹ *f x* = *exp*(*g x*)
using *0 contfS* **by** *blast*
have ⟦*continuous_on T f*; *f* ‘ *T* ⊆ −{*0*}⟧ ⟹ ∃ *g*. *continuous_on T g* ∧ (∀ *x* ∈

T. *f x* = *exp*(*g x*))
  using *BT* **by** (*auto simp*: *Borsukian_continuous_logarithm*)
 then obtain *h* **where** *conth*: *continuous_on T h* **and** *fh*: $\bigwedge x.\ x \in T \Longrightarrow f\ x =$ *exp*(*h x*)
  using *0 contfT* **by** *blast*
 show ∃ *g*. *continuous_on* (*S* ∪ *T*) *g* ∧ (∀ *x*∈*S* ∪ *T*. *f x* = *exp* (*g x*))
 proof (*cases S* ∩ *T* = {})
  case *True*
  show *?thesis*
  proof (*intro exI conjI*)
   show *continuous_on* (*S* ∪ *T*) (λ*x*. *if x* ∈ *S then g x else h x*)
    using *True* ∗ [*OF contg conth*]
    by (*meson disjoint_iff*)
   show ∀ *x*∈*S* ∪ *T*. *f x* = *exp* (*if x* ∈ *S then g x else h x*)
    using *fg fh* **by** *auto*
  qed
 next
  case *False*
  have (λ*x*. *g x* − *h x*) *constant_on S* ∩ *T*
  proof (*rule continuous_discrete_range_constant* [*OF ST*])
   show *continuous_on* (*S* ∩ *T*) (λ*x*. *g x* − *h x*)
   proof (*intro continuous_intros*)
    show *continuous_on* (*S* ∩ *T*) *g*
     by (*meson contg continuous_on_subset inf_le1*)
    show *continuous_on* (*S* ∩ *T*) *h*
     by (*meson conth continuous_on_subset inf_sup_ord*(*2*))
   qed
   show ∃ *e*>*0*. ∀ *y*. *y* ∈ *S* ∩ *T* ∧ *g y* − *h y* ≠ *g x* − *h x* ⟶ *e* ≤ *cmod* (*g y* − *h y* − (*g x* − *h x*))
     if *x* ∈ *S* ∩ *T* **for** *x*
   proof −
    have *g y* − *g x* = *h y* − *h x*
      if *y* ∈ *S y* ∈ *T cmod* (*g y* − *g x* − (*h y* − *h x*)) < *2* ∗ *pi* **for** *y*
    proof (*rule exp_complex_eqI*)
     have |*Im* (*g y* − *g x*) − *Im* (*h y* − *h x*)| ≤ *cmod* (*g y* − *g x* − (*h y* − *h x*))
      by (*metis abs_Im_le_cmod minus_complex.simps*(*2*))
     then show |*Im* (*g y* − *g x*) − *Im* (*h y* − *h x*)| < *2* ∗ *pi*
      using *that* **by** *linarith*
     have *exp* (*g x*) = *exp* (*h x*) *exp* (*g y*) = *exp* (*h y*)
      using *fg fh that* ‹*x* ∈ *S* ∩ *T*› **by** *fastforce*+
     then show *exp* (*g y* − *g x*) = *exp* (*h y* − *h x*)
      by (*simp add*: *exp_diff*)
    qed
    then show *?thesis*
     by (*rule_tac x*=*2*∗*pi* **in** *exI*) (*fastforce simp add*: *algebra_simps*)
   qed
  qed
  then obtain *a* **where** *a*: $\bigwedge x.\ x \in S \cap T \Longrightarrow g\ x - h\ x = a$

```
      by (auto simp: constant_on_def)
    with False have exp a = 1
     by (metis IntI disjoint_iff_not_equal divide_self_if exp_diff exp_not_eq_zero fg fh)
    with a show ?thesis
      apply (rule_tac x=λx. if x ∈ S then g x else a + h x in exI)
      apply (intro * contg conth continuous_intros conjI)
       apply (auto simp: algebra_simps fg fh exp_add)
      done
  qed
qed

proposition Borsukian_open_Un:
  fixes S :: 'a::real_normed_vector set
  assumes opeS: openin (top_of_set (S ∪ T)) S
      and opeT: openin (top_of_set (S ∪ T)) T
      and BS: Borsukian S and BT: Borsukian T and ST: connected(S ∩ T)
    shows Borsukian(S ∪ T)
 by (force intro: Borsukian_Un_lemma [OF BS BT ST] continuous_on_cases_local_open
[OF opeS opeT])

lemma Borsukian_closed_Un:
  fixes S :: 'a::real_normed_vector set
  assumes cloS: closedin (top_of_set (S ∪ T)) S
      and cloT: closedin (top_of_set (S ∪ T)) T
      and BS: Borsukian S and BT: Borsukian T and ST: connected(S ∩ T)
    shows Borsukian(S ∪ T)
  by (force intro: Borsukian_Un_lemma [OF BS BT ST] continuous_on_cases_local
[OF cloS cloT])

lemma Borsukian_separation_compact:
  fixes S :: complex set
  assumes compact S
    shows Borsukian S ⟷ connected(− S)
  by (simp add: Borsuk_separation_theorem Borsukian_circle assms)

lemma Borsukian_monotone_image_compact:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes Borsukian S and contf: continuous_on S f and fim: f ` S = T
      and compact S and conn: ⋀y. y ∈ T ⟹ connected {x. x ∈ S ∧ f x = y}
    shows Borsukian T
proof (clarsimp simp add: Borsukian_continuous_logarithm)
  fix g :: 'b ⇒ complex
  assume contg: continuous_on T g and 0: 0 ∉ g ` T
  have continuous_on S (g ∘ f)
    using contf contg continuous_on_compose fim by blast
  moreover have (g ∘ f) ` S ⊆ −{0}
    using fim 0 by auto
  ultimately obtain h where conth: continuous_on S h and gfh: ⋀x. x ∈ S ⟹
(g ∘ f) x = exp(h x)
```

using ‹*Borsukian S*› **by** (*auto simp*: *Borsukian_continuous_logarithm*)
have $\bigwedge y. \exists x. y \in T \longrightarrow x \in S \wedge f\, x = y$
using *fim* **by** *auto*
then obtain f' **where** f': $\bigwedge y. y \in T \longrightarrow f'\, y \in S \wedge f\,(f'\, y) = y$
by *metis*
have ∗: $(\lambda x. h\, x - h(f'\, y))$ *constant_on* $\{x. x \in S \wedge f\, x = y\}$ **if** $y \in T$ **for** y
proof (*rule continuous_discrete_range_constant* [*OF conn* [*OF that*], *of* $\lambda x. h\, x - h\,(f'\, y)$], *simp_all add*: *algebra_simps*)
show *continuous_on* $\{x \in S. f\, x = y\}$ $(\lambda x. h\, x - h\,(f'\, y))$
by (*intro continuous_intros continuous_on_subset* [*OF conth*]) *auto*
show $\exists e>0. \forall u. u \in S \wedge f\, u = y \wedge h\, u \neq h\, x \longrightarrow e \le cmod\,(h\, u - h\, x)$
if x: $x \in S \wedge f\, x = y$ **for** x
proof −
have $h\, u = h\, x$ **if** $u \in S$ $f\, u = y$ $cmod\,(h\, u - h\, x) < 2 * pi$ **for** u
proof (*rule exp_complex_eqI*)
have $|Im\,(h\, u) - Im\,(h\, x)| \le cmod\,(h\, u - h\, x)$
by (*metis abs_Im_le_cmod minus_complex.simps*(*2*))
then show $|Im\,(h\, u) - Im\,(h\, x)| < 2 * pi$
using *that* **by** *linarith*
show $exp\,(h\, u) = exp\,(h\, x)$
by (*simp add*: *gfh* [*symmetric*] *x that*)
qed
then show *?thesis*
by (*rule_tac x=2∗pi* **in** *exI*) (*fastforce simp add*: *algebra_simps*)
qed
qed
show $\exists h.$ *continuous_on* $T\, h \wedge (\forall x \in T. g\, x = exp\,(h\, x))$
proof (*intro exI conjI*)
show *continuous_on* $T\,(h \circ f')$
proof (*rule continuous_from_closed_graph* [*of h* ‘ *S*])
show *compact* (*h* ‘ *S*)
by (*simp add*: ‹*compact S*› *compact_continuous_image conth*)
show $(h \circ f')$ ‘ $T \subseteq h$ ‘ S
by (*auto simp*: f')
have $h\, x = h\,(f'\,(f\, x))$ **if** $x \in S$ **for** x
using ∗ [*of f x*] *fim that* **unfolding** *constant_on_def* **by** *clarsimp* (*metis* f' *imageI right_minus_eq*)
moreover have $\bigwedge x. x \in T \Longrightarrow \exists u. u \in S \wedge x = f\, u \wedge h\,(f'\, x) = h\, u$
using f' **by** *fastforce*
ultimately
have *eq*: $((\lambda x. (x, (h \circ f')\, x))$ ‘ $T) =$
$\{p. \exists x. x \in S \wedge (x, p) \in (S \times UNIV) \cap ((\lambda z. snd\, z - ((f \circ fst)\, z, (h \circ fst)\, z))$ −‘ $\{0\})\}$
using *fim* **by** (*auto simp*: *image_iff*)
moreover have *closed* . . .
apply (*intro closed_compact_projection* [*OF* ‹*compact S*›] *continuous_closed_preimage continuous_intros continuous_on_subset* [*OF contf*] *continuous_on_subset* [*OF conth*])
by (*auto simp*: ‹*compact S*› *closed_Times compact_imp_closed*)

ultimately show *closed* $((\lambda x. (x, (h \circ f') x))$ ‘ $T)$
by *simp*
qed
qed (*use f' gfh* **in** *fastforce*)
qed

lemma *Borsukian_open_map_image_compact*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *Borsukian S* **and** *contf*: *continuous_on S f* **and** *fim*: f ‘ $S = T$ **and** *compact S*
and *ope*: $\bigwedge U$. *openin* (*top_of_set S*) U
⟹ *openin* (*top_of_set T*) (f ‘ U)
shows *Borsukian T*
proof (*clarsimp simp add*: *Borsukian_continuous_logarithm_circle_real*)
fix g :: *'b* ⇒ *complex*
assume *contg*: *continuous_on T g* **and** *gim*: g ‘ $T \subseteq$ *sphere 0 1*
have *continuous_on S* $(g \circ f)$
using *contf contg continuous_on_compose fim* **by** *blast*
moreover have $(g \circ f)$ ‘ $S \subseteq$ *sphere 0 1*
using *fim gim* **by** *auto*
ultimately obtain h **where** *cont_cxh*: *continuous_on S* (*complex_of_real* $\circ h$)
and *gfh*: $\bigwedge x. x \in S \Longrightarrow (g \circ f)\ x = exp(\mathrm{i} * of\_real(h\ x))$
using ‹*Borsukian S*› *Borsukian_continuous_logarithm_circle_real* **by** *metis*
then have *conth*: *continuous_on S h*
by *simp*
have $\exists x. x \in S \wedge f\ x = y \wedge (\forall x' \in S. f\ x' = y \longrightarrow h\ x \le h\ x')$ **if** $y \in T$ **for** y
proof −
have *1*: *compact* (h ‘ $\{x \in S. f\ x = y\}$)
proof (*rule compact_continuous_image*)
show *continuous_on* $\{x \in S. f\ x = y\}$ h
by (*rule continuous_on_subset* [*OF conth*]) *auto*
have *compact* $(S \cap f$ −‘ $\{y\})$
by (*rule proper_map_from_compact* [*OF contf _* ‹*compact S*›, *of T*]) (*simp_all add*: *fim that*)
then show *compact* $\{x \in S. f\ x = y\}$
by (*auto simp*: *vimage_def Int_def*)
qed
have *2*: h ‘ $\{x \in S. f\ x = y\} \neq \{\}$
using *fim that* **by** *auto*
have $\exists s \in h$ ‘ $\{x \in S. f\ x = y\}. \forall t \in h$ ‘ $\{x \in S. f\ x = y\}. s \le t$
using *compact_attains_inf* [*OF 1 2*] **by** *blast*
then show *?thesis* **by** *auto*
qed
then obtain k **where** *kTS*: $\bigwedge y. y \in T \Longrightarrow k\ y \in S$
and *fk*: $\bigwedge y. y \in T \Longrightarrow f\ (k\ y) = y$
and *hle*: $\bigwedge x'\ y. [\![y \in T; x' \in S; f\ x' = y]\!] \Longrightarrow h\ (k\ y) \le h\ x'$
by *metis*
have *continuous_on T* $(h \circ k)$

proof (*clarsimp simp add*: *continuous_on_iff*)
fix *y* **and** *e*::*real*
assume $y \in T$ $0 < e$
moreover have *uniformly_continuous_on S* (*complex_of_real* ∘ *h*)
using ‹*compact S*› *cont_cxh compact_uniformly_continuous* **by** *blast*
ultimately obtain *d* **where** $0 < d$
and *d*: $\bigwedge x\ x'.\ [\![x{\in}S;\ x'{\in}S;\ dist\ x'\ x < d]\!] \Longrightarrow dist\ (h\ x')\ (h\ x) < e$
by (*force simp*: *uniformly_continuous_on_def*)
obtain δ **where** $0 < \delta$ **and** δ:
$\bigwedge x'.\ [\![x' \in T;\ dist\ y\ x' < \delta]\!]$
$\Longrightarrow (\forall\, v \in \{z \in S.\ f\ z = y\}.\ \exists\, v'.\ v' \in \{z \in S.\ f\ z = x'\} \wedge dist\ v\ v' < d) \wedge$
$(\forall\, v' \in \{z \in S.\ f\ z = x'\}.\ \exists\, v.\ v \in \{z \in S.\ f\ z = y\} \wedge dist\ v'\ v < d)$
proof (*rule upper_lower_hemicontinuous_explicit* [*of T* λ*y*. {*z* ∈ *S*. *f z* = *y*} *S*])
show $\bigwedge U.$ *openin* (*top_of_set S*) *U*
⟹ *openin* (*top_of_set T*) {*x* ∈ *T*. {*z* ∈ *S*. *f z* = *x*} ⊆ *U*}
using *closed_map_iff_upper_hemicontinuous_preimage* [*OF fim* [*THEN equalityD1*]]
by (*simp add*: *Abstract_Topology_2.continuous_imp_closed_map* ‹*compact S*› *contf fim*)
show $\bigwedge U.$ *closedin* (*top_of_set S*) *U* ⟹
closedin (*top_of_set T*) {*x* ∈ *T*. {*z* ∈ *S*. *f z* = *x*} ⊆ *U*}
using *ope open_map_iff_lower_hemicontinuous_preimage* [*OF fim* [*THEN equalityD1*]]
by *meson*
show *bounded* {*z* ∈ *S*. *f z* = *y*}
by (*metis* (*no_types*, *lifting*) *compact_imp_bounded* [*OF* ‹*compact S*›] *bounded_subset mem_Collect_eq subsetI*)
qed (*use* ‹*y* ∈ *T*› ‹*0* < *d*› *fk kTS* **in** ‹*force*+›)
have *dist* (*h* (*k y*′)) (*h* (*k y*)) < *e* **if** *y*′ ∈ *T dist y y*′ < δ **for** *y*′
proof −
have *k1*: *k y* ∈ *S f* (*k y*) = *y* **and** *k2*: *k y*′ ∈ *S f* (*k y*′) = *y*′
by (*auto simp*: ‹*y* ∈ *T*› ‹*y*′ ∈ *T*› *kTS fk*)
have *1*: $\bigwedge v.\ [\![v \in S;\ f\ v = y]\!] \Longrightarrow \exists\, v'.\ v' \in \{z \in S.\ f\ z = y'\} \wedge dist\ v\ v' < d$
and *2*: $\bigwedge v'.\ [\![v' \in S;\ f\ v' = y']\!] \Longrightarrow \exists\, v.\ v \in \{z \in S.\ f\ z = y\} \wedge dist\ v'\ v < d$
using δ [*OF that*] **by** *auto*
then obtain *w*′ *w* **where** *w*′ ∈ *S f w*′ = *y*′ *dist* (*k y*) *w*′ < *d*
and *w* ∈ *S f w* = *y dist* (*k y*′) *w* < *d*
using *1* [*OF k1*] *2* [*OF k2*] **by** *auto*
then show *?thesis*
using *d* [*of w k y*′] *d* [*of w*′ *k y*] *k1 k2* ‹*y*′ ∈ *T*› ‹*y* ∈ *T*› *hle*
by (*fastforce simp*: *dist_norm abs_diff_less_iff algebra_simps*)
qed
then show $\exists\, d{>}0.\ \forall\, x'{\in}T.\ dist\ x'\ y < d \longrightarrow dist\ (h\ (k\ x'))\ (h\ (k\ y)) < e$
using ‹*0* < δ› **by** (*auto simp*: *dist_commute*)
qed
then show $\exists\, h.$ *continuous_on T h* ∧ (∀ *x*∈*T*. *g x* = *exp* (i ∗ *complex_of_real* (*h x*)))

using *fk gfh kTS* **by** *force*
qed

If two points are separated by a closed set, there's a minimal one.

proposition *closed_irreducible_separator*:
fixes a :: $'a$::*real_normed_vector*
assumes *closed S* **and** *ab*: ¬ *connected_component* (− S) a b
obtains T **where** $T \subseteq S$ *closed* T $T \neq \{\}$ ¬ *connected_component* (− T) a b
$\bigwedge U.\ U \subset T \Longrightarrow$ *connected_component* (− U) a b
proof (*cases* $a \in S \lor b \in S$)
case *True*
then show *?thesis*
proof
assume ∗: $a \in S$
show *?thesis*
proof
show $\{a\} \subseteq S$
using ∗ **by** *blast*
show ¬ *connected_component* (− $\{a\}$) a b
using *connected_component_in* **by** *auto*
show $\bigwedge U.\ U \subset \{a\} \Longrightarrow$ *connected_component* (− U) a b
by (*metis connected_component_UNIV UNIV_I compl_bot_eq connected_component_eq_eq less_le_not_le subset_singletonD*)
qed *auto*
next
assume ∗: $b \in S$
show *?thesis*
proof
show $\{b\} \subseteq S$
using ∗ **by** *blast*
show ¬ *connected_component* (− $\{b\}$) a b
using *connected_component_in* **by** *auto*
show $\bigwedge U.\ U \subset \{b\} \Longrightarrow$ *connected_component* (− U) a b
by (*metis connected_component_UNIV UNIV_I compl_bot_eq connected_component_eq_eq less_le_not_le subset_singletonD*)
qed *auto*
qed
next
case *False*
define A **where** $A \equiv$ *connected_component_set* (− S) a
define B **where** $B \equiv$ *connected_component_set* (− (*closure* A)) b
have $a \in A$
using *False A_def* **by** *auto*
have $b \in B$
unfolding *A_def B_def closure_Un_frontier*
using *ab False* ‹*closed S*› *frontier_complement frontier_of_connected_component_subset frontier_subset_closed* **by** *force*
have *frontier* $B \subseteq$ *frontier* (*connected_component_set* (− *closure* A) b)
using *B_def* **by** *blast*

also have *frsub*: ... ⊆ *frontier A*
proof −
have ⋀*A*. *closure* (− *closure* (− *A*)) ⊆ *closure A*
by (*metis* (*no_types*) *closure_mono closure_subset compl_le_compl_iff double_compl*)
then show *?thesis*
by (*metis* (*no_types*) *closure_closure double_compl frontier_closures frontier_of_connected_component_subset le_inf_iff subset_trans*)
qed
finally have *frBA*: *frontier B* ⊆ *frontier A* **.**
show *?thesis*
proof
show *frontier B* ⊆ *S*
proof −
have *frontier S* ⊆ *S*
by (*simp add*: ‹*closed S*› *frontier_subset_closed*)
then show *?thesis*
using *frsub frontier_complement frontier_of_connected_component_subset*
unfolding *A_def B_def* **by** *blast*
qed
show *closed* (*frontier B*)
by *simp*
show ¬ *connected_component* (− *frontier B*) *a b*
unfolding *connected_component_def*
proof *clarify*
fix *T*
assume *connected T* **and** *TB*: *T* ⊆ − *frontier B* **and** *a* ∈ *T* **and** *b* ∈ *T*
have *a* ∉ *B*
by (*metis A_def B_def ComplD* ‹*a* ∈ *A*› *assms*(*1*) *closed_open connected_component_subset in_closure_connected_component subsetD*)
have *T* ∩ *B* ≠ {}
using ‹*b* ∈ *B*› ‹*b* ∈ *T*› **by** *blast*
moreover have *T* − *B* ≠ {}
using ‹*a* ∉ *B*› ‹*a* ∈ *T*› **by** *blast*
ultimately show *False*
using *connected_Int_frontier* [*of T B*] *TB* ‹*connected T*› **by** *blast*
qed
moreover have *connected_component* (− *frontier B*) *a b* **if** *frontier B* = {}
using *connected_component_eq_UNIV that* **by** *auto*
ultimately show *frontier B* ≠ {}
by *blast*
show *connected_component* (− *U*) *a b* **if** *U* ⊂ *frontier B* **for** *U*
proof −
obtain *p* **where** *Usub*: *U* ⊆ *frontier B* **and** *p*: *p* ∈ *frontier B p* ∉ *U*
using ‹*U* ⊂ *frontier B*› **by** *blast*
show *?thesis*
unfolding *connected_component_def*
proof (*intro exI conjI*)
have *connected* ((*insert p A*) ∪ (*insert p B*))

```
        proof (rule connected_Un)
          show connected (insert p A)
           by (metis A_def IntD1 frBA ‹p ∈ frontier B› closure_insert closure_subset
connected_connected_component connected_intermediate_closure frontier_closures in-
sert_absorb subsetCE subset_insertI)
          show connected (insert p B)
           by (metis B_def IntD1 ‹p ∈ frontier B› closure_insert closure_subset con-
nected_connected_component connected_intermediate_closure frontier_closures insert_absorb
subset_insertI)
        qed blast
        then show connected (insert p (B ∪ A))
          by (simp add: sup.commute)
        have A ⊆ − U
        using A_def Usub ‹frontier B ⊆ S› connected_component_subset by fastforce
        moreover have B ⊆ − U
           using B_def Usub connected_component_subset frBA frontier_closures by
fastforce
        ultimately show insert p (B ∪ A) ⊆ − U
          using p by auto
      qed (auto simp: ‹a ∈ A› ‹b ∈ B›)
    qed
  qed
qed

lemma frontier_minimal_separating_closed_pointwise:
  fixes S :: 'a::real_normed_vector set
  assumes S: closed S a ∉ S and nconn: ¬ connected_component (− S) a b
      and conn: ⋀T. ⟦closed T; T ⊂ S⟧ ⟹ connected_component (− T) a b
    shows frontier(connected_component_set (− S) a) = S (is ?F = S)
proof −
  have ?F ⊆ S
    by (simp add: S componentsI frontier_of_components_closed_complement)
  moreover have False if ?F ⊂ S
  proof −
    have connected_component (− ?F) a b
      by (simp add: conn that)
    then obtain T where connected T T ⊆ −?F a ∈ T b ∈ T
      by (auto simp: connected_component_def)
    moreover have T ∩ ?F ≠ {}
    proof (rule connected_Int_frontier [OF ‹connected T›])
      show T ∩ connected_component_set (− S) a ≠ {}
        using ‹a ∉ S› ‹a ∈ T› by fastforce
      show T − connected_component_set (− S) a ≠ {}
        using ‹b ∈ T› nconn by blast
    qed
    ultimately show ?thesis
      by blast
  qed
  ultimately show ?thesis
```

by *blast*
qed

6.41.17 Unicoherence (closed)

definition *unicoherent* **where**
unicoherent U ≡
∀ *S T*. *connected S* ∧ *connected T* ∧ *S* ∪ *T* = *U* ∧
closedin (*top_of_set U*) *S* ∧ *closedin* (*top_of_set U*) *T*
⟶ *connected* (*S* ∩ *T*)

lemma *unicoherentI* [*intro?*]:
assumes ⋀*S T*. ⟦*connected S*; *connected T*; *U* = *S* ∪ *T*; *closedin* (*top_of_set U*) *S*; *closedin* (*top_of_set U*) *T*⟧
⟹ *connected* (*S* ∩ *T*)
shows *unicoherent U*
using *assms* **unfolding** *unicoherent_def* **by** *blast*

lemma *unicoherentD*:
assumes *unicoherent U connected S connected T U* = *S* ∪ *T closedin* (*top_of_set U*) *S closedin* (*top_of_set U*) *T*
shows *connected* (*S* ∩ *T*)
using *assms* **unfolding** *unicoherent_def* **by** *blast*

proposition *homeomorphic_unicoherent*:
assumes *ST*: *S homeomorphic T* **and** *S*: *unicoherent S*
shows *unicoherent T*
proof −
obtain *f g* **where** *gf*: ⋀*x*. *x* ∈ *S* ⟹ *g* (*f x*) = *x* **and** *fim*: *T* = *f* ‘ *S* **and** *gfim*: *g* ‘ *f* ‘ *S* = *S*
and *contf*: *continuous_on S f* **and** *contg*: *continuous_on* (*f* ‘ *S*) *g*
using *ST* **by** (*auto simp*: *homeomorphic_def homeomorphism_def*)
show *?thesis*
proof
fix *U V*
assume *connected U connected V* **and** *T*: *T* = *U* ∪ *V*
and *cloU*: *closedin* (*top_of_set T*) *U*
and *cloV*: *closedin* (*top_of_set T*) *V*
have *f* ‘ (*g* ‘ *U* ∩ *g* ‘ *V*) ⊆ *U f* ‘ (*g* ‘ *U* ∩ *g* ‘ *V*) ⊆ *V*
using *gf fim T* **by** *auto* (*metis UnCI image_iff*)+
moreover have *U* ∩ *V* ⊆ *f* ‘ (*g* ‘ *U* ∩ *g* ‘ *V*)
using *gf fim* **by** (*force simp*: *image_iff T*)
ultimately have *U* ∩ *V* = *f* ‘ (*g* ‘ *U* ∩ *g* ‘ *V*) **by** *blast*
moreover have *connected* (*f* ‘ (*g* ‘ *U* ∩ *g* ‘ *V*))
proof (*rule connected_continuous_image*)
show *continuous_on* (*g* ‘ *U* ∩ *g* ‘ *V*) *f*
using *T fim gfim* **by** (*metis Un_upper1 contf continuous_on_subset image_mono inf_le1*)
show *connected* (*g* ‘ *U* ∩ *g* ‘ *V*)

```
    proof (intro conjI unicoherentD [OF S])
      show connected (g ‘ U) connected (g ‘ V)
        using ‹connected U› cloU ‹connected V› cloV
     by (metis Topological_Spaces.connected_continuous_image closedin_imp_subset
contg continuous_on_subset fim)+
      show S = g ‘ U ∪ g ‘ V
        using T fim gfim by auto
      have hom: homeomorphism T S g f
        by (simp add: contf contg fim gf gfim homeomorphism_def)
      have closedin (top_of_set T) U closedin (top_of_set T) V
        by (simp_all add: cloU cloV)
      then show closedin (top_of_set S) (g ‘ U)
                closedin (top_of_set S) (g ‘ V)
        by (blast intro: homeomorphism_imp_closed_map [OF hom])+
    qed
  qed
  ultimately show connected (U ∩ V) by metis
 qed
qed


lemma homeomorphic_unicoherent_eq:
  S homeomorphic T ⟹ (unicoherent S ⟷ unicoherent T)
 by (meson homeomorphic_sym homeomorphic_unicoherent)

lemma unicoherent_translation:
 fixes S :: 'a::real_normed_vector set
 shows
  unicoherent (image (λx. a + x) S) ⟷ unicoherent S
 using homeomorphic_translation homeomorphic_unicoherent_eq by blast

lemma unicoherent_injective_linear_image:
 fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
 assumes linear f inj f
 shows (unicoherent(f ‘ S) ⟷ unicoherent S)
 using assms homeomorphic_unicoherent_eq linear_homeomorphic_image by blast


lemma Borsukian_imp_unicoherent:
 fixes U :: 'a::euclidean_space set
 assumes Borsukian U  shows unicoherent U
 unfolding unicoherent_def
proof clarify
 fix S T
 assume connected S connected T U = S ∪ T
   and cloS: closedin (top_of_set (S ∪ T)) S
   and cloT: closedin (top_of_set (S ∪ T)) T
 show connected (S ∩ T)
  unfolding connected_closedin_eq
```

proof *clarify*
fix *V W*
assume *closedin* (*top_of_set* (*S* ∩ *T*)) *V*
and *closedin* (*top_of_set* (*S* ∩ *T*)) *W*
and *VW*: *V* ∪ *W* = *S* ∩ *T V* ∩ *W* = {} **and** *V* ≠ {} *W* ≠ {}
then have *cloV*: *closedin* (*top_of_set U*) *V* **and** *cloW*: *closedin* (*top_of_set U*) *W*
using ‹*U* = *S* ∪ *T*› *cloS cloT closedin_trans* **by** *blast+*
obtain *q* **where** *contq*: *continuous_on U q*
and *q01*: ⋀*x*. *x* ∈ *U* ⟹ *q x* ∈ {*0*..*1*::*real*}
and *qV*: ⋀*x*. *x* ∈ *V* ⟹ *q x* = *0* **and** *qW*: ⋀*x*. *x* ∈ *W* ⟹ *q x* = *1*
by (*rule Urysohn_local* [*OF cloV cloW* ‹*V* ∩ *W* = {}›, *of 0 1*])
(*fastforce simp*: *closed_segment_eq_real_ivl*)
let *?h* = λ*x*. *if x* ∈ *S then exp*(*pi* ∗ i ∗ *q x*) *else 1* / *exp*(*pi* ∗ i ∗ *q x*)
have *eqST*: *exp*(*pi* ∗ i ∗ *q x*) = *1* / *exp*(*pi* ∗ i ∗ *q x*) **if** *x* ∈ *S* ∩ *T* **for** *x*
proof −
have *x* ∈ *V* ∪ *W*
using *that* ‹*V* ∪ *W* = *S* ∩ *T*› **by** *blast*
with *qV qW* **show** *?thesis* **by** *force*
qed
obtain *g* **where** *contg*: *continuous_on U g*
and *circle*: *g* ‘ *U* ⊆ *sphere 0 1*
and *S*: ⋀*x*. *x* ∈ *S* ⟹ *g x* = *exp*(*pi* ∗ i ∗ *q x*)
and *T*: ⋀*x*. *x* ∈ *T* ⟹ *g x* = *1* / *exp*(*pi* ∗ i ∗ *q x*)
proof
show *continuous_on U ?h*
unfolding ‹*U* = *S* ∪ *T*›
proof (*rule continuous_on_cases_local* [*OF cloS cloT*])
show *continuous_on S* (λ*x*. *exp* (*pi* ∗ i ∗ *q x*))
proof (*intro continuous_intros*)
show *continuous_on S q*
using ‹*U* = *S* ∪ *T*› *continuous_on_subset contq* **by** *blast*
qed
show *continuous_on T* (λ*x*. *1* / *exp* (*pi* ∗ i ∗ *q x*))
proof (*intro continuous_intros*)
show *continuous_on T q*
using ‹*U* = *S* ∪ *T*› *continuous_on_subset contq* **by** *auto*
qed *auto*
qed (*use eqST* **in** *auto*)
qed (*use eqST* **in** ‹*auto simp*: *norm_divide*›)
then obtain *h* **where** *conth*: *continuous_on U h* **and** *heq*: ⋀*x*. *x* ∈ *U* ⟹ *g x* = *exp* (*h x*)
by (*metis Borsukian_continuous_logarithm_circle assms*)
obtain *v w* **where** *v* ∈ *V w* ∈ *W*
using ‹*V* ≠ {}› ‹*W* ≠ {}› **by** *blast*
then have *vw*: *v* ∈ *S* ∩ *T w* ∈ *S* ∩ *T*
using *VW* **by** *auto*
have *iff*: *2* ∗ *pi* ≤ *cmod* (*2* ∗ *of_int m* ∗ *of_real pi* ∗ i − *2* ∗ *of_int n* ∗ *of_real pi* ∗ i)

⟷ *1* ≤ *abs* (*m* − *n*) **for** *m n*
proof −
have *2* ∗ *pi* ≤ *cmod* (*2* ∗ *of_int m* ∗ *of_real pi* ∗ i − *2* ∗ *of_int n* ∗ *of_real pi* ∗ i)
⟷ *2* ∗ *pi* ≤ *cmod* ((*2* ∗ *pi* ∗ i) ∗ (*of_int m* − *of_int n*))
by (*simp add*: *algebra_simps*)
also have ... ⟷ *2* ∗ *pi* ≤ *2* ∗ *pi* ∗ *cmod* (*of_int m* − *of_int n*)
by (*simp add*: *norm_mult*)
also have ... ⟷ *1* ≤ *abs* (*m* − *n*)
by *simp* (*metis norm_of_int of_int_1_le_iff of_int_abs of_int_diff*)
finally show *?thesis* **.**
qed
have ∗: ∃ *n*::*int*. *h x* − (*pi* ∗ i ∗ *q x*) = (*of_int*(*2*∗*n*) ∗ *pi*) ∗ i **if** *x* ∈ *S* **for** *x*
using *that S* ‹*U* = *S* ∪ *T*› *heq exp_eq* [*symmetric*] **by** (*simp add*: *algebra_simps*)
moreover have (λ*x*. *h x* − (*pi* ∗ i ∗ *q x*)) *constant_on S*
proof (*rule continuous_discrete_range_constant* [*OF* ‹*connected S*›])
have *continuous_on S h continuous_on S q*
using ‹*U* = *S* ∪ *T*› *continuous_on_subset conth contq* **by** *blast*+
then show *continuous_on S* (λ*x*. *h x* − (*pi* ∗ i ∗ *q x*))
by (*intro continuous_intros*)
have *2*∗*pi* ≤ *cmod* (*h y* − (*pi* ∗ i ∗ *q y*) − (*h x* − (*pi* ∗ i ∗ *q x*)))
if *x* ∈ *S y* ∈ *S* **and** *ne*: *h y* − (*pi* ∗ i ∗ *q y*) ≠ *h x* − (*pi* ∗ i ∗ *q x*) **for** *x y*
using ∗ [*OF* ‹*x* ∈ *S*›] ∗ [*OF* ‹*y* ∈ *S*›] *ne* **by** (*auto simp*: *iff*)
then show ⋀*x*. *x* ∈ *S* ⟹
∃ *e*>*0*. ∀ *y*. *y* ∈ *S* ∧ *h y* − (*pi* ∗ i ∗ *q y*) ≠ *h x* − (*pi* ∗ i ∗ *q x*) ⟶
e ≤ *cmod* (*h y* − (*pi* ∗ i ∗ *q y*) − (*h x* − (*pi* ∗ i ∗ *q x*)))
by (*rule_tac x*=*2*∗*pi* **in** *exI*) *auto*
qed
ultimately
obtain *m* **where** *m*: ⋀*x*. *x* ∈ *S* ⟹ *h x* − (*pi* ∗ i ∗ *q x*) = (*of_int*(*2*∗*m*) ∗ *pi*) ∗ i
using *vw* **by** (*force simp*: *constant_on_def*)
have ∗: ∃ *n*::*int*. *h x* = − (*pi* ∗ i ∗ *q x*) + (*of_int*(*2*∗*n*) ∗ *pi*) ∗ i **if** *x* ∈ *T* **for** *x*
unfolding *exp_eq* [*symmetric*]
using *that T* ‹*U* = *S* ∪ *T*› **by** (*simp add*: *exp_minus field_simps heq* [*symmetric*])
moreover have (λ*x*. *h x* + (*pi* ∗ i ∗ *q x*)) *constant_on T*
proof (*rule continuous_discrete_range_constant* [*OF* ‹*connected T*›])
have *continuous_on T h continuous_on T q*
using ‹*U* = *S* ∪ *T*› *continuous_on_subset conth contq* **by** *blast*+
then show *continuous_on T* (λ*x*. *h x* + (*pi* ∗ i ∗ *q x*))
by (*intro continuous_intros*)
have *2*∗*pi* ≤ *cmod* (*h y* + (*pi* ∗ i ∗ *q y*) − (*h x* + (*pi* ∗ i ∗ *q x*)))
if *x* ∈ *T y* ∈ *T* **and** *ne*: *h y* + (*pi* ∗ i ∗ *q y*) ≠ *h x* + (*pi* ∗ i ∗ *q x*) **for** *x y*
using ∗ [*OF* ‹*x* ∈ *T*›] ∗ [*OF* ‹*y* ∈ *T*›] *ne* **by** (*auto simp*: *iff*)
then show ⋀*x*. *x* ∈ *T* ⟹
∃ *e*>*0*. ∀ *y*. *y* ∈ *T* ∧ *h y* + (*pi* ∗ i ∗ *q y*) ≠ *h x* + (*pi* ∗ i ∗ *q x*) ⟶
e ≤ *cmod* (*h y* + (*pi* ∗ i ∗ *q y*) − (*h x* + (*pi* ∗ i ∗ *q x*)))
by (*rule_tac x*=*2*∗*pi* **in** *exI*) *auto*

qed
ultimately
obtain *n* **where** *n*: $\bigwedge x.\ x \in T \Longrightarrow h\ x + (pi * i * q\ x) = (of\_int(2*n) * pi) * i$
using *vw* **by** (*force simp*: *constant_on_def*)
show *False*
using *m* [*of v*] *m* [*of w*] *n* [*of v*] *n* [*of w*] *vw*
by (*auto simp*: *algebra_simps* ‹*v* ∈ *V*› ‹*w* ∈ *W*› *qV qW*)
qed
qed

corollary *contractible_imp_unicoherent*:
fixes *U* :: *'a*::*euclidean_space set*
assumes *contractible U* **shows** *unicoherent U*
by (*simp add*: *Borsukian_imp_unicoherent assms contractible_imp_Borsukian*)

corollary *convex_imp_unicoherent*:
fixes *U* :: *'a*::*euclidean_space set*
assumes *convex U* **shows** *unicoherent U*
by (*simp add*: *Borsukian_imp_unicoherent assms convex_imp_Borsukian*)

If the type class constraint can be relaxed, I don't know how!

corollary *unicoherent_UNIV*: *unicoherent* (*UNIV* :: *'a* :: *euclidean_space set*)
by (*simp add*: *convex_imp_unicoherent*)

lemma *unicoherent_monotone_image_compact*:
fixes *T* :: *'b* :: *t2_space set*
assumes *S*: *unicoherent S compact S* **and** *contf*: *continuous_on S f* **and** *fim*: *f* ‘ *S* = *T*
and *conn*: $\bigwedge y.\ y \in T \Longrightarrow$ *connected* (*S* ∩ *f* −‘ {*y*})
shows *unicoherent T*
proof
fix *U V*
assume *UV*: *connected U connected V T* = *U* ∪ *V*
and *cloU*: *closedin* (*top_of_set T*) *U*
and *cloV*: *closedin* (*top_of_set T*) *V*
moreover have *compact T*
using ‹*compact S*› *compact_continuous_image contf fim* **by** *blast*
ultimately have *closed U closed V*
by (*auto simp*: *closedin_closed_eq compact_imp_closed*)
let *?SUV* = (*S* ∩ *f* −‘ *U*) ∩ (*S* ∩ *f* −‘ *V*)
have *UV_eq*: *f* ‘ *?SUV* = *U* ∩ *V*
using ‹*T* = *U* ∪ *V*› *fim* **by** *force+*
have *connected* (*f* ‘ *?SUV*)
proof (*rule connected_continuous_image*)
show *continuous_on ?SUV f*
by (*meson contf continuous_on_subset inf_le1*)

show *connected ?SUV*
proof (*rule unicoherentD* [*OF* ‹*unicoherent S*›, *of S* ∩ *f* −‘ *U S* ∩ *f* −‘ *V*])
have $\bigwedge C$. *closedin* (*top_of_set S*) *C* ⟹ *closedin* (*top_of_set T*) (*f* ‘ *C*)
by (*metis* ‹*compact S*› *closed_subset closedin_compact closedin_imp_subset compact_continuous_image compact_imp_closed contf continuous_on_subset fim image_mono*)
then show *connected* (*S* ∩ *f* −‘ *U*) *connected* (*S* ∩ *f* −‘ *V*)
using *UV* **by** (*auto simp*: *conn intro*: *connected_closed_monotone_preimage* [*OF contf fim*])
show *S* = (*S* ∩ *f* −‘ *U*) ∪ (*S* ∩ *f* −‘ *V*)
using *UV fim* **by** *blast*
show *closedin* (*top_of_set S*) (*S* ∩ *f* −‘ *U*)
closedin (*top_of_set S*) (*S* ∩ *f* −‘ *V*)
by (*auto simp*: *continuous_on_imp_closedin cloU cloV contf fim*)
qed
qed
with *UV_eq* **show** *connected* (*U* ∩ *V*)
by *simp*
qed

6.41.18 Several common variants of unicoherence

lemma *connected_frontier_simple*:
fixes *S* :: ′*a* :: *euclidean_space set*
assumes *connected S connected*(− *S*) **shows** *connected*(*frontier S*)
unfolding *frontier_closures*
by (*rule unicoherentD* [*OF unicoherent_UNIV*]; *simp add*: *assms connected_imp_connected_closure flip*: *closure_Un*)

lemma *connected_frontier_component_complement*:
fixes *S* :: ′*a* :: *euclidean_space set*
assumes *connected S C* ∈ *components*(− *S*) **shows** *connected*(*frontier C*)
by (*meson assms component_complement_connected connected_frontier_simple in_components_connected*)

lemma *connected_frontier_disjoint*:
fixes *S* :: ′*a* :: *euclidean_space set*
assumes *connected S connected T disjnt S T* **and** *ST*: *frontier S* ⊆ *frontier T*
shows *connected*(*frontier S*)
proof (*cases S* = *UNIV*)
case *True* **then show** *?thesis*
by *simp*
next
case *False*
then have −*S* ≠ {}
by *blast*
then obtain *C* **where** *C*: *C* ∈ *components*(− *S*) **and** *T* ⊆ *C*
by (*metis ComplI disjnt_iff subsetI exists_component_superset* ‹*disjnt S T*› ‹*connected T*›)
moreover have *frontier S* = *frontier C*

proof −
have *frontier C* ⊆ *frontier S*
using *C frontier_complement frontier_of_components_subset* **by** *blast*
moreover have *x* ∈ *frontier C* **if** *x* ∈ *frontier S* **for** *x*
proof −
have *x* ∈ *closure C*
using *that* **unfolding** *frontier_def*
by (*metis* (*no_types*) *Diff_eq ST* ‹*T* ⊆ *C*› *closure_mono contra_subsetD frontier_def le_inf_iff that*)
moreover have *x* ∉ *interior C*
using *that* **unfolding** *frontier_def*
by (*metis C Compl_eq_Diff_UNIV Diff_iff subsetD in_components_subset interior_diff interior_mono*)
ultimately show *?thesis*
by (*auto simp*: *frontier_def*)
qed
ultimately show *?thesis*
by *blast*
qed
ultimately show *?thesis*
using ‹*connected S*› *connected_frontier_component_complement* **by** *auto*
qed

6.41.19 Some separation results

lemma *separation_by_component_closed_pointwise*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *closed S* ¬ *connected_component* (− *S*) *a b*
obtains *C* **where** *C* ∈ *components S* ¬ *connected_component*(− *C*) *a b*
proof (*cases a* ∈ *S* ∨ *b* ∈ *S*)
case *True*
then show *?thesis*
using *connected_component_in componentsI that* **by** *fastforce*
next
case *False*
obtain *T* **where** *T* ⊆ *S closed T T* ≠ {}
and *nab*: ¬ *connected_component* (− *T*) *a b*
and *conn*: $\bigwedge$*U*. *U* ⊂ *T* ⟹ *connected_component* (− *U*) *a b*
using *closed_irreducible_separator* [*OF assms*] **by** *metis*
moreover have *connected T*
proof −
have *ab*: *frontier*(*connected_component_set* (− *T*) *a*) = *T frontier*(*connected_component_set* (− *T*) *b*) = *T*
using *frontier_minimal_separating_closed_pointwise*
by (*metis False* ‹*T* ⊆ *S*› ‹*closed T*› *connected_component_sym conn connected_component_eq_empty connected_component_intermediate_subset empty_subsetI nab*)+
have *connected* (*frontier* (*connected_component_set* (− *T*) *a*))
proof (*rule connected_frontier_disjoint*)

show *disjnt* (*connected_component_set* (− *T*) *a*) (*connected_component_set* (− *T*) *b*)
unfolding *disjnt_iff*
by (*metis connected_component_eq connected_component_eq_empty connected_component_idemp mem_Collect_eq nab*)
show *frontier* (*connected_component_set* (− *T*) *a*) ⊆ *frontier* (*connected_component_set* (− *T*) *b*)
by (*simp add*: *ab*)
qed *auto*
with *ab* ‹*closed T*› **show** *?thesis*
by *simp*
qed
ultimately obtain *C* **where** *C* ∈ *components S T* ⊆ *C*
using *exists_component_superset* [*of T S*] **by** *blast*
then show *?thesis*
by (*meson Compl_anti_mono connected_component_of_subset nab that*)
qed

lemma *separation_by_component_closed*:
fixes *S* :: ′*a* :: *euclidean_space set*
assumes *closed S* ¬ *connected*(− *S*)
obtains *C* **where** *C* ∈ *components S* ¬ *connected*(− *C*)
proof −
obtain *x y* **where** *closed S x* ∉ *S y* ∉ *S* **and** ¬ *connected_component* (− *S*) *x y*
using *assms* **by** (*auto simp*: *connected_iff_connected_component*)
then obtain *C* **where** *C* ∈ *components S* ¬ *connected_component*(− *C*) *x y*
using *separation_by_component_closed_pointwise* **by** *metis*
then show *thesis*
by (*metis Compl_iff* ‹*x* ∉ *S*› ‹*y* ∉ *S*› *connected_component_eq_self in_components_subset mem_Collect_eq subsetD that*)
qed

lemma *separation_by_Un_closed_pointwise*:
fixes *S* :: ′*a* :: *euclidean_space set*
assumes *ST*: *closed S closed T S* ∩ *T* = {}
and *conS*: *connected_component* (− *S*) *a b* **and** *conT*: *connected_component* (− *T*) *a b*
shows *connected_component* (− (*S* ∪ *T*)) *a b*
proof (*rule ccontr*)
have *a* ∉ *S b* ∉ *S a* ∉ *T b* ∉ *T*
using *conS conT connected_component_in* **by** *auto*
assume ¬ *connected_component* (− (*S* ∪ *T*)) *a b*
then obtain *C* **where** *C* ∈ *components* (*S* ∪ *T*) **and** *C*: ¬ *connected_component*(− *C*) *a b*
using *separation_by_component_closed_pointwise assms* **by** *blast*
then have *C* ⊆ *S* ∨ *C* ⊆ *T*
proof −
have *connected C C* ⊆ *S* ∪ *T*

using ‹$C \in$ *components* $(S \cup T)$› *in_components_subset* **by** (*blast elim*: *componentsE*)+
moreover then have $C \cap T = \{\} \vee C \cap S = \{\}$
by (*metis Int_empty_right ST inf.commute connected_closed*)
ultimately show *?thesis*
by *blast*
qed
then show *False*
by (*meson Compl_anti_mono C conS conT connected_component_of_subset*)
qed

lemma *separation_by_Un_closed*:
fixes S :: *'a* :: *euclidean_space set*
assumes *ST*: *closed* S *closed* T $S \cap T = \{\}$ **and** *conS*: *connected*$(- S)$ **and** *conT*: *connected*$(- T)$
shows *connected*$(- (S \cup T))$
using *assms separation_by_Un_closed_pointwise*
by (*fastforce simp add*: *connected_iff_connected_component*)

lemma *open_unicoherent_UNIV*:
fixes S :: *'a* :: *euclidean_space set*
assumes *open* S *open* T *connected* S *connected* T $S \cup T = UNIV$
shows *connected*$(S \cap T)$
proof −
have *connected*$(- (-S \cup -T))$
by (*metis closed_Compl compl_sup compl_top_eq double_compl separation_by_Un_closed assms*)
then show *?thesis*
by *simp*
qed

lemma *separation_by_component_open_aux*:
fixes S :: *'a* :: *euclidean_space set*
assumes *ST*: *closed* S *closed* T $S \cap T = \{\}$
and $S \neq \{\}$ $T \neq \{\}$
obtains C **where** $C \in$ *components*$(-(S \cup T))$ $C \neq \{\}$ *frontier* $C \cap S \neq \{\}$ *frontier* $C \cap T \neq \{\}$
proof (*rule ccontr*)
let $?S = S \cup \bigcup\{C \in$ *components*$(- (S \cup T))$. *frontier* $C \subseteq S\}$
let $?T = T \cup \bigcup\{C \in$ *components*$(- (S \cup T))$. *frontier* $C \subseteq T\}$
assume ¬ *thesis*
with *that* **have** ∗: *frontier* $C \cap S = \{\} \vee$ *frontier* $C \cap T = \{\}$
if C: $C \in$ *components* $(- (S \cup T))$ $C \neq \{\}$ **for** C
using C **by** *blast*
have $\exists A\ B$::*'a set*. *closed* $A \wedge$ *closed* $B \wedge UNIV \subseteq A \cup B \wedge A \cap B = \{\} \wedge A \neq \{\} \wedge B \neq \{\}$
proof (*intro exI conjI*)
have *frontier* $(\bigcup\{C \in$ *components* $(- S \cap - T)$. *frontier* $C \subseteq S\}) \subseteq S$
using *subset_trans* [*OF frontier_Union_subset_closure*]

```
by (metis (no_types, lifting) SUP_least ‹closed S› closure_minimal mem_Collect_eq)
then have frontier ?S ⊆ S
  by (simp add: frontier_subset_eq assms  subset_trans [OF frontier_Un_subset])
then show closed ?S
  using frontier_subset_eq by fastforce
have frontier (⋃{C ∈ components (− S ∩ − T). frontier C ⊆ T}) ⊆ T
  using subset_trans [OF frontier_Union_subset_closure]
 by (metis (no_types, lifting) SUP_least ‹closed T› closure_minimal mem_Collect_eq)
then have frontier ?T ⊆ T
  by (simp add: frontier_subset_eq assms  subset_trans [OF frontier_Un_subset])
then show closed ?T
  using frontier_subset_eq by fastforce
have UNIV ⊆ (S ∪ T) ∪ ⋃(components(− (S ∪ T)))
  using Union_components by blast
also have ...  ⊆ ?S ∪ ?T
proof −
  have C ∈ components (−(S ∪ T)) ∧ frontier C ⊆ S ∨
        C ∈ components (−(S ∪ T)) ∧ frontier C ⊆ T
    if C ∈ components (− (S ∪ T)) C ≠ {} for C
    using ∗ [OF that] that
      by clarify (metis (no_types, lifting) UnE ‹closed S› ‹closed T› closed_Un
disjoint_iff_not_equal frontier_of_components_closed_complement subsetCE)
  then show ?thesis
    by blast
qed
finally show UNIV ⊆ ?S ∪ ?T .
have ⋃{C ∈ components (− (S ∪ T)). frontier C ⊆ S} ∪
      ⋃{C ∈ components (− (S ∪ T)). frontier C ⊆ T} ⊆ − (S ∪ T)
  using in_components_subset by fastforce
moreover have ⋃{C ∈ components (− (S ∪ T)). frontier C ⊆ S} ∩
              ⋃{C ∈ components (− (S ∪ T)). frontier C ⊆ T} = {}
proof −
  have C ∩ C' = {} if C ∈ components (− (S ∪ T)) frontier C ⊆ S
                     C' ∈ components (− (S ∪ T)) frontier C' ⊆ T for C C'
  proof −
    have NUN: − S ∩ − T ≠ UNIV
      using ‹T ≠ {}› by blast
    have C ≠ C'
    proof
      assume C = C'
      with that have frontier C' ⊆ S ∩ T
        by simp
      also have ... = {}
        using ‹S ∩ T = {}› by blast
      finally have C' = {} ∨ C' = UNIV
        using frontier_eq_empty by auto
      then show False
       using ‹C = C'› NUN that by (force simp: dest: in_components_nonempty
in_components_subset)
```

qed
with *that* **show** *?thesis*
by (*simp add*: *components_nonoverlap* [*of* _ −(*S* ∪ *T*)])
qed
then show *?thesis*
by *blast*
qed
ultimately show *?S* ∩ *?T* = {}
using *ST* **by** *blast*
show *?S* ≠ {} *?T* ≠ {}
using ‹*S* ≠ {}› ‹*T* ≠ {}› **by** *blast+*
qed
then show *False*
by (*metis Compl_disjoint connected_UNIV compl_bot_eq compl_unique connected_closedD inf_sup_absorb sup_compl_top_left1 top.extremum_uniqueI*)
qed

proposition *separation_by_component_open*:
fixes *S* :: ′*a* :: *euclidean_space set*
assumes *open S* **and** *non*: ¬ *connected*(− *S*)
obtains *C* **where** *C* ∈ *components S* ¬ *connected*(− *C*)
proof −
obtain *T U*
where *closed T closed U* **and** *TU*: *T* ∪ *U* = − *S* *T* ∩ *U* = {} *T* ≠ {} *U* ≠ {}
using *assms* **by** (*auto simp*: *connected_closed_set closed_def*)
then obtain *C* **where** *C*: *C* ∈ *components*(−(*T* ∪ *U*)) *C* ≠ {}
and *frontier C* ∩ *T* ≠ {} *frontier C* ∩ *U* ≠ {}
using *separation_by_component_open_aux* [*OF* ‹*closed T*› ‹*closed U*› ‹*T* ∩ *U* = {}›] **by** *force*
show *thesis*
proof
show *C* ∈ *components S*
using *C*(*1*) *TU*(*1*) **by** *auto*
show ¬ *connected* (− *C*)
proof
assume *connected* (− *C*)
then have *connected* (*frontier C*)
using *connected_frontier_simple* [*of C*] ‹*C* ∈ *components S*› *in_components_connected* **by** *blast*
then show *False*
unfolding *connected_closed*
by (*metis C*(*1*) *TU*(*2*) ‹*closed T*› ‹*closed U*› ‹*frontier C* ∩ *T* ≠ {}› ‹*frontier C* ∩ *U* ≠ {}› *closed_Un frontier_of_components_closed_complement inf_bot_right inf_commute*)
qed
qed
qed

lemma *separation_by_Un_open*:
fixes S :: $'a$ :: *euclidean_space set*
assumes *open* S *open* T $S \cap T = \{\}$ **and** *cS*: *connected*$(-S)$ **and** *cT*: *connected*$(-T)$
shows *connected*$(- (S \cup T))$
using *assms unicoherent_UNIV* **unfolding** *unicoherent_def* **by** *force*

lemma *nonseparation_by_component_eq*:
fixes S :: $'a$ :: *euclidean_space set*
assumes *open* $S \lor$ *closed* S
shows $((\forall C \in$ *components* S. *connected*$(-C)) \longleftrightarrow$ *connected*$(- S))$ (**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **with** *assms* **show** *?rhs*
by (*meson separation_by_component_closed separation_by_component_open*)
next
assume *?rhs* **with** *assms* **show** *?lhs*
using *component_complement_connected* **by** *force*
qed

Another interesting equivalent of an inessential mapping into C-0

proposition *inessential_eq_extensible*:
fixes f :: $'a$::*euclidean_space* $\Rightarrow$ *complex*
assumes *closed* S
shows $(\exists a.$ *homotopic_with_canon* $(\lambda h.$ *True*$)$ S $(-\{0\})$ f $(\lambda t.\ a)) \longleftrightarrow$
$(\exists g.$ *continuous_on UNIV* $g \land (\forall x \in S.\ g\ x = f\ x) \land (\forall x.\ g\ x \neq 0))$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then obtain a **where** a: *homotopic_with_canon* $(\lambda h.$ *True*$)$ S $(-\{0\})$ f $(\lambda t.\ a)$
..
show *?rhs*
proof (*cases* $S = \{\}$)
case *True*
with a **show** *?thesis* **by** *force*
next
case *False*
have *anr*: *ANR* $(-\{0$::*complex*$\})$
by (*simp add*: *ANR_delete open_Compl open_imp_ANR*)
obtain g **where** *contg*: *continuous_on UNIV* g **and** *gim*: g ' *UNIV* $\subseteq -\{0\}$
and *gf*: $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
proof (*rule Borsuk_homotopy_extension_homotopic* [*OF* _ _ *continuous_on_const* _ *homotopic_with_symD* [*OF* *a*]])
show *closedin* (*top_of_set UNIV*) S
using *assms* **by** *auto*
show *range* $(\lambda t.\ a) \subseteq - \{0\}$
using *a homotopic_with_imp_subset2 False* **by** *blast*

qed (*use anr that* **in** ‹*force*+›)
then show *?thesis*
by *force*
qed
next
assume *?rhs*
then obtain *g* **where** *contg*: *continuous_on UNIV g*
and *gf*: ⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x* **and** *non0*: ⋀*x*. *g x* ≠ *0*
by *metis*
obtain *h k*::′*a*⇒′*a* **where** *hk*: *homeomorphism* (*ball 0 1*) *UNIV h k*
using *homeomorphic_ball01_UNIV homeomorphic_def* **by** *blast*
then have *continuous_on* (*ball 0 1*) (*g* ∘ *h*)
by (*meson contg continuous_on_compose continuous_on_subset homeomorphism_cont1 top_greatest*)
then obtain *j* **where** *contj*: *continuous_on* (*ball 0 1*) *j*
and *j*: ⋀*z*. *z* ∈ *ball 0 1* ⟹ *exp*(*j z*) = (*g* ∘ *h*) *z*
by (*metis* (*mono_tags*, *hide_lams*) *continuous_logarithm_on_ball comp_apply non0*)
have [*simp*]: ⋀*x*. *x* ∈ *S* ⟹ *h* (*k x*) = *x*
using *hk homeomorphism_apply2* **by** *blast*
have ∃ζ. *continuous_on S* ζ∧ (∀ *x*∈*S*. *f x* = *exp* (ζ *x*))
proof (*intro exI conjI ballI*)
show *continuous_on S* (*j* ∘ *k*)
proof (*rule continuous_on_compose*)
show *continuous_on S k*
by (*meson continuous_on_subset hk homeomorphism_cont2 top_greatest*)
show *continuous_on* (*k* ‘ *S*) *j*
by (*auto intro*: *continuous_on_subset* [*OF contj*] *simp flip*: *homeomorphism_image2* [*OF hk*])
qed
show *f x* = *exp* ((*j* ∘ *k*) *x*) **if** *x* ∈ *S* **for** *x*
proof −
have *f x* = (*g* ∘ *h*) (*k x*)
by (*simp add*: *gf that*)
also have ... = *exp* (*j* (*k x*))
by (*metis rangeI homeomorphism_image2* [*OF hk*] *j*)
finally show *?thesis* **by** *simp*
qed
qed
then show *?lhs*
by (*simp add*: *inessential_eq_continuous_logarithm*)
qed

lemma *inessential_on_clopen_Union*:
fixes $\mathcal{F}$ :: ′*a*::*euclidean_space set set*
assumes *T*: *path_connected T*
and ⋀*S*. *S* ∈ $\mathcal{F}$ ⟹ *closedin* (*top_of_set* (⋃$\mathcal{F}$)) *S*
and ⋀*S*. *S* ∈ $\mathcal{F}$ ⟹ *openin* (*top_of_set* (⋃$\mathcal{F}$)) *S*
and *hom*: ⋀*S*. *S* ∈ $\mathcal{F}$ ⟹ ∃ *a*. *homotopic_with_canon* (λ*x*. *True*) *S T f* (λ*x*.

```
a)
  obtains a where homotopic_with_canon (λx. True) (⋃ℱ) T f (λx. a)
proof (cases ⋃ℱ = {})
  case True
  with that show ?thesis
    by force
next
  case False
  then obtain C where C ∈ ℱ C ≠ {}
    by blast
  then obtain a where clo: closedin (top_of_set (⋃ℱ)) C
    and ope: openin (top_of_set (⋃ℱ)) C
    and homotopic_with_canon (λx. True) C T f (λx. a)
    using assms by blast
  with ‹C ≠ {}› have f ‘ C ⊆ T a ∈ T
    using homotopic_with_imp_subset1 homotopic_with_imp_subset2 by blast+
  have homotopic_with_canon (λx. True) (⋃ℱ) T f (λx. a)
  proof (rule homotopic_on_clopen_Union)
    show ⋀S. S ∈ ℱ ⟹ closedin (top_of_set (⋃ℱ)) S
         ⋀S. S ∈ ℱ ⟹ openin (top_of_set (⋃ℱ)) S
      by (simp_all add: assms)
    show homotopic_with_canon (λx. True) S T f (λx. a) if S ∈ ℱ for S
    proof (cases S = {})
      case True
      then show ?thesis
        by auto
    next
      case False
      then obtain b where b ∈ S
        by blast
      obtain c where c: homotopic_with_canon (λx. True) S T f (λx. c)
        using ‹S ∈ ℱ› hom by blast
      then have c ∈ T
        using ‹b ∈ S› homotopic_with_imp_subset2 by blast
      then have homotopic_with_canon (λx. True) S T (λx. a) (λx. c)
        using T ‹a ∈ T› homotopic_constant_maps path_connected_component
        by (simp add: homotopic_constant_maps path_connected_component)
      then show ?thesis
        using c homotopic_with_symD homotopic_with_trans by blast
    qed
  qed
  then show ?thesis ..
qed

proposition Janiszewski_dual:
  fixes S :: complex set
  assumes
   compact S compact T connected S connected T connected(− (S ∪ T))
 shows connected(S ∩ T)
```

proof −
  have *ST*: *compact* (*S* ∪ *T*)
    by (*simp add*: *assms compact_Un*)
  with *Borsukian_imp_unicoherent* [*of S* ∪ *T*] *ST assms*
  show *?thesis*
    by (*auto simp*: *closed_subset compact_imp_closed Borsukian_separation_compact unicoherent_def*)
qed

end

6.42 The Jordan Curve Theorem and Applications

theory *Jordan_Curve*
  imports *Arcwise_Connected Further_Topology*
begin

6.42.1 Janiszewski's theorem

lemma *Janiszewski_weak*:
  fixes *a b*::*complex*
  assumes *compact S compact T* **and** *conST*: *connected*(*S* ∩ *T*)
    and *ccS*: *connected_component* (− *S*) *a b* **and** *ccT*: *connected_component* (− *T*) *a b*
    shows *connected_component* (− (*S* ∪ *T*)) *a b*
proof −
  have [*simp*]: *a* ∉ *S a* ∉ *T b* ∉ *S b* ∉ *T*
    by (*meson ComplD ccS ccT connected_component_in*)+
  have *clo*: *closedin* (*top_of_set* (*S* ∪ *T*)) *S closedin* (*top_of_set* (*S* ∪ *T*)) *T*
    by (*simp_all add*: *assms closed_subset compact_imp_closed*)
  obtain *g* **where** *contg*: *continuous_on S g*
          and *g*: ⋀*x*. *x* ∈ *S* ⟹ *exp* (i∗ *of_real* (*g x*)) = (*x* − *a*) $/_R$ *cmod* (*x* − *a*) / ((*x* − *b*) $/_R$ *cmod* (*x* − *b*))
    using *ccS* ‹*compact S*›
    apply (*simp add*: *Borsuk_maps_homotopic_in_connected_component_eq* [*symmetric*])
    apply (*subst* (*asm*) *homotopic_circlemaps_divide*)
    apply (*auto simp*: *inessential_eq_continuous_logarithm_circle*)
    done
  obtain *h* **where** *conth*: *continuous_on T h*
          and *h*: ⋀*x*. *x* ∈ *T* ⟹ *exp* (i∗ *of_real* (*h x*)) = (*x* − *a*) $/_R$ *cmod* (*x* − *a*) / ((*x* − *b*) $/_R$ *cmod* (*x* − *b*))
    using *ccT* ‹*compact T*›
    apply (*simp add*: *Borsuk_maps_homotopic_in_connected_component_eq* [*symmetric*])
    apply (*subst* (*asm*) *homotopic_circlemaps_divide*)
    apply (*auto simp*: *inessential_eq_continuous_logarithm_circle*)
    done
  have *continuous_on* (*S* ∪ *T*) (λ*x*. (*x* − *a*) $/_R$ *cmod* (*x* − *a*)) *continuous_on* (*S* ∪ *T*) (λ*x*. (*x* − *b*) $/_R$ *cmod* (*x* − *b*))
    by (*intro continuous_intros*; *force*)+

moreover have (λx. (x − a) /$_R$ *cmod* (x − a)) ' (S ∪ T) ⊆ *sphere 0 1* (λx. (x − b) /$_R$ *cmod* (x − b)) ' (S ∪ T) ⊆ *sphere 0 1*
  by (*auto simp*: *divide_simps*)
moreover have ∃ g. *continuous_on* (S ∪ T) g ∧
  (∀ x∈S ∪ T. (x − a) /$_R$ *cmod* (x − a) / ((x − b) /$_R$ *cmod* (x − b)) = *exp* (i∗*complex_of_real* (g x)))
proof (*cases* S ∩ T = {})
  case *True*
  have *continuous_on* (S ∪ T) (λx. *if* x ∈ S *then* g x *else* h x)
    apply (*rule continuous_on_cases_local* [*OF clo contg conth*])
    using *True* **by** *auto*
  then show *?thesis*
    by (*rule_tac* x=(λx. *if* x ∈ S *then* g x *else* h x) **in** *exI*) (*auto simp*: g h)
next
  case *False*
  have *diffpi*: ∃ n. g x = h x + *2*∗ *of_int* n∗*pi* **if** x ∈ S ∩ T **for** x
  proof −
    have *exp* (i∗ *of_real* (g x)) = *exp* (i∗ *of_real* (h x))
      using *that* **by** (*simp add*: g h)
    then obtain n **where** *complex_of_real* (g x) = *complex_of_real* (h x) + *2*∗ *of_int* n∗*complex_of_real pi*
      apply (*auto simp*: *exp_eq*)
      by (*metis complex_i_not_zero distrib_left mult.commute mult_cancel_left*)
    then show *?thesis*
      apply (*rule_tac* x=n **in** *exI*)
      using *of_real_eq_iff* **by** *fastforce*
  qed
  have *contgh*: *continuous_on* (S ∩ T) (λx. g x − h x)
    by (*intro continuous_intros continuous_on_subset* [*OF contg*] *continuous_on_subset* [*OF conth*]) *auto*
  moreover have *disc*:
      ∃ e>*0*. ∀ y. y ∈ S ∩ T ∧ g y − h y ≠ g x − h x ⟶ e ≤ *norm* ((g y − h y) − (g x − h x))
      if x ∈ S ∩ T **for** x
  proof −
    obtain nx **where** nx: g x = h x + *2*∗ *of_int* nx∗*pi*
      using ‹x ∈ S ∩ T› *diffpi* **by** *blast*
    have *2*∗*pi* ≤ *norm* (g y − h y − (g x − h x)) **if** y: y ∈ S ∩ T **and** *neq*: g y − h y ≠ g x − h x **for** y
    proof −
      obtain ny **where** ny: g y = h y + *2*∗ *of_int* ny∗*pi*
        using ‹y ∈ S ∩ T› *diffpi* **by** *blast*
      { **assume** nx ≠ ny
        then have *1* ≤ |*real_of_int* ny − *real_of_int* nx|
          by *linarith*
        then have (*2*∗*pi*)∗*1* ≤ (*2*∗*pi*)∗|*real_of_int* ny − *real_of_int* nx|
          by *simp*
        also have ... = |*2*∗*real_of_int* ny∗*pi* − *2*∗*real_of_int* nx∗*pi*|
          by (*simp add*: *algebra_simps abs_if*)

```
        finally have 2∗pi ≤ |2∗real_of_int ny∗pi − 2∗real_of_int nx∗pi| by simp
       }
       with neq show ?thesis
         by (simp add: nx ny)
     qed
     then show ?thesis
       by (rule_tac x=2∗pi in exI) auto
   qed
   ultimately have (λx. g x − h x) constant_on S ∩ T
     using continuous_discrete_range_constant [OF conST contgh] by blast
   then obtain z where z: ⋀x. x ∈ S ∩ T ⟹ g x − h x = z
     by (auto simp: constant_on_def)
   obtain w where exp(i ∗ of_real(h w)) = exp (i ∗ of_real(z + h w))
     using disc z False
     by auto (metis diff_add_cancel g h of_real_add)
   then have [simp]: exp (i∗ of_real z) = 1
     by (metis cis_conv_exp cis_mult exp_not_eq_zero mult_cancel_right1)
   show ?thesis
   proof (intro exI conjI)
     show continuous_on (S ∪ T) (λx. if x ∈ S then g x else z + h x)
         apply (intro continuous_intros continuous_on_cases_local [OF clo contg]
conth)
       using z by fastforce
   qed (auto simp: g h algebra_simps exp_add)
 qed
 ultimately have ∗: homotopic_with_canon (λx. True) (S ∪ T) (sphere 0 1)
                    (λx. (x − a) /R cmod (x − a)) (λx. (x − b) /R cmod (x −
b))
  by (subst homotopic_circlemaps_divide) (auto simp: inessential_eq_continuous_logarithm_circle)
 show ?thesis
  apply (rule Borsuk_maps_homotopic_in_connected_component_eq [THEN iffD1])
   using assms by (auto simp: ∗)
qed


theorem Janiszewski:
 fixes a b :: complex
 assumes compact S closed T and conST: connected (S ∩ T)
    and ccS: connected_component (− S) a b and ccT: connected_component (−
T) a b
   shows connected_component (− (S ∪ T)) a b
proof −
 have path_component(− T) a b
   by (simp add: ‹closed T› ccT open_Compl open_path_connected_component)
 then obtain g where g: path g path_image g ⊆ − T pathstart g = a pathfinish
g = b
   by (auto simp: path_component_def)
 obtain C where C: compact C connected C a ∈ C b ∈ C C ∩ T = {}
 proof
```

```
    show compact (path_image g)
      by (simp add: ‹path g› compact_path_image)
    show connected (path_image g)
      by (simp add: ‹path g› connected_path_image)
  qed (use g in auto)
  obtain r where 0 < r and r: C ∪ S ⊆ ball 0 r
   by (metis ‹compact C› ‹compact S› bounded_Un compact_imp_bounded bounded_subset_ballD)
  have connected_component (− (S ∪ (T ∩ cball 0 r ∪ sphere 0 r))) a b
  proof (rule Janiszewski_weak [OF ‹compact S›])
    show comT': compact ((T ∩ cball 0 r) ∪ sphere 0 r)
      by (simp add: ‹closed T› closed_Int_compact compact_Un)
    have S ∩ (T ∩ cball 0 r ∪ sphere 0 r) = S ∩ T
      using r by auto
    with conST show connected (S ∩ (T ∩ cball 0 r ∪ sphere 0 r))
      by simp
    show connected_component (− (T ∩ cball 0 r ∪ sphere 0 r)) a b
      using conST C r
      apply (simp add: connected_component_def)
      apply (rule_tac x=C in exI)
      by auto
  qed (simp add: ccS)
  then obtain U where U: connected U U ⊆ − S U ⊆ − T ∪ − cball 0 r U ⊆
− sphere 0 r a ∈ U b ∈ U
    by (auto simp: connected_component_def)
  show ?thesis
    unfolding connected_component_def
  proof (intro exI conjI)
    show U ⊆ − (S ∪ T)
      using U r ‹0 < r› ‹a ∈ C› connected_Int_frontier [of U cball 0 r]
      apply simp
     by (metis ball_subset_cball compl_inf disjoint_eq_subset_Compl disjoint_iff_not_equal
inf.orderE inf_sup_aci(3) subsetCE)
  qed (auto simp: U)
qed

lemma Janiszewski_connected:
  fixes S :: complex set
  assumes ST: compact S closed T connected(S ∩ T)
      and notST: connected (− S) connected (− T)
    shows connected(− (S ∪ T))
using Janiszewski [OF ST]
by (metis IntD1 IntD2 notST compl_sup connected_iff_connected_component)
```

6.42.2 The Jordan Curve theorem

```
lemma exists_double_arc:
  fixes g :: real ⇒ 'a::real_normed_vector
  assumes simple_path g pathfinish g = pathstart g a ∈ path_image g b ∈ path_image
g a ≠ b
```

obtains *u d* **where** *arc u arc d pathstart u = a pathfinish u = b*
pathstart d = b pathfinish d = a
(path_image u) ∩ (path_image d) = {a,b}
(path_image u) ∪ (path_image d) = path_image g
proof −
obtain *u* **where** *u*: *0 ≤ u u ≤ 1 g u = a*
using *assms* **by** (*auto simp*: *path_image_def*)
define *h* **where** *h ≡ shiftpath u g*
have *simple_path h*
using ‹*simple_path g*› *simple_path_shiftpath* ‹*0 ≤ u*› ‹*u ≤ 1*› *assms*(*2*) *h_def* **by** *blast*
have *pathstart h = g u*
by (*simp add*: ‹*u ≤ 1*› *h_def pathstart_shiftpath*)
have *pathfinish h = g u*
by (*simp add*: ‹*0 ≤ u*› *assms h_def pathfinish_shiftpath*)
have *pihg*: *path_image h = path_image g*
by (*simp add*: ‹*0 ≤ u*› ‹*u ≤ 1*› *assms h_def path_image_shiftpath*)
then obtain *v* **where** *v*: *0 ≤ v v ≤ 1 h v = b*
using *assms* **by** (*metis* (*mono_tags*, *lifting*) *atLeastAtMost_iff imageE path_image_def*)
show *?thesis*
proof
show *arc* (*subpath 0 v h*)
by (*metis* (*no_types*) ‹*pathstart h = g u*› ‹*simple_path h*› *arc_simple_path_subpath* ‹*a ≠ b*› *atLeastAtMost_iff zero_le_one order_refl pathstart_def u*(*3*) *v*)
show *arc* (*subpath v 1 h*)
by (*metis* (*no_types*) ‹*pathfinish h = g u*› ‹*simple_path h*› *arc_simple_path_subpath* ‹*a ≠ b*› *atLeastAtMost_iff zero_le_one order_refl pathfinish_def u*(*3*) *v*)
show *pathstart* (*subpath 0 v h*) = *a*
by (*metis* ‹*pathstart h = g u*› *pathstart_def pathstart_subpath u*(*3*))
show *pathfinish* (*subpath 0 v h*) = *b* *pathstart* (*subpath v 1 h*) = *b*
by (*simp_all add*: *v*(*3*))
show *pathfinish* (*subpath v 1 h*) = *a*
by (*metis* ‹*pathfinish h = g u*› *pathfinish_def pathfinish_subpath u*(*3*))
show *path_image* (*subpath 0 v h*) ∩ *path_image* (*subpath v 1 h*) = {*a*, *b*}
proof
show *path_image* (*subpath 0 v h*) ∩ *path_image* (*subpath v 1 h*) ⊆ {*a*, *b*}
using *v* ‹*pathfinish* (*subpath v 1 h*) = *a*› ‹*simple_path h*›
apply (*auto simp*: *simple_path_def path_image_subpath image_iff Ball_def*)
by (*metis* (*full_types*) *less_eq_real_def less_irrefl less_le_trans*)
show {*a*, *b*} ⊆ *path_image* (*subpath 0 v h*) ∩ *path_image* (*subpath v 1 h*)
using *v* ‹*pathstart* (*subpath 0 v h*) = *a*› ‹*pathfinish* (*subpath v 1 h*) = *a*›
apply (*auto simp*: *path_image_subpath image_iff*)
by (*metis atLeastAtMost_iff order_refl*)
qed
show *path_image* (*subpath 0 v h*) ∪ *path_image* (*subpath v 1 h*) = *path_image g*
using *v* **apply** (*simp add*: *path_image_subpath pihg* [*symmetric*])
using *path_image_def* **by** *fastforce*
qed
qed

theorem *Jordan_curve*:
fixes *c* :: *real* ⇒ *complex*
assumes *simple_path c* **and** *loop*: *pathfinish c* = *pathstart c*
obtains *inner outer* **where**
inner ≠ {} *open inner connected inner*
outer ≠ {} *open outer connected outer*
bounded inner ¬ *bounded outer inner* ∩ *outer* = {}
inner ∪ *outer* = − *path_image c*
frontier inner = *path_image c*
frontier outer = *path_image c*
proof −
have *path c*
by (*simp add*: *assms simple_path_imp_path*)
have *hom*: (*path_image c*) *homeomorphic* (*sphere*(*0*::*complex*) *1*)
by (*simp add*: *assms homeomorphic_simple_path_image_circle*)
with *Jordan_Brouwer_separation* **have** ¬ *connected* (− (*path_image c*))
by *fastforce*
then obtain *inner* **where** *inner*: *inner* ∈ *components* (− *path_image c*) **and** *bounded inner*
using *cobounded_has_bounded_component* [*of* − (*path_image c*)]
using ‹¬ *connected* (− *path_image c*)› ‹*simple_path c*› *bounded_simple_path_image* **by** *force*
obtain *outer* **where** *outer*: *outer* ∈ *components* (− *path_image c*) **and** ¬ *bounded outer*
using *cobounded_unbounded_components* [*of* − (*path_image c*)]
using ‹*path c*› *bounded_path_image* **by** *auto*
show *?thesis*
proof
show *inner* ≠ {}
using *inner in_components_nonempty* **by** *auto*
show *open inner*
by (*meson* ‹*simple_path c*› *compact_imp_closed compact_simple_path_image inner open_Compl open_components*)
show *connected inner*
using *in_components_connected inner* **by** *blast*
show *outer* ≠ {}
using *outer in_components_nonempty* **by** *auto*
show *open outer*
by (*meson* ‹*simple_path c*› *compact_imp_closed compact_simple_path_image outer open_Compl open_components*)
show *connected outer*
using *in_components_connected outer* **by** *blast*
show *inner* ∩ *outer* = {}
by (*meson* ‹¬ *bounded outer*› ‹*bounded inner*› ‹*connected outer*› *bounded_subset components_maximal in_components_subset inner outer*)
show *fro_inner*: *frontier inner* = *path_image c*
by (*simp add*: *Jordan_Brouwer_frontier* [*OF hom inner*])

show *fro_outer*: *frontier outer* = *path_image c*
by (*simp add*: *Jordan_Brouwer_frontier* [*OF hom outer*])
have *False* **if** *m*: *middle* ∈ *components* (− *path_image c*) **and** *middle* ≠ *inner middle* ≠ *outer* **for** *middle*
proof −
have *frontier middle* = *path_image c*
by (*simp add*: *Jordan_Brouwer_frontier* [*OF hom*] *that*)
have *middle*: *open middle connected middle middle* ≠ {}
apply (*meson* ‹*simple_path c*› *compact_imp_closed compact_simple_path_image m open_Compl open_components*)
using *in_components_connected in_components_nonempty m* **by** *blast+*
obtain *a0 b0* **where** *a0* ∈ *path_image c b0* ∈ *path_image c a0* ≠ *b0*
using *simple_path_image_uncountable* [*OF* ‹*simple_path c*›]
by (*metis Diff_cancel countable_Diff_eq countable_empty insert_iff subsetI subset_singleton_iff*)
obtain *a b g* **where** *ab*: *a* ∈ *path_image c b* ∈ *path_image c a* ≠ *b*
and *arc g pathstart g* = *a pathfinish g* = *b*
and *pag_sub*: *path_image g* − {*a*,*b*} ⊆ *middle*
proof (*rule dense_accessible_frontier_point_pairs* [*OF* ‹*open middle*› ‹*connected middle*›, *of path_image c* ∩ *ball a0* (*dist a0 b0*) *path_image c* ∩ *ball b0* (*dist a0 b0*)])
show *openin* (*top_of_set* (*frontier middle*)) (*path_image c* ∩ *ball a0* (*dist a0 b0*))
openin (*top_of_set* (*frontier middle*)) (*path_image c* ∩ *ball b0* (*dist a0 b0*))
by (*simp_all add*: ‹*frontier middle* = *path_image c*› *openin_open_Int*)
show *path_image c* ∩ *ball a0* (*dist a0 b0*) ≠ *path_image c* ∩ *ball b0* (*dist a0 b0*)
using ‹*a0* ≠ *b0*› ‹*b0* ∈ *path_image c*› **by** *auto*
show *path_image c* ∩ *ball a0* (*dist a0 b0*) ≠ {}
using ‹*a0* ∈ *path_image c*› ‹*a0* ≠ *b0*› **by** *auto*
show *path_image c* ∩ *ball b0* (*dist a0 b0*) ≠ {}
using ‹*b0* ∈ *path_image c*› ‹*a0* ≠ *b0*› **by** *auto*
qed (*use arc_distinct_ends arc_imp_simple_path simple_path_endless that* **in** *fastforce*)
obtain *u d* **where** *arc u arc d*
and *pathstart u* = *a pathfinish u* = *b pathstart d* = *b pathfinish d* = *a*
and *ud_ab*: (*path_image u*) ∩ (*path_image d*) = {*a*,*b*}
and *ud_Un*: (*path_image u*) ∪ (*path_image d*) = *path_image c*
using *exists_double_arc* [*OF assms ab*] **by** *blast*
obtain *x y* **where** *x* ∈ *inner y* ∈ *outer*
using ‹*inner* ≠ {}› ‹*outer* ≠ {}› **by** *auto*
have *inner* ∩ *middle* = {} *middle* ∩ *outer* = {}
using *components_nonoverlap inner outer m that* **by** *blast+*
have *connected_component* (− (*path_image u* ∪ *path_image g* ∪ (*path_image d* ∪ *path_image g*))) *x y*
proof (*rule Janiszewski*)
show *compact* (*path_image u* ∪ *path_image g*)

by (*simp add*: ‹*arc g*› ‹*arc u*› *compact_Un compact_arc_image*)
show *closed* (*path_image d* ∪ *path_image g*)
by (*simp add*: ‹*arc d*› ‹*arc g*› *closed_Un closed_arc_image*)
show *connected* ((*path_image u* ∪ *path_image g*) ∩ (*path_image d* ∪ *path_image g*))
by (*metis Un_Diff_cancel* ‹*arc g*› ‹*path_image u* ∩ *path_image d* = {*a*, *b*}› ‹*pathfinish g* = *b*› ‹*pathstart g* = *a*› *connected_arc_image insert_Diff1 pathfinish_in_path_image pathstart_in_path_image sup_bot.right_neutral sup_commute sup_inf_distrib1*)
show *connected_component* (− (*path_image u* ∪ *path_image g*)) *x y*
unfolding *connected_component_def*
proof (*intro exI conjI*)
have *connected* ((*inner* ∪ (*path_image c* − *path_image u*)) ∪ (*outer* ∪ (*path_image c* − *path_image u*)))
proof (*rule connected_Un*)
show *connected* (*inner* ∪ (*path_image c* − *path_image u*))
apply (*rule connected_intermediate_closure* [*OF* ‹*connected inner*›])
using *fro_inner* [*symmetric*] **apply** (*auto simp*: *closure_subset frontier_def*)
done
show *connected* (*outer* ∪ (*path_image c* − *path_image u*))
apply (*rule connected_intermediate_closure* [*OF* ‹*connected outer*›])
using *fro_outer* [*symmetric*] **apply** (*auto simp*: *closure_subset frontier_def*)
done
have (*inner* ∩ *outer*) ∪ (*path_image c* − *path_image u*) ≠ {}
by (*metis* ‹*arc d*› *ud_ab Diff_Int Diff_cancel Un_Diff* ‹*inner* ∩ *outer* = {}› ‹*pathfinish d* = *a*› ‹*pathstart d* = *b*› *arc_simple_path insert_commute nonempty_simple_path_endless sup_bot_left ud_Un*)
then show (*inner* ∪ (*path_image c* − *path_image u*)) ∩ (*outer* ∪ (*path_image c* − *path_image u*)) ≠ {}
by *auto*
qed
then show *connected* (*inner* ∪ *outer* ∪ (*path_image c* − *path_image u*))
by (*metis sup.right_idem sup_assoc sup_commute*)
have *inner* ⊆ − *path_image u outer* ⊆ − *path_image u*
using *in_components_subset inner outer ud_Un* **by** *auto*
moreover have *inner* ⊆ − *path_image g outer* ⊆ − *path_image g*
using ‹*inner* ∩ *middle* = {}› ‹*inner* ⊆ − *path_image u*›
using ‹*middle* ∩ *outer* = {}› ‹*outer* ⊆ − *path_image u*› *pag_sub ud_ab* **by** *fastforce*+
moreover have *path_image c* − *path_image u* ⊆ − *path_image g*
using *in_components_subset m pag_sub ud_ab* **by** *fastforce*
ultimately show *inner* ∪ *outer* ∪ (*path_image c* − *path_image u*) ⊆ − (*path_image u* ∪ *path_image g*)
by *force*
show *x* ∈ *inner* ∪ *outer* ∪ (*path_image c* − *path_image u*)
by (*auto simp*: ‹*x* ∈ *inner*›)
show *y* ∈ *inner* ∪ *outer* ∪ (*path_image c* − *path_image u*)
by (*auto simp*: ‹*y* ∈ *outer*›)

qed
show *connected_component* (− (*path_image d* ∪ *path_image g*)) *x y*
unfolding *connected_component_def*
proof (*intro exI conjI*)
have *connected* ((*inner* ∪ (*path_image c* − *path_image d*)) ∪ (*outer* ∪ (*path_image c* − *path_image d*)))
proof (*rule connected_Un*)
show *connected* (*inner* ∪ (*path_image c* − *path_image d*))
apply (*rule connected_intermediate_closure* [*OF* ‹*connected inner*›])
using *fro_inner* [*symmetric*] **apply** (*auto simp*: *closure_subset frontier_def*)
done
show *connected* (*outer* ∪ (*path_image c* − *path_image d*))
apply (*rule connected_intermediate_closure* [*OF* ‹*connected outer*›])
using *fro_outer* [*symmetric*] **apply** (*auto simp*: *closure_subset frontier_def*)
done
have (*inner* ∩ *outer*) ∪ (*path_image c* − *path_image d*) ≠ {}
using ‹*arc u*› ‹*pathfinish u* = *b*› ‹*pathstart u* = *a*› *arc_imp_simple_path nonempty_simple_path_endless ud_Un ud_ab* **by** *fastforce*
then show (*inner* ∪ (*path_image c* − *path_image d*)) ∩ (*outer* ∪ (*path_image c* − *path_image d*)) ≠ {}
by *auto*
qed
then show *connected* (*inner* ∪ *outer* ∪ (*path_image c* − *path_image d*))
by (*metis sup.right_idem sup_assoc sup_commute*)
have *inner* ⊆ − *path_image d outer* ⊆ − *path_image d*
using *in_components_subset inner outer ud_Un* **by** *auto*
moreover have *inner* ⊆ − *path_image g outer* ⊆ − *path_image g*
using ‹*inner* ∩ *middle* = {}› ‹*inner* ⊆ − *path_image d*›
using ‹*middle* ∩ *outer* = {}› ‹*outer* ⊆ − *path_image d*› *pag_sub ud_ab* **by** *fastforce*+
moreover have *path_image c* − *path_image d* ⊆ − *path_image g*
using *in_components_subset m pag_sub ud_ab* **by** *fastforce*
ultimately show *inner* ∪ *outer* ∪ (*path_image c* − *path_image d*) ⊆ −(*path_image d* ∪ *path_image g*)
by *force*
show *x* ∈ *inner* ∪ *outer* ∪ (*path_image c* − *path_image d*)
by (*auto simp*: ‹*x* ∈ *inner*›)
show *y* ∈ *inner* ∪ *outer* ∪ (*path_image c* − *path_image d*)
by (*auto simp*: ‹*y* ∈ *outer*›)
qed
qed
then have *connected_component* (− (*path_image u* ∪ *path_image d* ∪ *path_image g*)) *x y*
by (*simp add*: *Un_ac*)
moreover have ¬(*connected_component* (− (*path_image c*)) *x y*)
by (*metis* (*no_types*, *lifting*) ‹¬ *bounded outer*› ‹*bounded inner*› ‹*x* ∈ *inner*› ‹*y* ∈ *outer*› *componentsE connected_component_eq inner mem_Collect_eq outer*)

```
      ultimately show False
        by (auto simp: ud_Un [symmetric] connected_component_def)
    qed
    then have components (− path_image c) = {inner,outer}
      using inner outer by blast
    then have Union (components (− path_image c)) = inner ∪ outer
      by simp
    then show inner ∪ outer = − path_image c
      by auto
  qed (auto simp: ‹bounded inner› ‹¬ bounded outer›)
qed


corollary Jordan_disconnected:
  fixes c :: real ⇒ complex
  assumes simple_path c pathfinish c = pathstart c
    shows ¬ connected(− path_image c)
using Jordan_curve [OF assms]
 by (metis Jordan_Brouwer_separation assms homeomorphic_simple_path_image_circle
zero_less_one)


corollary Jordan_inside_outside:
  fixes c :: real ⇒ complex
  assumes simple_path c pathfinish c = pathstart c
    shows inside(path_image c) ≠ {} ∧
          open(inside(path_image c)) ∧
          connected(inside(path_image c)) ∧
          outside(path_image c) ≠ {} ∧
          open(outside(path_image c)) ∧
          connected(outside(path_image c)) ∧
          bounded(inside(path_image c)) ∧
          ¬ bounded(outside(path_image c)) ∧
          inside(path_image c) ∩ outside(path_image c) = {} ∧
          inside(path_image c) ∪ outside(path_image c) =
          − path_image c ∧
          frontier(inside(path_image c)) = path_image c ∧
          frontier(outside(path_image c)) = path_image c
proof −
  obtain inner outer
    where ∗: inner ≠ {} open inner connected inner
             outer ≠ {} open outer connected outer
             bounded inner ¬ bounded outer inner ∩ outer = {}
             inner ∪ outer = − path_image c
             frontier inner = path_image c
             frontier outer = path_image c
    using Jordan_curve [OF assms] by blast
  then have inner: inside(path_image c) = inner
    by (metis dual_order.antisym inside_subset interior_eq interior_inside_frontier)
```

have *outer*: *outside*(*path_image c*) = *outer*
using ‹*inner* ∪ *outer* = − *path_image c*› ‹*inside* (*path_image c*) = *inner*›
outside_inside ‹*inner* ∩ *outer* = {}› **by** *auto*
show *?thesis*
using ∗ **by** (*auto simp*: *inner outer*)
qed

Triple-curve or "theta-curve" theorem

Proof that there is no fourth component taken from Kuratowski's Topology vol 2, para 61, II.

theorem *split_inside_simple_closed_curve*:
fixes *c* :: *real* ⇒ *complex*
assumes *simple_path c1* **and** *c1*: *pathstart c1* = *a pathfinish c1* = *b*
and *simple_path c2* **and** *c2*: *pathstart c2* = *a pathfinish c2* = *b*
and *simple_path c* **and** *c*: *pathstart c* = *a pathfinish c* = *b*
and *a* ≠ *b*
and *c1c2*: *path_image c1* ∩ *path_image c2* = {*a*,*b*}
and *c1c*: *path_image c1* ∩ *path_image c* = {*a*,*b*}
and *c2c*: *path_image c2* ∩ *path_image c* = {*a*,*b*}
and *ne_12*: *path_image c* ∩ *inside*(*path_image c1* ∪ *path_image c2*) ≠ {}
obtains *inside*(*path_image c1* ∪ *path_image c*) ∩ *inside*(*path_image c2* ∪ *path_image c*) = {}
inside(*path_image c1* ∪ *path_image c*) ∪ *inside*(*path_image c2* ∪ *path_image c*) ∪
(*path_image c* − {*a*,*b*}) = *inside*(*path_image c1* ∪ *path_image c2*)
proof −
let *?Θ* = *path_image c* **let** *?Θ1* = *path_image c1* **let** *?Θ2* = *path_image c2*
have *sp*: *simple_path* (*c1* +++ *reversepath c2*) *simple_path* (*c1* +++ *reversepath c*) *simple_path* (*c2* +++ *reversepath c*)
using *assms* **by** (*auto simp*: *simple_path_join_loop_eq arc_simple_path simple_path_reversepath*)
then have *op_in12*: *open* (*inside* (*?Θ1* ∪ *?Θ2*))
and *op_out12*: *open* (*outside* (*?Θ1* ∪ *?Θ2*))
and *op_in1c*: *open* (*inside* (*?Θ1* ∪ *?Θ*))
and *op_in2c*: *open* (*inside* (*?Θ2* ∪ *?Θ*))
and *op_out1c*: *open* (*outside* (*?Θ1* ∪ *?Θ*))
and *op_out2c*: *open* (*outside* (*?Θ2* ∪ *?Θ*))
and *co_in1c*: *connected* (*inside* (*?Θ1* ∪ *?Θ*))
and *co_in2c*: *connected* (*inside* (*?Θ2* ∪ *?Θ*))
and *co_out12c*: *connected* (*outside* (*?Θ1* ∪ *?Θ2*))
and *co_out1c*: *connected* (*outside* (*?Θ1* ∪ *?Θ*))
and *co_out2c*: *connected* (*outside* (*?Θ2* ∪ *?Θ*))
and *pa_c*: *?Θ* − {*pathstart c*, *pathfinish c*} ⊆ − *?Θ1*
?Θ − {*pathstart c*, *pathfinish c*} ⊆ − *?Θ2*
and *pa_c1*: *?Θ1* − {*pathstart c1*, *pathfinish c1*} ⊆ − *?Θ2*
?Θ1 − {*pathstart c1*, *pathfinish c1*} ⊆ − *?Θ*
and *pa_c2*: *?Θ2* − {*pathstart c2*, *pathfinish c2*} ⊆ − *?Θ1*
?Θ2 − {*pathstart c2*, *pathfinish c2*} ⊆ − *?Θ*

and *co_c*: *connected*(*?Θ* − {*pathstart c*,*pathfinish c*})
and *co_c1*: *connected*(*?Θ1* − {*pathstart c1*,*pathfinish c1*})
and *co_c2*: *connected*(*?Θ2* − {*pathstart c2*,*pathfinish c2*})
and *fr_in*: *frontier*(*inside*(*?Θ1* ∪ *?Θ2*)) = *?Θ1* ∪ *?Θ2*
frontier(*inside*(*?Θ2* ∪ *?Θ*)) = *?Θ2* ∪ *?Θ*
frontier(*inside*(*?Θ1* ∪ *?Θ*)) = *?Θ1* ∪ *?Θ*
and *fr_out*: *frontier*(*outside*(*?Θ1* ∪ *?Θ2*)) = *?Θ1* ∪ *?Θ2*
frontier(*outside*(*?Θ2* ∪ *?Θ*)) = *?Θ2* ∪ *?Θ*
frontier(*outside*(*?Θ1* ∪ *?Θ*)) = *?Θ1* ∪ *?Θ*
using *Jordan_inside_outside* [*of c1* +++ *reversepath c2*]
using *Jordan_inside_outside* [*of c1* +++ *reversepath c*]
using *Jordan_inside_outside* [*of c2* +++ *reversepath c*] *assms*
apply (*simp_all add*: *path_image_join closed_Un closed_simple_path_image open_inside open_outside*)
apply (*blast elim*: | *metis connected_simple_path_endless*)+
done
have *inout_12*: *inside* (*?Θ1* ∪ *?Θ2*) ∩ (*?Θ* − {*pathstart c*, *pathfinish c*}) ≠ {}
by (*metis* (*no_types*, *lifting*) *c c1c ne_12 Diff_Int_distrib Diff_empty Int_empty_right Int_left_commute inf_sup_absorb inf_sup_aci*(*1*) *inside_no_overlap*)
have *pi_disjoint*: *?Θ* ∩ *outside*(*?Θ1* ∪ *?Θ2*) = {}
proof (*rule ccontr*)
assume *?Θ* ∩ *outside* (*?Θ1* ∪ *?Θ2*) ≠ {}
then show *False*
using *connectedD* [*OF co_c*, *of inside*(*?Θ1* ∪ *?Θ2*) *outside*(*?Θ1* ∪ *?Θ2*)]
using *c c1c2 pa_c op_in12 op_out12 inout_12*
apply *auto*
apply (*metis Un_Diff_cancel2 Un_iff compl_sup disjoint_insert*(*1*) *inf_commute inf_compl_bot_left2 inside_Un_outside mk_disjoint_insert sup_inf_absorb*)
done
qed
have *out_sub12*: *outside*(*?Θ1* ∪ *?Θ2*) ⊆ *outside*(*?Θ1* ∪ *?Θ*) *outside*(*?Θ1* ∪ *?Θ2*) ⊆ *outside*(*?Θ2* ∪ *?Θ*)
by (*metis Un_commute pi_disjoint outside_Un_outside_Un*)+
have *pa1_disj_in2*: *?Θ1* ∩ *inside* (*?Θ2* ∪ *?Θ*) = {}
proof (*rule ccontr*)
assume *ne*: *?Θ1* ∩ *inside* (*?Θ2* ∪ *?Θ*) ≠ {}
have *1*: *inside* (*?Θ* ∪ *?Θ2*) ∩ *?Θ* = {}
by (*metis* (*no_types*) *Diff_Int_distrib Diff_cancel inf_sup_absorb inf_sup_aci*(*3*) *inside_no_overlap*)
have *2*: *outside* (*?Θ* ∪ *?Θ2*) ∩ *?Θ* = {}
by (*metis* (*no_types*) *Int_empty_right Int_left_commute inf_sup_absorb outside_no_overlap*)
have *outside* (*?Θ2* ∪ *?Θ*) ⊆ *outside* (*?Θ1* ∪ *?Θ2*)
apply (*subst Un_commute*, *rule outside_Un_outside_Un*)
using *connectedD* [*OF co_c1*, *of inside*(*?Θ2* ∪ *?Θ*) *outside*(*?Θ2* ∪ *?Θ*)]
pa_c1 op_in2c op_out2c ne c1 c2c 1 2 **by** (*auto simp*: *inf_sup_aci*)
with *out_sub12*
have *outside*(*?Θ1* ∪ *?Θ2*) = *outside*(*?Θ2* ∪ *?Θ*) **by** *blast*
then have *frontier*(*outside*(*?Θ1* ∪ *?Θ2*)) = *frontier*(*outside*(*?Θ2* ∪ *?Θ*))

by *simp*
then show *False*
using *inout_12 pi_disjoint c c1c c2c fr_out* **by** *auto*
qed
have *pa2_disj_in1*: *?Θ2* ∩ *inside*(*?Θ1* ∪ *?Θ*) = {}
proof (*rule ccontr*)
assume *ne*: *?Θ2* ∩ *inside* (*?Θ1* ∪ *?Θ*) ≠ {}
have *1*: *inside* (*?Θ* ∪ *?Θ1*) ∩ *?Θ* = {}
by (*metis* (*no_types*) *Diff_Int_distrib Diff_cancel inf_sup_absorb inf_sup_aci*(*3*) *inside_no_overlap*)
have *2*: *outside* (*?Θ* ∪ *?Θ1*) ∩ *?Θ* = {}
by (*metis* (*no_types*) *Int_empty_right Int_left_commute inf_sup_absorb outside_no_overlap*)
have *outside* (*?Θ1* ∪ *?Θ*) ⊆ *outside* (*?Θ1* ∪ *?Θ2*)
apply (*rule outside_Un_outside_Un*)
using *connectedD* [*OF co_c2*, *of inside*(*?Θ1* ∪ *?Θ*) *outside*(*?Θ1* ∪ *?Θ*)]
pa_c2 op_in1c op_out1c ne c2 c1c 1 2 **by** (*auto simp*: *inf_sup_aci*)
with *out_sub12*
have *outside*(*?Θ1* ∪ *?Θ2*) = *outside*(*?Θ1* ∪ *?Θ*)
by *blast*
then have *frontier*(*outside*(*?Θ1* ∪ *?Θ2*)) = *frontier*(*outside*(*?Θ1* ∪ *?Θ*))
by *simp*
then show *False*
using *inout_12 pi_disjoint c c1c c2c fr_out* **by** *auto*
qed
have *in_sub_in1*: *inside*(*?Θ1* ∪ *?Θ*) ⊆ *inside*(*?Θ1* ∪ *?Θ2*)
using *pa2_disj_in1 out_sub12* **by** (*auto simp*: *inside_outside*)
have *in_sub_in2*: *inside*(*?Θ2* ∪ *?Θ*) ⊆ *inside*(*?Θ1* ∪ *?Θ2*)
using *pa1_disj_in2 out_sub12* **by** (*auto simp*: *inside_outside*)
have *in_sub_out12*: *inside*(*?Θ1* ∪ *?Θ*) ⊆ *outside*(*?Θ2* ∪ *?Θ*)
proof
fix *x*
assume *x*: *x* ∈ *inside* (*?Θ1* ∪ *?Θ*)
then have *xnot*: *x* ∉ *?Θ*
by (*simp add*: *inside_def*)
obtain *z* **where** *zim*: *z* ∈ *?Θ1* **and** *zout*: *z* ∈ *outside*(*?Θ2* ∪ *?Θ*)
apply (*auto simp*: *outside_inside*)
using *nonempty_simple_path_endless* [*OF* ‹*simple_path c1*›]
by (*metis Diff_Diff_Int Diff_iff ex_in_conv c1 c1c c1c2 pa1_disj_in2*)
obtain *e* **where** *e* > *0* **and** *e*: *ball z e* ⊆ *outside*(*?Θ2* ∪ *?Θ*)
using *zout op_out2c open_contains_ball_eq* **by** *blast*
have *z* ∈ *frontier* (*inside* (*?Θ1* ∪ *?Θ*))
using *zim* **by** (*auto simp*: *fr_in*)
then obtain *w* **where** *w1*: *w* ∈ *inside* (*?Θ1* ∪ *?Θ*) **and** *dwz*: *dist w z* < *e*
using *zim* ‹*e* > *0*› **by** (*auto simp*: *frontier_def closure_approachable*)
then have *w2*: *w* ∈ *outside* (*?Θ2* ∪ *?Θ*)
by (*metis e dist_commute mem_ball subsetCE*)
then have *connected_component* (− *?Θ2* ∩ − *?Θ*) *z w*
apply (*simp add*: *connected_component_def*)

```
      apply (rule_tac x = outside(?Θ2 ∪ ?Θ) in exI)
      using zout apply (auto simp: co_out2c)
       apply (simp_all add: outside_inside)
      done
    moreover have connected_component (− ?Θ2 ∩ − ?Θ) w x
      unfolding connected_component_def
      using pa2_disj_in1 co_in1c x w1 union_with_outside by fastforce
    ultimately have eq: connected_component_set (− ?Θ2 ∩ − ?Θ) x =
                    connected_component_set (− ?Θ2 ∩ − ?Θ) z
      by (metis (mono_tags, lifting) connected_component_eq mem_Collect_eq)
    show x ∈ outside (?Θ2 ∪ ?Θ)
      using zout x pa2_disj_in1 by (auto simp: outside_def eq xnot)
  qed
  have in_sub_out21: inside(?Θ2 ∪ ?Θ) ⊆ outside(?Θ1 ∪ ?Θ)
  proof
    fix x
    assume x: x ∈ inside (?Θ2 ∪ ?Θ)
    then have xnot: x ∉ ?Θ
      by (simp add: inside_def)
    obtain z where zim: z ∈ ?Θ2 and zout: z ∈ outside(?Θ1 ∪ ?Θ)
      apply (auto simp: outside_inside)
      using nonempty_simple_path_endless [OF ‹simple_path c2›]
       by (metis (no_types, hide_lams) Diff_Diff_Int Diff_iff c1c2 c2 c2c ex_in_conv
pa2_disj_in1)
    obtain e where e > 0 and e: ball z e ⊆ outside(?Θ1 ∪ ?Θ)
      using zout op_out1c open_contains_ball_eq by blast
    have z ∈ frontier (inside (?Θ2 ∪ ?Θ))
      using zim by (auto simp: fr_in)
    then obtain w where w2: w ∈ inside (?Θ2 ∪ ?Θ) and dwz: dist w z < e
      using zim ‹e > 0› by (auto simp: frontier_def closure_approachable)
    then have w1: w ∈ outside (?Θ1 ∪ ?Θ)
      by (metis e dist_commute mem_ball subsetCE)
    then have connected_component (− ?Θ1 ∩ − ?Θ) z w
      apply (simp add: connected_component_def)
      apply (rule_tac x = outside(?Θ1 ∪ ?Θ) in exI)
      using zout apply (auto simp: co_out1c)
       apply (simp_all add: outside_inside)
      done
    moreover have connected_component (− ?Θ1 ∩ − ?Θ) w x
      unfolding connected_component_def
      using pa1_disj_in2 co_in2c x w2 union_with_outside by fastforce
    ultimately have eq: connected_component_set (− ?Θ1 ∩ − ?Θ) x =
                      connected_component_set (− ?Θ1 ∩ − ?Θ) z
      by (metis (no_types, lifting) connected_component_eq mem_Collect_eq)
    show x ∈ outside (?Θ1 ∪ ?Θ)
      using zout x pa1_disj_in2 by (auto simp: outside_def eq xnot)
  qed
  show ?thesis
  proof
```

show *inside* (*?Θ1* ∪ *?Θ*) ∩ *inside* (*?Θ2* ∪ *?Θ*) = {}
by (*metis Int_Un_distrib in_sub_out12 bot_eq_sup_iff disjoint_eq_subset_Compl outside_inside*)
have ∗: *outside* (*?Θ1* ∪ *?Θ*) ∩ *outside* (*?Θ2* ∪ *?Θ*) ⊆ *outside* (*?Θ1* ∪ *?Θ2*)
proof (*rule components_maximal*)
show *out_in*: *outside* (*?Θ1* ∪ *?Θ2*) ∈ *components* (− (*?Θ1* ∪ *?Θ2*))
apply (*simp only*: *outside_in_components co_out12c*)
by (*metis bounded_empty fr_out*(*1*) *frontier_empty unbounded_outside*)
have *conn_U*: *connected* (− (*closure* (*inside* (*?Θ1* ∪ *?Θ*)) ∪ *closure* (*inside* (*?Θ2* ∪ *?Θ*))))
proof (*rule Janiszewski_connected*, *simp_all*)
show *bounded* (*inside* (*?Θ1* ∪ *?Θ*))
by (*simp add*: ‹*simple_path c1*› ‹*simple_path c*› *bounded_inside bounded_simple_path_image*)
have *if1*: − (*inside* (*?Θ1* ∪ *?Θ*) ∪ *frontier* (*inside* (*?Θ1* ∪ *?Θ*))) = − *?Θ1* ∩ − *?Θ* ∩ − *inside* (*?Θ1* ∪ *?Θ*)
by (*metis* (*no_types*, *lifting*) *Int_commute Jordan_inside_outside c c1 compl_sup path_image_join path_image_reversepath pathfinish_join pathfinish_reversepath pathstart_join pathstart_reversepath sp*(*2*) *closure_Un_frontier fr_out*(*3*))
then show *connected* (− *closure* (*inside* (*?Θ1* ∪ *?Θ*)))
by (*metis Compl_Un outside_inside co_out1c closure_Un_frontier*)
have *if2*: − (*inside* (*?Θ2* ∪ *?Θ*) ∪ *frontier* (*inside* (*?Θ2* ∪ *?Θ*))) = − *?Θ2* ∩ − *?Θ* ∩ − *inside* (*?Θ2* ∪ *?Θ*)
by (*metis* (*no_types*, *lifting*) *Int_commute Jordan_inside_outside c c2 compl_sup path_image_join path_image_reversepath pathfinish_join pathfinish_reversepath pathstart_join pathstart_reversepath sp*(*3*) *closure_Un_frontier fr_out*(*2*))
then show *connected* (− *closure* (*inside* (*?Θ2* ∪ *?Θ*)))
by (*metis Compl_Un outside_inside co_out2c closure_Un_frontier*)
have *connected*(*?Θ*)
by (*metis* ‹*simple_path c*› *connected_simple_path_image*)
moreover
have *closure* (*inside* (*?Θ1* ∪ *?Θ*)) ∩ *closure* (*inside* (*?Θ2* ∪ *?Θ*)) = *?Θ*
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
proof *clarify*
fix *x*
assume *x*: *x* ∈ *closure* (*inside* (*?Θ1* ∪ *?Θ*)) *x* ∈ *closure* (*inside* (*?Θ2* ∪ *?Θ*))
then have *x* ∉ *inside* (*?Θ1* ∪ *?Θ*)
by (*meson closure_iff_nhds_not_empty in_sub_out12 inside_Int_outside op_in1c*)
with *fr_in x* **show** *x* ∈ *?Θ*
by (*metis c1c c1c2 closure_Un_frontier pa1_disj_in2 Int_iff Un_iff insert_disjoint*(*2*) *insert_subset subsetI subset_antisym*)
qed
show *?rhs* ⊆ *?lhs*
using *if1 if2 closure_Un_frontier* **by** *fastforce*
qed
ultimately

show *connected* (*closure* (*inside* (*?Θ1* ∪ *?Θ*)) ∩ *closure* (*inside* (*?Θ2* ∪ *?Θ*)))
by *auto*
qed
show *connected* (*outside* (*?Θ1* ∪ *?Θ*) ∩ *outside* (*?Θ2* ∪ *?Θ*))
using *fr_in conn_U* **by** (*simp add*: *closure_Un_frontier outside_inside Un_commute*)
show *outside* (*?Θ1* ∪ *?Θ*) ∩ *outside* (*?Θ2* ∪ *?Θ*) ⊆ − (*?Θ1* ∪ *?Θ2*)
by *clarify* (*metis Diff_Compl Diff_iff Un_iff inf_sup_absorb outside_inside*)
show *outside* (*?Θ1* ∪ *?Θ2*) ∩
(*outside* (*?Θ1* ∪ *?Θ*) ∩ *outside* (*?Θ2* ∪ *?Θ*)) ≠ {}
by (*metis Int_assoc out_in inf.orderE out_sub12*(*1*) *out_sub12*(*2*) *outside_in_components*)
qed
show *inside* (*?Θ1* ∪ *?Θ*) ∪ *inside* (*?Θ2* ∪ *?Θ*) ∪ (*?Θ* − {*a*, *b*}) = *inside* (*?Θ1* ∪ *?Θ2*)
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
apply (*simp add*: *in_sub_in1 in_sub_in2*)
using *c1c c2c inside_outside pi_disjoint* **by** *fastforce*
have *inside* (*?Θ1* ∪ *?Θ2*) ⊆ *inside* (*?Θ1* ∪ *?Θ*) ∪ *inside* (*?Θ2* ∪ *?Θ*) ∪ (*?Θ*)
using *Compl_anti_mono* [*OF* ∗] **by** (*force simp*: *inside_outside*)
moreover have *inside* (*?Θ1* ∪ *?Θ2*) ⊆ −{*a*,*b*}
using *c1 union_with_outside* **by** *fastforce*
ultimately show *?rhs* ⊆ *?lhs* **by** *auto*
qed
qed
qed

end

6.43 Polynomial Functions: Extremal Behaviour and Root Counts

theory *Poly_Roots*
imports *Complex_Main*
begin

6.43.1 Basics about polynomial functions: extremal behaviour and root counts

lemma *sub_polyfun*:
fixes *x* :: *'a*::{*comm_ring*,*monoid_mult*}
shows ($\sum$ *i*≤*n*. *a i* ∗ *x*ˆ*i*) − ($\sum$ *i*≤*n*. *a i* ∗ *y*ˆ*i*) =
(*x* − *y*) ∗ ($\sum$ *j*<*n*. $\sum$ *k*= *Suc j*..*n*. *a k* ∗ *y*ˆ(*k* − *Suc j*) ∗ *x*ˆ*j*)
proof −
have ($\sum$ *i*≤*n*. *a i* ∗ *x*ˆ*i*) − ($\sum$ *i*≤*n*. *a i* ∗ *y*ˆ*i*) =

$(\sum i{\le}n.\ a\ i * (x\hat{}i - y\hat{}i))$
by (*simp add*: *algebra_simps sum_subtractf* [*symmetric*])
also have ... = $(\sum i{\le}n.\ a\ i * (x - y) * (\sum j{<}i.\ y\hat{}(i - Suc\ j) * x\hat{}j))$
by (*simp add*: *power_diff_sumr2 ac_simps*)
also have ... = $(x - y) * (\sum i{\le}n.\ (\sum j{<}i.\ a\ i * y\hat{}(i - Suc\ j) * x\hat{}j))$
by (*simp add*: *sum_distrib_left ac_simps*)
also have ... = $(x - y) * (\sum j{<}n.\ (\sum i{=}Suc\ j..n.\ a\ i * y\hat{}(i - Suc\ j) * x\hat{}j))$
by (*simp add*: *sum.nested_swap'*)
finally show *?thesis* **.**
qed

lemma *sub_polyfun_alt*:
fixes x :: $'a$::{*comm_ring*,*monoid_mult*}
shows $(\sum i{\le}n.\ a\ i * x\hat{}i) - (\sum i{\le}n.\ a\ i * y\hat{}i) =$
$(x - y) * (\sum j{<}n.\ \sum k{<}n{-}j.\ a\ (j{+}k{+}1) * y\hat{}k * x\hat{}j)$
proof −
{ fix j
have $(\sum k = Suc\ j..n.\ a\ k * y\hat{}(k - Suc\ j) * x\hat{}j) =$
$(\sum k {<}n - j.\ a\ (Suc\ (j + k)) * y\hat{}k * x\hat{}j)$
by (*rule sum.reindex_bij_witness*[**where** $i{=}\lambda i.\ i + Suc\ j$ **and** $j{=}\lambda i.\ i - Suc$
j]) *auto* **}**
then show *?thesis*
by (*simp add*: *sub_polyfun*)
qed

lemma *polyfun_linear_factor*:
fixes a :: $'a$::{*comm_ring*,*monoid_mult*}
shows $\exists\, b.\ \forall\, z.\ (\sum i{\le}n.\ c\ i * z\hat{}i) =$
$(z{-}a) * (\sum i{<}n.\ b\ i * z\hat{}i) + (\sum i{\le}n.\ c\ i * a\hat{}i)$
proof −
{ fix z
have $(\sum i{\le}n.\ c\ i * z\hat{}i) - (\sum i{\le}n.\ c\ i * a\hat{}i) =$
$(z - a) * (\sum j{<}n.\ (\sum k = Suc\ j..n.\ c\ k * a\hat{}(k - Suc\ j)) * z\hat{}j)$
by (*simp add*: *sub_polyfun sum_distrib_right*)
then have $(\sum i{\le}n.\ c\ i * z\hat{}i) =$
$(z - a) * (\sum j{<}n.\ (\sum k = Suc\ j..n.\ c\ k * a\hat{}(k - Suc\ j)) * z\hat{}j)$
$+ (\sum i{\le}n.\ c\ i * a\hat{}i)$
by (*simp add*: *algebra_simps*) **}**
then show *?thesis*
by (*intro exI allI*)
qed

lemma *polyfun_linear_factor_root*:
fixes a :: $'a$::{*comm_ring*,*monoid_mult*}
assumes $(\sum i{\le}n.\ c\ i * a\hat{}i) = 0$
shows $\exists\, b.\ \forall\, z.\ (\sum i{\le}n.\ c\ i * z\hat{}i) = (z{-}a) * (\sum i{<}n.\ b\ i * z\hat{}i)$
using *polyfun_linear_factor* [*of c n a*] *assms*
by *simp*

lemma *adhoc_norm_triangle*: $a + norm(y) \leq b$ ==> $norm(x) \leq a$ ==> $norm(x + y) \leq b$
  by (*metis norm_triangle_mono order.trans order_refl*)

proposition *polyfun_extremal_lemma*:
  fixes $c :: nat \Rightarrow 'a::real\_normed\_div\_algebra$
  assumes $e > 0$
    shows $\exists M. \forall z. M \leq norm\ z \longrightarrow norm(\sum i \leq n.\ c\ i * z \hat{}\ i) \leq e * norm(z)\ \hat{}\ Suc\ n$
proof (*induction n*)
  case *0*
  show *?case*
   by (*rule exI* [**where** $x=norm\ (c\ 0) / e$]) (*auto simp*: *mult.commute pos_divide_le_eq assms*)
next
  case (*Suc n*)
  then obtain *M* **where** $M: \forall z. M \leq norm\ z \longrightarrow norm\ (\sum i \leq n.\ c\ i * z \hat{}\ i) \leq e * norm\ z\ \hat{}\ Suc\ n$ **..**
  show *?case*
  proof (*rule exI* [**where** $x=max\ 1\ (max\ M\ ((e + norm(c(Suc\ n))) / e))$], *clarify*)
    fix $z::'a$
    assume $max\ 1\ (max\ M\ ((e + norm\ (c\ (Suc\ n))) / e)) \leq norm\ z$
    then have *norm1*: $0 < norm\ z\ M \leq norm\ z\ (e + norm\ (c\ (Suc\ n))) / e \leq norm\ z$
      by *auto*
    then have *norm2*: $(e + norm\ (c\ (Suc\ n))) \leq e * norm\ z\ \ (norm\ z * norm\ z\ \hat{}\ n) > 0$
      apply (*metis assms less_divide_eq mult.commute not_le*)
      using *norm1* **apply** (*metis mult_pos_pos zero_less_power*)
      done
    have $e * (norm\ z * norm\ z\ \hat{}\ n) + norm\ (c\ (Suc\ n) * (z * z\ \hat{}\ n)) =$
          $(e + norm\ (c\ (Suc\ n))) * (norm\ z * norm\ z\ \hat{}\ n)$
      by (*simp add*: *norm_mult norm_power algebra_simps*)
    also have ... $\leq (e * norm\ z) * (norm\ z * norm\ z\ \hat{}\ n)$
      using *norm2*
      using *assms mult_mono* **by** *fastforce*
    also have ... $= e * (norm\ z * (norm\ z * norm\ z\ \hat{}\ n))$
      by (*simp add*: *algebra_simps*)
    finally have $e * (norm\ z * norm\ z\ \hat{}\ n) + norm\ (c\ (Suc\ n) * (z * z\ \hat{}\ n))$
                $\leq e * (norm\ z * (norm\ z * norm\ z\ \hat{}\ n))$ **.**
    then show $norm\ (\sum i \leq Suc\ n.\ c\ i * z \hat{}\ i) \leq e * norm\ z\ \hat{}\ Suc\ (Suc\ n)$ **using** *M norm1*
      by (*drule_tac x=z* **in** *spec*) (*auto simp*: *intro!*: *adhoc_norm_triangle*)
    qed
qed

lemma *norm_lemma_xy*: **assumes** $|b| + 1 \leq norm(y) - a\ norm(x) \leq a$ **shows** $b \leq norm(x + y)$
proof −

have $b \le norm\ y - norm\ x$
using *assms* **by** *linarith*
then show *?thesis*
by (*metis* (*no_types*) *add.commute norm_diff_ineq order_trans*)
qed

proposition *polyfun_extremal*:
fixes $c :: nat \Rightarrow 'a::real\_normed\_div\_algebra$
assumes $\exists k.\ k \neq 0 \wedge k \le n \wedge c\ k \neq 0$
shows *eventually* $(\lambda z.\ norm(\sum i{\le}n.\ c\ i * z\hat{}i) \ge B)$ *at_infinity*
using *assms*
proof (*induction n*)
case *0* **then show** *?case*
by *simp*
next
case (*Suc n*)
show *?case*
proof (*cases* $c\ (Suc\ n) = 0$)
case *True*
with *Suc* **show** *?thesis*
by *auto* (*metis diff_is_0_eq diffs0_imp_equal less_Suc_eq_le not_less_eq*)
next
case *False*
with *polyfun_extremal_lemma* [*of* $norm(c\ (Suc\ n))\ /\ 2\ c\ n$]
obtain *M* **where** M: $\bigwedge z.\ M \le norm\ z \Longrightarrow$
$norm\ (\sum i{\le}n.\ c\ i * z\hat{}i) \le norm\ (c\ (Suc\ n))\ /\ 2 * norm\ z\ \hat{}\ Suc\ n$
by *auto*
show *?thesis*
unfolding *eventually_at_infinity*
proof (*rule exI* [**where** $x{=}max\ M\ (max\ 1\ ((|B| + 1)\ /\ (norm\ (c\ (Suc\ n))\ /\ 2)))$], *clarsimp*)
fix $z::'a$
assume *les*: $M \le norm\ z\ \ 1 \le norm\ z\ \ (|B| * 2 + 2)\ /\ norm\ (c\ (Suc\ n)) \le norm\ z$
then have $|B| * 2 + 2 \le norm\ z * norm\ (c\ (Suc\ n))$
by (*metis False pos_divide_le_eq zero_less_norm_iff*)
then have $|B| * 2 + 2 \le norm\ z\ \hat{}\ (Suc\ n) * norm\ (c\ (Suc\ n))$
by (*metis* ‹$1 \le norm\ z$› *order.trans mult_right_mono norm_ge_zero self_le_power zero_less_Suc*)
then show $B \le norm\ ((\sum i{\le}n.\ c\ i * z\hat{}i) + c\ (Suc\ n) * (z * z\ \hat{}\ n))$ **using** *M les*
apply *auto*
apply (*rule norm_lemma_xy* [**where** $a = norm\ (c\ (Suc\ n)) * norm\ z\ \hat{}\ (Suc\ n)\ /\ 2$])
apply (*simp_all add*: *norm_mult norm_power*)
done
qed
qed
qed

proposition *polyfun_rootbound*:
fixes $c :: nat \Rightarrow 'a::\{comm\_ring,real\_normed\_div\_algebra\}$
assumes $\exists k.\ k \le n \wedge c\ k \ne 0$
shows $finite\ \{z.\ (\sum i\le n.\ c\ i * z\hat{}i) = 0\} \wedge card\ \{z.\ (\sum i\le n.\ c\ i * z\hat{}i) = 0\} \le n$
using *assms*
proof (*induction n arbitrary*: *c*)
case (*Suc n*) **show** *?case*
proof (*cases* $\{z.\ (\sum i\le Suc\ n.\ c\ i * z\hat{}i) = 0\} = \{\}$)
case *False*
then obtain *a* **where** *a*: $(\sum i\le Suc\ n.\ c\ i * a\hat{}i) = 0$
by *auto*
from *polyfun_linear_factor_root* [*OF this*]
obtain *b* **where** $\bigwedge z.\ (\sum i\le Suc\ n.\ c\ i * z\hat{}i) = (z - a) * (\sum i< Suc\ n.\ b\ i * z\hat{}i)$
by *auto*
then have *b*: $\bigwedge z.\ (\sum i\le Suc\ n.\ c\ i * z\hat{}i) = (z - a) * (\sum i\le n.\ b\ i * z\hat{}i)$
by (*metis lessThan_Suc_atMost*)
then have *ins_ab*: $\{z.\ (\sum i\le Suc\ n.\ c\ i * z\hat{}i) = 0\} = insert\ a\ \{z.\ (\sum i\le n.\ b\ i * z\hat{}i) = 0\}$
by *auto*
have *c0*: $c\ 0 = -\ (a * b\ 0)$ **using** *b* [*of 0*]
by *simp*
then have *extr_prem*: $\neg (\exists k\le n.\ b\ k \ne 0) \Longrightarrow \exists k.\ k \ne 0 \wedge k \le Suc\ n \wedge c\ k \ne 0$
by (*metis Suc.prems le0 minus_zero mult_zero_right*)
have $\exists k\le n.\ b\ k \ne 0$
apply (*rule ccontr*)
using *polyfun_extremal* [*OF extr_prem*, *of 1*]
apply (*auto simp*: *eventually_at_infinity b simp del*: *sum.atMost_Suc*)
apply (*drule_tac x=of_real ba* **in** *spec*, *simp*)
done
then show *?thesis* **using** *Suc.IH* [*of b*] *ins_ab*
by (*auto simp*: *card_insert_if*)
qed *simp*
qed *simp*

corollary
fixes $c :: nat \Rightarrow 'a::\{comm\_ring,real\_normed\_div\_algebra\}$
assumes $\exists k.\ k \le n \wedge c\ k \ne 0$
shows *polyfun_rootbound_finite*: $finite\ \{z.\ (\sum i\le n.\ c\ i * z\hat{}i) = 0\}$
and *polyfun_rootbound_card*: $card\ \{z.\ (\sum i\le n.\ c\ i * z\hat{}i) = 0\} \le n$
using *polyfun_rootbound* [*OF assms*] **by** *auto*

proposition *polyfun_finite_roots*:
fixes $c :: nat \Rightarrow 'a::\{comm\_ring,real\_normed\_div\_algebra\}$
shows $finite\ \{z.\ (\sum i\le n.\ c\ i * z\hat{}i) = 0\} \longleftrightarrow (\exists k.\ k \le n \wedge c\ k \ne 0)$
proof (*cases* $\exists k\le n.\ c\ k \ne 0$)

case *True* **then show** *?thesis*
  by (*blast intro*: *polyfun_rootbound_finite*)
next
  case *False* **then show** *?thesis*
  by (*auto simp*: *infinite_UNIV_char_0*)
qed

lemma *polyfun_eq_0*:
  fixes c :: *nat* ⇒ ′*a*::{*comm_ring*,*real_normed_div_algebra*}
  shows $(\forall z. (\sum i{\le}n.\ c\ i * z\hat{}i) = 0) \longleftrightarrow (\forall k.\ k \le n \longrightarrow c\ k = 0)$
proof (*cases* $(\forall z. (\sum i{\le}n.\ c\ i * z\hat{}i) = 0)$)
  case *True*
  then have ¬ *finite* $\{z. (\sum i{\le}n.\ c\ i * z\hat{}i) = 0\}$
  by (*simp add*: *infinite_UNIV_char_0*)
  with *True* **show** *?thesis*
  by (*metis* (*poly_guards_query*) *polyfun_rootbound_finite*)
next
  case *False*
  then show *?thesis*
  by *auto*
qed

theorem *polyfun_eq_const*:
  fixes c :: *nat* ⇒ ′*a*::{*comm_ring*,*real_normed_div_algebra*}
  shows $(\forall z. (\sum i{\le}n.\ c\ i * z\hat{}i) = k) \longleftrightarrow c\ 0 = k \wedge (\forall k.\ k \neq 0 \wedge k \le n \longrightarrow c\ k = 0)$
proof −
  {**fix** z
  have $(\sum i{\le}n.\ c\ i * z\hat{}i) = (\sum i{\le}n.\ (\mathit{if}\ i = 0\ \mathit{then}\ c\ 0 - k\ \mathit{else}\ c\ i) * z\hat{}i) + k$
  by (*induct n*) *auto*
  } **then**
  have $(\forall z. (\sum i{\le}n.\ c\ i * z\hat{}i) = k) \longleftrightarrow (\forall z. (\sum i{\le}n.\ (\mathit{if}\ i = 0\ \mathit{then}\ c\ 0 - k\ \mathit{else}\ c\ i) * z\hat{}i) = 0)$
  by *auto*
  also have ... $\longleftrightarrow\ c\ 0 = k \wedge (\forall k.\ k \neq 0 \wedge k \le n \longrightarrow c\ k = 0)$
  by (*auto simp*: *polyfun_eq_0*)
  finally show *?thesis* **.**
qed

end

6.44 Generalised Binomial Theorem

The proof of the Generalised Binomial Theorem and related results. We prove the generalised binomial theorem for complex numbers, following the proof at: https://proofwiki.org/wiki/Binomial_Theorem/General_Binomial_Theorem

theory *Generalised_Binomial_Theorem*

imports
Complex_Main
Complex_Transcendental
Summation_Tests
begin

lemma *gbinomial_ratio_limit*:
fixes *a* :: *'a* :: *real_normed_field*
assumes $a \notin \mathbb{N}$
shows (λ*n*. (*a gchoose n*) / (*a gchoose Suc n*)) ⟶ −*1*
proof (*rule Lim_transform_eventually*)
let *?f* = λ*n*. *inverse* (*a* / *of_nat* (*Suc n*) − *of_nat n* / *of_nat* (*Suc n*))
from *eventually_gt_at_top*[*of 0*::*nat*]
show *eventually* (λ*n*. *?f n* = (*a gchoose n*) /(*a gchoose Suc n*)) *sequentially*
proof *eventually_elim*
fix *n* :: *nat* **assume** *n*: *n* > *0*
then obtain *q* **where** *q*: *n* = *Suc q* **by** (*cases n*) *blast*
let *?P* = ∏ *i*=*0*..<*n*. *a* − *of_nat i*
from *n* **have** (*a gchoose n*) / (*a gchoose Suc n*) = (*of_nat* (*Suc n*) :: *'a*) ∗
(*?P* / (∏ *i*=*0*..*n*. *a* − *of_nat i*))
by (*simp add*: *gbinomial_prod_rev atLeastLessThanSuc_atLeastAtMost*)
also from *q* **have** (∏ *i*=*0*..*n*. *a* − *of_nat i*) = *?P* ∗ (*a* − *of_nat n*)
by (*simp add*: *prod.atLeast0_atMost_Suc atLeastLessThanSuc_atLeastAtMost*)
also have *?P* / ... = (*?P* / *?P*) / (*a* − *of_nat n*) **by** (*rule divide_divide_eq_left*[*symmetric*])
also from *assms* **have** *?P* / *?P* = *1* **by** *auto*
also have *of_nat* (*Suc n*) ∗ (*1* / (*a* − *of_nat n*)) =
inverse (*inverse* (*of_nat* (*Suc n*)) ∗ (*a* − *of_nat n*)) **by** (*simp add*:
field_simps)
also have *inverse* (*of_nat* (*Suc n*)) ∗ (*a* − *of_nat n*) = *a* / *of_nat* (*Suc n*) −
of_nat n / *of_nat* (*Suc n*)
by (*simp add*: *field_simps del*: *of_nat_Suc*)
finally show *?f n* = (*a gchoose n*) / (*a gchoose Suc n*) **by** *simp*
qed

have (λ*n*. *norm a* / (*of_nat* (*Suc n*))) ⟶ *0*
unfolding *divide_inverse*
by (*intro tendsto_mult_right_zero LIMSEQ_inverse_real_of_nat*)
hence (λ*n*. *a* / *of_nat* (*Suc n*)) ⟶ *0*
by (*subst tendsto_norm_zero_iff*[*symmetric*]) (*simp add*: *norm_divide del*: *of_nat_Suc*)
hence *?f* ⟶ *inverse* (*0* − *1*)
by (*intro tendsto_inverse tendsto_diff LIMSEQ_n_over_Suc_n*) *simp_all*
thus *?f* ⟶ −*1* **by** *simp*
qed

lemma *conv_radius_gchoose*:
fixes *a* :: *'a* :: {*real_normed_field*,*banach*}
shows *conv_radius* (λ*n*. *a gchoose n*) = (*if* $a \in \mathbb{N}$ *then* ∞ *else 1*)
proof (*cases* $a \in \mathbb{N}$)
assume *a*: $a \in \mathbb{N}$

have *eventually* (λ*n*. (*a gchoose n*) = *0*) *sequentially*
using *eventually_gt_at_top*[*of nat* ⌊*norm a*⌋]
by *eventually_elim* (*insert a*, *auto elim*!: *Nats_cases simp*: *binomial_gbinomial*[*symmetric*])
from *conv_radius_cong'*[*OF this*] *a* **show** *?thesis* **by** *simp*
next
assume *a*: *a* ∉ ℕ
from *tendsto_norm*[*OF gbinomial_ratio_limit*[*OF this*]]
have *conv_radius* (λ*n*. *a gchoose n*) = *1*
by (*intro conv_radius_ratio_limit_nonzero*[*of _ 1*]) (*simp_all add*: *norm_divide*)
with *a* **show** *?thesis* **by** *simp*
qed

theorem *gen_binomial_complex*:
fixes *z* :: *complex*
assumes *norm z* < *1*
shows (λ*n*. (*a gchoose n*) ∗ *z*ˆ*n*) *sums* (*1* + *z*) *powr a*
proof −
define *K* **where** *K* = *1* − (*1* − *norm z*) / *2*
from *assms* **have** *K*: *K* > *0 K* < *1 norm z* < *K*
unfolding *K_def* **by** (*auto simp*: *field_simps intro*!: *add_pos_nonneg*)
let *?f* = λ*n*. *a gchoose n* **and** *?f'* = *diffs* (λ*n*. *a gchoose n*)
have *summable_strong*: *summable* (λ*n*. *?f n* ∗ *z* ˆ *n*) **if** *norm z* < *1* **for** *z* **using** *that*
by (*intro summable_in_conv_radius*) (*simp_all add*: *conv_radius_gchoose*)
with *K* **have** *summable*: *summable* (λ*n*. *?f n* ∗ *z* ˆ *n*) **if** *norm z* < *K* **for** *z* **using** *that* **by** *auto*
hence *summable'*: *summable* (λ*n*. *?f' n* ∗ *z* ˆ *n*) **if** *norm z* < *K* **for** *z* **using** *that*
by (*intro termdiff_converges*[*of _ K*]) *simp_all*

define *f f'* **where** [*abs_def*]: *f z* = (∑*n*. *?f n* ∗ *z* ˆ *n*) *f' z* = (∑*n*. *?f' n* ∗ *z* ˆ *n*) **for** *z*
{
fix *z* :: *complex* **assume** *z*: *norm z* < *K*
from *summable_mult2*[*OF summable'*[*OF z*], *of z*]
have *summable1*: *summable* (λ*n*. *?f' n* ∗ *z* ˆ *Suc n*) **by** (*simp add*: *mult_ac*)
hence *summable2*: *summable* (λ*n*. *of_nat n* ∗ *?f n* ∗ *z*ˆ*n*)
unfolding *diffs_def* **by** (*subst* (*asm*) *summable_Suc_iff*)

have (*1* + *z*) ∗ *f' z* = (∑*n*. *?f' n* ∗ *z*ˆ*n*) + (∑*n*. *?f' n* ∗ *z*ˆ*Suc n*)
unfolding *f_f'_def* **using** *summable' z* **by** (*simp add*: *algebra_simps suminf_mult*)
also have (∑*n*. *?f' n* ∗ *z*ˆ*n*) = (∑*n*. *of_nat* (*Suc n*) ∗ *?f* (*Suc n*) ∗ *z*ˆ*n*)
by (*intro suminf_cong*) (*simp add*: *diffs_def*)
also have (∑*n*. *?f' n* ∗ *z*ˆ*Suc n*) = (∑*n*. *of_nat n* ∗ *?f n* ∗ *z* ˆ *n*)
using *summable1 suminf_split_initial_segment*[*OF summable1*] **unfolding** *diffs_def*
by (*subst suminf_split_head*, *subst* (*asm*) *summable_Suc_iff*) *simp_all*
also have (∑*n*. *of_nat* (*Suc n*) ∗ *?f* (*Suc n*) ∗ *z*ˆ*n*) + (∑*n*. *of_nat n* ∗ *?f n*

∗ z^n) =
(∑n. a ∗ ?f n ∗ z^n)
by (subst gbinomial_mult_1, subst suminf_add)
(insert summable'[OF z] summable2,
simp_all add: summable_powser_split_head algebra_simps diffs_def)
also have ... = a ∗ f z unfolding f_f'_def
by (subst suminf_mult[symmetric]) (simp_all add: summable[OF z] mult_ac)
finally have a ∗ f z = (1 + z) ∗ f' z by simp
} note deriv = this

have [derivative_intros]: (f has_field_derivative f' z) (at z) if norm z < of_real K for z
unfolding f_f'_def using K that
by (intro termdiffs_strong[of ?f K z] summable_strong) simp_all
have f 0 = (∑n. if n = 0 then 1 else 0) unfolding f_f'_def by (intro suminf_cong) simp
also have ... = 1 using sums_single[of 0 λ_. 1::complex] unfolding sums_iff by simp
finally have [simp]: f 0 = 1 .

have ∃c. ∀z∈ball 0 K. f z ∗ (1 + z) powr (−a) = c
proof (rule has_field_derivative_zero_constant)
fix z :: complex assume z': z ∈ ball 0 K
hence z: norm z < K by simp
with K have nz: 1 + z ≠ 0 by (auto dest!: minus_unique)
from z K have norm z < 1 by simp
hence (1 + z) ∉ $\mathbb{R}_{\leq 0}$ by (cases z) (auto simp: Complex_eq complex_nonpos_Reals_iff)
hence ((λz. f z ∗ (1 + z) powr (−a)) has_field_derivative
f' z ∗ (1 + z) powr (−a) − a ∗ f z ∗ (1 + z) powr (−a−1)) (at z) using z
by (auto intro!: derivative_eq_intros)
also from z have a ∗ f z = (1 + z) ∗ f' z by (rule deriv)
finally show ((λz. f z ∗ (1 + z) powr (−a)) has_field_derivative 0) (at z within ball 0 K)
using nz by (simp add: field_simps powr_diff at_within_open[OF z'])
qed simp_all
then obtain c where c: ⋀z. z ∈ ball 0 K ⟹ f z ∗ (1 + z) powr (−a) = c by blast
from c[of 0] and K have c = 1 by simp
with c[of z] have f z = (1 + z) powr a using K
by (simp add: powr_minus field_simps dist_complex_def)
with summable K show ?thesis unfolding f_f'_def by (simp add: sums_iff)
qed

lemma gen_binomial_complex':
fixes x y :: real and a :: complex
assumes |x| < |y|
shows (λn. (a gchoose n) ∗ of_real x^n ∗ of_real y powr (a − of_nat n)) sums
of_real (x + y) powr a (is ?P x y)

proof −
{
fix *x y* :: *real* **assume** *xy*: |*x*| < |*y*| *y* ≥ *0*
hence *y* > *0* **by** *simp*
note *xy* = *xy this*
from *xy* **have** (λ*n*. (*a gchoose n*) ∗ *of_real* (*x* / *y*) ˆ *n*) *sums* (*1* + *of_real* (*x* / *y*)) *powr a*
by (*intro gen_binomial_complex*) (*simp add*: *norm_divide*)
hence (λ*n*. (*a gchoose n*) ∗ *of_real* (*x* / *y*) ˆ *n* ∗ *y powr a*) *sums*
((*1* + *of_real* (*x* / *y*)) *powr a* ∗ *y powr a*)
by (*rule sums_mult2*)
also have (*1* + *complex_of_real* (*x* / *y*)) = *complex_of_real* (*1* + *x*/*y*) **by** *simp*
also from *xy* **have** . . . *powr a* ∗ *of_real y powr a* = (. . . ∗ *y*) *powr a*
by (*subst powr_times_real*[*symmetric*]) (*simp_all add*: *field_simps*)
also from *xy* **have** *complex_of_real* (*1* + *x* / *y*) ∗ *complex_of_real y* = *of_real* (*x* + *y*)
by (*simp add*: *field_simps*)
finally have *?P x y* **using** *xy* **by** (*simp add*: *field_simps powr_diff powr_nat*)
} **note** *A* = *this*

show *?thesis*
proof (*cases y* < *0*)
assume *y*: *y* < *0*
with *assms* **have** *xy*: *x* + *y* < *0* **by** *simp*
with *assms* **have** |−*x*| < |−*y*| −*y* ≥ *0* **by** *simp_all*
note *A*[*OF this*]
also have *complex_of_real* (−*x* + −*y*) = − *complex_of_real* (*x* + *y*) **by** *simp*
also from *xy assms* **have** ... *powr a* = (−*1*) *powr* −*a* ∗ *of_real* (*x* + *y*) *powr a*
by (*subst powr_neg_real_complex*) (*simp add*: *abs_real_def split*: *if_split_asm*)
also {
fix *n* :: *nat*
from *y* **have** (*a gchoose n*) ∗ *of_real* (−*x*) ˆ *n* ∗ *of_real* (−*y*) *powr* (*a* − *of_nat n*) =
(*a gchoose n*) ∗ (−*of_real x* / −*of_real y*) ˆ *n* ∗ (− *of_real y*) *powr a*
by (*subst power_divide*) (*simp add*: *powr_diff powr_nat*)
also from *y* **have** (− *of_real y*) *powr a* = (−*1*) *powr* −*a* ∗ *of_real y powr a*
by (*subst powr_neg_real_complex*) *simp*
also have −*complex_of_real x* / −*complex_of_real y* = *complex_of_real x* / *complex_of_real y*
by *simp*
also have ... ˆ *n* = *of_real x* ˆ *n* / *of_real y* ˆ *n* **by** (*simp add*: *power_divide*)
also have (*a gchoose n*) ∗ ... ∗ ((−*1*) *powr* −*a* ∗ *of_real y powr a*) =
(−*1*) *powr* −*a* ∗ ((*a gchoose n*) ∗ *of_real x* ˆ *n* ∗ *of_real y powr* (*a* − *n*))
by (*simp add*: *algebra_simps powr_diff powr_nat*)
finally have (*a gchoose n*) ∗ *of_real* (− *x*) ˆ *n* ∗ *of_real* (− *y*) *powr* (*a* − *of_nat n*) =
(−*1*) *powr* −*a* ∗ ((*a gchoose n*) ∗ *of_real x* ˆ *n* ∗ *of_real y powr*

```
(a − of_nat n)) .
  }
  note sums_cong[OF this]
  finally show ?thesis by (simp add: sums_mult_iff)
 qed (insert A[of x y] assms, simp_all add: not_less)
qed

lemma gen_binomial_complex'':
 fixes x y :: real and a :: complex
 assumes |y| < |x|
 shows  (λn. (a gchoose n) ∗ of_real x powr (a − of_nat n) ∗ of_real y ˆ n) sums
          of_real (x + y) powr a
 using gen_binomial_complex'[OF assms] by (simp add: mult_ac add.commute)

lemma gen_binomial_real:
 fixes z :: real
 assumes |z| < 1
 shows  (λn. (a gchoose n) ∗ zˆn) sums (1 + z) powr a
proof −
 from assms have norm (of_real z :: complex) < 1 by simp
 from gen_binomial_complex[OF this]
  have (λn. (of_real a gchoose n :: complex) ∗ of_real z ˆ n) sums
         (of_real (1 + z)) powr (of_real a) by simp
 also have (of_real (1 + z) :: complex) powr (of_real a) = of_real ((1 + z) powr
a)
  using assms by (subst powr_of_real) simp_all
 also have (of_real a gchoose n :: complex) = of_real (a gchoose n) for n
  by (simp add: gbinomial_prod_rev)
 hence (λn. (of_real a gchoose n :: complex) ∗ of_real z ˆ n) =
        (λn. of_real ((a gchoose n) ∗ z ˆ n)) by (intro ext) simp
 finally show ?thesis by (simp only: sums_of_real_iff)
qed

lemma gen_binomial_real':
 fixes x y a :: real
 assumes |x| < y
 shows  (λn. (a gchoose n) ∗ xˆn ∗ y powr (a − of_nat n)) sums (x + y) powr a
proof −
 from assms have y > 0 by simp
 note xy = this assms
 from assms have |x / y| < 1 by simp
 hence (λn. (a gchoose n) ∗ (x / y) ˆ n) sums (1 + x / y) powr a
  by (rule gen_binomial_real)
 hence (λn. (a gchoose n) ∗ (x / y) ˆ n ∗ y powr a) sums ((1 + x / y) powr a
∗ y powr a)
  by (rule sums_mult2)
 with xy show ?thesis
  by (simp add: field_simps powr_divide powr_diff powr_realpow)
qed
```

lemma *one_plus_neg_powr_powser*:
fixes *z s* :: *complex*
assumes *norm* (*z* :: *complex*) < *1*
shows (λ*n*. (−*1*) ^*n* ∗ ((*s* + *n* − *1*) *gchoose n*) ∗ *z*^*n*) *sums* (*1* + *z*) *powr* (−*s*)
using *gen_binomial_complex*[*OF assms*, *of* −*s*] **by** (*simp add*: *gbinomial_minus*)

lemma *gen_binomial_real''*:
fixes *x y a* :: *real*
assumes |*y*| < *x*
shows (λ*n*. (*a gchoose n*) ∗ *x powr* (*a* − *of_nat n*) ∗ *y*^*n*) *sums* (*x* + *y*) *powr a*
using *gen_binomial_real'*[*OF assms*] **by** (*simp add*: *mult_ac add.commute*)

lemma *sqrt_series'*:
|*z*| < *a* ⟹ (λ*n*. ((*1*/*2*) *gchoose n*) ∗ *a powr* (*1*/*2* − *real_of_nat n*) ∗ *z* ^ *n*) *sums*
sqrt (*a* + *z* :: *real*)
using *gen_binomial_real''*[*of z a 1*/*2*] **by** (*simp add*: *powr_half_sqrt*)

lemma *sqrt_series*:
|*z*| < *1* ⟹ (λ*n*. ((*1*/*2*) *gchoose n*) ∗ *z* ^ *n*) *sums sqrt* (*1* + *z*)
using *gen_binomial_real*[*of z 1*/*2*] **by** (*simp add*: *powr_half_sqrt*)

end

6.45 Vitali Covering Theorem and an Application to Negligibility

theory *Vitali_Covering_Theorem*
imports *Equivalence_Lebesgue_Henstock_Integration HOL*−*Library.Permutations*

begin

lemma *stretch_Galois*:
fixes *x* :: *real*^'*n*
shows (⋀*k*. *m k* ≠ *0*) ⟹ ((*y* = (χ *k*. *m k* ∗ *x*$*k*)) ⟷ (χ *k*. *y*$*k* / *m k*) = *x*)
by *auto*

lemma *lambda_swap_Galois*:
(*x* = (χ *i*. *y* $ *Fun.swap m n id i*) ⟷ (χ *i*. *x* $ *Fun.swap m n id i*) = *y*)
by (*auto*; *simp add*: *pointfree_idE vec_eq_iff*)

lemma *lambda_add_Galois*:
fixes *x* :: *real*^'*n*
shows *m* ≠ *n* ⟹ (*x* = (χ *i*. *if i* = *m then y*$*m* + *y*$*n else y*$*i*) ⟷ (χ *i*. *if i* = *m then x*$*m* − *x*$*n else x*$*i*) = *y*)
by (*safe*; *simp add*: *vec_eq_iff*)

```
lemma Vitali_covering_lemma_cballs_balls:
  fixes a :: 'a ⇒ 'b::euclidean_space
  assumes ⋀i. i ∈ K ⟹ 0 < r i ∧ r i ≤ B
  obtains C where countable C C ⊆ K
    pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j))) C
    ⋀i. i ∈ K ⟹ ∃j. j ∈ C ∧
                      ¬ disjnt (cball (a i) (r i)) (cball (a j) (r j)) ∧
                      cball (a i) (r i) ⊆ ball (a j) (5 ∗ r j)
proof (cases K = {})
  case True
  with that show ?thesis
    by auto
next
  case False
  then have B > 0
    using assms less_le_trans by auto
  have rgt0[simp]: ⋀i. i ∈ K ⟹ 0 < r i
    using assms by auto
  let ?djnt = pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j)))
  have ∃ C. ∀ n. (C n ⊆ K ∧
           (∀ i ∈ C n. B/2 ^ n ≤ r i) ∧ ?djnt (C n) ∧
           (∀ i ∈ K. B/2 ^ n < r i
              ⟶ (∃j. j ∈ C n ∧
                      ¬ disjnt (cball (a i) (r i)) (cball (a j) (r j)) ∧
                      cball (a i) (r i) ⊆ ball (a j) (5 ∗ r j)))) ∧ (C n ⊆ C(Suc n))
  proof (rule dependent_nat_choice, safe)
    fix C n
     define D where D ≡ {i ∈ K. B/2 ^ Suc n < r i ∧ (∀j∈C. disjnt (cball(a
i)(r i)) (cball (a j) (r j)))}
    let ?cover_ar = λi j. ¬ disjnt (cball (a i) (r i)) (cball (a j) (r j)) ∧
                          cball (a i) (r i) ⊆ ball (a j) (5 ∗ r j)
    assume C ⊆ K
      and Ble: ∀i∈C. B/2 ^ n ≤ r i
      and djntC: ?djnt C
      and cov_n: ∀i∈K. B/2 ^ n < r i ⟶ (∃j. j ∈ C ∧ ?cover_ar i j)
    have ∗: ∀C∈chains {C. C ⊆ D ∧ ?djnt C}. ⋃C ∈ {C. C ⊆ D ∧ ?djnt C}
    proof (clarsimp simp: chains_def)
      fix C
      assume C: C ⊆ {C. C ⊆ D ∧ ?djnt C} and chain⊆ C
      show ⋃C ⊆ D ∧ ?djnt (⋃C)
        unfolding pairwise_def
      proof (intro ballI conjI impI)
        show ⋃C ⊆ D
          using C by blast
      next
        fix x y
        assume x ∈ ⋃C and y ∈ ⋃C and x ≠ y
        then obtain X Y where XY: x ∈ X X ∈ C y ∈ Y Y ∈ C
          by blast
```

```
        then consider X ⊆ Y | Y ⊆ X
          by (meson ‹chain⊆ C› chain_subset_def)
        then show disjnt (cball (a x) (r x)) (cball (a y) (r y))
        proof cases
          case 1
          with C XY ‹x ≠ y› show ?thesis
            unfolding pairwise_def by blast
        next
          case 2
          with C XY ‹x ≠ y› show ?thesis
            unfolding pairwise_def by blast
        qed
      qed
    qed
    obtain E where E ⊆ D and djntE: ?djnt E and maximalE: ⋀X. ⟦X ⊆ D;
?djnt X; E ⊆ X⟧ ⟹ X = E
      using Zorn_Lemma [OF *] by safe blast
    show ∃L. (L ⊆ K ∧
              (∀i∈L. B/2 ^ Suc n ≤ r i) ∧ ?djnt L ∧
             (∀i∈K. B/2 ^ Suc n < r i ⟶ (∃j. j ∈ L ∧ ?cover_ar i j))) ∧ C ⊆ L
    proof (intro exI conjI ballI)
      show C ∪ E ⊆ K
        using D_def ‹C ⊆ K› ‹E ⊆ D› by blast
      show B/2 ^ Suc n ≤ r i if i: i ∈ C ∪ E for i
        using i
      proof
        assume i ∈ C
        have B/2 ^ Suc n ≤ B/2 ^ n
          using ‹B > 0› by (simp add: field_split_simps)
        also have ... ≤ r i
          using Ble ‹i ∈ C› by blast
        finally show ?thesis .
      qed (use D_def ‹E ⊆ D› in auto)
      show ?djnt (C ∪ E)
        using D_def ‹C ⊆ K› ‹E ⊆ D› djntC djntE
        unfolding pairwise_def disjnt_def by blast
    next
      fix i
      assume i ∈ K
      show B/2 ^ Suc n < r i ⟶ (∃j. j ∈ C ∪ E ∧ ?cover_ar i j)
      proof (cases r i ≤ B/2^n)
        case False
        then show ?thesis
          using cov_n ‹i ∈ K› by auto
      next
        case True
        have cball (a i) (r i) ⊆ ball (a j) (5 ∗ r j)
          if less: B/2 ^ Suc n < r i and j: j ∈ C ∪ E
            and nondis: ¬ disjnt (cball (a i) (r i)) (cball (a j) (r j)) for j
```

```
    proof –
      obtain x where x: dist (a i) x ≤ r i dist (a j) x ≤ r j
        using nondis by (force simp: disjnt_def)
      have dist (a i) (a j) ≤ dist (a i) x + dist x (a j)
        by (simp add: dist_triangle)
      also have ... ≤ r i + r j
        by (metis add_mono_thms_linordered_semiring(1) dist_commute x)
      finally have aij: dist (a i) (a j) + r i < 5 ∗ r j if r i < 2 ∗ r j
        using that by auto
      show ?thesis
        using j
      proof
        assume j ∈ C
        have B/2^n < 2 ∗ r j
          using Ble True ‹j ∈ C› less by auto
        with aij True show cball (a i) (r i) ⊆ ball (a j) (5 ∗ r j)
          by (simp add: cball_subset_ball_iff)
      next
        assume j ∈ E
        then have B/2 ^ n < 2 ∗ r j
          using D_def ‹E ⊆ D› by auto
        with True have r i < 2 ∗ r j
          by auto
        with aij show cball (a i) (r i) ⊆ ball (a j) (5 ∗ r j)
          by (simp add: cball_subset_ball_iff)
      qed
    qed
  moreover have ∃j. j ∈ C ∪ E ∧ ¬ disjnt (cball (a i) (r i)) (cball (a j) (r
j))
    if B/2 ^ Suc n < r i
  proof (rule classical)
    assume NON: ¬ ?thesis
    show ?thesis
    proof (cases i ∈ D)
      case True
      have insert i E = E
      proof (rule maximalE)
        show insert i E ⊆ D
          by (simp add: True ‹E ⊆ D›)
        show pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j))) (insert
i E)
          using False NON by (auto simp: pairwise_insert djntE disjnt_sym)
      qed auto
      then show ?thesis
        using ‹i ∈ K› assms by fastforce
    next
      case False
      with that show ?thesis
        by (auto simp: D_def disjnt_def ‹i ∈ K›)
```

qed
qed
ultimately
show $B/2 \hat{} Suc\ n < r\ i \longrightarrow$
$(\exists j.\ j \in C \cup E \wedge$
$\neg\ disjnt\ (cball\ (a\ i)\ (r\ i))\ (cball\ (a\ j)\ (r\ j)) \wedge$
$cball\ (a\ i)\ (r\ i) \subseteq ball\ (a\ j)\ (5 * r\ j))$
by *blast*
qed
qed *auto*
qed (*use assms* **in** *force*)
then obtain F **where** FK: $\bigwedge n.\ F\ n \subseteq K$
and Fle: $\bigwedge n\ i.\ i \in F\ n \Longrightarrow B/2 \hat{} n \le r\ i$
and $Fdjnt$: $\bigwedge n.\ ?djnt\ (F\ n)$
and FF: $\bigwedge n\ i.\ [\![i \in K;\ B/2 \hat{} n < r\ i]\!]$
$\Longrightarrow \exists j.\ j \in F\ n \wedge \neg\ disjnt\ (cball\ (a\ i)\ (r\ i))\ (cball\ (a\ j)\ (r\ j))$
$\wedge$
$cball\ (a\ i)\ (r\ i) \subseteq ball\ (a\ j)\ (5 * r\ j)$
and inc: $\bigwedge n.\ F\ n \subseteq F(Suc\ n)$
by (*force simp*: *all_conj_distrib*)
show *thesis*
proof
have $*$: *countable* I
if $I \subseteq K$ **and** pw: *pairwise* $(\lambda i\ j.\ disjnt\ (cball\ (a\ i)\ (r\ i))\ (cball\ (a\ j)\ (r\ j)))$
I **for** I
proof −
show *?thesis*
proof (*rule countable_image_inj_on* [*of* $\lambda i.\ cball(a\ i)(r\ i)$])
show *countable* $((\lambda i.\ cball\ (a\ i)\ (r\ i))$ ‘ $I)$
proof (*rule countable_disjoint_nonempty_interior_subsets*)
show *disjoint* $((\lambda i.\ cball\ (a\ i)\ (r\ i))$ ‘ $I)$
by (*auto simp*: *dest*: *pairwiseD* [*OF pw*] *intro*: *pairwise_imageI*)
show $\bigwedge S.\ [\![S \in (\lambda i.\ cball\ (a\ i)\ (r\ i))$ ‘ $I;\ interior\ S = \{\}]\!] \Longrightarrow S = \{\}$
using ‹$I \subseteq K$›
by (*auto simp*: *not_less* [*symmetric*])
qed
next
have $\bigwedge x\ y.\ [\![x \in I;\ y \in I;\ a\ x = a\ y;\ r\ x = r\ y]\!] \Longrightarrow x = y$
using *pw* ‹$I \subseteq K$› *assms*
apply (*clarsimp simp*: *pairwise_def disjnt_def*)
by (*metis assms centre_in_cball subsetD empty_iff inf.idem less_eq_real_def*)
then show *inj_on* $(\lambda i.\ cball\ (a\ i)\ (r\ i))\ I$
using ‹$I \subseteq K$› **by** (*fastforce simp*: *inj_on_def cball_eq_cball_iff dest*: *assms*)
qed
qed
show $(Union(range\ F)) \subseteq K$
using FK **by** *blast*
moreover show *pairwise* $(\lambda i\ j.\ disjnt\ (cball\ (a\ i)\ (r\ i))\ (cball\ (a\ j)\ (r\ j)))$
$(Union(range\ F))$

```
    proof (rule pairwise_chain_Union)
      show chain⊆ (range F)
         unfolding chain_subset_def by clarify (meson inc lift_Suc_mono_le linear
subsetCE)
    qed (use Fdjnt in blast)
    ultimately show countable (Union(range F))
      by (blast intro: ∗)
  next
    fix i assume i ∈ K
    then obtain n where (1/2) ^ n < r i / B
      using ‹B > 0› assms real_arch_pow_inv by fastforce
    then have B2: B/2 ^ n < r i
      using ‹B > 0› by (simp add: field_split_simps)
    have 0 < r i r i ≤ B
      by (auto simp: ‹i ∈ K› assms)
    show ∃j. j ∈ (Union(range F)) ∧
          ¬ disjnt (cball (a i) (r i)) (cball (a j) (r j)) ∧
          cball (a i) (r i) ⊆ ball (a j) (5 ∗ r j)
      using FF [OF ‹i ∈ K› B2] by auto
  qed
qed
```

6.45.1 Vitali covering theorem

```
lemma Vitali_covering_lemma_cballs:
  fixes a :: 'a ⇒ 'b::euclidean_space
  assumes S: S ⊆ (⋃i∈K. cball (a i) (r i))
      and r: ⋀i. i ∈ K ⟹ 0 < r i ∧ r i ≤ B
  obtains C where countable C C ⊆ K
     pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j))) C
     S ⊆ (⋃i∈C. cball (a i) (5 ∗ r i))
proof −
  obtain C where C: countable C C ⊆ K
                    pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j))) C
          and cov: ⋀i. i ∈ K ⟹ ∃j. j ∈ C ∧ ¬ disjnt (cball (a i) (r i)) (cball (a
j) (r j)) ∧
                        cball (a i) (r i) ⊆ ball (a j) (5 ∗ r j)
     by (rule Vitali_covering_lemma_cballs_balls [OF r, where a=a]) (blast intro:
that)+
  show ?thesis
  proof
    have (⋃i∈K. cball (a i) (r i)) ⊆ (⋃i∈C. cball (a i) (5 ∗ r i))
      using cov subset_iff by fastforce
    with S show S ⊆ (⋃i∈C. cball (a i) (5 ∗ r i))
      by blast
  qed (use C in auto)
qed

lemma Vitali_covering_lemma_balls:
```

```
  fixes a :: 'a ⇒ 'b::euclidean_space
  assumes S: S ⊆ (⋃i∈K. ball (a i) (r i))
      and r: ⋀i. i ∈ K ⟹ 0 < r i ∧ r i ≤ B
  obtains C where countable C C ⊆ K
     pairwise (λi j. disjnt (ball (a i) (r i)) (ball (a j) (r j))) C
     S ⊆ (⋃i∈C. ball (a i) (5 ∗ r i))
proof −
  obtain C where C: countable C C ⊆ K
           and pw: pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j))) C
          and cov: ⋀i. i ∈ K ⟹ ∃j. j ∈ C ∧ ¬ disjnt (cball (a i) (r i)) (cball (a
j) (r j)) ∧
                         cball (a i) (r i) ⊆ ball (a j) (5 ∗ r j)
     by (rule Vitali_covering_lemma_cballs_balls [OF r, where a=a]) (blast intro:
that)+
  show ?thesis
  proof
    have (⋃i∈K. ball (a i) (r i)) ⊆ (⋃i∈C. ball (a i) (5 ∗ r i))
      using cov subset_iff
      by clarsimp (meson less_imp_le mem_ball mem_cball subset_eq)
    with S show S ⊆ (⋃i∈C. ball (a i) (5 ∗ r i))
      by blast
    show pairwise (λi j. disjnt (ball (a i) (r i)) (ball (a j) (r j))) C
      using pw
       by (clarsimp simp: pairwise_def) (meson ball_subset_cball disjnt_subset1 dis-
jnt_subset2)
  qed (use C in auto)
qed


theorem Vitali_covering_theorem_cballs:
  fixes a :: 'a ⇒ 'n::euclidean_space
  assumes r: ⋀i. i ∈ K ⟹ 0 < r i
      and S: ⋀x d. ⟦x ∈ S; 0 < d⟧
                    ⟹ ∃i. i ∈ K ∧ x ∈ cball (a i) (r i) ∧ r i < d
  obtains C where countable C C ⊆ K
     pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j))) C
     negligible(S − (⋃i ∈ C. cball (a i) (r i)))
proof −
  let ?μ = measure lebesgue
  have ∗: ∃C. countable C ∧ C ⊆ K ∧
            pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j))) C ∧
            negligible(S − (⋃i ∈ C. cball (a i) (r i)))
    if r01: ⋀i. i ∈ K ⟹ 0 < r i ∧ r i ≤ 1
       and Sd: ⋀x d. ⟦x ∈ S; 0 < d⟧ ⟹ ∃i. i ∈ K ∧ x ∈ cball (a i) (r i) ∧ r i
< d
     for K r and a :: 'a ⇒ 'n
  proof −
    obtain C where C: countable C C ⊆ K
      and pwC: pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j))) C
```

```
      and cov: ⋀i. i ∈ K ⟹ ∃j. j ∈ C ∧ ¬ disjnt (cball (a i) (r i)) (cball (a j)
(r j)) ∧
                    cball (a i) (r i) ⊆ ball (a j) (5 * r j)
      by (rule Vitali_covering_lemma_cballs_balls [of K r 1 a]) (auto simp: r01)
    have ar_injective: ⋀x y. ⟦x ∈ C; y ∈ C; a x = a y; r x = r y⟧ ⟹ x = y
      using ‹C ⊆ K› pwC cov
      by (force simp: pairwise_def disjnt_def)
    show ?thesis
    proof (intro exI conjI)
      show negligible (S − (⋃i∈C. cball (a i) (r i)))
      proof (clarsimp simp: negligible_on_intervals [of S−T for T])
        fix l u
        show negligible ((S − (⋃i∈C. cball (a i) (r i))) ∩ cbox l u)
          unfolding negligible_outer_le
        proof (intro allI impI)
          fix e::real
          assume e > 0
          define D where D ≡ {i ∈ C. ¬ disjnt (ball(a i) (5 * r i)) (cbox l u)}
          then have D ⊆ C
            by auto
          have countable D
            unfolding D_def using ‹countable C› by simp
          have UD: (⋃i∈D. cball (a i) (r i)) ∈ lmeasurable
          proof (rule fmeasurableI2)
            show cbox (l − 6 *R One) (u + 6 *R One) ∈ lmeasurable
              by blast
            have y ∈ cbox (l − 6 *R One) (u + 6 *R One)
              if i ∈ C and x: x ∈ cbox l u and ai: dist (a i) y ≤ r i dist (a i) x <
5 * r i
              for i x y
            proof −
              have d6: dist y x < 6 * r i
                using dist_triangle3 [of y x a i] that by linarith
              show ?thesis
              proof (clarsimp simp: mem_box algebra_simps)
                fix j::'n
                assume j: j ∈ Basis
                then have xyj: |x · j − y · j| ≤ dist y x
                  by (metis Basis_le_norm dist_commute dist_norm inner_diff_left)
                have l · j ≤ x · j
                  using ‹j ∈ Basis› mem_box ‹x ∈ cbox l u› by blast
                also have ... ≤ y · j + 6 * r i
                  using d6 xyj by (auto simp: algebra_simps)
                also have ... ≤ y · j + 6
                  using r01 [of i] ‹C ⊆ K› ‹i ∈ C› by auto
                finally have l: l · j ≤ y · j + 6 .
                have y · j ≤ x · j + 6 * r i
                  using d6 xyj by (auto simp: algebra_simps)
                also have ... ≤ u · j + 6 * r i
```

using *j x* **by** (*auto simp*: *mem_box*)
also have ... $\leq u \cdot j + 6$
using *r01* [*of i*] ‹$C \subseteq K$› ‹$i \in C$› **by** *auto*
finally have *u*: $y \cdot j \leq u \cdot j + 6$.
show $l \cdot j \leq y \cdot j + 6 \wedge y \cdot j \leq u \cdot j + 6$
using *l u* **by** *blast*
qed
qed
then show $(\bigcup i{\in}D.\ cball\ (a\ i)\ (r\ i)) \subseteq cbox\ (l - 6 *_R One)\ (u + 6 *_R One)$
by (*force simp*: *D_def disjnt_def*)
show $(\bigcup i{\in}D.\ cball\ (a\ i)\ (r\ i)) \in sets\ lebesgue$
using ‹*countable D*› **by** *auto*
qed
obtain *D1* **where** $D1 \subseteq D$ *finite D1*
and *measD1*: $?\mu\ (\bigcup i{\in}D.\ cball\ (a\ i)\ (r\ i)) - e\ /\ 5\ \hat{}\ DIM('n) < ?\mu\ (\bigcup i{\in}D1.\ cball\ (a\ i)\ (r\ i))$
proof (*rule measure_countable_Union_approachable* [**where** $e = e\ /\ 5\ \hat{}\ (DIM('n))$])
show *countable* $((\lambda i.\ cball\ (a\ i)\ (r\ i))$ ‘ $D)$
using ‹*countable D*› **by** *auto*
show $\bigwedge d.\ d \in (\lambda i.\ cball\ (a\ i)\ (r\ i))$ ‘ $D \Longrightarrow d \in lmeasurable$
by *auto*
show $\bigwedge D'.\ [\![D' \subseteq (\lambda i.\ cball\ (a\ i)\ (r\ i))$ ‘ $D;\ finite\ D']\!] \Longrightarrow ?\mu\ (\bigcup D') \leq ?\mu\ (\bigcup i{\in}D.\ cball\ (a\ i)\ (r\ i))$
by (*fastforce simp add*: *intro*!: *measure_mono_fmeasurable UD*)
qed (*use* ‹$e > 0$› **in** ‹*auto dest*: *finite_subset_image*›)
show $\exists\ T.\ (S - (\bigcup i{\in}C.\ cball\ (a\ i)\ (r\ i))) \cap$
$cbox\ l\ u \subseteq T \wedge T \in lmeasurable \wedge ?\mu\ T \leq e$
proof (*intro exI conjI*)
show $(S - (\bigcup i{\in}C.\ cball\ (a\ i)\ (r\ i))) \cap cbox\ l\ u \subseteq (\bigcup i{\in}D - D1.\ ball\ (a\ i)\ (5 * r\ i))$
proof *clarify*
fix *x*
assume *x*: $x \in cbox\ l\ u$ $x \in S$ $x \notin (\bigcup i{\in}C.\ cball\ (a\ i)\ (r\ i))$
have *closed* $(\bigcup i{\in}D1.\ cball\ (a\ i)\ (r\ i))$
using ‹*finite D1*› **by** *blast*
moreover have $x \notin (\bigcup j{\in}D1.\ cball\ (a\ j)\ (r\ j))$
using *x* ‹$D1 \subseteq D$› **unfolding** *D_def* **by** *blast*
ultimately obtain *q* **where** $q > 0$ **and** *q*: $ball\ x\ q \subseteq - (\bigcup i{\in}D1.\ cball\ (a\ i)\ (r\ i))$
by (*metis* (*no_types*, *lifting*) *ComplI open_contains_ball closed_def*)
obtain *i* **where** $i \in K$ **and** *xi*: $x \in cball\ (a\ i)\ (r\ i)$ **and** *ri*: $r\ i < q/2$
using *Sd* [*OF* ‹$x \in S$›] ‹$q > 0$› *half_gt_zero* **by** *blast*
then obtain *j* **where** $j \in C$
and *nondisj*: $\neg$ *disjnt* (*cball* (*a i*) (*r i*)) (*cball* (*a j*) (*r j*))
and *sub5j*: $cball\ (a\ i)\ (r\ i) \subseteq ball\ (a\ j)\ (5 * r\ j)$
using *cov* [*OF* ‹$i \in K$›] **by** *metis*
show $x \in (\bigcup i{\in}D - D1.\ ball\ (a\ i)\ (5 * r\ i))$

```
        proof
          show j ∈ D − D1
          proof
            show j ∈ D
                using ‹j ∈ C› sub5j ‹x ∈ cbox l u› xi by (auto simp: D_def
disjnt_def)
            obtain y where yi: dist (a i) y ≤ r i and yj: dist (a j) y ≤ r j
              using disjnt_def nondisj by fastforce
            have dist x y ≤ r i + r i
             by (metis add_mono dist_commute dist_triangle_le mem_cball xi yi)
            also have ... < q
              using ri by linarith
            finally have y ∈ ball x q
              by simp
            with yj q show j ∉ D1
              by (auto simp: disjoint_UN_iff)
          qed
          show x ∈ ball (a j) (5 ∗ r j)
            using xi sub5j by blast
        qed
      qed
      have 3: ?μ (⋃i∈D2. ball (a i) (5 ∗ r i)) ≤ e
        if D2: D2 ⊆ D − D1 and finite D2 for D2
      proof −
        have rgt0: 0 < r i if i ∈ D2 for i
          using ‹C ⊆ K› D_def ‹i ∈ D2› D2 r01
          by (simp add: subset_iff)
        then have inj: inj_on (λi. ball (a i) (5 ∗ r i)) D2
         using ‹C ⊆ K› D2 by (fastforce simp: inj_on_def D_def ball_eq_ball_iff
intro: ar_injective)
         have ?μ (⋃i∈D2. ball (a i) (5 ∗ r i)) ≤ sum (?μ) ((λi. ball (a i) (5
∗ r i)) ‘ D2)
          using that by (force intro: measure_Union_le)
        also have ... = (∑i∈D2. ?μ (ball (a i) (5 ∗ r i)))
          by (simp add: comm_monoid_add_class.sum.reindex [OF inj])
        also have ... = (∑i∈D2. 5 ^ DIM('n) ∗ ?μ (ball (a i) (r i)))
        proof (rule sum.cong [OF refl])
          fix i assume i ∈ D2
          thus ?μ (ball (a i) (5 ∗ r i)) = 5 ^ DIM('n) ∗ ?μ (ball (a i) (r i))
            using content_ball_conv_unit_ball[of 5 ∗ r i a i]
                content_ball_conv_unit_ball[of r i a i] rgt0[of i] by auto
        qed
        also have ... = (∑i∈D2. ?μ (ball (a i) (r i))) ∗ 5 ^ DIM('n)
          by (simp add: sum_distrib_left mult.commute)
       finally have ?μ (⋃i∈D2. ball (a i) (5 ∗ r i)) ≤ (∑i∈D2. ?μ (ball (a
i) (r i))) ∗ 5 ^ DIM('n) .
        moreover have (∑i∈D2. ?μ (ball (a i) (r i))) ≤ e / 5 ^ DIM('n)
        proof −
         have D12_dis: ((⋃x∈D1. cball (a x) (r x)) ∩ (⋃x∈D2. cball (a x)
```

$(r\ x))) \leq \{\}$
proof *clarify*
fix *w d1 d2*
assume $d1 \in D1$ *w d1 d2* $\in$ *cball* (*a d1*) (*r d1*) $d2 \in D2$ *w d1 d2* $\in$ *cball* (*a d2*) (*r d2*)
then show *w d1 d2* $\in \{\}$
by (*metis DiffE disjnt_iff subsetCE D2* ‹$D1 \subseteq D$› ‹$D \subseteq C$› *pairwiseD* [*OF pwC*, *of d1 d2*])
qed
have *inj*: *inj_on* (λi. *cball* (*a i*) (*r i*)) *D2*
using *rgt0 D2* ‹$D \subseteq C$› **by** (*force simp*: *inj_on_def cball_eq_cball_iff intro*!: *ar_injective*)
have *ds*: *disjoint* ((λi. *cball* (*a i*) (*r i*)) ‘ *D2*)
using *D2* ‹$D \subseteq C$› **by** (*auto intro*: *pairwiseI pairwiseD* [*OF pwC*])
have ($\sum i \in D2$. *?μ* (*ball* (*a i*) (*r i*))) = ($\sum i \in D2$. *?μ* (*cball* (*a i*) (*r i*)))
by (*simp add*: *content_cball_conv_ball*)
also have ... = *sum ?μ* ((λi. *cball* (*a i*) (*r i*)) ‘ *D2*)
by (*simp add*: *comm_monoid_add_class.sum.reindex* [*OF inj*])
also have ... = *?μ* ($\bigcup i \in D2$. *cball* (*a i*) (*r i*))
by (*auto intro*: *measure_Union′* [*symmetric*] *ds simp add*: ‹*finite D2*›)
finally have *?μ* ($\bigcup i \in D1$. *cball* (*a i*) (*r i*)) + ($\sum i \in D2$. *?μ* (*ball* (*a i*) (*r i*))) =
?μ ($\bigcup i \in D1$. *cball* (*a i*) (*r i*)) + *?μ* ($\bigcup i \in D2$. *cball* (*a i*) (*r i*))
by *simp*
also have ... = *?μ* ($\bigcup i \in D1 \cup D2$. *cball* (*a i*) (*r i*))
using *D12_dis* **by** (*simp add*: *measure_Un3* ‹*finite D1*› ‹*finite D2*› *fmeasurable.finite_UN*)
also have ... $\leq$ *?μ* ($\bigcup i \in D$. *cball* (*a i*) (*r i*))
using *D2* ‹$D1 \subseteq D$› **by** (*fastforce intro*!: *measure_mono_fmeasurable* [*OF _ _ UD*] ‹*finite D1*› ‹*finite D2*›)
finally have *?μ* ($\bigcup i \in D1$. *cball* (*a i*) (*r i*)) + ($\sum i \in D2$. *?μ* (*ball* (*a i*) (*r i*))) $\leq$ *?μ* ($\bigcup i \in D$. *cball* (*a i*) (*r i*)) **.**
with *measD1* **show** *?thesis*
by *simp*
qed
ultimately show *?thesis*
by (*simp add*: *field_split_simps*)
qed
have *co*: *countable* ($D - D1$)
by (*simp add*: ‹*countable D*›)
show ($\bigcup i \in D - D1$. *ball* (*a i*) ($5 * r\ i$)) $\in$ *lmeasurable*
using ‹$e > 0$› **by** (*auto simp*: *fmeasurable_UN_bound* [*OF co _ 3*])
show *?μ* ($\bigcup i \in D - D1$. *ball* (*a i*) ($5 * r\ i$)) $\leq e$
using ‹$e > 0$› **by** (*auto simp*: *measure_UN_bound* [*OF co _ 3*])
qed
qed
qed

```
    qed (use C pwC in auto)
  qed
  define K' where K' ≡ {i ∈ K. r i ≤ 1}
  have 1: ⋀i. i ∈ K' ⟹ 0 < r i ∧ r i ≤ 1
    using K'_def r by auto
  have 2: ∃i. i ∈ K' ∧ x ∈ cball (a i) (r i) ∧ r i < d
    if x ∈ S ∧ 0 < d for x d
    using that by (auto simp: K'_def dest!: S [where d = min d 1])
  have K' ⊆ K
    using K'_def by auto
  then show thesis
    using ∗ [OF 1 2] that by fastforce
qed


theorem Vitali_covering_theorem_balls:
  fixes a :: 'a ⇒ 'b::euclidean_space
  assumes S: ⋀x d. ⟦x ∈ S; 0 < d⟧ ⟹ ∃i. i ∈ K ∧ x ∈ ball (a i) (r i) ∧ r i <
d
  obtains C where countable C C ⊆ K
     pairwise (λi j. disjnt (ball (a i) (r i)) (ball (a j) (r j))) C
     negligible(S − (⋃i ∈ C. ball (a i) (r i)))
proof −
  have 1: ∃i. i ∈ {i ∈ K. 0 < r i} ∧ x ∈ cball (a i) (r i) ∧ r i < d
         if xd: x ∈ S d > 0 for x d
   by (metis (mono_tags, lifting) assms ball_eq_empty less_eq_real_def mem_Collect_eq
mem_ball mem_cball not_le xd(1) xd(2))
  obtain C where C: countable C C ⊆ K
             and pw: pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j))) C
             and neg: negligible(S − (⋃i ∈ C. cball (a i) (r i)))
    by (rule Vitali_covering_theorem_cballs [of {i ∈ K. 0 < r i} r S a, OF _ 1])
auto
  show thesis
  proof
    show pairwise (λi j. disjnt (ball (a i) (r i)) (ball (a j) (r j))) C
      apply (rule pairwise_mono [OF pw])
      apply (auto simp: disjnt_def)
      by (meson disjoint_iff_not_equal less_imp_le mem_cball)
    have negligible (⋃i∈C. sphere (a i) (r i))
      by (auto intro: negligible_sphere ‹countable C›)
    then have negligible (S − (⋃i ∈ C. cball(a i)(r i)) ∪ (⋃i ∈ C. sphere (a i)
(r i)))
      by (rule negligible_Un [OF neg])
    then show negligible (S − (⋃i∈C. ball (a i) (r i)))
      by (rule negligible_subset) force
  qed (use C in auto)
qed
```

```
lemma negligible_eq_zero_density_alt:
    negligible S ⟷
     (∀ x ∈ S. ∀ e > 0.
       ∃ d U. 0 < d ∧ d ≤ e ∧ S ∩ ball x d ⊆ U ∧
            U ∈ lmeasurable ∧ measure lebesgue U < e ∗ measure lebesgue (ball x
d))
    (is _ = (∀ x ∈ S. ∀ e > 0. ?Q x e))
proof (intro iffI ballI allI impI)
  fix x and e :: real
  assume negligible S and x ∈ S and e > 0
  then
  show ∃ d U. 0 < d ∧ d ≤ e ∧ S ∩ ball x d ⊆ U ∧ U ∈ lmeasurable ∧
            measure lebesgue U < e ∗ measure lebesgue (ball x d)
    apply (rule_tac x=e in exI)
    apply (rule_tac x=S ∩ ball x e in exI)
    apply (auto simp: negligible_imp_measurable negligible_Int negligible_imp_measure0
zero_less_measure_iff
              intro: mult_pos_pos content_ball_pos)
    done
next
  assume R [rule_format]: ∀ x ∈ S. ∀ e > 0. ?Q x e
  let ?μ = measure lebesgue
  have ∃ U. openin (top_of_set S) U ∧ z ∈ U ∧ negligible U
    if z ∈ S for z
  proof (intro exI conjI)
    show openin (top_of_set S) (S ∩ ball z 1)
      by (simp add: openin_open_Int)
    show z ∈ S ∩ ball z 1
      using ‹z ∈ S› by auto
    show negligible (S ∩ ball z 1)
    proof (clarsimp simp: negligible_outer_le)
      fix e :: real
      assume e > 0
      let ?K = {(x,d). x ∈ S ∧ 0 < d ∧ ball x d ⊆ ball z 1 ∧
                  (∃ U. S ∩ ball x d ⊆ U ∧ U ∈ lmeasurable ∧
                     ?μ U < e / ?μ (ball z 1) ∗ ?μ (ball x d))}
      obtain C where countable C and Csub: C ⊆ ?K
       and pwC: pairwise (λi j. disjnt (ball (fst i) (snd i)) (ball (fst j) (snd j))) C
       and negC: negligible((S ∩ ball z 1) − (⋃ i ∈ C. ball (fst i) (snd i)))
      proof (rule Vitali_covering_theorem_balls [of S ∩ ball z 1 ?K fst snd])
        fix x and d :: real
        assume x: x ∈ S ∩ ball z 1 and d > 0
        obtain k where k > 0 and k: ball x k ⊆ ball z 1
          by (meson Int_iff open_ball openE x)
        let ?ε = min (e / ?μ (ball z 1) / 2) (min (d / 2) k)
       obtain r U where r: r > 0 r ≤ ?ε and U: S ∩ ball x r ⊆ U U ∈ lmeasurable
          and mU: ?μ U < ?ε ∗ ?μ (ball x r)
         using R [of x ?ε] ‹d > 0› ‹e > 0› ‹k > 0› x by (auto simp: content_ball_pos)
        show ∃ i. i ∈ ?K ∧ x ∈ ball (fst i) (snd i) ∧ snd i < d
```

```
        proof (rule exI [of _ (x,r)], simp, intro conjI exI)
          have ball x r ⊆ ball x k
            using r by (simp add: ball_subset_ball_iff)
          also have ... ⊆ ball z 1
            using ball_subset_ball_iff k by auto
          finally show ball x r ⊆ ball z 1 .
          have ?ε * ?μ (ball x r) ≤ e * content (ball x r) / content (ball z 1)
                using r ‹e > 0› by (simp add: ord_class.min_def field_split_simps
content_ball_pos)
          with mU show ?μ U < e * content (ball x r) / content (ball z 1)
            by auto
        qed (use r U x in auto)
      qed
      have ∃ U. case p of (x,d) ⇒ S ∩ ball x d ⊆ U ∧
                      U ∈ lmeasurable ∧ ?μ U < e / ?μ (ball z 1) * ?μ (ball x d)
        if p ∈ C for p
        using that Csub unfolding case_prod_unfold by blast
      then obtain U where U:
                ⋀p. p ∈ C ⟹
                      case p of (x,d) ⇒ S ∩ ball x d ⊆ U p ∧
                     U p ∈ lmeasurable ∧ ?μ (U p) < e / ?μ (ball z 1) * ?μ (ball x d)
        by (rule that [OF someI_ex])
      let ?T = ((S ∩ ball z 1) − (⋃(x,d)∈C. ball x d)) ∪ ⋃(U ` C)
      show ∃ T. S ∩ ball z 1 ⊆ T ∧ T ∈ lmeasurable ∧ ?μ T ≤ e
      proof (intro exI conjI)
        show S ∩ ball z 1 ⊆ ?T
          using U by fastforce
        { have Um: U i ∈ lmeasurable if i ∈ C for i
            using that U by blast
          have lee: ?μ (⋃i∈I. U i) ≤ e if I ⊆ C finite I for I
          proof −
            have ?μ (⋃(x,d)∈I. ball x d) ≤ ?μ (ball z 1)
              apply (rule measure_mono_fmeasurable)
                  using ‹I ⊆ C› ‹finite I› Csub by (force simp: prod.case_eq_if
sets.finite_UN)+
            then have le1: (?μ (⋃(x,d)∈I. ball x d) / ?μ (ball z 1)) ≤ 1
              by (simp add: content_ball_pos)
            have ?μ (⋃i∈I. U i) ≤ (∑i∈I. ?μ (U i))
              using that U by (blast intro: measure_UNION_le)
            also have ... ≤ (∑(x,r)∈I. e / ?μ (ball z 1) * ?μ (ball x r))
              by (rule sum_mono) (use ‹I ⊆ C› U in force)
            also have ... = (e / ?μ (ball z 1)) * (∑(x,r)∈I. ?μ (ball x r))
              by (simp add: case_prod_app prod.case_distrib sum_distrib_left)
            also have ... = e * (?μ (⋃(x,r)∈I. ball x r) / ?μ (ball z 1))
              apply (subst measure_UNION')
              using that pwC by (auto simp: case_prod_unfold elim: pairwise_mono)
            also have ... ≤ e
                by (metis mult.commute mult.left_neutral mult_le_cancel_iff1 ‹e > 0›
le1)
```

```
          finally show ?thesis .
        qed
        have ⋃(U ‘ C) ∈ lmeasurable ?μ (⋃(U ‘ C)) ≤ e
          using ‹e > 0› Um lee
      by(auto intro!: fmeasurable_UN_bound [OF ‹countable C›] measure_UN_bound
[OF ‹countable C›])
      }
      moreover have ?μ ?T = ?μ (⋃(U ‘ C))
      proof (rule measure_negligible_symdiff [OF ‹⋃(U ‘ C) ∈ lmeasurable›])
        show negligible((⋃(U ‘ C) − ?T) ∪ (?T − ⋃(U ‘ C)))
          by (force intro!: negligible_subset [OF negC])
      qed
      ultimately show ?T ∈ lmeasurable  ?μ ?T ≤ e
       by (simp_all add: fmeasurable.Un negC negligible_imp_measurable split_def)
    qed
  qed
qed
with locally_negligible_alt show negligible S
  by metis
qed

proposition negligible_eq_zero_density:
   negligible S ⟷
    (∀x∈S. ∀r>0. ∀e>0. ∃d. 0 < d ∧ d ≤ r ∧
                (∃U. S ∩ ball x d ⊆ U ∧ U ∈ lmeasurable ∧ measure lebesgue U
< e ∗ measure lebesgue (ball x d)))
proof −
  let ?Q = λx d e. ∃U. S ∩ ball x d ⊆ U ∧ U ∈ lmeasurable ∧ measure lebesgue
U < e ∗ content (ball x d)
  have (∀e>0. ∃d>0. d ≤ e ∧ ?Q x d e) = (∀r>0. ∀e>0. ∃d>0. d ≤ r ∧ ?Q
x d e)
    if x ∈ S for x
  proof (intro iffI allI impI)
    fix r :: real and e :: real
    assume L [rule_format]: ∀e>0. ∃d>0. d ≤ e ∧ ?Q x d e and r > 0 e > 0
    show ∃d>0. d ≤ r ∧ ?Q x d e
      using L [of min r e] apply (rule ex_forward)
       using ‹r > 0› ‹e > 0›  by (auto intro: less_le_trans elim!: ex_forward simp:
content_ball_pos)
  qed auto
  then show ?thesis
    by (force simp: negligible_eq_zero_density_alt)
qed

end
```

6.46 Change of Variables Theorems

```
theory Change_Of_Vars
```

imports *Vitali_Covering_Theorem Determinants*

begin

6.46.1 Measurable Shear and Stretch

proposition
fixes $a :: real^{\wedge\prime}n$
assumes $m \neq n$ **and** *ab_ne*: *cbox a b* $\neq \{\}$ **and** *an*: $0 \leq a\$n$
shows *measurable_shear_interval*: $(\lambda x.\ \chi\ i.\ if\ i = m\ then\ x\$m + x\$n\ else\ x\$i)$ ‘ $(cbox\ a\ b) \in lmeasurable$
(**is** *?f* ‘ _ ∈ _)
and *measure_shear_interval*: *measure lebesgue* $((\lambda x.\ \chi\ i.\ if\ i = m\ then\ x\$m + x\$n\ else\ x\$i)$ ‘ *cbox a b*)
= *measure lebesgue* (*cbox a b*) (**is** *?Q*)
proof –
have *lin*: *linear ?f*
by (*rule linearI*) (*auto simp*: *plus_vec_def scaleR_vec_def algebra_simps*)
show *fab*: *?f* ‘ *cbox a b* ∈ *lmeasurable*
by (*simp add*: *lin measurable_linear_image_interval*)
let $?c = \chi\ i.\ if\ i = m\ then\ b\$m + b\$n\ else\ b\$i$
let $?mn = axis\ m\ 1 - axis\ n\ (1::real)$
have *eq1*: *measure lebesgue* (*cbox a ?c*)
= *measure lebesgue* (*?f* ‘ *cbox a b*)
+ *measure lebesgue* $(cbox\ a\ ?c \cap \{x.\ ?mn \cdot x \leq a\$m\})$
+ *measure lebesgue* $(cbox\ a\ ?c \cap \{x.\ ?mn \cdot x \geq b\$m\})$
proof (*rule measure_Un3_negligible*)
show $cbox\ a\ ?c \cap \{x.\ ?mn \cdot x \leq a\$m\} \in lmeasurable$ $cbox\ a\ ?c \cap \{x.\ ?mn \cdot x \geq b\$m\} \in lmeasurable$
by (*auto simp*: *convex_Int convex_halfspace_le convex_halfspace_ge bounded_Int measurable_convex*)
have *negligible* $\{x.\ ?mn \cdot x = a\$m\}$
by (*metis* ‹$m \neq n$› *axis_index_axis eq_iff_diff_eq_0 negligible_hyperplane*)
moreover have *?f* ‘ $cbox\ a\ b \cap (cbox\ a\ ?c \cap \{x.\ ?mn \cdot x \leq a\ \$\ m\}) \subseteq \{x.\ ?mn \cdot x = a\$m\}$
using ‹$m \neq n$› *antisym_conv* **by** (*fastforce simp*: *algebra_simps mem_box_cart inner_axis′*)
ultimately show *negligible* ((*?f* ‘ *cbox a b*) $\cap (cbox\ a\ ?c \cap \{x.\ ?mn \cdot x \leq a\ \$\ m\}))$
by (*rule negligible_subset*)
have *negligible* $\{x.\ ?mn \cdot x = b\$m\}$
by (*metis* ‹$m \neq n$› *axis_index_axis eq_iff_diff_eq_0 negligible_hyperplane*)
moreover have (*?f* ‘ *cbox a b*) $\cap (cbox\ a\ ?c \cap \{x.\ ?mn \cdot x \geq b\$m\}) \subseteq \{x.\ ?mn \cdot x = b\$m\}$
using ‹$m \neq n$› *antisym_conv* **by** (*fastforce simp*: *algebra_simps mem_box_cart inner_axis′*)
ultimately show *negligible* (*?f* ‘ $cbox\ a\ b \cap (cbox\ a\ ?c \cap \{x.\ ?mn \cdot x \geq b\$m\}))$
by (*rule negligible_subset*)
have *negligible* $\{x.\ ?mn \cdot x = b\$m\}$

by (*metis* ‹$m \neq n$› *axis_index_axis eq_iff_diff_eq_0 negligible_hyperplane*)
moreover have (*cbox a ?c* ∩ {*x. ?mn* · $x \leq a$ \$ *m*} ∩ (*cbox a ?c* ∩ {*x. ?mn* · $x \geq b$\$*m*})) ⊆ {*x. ?mn* · $x = b$\$*m*}
using ‹$m \neq n$› *ab_ne*
apply (*auto simp*: *algebra_simps mem_box_cart inner_axis'*)
apply (*drule_tac x=m* **in** *spec*)+
apply *simp*
done
ultimately show *negligible* (*cbox a ?c* ∩ {*x. ?mn* · $x \leq a$ \$ *m*} ∩ (*cbox a ?c* ∩ {*x. ?mn* · $x \geq b$\$*m*}))
by (*rule negligible_subset*)
show *?f* ‘ *cbox a b* ∪ *cbox a ?c* ∩ {*x. ?mn* · $x \leq a$ \$ *m*} ∪ *cbox a ?c* ∩ {*x. ?mn* · $x \geq b$\$*m*} = *cbox a ?c* (**is** *?lhs* = _)
proof
show *?lhs* ⊆ *cbox a ?c*
by (*auto simp*: *mem_box_cart add_mono*) (*meson add_increasing2 an order_trans*)
show *cbox a ?c* ⊆ *?lhs*
apply (*auto simp*: *algebra_simps image_iff inner_axis' lambda_add_Galois* [*OF* ‹$m \neq n$›])
apply (*auto simp*: *mem_box_cart split*: *if_split_asm*)
done
qed
qed (*fact fab*)
let *?d* = χ *i. if i = m then a* \$ *m* − *b* \$ *m else 0*
have *eq2*: *measure lebesgue* (*cbox a ?c* ∩ {*x. ?mn* · $x \leq a$ \$ *m*}) + *measure lebesgue* (*cbox a ?c* ∩ {*x. ?mn* · $x \geq b$\$*m*})
= *measure lebesgue* (*cbox a* (χ *i. if i = m then a* \$ *m* + *b* \$ *n else b* \$ *i*))
proof (*rule measure_translate_add*[*of cbox a ?c* ∩ {*x. ?mn* · $x \leq a$\$*m*} *cbox a ?c* ∩ {*x. ?mn* · $x \geq b$\$*m*}
(χ *i. if i = m then a*\$*m* − *b*\$*m else 0*) *cbox a* (χ *i. if i = m then a*\$*m* + *b*\$*n else b*\$*i*)])
show (*cbox a ?c* ∩ {*x. ?mn* · $x \leq a$\$*m*}) ∈ *lmeasurable*
cbox a ?c ∩ {*x. ?mn* · $x \geq b$\$*m*} ∈ *lmeasurable*
by (*auto simp*: *convex_Int convex_halfspace_le convex_halfspace_ge bounded_Int measurable_convex*)
have ⋀*x*. ⟦*x* \$ *n* + *a* \$ *m* ≤ *x* \$ *m*⟧
⟹ *x* ∈ (+) (χ *i. if i = m then a* \$ *m* − *b* \$ *m else 0*) ‘ {*x. x* \$ *n* + *b* \$ *m* ≤ *x* \$ *m*}
using ‹$m \neq n$›
by (*rule_tac x=x* − (χ *i. if i = m then a*\$*m* − *b*\$*m else 0*) **in** *image_eqI*)
(*simp_all add*: *mem_box_cart*)
then have *imeq*: (+) *?d* ‘ {*x. b* \$ *m* ≤ *?mn* · *x*} = {*x. a* \$ *m* ≤ *?mn* · *x*}
using ‹$m \neq n$› **by** (*auto simp*: *mem_box_cart inner_axis' algebra_simps*)
have ⋀*x*. ⟦*0* ≤ *a* \$ *n*; *x* \$ *n* + *a* \$ *m* ≤ *x* \$ *m*;
∀ *i. i* ≠ *m* ⟶ *a* \$ *i* ≤ *x* \$ *i* ∧ *x* \$ *i* ≤ *b* \$ *i*⟧
⟹ *a* \$ *m* ≤ *x* \$ *m*
using ‹$m \neq n$› **by** *force*
then have (+) *?d* ‘ (*cbox a ?c* ∩ {*x. b* \$ *m* ≤ *?mn* · *x*})

= *cbox a* (χ *i*. *if i* = *m then a* $ *m* + *b* $ *n else b* $ *i*) ∩ {*x*. *a* $ *m* ≤ *?mn* · *x*}
using *an ab_ne*
apply (*simp add*: *cbox_translation* [*symmetric*] *translation_Int interval_ne_empty_cart imeq*)
apply (*auto simp*: *mem_box_cart inner_axis′ algebra_simps if_distrib all_if_distrib*)
by (*metis* (*full_types*) *add_mono mult_2_right*)
then show *cbox a ?c* ∩ {*x*. *?mn* · *x* ≤ *a* $ *m*} ∪
(+) *?d* ‘ (*cbox a ?c* ∩ {*x*. *b* $ *m* ≤ *?mn* · *x*}) =
cbox a (χ *i*. *if i* = *m then a* $ *m* + *b* $ *n else b* $ *i*) (**is** *?lhs* = *?rhs*)
using *an* ‹*m* ≠ *n*›
apply (*auto simp*: *mem_box_cart inner_axis′ algebra_simps if_distrib all_if_distrib*, *force*)
apply (*drule_tac x*=*n* **in** *spec*)+
by (*meson ab_ne add_mono_thms_linordered_semiring*(*3*) *dual_order.trans interval_ne_empty_cart*(*1*))
have *negligible*{*x*. *?mn* · *x* = *a*$*m*}
by (*metis* ‹*m* ≠ *n*› *axis_index_axis eq_iff_diff_eq_0 negligible_hyperplane*)
moreover have (*cbox a ?c* ∩ {*x*. *?mn* · *x* ≤ *a* $ *m*} ∩
(+) *?d* ‘ (*cbox a ?c* ∩ {*x*. *b* $ *m* ≤ *?mn* · *x*})) ⊆ {*x*. *?mn* · *x* = *a*$*m*}
using ‹*m* ≠ *n*› *antisym_conv* **by** (*fastforce simp*: *algebra_simps mem_box_cart inner_axis′*)
ultimately show *negligible* (*cbox a ?c* ∩ {*x*. *?mn* · *x* ≤ *a* $ *m*} ∩
(+) *?d* ‘ (*cbox a ?c* ∩ {*x*. *b* $ *m* ≤ *?mn* · *x*}))
by (*rule negligible_subset*)
qed
have *ac_ne*: *cbox a ?c* ≠ {}
using *ab_ne an*
by (*clarsimp simp*: *interval_eq_empty_cart*) (*meson add_less_same_cancel1 le_less_linear less_le_trans*)
have *ax_ne*: *cbox a* (χ *i*. *if i* = *m then a* $ *m* + *b* $ *n else b* $ *i*) ≠ {}
using *ab_ne an*
by (*clarsimp simp*: *interval_eq_empty_cart*) (*meson add_less_same_cancel1 le_less_linear less_le_trans*)
have *eq3*: *measure lebesgue* (*cbox a ?c*) = *measure lebesgue* (*cbox a* (χ *i*. *if i* = *m then a*$*m* + *b*$*n else b*$*i*)) + *measure lebesgue* (*cbox a b*)
by (*simp add*: *content_cbox_if_cart ab_ne ac_ne ax_ne algebra_simps prod.delta_remove if_distrib* [*of* λ*u*. *u* − *z* **for** *z*] *prod.remove*)
show *?Q*
using *eq1 eq2 eq3*
by (*simp add*: *algebra_simps*)
qed

proposition
fixes *S* :: (*real*ˆ′*n*) *set*
assumes *S* ∈ *lmeasurable*
shows *measurable_stretch*: ((λ*x*. χ *k*. *m k* ∗ *x*$*k*) ‘ *S*) ∈ *lmeasurable* (**is** *?f* ‘ *S*

∈ _)
  and *measure_stretch*: *measure lebesgue* ((λ*x*. χ *k*. *m k* ∗ *x*$*k*) ' *S*) = |*prod m UNIV*| ∗ *measure lebesgue S*
  (**is** *?MEQ*)
proof −
 have (*?f* ' *S*) ∈ *lmeasurable* ∧ *?MEQ*
 proof (*cases* ∀ *k*. *m k* ≠ *0*)
  case *True*
  have *m0*: *0* < |*prod m UNIV*|
   using *True* **by** *simp*
  have (*indicat_real* (*?f* ' *S*) *has_integral* |*prod m UNIV*| ∗ *measure lebesgue S*) *UNIV*
  proof (*clarsimp simp add*: *has_integral_alt* [**where** *i*=*UNIV*])
   fix *e* :: *real*
   assume *e* > *0*
   have (*indicat_real S has_integral* (*measure lebesgue S*)) *UNIV*
    using *assms lmeasurable_iff_has_integral* **by** *blast*
   then obtain *B* **where** *B*>*0*
    and *B*: ⋀*a b*. *ball 0 B* ⊆ *cbox a b* ⟹
         ∃ *z*. (*indicat_real S has_integral z*) (*cbox a b*) ∧
          |*z* − *measure lebesgue S*| < *e* / |*prod m UNIV*|
     by (*simp add*: *has_integral_alt* [**where** *i*=*UNIV*]) (*metis* (*full_types*) *divide_pos_pos m0 m0* ‹*e* > *0*›)
   show ∃ *B*>*0*. ∀ *a b*. *ball 0 B* ⊆ *cbox a b* ⟶
      (∃ *z*. (*indicat_real* (*?f* ' *S*) *has_integral z*) (*cbox a b*) ∧
       |*z* − |*prod m UNIV*| ∗ *measure lebesgue S*| < *e*)
   proof (*intro exI conjI allI*)
    let *?C* = *Max* (*range* (λ*k*. |*m k*|)) ∗ *B*
    show *?C* > *0*
     using *True* ‹*B* > *0*› **by** (*simp add*: *Max_gr_iff*)
    show *ball 0 ?C* ⊆ *cbox u v* ⟶
      (∃ *z*. (*indicat_real* (*?f* ' *S*) *has_integral z*) (*cbox u v*) ∧
       |*z* − |*prod m UNIV*| ∗ *measure lebesgue S*| < *e*) **for** *u v*
    proof
     assume *uv*: *ball 0 ?C* ⊆ *cbox u v*
     with ‹*?C* > *0*› **have** *cbox_ne*: *cbox u v* ≠ {}
      using *centre_in_ball* **by** *blast*
     let *?α* = λ*k*. *u*$*k* / *m k*
     let *?β* = λ*k*. *v*$*k* / *m k*
     have *invm0*: ⋀*k*. *inverse* (*m k*) ≠ *0*
      using *True* **by** *auto*
     have *ball 0 B* ⊆ (λ*x*. χ *k*. *x* $ *k* / *m k*) ' *ball 0 ?C*
     proof *clarsimp*
      fix *x* :: *real*^'*n*
      assume *x*: *norm x* < *B*
      have [*simp*]: |*Max* (*range* (λ*k*. |*m k*|))| = *Max* (*range* (λ*k*. |*m k*|))
        by (*meson Max_ge abs_ge_zero abs_of_nonneg finite finite_imageI order_trans rangeI*)
      have *norm* (χ *k*. *m k* ∗ *x* $ *k*) ≤ *norm* (*Max* (*range* (λ*k*. |*m k*|)) $*_R$ *x*)

```
              by (rule norm_le_componentwise_cart) (auto simp: abs_mult intro:
mult_right_mono)
          also have ... < ?C
            using x ‹0 < (MAX k. |m k|) * B› ‹0 < B› zero_less_mult_pos2 by
fastforce
          finally have norm (χ k. m k * x $ k) < ?C .
          then show x ∈ (λx. χ k. x $ k / m k) ` ball 0 ?C
            using stretch_Galois [of inverse ∘ m] True by (auto simp: image_iff
field_simps)
         qed
        then have Bsub: ball 0 B ⊆ cbox (χ k. min (?α k) (?β k)) (χ k. max (?α
k) (?β k))
        using cbox_ne uv image_stretch_interval_cart [of inverse ∘ m u v, symmetric]
          by (force simp: field_simps)
         obtain z where zint: (indicat_real S has_integral z) (cbox (χ k. min (?α
k) (?β k)) (χ k. max (?α k) (?β k)))
              and zless: |z − measure lebesgue S| < e / |prod m UNIV|
          using B [OF Bsub] by blast
         have ind: indicat_real (?f ` S) = (λx. indicator S (χ k. x$k / m k))
          using True stretch_Galois [of m] by (force simp: indicator_def)
         show ∃ z. (indicat_real (?f ` S) has_integral z) (cbox u v) ∧
                     |z − |prod m UNIV| * measure lebesgue S| < e
         proof (simp add: ind, intro conjI exI)
           have ((λx. indicat_real S (χ k. x $ k/ m k)) has_integral z *R |prod m
UNIV|)
                ((λx. χ k. x $ k * m k) ` cbox (χ k. min (?α k) (?β k)) (χ k. max
(?α k) (?β k)))
             using True has_integral_stretch_cart [OF zint, of inverse ∘ m]
             by (simp add: field_simps prod_dividef)
           moreover have ((λx. χ k. x $ k * m k) ` cbox (χ k. min (?α k) (?β
k)) (χ k. max (?α k) (?β k))) = cbox u v
             using True image_stretch_interval_cart [of inverse ∘ m u v, symmetric]
               image_stretch_interval_cart [of λk. 1 u v, symmetric] ‹cbox u v ≠ {}›
             by (simp add: field_simps image_comp o_def)
          ultimately show ((λx. indicat_real S (χ k. x $ k/ m k)) has_integral z
*R |prod m UNIV|) (cbox u v)
             by simp
          have |z *R |prod m UNIV| − |prod m UNIV| * measure lebesgue S|
                = |prod m UNIV| * |z − measure lebesgue S|
              by (metis (no_types, hide_lams) abs_abs abs_scaleR mult.commute
real_scaleR_def right_diff_distrib')
          also have ... < e
            using zless True by (simp add: field_simps)
          finally show |z *R |prod m UNIV| − |prod m UNIV| * measure lebesgue
S| < e .
         qed
       qed
     qed
   qed
```

then show *?thesis*
by (*auto simp*: *has_integral_integrable integral_unique lmeasure_integral_UNIV measurable_integrable*)
next
case *False*
then obtain *k* **where** *m k* = *0* **and** *prm*: *prod m UNIV* = *0*
by *auto*
have *nfS*: *negligible* (*?f ' S*)
by (*rule negligible_subset* [*OF negligible_standard_hyperplane_cart*]) (*use* ‹*m k* = *0*› **in** *auto*)
then have (*?f ' S*) ∈ *lmeasurable*
by (*simp add*: *negligible_iff_measure*)
with *nfS* **show** *?thesis*
by (*simp add*: *prm negligible_iff_measure0*)
qed
then show (*?f ' S*) ∈ *lmeasurable ?MEQ*
by *metis*+
qed

proposition
fixes *f* :: *real^'n::{finite,wellorder}* ⇒ *real^'n::_*
assumes *linear f S* ∈ *lmeasurable*
shows *measurable_linear_image*: (*f ' S*) ∈ *lmeasurable*
and *measure_linear_image*: *measure lebesgue* (*f ' S*) = |*det* (*matrix f*)| ∗ *measure lebesgue S* (**is** *?Q f S*)
proof −
have ∀ *S* ∈ *lmeasurable*. (*f ' S*) ∈ *lmeasurable* ∧ *?Q f S*
proof (*rule induct_linear_elementary* [*OF* ‹*linear f*›]; *intro ballI*)
fix *f g* **and** *S* :: (*real,'n*) *vec set*
assume *linear f* **and** *linear g*
and *f* [*rule_format*]: ∀ *S* ∈ *lmeasurable*. *f ' S* ∈ *lmeasurable* ∧ *?Q f S*
and *g* [*rule_format*]: ∀ *S* ∈ *lmeasurable*. *g ' S* ∈ *lmeasurable* ∧ *?Q g S*
and *S*: *S* ∈ *lmeasurable*
then have *gS*: *g ' S* ∈ *lmeasurable*
by *blast*
show (*f* ∘ *g*) *' S* ∈ *lmeasurable* ∧ *?Q* (*f* ∘ *g*) *S*
using *f* [*OF gS*] *g* [*OF S*] *matrix_compose* [*OF* ‹*linear g*› ‹*linear f*›]
by (*simp add*: *o_def image_comp abs_mult det_mul*)
next
fix *f* :: *real^'n::_* ⇒ *real^'n::_* **and** *i* **and** *S* :: (*real^'n::_*) *set*
assume *linear f* **and** *0*: ⋀*x*. *f x* $ *i* = *0* **and** *S* ∈ *lmeasurable*
then have ¬ *inj f*
by (*metis* (*full_types*) *linear_injective_imp_surjective one_neq_zero surjE vec_component*)
have *detf*: *det* (*matrix f*) = *0*
using ‹¬ *inj f*› *det_nz_iff_inj*[*OF* ‹*linear f*›] **by** *blast*
show *f ' S* ∈ *lmeasurable* ∧ *?Q f S*
proof
show *f ' S* ∈ *lmeasurable*

```
      using lmeasurable_iff_indicator_has_integral ‹linear f› ‹¬ inj f› negligible_UNIV
negligible_linear_singular_image by blast
      have measure lebesgue (f ` S) = 0
      by (meson ‹¬ inj f› ‹linear f› negligible_imp_measure0 negligible_linear_singular_image)
      also have ... = |det (matrix f)| * measure lebesgue S
        by (simp add: detf)
      finally show ?Q f S .
    qed
  next
    fix c and S :: (real^'n::_) set
    assume S ∈ lmeasurable
    show (λa. χ i. c i * a $ i) ` S ∈ lmeasurable ∧ ?Q (λa. χ i. c i * a $ i) S
    proof
      show (λa. χ i. c i * a $ i) ` S ∈ lmeasurable
        by (simp add: ‹S ∈ lmeasurable› measurable_stretch)
      show ?Q (λa. χ i. c i * a $ i) S
      by (simp add: measure_stretch [OF ‹S ∈ lmeasurable›, of c] axis_def matrix_def
det_diagonal)
    qed
  next
    fix m :: 'n and n :: 'n and S :: (real, 'n) vec set
    assume m ≠ n and S ∈ lmeasurable
    let ?h = λv::(real, 'n) vec. χ i. v $ Fun.swap m n id i
    have lin: linear ?h
      by (rule linearI) (simp_all add: plus_vec_def scaleR_vec_def)
    have meq: measure lebesgue ((λv::(real, 'n) vec. χ i. v $ Fun.swap m n id i) `
cbox a b)
            = measure lebesgue (cbox a b) for a b
    proof (cases cbox a b = {})
      case True then show ?thesis
        by simp
    next
      case False
      then have him: ?h ` (cbox a b) ≠ {}
        by blast
      have eq: ?h ` (cbox a b) = cbox (?h a) (?h b)
      by (auto simp: image_iff lambda_swap_Galois mem_box_cart) (metis swap_id_eq)+
      show ?thesis
       using him prod.permute [OF permutes_swap_id, where S=UNIV and g=λi.
(b − a)$i, symmetric]
        by (simp add: eq content_cbox_cart False)
    qed
    have (χ i j. if Fun.swap m n id i = j then 1 else 0) = (χ i j. if j = Fun.swap
m n id i then 1 else (0::real))
      by (auto intro!: Cart_lambda_cong)
    then have matrix ?h = transpose(χ i j. mat 1 $ i $ Fun.swap m n id j)
      by (auto simp: matrix_eq transpose_def axis_def mat_def matrix_def)
    then have 1: |det (matrix ?h)| = 1
      by (simp add: det_permute_columns permutes_swap_id sign_swap_id abs_mult)
```

show *?h ‘ S ∈ lmeasurable ∧ ?Q ?h S*
proof
show *?h ‘ S ∈ lmeasurable ?Q ?h S*
using *measure_linear_sufficient* [*OF lin* ‹*S ∈ lmeasurable*›] *meq 1* **by** *force+*
qed
next
fix *m n* :: *′n* **and** *S* :: (*real*, *′n*) *vec set*
assume *m ≠ n* **and** *S ∈ lmeasurable*
let *?h* = *λv*::(*real*, *′n*) *vec. χ i. if i = m then v $ m + v $ n else v $ i*
have *lin*: *linear ?h*
by (*rule linearI*) (*auto simp*: *algebra_simps plus_vec_def scaleR_vec_def vec_eq_iff*)
consider *m < n* | *n < m*
using ‹*m ≠ n*› *less_linear* **by** *blast*
then have *1*: *det*(*matrix ?h*) = *1*
proof *cases*
assume *m < n*
have ∗: *matrix ?h $ i $ j* = (*0*::*real*) **if** *j < i* **for** *i j* :: *′n*
proof −
have *axis j 1* = (*χ n. if n = j then 1 else* (*0*::*real*))
using *axis_def* **by** *blast*
then have (*χ p q. if p = m then axis q 1 $ m + axis q 1 $ n else axis q 1 $ p*) *$ i $ j* = (*0*::*real*)
using ‹*j < i*› *axis_def* ‹*m < n*› **by** *auto*
with ‹*m < n*› **show** *?thesis*
by (*auto simp*: *matrix_def axis_def cong*: *if_cong*)
qed
show *?thesis*
using ‹*m ≠ n*› **by** (*subst det_upperdiagonal* [*OF* ∗]) (*auto simp*: *matrix_def axis_def cong*: *if_cong*)
next
assume *n < m*
have ∗: *matrix ?h $ i $ j* = (*0*::*real*) **if** *j > i* **for** *i j* :: *′n*
proof −
have *axis j 1* = (*χ n. if n = j then 1 else* (*0*::*real*))
using *axis_def* **by** *blast*
then have (*χ p q. if p = m then axis q 1 $ m + axis q 1 $ n else axis q 1 $ p*) *$ i $ j* = (*0*::*real*)
using ‹*j > i*› *axis_def* ‹*m > n*› **by** *auto*
with ‹*m > n*› **show** *?thesis*
by (*auto simp*: *matrix_def axis_def cong*: *if_cong*)
qed
show *?thesis*
using ‹*m ≠ n*›
by (*subst det_lowerdiagonal* [*OF* ∗]) (*auto simp*: *matrix_def axis_def cong*: *if_cong*)
qed
have *meq*: *measure lebesgue* (*?h ‘* (*cbox a b*)) = *measure lebesgue* (*cbox a b*) **for** *a b*
proof (*cases cbox a b* = {})

```
      case True then show ?thesis by simp
    next
      case False
      then have ne: (+) (χ i. if i = n then − a $ n else 0) ` cbox a b ≠ {}
        by auto
      let ?v = χ i. if i = n then − a $ n else 0
      have ?h ` cbox a b
           = (+) (χ i. if i = m ∨ i = n then a $ n else 0) ` ?h ` (+) ?v ` (cbox a b)
        using ‹m ≠ n› unfolding image_comp o_def by (force simp: vec_eq_iff)
      then have measure lebesgue (?h ` (cbox a b))
               = measure lebesgue ((λv. χ i. if i = m then v $ m + v $ n else v $ i) `
                                   (+) ?v ` cbox a b)
        by (rule ssubst) (rule measure_translation)
       also have ... = measure lebesgue ((λv. χ i. if i = m then v $ m + v $ n
else v $ i) ` cbox (?v +a) (?v + b))
        by (metis (no_types, lifting) cbox_translation)
      also have ... = measure lebesgue ((+) (χ i. if i = n then − a $ n else 0) `
cbox a b)
        apply (subst measure_shear_interval)
        using ‹m ≠ n› ne apply auto
        apply (simp add: cbox_translation)
        by (metis cbox_borel cbox_translation measure_completion sets_lborel)
      also have ... = measure lebesgue (cbox a b)
        by (rule measure_translation)
        finally show ?thesis .
      qed
    show ?h ` S ∈ lmeasurable ∧ ?Q ?h S
      using measure_linear_sufficient [OF lin ‹S ∈ lmeasurable›] meq 1 by force
  qed
  with assms show (f ` S) ∈ lmeasurable ?Q f S
    by metis+
qed


lemma
 fixes f :: real^'n::{finite,wellorder} ⇒ real^'n::_
  assumes f: orthogonal_transformation f and S: S ∈ lmeasurable
  shows measurable_orthogonal_image: f ` S ∈ lmeasurable
    and measure_orthogonal_image: measure lebesgue (f ` S) = measure lebesgue S
proof −
  have linear f
    by (simp add: f orthogonal_transformation_linear)
  then show f ` S ∈ lmeasurable
    by (metis S measurable_linear_image)
  show measure lebesgue (f ` S) = measure lebesgue S
    by (simp add: measure_linear_image ‹linear f› S f)
qed

proposition measure_semicontinuous_with_hausdist_explicit:
```

assumes *bounded S* **and** *neg*: *negligible*(*frontier S*) **and** $e > 0$
obtains *d* **where** $d > 0$
$\bigwedge T.$ ⟦$T \in$ *lmeasurable*; $\bigwedge y.\ y \in T \Longrightarrow \exists x.\ x \in S \wedge$ *dist x y* $< d$⟧
$\Longrightarrow$ *measure lebesgue T* $<$ *measure lebesgue S* $+ e$
proof (*cases* $S = \{\}$)
case *True*
with *that* ‹$e > 0$› **show** *?thesis* **by** *force*
next
case *False*
then have *frS*: *frontier S* $\neq \{\}$
using ‹*bounded S*› *frontier_eq_empty not_bounded_UNIV* **by** *blast*
have $S \in$ *lmeasurable*
by (*simp add*: ‹*bounded S*› *measurable_Jordan neg*)
have *null*: (*frontier S*) $\in$ *null_sets lebesgue*
by (*metis neg negligible_iff_null_sets*)
have *frontier S* $\in$ *lmeasurable* **and** *mS0*: *measure lebesgue* (*frontier S*) $= 0$
using *neg negligible_imp_measurable negligible_iff_measure* **by** *blast+*
with ‹$e > 0$› *sets_lebesgue_outer_open*
obtain *U* **where** *open U*
and *U*: *frontier S* $\subseteq$ *U U* $-$ *frontier S* $\in$ *lmeasurable emeasure lebesgue* (*U* $-$ *frontier S*) $< e$
by (*metis fmeasurableD*)
with *null* **have** $U \in$ *lmeasurable*
by (*metis borel_open measurable_Diff_null_set sets_completionI_sets sets_lborel*)
have *measure lebesgue* (*U* $-$ *frontier S*) $=$ *measure lebesgue U*
using *mS0* **by** (*simp add*: ‹$U \in$ *lmeasurable*› *fmeasurableD measure_Diff_null_set null*)
with *U* **have** *mU*: *measure lebesgue U* $< e$
by (*simp add*: *emeasure_eq_measure2 ennreal_less_iff*)
show *?thesis*
proof
have $U \neq$ *UNIV*
using ‹$U \in$ *lmeasurable*› **by** *auto*
then have $- U \neq \{\}$
by *blast*
with ‹*open U*› ‹*frontier S* $\subseteq U$› **show** *setdist* (*frontier S*) $(- U) > 0$
by (*auto simp*: ‹*bounded S*› *open_closed compact_frontier_bounded setdist_gt_0_compact_closed frS*)
fix *T*
assume $T \in$ *lmeasurable*
and *T*: $\bigwedge t.\ t \in T \Longrightarrow \exists y.\ y \in S \wedge$ *dist y t* $<$ *setdist* (*frontier S*) $(- U)$
then have *measure lebesgue T* $-$ *measure lebesgue S* $\leq$ *measure lebesgue* $(T - S)$
by (*simp add*: ‹$S \in$ *lmeasurable*› *measure_diff_le_measure_setdiff*)
also have ... $\leq$ *measure lebesgue U*
proof $-$
have $T - S \subseteq U$
proof *clarify*
fix *x*

```
        assume x ∈ T and x ∉ S
        then obtain y where y ∈ S and y: dist y x < setdist (frontier S) (− U)
          using T by blast
        have closed_segment x y ∩ frontier S ≠ {}
          using connected_Int_frontier ‹x ∉ S› ‹y ∈ S› by blast
        then obtain z where z: z ∈ closed_segment x y z ∈ frontier S
          by auto
        with y have dist z x < setdist(frontier S) (− U)
          by (auto simp: dist_commute dest!: dist_in_closed_segment)
        with z have False if x ∈ −U
          using setdist_le_dist [OF ‹z ∈ frontier S› that] by auto
        then show x ∈ U
          by blast
      qed
      then show ?thesis
         by (simp add: ‹S ∈ lmeasurable› ‹T ∈ lmeasurable› ‹U ∈ lmeasurable›
fmeasurableD measure_mono_fmeasurable sets.Diff)
    qed
    finally have measure lebesgue T − measure lebesgue S ≤ measure lebesgue U
.
    with mU show measure lebesgue T < measure lebesgue S + e
      by linarith
  qed
qed

proposition
 fixes f :: real^'n::{finite,wellorder} ⇒ real^'n::_
  assumes S: S ∈ lmeasurable
  and deriv: ⋀x. x ∈ S ⟹ (f has_derivative f' x) (at x within S)
  and int: (λx. |det (matrix (f' x))|) integrable_on S
  and bounded: ⋀x. x ∈ S ⟹ |det (matrix (f' x))| ≤ B
  shows measurable_bounded_differentiable_image:
       f ` S ∈ lmeasurable
    and measure_bounded_differentiable_image:
       measure lebesgue (f ` S) ≤ B * measure lebesgue S (is ?M)
proof −
  have f ` S ∈ lmeasurable ∧ measure lebesgue (f ` S) ≤ B * measure lebesgue S
  proof (cases B < 0)
    case True
    then have S = {}
     by (meson abs_ge_zero bounded empty_iff equalityI less_le_trans linorder_not_less
subsetI)
    then show ?thesis
      by auto
  next
    case False
    then have B ≥ 0
      by arith
    let ?μ = measure lebesgue
```

have *f_diff*: *f differentiable_on S*
using *deriv* **by** (*auto simp*: *differentiable_on_def differentiable_def*)
have *eps*: $f` S \in lmeasurable\ ?\mu\ (f` S) \leq (B+e) * ?\mu\ S$ (**is** *?ME*)
if $e > 0$ **for** *e*
proof −
have *eps_d*: $f` S \in lmeasurable\ \ ?\mu\ (f` S) \leq (B+e) * (?\mu\ S + d)$ (**is** *?MD*)
if $d > 0$ **for** *d*
proof −
obtain *T* **where** *T*: *open T* $S \subseteq T$ **and** *TS*: $(T-S) \in lmeasurable$ **and** *emeasure lebesgue* $(T-S) <$ *ennreal d*
using *S* ‹$d > 0$› *sets_lebesgue_outer_open* **by** *blast*
then have $?\mu\ (T-S) < d$
by (*metis emeasure_eq_measure2 ennreal_leI not_less*)
with *S T TS* **have** $T \in lmeasurable$ **and** *Tless*: $?\mu\ T < ?\mu\ S + d$
by (*auto simp*: *measurable_measure_Diff dest*!: *fmeasurable_Diff_D*)
have $\exists r.\ 0 < r \wedge r < d \wedge ball\ x\ r \subseteq T \wedge f` (S \cap ball\ x\ r) \in lmeasurable \wedge$
$?\mu\ (f` (S \cap ball\ x\ r)) \leq (B + e) * ?\mu\ (ball\ x\ r)$
if $x \in S$ $d > 0$ **for** *x d*
proof −
have *lin*: *linear* $(f'\ x)$
and *lim0*: $((\lambda y.\ (f\ y - (f\ x + f'\ x\ (y - x)))\ /_R\ norm(y - x)) \longrightarrow 0)$ (*at x within S*)
using *deriv* ‹$x \in S$› **by** (*auto simp*: *has_derivative_within bounded_linear.linear field_simps*)
have *bo*: *bounded* $(f'\ x` ball\ 0\ 1)$
by (*simp add*: *bounded_linear_image linear_linear lin*)
have *neg*: *negligible* (*frontier* $(f'\ x` ball\ 0\ 1)$)
using *deriv has_derivative_linear* ‹$x \in S$›
by (*auto intro*!: *negligible_convex_frontier* [*OF convex_linear_image*])
let *?unit_vol* = *content* (*ball* (0 :: *real* ^ '*n* :: {*finite*, *wellorder*}) 1))
have *0*: $0 < e * ?unit\_vol$
using ‹$e > 0$› **by** (*simp add*: *content_ball_pos*)
obtain *k* **where** $k > 0$ **and** *k*:
$\bigwedge U.\ [\![U \in lmeasurable;\ \bigwedge y.\ y \in U \Longrightarrow \exists z.\ z \in f'\ x` ball\ 0\ 1 \wedge dist\ z\ y < k]\!]$
$\Longrightarrow ?\mu\ U < ?\mu\ (f'\ x` ball\ 0\ 1) + e * ?unit\_vol$
using *measure_semicontinuous_with_hausdist_explicit* [*OF bo neg 0*] **by** *blast*
obtain *l* **where** $l > 0$ **and** *l*: $ball\ x\ l \subseteq T$
using ‹$x \in S$› ‹*open T*› ‹$S \subseteq T$› *openE* **by** *blast*
obtain ζ **where** $0 < \zeta$
and ζ: $\bigwedge y.\ [\![y \in S;\ y \neq x;\ dist\ y\ x < \zeta]\!]$
$\Longrightarrow norm\ (f\ y - (f\ x + f'\ x\ (y - x)))\ /\ norm\ (y - x) < k$
using *lim0* ‹$k > 0$› **by** (*simp add*: *Lim_within*) (*auto simp add*: *field_simps*)
define *r* **where** $r \equiv min\ (min\ l\ (\zeta/2))\ (min\ 1\ (d/2))$
show *?thesis*
proof (*intro exI conjI*)
show $r > 0$ $r < d$
using ‹$l > 0$› ‹$\zeta > 0$› ‹$d > 0$› **by** (*auto simp*: *r_def*)

```
    have r ≤ l
      by (auto simp: r_def)
    with l show ball x r ⊆ T
      by auto
    have ex_lessK: ∃x' ∈ ball 0 1. dist (f' x x') ((f y − f x) /R r) < k
      if y ∈ S and dist x y < r for y
    proof (cases y = x)
      case True
      with lin linear_0 ‹k > 0› that show ?thesis
        by (rule_tac x=0 in bexI) (auto simp: linear_0)
    next
      case False
      then show ?thesis
      proof (rule_tac x=(y − x) /R r in bexI)
        have f' x ((y − x) /R r) = f' x (y − x) /R r
          by (simp add: lin linear_scale)
        then have dist (f' x ((y − x) /R r)) ((f y − f x) /R r) = norm (f'
x (y − x) /R r − (f y − f x) /R r)
          by (simp add: dist_norm)
        also have ... = norm (f' x (y − x) − (f y − f x)) / r
            using ‹r > 0› by (simp add: divide_simps scale_right_diff_distrib
[symmetric])
        also have ... ≤ norm (f y − (f x + f' x (y − x))) / norm (y − x)
          using that ‹r > 0› False by (simp add: field_split_simps dist_norm
norm_minus_commute mult_right_mono)
        also have ... < k
          using that ‹0 < ζ› by (simp add: dist_commute r_def ζ [OF ‹y ∈
S› False])
        finally show dist (f' x ((y − x) /R r)) ((f y − f x) /R r) < k .
        show (y − x) /R r ∈ ball 0 1
              using that ‹r > 0› by (simp add: dist_norm divide_simps
norm_minus_commute)
      qed
    qed
    let ?rfs = (λx. x /R r) ` (+) (− f x) ` f ` (S ∩ ball x r)
    have rfs_mble: ?rfs ∈ lmeasurable
    proof (rule bounded_set_imp_lmeasurable)
      have f differentiable_on S ∩ ball x r
        using f_diff by (auto simp: fmeasurableD differentiable_on_subset)
      with S show ?rfs ∈ sets lebesgue
          by (auto simp: sets.Int intro!: lebesgue_sets_translation differen-
tiable_image_in_sets_lebesgue)
      let ?B = (λ(x, y). x + y) ` (f' x ` ball 0 1 × ball 0 k)
      have bounded ?B
        by (simp add: bounded_plus [OF bo])
      moreover have ?rfs ⊆ ?B
        apply (auto simp: dist_norm image_iff dest!: ex_lessK)
    by (metis (no_types, hide_lams) add.commute diff_add_cancel dist_0_norm
dist_commute dist_norm mem_ball)
```

ultimately show *bounded* (*?rfs*)
by (*rule bounded_subset*)
qed
then have ($\lambda x.\ r *_R x$) ' *?rfs* ∈ *lmeasurable*
by (*simp add*: *measurable_linear_image*)
with ‹*r* > *0*› **have** (+) (− *f x*) ' *f* ' (*S* ∩ *ball x r*) ∈ *lmeasurable*
by (*simp add*: *image_comp o_def*)
then have (+) (*f x*) ' (+) (− *f x*) ' *f* ' (*S* ∩ *ball x r*) ∈ *lmeasurable*
using *measurable_translation* **by** *blast*
then show *fsb*: *f* ' (*S* ∩ *ball x r*) ∈ *lmeasurable*
by (*simp add*: *image_comp o_def*)
have *?μ* (*f* ' (*S* ∩ *ball x r*)) = *?μ* (*?rfs*) ∗ *r* ˆ *CARD*(′*n*)
using ‹*r* > *0*› *fsb*
by (*simp add*: *measure_linear_image measure_translation_subtract measurable_translation_subtract field_simps cong*: *image_cong_simp*)
also have ... ≤ (|*det* (*matrix* (*f′ x*))| ∗ *?unit_vol* + *e* ∗ *?unit_vol*) ∗ *r* ˆ *CARD*(′*n*)
proof −
have *?μ* (*?rfs*) < *?μ* (*f′ x* ' *ball 0 1*) + *e* ∗ *?unit_vol*
using *rfs_mble* **by** (*force intro*: *k dest*!: *ex_lessK*)
then have *?μ* (*?rfs*) < |*det* (*matrix* (*f′ x*))| ∗ *?unit_vol* + *e* ∗ *?unit_vol*
by (*simp add*: *lin measure_linear_image* [*of f′ x*])
with ‹*r* > *0*› **show** *?thesis*
by *auto*
qed
also have ... ≤ (*B* + *e*) ∗ *?μ* (*ball x r*)
using *bounded* [*OF* ‹*x* ∈ *S*›] ‹*r* > *0*›
by (*simp add*: *algebra_simps content_ball_conv_unit_ball*[*of r*] *content_ball_pos*)
finally show *?μ* (*f* ' (*S* ∩ *ball x r*)) ≤ (*B* + *e*) ∗ *?μ* (*ball x r*) **.**
qed
qed
then obtain *r* **where**
r0d: ⋀*x d*. ⟦*x* ∈ *S*; *d* > *0*⟧ ⟹ *0* < *r x d* ∧ *r x d* < *d*
and *rT*: ⋀*x d*. ⟦*x* ∈ *S*; *d* > *0*⟧ ⟹ *ball x* (*r x d*) ⊆ *T*
and *r*: ⋀*x d*. ⟦*x* ∈ *S*; *d* > *0*⟧ ⟹
(*f* ' (*S* ∩ *ball x* (*r x d*))) ∈ *lmeasurable* ∧
?μ (*f* ' (*S* ∩ *ball x* (*r x d*))) ≤ (*B* + *e*) ∗ *?μ* (*ball x* (*r x d*))
by *metis*
obtain *C* **where** *countable C* **and** *Csub*: *C* ⊆ {(*x*,*r x t*) |*x t*. *x* ∈ *S* ∧ *0* < *t*}
and *pwC*: *pairwise* (λ*i j*. *disjnt* (*ball* (*fst i*) (*snd i*)) (*ball* (*fst j*) (*snd j*))) *C*
and *negC*: *negligible*(*S* − (⋃*i* ∈ *C*. *ball* (*fst i*) (*snd i*)))
apply (*rule Vitali_covering_theorem_balls* [*of S* {(*x*,*r x t*) |*x t*. *x* ∈ *S* ∧ *0* < *t*} *fst snd*])
apply *auto*
by (*metis dist_eq_0_iff r0d*)
let *?UB* = (⋃(*x*,*s*) ∈ *C*. *ball x s*)

```
    have eq: f ` (S ∩ ?UB) = (⋃(x,s) ∈ C. f ` (S ∩ ball x s))
      by auto
    have mle: ?μ (⋃(x,s) ∈ K. f ` (S ∩ ball x s)) ≤ (B + e) ∗ (?μ S + d)  (is
?l ≤ ?r)
      if K ⊆ C and finite K for K
    proof –
      have gt0: b > 0 if (a, b) ∈ K for a b
        using Csub that ‹K ⊆ C› r0d by auto
      have inj: inj_on (λ(x, y). ball x y) K
        by (force simp: inj_on_def ball_eq_ball_iff dest: gt0)
      have disjnt: disjoint ((λ(x, y). ball x y) ` K)
        using pwC that
        apply (clarsimp simp: pairwise_def case_prod_unfold ball_eq_ball_iff)
        by (metis subsetD fst_conv snd_conv)
      have ?l ≤ (∑i∈K. ?μ (case i of (x, s) ⇒ f ` (S ∩ ball x s)))
      proof (rule measure_UNION_le [OF ‹finite K›], clarify)
        fix x r
        assume (x,r) ∈ K
        then have x ∈ S
          using Csub ‹K ⊆ C› by auto
        show f ` (S ∩ ball x r) ∈ sets lebesgue
            by (meson Int_lower1 S differentiable_on_subset f_diff fmeasurableD
lmeasurable_ball order_refl sets.Int differentiable_image_in_sets_lebesgue)
      qed
      also have ... ≤ (∑(x,s) ∈ K. (B + e) ∗ ?μ (ball x s))
        apply (rule sum_mono)
        using Csub r ‹K ⊆ C› by auto
      also have ... = (B + e) ∗ (∑(x,s) ∈ K. ?μ (ball x s))
        by (simp add: prod.case_distrib sum_distrib_left)
      also have ... = (B + e) ∗ sum ?μ ((λ(x, y). ball x y) ` K)
        using ‹B ≥ 0› ‹e > 0› by (simp add: inj sum.reindex prod.case_distrib)
      also have ... = (B + e) ∗ ?μ (⋃(x,s) ∈ K. ball x s)
        using ‹B ≥ 0› ‹e > 0› that
        by (subst measure_Union') (auto simp: disjnt measure_Union')
      also have ... ≤ (B + e) ∗ ?μ T
        using ‹B ≥ 0› ‹e > 0› that apply simp
        apply (rule measure_mono_fmeasurable [OF _ _ ‹T ∈ lmeasurable›])
        using Csub rT by force+
      also have ... ≤ (B + e) ∗ (?μ S + d)
        using ‹B ≥ 0› ‹e > 0› Tless by simp
      finally show ?thesis .
    qed
    have fSUB_mble: (f ` (S ∩ ?UB)) ∈ lmeasurable
      unfolding eq using Csub r False ‹e > 0› that
      by (auto simp: intro!: fmeasurable_UN_bound [OF ‹countable C› _ mle])
    have fSUB_meas: ?μ (f ` (S ∩ ?UB)) ≤ (B + e) ∗ (?μ S + d)  (is ?MUB)
      unfolding eq using Csub r False ‹e > 0› that
      by (auto simp: intro!: measure_UN_bound [OF ‹countable C› _ mle])
    have neg: negligible ((f ` (S ∩ ?UB) − f ` S) ∪ (f ` S − f ` (S ∩ ?UB)))
```

proof (*rule negligible_subset* [*OF negligible_differentiable_image_negligible* [*OF order_refl negC*, **where** *f*=*f*]])
show *f differentiable_on S* − (⋃*i*∈*C*. *ball* (*fst i*) (*snd i*))
by (*meson DiffE differentiable_on_subset subsetI f_diff*)
qed *force*
show *f* ‘ *S* ∈ *lmeasurable*
by (*rule lmeasurable_negligible_symdiff* [*OF fSUB_mble neg*])
show *?MD*
using *fSUB_meas measure_negligible_symdiff* [*OF fSUB_mble neg*] **by** *simp*
qed
show *f* ‘ *S* ∈ *lmeasurable*
using *eps_d* [*of 1*] **by** *simp*
show *?ME*
proof (*rule field_le_epsilon*)
fix *δ* :: *real*
assume *0* < *δ*
then show *?μ* (*f* ‘ *S*) ≤ (*B* + *e*) ∗ *?μ S* + *δ*
using *eps_d* [*of δ* / (*B*+*e*)] ‹*e* > *0*› ‹*B* ≥ *0*› **by** (*auto simp*: *divide_simps mult_ac*)
qed
qed
show *?thesis*
proof (*cases ?μ S* = *0*)
case *True*
with *eps* **have** *?μ* (*f* ‘ *S*) = *0*
by (*metis mult_zero_right not_le zero_less_measure_iff*)
then show *?thesis*
using *eps* [*of 1*] **by** (*simp add*: *True*)
next
case *False*
have *?μ* (*f* ‘ *S*) ≤ *B* ∗ *?μ S*
proof (*rule field_le_epsilon*)
fix *e* :: *real*
assume *e* > *0*
then show *?μ* (*f* ‘ *S*) ≤ *B* ∗ *?μ S* + *e*
using *eps* [*of e* / *?μ S*] *False* **by** (*auto simp*: *algebra_simps zero_less_measure_iff*)
qed
with *eps* [*of 1*] **show** *?thesis* **by** *auto*
qed
qed
then show *f* ‘ *S* ∈ *lmeasurable ?M* **by** *blast*+
qed

lemma *m_diff_image_weak*:
fixes *f* :: *real*^'*n*::{*finite*,*wellorder*} ⇒ *real*^'*n*::_
assumes *S*: *S* ∈ *lmeasurable*
and *deriv*: ⋀*x*. *x* ∈ *S* ⟹ (*f has_derivative f′ x*) (*at x within S*)
and *int*: (λ*x*. |*det* (*matrix* (*f′ x*))|) *integrable_on S*
shows *f* ‘ *S* ∈ *lmeasurable* ∧ *measure lebesgue* (*f* ‘ *S*) ≤ *integral S* (λ*x*. |*det*

(*matrix* (*f′ x*))|)
proof −
  let *?μ* = *measure lebesgue*
  have *aint_S*: (λ*x*. |*det* (*matrix* (*f′ x*))|) *absolutely_integrable_on S*
    using *int* **unfolding** *absolutely_integrable_on_def* **by** *auto*
  define *m* **where** *m* ≡ *integral S* (λ*x*. |*det* (*matrix* (*f′ x*))|)
  have ∗: *f ‘ S* ∈ *lmeasurable ?μ* (*f ‘ S*) ≤ *m* + *e* ∗ *?μ S*
    if *e* > *0* **for** *e*
  proof −
    define *T* **where** *T* ≡ λ*n*. {*x* ∈ *S*. *n* ∗ *e* ≤ |*det* (*matrix* (*f′ x*))| ∧
                                  |*det* (*matrix* (*f′ x*))| < (*Suc n*) ∗ *e*}
    have *meas_t*: *T n* ∈ *lmeasurable* **for** *n*
    proof −
      have ∗: (λ*x*. |*det* (*matrix* (*f′ x*))|) ∈ *borel_measurable* (*lebesgue_on S*)
       using *aint_S* **by** (*simp add*: *S borel_measurable_restrict_space_iff fmeasurableD set_integrable_def*)
      have [*intro*]: *x* ∈ *sets* (*lebesgue_on S*) ⟹ *x* ∈ *sets lebesgue* **for** *x*
        using *S sets_restrict_space_subset* **by** *blast*
      have {*x* ∈ *S*. *real n* ∗ *e* ≤ |*det* (*matrix* (*f′ x*))|} ∈ *sets lebesgue*
       using ∗ **by** (*auto simp*: *borel_measurable_iff_halfspace_ge space_restrict_space*)
      then have *1*: {*x* ∈ *S*. *real n* ∗ *e* ≤ |*det* (*matrix* (*f′ x*))|} ∈ *lmeasurable*
        using *S* **by** (*simp add*: *fmeasurableI2*)
      have {*x* ∈ *S*. |*det* (*matrix* (*f′ x*))| < (*1* + *real n*) ∗ *e*} ∈ *sets lebesgue*
      using ∗ **by** (*auto simp*: *borel_measurable_iff_halfspace_less space_restrict_space*)
     then have *2*: {*x* ∈ *S*. |*det* (*matrix* (*f′ x*))| < (*1* + *real n*) ∗ *e*} ∈ *lmeasurable*
        using *S* **by** (*simp add*: *fmeasurableI2*)
      show *?thesis*
       using *fmeasurable.Int* [*OF 1 2*] **by** (*simp add*: *T_def Int_def cong*: *conj_cong*)
    qed
    have *aint_T*: ⋀*k*. (λ*x*. |*det* (*matrix* (*f′ x*))|) *absolutely_integrable_on T k*
      using *set_integrable_subset* [*OF aint_S*] *meas_t T_def* **by** *blast*
    have *Seq*: *S* = (⋃*n*. *T n*)
      apply (*auto simp*: *T_def*)
      apply (*rule_tac x*=*nat*(*floor*(*abs*(*det*(*matrix*(*f′ x*))) / *e*)) **in** *exI*)
      using *that* **apply** *auto*
      using *of_int_floor_le pos_le_divide_eq* **apply** *blast*
      by (*metis add.commute pos_divide_less_eq real_of_int_floor_add_one_gt*)
    have *meas_ft*: *f ‘ T n* ∈ *lmeasurable* **for** *n*
    proof (*rule measurable_bounded_differentiable_image*)
      show *T n* ∈ *lmeasurable*
        by (*simp add*: *meas_t*)
    next
      fix *x* :: (*real*,′*n*) *vec*
      assume *x* ∈ *T n*
      show (*f has_derivative f′ x*) (*at x within T n*)
      by (*metis* (*no_types*, *lifting*) ‹*x* ∈ *T n*› *deriv has_derivative_subset mem_Collect_eq subsetI T_def*)
      show |*det* (*matrix* (*f′ x*))| ≤ (*Suc n*) ∗ *e*
        using ‹*x* ∈ *T n*› *T_def* **by** *auto*

next
  show (*λx.* |*det* (*matrix* (*f′ x*))|) *integrable_on T n*
    using *aint_T absolutely_integrable_on_def* **by** *blast*
qed
have *disT*: *disjoint* (*range T*)
  unfolding *disjoint_def*
proof *clarsimp*
  show *T m* ∩ *T n* = {} **if** *T m* ≠ *T n* **for** *m n*
    using *that*
  proof (*induction m n rule*: *linorder_less_wlog*)
    case (*less m n*)
    with ‹*e* > *0*› **show** *?case*
      unfolding *T_def*
      proof (*clarsimp simp add*: *Collect_conj_eq* [*symmetric*])
        fix *x*
        assume *e* > *0* *m* < *n* *n* ∗ *e* ≤ |*det* (*matrix* (*f′ x*))| |*det* (*matrix* (*f′ x*))| < (*1* + *real m*) ∗ *e*
        then have *n* < *1* + *real m*
          by (*metis* (*no_types*, *hide_lams*) *less_le_trans mult.commute not_le mult_le_cancel_iff2*)
        then show *False*
          using *less.hyps* **by** *linarith*
      qed
  qed *auto*
qed
have *injT*: *inj_on T* ({*n*. *T n* ≠ {}})
  unfolding *inj_on_def*
proof *clarsimp*
  show *m* = *n* **if** *T m* = *T n T n* ≠ {} **for** *m n*
    using *that*
  proof (*induction m n rule*: *linorder_less_wlog*)
    case (*less m n*)
    have *False* **if** *T n* ⊆ *T m x* ∈ *T n* **for** *x*
      using ‹*e* > *0*› ‹*m* < *n*› *that*
     apply (*auto simp*: *T_def* *mult.commute intro*: *less_le_trans dest*!: *subsetD*)
    by (*metis add.commute less_le_trans nat_less_real_le not_le mult_le_cancel_iff2*)
    then show *?case*
      using *less.prems* **by** *blast*
  qed *auto*
qed
have *sum_eq_Tim*: ($\sum$ *k*≤*n*. *f* (*T k*)) = *sum f* (*T* ‘ {..*n*}) **if** *f* {} = *0* **for** *f* :: _ ⇒ *real* **and** *n*
proof (*subst sum.reindex_nontrivial*)
  fix *i j* **assume** *i* ∈ {..*n*} *j* ∈ {..*n*} *i* ≠ *j T i* = *T j*
  with *that* *injT* [*unfolded inj_on_def*] **show** *f* (*T i*) = *0*
    by *simp metis*
qed (*use atMost_atLeast0* **in** *auto*)
let *?B* = *m* + *e* ∗ *?μ S*
have ($\sum$ *k*≤*n*. *?μ* (*f* ‘ *T k*)) ≤ *?B* **for** *n*

```
  proof –
    have (∑k≤n. ?μ (f ` T k)) ≤ (∑k≤n. ((k+1) ∗ e) ∗ ?μ(T k))
    proof (rule sum_mono [OF measure_bounded_differentiable_image])
      show (f has_derivative f' x) (at x within T k) if x ∈ T k for k x
        using that unfolding T_def by (blast intro: deriv has_derivative_subset)
      show (λx. |det (matrix (f' x))|) integrable_on T k for k
        using absolutely_integrable_on_def aint_T by blast
      show |det (matrix (f' x))| ≤ real (k + 1) ∗ e if x ∈ T k for k x
        using T_def that by auto
    qed (use meas_t in auto)
    also have . . . ≤ (∑k≤n. (k ∗ e) ∗ ?μ(T k)) + (∑k≤n. e ∗ ?μ(T k))
      by (simp add: algebra_simps sum.distrib)
    also have . . . ≤ ?B
    proof (rule add_mono)
     have (∑k≤n. real k ∗ e ∗ ?μ (T k)) = (∑k≤n. integral (T k) (λx. k ∗ e))
        by (simp add: lmeasure_integral [OF meas_t]
                flip: integral_mult_right integral_mult_left)
     also have . . . ≤ (∑k≤n. integral (T k) (λx.  (abs (det (matrix (f' x))))))
      proof (rule sum_mono)
        fix k
        assume k ∈ {..n}
         show integral (T k) (λx. k ∗ e) ≤ integral (T k) (λx. |det (matrix (f'
x))|)
        proof (rule integral_le [OF integrable_on_const [OF meas_t]])
          show (λx. |det (matrix (f' x))|) integrable_on T k
            using absolutely_integrable_on_def aint_T by blast
        next
          fix x assume x ∈ T k
          show k ∗ e ≤ |det (matrix (f' x))|
            using ‹x ∈ T k› T_def by blast
        qed
      qed
     also have . . . = sum (λT. integral T (λx. |det (matrix (f' x))|)) (T ` {..n})
        by (auto intro: sum_eq_Tim)
      also have . . . = integral (⋃k≤n. T k) (λx. |det (matrix (f' x))|)
      proof (rule integral_unique [OF has_integral_Union, symmetric])
        fix S  assume S ∈ T ` {..n}
         then show ((λx. |det (matrix (f' x))|) has_integral integral S (λx. |det
(matrix (f' x))|)) S
        using absolutely_integrable_on_def aint_T by blast
      next
        show pairwise (λS S'. negligible (S ∩ S')) (T ` {..n})
            using disT unfolding disjnt_iff by (auto simp: pairwise_def intro!:
empty_imp_negligible)
      qed auto
      also have . . . ≤ m
        unfolding m_def
      proof (rule integral_subset_le)
        have (λx. |det (matrix (f' x))|) absolutely_integrable_on (⋃k≤n. T k)
```

apply (*rule set_integrable_subset* [*OF aint_S*])
apply (*intro measurable meas_t fmeasurableD*)
apply (*force simp*: *Seq*)
done
then show (λ*x*. |*det* (*matrix* (*f′ x*))|) *integrable_on* (⋃*k*≤*n*. *T k*)
using *absolutely_integrable_on_def* **by** *blast*
qed (*use Seq int* **in** *auto*)
finally show (∑*k*≤*n*. *real k* ∗ *e* ∗ *?μ* (*T k*)) ≤ *m* **.**
next
have (∑*k*≤*n*. *?μ* (*T k*)) = *sum ?μ* (*T* ‘ {..*n*})
by (*auto intro*: *sum_eq_Tim*)
also have ... = *?μ* (⋃*k*≤*n*. *T k*)
using *S disT* **by** (*auto simp*: *pairwise_def meas_t intro*: *measure_Union′* [*symmetric*])
also have ... ≤ *?μ S*
using *S* **by** (*auto simp*: *Seq intro*: *meas_t fmeasurableD measure_mono_fmeasurable*)
finally have (∑*k*≤*n*. *?μ* (*T k*)) ≤ *?μ S* **.**
then show (∑*k*≤*n*. *e* ∗ *?μ* (*T k*)) ≤ *e* ∗ *?μ S*
by (*metis less_eq_real_def ordered_comm_semiring_class.comm_mult_left_mono sum_distrib_left that*)
qed
finally show (∑*k*≤*n*. *?μ* (*f* ‘ *T k*)) ≤ *?B* **.**
qed
moreover have *measure lebesgue* (⋃*k*≤*n*. *f* ‘ *T k*) ≤ (∑*k*≤*n*. *?μ* (*f* ‘ *T k*)) **for** *n*
by (*simp add*: *fmeasurableD meas_ft measure_UNION_le*)
ultimately have *B_ge_m*: *?μ* (⋃*k*≤*n*. (*f* ‘ *T k*)) ≤ *?B* **for** *n*
by (*meson order_trans*)
have (⋃*n*. *f* ‘ *T n*) ∈ *lmeasurable*
by (*rule fmeasurable_countable_Union* [*OF meas_ft B_ge_m*])
moreover have *?μ* (⋃*n*. *f* ‘ *T n*) ≤ *m* + *e* ∗ *?μ S*
by (*rule measure_countable_Union_le* [*OF meas_ft B_ge_m*])
ultimately show *f* ‘ *S* ∈ *lmeasurable ?μ* (*f* ‘ *S*) ≤ *m* + *e* ∗ *?μ S*
by (*auto simp*: *Seq image_Union*)
qed
show *?thesis*
proof
show *f* ‘ *S* ∈ *lmeasurable*
using ∗ *linordered_field_no_ub* **by** *blast*
let *?x* = *m* − *?μ* (*f* ‘ *S*)
have *False* **if** *?μ* (*f* ‘ *S*) > *integral S* (λ*x*. |*det* (*matrix* (*f′ x*))|)
proof −
have *ml*: *m* < *?μ* (*f* ‘ *S*)
using *m_def that* **by** *blast*
then have *?μ S* ≠ *0*
using ∗(*2*) *bgauge_existence_lemma* **by** *fastforce*
with *ml* **have** *0*: *0* < − (*m* − *?μ* (*f* ‘ *S*))/*2* / *?μ S*
using *that zero_less_measure_iff* **by** *force*
then show *?thesis*

using $*$ [*OF 0*] *that* **by** (*auto simp*: *field_split_simps m_def split*: *if_split_asm*)
qed
then show *?μ* (*f* ‘ *S*) ≤ *integral S* (λ*x*. |*det* (*matrix* (*f′ x*))|)
by *fastforce*
qed
qed

theorem
fixes *f* :: *real^′n*::{*finite*,*wellorder*} ⇒ *real^′n*::_
assumes *S*: *S* ∈ *sets lebesgue*
and *deriv*: ⋀*x*. *x* ∈ *S* ⟹ (*f has_derivative f′ x*) (*at x within S*)
and *int*: (λ*x*. |*det* (*matrix* (*f′ x*))|) *integrable_on S*
shows *measurable_differentiable_image*: *f* ‘ *S* ∈ *lmeasurable*
and *measure_differentiable_image*:
measure lebesgue (*f* ‘ *S*) ≤ *integral S* (λ*x*. |*det* (*matrix* (*f′ x*))|) (**is** *?M*)
proof −
let *?I* = λ*n*::*nat*. *cbox* (*vec* (−*n*)) (*vec n*) ∩ *S*
let *?μ* = *measure lebesgue*
have *x* ∈ *cbox* (*vec* (− *real* (*nat* ⌈*norm x*⌉))) (*vec* (*real* (*nat* ⌈*norm x*⌉))) **for** *x* :: *real^′n*::_
apply (*auto simp*: *mem_box_cart*)
apply (*metis abs_le_iff component_le_norm_cart minus_le_iff of_nat_ceiling order.trans*)
by (*meson abs_le_D1 norm_bound_component_le_cart real_nat_ceiling_ge*)
then have *Seq*: *S* = (⋃*n*. *?I n*)
by *auto*
have *fIn*: *f* ‘ *?I n* ∈ *lmeasurable*
and *mfIn*: *?μ* (*f* ‘ *?I n*) ≤ *integral S* (λ*x*. |*det* (*matrix* (*f′ x*))|) (**is** *?MN*)
for *n*
proof −
have *In*: *?I n* ∈ *lmeasurable*
by (*simp add*: *S bounded_Int bounded_set_imp_lmeasurable sets.Int*)
moreover have ⋀*x*. *x* ∈ *?I n* ⟹ (*f has_derivative f′ x*) (*at x within ?I n*)
by (*meson Int_iff deriv has_derivative_subset subsetI*)
moreover have *int_In*: (λ*x*. |*det* (*matrix* (*f′ x*))|) *integrable_on ?I n*
proof −
have (λ*x*. |*det* (*matrix* (*f′ x*))|) *absolutely_integrable_on S*
using *int absolutely_integrable_integrable_bound* **by** *force*
then have (λ*x*. |*det* (*matrix* (*f′ x*))|) *absolutely_integrable_on ?I n*
by (*metis* (*no_types*) *Int_lower1 In fmeasurableD inf_commute set_integrable_subset*)
then show *?thesis*
using *absolutely_integrable_on_def* **by** *blast*
qed
ultimately have *f* ‘ *?I n* ∈ *lmeasurable ?μ* (*f* ‘ *?I n*) ≤ *integral* (*?I n*) (λ*x*. |*det* (*matrix* (*f′ x*))|)
using *m_diff_image_weak* **by** *metis+*
moreover have *integral* (*?I n*) (λ*x*. |*det* (*matrix* (*f′ x*))|) ≤ *integral S* (λ*x*. |*det* (*matrix* (*f′ x*))|)

by (*simp add*: *int_In int integral_subset_le*)
ultimately show *f* ‘ *?I n* ∈ *lmeasurable ?MN*
by *auto*
qed
have *?I k* ⊆ *?I n* **if** *k* ≤ *n* **for** *k n*
by (*rule Int_mono*) (*use that* **in** ‹*auto simp*: *subset_interval_imp_cart*›)
then have (⋃*k*≤*n*. *f* ‘ *?I k*) = *f* ‘ *?I n* **for** *n*
by (*fastforce simp add*:)
with *mfIn* **have** *?μ* (⋃*k*≤*n*. *f* ‘ *?I k*) ≤ *integral S* (λ*x*. |*det* (*matrix* (*f′ x*))|) **for** *n*
by *simp*
then have (⋃*n*. *f* ‘ *?I n*) ∈ *lmeasurable ?μ* (⋃*n*. *f* ‘ *?I n*) ≤ *integral S* (λ*x*. |*det* (*matrix* (*f′ x*))|)
by (*rule fmeasurable_countable_Union* [*OF fIn*] *measure_countable_Union_le* [*OF fIn*])+
then show *f* ‘ *S* ∈ *lmeasurable ?M*
by (*metis Seq image_UN*)+
qed

lemma *borel_measurable_simple_function_limit_increasing*:
fixes *f* :: *′a*::*euclidean_space* ⇒ *real*
shows (*f* ∈ *borel_measurable lebesgue* ∧ (∀*x*. *0* ≤ *f x*)) ⟷
(∃*g*. (∀*n x*. *0* ≤ *g n x* ∧ *g n x* ≤ *f x*) ∧ (∀*n x*. *g n x* ≤ (*g*(*Suc n*) *x*)) ∧
(∀*n*. *g n* ∈ *borel_measurable lebesgue*) ∧ (∀*n*. *finite*(*range* (*g n*))) ∧
(∀*x*. (λ*n*. *g n x*) ⟶ *f x*))
(**is** *?lhs* = *?rhs*)
proof
assume *f*: *?lhs*
have *leb_f*: {*x*. *a* ≤ *f x* ∧ *f x* < *b*} ∈ *sets lebesgue* **for** *a b*
proof −
have {*x*. *a* ≤ *f x* ∧ *f x* < *b*} = {*x*. *f x* < *b*} − {*x*. *f x* < *a*}
by *auto*
also have ... ∈ *sets lebesgue*
using *borel_measurable_vimage_halfspace_component_lt* [*of f UNIV*] *f* **by** *auto*
finally show *?thesis* **.**
qed
have *g n x* ≤ *f x*
if *inc_g*: ⋀*n x*. *0* ≤ *g n x* ∧ *g n x* ≤ *g* (*Suc n*) *x*
and *meas_g*: ⋀*n*. *g n* ∈ *borel_measurable lebesgue*
and *fin*: ⋀*n*. *finite*(*range* (*g n*)) **and** *lim*: ⋀*x*. (λ*n*. *g n x*) ⟶ *f x* **for** *g n x*
proof −
have ∃*r*>*0*. ∀*N*. ∃*n*≥*N*. *dist* (*g n x*) (*f x*) ≥ *r* **if** *g n x* > *f x*
proof −
have *g*: *g n x* ≤ *g* (*N* + *n*) *x* **for** *N*
by (*rule transitive_stepwise_le*) (*use inc_g* **in** *auto*)
have ∃*na*≥*N*. *g n x* − *f x* ≤ *dist* (*g na x*) (*f x*) **for** *N*
apply (*rule_tac x*=*N*+*n* **in** *exI*)

using *g* [*of N*] **by** (*auto simp*: *dist_norm*)
with *that* **show** *?thesis*
using *diff_gt_0_iff_gt* **by** *blast*
qed
with *lim* **show** *?thesis*
apply (*auto simp*: *lim_sequentially*)
by (*meson less_le_not_le not_le_imp_less*)
qed
moreover
let $?\Omega = \lambda n\ k.\ indicator\ \{y.\ k/2\hat{}n \le f\ y \wedge f\ y < (k+1)/2\hat{}n\}$
let $?g = \lambda n\ x.\ (\sum k::real \mid k \in \mathbb{Z} \wedge |k| \le 2\ \hat{}\ (2*n).\ k/2\hat{}n * ?\Omega\ n\ k\ x)$
have $\exists g.\ (\forall n\ x.\ 0 \le g\ n\ x \wedge g\ n\ x \le (g(Suc\ n)\ x)) \wedge$
$(\forall n.\ g\ n \in borel\_measurable\ lebesgue) \wedge (\forall n.\ finite(range\ (g\ n))) \wedge (\forall x.$
$(\lambda n.\ g\ n\ x) \longrightarrow f\ x)$
proof (*intro exI allI conjI*)
show $0 \le ?g\ n\ x$ **for** *n x*
proof (*clarify intro*!: *ordered_comm_monoid_add_class.sum_nonneg*)
fix *k*::*real*
assume $k \in \mathbb{Z}$ **and** *k*: $|k| \le 2\ \hat{}\ (2*n)$
show $0 \le k/2\hat{}n * ?\Omega\ n\ k\ x$
using *f* ‹$k \in \mathbb{Z}$› **apply** (*auto simp*: *indicator_def field_split_simps Ints_def*)
apply (*drule spec* [**where** *x*=*x*])
using *zero_le_power* [*of 2*::*real n*] *mult_nonneg_nonneg* [*of f x 2^n*]
by *linarith*
qed
show $?g\ n\ x \le ?g\ (Suc\ n)\ x$ **for** *n x*
proof −
have $?g\ n\ x =$
$(\sum k \mid k \in \mathbb{Z} \wedge |k| \le 2\ \hat{}\ (2*n).$
$k/2\hat{}n * (indicator\ \{y.\ k/2\hat{}n \le f\ y \wedge f\ y < (k+1/2)/2\hat{}n\}\ x +$
$indicator\ \{y.\ (k+1/2)/2\hat{}n \le f\ y \wedge f\ y < (k+1)/2\hat{}n\}\ x))$
by (*rule sum.cong* [*OF refl*]) (*simp add*: *indicator_def field_split_simps*)
also have $\ldots = (\sum k \mid k \in \mathbb{Z} \wedge |k| \le 2\ \hat{}\ (2*n).\ k/2\hat{}n * indicator\ \{y.$
$k/2\hat{}n \le f\ y \wedge f\ y < (k+1/2)/2\hat{}n\}\ x) +$
$(\sum k \mid k \in \mathbb{Z} \wedge |k| \le 2\ \hat{}\ (2*n).\ k/2\hat{}n * indicator\ \{y.$
$(k+1/2)/2\hat{}n \le f\ y \wedge f\ y < (k+1)/2\hat{}n\}\ x)$
by (*simp add*: *comm_monoid_add_class.sum.distrib algebra_simps*)
also have $\ldots = (\sum k \mid k \in \mathbb{Z} \wedge |k| \le 2\ \hat{}\ (2*n).\ (2 * k)/2\ \hat{}\ Suc\ n *$
$indicator\ \{y.\ (2 * k)/2\ \hat{}\ Suc\ n \le f\ y \wedge f\ y < (2 * k+1)/2\ \hat{}\ Suc\ n\}\ x) +$
$(\sum k \mid k \in \mathbb{Z} \wedge |k| \le 2\ \hat{}\ (2*n).\ (2 * k)/2\ \hat{}\ Suc\ n * indicator$
$\{y.\ (2 * k+1)/2\ \hat{}\ Suc\ n \le f\ y \wedge f\ y < ((2 * k+1) + 1)/2\ \hat{}\ Suc\ n\}\ x)$
by (*force simp*: *field_simps indicator_def intro*: *sum.cong*)
also have $\ldots \le (\sum k \mid k \in \mathbb{Z} \wedge |k| \le 2\ \hat{}\ (2 * Suc\ n).\ k/2\ \hat{}\ Suc\ n *$
$(indicator\ \{y.\ k/2\ \hat{}\ Suc\ n \le f\ y \wedge f\ y < (k+1)/2\ \hat{}\ Suc\ n\}\ x))$
(**is** $?a + \_ \le ?b$)
proof −
have $*$: $[\![sum\ f\ I \le sum\ h\ I;\ a + sum\ h\ I \le b]\!] \Longrightarrow a + sum\ f\ I \le b$ **for** *I*
a b f **and** *h* :: *real*⇒*real*
by *linarith*

let *?h* = λ*k*. (*2*∗*k*+*1*)/*2* ^ *Suc n* ∗
(*indicator* {*y*. (*2* ∗ *k*+*1*)/*2* ^ *Suc n* ≤ *f y* ∧ *f y* < ((*2*∗*k*+*1*) + *1*)/*2* ^ *Suc n*} *x*)
show *?thesis*
proof (*rule* ∗)
show (∑*k* | *k* ∈ ℤ ∧ |*k*| ≤ *2* ^ (*2*∗*n*).
2 ∗ *k*/*2* ^ *Suc n* ∗ *indicator* {*y*. (*2* ∗ *k*+*1*)/*2* ^ *Suc n* ≤ *f y* ∧ *f y* < (*2* ∗ *k*+*1* + *1*)/*2* ^ *Suc n*} *x*)
≤ *sum ?h* {*k* ∈ ℤ. |*k*| ≤ *2* ^ (*2*∗*n*)}
by (*rule sum_mono*) (*simp add*: *indicator_def field_split_simps*)
next
have α: *?a* = (∑*k* ∈ (∗)*2* ‘ {*k* ∈ ℤ. |*k*| ≤ *2* ^ (*2*∗*n*)}.
k/*2* ^ *Suc n* ∗ *indicator* {*y*. *k*/*2* ^ *Suc n* ≤ *f y* ∧ *f y* < (*k*+*1*)/*2* ^ *Suc n*} *x*)
by (*auto simp*: *inj_on_def field_simps comm_monoid_add_class.sum.reindex*)
have β: *sum ?h* {*k* ∈ ℤ. |*k*| ≤ *2* ^ (*2*∗*n*)}
= (∑*k* ∈ (λ*x*. *2*∗*x* + *1*) ‘ {*k* ∈ ℤ. |*k*| ≤ *2* ^ (*2*∗*n*)}.
k/*2* ^ *Suc n* ∗ *indicator* {*y*. *k*/*2* ^ *Suc n* ≤ *f y* ∧ *f y* < (*k*+*1*)/*2* ^ *Suc n*} *x*)
by (*auto simp*: *inj_on_def field_simps comm_monoid_add_class.sum.reindex*)
have *0*: (∗) *2* ‘ {*k* ∈ ℤ. *P k*} ∩ (λ*x*. *2* ∗ *x* + *1*) ‘ {*k* ∈ ℤ. *P k*} = {} **for** *P* :: *real* ⇒ *bool*
proof −
have *2* ∗ *i* ≠ *2* ∗ *j* + *1* **for** *i j* :: *int* **by** *arith*
thus *?thesis*
unfolding *Ints_def* **by** *auto* (*use of_int_eq_iff* **in** *fastforce*)
qed
have *?a* + *sum ?h* {*k* ∈ ℤ. |*k*| ≤ *2* ^ (*2*∗*n*)}
= (∑*k* ∈ (∗)*2* ‘ {*k* ∈ ℤ. |*k*| ≤ *2* ^ (*2*∗*n*)} ∪ (λ*x*. *2*∗*x* + *1*) ‘ {*k* ∈ ℤ. |*k*| ≤ *2* ^ (*2*∗*n*)}.
k/*2* ^ *Suc n* ∗ *indicator* {*y*. *k*/*2* ^ *Suc n* ≤ *f y* ∧ *f y* < (*k*+*1*)/*2* ^ *Suc n*} *x*)
unfolding α β
using *finite_abs_int_segment* [*of 2* ^ (*2*∗*n*)]
by (*subst sum_Un*) (*auto simp*: *0*)
also have ... ≤ *?b*
proof (*rule sum_mono2*)
show *finite* {*k*::*real*. *k* ∈ ℤ ∧ |*k*| ≤ *2* ^ (*2* ∗ *Suc n*)}
by (*rule finite_abs_int_segment*)
show (∗) *2* ‘ {*k*::*real*. *k* ∈ ℤ ∧ |*k*| ≤ *2*^(*2*∗*n*)} ∪ (λ*x*. *2*∗*x* + *1*) ‘ {*k* ∈ ℤ. |*k*| ≤ *2*^(*2*∗*n*)} ⊆ {*k* ∈ ℤ. |*k*| ≤ *2* ^ (*2* ∗ *Suc n*)}
apply *auto*
using *one_le_power* [*of 2*::*real 2*∗*n*] **by** *linarith*
have ∗: ⟦*x* ∈ (*S* ∪ *T*) − *U*; ⋀*x*. *x* ∈ *S* ⟹ *x* ∈ *U*; ⋀*x*. *x* ∈ *T* ⟹ *x* ∈ *U*⟧ ⟹ *P x* **for** *S T U P*
by *blast*
have *0* ≤ *b* **if** *b* ∈ ℤ *f x* ∗ (*2* ∗ *2*^*n*) < *b* + *1* **for** *b*
proof −
have *0* ≤ *f x* ∗ (*2* ∗ *2*^*n*)

```
            by (simp add: f)
          also have ... < b+1
            by (simp add: that)
          finally show 0 ≤ b
            using ‹b ∈ ℤ› by (auto simp: elim!: Ints_cases)
        qed
        then show 0 ≤ b/2 ^ Suc n ∗ indicator {y. b/2 ^ Suc n ≤ f y ∧ f y <
(b + 1)/2 ^ Suc n} x
              if b ∈ {k ∈ ℤ. |k| ≤ 2 ^ (2 ∗ Suc n)} −
                    ((∗) 2 ‘ {k ∈ ℤ. |k| ≤ 2 ^ (2∗n)} ∪ (λx. 2∗x + 1) ‘ {k ∈ ℤ.
|k| ≤ 2 ^ (2∗n)}) for b
          using that by (simp add: indicator_def divide_simps)
      qed
      finally show ?a + sum ?h {k ∈ ℤ. |k| ≤ 2 ^ (2∗n)} ≤ ?b .
    qed
  qed
  finally show ?thesis .
qed
show ?g n ∈ borel_measurable lebesgue for n
 apply (intro borel_measurable_indicator borel_measurable_times borel_measurable_sum)
  using leb_f sets_restrict_UNIV by auto
show finite (range (?g n)) for n
proof −
  have (∑ k | k ∈ ℤ ∧ |k| ≤ 2 ^ (2∗n). k/2^n ∗ ?Ω n k x)
          ∈ (λk. k/2^n) ‘ {k ∈ ℤ. |k| ≤ 2 ^ (2∗n)} for x
 proof (cases ∃ k. k ∈ ℤ ∧ |k| ≤ 2 ^ (2∗n) ∧ k/2^n ≤ f x ∧ f x < (k+1)/2^n)
    case True
    then show ?thesis
      by (blast intro: indicator_sum_eq)
  next
    case False
    then have (∑ k | k ∈ ℤ ∧ |k| ≤ 2 ^ (2∗n). k/2^n ∗ ?Ω n k x) = 0
      by auto
    then show ?thesis by force
  qed
  then have range (?g n) ⊆ ((λk. (k/2^n)) ‘ {k. k ∈ ℤ ∧ |k| ≤ 2 ^ (2∗n)})
    by auto
  moreover have finite ((λk::real. (k/2^n)) ‘ {k ∈ ℤ. |k| ≤ 2 ^ (2∗n)})
    by (intro finite_imageI finite_abs_int_segment)
  ultimately show ?thesis
    by (rule finite_subset)
qed
show (λn. ?g n x) ⟶ f x for x
proof (clarsimp simp add: lim_sequentially)
  fix e::real
  assume e > 0
  obtain N1 where N1: 2 ^ N1 > abs(f x)
    using real_arch_pow by fastforce
  obtain N2 where N2: (1/2) ^ N2 < e
```

using *real_arch_pow_inv* ‹$e > 0$› **by** *fastforce*
have *dist* ($\sum k \mid k \in \mathbb{Z} \wedge |k| \leq 2$ ^ $(2*n)$. $k/2$^$n * ?\Omega\ n\ k\ x$) ($f\ x$) $< e$ **if** $N1 + N2 \leq n$ **for** n
proof −
let $?m$ = *real_of_int* $\lfloor 2$^$n * f\ x \rfloor$
have $|?m| \leq 2$^$n * 2$^$N1$
using *N1* **apply** (*simp add*: *f*)
by (*meson floor_mono le_floor_iff less_le_not_le mult_le_cancel_left_pos zero_less_numeral zero_less_power*)
also have ... ≤ 2 ^ $(2*n)$
by (*metis that add_leD1 add_le_cancel_left mult.commute mult_2_right one_less_numeral_iff*
power_add power_increasing_iff semiring_norm(*76*))
finally have *m_le*: $|?m| \leq 2$ ^ $(2*n)$ **.**
have $?m/2$^$n \leq f\ x\ f\ x < (?m + 1)/2$^$n$
by (*auto simp*: *mult.commute pos_divide_le_eq mult_imp_less_div_pos*)
then have *eq*: *dist* ($\sum k \mid k \in \mathbb{Z} \wedge |k| \leq 2$ ^ $(2*n)$. $k/2$^$n * ?\Omega\ n\ k\ x$) (f x)
= *dist* ($?m/2$^n) ($f\ x$)
by (*subst indicator_sum_eq* [*of ?m*]) (*auto simp*: *m_le*)
have $|2$^$n| * |?m/2$^$n - f\ x| = |2$^$n * (?m/2$^$n - f\ x)|$
by (*simp add*: *abs_mult*)
also have ... < 2 ^ $N2 * e$
using *N2* **by** (*simp add*: *divide_simps mult.commute*) *linarith*
also have ... $\leq |2$^$n| * e$
using *that* ‹$e > 0$› **by** *auto*
finally have *dist* ($?m/2$^n) ($f\ x$) $< e$
by (*simp add*: *dist_norm*)
then show *?thesis*
using *eq* **by** *linarith*
qed
then show $\exists no. \forall n \geq no.$ *dist* ($\sum k \mid k \in \mathbb{Z} \wedge |k| \leq 2$ ^ $(2*n)$. $k * ?\Omega\ n\ k\ x/2$^n) ($f\ x$) $< e$
by *force*
qed
qed
ultimately show *?rhs*
by *metis*
next
assume *RHS*: *?rhs*
with *borel_measurable_simple_function_limit* [*of f UNIV*, *unfolded lebesgue_on_UNIV_eq*]
show *?lhs*
by (*blast intro*: *order_trans*)
qed

6.46.2 Borel measurable Jacobian determinant

lemma *lemma_partial_derivatives0*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*

```
  assumes linear f and lim0: ((λx. f x /R norm x) ⟶ 0) (at 0 within S)
    and lb: ⋀v. v ≠ 0 ⟹ (∃k>0. ∀e>0. ∃x. x ∈ S − {0} ∧ norm x < e ∧ k ∗
norm x ≤ |v · x|)
  shows f x = 0
proof −
  interpret linear f by fact
  have dim {x. f x = 0} ≤ DIM('a)
    by (rule dim_subset_UNIV)
  moreover have False if less: dim {x. f x = 0} < DIM('a)
  proof −
    obtain d where d ≠ 0 and d: ⋀y. f y = 0 ⟹ d · y = 0
      using orthogonal_to_subspace_exists [OF less] orthogonal_def
      by (metis (mono_tags, lifting) mem_Collect_eq span_base)
    then obtain k where k > 0
      and k: ⋀e. e > 0 ⟹ ∃y. y ∈ S − {0} ∧ norm y < e ∧ k ∗ norm y ≤ |d ·
y|
      using lb by blast
    have ∃h. ∀n. ((h n ∈ S ∧ h n ≠ 0 ∧ k ∗ norm (h n) ≤ |d · h n|) ∧ norm (h
n) < 1 / real (Suc n)) ∧
               norm (h (Suc n)) < norm (h n)
    proof (rule dependent_nat_choice)
      show ∃y. (y ∈ S ∧ y ≠ 0 ∧ k ∗ norm y ≤ |d · y|) ∧ norm y < 1 / real
(Suc 0)
        by simp (metis DiffE insertCI k not_less not_one_le_zero)
    qed (use k [of min (norm x) (1/(Suc n + 1)) for x n] in auto)
    then obtain α where α: ⋀n. α n ∈ S − {0} and kd: ⋀n. k ∗ norm(α n) ≤
|d · α n|
         and norm_lt: ⋀n. norm(α n) < 1/(Suc n)
      by force
    let ?β = λn. α n /R norm (α n)
    have com: ⋀g. (∀n. g n ∈ sphere (0::'a) 1)
              ⟹ ∃l ∈ sphere 0 1. ∃ϱ::nat⇒nat. strict_mono ϱ ∧ (g ∘ ϱ) ⟶ l
      using compact_sphere compact_def by metis
    moreover have ∀n. ?β n ∈ sphere 0 1
      using α by auto
    ultimately obtain l::'a and ϱ::nat⇒nat
       where l: l ∈ sphere 0 1 and strict_mono ϱ and to_l: (?β ∘ ϱ) ⟶ l
      by meson
    moreover have continuous (at l) (λx. (|d · x| − k))
      by (intro continuous_intros)
    ultimately have lim_dl: ((λx. (|d · x| − k)) ∘ (?β ∘ ϱ)) ⟶ (|d · l| − k)
      by (meson continuous_imp_tendsto)
    have ∀F i in sequentially. 0 ≤ ((λx. |d · x| − k) ∘ ((λn. α n /R norm (α n))
∘ ϱ)) i
      using α kd by (auto simp: field_split_simps)
    then have k ≤ |d · l|
      using tendsto_lowerbound [OF lim_dl, of 0] by auto
    moreover have d · l = 0
    proof (rule d)
```

show *f l* = *0*
proof (*rule LIMSEQ_unique* [*of f* ∘ *?β* ∘ *ϱ*])
have *isCont f l*
using ‹*linear f*› *linear_continuous_at linear_conv_bounded_linear* **by** *blast*
then show (*f* ∘ (*λn. α n* $/_R$ *norm* (*α n*)) ∘ *ϱ*) ⟶ *f l*
unfolding *comp_assoc*
using *to_l continuous_imp_tendsto* **by** *blast*
have *α* ⟶ *0*
using *norm_lt LIMSEQ_norm_0* **by** *metis*
with ‹*strict_mono ϱ*› **have** (*α* ∘ *ϱ*) ⟶ *0*
by (*metis LIMSEQ_subseq_LIMSEQ*)
with *lim0 α* **have** ((*λx. f x* $/_R$ *norm x*) ∘ (*α* ∘ *ϱ*)) ⟶ *0*
by (*force simp*: *tendsto_at_iff_sequentially*)
then show (*f* ∘ (*λn. α n* $/_R$ *norm* (*α n*)) ∘ *ϱ*) ⟶ *0*
by (*simp add*: *o_def scale*)
qed
qed
ultimately show *False*
using ‹*k* > *0*› **by** *auto*
qed
ultimately have *dim*: *dim* {*x. f x* = *0*} = *DIM*(*'a*)
by *force*
then show *?thesis*
using *dim_eq_full*
by (*metis* (*mono_tags*, *lifting*) *eq_0_on_span eucl.span_Basis linear_axioms linear_eq_stdbasis*
mem_Collect_eq module_hom_zero span_base span_raw_def)
qed

lemma *lemma_partial_derivatives*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *linear f* **and** *lim*: ((*λx. f* (*x* − *a*) $/_R$ *norm* (*x* − *a*)) ⟶ *0*) (*at a within S*)
and *lb*: ⋀*v. v* ≠ *0* ⟹ (∃ *k*>*0*. ∀ *e*>*0*. ∃ *x* ∈ *S* − {*a*}. *norm*(*a* − *x*) < *e* ∧ *k* ∗ *norm*(*a* − *x*) ≤ |*v* • (*x* − *a*)|)
shows *f x* = *0*
proof −
have ((*λx. f x* $/_R$ *norm x*) ⟶ *0*) (*at 0 within* (*λx. x*−*a*) ‘ *S*)
using *lim* **by** (*simp add*: *Lim_within dist_norm*)
then show *?thesis*
proof (*rule lemma_partial_derivatives0* [*OF* ‹*linear f*›])
fix *v* :: *'a*
assume *v*: *v* ≠ *0*
show ∃ *k*>*0*. ∀ *e*>*0*. ∃ *x. x* ∈ (*λx. x* − *a*) ‘ *S* − {*0*} ∧ *norm x* < *e* ∧ *k* ∗ *norm x* ≤ |*v* • *x*|
using *lb* [*OF v*] **by** (*force simp*: *norm_minus_commute*)
qed
qed

proposition *borel_measurable_partial_derivatives*:
  fixes f :: *real^'m::{finite,wellorder}* ⇒ *real^'n*
  assumes S: $S \in$ *sets lebesgue*
    and f: $\bigwedge x.\ x \in S \Longrightarrow$ (f *has_derivative* f' x) (*at* x *within* S)
  shows ($\lambda x.$ (*matrix*(f' x)\$$m$\$$n$)) ∈ *borel_measurable* (*lebesgue_on* S)
proof −
  have *contf*: *continuous_on* S f
    using *continuous_on_eq_continuous_within f has_derivative_continuous* **by** *blast*
  have {$x \in S$. (*matrix* (f' x)\$$m$\$$n$) ≤ b} ∈ *sets lebesgue* **for** b
  proof (*rule sets_negligible_symdiff*)
    let *?T* = {$x \in S$. ∀ e>*0*. ∃ d>*0*. ∃ A. A\$$m$\$$n$ < b ∧ (∀ i j. A\$$i$\$$j$ ∈ ℚ) ∧
                    (∀ $y \in S$. *norm*($y - x$) < d ⟶ *norm*(f y − f x − A ∗v (y −
x)) ≤ e ∗ *norm*($y - x$))}
    let *?U* = S ∩
          (⋂ e ∈ {e ∈ ℚ. e > *0*}.
            ⋃ A ∈ {A. A\$$m$\$$n$ < b ∧ (∀ i j. A\$$i$\$$j$ ∈ ℚ)}.
              ⋃ d ∈ {d ∈ ℚ. *0* < d}.
                S ∩ (⋂ $y \in S$. {$x \in S$. *norm*($y - x$) < d ⟶ *norm*(f y − f x −
A ∗v ($y - x$)) ≤ e ∗ *norm*($y - x$)}))
    have *?T* = *?U*
    proof (*intro set_eqI iffI*)
      fix x
      assume *xT*: x ∈ *?T*
      then show x ∈ *?U*
      proof (*clarsimp simp add*:)
        fix q :: *real*
        assume q ∈ ℚ q > *0*
        then obtain d A **where** d > *0* **and** A: A \$ m \$ n < b ⋀i j. A \$ i \$ j ∈ ℚ
          ⋀y. ⟦y∈S; *norm* ($y - x$) < d⟧ ⟹ *norm* (f y − f x − A ∗v ($y - x$)) ≤
q ∗ *norm* ($y - x$)
          using *xT* **by** *auto*
        then obtain δ **where** $d > \delta$ δ > *0* δ ∈ ℚ
          using *Rats_dense_in_real* **by** *blast*
        with A **show** ∃ A. A \$ m \$ n < b ∧ (∀ i j. A \$ i \$ j ∈ ℚ) ∧
                      (∃ s. s ∈ ℚ ∧ *0* < s ∧ (∀ y∈S. *norm* ($y - x$) < s ⟶ *norm*
(f y − f x − A ∗v ($y - x$)) ≤ q ∗ *norm* ($y - x$)))
          by *force*
      qed
    next
      fix x
      assume *xU*: x ∈ *?U*
      then show x ∈ *?T*
      proof *clarsimp*
        fix e :: *real*
        assume e > *0*
        then obtain ε **where** ε: $e > \varepsilon$ ε > *0* ε ∈ ℚ
          using *Rats_dense_in_real* **by** *blast*
        with *xU* **obtain** A r **where** $x \in S$ **and** *Ar*: A \$ m \$ n < b ∀ i j. A \$ i \$

$j \in \mathbb{Q}$ $r \in \mathbb{Q}$ $r > 0$
and $\forall y \in S.$ *norm* $(y - x) < r \longrightarrow$ *norm* $(f\, y - f\, x - A *v\ (y - x)) \leq \varepsilon$ $*$ *norm* $(y - x)$
by (*auto simp*: *split*: *if_split_asm*)
then have $\forall y \in S.$ *norm* $(y - x) < r \longrightarrow$ *norm* $(f\, y - f\, x - A *v\ (y - x)) \leq e *$ *norm* $(y - x)$
by (*meson* ‹$e > \varepsilon$› *less_eq_real_def mult_right_mono norm_ge_zero order_trans*)
then show $\exists d>0.\ \exists A.\ A\ \$\ m\ \$\ n < b \wedge (\forall i\ j.\ A\ \$\ i\ \$\ j \in \mathbb{Q}) \wedge (\forall y \in S.$ *norm* $(y - x) < d \longrightarrow$ *norm* $(f\, y - f\, x - A *v\ (y - x)) \leq e *$ *norm* $(y - x))$
using ‹$x \in S$› *Ar* **by** *blast*
qed
qed
moreover have *?U* $\in$ *sets lebesgue*
proof −
have *coQ*: *countable* $\{e \in \mathbb{Q}.\ 0 < e\}$
using *countable_Collect countable_rat* **by** *blast*
have *ne*: $\{e \in \mathbb{Q}.\ (0::real) < e\} \neq \{\}$
using *zero_less_one Rats_1* **by** *blast*
have *coA*: *countable* $\{A.\ A\ \$\ m\ \$\ n < b \wedge (\forall i\ j.\ A\ \$\ i\ \$\ j \in \mathbb{Q})\}$
proof (*rule countable_subset*)
show *countable* $\{A.\ \forall i\ j.\ A\ \$\ i\ \$\ j \in \mathbb{Q}\}$
using *countable_vector* [*OF countable_vector*, *of* $\lambda i\ j.\ \mathbb{Q}$] **by** (*simp add*: *countable_rat*)
qed *blast*
have $*$: $[\![U \neq \{\} \Longrightarrow$ *closedin* (*top_of_set S*) $(S \cap \bigcap\ U)]\!]$
$\Longrightarrow$ *closedin* (*top_of_set S*) $(S \cap \bigcap\ U)$ **for** *U*
by *fastforce*
have *eq*: $\{x::(real,'m)vec.\ P\ x \wedge (Q\ x \longrightarrow R\ x)\} = \{x.\ P\ x \wedge \neg\ Q\ x\} \cup \{x.\ P\ x \wedge R\ x\}$ **for** *P Q R*
by *auto*
have *sets*: $S \cap (\bigcap y \in S.\ \{x \in S.$ *norm* $(y - x) < d \longrightarrow$ *norm* $(f\, y - f\, x - A *v\ (y - x)) \leq e *$ *norm* $(y - x)\})$
$\in$ *sets lebesgue* **for** *e A d*
proof −
have *clo*: *closedin* (*top_of_set S*)
$\{x \in S.$ *norm* $(y - x) < d \longrightarrow$ *norm* $(f\, y - f\, x - A *v\ (y - x)) \leq e *$ *norm* $(y - x)\}$
for *y*
proof −
have *cont1*: *continuous_on S* ($\lambda x.$ *norm* $(y - x)$)
and *cont2*: *continuous_on S* ($\lambda x.\ e *$ *norm* $(y - x) -$ *norm* $(f\, y - f\, x - (A *v\ y - A *v\ x)))$
by (*force intro*: *contf continuous_intros*)+
have *clo1*: *closedin* (*top_of_set S*) $\{x \in S.\ d \leq$ *norm* $(y - x)\}$
using *continuous_closedin_preimage* [*OF cont1*, *of* $\{d..\}$] **by** (*simp add*: *vimage_def Int_def*)
have *clo2*: *closedin* (*top_of_set S*)
$\{x \in S.$ *norm* $(f\, y - f\, x - (A *v\ y - A *v\ x)) \leq e *$ *norm* $(y - x)\}$

```
            using continuous_closedin_preimage [OF cont2, of {0..}] by (simp add:
vimage_def Int_def)
          show ?thesis
            by (auto simp: eq not_less matrix_vector_mult_diff_distrib intro: clo1 clo2)
        qed
        show ?thesis
          by (rule lebesgue_closedin [of S]) (force intro: ∗ S clo)+
      qed
      show ?thesis
        by (intro sets sets.Int S sets.countable_UN'' sets.countable_INT'' coQ coA)
auto
    qed
    ultimately show ?T ∈ sets lebesgue
      by simp
    let ?M = (?T − {x ∈ S. matrix (f' x) $ m $ n ≤ b} ∪ ({x ∈ S. matrix (f'
x) $ m $ n ≤ b} − ?T))
    let ?Θ = λx v. ∀ξ>0. ∃e>0. ∀y ∈ S−{x}. norm (x − y) < e ⟶ |v · (y −
x)| < ξ ∗ norm (x − y)
    have nN: negligible {x ∈ S. ∃v≠0. ?Θ x v}
      unfolding negligible_eq_zero_density
    proof clarsimp
      fix x v and r e :: real
      assume x ∈ S v ≠ 0 r > 0 e > 0
      and Theta [rule_format]: ?Θ x v
      moreover have (norm v ∗ e / 2) / CARD('m) ^ CARD('m) > 0
        by (simp add: ‹v ≠ 0› ‹e > 0›)
      ultimately obtain d where d > 0
         and dless: ⋀y. ⟦y ∈ S − {x}; norm (x − y) < d⟧ ⟹
                      |v · (y − x)| < ((norm v ∗ e / 2) / CARD('m) ^ CARD('m))
∗ norm (x − y)
        by metis
      let ?W = ball x (min d r) ∩ {y. |v · (y − x)| < (norm v ∗ e/2 ∗ min d r)
/ CARD('m) ^ CARD('m)}
      have open {x. |v · (x − a)| < b} for a b
        by (intro open_Collect_less continuous_intros)
      show ∃d>0. d ≤ r ∧
            (∃U. {x' ∈ S. ∃v≠0. ?Θ x' v} ∩ ball x d ⊆ U ∧
                 U ∈ lmeasurable ∧ measure lebesgue U < e ∗ content (ball x d))
      proof (intro exI conjI)
        show 0 < min d r min d r ≤ r
          using ‹r > 0› ‹d > 0› by auto
        show {x' ∈ S. ∃v. v ≠ 0 ∧ (∀ξ>0. ∃e>0. ∀z∈S − {x'}. norm (x' − z)
< e ⟶ |v · (z − x')| < ξ ∗ norm (x' − z))} ∩ ball x (min d r) ⊆ ?W
          proof (clarsimp simp: dist_norm norm_minus_commute)
            fix y w
            assume y ∈ S w ≠ 0
              and less [rule_format]:
                   ∀ξ>0. ∃e>0. ∀z∈S − {y}. norm (y − z) < e ⟶ |w · (z − y)|
< ξ ∗ norm (y − z)
```

and *d*: *norm* (*y* − *x*) < *d* **and** *r*: *norm* (*y* − *x*) < *r*
show |*v* · (*y* − *x*)| < *norm v* ∗ *e* ∗ *min d r* / (*2* ∗ *real CARD*(′*m*) ^ *CARD*(′*m*))
proof (*cases y* = *x*)
case *True*
with ‹*r* > *0*› ‹*d* > *0*› ‹*e* > *0*› ‹*v* ≠ *0*› **show** *?thesis*
by *simp*
next
case *False*
have |*v* · (*y* − *x*)| < *norm v* ∗ *e* / *2* / *real* (*CARD*(′*m*) ^ *CARD*(′*m*)) ∗ *norm* (*x* − *y*)
apply (*rule dless*)
using *False* ‹*y* ∈ *S*› *d* **by** (*auto simp*: *norm_minus_commute*)
also have ... ≤ *norm v* ∗ *e* ∗ *min d r* / (*2* ∗ *real CARD*(′*m*) ^ *CARD*(′*m*))
using *d r* ‹*e* > *0*› **by** (*simp add*: *field_simps norm_minus_commute mult_left_mono*)
finally show *?thesis* **.**
qed
qed
show *?W* ∈ *lmeasurable*
by (*simp add*: *fmeasurable_Int_fmeasurable borel_open*)
obtain *k*::′*m* **where** *True*
by *metis*
obtain *T* **where** *T*: *orthogonal_transformation T* **and** *v*: *v* = *T*(*norm v* $*_R$ *axis k* (*1*::*real*))
using *rotation_rightward_line* **by** *metis*
define *b* **where** *b* ≡ *norm v*
have *b* > *0*
using ‹*v* ≠ *0*› **by** (*auto simp*: *b_def*)
obtain *eqb*: *inv T v* = *b* $*_R$ *axis k* (*1*::*real*) **and** *inj T bij T* **and** *invT*: *orthogonal_transformation* (*inv T*)
by (*metis UNIV_I b_def T v bij_betw_inv_into_left orthogonal_transformation_inj orthogonal_transformation_bij orthogonal_transformation_inv*)
let *?v* = χ *i*. *min d r* / *CARD*(′*m*)
let *?v*′ = χ *i*. *if i* = *k then* (*e*/*2* ∗ *min d r*) / *CARD*(′*m*) ^ *CARD*(′*m*) *else min d r*
let *?x*′ = *inv T x*
let *?W*′ = (*ball ?x*′ (*min d r*) ∩ {*y*. |(*y* − *?x*′)$*k*| < *e* ∗ *min d r* / (*2* ∗ *CARD*(′*m*) ^ *CARD*(′*m*))})
have *abs*: *x* − *e* ≤ *y* ∧ *y* ≤ *x* + *e* ⟷ *abs*(*y* − *x*) ≤ *e* **for** *x y e*::*real*
by *auto*
have *?W* = *T* ‘ *?W*′
proof −
have *1*: *T* ‘ (*ball* (*inv T x*) (*min d r*)) = *ball x* (*min d r*)
by (*simp add*: *T image_orthogonal_transformation_ball orthogonal_transformation_surj surj_f_inv_f*)
have *2*: {*y*. |*v* · (*y* − *x*)| < *b* ∗ *e* ∗ *min d r* / (*2* ∗ *real CARD*(′*m*) ^ *CARD*(′*m*))} =

```
                T ` {y. |y $ k − ?x' $ k| < e * min d r / (2 * real CARD('m)
^ CARD('m))}
          proof −
            have *: |T (b *R axis k 1) · (y − x)| = b * |inv T y $ k − ?x' $ k|
for y
              proof −
              have |T (b *R axis k 1) · (y − x)| = |(b *R axis k 1) · inv T (y − x)|
            by (metis (no_types, hide_lams) b_def eqb invT orthogonal_transformation_def
v)
                also have ... = b * |(axis k 1) · inv T (y − x)|
                  using ‹b > 0› by (simp add: abs_mult)
                also have ... = b * |inv T y $ k − ?x' $ k|
                  using orthogonal_transformation_linear [OF invT]
                  by (simp add: inner_axis' linear_diff)
                finally show ?thesis
                  by simp
              qed
              show ?thesis
                using v b_def [symmetric]
                   using ‹b > 0› by (simp add: * bij_image_Collect_eq [OF ‹bij T›]
mult_less_cancel_left_pos times_divide_eq_right [symmetric] del: times_divide_eq_right)
          qed
          show ?thesis
            using ‹b > 0› by (simp add: image_Int ‹inj T› 1 2 b_def [symmetric])
        qed
        moreover have ?W' ∈ lmeasurable
          by (auto intro: fmeasurable_Int_fmeasurable)
        ultimately have measure lebesgue ?W = measure lebesgue ?W'
          by (metis measure_orthogonal_image T)
        also have ... ≤ measure lebesgue (cbox (?x' − ?v') (?x' + ?v'))
        proof (rule measure_mono_fmeasurable)
          show ?W' ⊆ cbox (?x' − ?v') (?x' + ?v')
        apply (clarsimp simp add: mem_box_cart abs dist_norm norm_minus_commute
simp del: min_less_iff_conj min.bounded_iff)
                by (metis component_le_norm_cart less_eq_real_def le_less_trans vec-
tor_minus_component)
        qed auto
        also have ... ≤ e/2 * measure lebesgue (cbox (?x' − ?v) (?x' + ?v))
        proof −
          have cbox (?x' − ?v) (?x' + ?v) ≠ {}
                using ‹r > 0› ‹d > 0› by (auto simp: interval_eq_empty_cart di-
vide_less_0_iff)
          with ‹r > 0› ‹d > 0› ‹e > 0› show ?thesis
            apply (simp add: content_cbox_if_cart mem_box_cart)
            apply (auto simp: prod_nonneg)
           apply (simp add: abs if_distrib prod.delta_remove field_simps power_diff
split: if_split_asm)
            done
        qed
```

also have ... ≤ *e/2* ∗ *measure lebesgue* (*cball ?x′* (*min d r*))
proof (*rule mult_left_mono* [*OF measure_mono_fmeasurable*])
have ∗: *norm* (*?x′* − *y*) ≤ *min d r*
if *y*: ⋀*i*. |*?x′* $ *i* − *y* $ *i*| ≤ *min d r* / *real CARD*(′*m*) **for** *y*
proof −
have *norm* (*?x′* − *y*) ≤ (∑ *i*∈*UNIV*. |(*?x′* − *y*) $ *i*|)
by (*rule norm_le_l1_cart*)
also have ... ≤ *real CARD*(′*m*) ∗ (*min d r* / *real CARD*(′*m*))
by (*rule sum_bounded_above*) (*use y* **in** *auto*)
finally show *?thesis*
by *simp*
qed
show *cbox* (*?x′* − *?v*) (*?x′* + *?v*) ⊆ *cball ?x′* (*min d r*)
apply (*clarsimp simp only*: *mem_box_cart dist_norm mem_cball intro*!:
∗)
by (*simp add*: *abs_diff_le_iff abs_minus_commute*)
qed (*use* ‹*e* > *0*› **in** *auto*)
also have ... < *e* ∗ *content* (*cball ?x′* (*min d r*))
using ‹*r* > *0*› ‹*d* > *0*› ‹*e* > *0*› **by** (*auto intro*: *content_cball_pos*)
also have ... = *e* ∗ *content* (*ball x* (*min d r*))
using ‹*r* > *0*› ‹*d* > *0*› *content_ball_conv_unit_ball*[*of min d r inv T x*]
content_ball_conv_unit_ball[*of min d r x*]
by (*simp add*: *content_cball_conv_ball*)
finally show *measure lebesgue ?W* < *e* ∗ *content* (*ball x* (*min d r*)) **.**
qed
qed
have ∗: (⋀*x*. (*x* ∉ *S*) ⟹ (*x* ∈ *T* ⟷ *x* ∈ *U*)) ⟹ (*T* − *U*) ∪ (*U* − *T*) ⊆
S **for** *S T U* :: (*real*,′*m*) *vec set*
by *blast*
have *MN*: *?M* ⊆ {*x* ∈ *S*. ∃ *v*≠*0*. *?Θ x v*}
proof (*rule* ∗)
fix *x*
assume *x*: *x* ∉ {*x* ∈ *S*. ∃ *v*≠*0*. *?Θ x v*}
show (*x* ∈ *?T*) ⟷ (*x* ∈ {*x* ∈ *S*. *matrix* (*f′ x*) $ *m* $ *n* ≤ *b*})
proof (*cases x* ∈ *S*)
case *True*
then have *x*: ¬ *?Θ x v* **if** *v* ≠ *0* **for** *v*
using *x that* **by** *force*
show *?thesis*
proof (*rule iffI*; *clarsimp*)
assume *b*: ∀ *e*>*0*. ∃ *d*>*0*. ∃ *A*. *A* $ *m* $ *n* < *b* ∧ (∀ *i j*. *A* $ *i* $ *j* ∈ ℚ) ∧
(∀ *y*∈*S*. *norm* (*y* − *x*) < *d* ⟶ *norm* (*f y* − *f x* − *A*
∗*v* (*y* − *x*)) ≤ *e* ∗ *norm* (*y* − *x*))
(**is** ∀ *e*>*0*. ∃ *d*>*0*. ∃ *A*. *?Φ e d A*)
then have ∀ *k*. ∃ *d*>*0*. ∃ *A*. *?Φ* (*1* / *Suc k*) *d A*
by (*metis* (*no_types*, *hide_lams*) *less_Suc_eq_0_disj of_nat_0_less_iff*
zero_less_divide_1_iff)
then obtain *δ A* **where** *δ*: ⋀*k*. *δ k* > *0*
and *Ab*: ⋀*k*. *A k* $ *m* $ *n* < *b*

and A: $\bigwedge k\ y.\ [\![y \in S;\ norm\ (y - x) < \delta\ k]\!] \Longrightarrow$
$norm\ (f\ y - f\ x - A\ k *v\ (y - x)) \leq 1/(Suc\ k)$
$*\ norm\ (y - x)$
by *metis*
have $\forall\, i\ j.\ \exists\, a.\ (\lambda n.\ A\ n\ \$\ i\ \$\ j) \longrightarrow a$
proof (*intro allI*)
fix $i\ j$
have *vax*: $(A\ n *v\ axis\ j\ 1)\ \$\ i = A\ n\ \$\ i\ \$\ j$ **for** n
by (*metis cart_eq_inner_axis matrix_vector_mul_component*)
let $?CA = \{x.\ Cauchy\ (\lambda n.\ (A\ n) *v\ x)\}$
have *subspace ?CA*
unfolding *subspace_def convergent_eq_Cauchy* [*symmetric*]
by (*force simp*: *algebra_simps intro*: *tendsto_intros*)
then have *CA_eq*: $?CA = span\ ?CA$
by (*metis span_eq_iff*)
also have $\ldots = UNIV$
proof −
have $dim\ ?CA \leq CARD('m)$
using *dim_subset_UNIV*[*of ?CA*]
by *auto*
moreover have *False* **if** *less*: $dim\ ?CA < CARD('m)$
proof −
obtain d **where** $d \neq 0$ **and** d: $\bigwedge y.\ y \in span\ ?CA \Longrightarrow orthogonal\ d\ y$
using *less* **by** (*force intro*: *orthogonal_to_subspace_exists* [*of ?CA*])
with x [*OF* ‹$d \neq 0$›] **obtain** ξ **where** $\xi > 0$
and ξ: $\bigwedge e.\ e > 0 \Longrightarrow \exists\, y \in S - \{x\}.\ norm\ (x - y) < e \wedge \xi *$
$norm\ (x - y) \leq |d \cdot (y - x)|$
by (*fastforce simp*: *not_le Bex_def*)
obtain $\gamma\ z$ **where** γSx: $\bigwedge i.\ \gamma\ i \in S - \{x\}$
and γle: $\bigwedge i.\ \xi * norm(\gamma\ i - x) \leq |d \cdot (\gamma\ i - x)|$
and γx: $\gamma \longrightarrow x$
and z: $(\lambda n.\ (\gamma\ n - x)\ /_R\ norm\ (\gamma\ n - x)) \longrightarrow z$
proof −
have $\exists \gamma.\ (\forall\, i.\ (\gamma\ i \in S - \{x\} \wedge$
$\xi * norm(\gamma\ i - x) \leq |d \cdot (\gamma\ i - x)| \wedge norm(\gamma\ i - x)$
$< 1/Suc\ i) \wedge$
$norm(\gamma(Suc\ i) - x) < norm(\gamma\ i - x))$
proof (*rule dependent_nat_choice*)
show $\exists\, y.\ y \in S - \{x\} \wedge \xi * norm\ (y - x) \leq |d \cdot (y - x)| \wedge$
$norm\ (y - x) < 1\ /\ Suc\ 0$
using ξ [*of 1*] **by** (*auto simp*: *dist_norm norm_minus_commute*)
next
fix $y\ i$
assume $y \in S - \{x\} \wedge \xi * norm\ (y - x) \leq |d \cdot (y - x)| \wedge norm$
$(y - x) < 1/Suc\ i$
then have $min\ (norm(y - x))\ (1/((Suc\ i) + 1)) > 0$
by *auto*
then obtain y' **where** $y' \in S - \{x\}$ **and** y': $norm\ (x - y') <$
$min\ (norm\ (y - x))\ (1/((Suc\ i) + 1))$

$\xi * norm\ (x - y') \leq |d \cdot (y' - x)|$
using ξ **by** *metis*
with ξ **show** $\exists\, y'.\ (y' \in S - \{x\} \wedge \xi * norm\ (y' - x) \leq |d \cdot (y' - x)| \wedge$
$norm\ (y' - x) < 1/(Suc\ (Suc\ i))) \wedge norm\ (y' - x) < norm\ (y - x)$
by (*auto simp*: *dist_norm norm_minus_commute*)
qed
then obtain γ **where**
γSx: $\bigwedge i.\ \gamma\ i \in S - \{x\}$
and γle: $\bigwedge i.\ \xi * norm(\gamma\ i - x) \leq |d \cdot (\gamma\ i - x)|$
and $\gamma conv$: $\bigwedge i.\ norm(\gamma\ i - x) < 1/(Suc\ i)$
by *blast*
let $?f = \lambda i.\ (\gamma\ i - x)\ /_R\ norm\ (\gamma\ i - x)$
have $?f\ i \in sphere\ 0\ 1$ **for** i
using γSx **by** *auto*
then obtain $l\ \varrho$ **where** $l \in sphere\ 0\ 1\ strict\_mono\ \varrho$ **and** l: $(?f \circ \varrho) \longrightarrow l$
using *compact_sphere* [*of* 0::(*real*,'*m*) *vec* 1] **unfolding** *compact_def* **by** *meson*
show *thesis*
proof
show $(\gamma \circ \varrho)\ i \in S - \{x\}\ \xi * norm\ ((\gamma \circ \varrho)\ i - x) \leq |d \cdot ((\gamma \circ \varrho)\ i - x)|$ **for** i
using $\gamma Sx\ \gamma le$ **by** *auto*
have $\gamma \longrightarrow x$
proof (*clarsimp simp add*: *LIMSEQ_def dist_norm*)
fix r :: *real*
assume $r > 0$
with *real_arch_invD* **obtain** *no* **where** $no \neq 0$ *real* $no > 1/r$
by (*metis divide_less_0_1_iff not_less_iff_gr_or_eq of_nat_0_eq_iff reals_Archimedean2*)
with $\gamma conv$ **show** $\exists\, no.\ \forall\, n \geq no.\ norm\ (\gamma\ n - x) < r$
by (*metis* ‹$r > 0$› *add.commute divide_inverse inverse_inverse_eq inverse_less_imp_less less_trans mult.left_neutral nat_le_real_less of_nat_Suc*)
qed
with ‹*strict_mono* ϱ› **show** $(\gamma \circ \varrho) \longrightarrow x$
by (*metis LIMSEQ_subseq_LIMSEQ*)
show $(\lambda n.\ ((\gamma \circ \varrho)\ n - x)\ /_R\ norm\ ((\gamma \circ \varrho)\ n - x)) \longrightarrow l$
using l **by** (*auto simp*: *o_def*)
qed
qed
have *isCont* $(\lambda x.\ (|d \cdot x| - \xi))\ z$
by (*intro continuous_intros*)
from *isCont_tendsto_compose* [*OF this z*]
have *lim*: $(\lambda y.\ |d \cdot ((\gamma\ y - x)\ /_R\ norm\ (\gamma\ y - x))| - \xi) \longrightarrow |d \cdot z| - \xi$
by *auto*
moreover have $\forall_F\ i$ *in sequentially*. $0 \leq |d \cdot ((\gamma\ i - x)\ /_R\ norm$

(γ i − x))| − ξ
proof (*rule eventuallyI*)
fix *n*
show *0* ≤ |*d* · ((γ *n* − *x*) /$_R$ *norm* (γ *n* − *x*))| − ξ
using γ*le* [*of n*] γ*Sx* **by** (*auto simp*: *abs_mult divide_simps*)
qed
ultimately have ξ ≤ |*d* · *z*|
using *tendsto_lowerbound* [**where** *a*=*0*] **by** *fastforce*
have *Cauchy* (λ*n*. (*A n*) ∗*v z*)
proof (*clarsimp simp add*: *Cauchy_def*)
fix ε :: *real*
assume *0* < ε
then obtain *N*::*nat* **where** *N* > *0* **and** *N*: ε/*2* > *1*/*N*
by (*metis half_gt_zero inverse_eq_divide neq0_conv real_arch_inverse*)
show ∃ *M*. ∀ *m*≥*M*. ∀ *n*≥*M*. *dist* (*A m* ∗*v z*) (*A n* ∗*v z*) < ε
proof (*intro exI allI impI*)
fix *i j*
assume *ij*: *N* ≤ *i N* ≤ *j*
let *?V* = λ*i k*. *A i* ∗*v* ((γ *k* − *x*) /$_R$ *norm* (γ *k* − *x*))
have ∀$_F$ *k in sequentially*. *dist* (γ *k*) *x* < *min* (δ *i*) (δ *j*)
using γ*x* [*unfolded tendsto_iff*] **by** (*meson min_less_iff_conj* δ)
then have *even*: ∀$_F$ *k in sequentially*. *norm* (*?V i k* − *?V j k*) −
2 / *N* ≤ *0*
proof (*rule eventually_mono*, *clarsimp*)
fix *p*
assume *p*: *dist* (γ *p*) *x* < δ *i dist* (γ *p*) *x* < δ *j*
let *?C* = λ*k*. *f* (γ *p*) − *f x* − *A k* ∗*v* (γ *p* − *x*)
have *norm* ((*A i* − *A j*) ∗*v* (γ *p* − *x*)) = *norm* (*?C j* − *?C i*)
by (*simp add*: *algebra_simps*)
also have ... ≤ *norm* (*?C j*) + *norm* (*?C i*)
using *norm_triangle_ineq4* **by** *blast*
also have ... ≤ *1*/(*Suc j*) ∗ *norm* (γ *p* − *x*) + *1*/(*Suc i*) ∗
norm (γ *p* − *x*)
by (*metis A Diff_iff* γ*Sx dist_norm p add_mono*)
also have ... ≤ *1*/*N* ∗ *norm* (γ *p* − *x*) + *1*/*N* ∗ *norm* (γ *p* −
x)
apply (*intro add_mono mult_right_mono*)
using *ij* ‹*N* > *0*› **by** (*auto simp*: *field_simps*)
also have ... = *2* / *N* ∗ *norm* (γ *p* − *x*)
by *simp*
finally have *no_le*: *norm* ((*A i* − *A j*) ∗*v* (γ *p* − *x*)) ≤ *2* / *N*
∗ *norm* (γ *p* − *x*) .
have *norm* (*?V i p* − *?V j p*) =
norm ((*A i* − *A j*) ∗*v* ((γ *p* − *x*) /$_R$ *norm* (γ *p* − *x*)))
by (*simp add*: *algebra_simps*)
also have ... = *norm* ((*A i* − *A j*) ∗*v* (γ *p* − *x*)) / *norm* (γ *p*
− *x*)
by (*simp add*: *divide_inverse matrix_vector_mult_scaleR*)
also have ... ≤ *2* / *N*

```
              using no_le by (auto simp: field_split_simps)
             finally show norm (?V i p − ?V j p) ≤ 2 / N .
            qed
            have isCont (λw. (norm(A i *v w − A j *v w) − 2 / N)) z
             by (intro continuous_intros)
            from isCont_tendsto_compose [OF this z]
            have lim: (λw. norm (A i *v ((γ w − x) /R norm (γ w − x)) −
                       A j *v ((γ w − x) /R norm (γ w − x))) − 2 / N)
                    ⟶ norm (A i *v z − A j *v z) − 2 / N
             by auto
            have dist (A i *v z) (A j *v z) ≤ 2 / N
           using tendsto_upperbound [OF lim even] by (auto simp: dist_norm)
            with N show dist (A i *v z) (A j *v z) < ε
             by linarith
           qed
          qed
          then have d · z = 0
           using CA_eq d orthogonal_def by auto
          then show False
           using ‹0 < ξ› ‹ξ ≤ |d · z|› by auto
         qed
         ultimately show ?thesis
          using dim_eq_full by fastforce
        qed
        finally have ?CA = UNIV .
        then have Cauchy (λn. (A n) *v axis j 1)
         by auto
        then obtain L where (λn. A n *v axis j 1) ⟶ L
         by (auto simp: Cauchy_convergent_iff convergent_def)
        then have (λx. (A x *v axis j 1) $ i) ⟶ L $ i
         by (rule tendsto_vec_nth)
        then show ∃ a. (λn. A n $ i $ j) ⟶ a
         by (force simp: vax)
       qed
       then obtain B where B: ⋀i j. (λn. A n $ i $ j) ⟶ B $ i $ j
        by (auto simp: lambda_skolem)
       have lin_df: linear (f' x)
          and lim_df: ((λy. (1 / norm (y − x)) *R (f y − (f x + f' x (y −
x)))) ⟶ 0) (at x within S)
        using ‹x ∈ S› assms by (auto simp: has_derivative_within linear_linear)
       moreover
       interpret linear f' x by fact
       have (matrix (f' x) − B) *v w = 0 for w
       proof (rule lemma_partial_derivatives [of (*v) (matrix (f' x) − B)])
        show linear ((*v) (matrix (f' x) − B))
         by (rule matrix_vector_mul_linear)
        have ((λy. ((f x + f' x (y − x)) − f y) /R norm (y − x)) ⟶ 0) (at
x within S)
         using tendsto_minus [OF lim_df] by (simp add: field_split_simps)
```

then show $((\lambda y.\ (matrix\ (f'\ x) - B) *v\ (y - x)\ /_R\ norm\ (y - x)) \longrightarrow 0)\ (at\ x\ within\ S)$
proof (*rule Lim_transform*)
have $((\lambda y.\ ((f\ y + B *v\ x - (f\ x + B *v\ y))\ /_R\ norm\ (y - x))) \longrightarrow 0)\ (at\ x\ within\ S)$
proof (*clarsimp simp add: Lim_within dist_norm*)
fix $e :: real$
assume $e > 0$
then obtain $q::nat$ **where** $q \neq 0$ **and** $qe2: 1/q < e/2$
by (*metis divide_pos_pos inverse_eq_divide real_arch_inverse zero_less_numeral*)
let $?g = \lambda p.\ sum\ (\lambda i.\ sum\ (\lambda j.\ abs((A\ p - B)\$i\$j))\ UNIV)\ UNIV$
have $(\lambda k.\ onorm\ (\lambda y.\ (A\ k - B) *v\ y)) \longrightarrow 0$
proof (*rule Lim_null_comparison*)
show $\forall_F\ k\ in\ sequentially.\ norm\ (onorm\ (\lambda y.\ (A\ k - B) *v\ y)) \leq ?g\ k$
proof (*rule eventually_sequentiallyI*)
fix $k :: nat$
assume $0 \leq k$
have $0 \leq onorm\ ((*v)\ (A\ k - B))$
using *matrix_vector_mul_bounded_linear*
by (*rule onorm_pos_le*)
then show $norm\ (onorm\ ((*v)\ (A\ k - B))) \leq (\sum i\in UNIV.\ \sum j\in UNIV.\ |(A\ k - B)\ \$\ i\ \$\ j|)$
by (*simp add: onorm_le_matrix_component_sum del: vector_minus_component*)
qed
next
show $?g \longrightarrow 0$
using *B Lim_null tendsto_rabs_zero_iff* **by** (*fastforce intro!: tendsto_null_sum*)
qed
with ‹$e > 0$› **obtain** p **where** $\bigwedge n.\ n \geq p \Longrightarrow |onorm\ ((*v)\ (A\ n - B))| < e/2$
unfolding *lim_sequentially* **by** (*metis diff_zero dist_real_def divide_pos_pos zero_less_numeral*)
then have $pqe2: |onorm\ ((*v)\ (A\ (p + q) - B))| < e/2$
using *le_add1* **by** *blast*
show $\exists d>0.\ \forall y\in S.\ y \neq x \wedge norm\ (y - x) < d \longrightarrow inverse\ (norm\ (y - x)) * norm\ (f\ y + B *v\ x - (f\ x + B *v\ y)) < e$
proof (*intro exI, safe*)
show $0 < \delta(p + q)$
by (*simp add: δ*)
next
fix y
assume $y: y \in S\ norm\ (y - x) < \delta(p + q)$ **and** $y \neq x$
have $*: [\![norm(b - c) < e - d;\ norm(y - x - b) \leq d]\!] \Longrightarrow norm(y - x - c) < e$

for *b c d e x* **and** *y*:: *real^'n*
using *norm_triangle_ineq2* [*of y* − *x* − *c y* − *x* − *b*] **by** *simp*
have *norm* (*f y* − *f x* − *B* ∗*v* (*y* − *x*)) < *e* ∗ *norm* (*y* − *x*)
proof (*rule* ∗)
show *norm* (*f y* − *f x* − *A* (*p* + *q*) ∗*v* (*y* − *x*)) ≤ *norm* (*y* − *x*) / (*Suc* (*p* + *q*))
using *A* [*OF y*] **by** *simp*
have *norm* (*A* (*p* + *q*) ∗*v* (*y* − *x*) − *B* ∗*v* (*y* − *x*)) ≤ *onorm*(λ*x*. (*A*(*p* + *q*) − *B*) ∗*v x*) ∗ *norm*(*y* − *x*)
by (*metis linear_linear matrix_vector_mul_linear matrix_vector_mult_diff_rdistrib onorm*)
also have ... < (*e*/*2*) ∗ *norm* (*y* − *x*)
using ‹*y* ≠ *x*› *pqe2* **by** *auto*
also have ... ≤ (*e* − *1* / (*Suc* (*p* + *q*))) ∗ *norm* (*y* − *x*)
proof (*rule mult_right_mono*)
have *1* / *Suc* (*p* + *q*) ≤ *1* / *q*
using ‹*q* ≠ *0*› **by** (*auto simp*: *field_split_simps*)
also have ... < *e*/*2*
using *qe2* **by** *auto*
finally show *e* / *2* ≤ *e* − *1* / *real* (*Suc* (*p* + *q*))
by *linarith*
qed *auto*
finally show *norm* (*A* (*p* + *q*) ∗*v* (*y* − *x*) − *B* ∗*v* (*y* − *x*)) < *e* ∗ *norm* (*y* − *x*) − *norm* (*y* − *x*) / *real* (*Suc* (*p* + *q*))
by (*simp add*: *algebra_simps*)
qed
then show *inverse* (*norm* (*y* − *x*)) ∗ *norm* (*f y* + *B* ∗*v x* − (*f x* + *B* ∗*v y*)) < *e*
using ‹*y* ≠ *x*› **by** (*simp add*: *field_split_simps algebra_simps*)
qed
qed
then show ((λ*y*. (*matrix* (*f*′ *x*) − *B*) ∗*v* (*y* − *x*) /$_R$ *norm* (*y* − *x*) − (*f x* + *f*′ *x* (*y* − *x*) − *f y*) /$_R$ *norm* (*y* − *x*)) ⟶ *0*)
(*at x within S*)
by (*simp add*: *algebra_simps diff lin_df scalar_mult_eq_scaleR*)
qed
qed (*use x* **in** ‹*simp*; *auto simp*: *not_less*›)
ultimately have *f*′ *x* = (∗*v*) *B*
by (*force simp*: *algebra_simps scalar_mult_eq_scaleR*)
show *matrix* (*f*′ *x*) $ *m* $ *n* ≤ *b*
proof (*rule tendsto_upperbound* [*of* λ*i*. (*A i* $ *m* $ *n*) _ *sequentially*])
show (λ*i*. *A i* $ *m* $ *n*) ⟶ *matrix* (*f*′ *x*) $ *m* $ *n*
by (*simp add*: *B* ‹*f*′ *x* = (∗*v*) *B*›)
show ∀$_F$ *i in sequentially*. *A i* $ *m* $ *n* ≤ *b*
by (*simp add*: *Ab less_eq_real_def*)
qed *auto*
next
fix *e* :: *real*

assume $x \in S$ **and** b: *matrix* $(f'\ x)$ \$ m \$ $n \leq b$ **and** $e > 0$
then obtain d **where** $d>0$
and d: $\bigwedge y.\ y{\in}S \Longrightarrow 0 < dist\ y\ x \wedge dist\ y\ x < d \longrightarrow norm\ (f\ y - f\ x - f'\ x\ (y - x))\ /\ (norm\ (y - x))$
$< e/2$
using f [*OF* ‹$x \in S$›]
by (*simp add*: *Deriv.has_derivative_at_within Lim_within*)
(*auto simp add*: *field_simps dest*: *spec* [*of* _ $e/2$])
let $?A = matrix(f'\ x) - (\chi\ i\ j.\ if\ i = m \wedge j = n\ then\ e\ /\ 4\ else\ 0)$
obtain B **where** *BRats*: $\bigwedge i\ j.\ B\$i\$j \in \mathbb{Q}$ **and** *Bo_e6*: $onorm((*v)\ (?A - B)) < e/6$
using *matrix_rational_approximation* ‹$e > 0$›
by (*metis zero_less_divide_iff zero_less_numeral*)
show $\exists d>0.\ \exists A.\ A\ \$\ m\ \$\ n < b \wedge (\forall i\ j.\ A\ \$\ i\ \$\ j \in \mathbb{Q}) \wedge$
$(\forall y{\in}S.\ norm\ (y - x) < d \longrightarrow norm\ (f\ y - f\ x - A *v\ (y - x)) \leq e * norm\ (y - x))$
proof (*intro exI conjI ballI allI impI*)
show $d>0$
by (*rule* ‹$d>0$›)
show $B\ \$\ m\ \$\ n < b$
proof −
have $|matrix\ ((*v)\ (?A - B))\ \$\ m\ \$\ n| \leq onorm\ ((*v)\ (?A - B))$
using *component_le_onorm* [*OF matrix_vector_mul_linear*, *of* _ m n] **by** *metis*
then show *?thesis*
using b *Bo_e6* **by** *simp*
qed
show $B\ \$\ i\ \$\ j \in \mathbb{Q}$ **for** $i\ j$
using *BRats* **by** *auto*
show $norm\ (f\ y - f\ x - B *v\ (y - x)) \leq e * norm\ (y - x)$
if $y \in S$ **and** y: $norm\ (y - x) < d$ **for** y
proof (*cases* $y = x$)
case *True* **then show** *?thesis*
by *simp*
next
case *False*
have $*$: $norm(d' - d) \leq e/2 \Longrightarrow norm(y - (x + d')) < e/2 \Longrightarrow norm(y - x - d) \leq e$ **for** $d\ d'\ e$ **and** $x\ y$::*real^'n*
using *norm_triangle_le* [*of* $d' - d\ y - (x + d')$] **by** *simp*
show *?thesis*
proof (*rule* $*$)
have *split246*: $[\![norm\ y \leq e\ /\ 6;\ norm(x - y) \leq e\ /\ 4]\!] \Longrightarrow norm\ x \leq e/2$ **if** $e > 0$ **for** e **and** $x\ y$ :: *real^'n*
using *norm_triangle_le* [*of* y $x{-}y$ $e/2$] ‹$e > 0$› **by** *simp*
have *linear* $(f'\ x)$
using *True f has_derivative_linear* **by** *blast*
then have $norm\ (f'\ x\ (y - x) - B *v\ (y - x)) = norm\ ((matrix\ (f'\ x) - B) *v\ (y - x))$
by (*simp add*: *matrix_vector_mult_diff_rdistrib*)

also have ... $\leq$ (*e* $*$ *norm* (*y* $-$ *x*)) / *2*
proof (*rule split246*)
have *norm* ((*?A* $-$ *B*) $*v$ (*y* $-$ *x*)) / *norm* (*y* $-$ *x*) $\leq$ *onorm*(λx. (*?A* $-$ *B*) $*v$ *x*)
by (*rule le_onorm*) *auto*
also have ... $<$ *e*/*6*
by (*rule Bo_e6*)
finally have *norm* ((*?A* $-$ *B*) $*v$ (*y* $-$ *x*)) / *norm* (*y* $-$ *x*) $<$ *e* / *6* **.**
then show *norm* ((*?A* $-$ *B*) $*v$ (*y* $-$ *x*)) $\leq$ *e* $*$ *norm* (*y* $-$ *x*) / *6*
by (*simp add*: *field_split_simps False*)
have *norm* ((*matrix* (*f'* *x*) $-$ *B*) $*v$ (*y* $-$ *x*) $-$ ((*?A* $-$ *B*) $*v$ (*y* $-$ *x*))) = *norm* ((χ *i j*. *if i* = *m* $\wedge$ *j* = *n then e* / *4 else 0*) $*v$ (*y* $-$ *x*))
by (*simp add*: *algebra_simps*)
also have ... = *norm*((*e*/*4*) $*_R$ (*y* $-$ *x*)\$*n* $*_R$ *axis m* (*1*::*real*))
proof $-$
have ($\sum$*j*$\in$*UNIV*. (*if i* = *m* $\wedge$ *j* = *n then e* / *4 else 0*) $*$ (*y* \$ *j* $-$ *x* \$ *j*)) $*$ *4* = *e* $*$ (*y* \$ *n* $-$ *x* \$ *n*) $*$ *axis m 1* \$ *i* **for** *i*
proof (*cases i*=*m*)
case *True* **then show** *?thesis*
by (*auto simp*: *if_distrib* [*of* λz. *z* $*$ _] *cong*: *if_cong*)
next
case *False* **then show** *?thesis*
by (*simp add*: *axis_def*)
qed
then have (χ *i j*. *if i* = *m* $\wedge$ *j* = *n then e* / *4 else 0*) $*v$ (*y* $-$ *x*) = (*e*/*4*) $*_R$ (*y* $-$ *x*)\$*n* $*_R$ *axis m* (*1*::*real*)
by (*auto simp*: *vec_eq_iff matrix_vector_mult_def*)
then show *?thesis*
by *metis*
qed
also have ... $\leq$ *e* $*$ *norm* (*y* $-$ *x*) / *4*
using ‹*e* $>$ *0*› **apply** (*simp add*: *norm_mult abs_mult*)
by (*metis component_le_norm_cart vector_minus_component*)
finally show *norm* ((*matrix* (*f'* *x*) $-$ *B*) $*v$ (*y* $-$ *x*) $-$ ((*?A* $-$ *B*) $*v$ (*y* $-$ *x*))) $\leq$ *e* $*$ *norm* (*y* $-$ *x*) / *4* **.**
show *0* $<$ *e* $*$ *norm* (*y* $-$ *x*)
by (*simp add*: *False* ‹*e* $>$ *0*›)
qed
finally show *norm* (*f'* *x* (*y* $-$ *x*) $-$ *B* $*v$ (*y* $-$ *x*)) $\leq$ (*e* $*$ *norm* (*y* $-$ *x*)) / *2* **.**
show *norm* (*f y* $-$ (*f x* + *f'* *x* (*y* $-$ *x*))) $<$ (*e* $*$ *norm* (*y* $-$ *x*)) / *2*
using *False d* [*OF* ‹*y* $\in$ *S*›] *y* **by** (*simp add*: *dist_norm field_simps*)
qed
qed
qed
qed
qed *auto*
qed

```
    show negligible ?M
      using negligible_subset [OF nN MN] .
  qed
  then show ?thesis
   by (simp add: borel_measurable_vimage_halfspace_component_le sets_restrict_space_iff
assms)
qed


theorem borel_measurable_det_Jacobian:
 fixes f :: real^'n::{finite,wellorder} ⇒ real^'n::_
  assumes S: S ∈ sets lebesgue and f: ⋀x. x ∈ S ⟹ (f has_derivative f' x) (at
x within S)
  shows (λx. det(matrix(f' x))) ∈ borel_measurable (lebesgue_on S)
  unfolding det_def
  by (intro measurable) (auto intro: f borel_measurable_partial_derivatives [OF S])
```

The localisation wrt S uses the same argument for many similar results.

```
theorem borel_measurable_lebesgue_on_preimage_borel:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes S ∈ sets lebesgue
  shows f ∈ borel_measurable (lebesgue_on S) ⟷
         (∀ T. T ∈ sets borel ⟶ {x ∈ S. f x ∈ T} ∈ sets lebesgue)
proof −
  have {x. (if x ∈ S then f x else 0) ∈ T} ∈ sets lebesgue ⟷ {x ∈ S. f x ∈ T}
∈ sets lebesgue
         if T ∈ sets borel for T
    proof (cases 0 ∈ T)
      case True
      then have {x ∈ S. f x ∈ T} = {x. (if x ∈ S then f x else 0) ∈ T} ∩ S
                {x. (if x ∈ S then f x else 0) ∈ T} = {x ∈ S. f x ∈ T} ∪ −S
        by auto
      then show ?thesis
        by (metis (no_types, lifting) Compl_in_sets_lebesgue assms sets.Int sets.Un)
    next
      case False
      then have {x. (if x ∈ S then f x else 0) ∈ T} = {x ∈ S. f x ∈ T}
        by auto
      then show ?thesis
        by auto
    qed
    then show ?thesis
       unfolding borel_measurable_lebesgue_preimage_borel borel_measurable_if [OF
assms, symmetric]
      by blast
qed

lemma sets_lebesgue_almost_borel:
  assumes S ∈ sets lebesgue
```

obtains *B N* **where** $B \in$ *sets borel negligible N* $B \cup N = S$
proof –
obtain *T N N′* **where** $S = T \cup N$ $N \subseteq N'$ $N' \in$ *null_sets lborel* $T \in$ *sets borel*
using *sets_completionE* [*OF assms*] **by** *auto*
then show *thesis*
by (*metis negligible_iff_null_sets negligible_subset null_sets_completionI that*)
qed

lemma *double_lebesgue_sets*:
assumes *S*: $S \in$ *sets lebesgue* **and** *T*: $T \in$ *sets lebesgue* **and** *fim*: *f* ‘ $S \subseteq T$
shows ($\forall U.\ U \in$ *sets lebesgue* $\wedge U \subseteq T \longrightarrow \{x \in S.\ f\,x \in U\} \in$ *sets lebesgue*)
$\longleftrightarrow$
$f \in$ *borel_measurable* (*lebesgue_on S*) $\wedge$
($\forall U.$ *negligible U* $\wedge U \subseteq T \longrightarrow \{x \in S.\ f\,x \in U\} \in$ *sets lebesgue*)
(**is** *?lhs* $\longleftrightarrow$ _ $\wedge$ *?rhs*)
unfolding *borel_measurable_lebesgue_on_preimage_borel* [*OF S*]
proof (*intro iffI allI conjI impI*, *safe*)
fix *V* :: *′b set*
assume ∗: $\forall U.\ U \in$ *sets lebesgue* $\wedge U \subseteq T \longrightarrow \{x \in S.\ f\,x \in U\} \in$ *sets lebesgue*
and $V \in$ *sets borel*
then have *V*: $V \in$ *sets lebesgue*
by *simp*
have $\{x \in S.\ f\,x \in V\} = \{x \in S.\ f\,x \in T \cap V\}$
using *fim* **by** *blast*
also have $\{x \in S.\ f\,x \in T \cap V\} \in$ *sets lebesgue*
using *T V* ∗ *le_inf_iff* **by** *blast*
finally show $\{x \in S.\ f\,x \in V\} \in$ *sets lebesgue* **.**
next
fix *U* :: *′b set*
assume $\forall U.\ U \in$ *sets lebesgue* $\wedge U \subseteq T \longrightarrow \{x \in S.\ f\,x \in U\} \in$ *sets lebesgue*
negligible U $U \subseteq T$
then show $\{x \in S.\ f\,x \in U\} \in$ *sets lebesgue*
using *negligible_imp_sets* **by** *blast*
next
fix *U* :: *′b set*
assume *1* [*rule_format*]: ($\forall T.\ T \in$ *sets borel* $\longrightarrow \{x \in S.\ f\,x \in T\} \in$ *sets lebesgue*)
and *2* [*rule_format*]: $\forall U.$ *negligible U* $\wedge U \subseteq T \longrightarrow \{x \in S.\ f\,x \in U\} \in$ *sets lebesgue*
and $U \in$ *sets lebesgue* $U \subseteq T$
then obtain *C N* **where** *C*: $C \in$ *sets borel* $\wedge$ *negligible N* $\wedge C \cup N = U$
using *sets_lebesgue_almost_borel*
by *metis*
then have $\{x \in S.\ f\,x \in C\} \in$ *sets lebesgue*
by (*blast intro*: *1*)
moreover have $\{x \in S.\ f\,x \in N\} \in$ *sets lebesgue*
using *C* ‹$U \subseteq T$› **by** (*blast intro*: *2*)
moreover have $\{x \in S.\ f\,x \in C \cup N\} = \{x \in S.\ f\,x \in C\} \cup \{x \in S.\ f\,x \in N\}$
by *auto*

ultimately show $\{x \in S.\ f\,x \in U\} \in$ *sets lebesgue*
using C **by** *auto*
qed

6.46.3 Simplest case of Sard's theorem (we don't need continuity of derivative)

lemma *Sard_lemma00*:
fixes P :: *'b::euclidean_space set*
assumes $a \geq 0$ **and** a: $a *_R i \neq 0$ **and** i: $i \in$ *Basis*
and P: $P \subseteq \{x.\ a *_R i \cdot x = 0\}$
and $0 \leq m$ $0 \leq e$
obtains S **where** $S \in$ *lmeasurable*
and $\{z.\ norm\ z \leq m \wedge (\exists\, t \in P.\ norm(z - t) \leq e)\} \subseteq S$
and *measure lebesgue* $S \leq (2 * e) * (2 * m)$ ^ $(DIM('b) - 1)$
proof −
have $a > 0$
using *assms* **by** *simp*
let $?v = (\sum j{\in}Basis.\ (if\ j = i\ then\ e\ else\ m) *_R j)$
show *thesis*
proof
have $- e \leq x \cdot i$ $x \cdot i \leq e$
if $t \in P$ *norm* $(x - t) \leq e$ **for** $x\ t$
using ‹$a > 0$› *that Basis_le_norm* [*of i x−t*] P i
by (*auto simp*: *inner_commute algebra_simps*)
moreover have $- m \leq x \cdot j$ $x \cdot j \leq m$
if *norm* $x \leq m$ $t \in P$ *norm* $(x - t) \leq e$ $j \in$ *Basis* **and** $j \neq i$
for $x\ t\ j$
using *that Basis_le_norm* [*of j x*] **by** *auto*
ultimately
show $\{z.\ norm\ z \leq m \wedge (\exists\, t{\in}P.\ norm\ (z - t) \leq e)\} \subseteq cbox\ (-?v)\ ?v$
by (*auto simp*: *mem_box*)
have $*$: $\forall\, k{\in}Basis.\ - ?v \cdot k \leq ?v \cdot k$
using ‹$0 \leq m$› ‹$0 \leq e$› **by** (*auto simp*: *inner_Basis*)
have 2: 2 ^ $DIM('b) = 2 * 2$ ^ $(DIM('b) - Suc\ 0)$
by (*metis DIM_positive Suc_pred power_Suc*)
show *measure lebesgue* (*cbox* $(-?v)$ $?v$) $\leq 2 * e * (2 * m)$ ^ $(DIM('b) - 1)$
using ‹$i \in$ *Basis*›
by (*simp add*: *content_cbox* [*OF* $*$] *prod.distrib prod.If_cases Diff_eq* [*symmetric*] *2*)
qed *blast*
qed

As above, but reorienting the vector (HOL Light's @textGEOM_BASIS_MULTIPLE_TAC)

lemma *Sard_lemma0*:
fixes P :: (*real^'n::{finite,wellorder}*) *set*
assumes $a \neq 0$
and P: $P \subseteq \{x.\ a \cdot x = 0\}$ **and** $0 \leq m$ $0 \leq e$
obtains S **where** $S \in$ *lmeasurable*

```
    and {z. norm z ≤ m ∧ (∃ t ∈ P. norm(z − t) ≤ e)} ⊆ S
    and measure lebesgue S ≤ (2 ∗ e) ∗ (2 ∗ m) ^ (CARD('n) − 1)
proof −
  obtain T and k::'n where T: orthogonal_transformation T and a: a = T (norm
a ∗R axis k (1::real))
    using rotation_rightward_line by metis
  have Tinv [simp]: T (inv T x) = x for x
    by (simp add: T orthogonal_transformation_surj surj_f_inv_f)
  obtain S where S: S ∈ lmeasurable
    and subS: {z. norm z ≤ m ∧ (∃ t ∈ T−`P. norm(z − t) ≤ e)} ⊆ S
    and mS: measure lebesgue S ≤ (2 ∗ e) ∗ (2 ∗ m) ^ (CARD('n) − 1)
  proof (rule Sard_lemma00 [of norm a axis k (1::real) T−`P m e])
    have norm a ∗R axis k 1 · x = 0 if T x ∈ P for x
    proof −
      have a · T x = 0
        using P that by blast
      then show ?thesis
          by (metis (no_types, lifting) T a orthogonal_orthogonal_transformation
orthogonal_def)
    qed
    then show T −` P ⊆ {x. norm a ∗R axis k 1 · x = 0}
      by auto
  qed (use assms T in auto)
  show thesis
  proof
    show T ` S ∈ lmeasurable
      using S measurable_orthogonal_image T by blast
    have {z. norm z ≤ m ∧ (∃ t∈P. norm (z − t) ≤ e)} ⊆ T ` {z. norm z ≤ m
∧ (∃ t∈T −` P. norm (z − t) ≤ e)}
    proof clarsimp
      fix x t
      assume norm x ≤ m t ∈ P norm (x − t) ≤ e
      then have norm (inv T x) ≤ m
          using orthogonal_transformation_inv [OF T] by (simp add: orthogo-
nal_transformation_norm)
      moreover have ∃ t∈T −` P. norm (inv T x − t) ≤ e
      proof
        have T (inv T x − inv T t) = x − t
          using T linear_diff orthogonal_transformation_def
          by (metis (no_types, hide_lams) Tinv)
        then have norm (inv T x − inv T t) = norm (x − t)
          by (metis T orthogonal_transformation_norm)
        then show norm (inv T x − inv T t) ≤ e
          using ‹norm (x − t) ≤ e› by linarith
       next
         show inv T t ∈ T −` P
           using ‹t ∈ P› by force
      qed
      ultimately show x ∈ T ` {z. norm z ≤ m ∧ (∃ t∈T −` P. norm (z − t) ≤
```

```
e)}
        by force
    qed
    then show {z. norm z ≤ m ∧ (∃t∈P. norm (z − t) ≤ e)} ⊆ T ‘ S
      using image_mono [OF subS] by (rule order_trans)
    show measure lebesgue (T ‘ S) ≤ 2 ∗ e ∗ (2 ∗ m) ^ (CARD('n) − 1)
      using mS T by (simp add: S measure_orthogonal_image)
  qed
qed
```

As above, but translating the sets (HOL Light's @textGEN_GEOM_ORIGIN_TAC)

```
lemma Sard_lemma1:
  fixes P :: (real^'n::{finite,wellorder}) set
   assumes P: dim P < CARD('n) and 0 ≤ m 0 ≤ e
 obtains S where S ∈ lmeasurable
            and {z. norm(z − w) ≤ m ∧ (∃t ∈ P. norm(z − w − t) ≤ e)} ⊆ S
            and measure lebesgue S ≤ (2 ∗ e) ∗ (2 ∗ m) ^ (CARD('n) − 1)
proof −
  obtain a where a ≠ 0 P ⊆ {x. a · x = 0}
    using lowdim_subset_hyperplane [of P] P span_base by auto
  then obtain S where S: S ∈ lmeasurable
    and subS: {z. norm z ≤ m ∧ (∃t ∈ P. norm(z − t) ≤ e)} ⊆ S
    and mS: measure lebesgue S ≤ (2 ∗ e) ∗ (2 ∗ m) ^ (CARD('n) − 1)
    by (rule Sard_lemma0 [OF _ _ ‹0 ≤ m› ‹0 ≤ e›])
  show thesis
  proof
    show (+)w ‘ S ∈ lmeasurable
      by (metis measurable_translation S)
    show {z. norm (z − w) ≤ m ∧ (∃t∈P. norm (z − w − t) ≤ e)} ⊆ (+)w ‘ S
      using subS by force
    show measure lebesgue ((+)w ‘ S) ≤ 2 ∗ e ∗ (2 ∗ m) ^ (CARD('n) − 1)
      by (metis measure_translation mS)
  qed
qed
```

```
lemma Sard_lemma2:
  fixes f :: real^'m::{finite,wellorder} ⇒ real^'n::{finite,wellorder}
  assumes mlen: CARD('m) ≤ CARD('n) (is ?m ≤ ?n)
    and B > 0 bounded S
    and derS: ⋀x. x ∈ S ⟹ (f has_derivative f' x) (at x within S)
    and rank: ⋀x. x ∈ S ⟹ rank(matrix(f' x)) < CARD('n)
    and B: ⋀x. x ∈ S ⟹ onorm(f' x) ≤ B
  shows negligible(f ‘ S)
proof −
  have lin_f': ⋀x. x ∈ S ⟹ linear(f' x)
    using derS has_derivative_linear by blast
  show ?thesis
  proof (clarsimp simp add: negligible_outer_le)
    fix e :: real
```

assume *e* > *0*
obtain *c* **where** *csub*: *S* ⊆ *cbox* (− (*vec c*)) (*vec c*) **and** *c* > *0*
proof −
obtain *b* **where** *b*: ⋀*x*. *x* ∈ *S* ⟹ *norm x* ≤ *b*
using ‹*bounded S*› **by** (*auto simp*: *bounded_iff*)
show *thesis*
proof
have − |*b*| − *1* ≤ *x* $ *i* ∧ *x* $ *i* ≤ |*b*| + *1* **if** *x* ∈ *S* **for** *x i*
using *component_le_norm_cart* [*of x i*] *b* [*OF that*] **by** *auto*
then show *S* ⊆ *cbox* (− *vec* (|*b*| + *1*)) (*vec* (|*b*| + *1*))
by (*auto simp*: *mem_box_cart*)
qed *auto*
qed
then have *box_cc*: *box* (− (*vec c*)) (*vec c*) ≠ {} **and** *cbox_cc*: *cbox* (− (*vec c*)) (*vec c*) ≠ {}
by (*auto simp*: *interval_eq_empty_cart*)
obtain *d* **where** *d* > *0* *d* ≤ *B*
and *d*: (*d* ∗ *2*) ∗ (*4* ∗ *B*) ^ (*?n* − *1*) ≤ *e* / (*2*∗*c*) ^ *?m* / *?m* ^ *?m*
apply (*rule that* [*of min B* (*e* / (*2*∗*c*) ^ *?m* / *?m* ^ *?m* / (*4* ∗ *B*) ^ (*?n* − *1*) / *2*)])
using ‹*B* > *0*› ‹*c* > *0*› ‹*e* > *0*›
by (*simp_all add*: *divide_simps min_mult_distrib_right*)
have ∃ *r*. *0* < *r* ∧ *r* ≤ *1/2* ∧
(*x* ∈ *S*
⟶ (∀ *y*. *y* ∈ *S* ∧ *norm*(*y* − *x*) < *r*
⟶ *norm*(*f y* − *f x* − *f′ x* (*y* − *x*)) ≤ *d* ∗ *norm*(*y* − *x*))) **for** *x*
proof (*cases x* ∈ *S*)
case *True*
then obtain *r* **where** *r* > *0*
and ⋀*y*. ⟦*y* ∈ *S*; *norm* (*y* − *x*) < *r*⟧
⟹ *norm* (*f y* − *f x* − *f′ x* (*y* − *x*)) ≤ *d* ∗ *norm* (*y* − *x*)
using *derS* ‹*d* > *0*› **by** (*force simp*: *has_derivative_within_alt*)
then show *?thesis*
by (*rule_tac x*=*min r* (*1/2*) **in** *exI*) *simp*
next
case *False*
then show *?thesis*
by (*rule_tac x*=*1/2* **in** *exI*) *simp*
qed
then obtain *r* **where** *r12*: ⋀*x*. *0* < *r x* ∧ *r x* ≤ *1/2*
and *r*: ⋀*x y*. ⟦*x* ∈ *S*; *y* ∈ *S*; *norm*(*y* − *x*) < *r x*⟧
⟹ *norm*(*f y* − *f x* − *f′ x* (*y* − *x*)) ≤ *d* ∗ *norm*(*y* − *x*)
by *metis*
then have *ga*: *gauge* (λ*x*. *ball x* (*r x*))
by (*auto simp*: *gauge_def*)
obtain 𝒟 **where** 𝒟: *countable* 𝒟 **and** *sub_cc*: ⋃𝒟 ⊆ *cbox* (− *vec c*) (*vec c*)
and *cbox*: ⋀*K*. *K* ∈ 𝒟 ⟹ *interior K* ≠ {} ∧ (∃ *u v*. *K* = *cbox u v*)
and *djointish*: *pairwise* (λ*A B*. *interior A* ∩ *interior B* = {}) 𝒟
and *covered*: ⋀*K*. *K* ∈ 𝒟 ⟹ ∃ *x* ∈ *S* ∩ *K*. *K* ⊆ *ball x* (*r x*)

```
      and close: ⋀u v. cbox u v ∈ 𝒟 ⟹ ∃n. ∀i::'m. v $ i − u $ i = 2*c / 2^n
      and covers: S ⊆ ⋃𝒟
      apply (rule covering_lemma [OF csub box_cc ga])
      apply (auto simp: Basis_vec_def cart_eq_inner_axis [symmetric])
      done
    let ?μ = measure lebesgue
    have ∃T. T ∈ lmeasurable ∧ f ` (K ∩ S) ⊆ T ∧ ?μ T ≤ e / (2*c) ^ ?m *
?μ K
      if K ∈ 𝒟 for K
    proof −
      obtain u v where uv: K = cbox u v
        using cbox ‹K ∈ 𝒟› by blast
      then have uv_ne: cbox u v ≠ {}
        using cbox that by fastforce
      obtain x where x: x ∈ S ∩ cbox u v cbox u v ⊆ ball x (r x)
        using ‹K ∈ 𝒟› covered uv by blast
      then have dim (range (f' x)) < ?n
        using rank_dim_range [of matrix (f' x)] x rank[of x]
        by (auto simp: matrix_works scalar_mult_eq_scaleR lin_f')
      then obtain T where T: T ∈ lmeasurable
           and subT: {z. norm(z − f x) ≤ (2 * B) * norm(v − u) ∧ (∃t ∈ range
(f' x). norm(z − f x − t) ≤ d * norm(v − u))} ⊆ T
           and measT: ?μ T ≤ (2 * (d * norm(v − u))) * (2 * ((2 * B) * norm(v
− u))) ^ (?n − 1)
                       (is _ ≤ ?DVU)
         apply (rule Sard_lemma1 [of range (f' x) (2 * B) * norm(v − u) d *
norm(v − u) f x])
        using ‹B > 0› ‹d > 0› by simp_all
      show ?thesis
      proof (intro exI conjI)
        have f ` (K ∩ S) ⊆ {z. norm(z − f x) ≤ (2 * B) * norm(v − u) ∧ (∃t ∈
range (f' x). norm(z − f x − t) ≤ d * norm(v − u))}
          unfolding uv
        proof (clarsimp simp: mult.assoc, intro conjI)
          fix y
          assume y: y ∈ cbox u v and y ∈ S
          then have norm (y − x) < r x
            by (metis dist_norm mem_ball norm_minus_commute subsetCE x(2))
          then have le_dyx: norm (f y − f x − f' x (y − x)) ≤ d * norm (y − x)
            using r [of x y] x ‹y ∈ S› by blast
          have yx_le: norm (y − x) ≤ norm (v − u)
          proof (rule norm_le_componentwise_cart)
            show norm ((y − x) $ i) ≤ norm ((v − u) $ i) for i
              using x y by (force simp: mem_box_cart dest!: spec [where x=i])
          qed
          have *: ⟦norm(y − x − z) ≤ d; norm z ≤ B; d ≤ B⟧ ⟹ norm(y − x)
≤ 2 * B
            for x y z :: real^'n::_ and d B
            using norm_triangle_ineq2 [of y − x z] by auto
```

```
      show norm (f y − f x) ≤ 2 ∗ (B ∗ norm (v − u))
      proof (rule ∗ [OF le_dyx])
        have norm (f′ x (y − x)) ≤ onorm (f′ x) ∗ norm (y − x)
          using onorm [of f′ x y−x] by (meson IntE lin_f′ linear_linear x(1))
        also have . . . ≤ B ∗ norm (v − u)
        proof (rule mult_mono)
          show onorm (f′ x) ≤ B
            using B x by blast
        qed (use ‹B > 0› yx_le in auto)
        finally show norm (f′ x (y − x)) ≤ B ∗ norm (v − u) .
        show d ∗ norm (y − x) ≤ B ∗ norm (v − u)
          using ‹B > 0› by (auto intro: mult_mono [OF ‹d ≤ B› yx_le])
      qed
      show ∃ t. norm (f y − f x − f′ x t) ≤ d ∗ norm (v − u)
        apply (rule_tac x=y−x in exI)
        using ‹d > 0› yx_le le_dyx mult_left_mono [where c=d]
        by (meson order_trans mult_le_cancel_iff2)
    qed
    with subT show f ‘ (K ∩ S) ⊆ T by blast
    show ?μ T ≤ e / (2∗c) ˆ ?m ∗ ?μ K
    proof (rule order_trans [OF measT])
      have ?DVU = (d ∗ 2 ∗ (4 ∗ B) ˆ (?n − 1)) ∗ norm (v − u)ˆ?n
        using ‹c > 0›
        apply (simp add: algebra_simps)
        by (metis Suc_pred power_Suc zero_less_card_finite)
      also have . . . ≤ (e / (2∗c) ˆ ?m / (?m ˆ ?m)) ∗ norm(v − u) ˆ ?n
        by (rule mult_right_mono [OF d]) auto
      also have . . . ≤ e / (2∗c) ˆ ?m ∗ ?μ K
      proof −
        have u ∈ ball (x) (r x) v ∈ ball x (r x)
           using box_ne_empty(1) contra_subsetD [OF x(2)] mem_box(2) uv_ne
by fastforce+
        moreover have r x ≤ 1/2
          using r12 by auto
        ultimately have norm (v − u) ≤ 1
          using norm_triangle_half_r [of x u 1 v]
         by (metis (no_types, hide_lams) dist_commute dist_norm less_eq_real_def
less_le_trans mem_ball)
        then have norm (v − u) ˆ ?n ≤ norm (v − u) ˆ ?m
          by (simp add: power_decreasing [OF mlen])
        also have . . . ≤ ?μ K ∗ real (?m ˆ ?m)
        proof −
          obtain n where n: ⋀i. v$i − u$i = 2 ∗ c / 2ˆn
            using close [of u v] ‹K ∈ 𝒟› uv by blast
          have norm (v − u) ˆ ?m ≤ (∑i∈UNIV. |(v − u) $ i|) ˆ ?m
            by (intro norm_le_l1_cart power_mono) auto
             also have . . . ≤ (∏i∈UNIV. v $ i − u $ i) ∗ real CARD(′m) ˆ
CARD(′m)
            by (simp add: n field_simps ‹c > 0› less_eq_real_def)
```

```
              also have ... = ?μ K * real (?m ^ ?m)
                by (simp add: uv uv_ne content_cbox_cart)
              finally show ?thesis .
            qed
            finally have *: 1 / real (?m ^ ?m) * norm (v − u) ^ ?n ≤ ?μ K
              by (simp add: field_split_simps)
            show ?thesis
              using mult_left_mono [OF *, of e / (2*c) ^ ?m] ‹c > 0› ‹e > 0› by
auto
          qed
          finally show ?DVU ≤ e / (2*c) ^ ?m * ?μ K .
        qed
      qed (use T in auto)
    qed
    then obtain g where meas_g: ⋀K. K ∈ 𝒟 ⟹ g K ∈ lmeasurable
                    and sub_g: ⋀K. K ∈ 𝒟 ⟹ f ` (K ∩ S) ⊆ g K
                    and le_g: ⋀K. K ∈ 𝒟 ⟹ ?μ (g K) ≤ e / (2*c)^?m * ?μ K
      by metis
    have le_e: ?μ (⋃i∈ℱ. g i) ≤ e
      if ℱ ⊆ 𝒟 finite ℱ for ℱ
    proof −
      have ?μ (⋃i∈ℱ. g i) ≤ (∑i∈ℱ. ?μ (g i))
        using meas_g ‹ℱ ⊆ 𝒟› by (auto intro: measure_UNION_le [OF ‹finite ℱ›])
      also have ... ≤ (∑K∈ℱ. e / (2*c) ^ ?m * ?μ K)
        using ‹ℱ ⊆ 𝒟› sum_mono [OF le_g] by (meson le_g subsetCE sum_mono)
      also have ... = e / (2*c) ^ ?m * (∑K∈ℱ. ?μ K)
        by (simp add: sum_distrib_left)
      also have ... ≤ e
      proof −
        have ℱ division_of ⋃ℱ
        proof (rule division_ofI)
          show K ⊆ ⋃ℱ  K ≠ {} ∃a b. K = cbox a b if K ∈ ℱ for K
            using ‹K ∈ ℱ› covered cbox ‹ℱ ⊆ 𝒟› by (auto simp: Union_upper)
          show interior K ∩ interior L = {} if K ∈ ℱ and L ∈ ℱ and K ≠ L for
K L
            by (metis (mono_tags, lifting) ‹ℱ ⊆ 𝒟› pairwiseD djointish pairwise_subset
that)
        qed (use that in auto)
        then have sum ?μ ℱ ≤ ?μ (⋃ℱ)
          by (simp add: content_division)
        also have ... ≤ ?μ (cbox (− vec c) (vec c) :: (real, 'm) vec set)
        proof (rule measure_mono_fmeasurable)
          show ⋃ℱ ⊆ cbox (− vec c) (vec c)
            by (meson Sup_subset_mono sub_cc order_trans ‹ℱ ⊆ 𝒟›)
        qed (use ‹ℱ division_of ⋃ℱ› lmeasurable_division in auto)
        also have ... = content (cbox (− vec c) (vec c) :: (real, 'm) vec set)
          by simp
        also have ... ≤ (2 ^ ?m * c ^ ?m)
          using ‹c > 0› by (simp add: content_cbox_if_cart)
```

```
      finally have sum ?μ ℱ ≤ (2 ^ ?m * c ^ ?m) .
      then show ?thesis
        using ‹e > 0› ‹c > 0› by (auto simp: field_split_simps)
    qed
    finally show ?thesis .
  qed
  show ∃ T. f ` S ⊆ T ∧ T ∈ lmeasurable ∧ ?μ T ≤ e
  proof (intro exI conjI)
    show f ` S ⊆ ⋃ (g ` 𝒟)
      using covers sub_g by force
    show ⋃ (g ` 𝒟) ∈ lmeasurable
      by (rule fmeasurable_UN_bound [OF ‹countable 𝒟› meas_g le_e])
    show ?μ (⋃ (g ` 𝒟)) ≤ e
      by (rule measure_UN_bound [OF ‹countable 𝒟› meas_g le_e])
  qed
 qed
qed
```

```
theorem baby_Sard:
  fixes f :: real^'m::{finite,wellorder} ⇒ real^'n::{finite,wellorder}
  assumes mlen: CARD('m) ≤ CARD('n)
    and der: ⋀x. x ∈ S ⟹ (f has_derivative f' x) (at x within S)
    and rank: ⋀x. x ∈ S ⟹ rank(matrix(f' x)) < CARD('n)
  shows negligible(f ` S)
proof -
  let ?U = λn. {x ∈ S. norm(x) ≤ n ∧ onorm(f' x) ≤ real n}
  have ⋀x. x ∈ S ⟹ ∃ n. norm x ≤ real n ∧ onorm (f' x) ≤ real n
    by (meson linear order_trans real_arch_simple)
  then have eq: S = (⋃ n. ?U n)
    by auto
  have negligible (f ` ?U n) for n
  proof (rule Sard_lemma2 [OF mlen])
    show 0 < real n + 1
      by auto
    show bounded (?U n)
      using bounded_iff by blast
    show (f has_derivative f' x) (at x within ?U n) if x ∈ ?U n for x
      using der that by (force intro: has_derivative_subset)
  qed (use rank in auto)
  then show ?thesis
    by (subst eq) (simp add: image_Union negligible_Union_nat)
qed
```

6.46.4 A one-way version of change-of-variables not assuming injectivity.

```
lemma integral_on_image_ubound_weak:
  fixes f :: real^'n::{finite,wellorder} ⇒ real
```

assumes *S*: *S* ∈ *sets lebesgue*
and *f*: *f* ∈ *borel_measurable* (*lebesgue_on* (*g* ‘ *S*))
and *nonneg_fg*: ⋀*x*. *x* ∈ *S* ⟹ *0* ≤ *f*(*g x*)
and *der_g*: ⋀*x*. *x* ∈ *S* ⟹ (*g has_derivative g′ x*) (*at x within S*)
and *det_int_fg*: (λ*x*. |*det* (*matrix* (*g′ x*))| ∗ *f*(*g x*)) *integrable_on S*
and *meas_gim*: ⋀*T*. ⟦*T* ⊆ *g* ‘ *S*; *T* ∈ *sets lebesgue*⟧ ⟹ {*x* ∈ *S*. *g x* ∈ *T*} ∈ *sets lebesgue*
shows *f integrable_on* (*g* ‘ *S*) ∧
integral (*g* ‘ *S*) *f* ≤ *integral S* (λ*x*. |*det* (*matrix* (*g′ x*))| ∗ *f*(*g x*))
(**is** _ ∧ _ ≤ *?b*)
proof −
let *?D* = λ*x*. |*det* (*matrix* (*g′ x*))|
have *cont_g*: *continuous_on S g*
using *der_g has_derivative_continuous_on* **by** *blast*
have [*simp*]: *space* (*lebesgue_on S*) = *S*
by (*simp add*: *S*)
have *gS_in_sets_leb*: *g* ‘ *S* ∈ *sets lebesgue*
apply (*rule differentiable_image_in_sets_lebesgue*)
using *der_g* **by** (*auto simp*: *S differentiable_def differentiable_on_def*)
obtain *h* **where** *nonneg_h*: ⋀*n x*. *0* ≤ *h n x*
and *h_le_f*: ⋀*n x*. *x* ∈ *S* ⟹ *h n* (*g x*) ≤ *f* (*g x*)
and *h_inc*: ⋀*n x*. *h n x* ≤ *h* (*Suc n*) *x*
and *h_meas*: ⋀*n*. *h n* ∈ *borel_measurable lebesgue*
and *fin_R*: ⋀*n*. *finite*(*range* (*h n*))
and *lim*: ⋀*x*. *x* ∈ *g* ‘ *S* ⟹ (λ*n*. *h n x*) ⟶ *f x*
proof −
let *?f* = λ*x*. *if x* ∈ *g* ‘ *S then f x else 0*
have *?f* ∈ *borel_measurable lebesgue* ∧ (∀*x*. *0* ≤ *?f x*)
by (*auto simp*: *gS_in_sets_leb f nonneg_fg measurable_restrict_space_iff* [*symmetric*])
then show *?thesis*
apply (*clarsimp simp add*: *borel_measurable_simple_function_limit_increasing*)
apply (*rename_tac h*)
by (*rule_tac h*=*h* **in** *that*) (*auto split*: *if_split_asm*)
qed
have *h_lmeas*: {*t*. *h n* (*g t*) = *y*} ∩ *S* ∈ *sets lebesgue* **for** *y n*
proof −
have *space* (*lebesgue_on* (*UNIV*::(*real*,′*n*) *vec set*)) = *UNIV*
by *simp*
then have ((*h n*) −‘{*y*} ∩ *g* ‘ *S*) ∈ *sets* (*lebesgue_on* (*g* ‘ *S*))
by (*metis Int_commute borel_measurable_vimage h_meas image_eqI inf_top.right_neutral sets_restrict_space space_borel space_completion space_lborel*)
then have ({*u*. *h n u* = *y*} ∩ *g* ‘ *S*) ∈ *sets lebesgue*
using *gS_in_sets_leb*
by (*simp add*: *integral_indicator fmeasurableI2 sets_restrict_space_iff vimage_def*)
then have {*x* ∈ *S*. *g x* ∈ ({*u*. *h n u* = *y*} ∩ *g* ‘ *S*)} ∈ *sets lebesgue*
using *meas_gim*[*of* ({*u*. *h n u* = *y*} ∩ *g* ‘ *S*)] **by** *force*
moreover have {*t*. *h n* (*g t*) = *y*} ∩ *S* = {*x* ∈ *S*. *g x* ∈ ({*u*. *h n u* = *y*} ∩ *g* ‘ *S*)}

by *blast*
ultimately show *?thesis*
by *auto*
qed
have *hint*: *h n integrable_on g ‘ S* ∧ *integral* (*g ‘ S*) (*h n*) ≤ *integral S* (λ*x*. *?D x* ∗ *h n* (*g x*))
(**is** *?INT* ∧ *?lhs* ≤ *?rhs*) **for** *n*
proof −
let *?R* = *range* (*h n*)
have *hn_eq*: *h n* = (λ*x*. ∑*y*∈*?R*. *y* ∗ *indicat_real* {*x*. *h n x* = *y*} *x*)
by (*simp add*: *indicator_def if_distrib fin_R cong*: *if_cong*)
have *yind*: (λ*t*. *y* ∗ *indicator*{*x*. *h n x* = *y*} *t*) *integrable_on* (*g ‘ S*) ∧
(*integral* (*g ‘ S*) (λ*t*. *y* ∗ *indicator* {*x*. *h n x* = *y*} *t*))
≤ *integral S* (λ*t*. |*det* (*matrix* (*g′ t*))| ∗ *y* ∗ *indicator* {*x*. *h n x* = *y*} (*g t*))
if *y*: *y* ∈ *?R* **for** *y*::*real*
proof (*cases y*=*0*)
case *True*
then show *?thesis* **using** *gS_in_sets_leb integrable_0* **by** *force*
next
case *False*
with *that* **have** *y* > *0*
using *less_eq_real_def nonneg_h* **by** *fastforce*
have (λ*x*. *if x* ∈ {*t*. *h n* (*g t*) = *y*} *then ?D x else 0*) *integrable_on S*
proof (*rule measurable_bounded_by_integrable_imp_integrable*)
have (λ*x*. *?D x*) ∈ *borel_measurable* (*lebesgue_on* ({*t*. *h n* (*g t*) = *y*} ∩ *S*))
apply (*intro borel_measurable_abs borel_measurable_det_Jacobian* [*OF h_lmeas*, **where** *f*=*g*])
by (*meson der_g IntD2 has_derivative_subset inf_le2*)
then have (λ*x*. *if x* ∈ {*t*. *h n* (*g t*) = *y*} ∩ *S then ?D x else 0*) ∈ *borel_measurable lebesgue*
by (*rule borel_measurable_if_I* [*OF _ h_lmeas*])
then show (λ*x*. *if x* ∈ {*t*. *h n* (*g t*) = *y*} *then ?D x else 0*) ∈ *borel_measurable* (*lebesgue_on S*)
by (*simp add*: *if_if_eq_conj Int_commute borel_measurable_if* [*OF S*, *symmetric*])
show (λ*x*. *?D x* $*_R$ *f* (*g x*) $/_R$ *y*) *integrable_on S*
by (*rule integrable_cmul*) (*use det_int_fg* **in** *auto*)
show *norm* (*if x* ∈ {*t*. *h n* (*g t*) = *y*} *then ?D x else 0*) ≤ *?D x* $*_R$ *f* (*g x*) $/_R$ *y*
if *x* ∈ *S* **for** *x*
using *nonneg_h* [*of n x*] ‹*y* > *0*› *nonneg_fg* [*of x*] *h_le_f* [*of x n*] *that*
by (*auto simp*: *divide_simps mult_left_mono*)
qed (*use S* **in** *auto*)
then have *int_det*: (λ*t*. |*det* (*matrix* (*g′ t*))|) *integrable_on* ({*t*. *h n* (*g t*) = *y*} ∩ *S*)
using *integrable_restrict_Int* **by** *force*
have (*g ‘* ({*t*. *h n* (*g t*) = *y*} ∩ *S*)) ∈ *lmeasurable*
apply (*rule measurable_differentiable_image* [*OF h_lmeas*])

```
      apply (blast intro: has_derivative_subset [OF der_g])
     apply (rule int_det)
     done
   moreover have g ‘ ({t. h n (g t) = y} ∩ S) = {x. h n x = y} ∩ g ‘ S
     by blast
   moreover have measure lebesgue (g ‘ ({t. h n (g t) = y} ∩ S))
                 ≤ integral ({t. h n (g t) = y} ∩ S) (λt. |det (matrix (g′ t))|)
     apply (rule measure_differentiable_image [OF h_lmeas _ int_det])
     apply (blast intro: has_derivative_subset [OF der_g])
     done
   ultimately show ?thesis
     using ‹y > 0› integral_restrict_Int [of S {t. h n (g t) = y} λt. |det (matrix
(g′ t))| ∗ y]
     apply (simp add: integrable_on_indicator integral_indicator)
     apply (simp add: indicator_def if_distrib cong: if_cong)
     done
  qed
  have hn_int: h n integrable_on g ‘ S
    apply (subst hn_eq)
    using yind by (force intro: integrable_sum [OF fin_R])
  then show ?thesis
  proof
    have ?lhs = integral (g ‘ S) (λx. ∑y∈range (h n). y ∗ indicat_real {x. h n
x = y} x)
     by (metis hn_eq)
   also have ... = (∑y∈range (h n). integral (g ‘ S) (λx. y ∗ indicat_real {x.
h n x = y} x))
     by (rule integral_sum [OF fin_R]) (use yind in blast)
   also have ... ≤ (∑y∈range (h n). integral S (λu. |det (matrix (g′ u))| ∗ y
∗ indicat_real {x. h n x = y} (g u)))
     using yind by (force intro: sum_mono)
    also have ... = integral S (λu. ∑y∈range (h n). |det (matrix (g′ u))| ∗ y
∗ indicat_real {x. h n x = y} (g u))
   proof (rule integral_sum [OF fin_R, symmetric])
     fix y assume y: y ∈ ?R
     with nonneg_h have y ≥ 0
       by auto
      show (λu. |det (matrix (g′ u))| ∗ y ∗ indicat_real {x. h n x = y} (g u))
integrable_on S
     proof (rule measurable_bounded_by_integrable_imp_integrable)
     have (λx. indicat_real {x. h n x = y} (g x)) ∈ borel_measurable (lebesgue_on
S)
         using h_lmeas S
       by (auto simp: indicator_vimage [symmetric] borel_measurable_indicator_iff
sets_restrict_space_iff)
       then show (λu. |det (matrix (g′ u))| ∗ y ∗ indicat_real {x. h n x = y} (g
u)) ∈ borel_measurable (lebesgue_on S)
      by (intro borel_measurable_times borel_measurable_abs borel_measurable_const
borel_measurable_det_Jacobian [OF S der_g])
```

next
fix *x*
assume $x \in S$
have $y * indicat\_real\ \{x.\ h\ n\ x = y\}\ (g\ x) \le f\ (g\ x)$
by (*metis* (*full_types*) ‹$x \in S$› *h_le_f indicator_def mem_Collect_eq mult.right_neutral mult_zero_right nonneg_fg*)
with ‹$y \ge 0$› **show** *norm* $(?D\ x * y * indicat\_real\ \{x.\ h\ n\ x = y\}\ (g\ x)) \le ?D\ x * f(g\ x)$
by (*simp add*: *abs_mult mult.assoc mult_left_mono*)
qed (*use S det_int_fg* **in** *auto*)
qed
also have ... $= integral\ S\ (\lambda T.\ |det\ (matrix\ (g'\ T))| * (\sum y{\in}range\ (h\ n).\ y * indicat\_real\ \{x.\ h\ n\ x = y\}\ (g\ T)))$
by (*simp add*: *sum_distrib_left mult.assoc*)
also have ... = *?rhs*
by (*metis hn_eq*)
finally show *integral* (*g* ‘ *S*) (*h n*) ≤ *?rhs* **.**
qed
qed
have *le*: $integral\ S\ (\lambda T.\ |det\ (matrix\ (g'\ T))| * h\ n\ (g\ T)) \le ?b$ **for** *n*
proof (*rule integral_le*)
show $(\lambda T.\ |det\ (matrix\ (g'\ T))| * h\ n\ (g\ T))$ *integrable_on S*
proof (*rule measurable_bounded_by_integrable_imp_integrable*)
have $(\lambda T.\ |det\ (matrix\ (g'\ T))| *_R h\ n\ (g\ T)) \in borel\_measurable\ (lebesgue\_on\ S)$
proof (*intro borel_measurable_scaleR borel_measurable_abs borel_measurable_det_Jacobian* ‹$S \in sets\ lebesgue$›)
have *eq*: $\{x \in S.\ f\ x \le a\} = (\bigcup b \in (f\ `\ S) \cap atMost\ a.\ \{x.\ f\ x = b\} \cap S)$ **for** *f* **and** *a*::*real*
by *auto*
have *finite* $((\lambda x.\ h\ n\ (g\ x))\ `\ S \cap \{..a\})$ **for** *a*
by (*force intro*: *finite_subset* [*OF _ fin_R*])
with *h_lmeas* [*of n*] **show** $(\lambda x.\ h\ n\ (g\ x)) \in borel\_measurable\ (lebesgue\_on\ S)$
apply (*simp add*: *borel_measurable_vimage_halfspace_component_le* ‹$S \in sets\ lebesgue$› *sets_restrict_space_iff eq*)
by (*metis* (*mono_tags*) *SUP_inf sets.finite_UN*)
qed (*use der_g* **in** *blast*)
then show $(\lambda T.\ |det\ (matrix\ (g'\ T))| * h\ n\ (g\ T)) \in borel\_measurable\ (lebesgue\_on\ S)$
by *simp*
show *norm* $(?D\ x * h\ n\ (g\ x)) \le ?D\ x *_R f\ (g\ x)$
if $x \in S$ **for** *x*
by (*simp add*: *h_le_f mult_left_mono nonneg_h that*)
qed (*use S det_int_fg* **in** *auto*)
show $?D\ x * h\ n\ (g\ x) \le ?D\ x * f\ (g\ x)$ **if** $x \in S$ **for** *x*
by (*simp add*: ‹$x \in S$› *h_le_f mult_left_mono*)
show $(\lambda x.\ ?D\ x * f\ (g\ x))$ *integrable_on S*

```
      using det_int_fg by blast
  qed
  have f integrable_on g ` S ∧ (λk. integral (g ` S) (h k)) ⟶ integral (g ` S) f
  proof (rule monotone_convergence_increasing)
    have |integral (g ` S) (h n)| ≤ integral S (λx. ?D x ∗ f (g x)) for n
    proof −
      have |integral (g ` S) (h n)| = integral (g ` S) (h n)
        using hint by (simp add: integral_nonneg nonneg_h)
      also have . . . ≤ integral S (λx. ?D x ∗ f (g x))
        using hint le by (meson order_trans)
      finally show ?thesis .
    qed
    then show bounded (range (λk. integral (g ` S) (h k)))
      by (force simp: bounded_iff )
  qed (use h_inc lim hint in auto)
  moreover have integral (g ` S) (h n) ≤ integral S (λx. ?D x ∗ f (g x)) for n
    using hint by (blast intro: le order_trans)
  ultimately show ?thesis
    by (auto intro: Lim_bounded)
qed


lemma integral_on_image_ubound_nonneg:
  fixes f :: real^'n::{finite,wellorder} ⇒ real
  assumes nonneg_fg: ⋀x. x ∈ S =⇒ 0 ≤ f(g x)
      and der_g:  ⋀x. x ∈ S =⇒ (g has_derivative g' x) (at x within S)
      and intS: (λx. |det (matrix (g' x))| ∗ f(g x)) integrable_on S
  shows f integrable_on (g ` S) ∧ integral (g ` S) f ≤ integral S (λx. |det (matrix
(g' x))| ∗ f(g x))
         (is _ ∧ _ ≤ ?b)
proof −
  let ?D = λx. det (matrix (g' x))
  define S' where S' ≡ {x ∈ S. ?D x ∗ f(g x) ≠ 0}
  then have der_gS': ⋀x. x ∈ S' =⇒ (g has_derivative g' x) (at x within S')
    by (metis (mono_tags, lifting) der_g has_derivative_subset mem_Collect_eq sub-
set_iff )
  have (λx. if x ∈ S then |?D x| ∗ f (g x) else 0) integrable_on UNIV
    by (simp add: integrable_restrict_UNIV intS)
  then have Df_borel: (λx. if x ∈ S then |?D x| ∗ f (g x) else 0) ∈ borel_measurable
lebesgue
    using integrable_imp_measurable lebesgue_on_UNIV_eq by force
  have S': S' ∈ sets lebesgue
  proof −
    from Df_borel borel_measurable_vimage_open [of _ UNIV]
    have {x. (if x ∈ S then |?D x| ∗ f (g x) else 0) ∈ T} ∈ sets lebesgue
      if open T for T
      using that unfolding lebesgue_on_UNIV_eq
      by (fastforce simp add: dest!: spec)
    then have {x. (if x ∈ S then |?D x| ∗ f (g x) else 0) ∈ −{0}} ∈ sets lebesgue
```

using *open_Compl* **by** *blast*
then show *?thesis*
by (*simp add*: *S′_def conj_ac split*: *if_split_asm cong*: *conj_cong*)
qed
then have *gS′*: *g ‘ S′* ∈ *sets lebesgue*
proof (*rule differentiable_image_in_sets_lebesgue*)
show *g differentiable_on S′*
using *der_g* **unfolding** *S′_def differentiable_def differentiable_on_def*
by (*blast intro*: *has_derivative_subset*)
qed *auto*
have *f*: *f* ∈ *borel_measurable* (*lebesgue_on* (*g ‘ S′*))
proof (*clarsimp simp add*: *borel_measurable_vimage_open*)
fix *T* :: *real set*
assume *open T*
have {*x* ∈ *g ‘ S′*. *f x* ∈ *T*} = *g ‘* {*x* ∈ *S′*. *f*(*g x*) ∈ *T*}
by *blast*
moreover have *g ‘* {*x* ∈ *S′*. *f*(*g x*) ∈ *T*} ∈ *sets lebesgue*
proof (*rule differentiable_image_in_sets_lebesgue*)
let *?h* = λ*x*. |*?D x*| ∗ *f* (*g x*) $/_R$ |*?D x*|
have (λ*x*. *if x* ∈ *S′ then* |*?D x*| ∗ *f* (*g x*) *else 0*) = (λ*x*. *if x* ∈ *S then* |*?D x*| ∗ *f* (*g x*) *else 0*)
by (*auto simp*: *S′_def*)
also have . . . ∈ *borel_measurable lebesgue*
by (*rule Df_borel*)
finally have ∗: (λ*x*. |*?D x*| ∗ *f* (*g x*)) ∈ *borel_measurable* (*lebesgue_on S′*)
by (*simp add*: *borel_measurable_if_D*)
have *?h* ∈ *borel_measurable* (*lebesgue_on S′*)
by (*intro* ∗ *S′ der_gS′ borel_measurable_det_Jacobian measurable*) (*blast intro*: *der_gS′*)
moreover have *?h x* = *f*(*g x*) **if** *x* ∈ *S′* **for** *x*
using *that* **by** (*auto simp*: *S′_def*)
ultimately have (λ*x*. *f*(*g x*)) ∈ *borel_measurable* (*lebesgue_on S′*)
by (*metis* (*no_types*, *lifting*) *measurable_lebesgue_cong*)
then show {*x* ∈ *S′*. *f* (*g x*) ∈ *T*} ∈ *sets lebesgue*
by (*simp add*: ‹*S′* ∈ *sets lebesgue*› ‹*open T*› *borel_measurable_vimage_open sets_restrict_space_iff*)
show *g differentiable_on* {*x* ∈ *S′*. *f* (*g x*) ∈ *T*}
using *der_g* **unfolding** *S′_def differentiable_def differentiable_on_def*
by (*blast intro*: *has_derivative_subset*)
qed *auto*
ultimately have {*x* ∈ *g ‘ S′*. *f x* ∈ *T*} ∈ *sets lebesgue*
by *metis*
then show {*x* ∈ *g ‘ S′*. *f x* ∈ *T*} ∈ *sets* (*lebesgue_on* (*g ‘ S′*))
by (*simp add*: ‹*g ‘ S′* ∈ *sets lebesgue*› *sets_restrict_space_iff*)
qed
have *intS′*: (λ*x*. |*?D x*| ∗ *f* (*g x*)) *integrable_on S′*
using *intS*
by (*rule integrable_spike_set*) (*auto simp*: *S′_def intro*: *empty_imp_negligible*)
have *lebS′*: {*x* ∈ *S′*. *g x* ∈ *T*} ∈ *sets lebesgue* **if** *T* ⊆ *g ‘ S′ T* ∈ *sets lebesgue*

for *T*
proof −
have *g* ∈ *borel_measurable* (*lebesgue_on S′*)
using *der_gS′ has_derivative_continuous_on S′*
by (*blast intro*: *continuous_imp_measurable_on_sets_lebesgue*)
moreover have {*x* ∈ *S′*. *g x* ∈ *U*} ∈ *sets lebesgue* **if** *negligible U U* ⊆ *g ‘ S′*
for *U*
proof (*intro negligible_imp_sets negligible_differentiable_vimage that*)
fix *x*
assume *x*: *x* ∈ *S′*
then have *linear* (*g′ x*)
using *der_gS′ has_derivative_linear* **by** *blast*
with *x* **show** *inj* (*g′ x*)
by (*auto simp*: *S′_def det_nz_iff_inj*)
qed (*use der_gS′* **in** *auto*)
ultimately show *?thesis*
using *double_lebesgue_sets* [*OF S′ gS′ order_refl*] *that* **by** *blast*
qed
have *int_gS′*: *f integrable_on g ‘ S′* ∧ *integral* (*g ‘ S′*) *f* ≤ *integral S′* (λ*x*. |*?D x*| ∗ *f*(*g x*))
using *integral_on_image_ubound_weak* [*OF S′ f nonneg_fg der_gS′ intS′ lebS′*] *S′_def* **by** *blast*
have *negligible* (*g ‘* {*x* ∈ *S*. *det*(*matrix*(*g′ x*)) = *0*})
proof (*rule baby_Sard*, *simp_all*)
fix *x*
assume *x*: *x* ∈ *S* ∧ *det* (*matrix* (*g′ x*)) = *0*
then show (*g has_derivative g′ x*) (*at x within* {*x* ∈ *S*. *det* (*matrix* (*g′ x*)) = *0*})
by (*metis* (*no_types*, *lifting*) *der_g has_derivative_subset mem_Collect_eq subsetI*)
then show *rank* (*matrix* (*g′ x*)) < *CARD*(*′n*)
using *det_nz_iff_inj matrix_vector_mul_linear x*
by (*fastforce simp add*: *less_rank_noninjective*)
qed
then have *negg*: *negligible* (*g ‘ S* − *g ‘* {*x* ∈ *S*. *?D x* ≠ *0*})
by (*rule negligible_subset*) (*auto simp*: *S′_def*)
have *null*: *g ‘* {*x* ∈ *S*. *?D x* ≠ *0*} − *g ‘ S* = {}
by (*auto simp*: *S′_def*)
let *?F* = {*x* ∈ *S*. *f* (*g x*) ≠ *0*}
have *eq*: *g ‘ S′* = *g ‘ ?F* ∩ *g ‘* {*x* ∈ *S*. *?D x* ≠ *0*}
by (*auto simp*: *S′_def image_iff*)
show *?thesis*
proof
have ((λ*x*. *if x* ∈ *g ‘ ?F then f x else 0*) *integrable_on g ‘* {*x* ∈ *S*. *?D x* ≠ *0*})
using *int_gS′ eq integrable_restrict_Int* [**where** *f*=*f*]
by *simp*
then have *f integrable_on g ‘* {*x* ∈ *S*. *?D x* ≠ *0*}
by (*auto simp*: *image_iff elim*!: *integrable_eq*)
then show *f integrable_on g ‘ S*

apply (*rule integrable_spike_set* [*OF* _ *empty_imp_negligible negligible_subset*])
using *negg null* **by** *auto*
have *integral* (*g* ‘ *S*) *f* = *integral* (*g* ‘ {*x* ∈ *S*. *?D x* ≠ *0*}) *f*
using *negg* **by** (*auto intro*: *negligible_subset integral_spike_set*)
also have ... = *integral* (*g* ‘ {*x* ∈ *S*. *?D x* ≠ *0*}) (λ*x*. *if x* ∈ *g* ‘ *?F then f x else 0*)
by (*auto simp*: *image_iff intro*!: *integral_cong*)
also have ... = *integral* (*g* ‘ *S'*) *f*
using *eq integral_restrict_Int* **by** *simp*
also have ... ≤ *integral S'* (λ*x*. |*?D x*| ∗ *f*(*g x*))
by (*metis int_gS'*)
also have ... ≤ *?b*
by (*rule integral_subset_le* [*OF* _ *intS' intS*]) (*use nonneg_fg S'_def* **in** *auto*)
finally show *integral* (*g* ‘ *S*) *f* ≤ *?b* **.**
qed
qed

lemma *absolutely_integrable_on_image_real*:
fixes *f* :: *real^'n*::{*finite*,*wellorder*} ⇒ *real* **and** *g* :: *real^'n*::_ ⇒ *real^'n*::_
assumes *der_g*: ⋀*x*. *x* ∈ *S* ⟹ (*g has_derivative g' x*) (*at x within S*)
and *intS*: (λ*x*. |*det* (*matrix* (*g' x*))| ∗ *f*(*g x*)) *absolutely_integrable_on S*
shows *f absolutely_integrable_on* (*g* ‘ *S*)
proof −
let *?D* = λ*x*. |*det* (*matrix* (*g' x*))| ∗ *f* (*g x*)
let *?N* = {*x* ∈ *S*. *f* (*g x*) < *0*} **and** *?P* = {*x* ∈ *S*. *f* (*g x*) > *0*}
have *eq*: {*x*. (*if x* ∈ *S then ?D x else 0*) > *0*} = {*x* ∈ *S*. *?D x* > *0*}
{*x*. (*if x* ∈ *S then ?D x else 0*) < *0*} = {*x* ∈ *S*. *?D x* < *0*}
by *auto*
have *?D integrable_on S*
using *intS absolutely_integrable_on_def* **by** *blast*
then have (λ*x*. *if x* ∈ *S then ?D x else 0*) *integrable_on UNIV*
by (*simp add*: *integrable_restrict_UNIV*)
then have *D_borel*: (λ*x*. *if x* ∈ *S then ?D x else 0*) ∈ *borel_measurable* (*lebesgue_on UNIV*)
using *integrable_imp_measurable lebesgue_on_UNIV_eq* **by** *blast*
then have *Dlt*: {*x* ∈ *S*. *?D x* < *0*} ∈ *sets lebesgue*
unfolding *borel_measurable_vimage_halfspace_component_lt*
by (*drule_tac x=0* **in** *spec*) (*auto simp*: *eq*)
from *D_borel* **have** *Dgt*: {*x* ∈ *S*. *?D x* > *0*} ∈ *sets lebesgue*
unfolding *borel_measurable_vimage_halfspace_component_gt*
by (*drule_tac x=0* **in** *spec*) (*auto simp*: *eq*)

have *dfgbm*: *?D* ∈ *borel_measurable* (*lebesgue_on S*)
using *intS absolutely_integrable_on_def integrable_imp_measurable* **by** *blast*
have *der_gN*: (*g has_derivative g' x*) (*at x within ?N*) **if** *x* ∈ *?N* **for** *x*
using *der_g has_derivative_subset that* **by** *force*
have (λ*x*. − *f x*) *integrable_on g* ‘ *?N* ∧
integral (*g* ‘ *?N*) (λ*x*. − *f x*) ≤ *integral ?N* (λ*x*. |*det* (*matrix* (*g' x*))| ∗ −

f (*g x*))
  proof (*rule integral_on_image_ubound_nonneg* [*OF* _ *der_gN*])
    have *1*: *?D integrable_on* {*x* ∈ *S*. *?D x* < *0*}
      using *Dlt*
     by (*auto intro*: *set_lebesgue_integral_eq_integral* [*OF set_integrable_subset*] *intS*)
    have *uminus* ∘ (λ*x*. |*det* (*matrix* (*g' x*))| ∗ − *f* (*g x*)) *integrable_on ?N*
       by (*simp add*: *o_def mult_less_0_iff empty_imp_negligible integrable_spike_set* [*OF 1*])
    then show (λ*x*. |*det* (*matrix* (*g' x*))| ∗ − *f* (*g x*)) *integrable_on ?N*
      by (*simp add*: *integrable_neg_iff o_def*)
  qed *auto*
  then have *f integrable_on g* ‘ *?N*
    by (*simp add*: *integrable_neg_iff*)
  moreover have *g* ‘ *?N* = {*y* ∈ *g* ‘ *S*. *f y* < *0*}
    by *auto*
  ultimately have *f integrable_on* {*y* ∈ *g* ‘ *S*. *f y* < *0*}
    by *simp*
  then have *N*: *f absolutely_integrable_on* {*y* ∈ *g* ‘ *S*. *f y* < *0*}
    by (*rule absolutely_integrable_absolutely_integrable_ubound*) *auto*

  have *der_gP*: (*g has_derivative g' x*) (*at x within ?P*) **if** *x* ∈ *?P* **for** *x*
      using *der_g has_derivative_subset that* **by** *force*
  have *f integrable_on g* ‘ *?P* ∧ *integral* (*g* ‘ *?P*) *f* ≤ *integral ?P ?D*
  proof (*rule integral_on_image_ubound_nonneg* [*OF* _ *der_gP*])
    have *?D integrable_on* {*x* ∈ *S*. *0* < *?D x*}
      using *Dgt*
     by (*auto intro*: *set_lebesgue_integral_eq_integral* [*OF set_integrable_subset*] *intS*)
    then show *?D integrable_on ?P*
      apply (*rule integrable_spike_set*)
      by (*auto simp*: *zero_less_mult_iff empty_imp_negligible*)
  qed *auto*
  then have *f integrable_on g* ‘ *?P*
    by *metis*
  moreover have *g* ‘ *?P* = {*y* ∈ *g* ‘ *S*. *f y* > *0*}
    by *auto*
  ultimately have *f integrable_on* {*y* ∈ *g* ‘ *S*. *f y* > *0*}
    by *simp*
  then have *P*: *f absolutely_integrable_on* {*y* ∈ *g* ‘ *S*. *f y* > *0*}
    by (*rule absolutely_integrable_absolutely_integrable_lbound*) *auto*
  have (λ*x*. *if x* ∈ *g* ‘ *S* ∧ *f x* < *0* ∨ *x* ∈ *g* ‘ *S* ∧ *0* < *f x then f x else 0*) = (λ*x*. *if x* ∈ *g* ‘ *S then f x else 0*)
    by *auto*
  then show *?thesis*
     using *absolutely_integrable_Un* [*OF N P*] *absolutely_integrable_restrict_UNIV* [*symmetric*, **where** *f*=*f*]
    by *simp*
qed

proposition *absolutely_integrable_on_image*:
  fixes f :: *real*^'*m*::{*finite*,*wellorder*} ⇒ *real*^'*n* **and** g :: *real*^'*m*::_ ⇒ *real*^'*m*::_
  assumes *der_g*: $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative* g' *x*) (*at x within S*)
    and *intS*: ($\lambda x.$ |*det* (*matrix* (g' *x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on S*
  shows *f absolutely_integrable_on* (*g* ' *S*)
 apply (*rule absolutely_integrable_componentwise* [*OF absolutely_integrable_on_image_real* [*OF der_g*]])
  using *absolutely_integrable_component* [*OF intS*] **by** *auto*

proposition *integral_on_image_ubound*:
  fixes f :: *real*^'*n*::{*finite*,*wellorder*} ⇒ *real* **and** g :: *real*^'*n*::_ ⇒ *real*^'*n*::_
  assumes $\bigwedge x.\ x \in S \Longrightarrow 0 \le f(g\ x)$
    and $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative* g' *x*) (*at x within S*)
    and ($\lambda x.$ |*det* (*matrix* (g' *x*))| ∗ *f*(*g x*)) *integrable_on S*
  shows *integral* (*g* ' *S*) *f* ≤ *integral S* ($\lambda x.$ |*det* (*matrix* (g' *x*))| ∗ *f*(*g x*))
  using *integral_on_image_ubound_nonneg* [*OF assms*] **by** *simp*

6.46.5 Change-of-variables theorem

The classic change-of-variables theorem. We have two versions with quite general hypotheses, the first that the transforming function has a continuous inverse, the second that the base set is Lebesgue measurable.

lemma *cov_invertible_nonneg_le*:
  fixes f :: *real*^'*n*::{*finite*,*wellorder*} ⇒ *real* **and** g :: *real*^'*n*::_ ⇒ *real*^'*n*::_
  assumes *der_g*: $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative* g' *x*) (*at x within S*)
    and *der_h*: $\bigwedge y.\ y \in T \Longrightarrow$ (*h has_derivative* h' *y*) (*at y within T*)
    and *f0*: $\bigwedge y.\ y \in T \Longrightarrow 0 \le f\ y$
    and *hg*: $\bigwedge x.\ x \in S \Longrightarrow g\ x \in T \land h(g\ x) = x$
    and *gh*: $\bigwedge y.\ y \in T \Longrightarrow h\ y \in S \land g(h\ y) = y$
    and *id*: $\bigwedge y.\ y \in T \Longrightarrow h'\ y \circ g'(h\ y) = id$
  shows *f integrable_on T* ∧ (*integral T f*) ≤ *b* ⟷
        ($\lambda x.$ |*det* (*matrix* (g' *x*))| ∗ *f*(*g x*)) *integrable_on S* ∧
        integral S ($\lambda x.$ |*det* (*matrix* (g' *x*))| ∗ *f*(*g x*)) ≤ *b*
    (**is** *?lhs* = *?rhs*)
proof −
  have *Teq*: *T* = *g*'*S* **and** *Seq*: *S* = *h*'*T*
    using *hg gh image_iff* **by** *fastforce+*
  have *gS*: *g differentiable_on S*
    by (*meson der_g differentiable_def differentiable_on_def*)
  let *?D* = $\lambda x.$ |*det* (*matrix* (g' *x*))| ∗ *f* (*g x*)
  show *?thesis*
  proof
    assume *?lhs*
    then have *fT*: *f integrable_on T* **and** *intf*: *integral T f* ≤ *b*
      by *blast+*
    show *?rhs*
    proof
      let *?fgh* = $\lambda x.$ |*det* (*matrix* (h' *x*))| ∗ (|*det* (*matrix* (g' (*h x*)))| ∗ *f* (*g* (*h x*)))
      have *ddf*: *?fgh x* = *f x*

```
        if x ∈ T for x
      proof -
        have matrix (h' x) ** matrix (g' (h x)) = mat 1
            using that id[OF that] der_g[of h x] gh[OF that] left_inverse_linear
has_derivative_linear
            by (subst matrix_compose[symmetric]) (force simp: matrix_id_mat_1
has_derivative_linear)+
        then have |det (matrix (h' x))| * |det (matrix (g' (h x)))| = 1
          by (metis abs_1 abs_mult det_I det_mul)
        then show ?thesis
          by (simp add: gh that)
      qed
      have ?D integrable_on (h ‘ T)
    proof (intro set_lebesgue_integral_eq_integral absolutely_integrable_on_image_real)
        show (λx. ?fgh x) absolutely_integrable_on T
        proof (subst absolutely_integrable_on_iff_nonneg)
          show (λx. ?fgh x) integrable_on T
            using ddf fT integrable_eq by force
        qed (simp add: zero_le_mult_iff f0 gh)
      qed (use der_h in auto)
      with Seq show (λx. ?D x) integrable_on S
        by simp
      have integral S (λx. ?D x) ≤ integral T (λx. ?fgh x)
        unfolding Seq
      proof (rule integral_on_image_ubound)
        show (λx. ?fgh x) integrable_on T
        using ddf fT integrable_eq by force
      qed (use f0 gh der_h in auto)
      also have ... = integral T f
        by (force simp: ddf intro: integral_cong)
      also have ... ≤ b
        by (rule intf)
      finally show integral S (λx. ?D x) ≤ b .
    qed
  next
    assume R: ?rhs
    then have f integrable_on g ‘ S
      using der_g f0 hg integral_on_image_ubound_nonneg by blast
    moreover have integral (g ‘ S) f ≤ integral S (λx. ?D x)
      by (rule integral_on_image_ubound [OF f0 der_g]) (use R Teq in auto)
    ultimately show ?lhs
      using R by (simp add: Teq)
  qed
qed


lemma cov_invertible_nonneg_eq:
  fixes f :: real^'n::{finite,wellorder} ⇒ real and g :: real^'n::_ ⇒ real^'n::_
  assumes ⋀x. x ∈ S ⟹ (g has_derivative g' x) (at x within S)
```

and $\bigwedge y.\ y \in T \Longrightarrow (h\ has\_derivative\ h'\ y)\ (at\ y\ within\ T)$
and $\bigwedge y.\ y \in T \Longrightarrow 0 \leq f\ y$
and $\bigwedge x.\ x \in S \Longrightarrow g\ x \in T \wedge h(g\ x) = x$
and $\bigwedge y.\ y \in T \Longrightarrow h\ y \in S \wedge g(h\ y) = y$
and $\bigwedge y.\ y \in T \Longrightarrow h'\ y \circ g'(h\ y) = id$
shows $((\lambda x.\ |det\ (matrix\ (g'\ x))| * f(g\ x))\ has\_integral\ b)\ S \longleftrightarrow (f\ has\_integral\ b)\ T$
using *cov_invertible_nonneg_le* [*OF assms*]
by (*simp add*: *has_integral_iff*) (*meson eq_iff*)

lemma *cov_invertible_real*:
fixes $f :: real\hat{\ }'n::\{finite,wellorder\} \Rightarrow real$ **and** $g :: real\hat{\ }'n::\_ \Rightarrow real\hat{\ }'n::\_$
assumes *der_g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has\_derivative\ g'\ x)\ (at\ x\ within\ S)$
and *der_h*: $\bigwedge y.\ y \in T \Longrightarrow (h\ has\_derivative\ h'\ y)\ (at\ y\ within\ T)$
and *hg*: $\bigwedge x.\ x \in S \Longrightarrow g\ x \in T \wedge h(g\ x) = x$
and *gh*: $\bigwedge y.\ y \in T \Longrightarrow h\ y \in S \wedge g(h\ y) = y$
and *id*: $\bigwedge y.\ y \in T \Longrightarrow h'\ y \circ g'(h\ y) = id$
shows $(\lambda x.\ |det\ (matrix\ (g'\ x))| * f(g\ x))\ absolutely\_integrable\_on\ S \wedge$
$integral\ S\ (\lambda x.\ |det\ (matrix\ (g'\ x))| * f(g\ x)) = b \longleftrightarrow$
$f\ absolutely\_integrable\_on\ T \wedge integral\ T\ f = b$
(**is** *?lhs = ?rhs*)
proof −
have *Teq*: $T = g`S$ **and** *Seq*: $S = h`T$
using *hg gh image_iff* **by** *fastforce+*
let $?DP = \lambda x.\ |det\ (matrix\ (g'\ x))| * f(g\ x)$ **and** $?DN = \lambda x.\ |det\ (matrix\ (g'\ x))| * -f(g\ x)$
have +: $(?DP\ has\_integral\ b)\ \{x \in S.\ f\ (g\ x) > 0\} \longleftrightarrow (f\ has\_integral\ b)\ \{y \in T.\ f\ y > 0\}$ **for** b
proof (*rule cov_invertible_nonneg_eq*)
have ∗: $(\lambda x.\ f\ (g\ x))\ -`\ Y \cap \{x \in S.\ f\ (g\ x) > 0\}$
$= ((\lambda x.\ f\ (g\ x))\ -`\ Y \cap S) \cap \{x \in S.\ f\ (g\ x) > 0\}$ **for** Y
by *auto*
show $(g\ has\_derivative\ g'\ x)\ (at\ x\ within\ \{x \in S.\ f\ (g\ x) > 0\})$ **if** $x \in \{x \in S.\ f\ (g\ x) > 0\}$ **for** x
using *that der_g has_derivative_subset* **by** *fastforce*
show $(h\ has\_derivative\ h'\ y)\ (at\ y\ within\ \{y \in T.\ f\ y > 0\})$ **if** $y \in \{y \in T.\ f\ y > 0\}$ **for** y
using *that der_h has_derivative_subset* **by** *fastforce*
qed (*use gh hg id* **in** *auto*)
have −: $(?DN\ has\_integral\ b)\ \{x \in S.\ f\ (g\ x) < 0\} \longleftrightarrow ((\lambda x.\ -\ f\ x)\ has\_integral\ b)\ \{y \in T.\ f\ y < 0\}$ **for** b
proof (*rule cov_invertible_nonneg_eq*)
have ∗: $(\lambda x.\ -\ f\ (g\ x))\ -`\ y \cap \{x \in S.\ f\ (g\ x) < 0\}$
$= ((\lambda x.\ f\ (g\ x))\ -`\ uminus\ `\ y \cap S) \cap \{x \in S.\ f\ (g\ x) < 0\}$ **for** y
using *image_iff* **by** *fastforce*
show $(g\ has\_derivative\ g'\ x)\ (at\ x\ within\ \{x \in S.\ f\ (g\ x) < 0\})$ **if** $x \in \{x \in S.\ f\ (g\ x) < 0\}$ **for** x
using *that der_g has_derivative_subset* **by** *fastforce*

show (*h has_derivative h′ y*) (*at y within* {*y* ∈ *T*. *f y* < *0*}) **if** *y* ∈ {*y* ∈ *T*. *f y* < *0*} **for** *y*
using *that der_h has_derivative_subset* **by** *fastforce*
qed (*use gh hg id* **in** *auto*)
show *?thesis*
proof
assume *LHS*: *?lhs*
have *eq*: {*x*. (*if x* ∈ *S then ?DP x else 0*) > *0*} = {*x* ∈ *S*. *?DP x* > *0*}
{*x*. (*if x* ∈ *S then ?DP x else 0*) < *0*} = {*x* ∈ *S*. *?DP x* < *0*}
by *auto*
have *?DP integrable_on S*
using *LHS absolutely_integrable_on_def* **by** *blast*
then have (λ*x*. *if x* ∈ *S then ?DP x else 0*) *integrable_on UNIV*
by (*simp add*: *integrable_restrict_UNIV*)
then have *D_borel*: (λ*x*. *if x* ∈ *S then ?DP x else 0*) ∈ *borel_measurable* (*lebesgue_on UNIV*)
using *integrable_imp_measurable lebesgue_on_UNIV_eq* **by** *blast*
then have *SN*: {*x* ∈ *S*. *?DP x* < *0*} ∈ *sets lebesgue*
unfolding *borel_measurable_vimage_halfspace_component_lt*
by (*drule_tac x=0* **in** *spec*) (*auto simp*: *eq*)
from *D_borel* **have** *SP*: {*x* ∈ *S*. *?DP x* > *0*} ∈ *sets lebesgue*
unfolding *borel_measurable_vimage_halfspace_component_gt*
by (*drule_tac x=0* **in** *spec*) (*auto simp*: *eq*)
have *?DP absolutely_integrable_on* {*x* ∈ *S*. *?DP x* > *0*}
using *LHS* **by** (*fast intro*!: *set_integrable_subset* [*OF _*, *of _ S*] *SP*)
then have *aP*: *?DP absolutely_integrable_on* {*x* ∈ *S*. *f* (*g x*) > *0*}
by (*rule absolutely_integrable_spike_set*) (*auto simp*: *zero_less_mult_iff empty_imp_negligible*)
have *?DP absolutely_integrable_on* {*x* ∈ *S*. *?DP x* < *0*}
using *LHS* **by** (*fast intro*!: *set_integrable_subset* [*OF _*, *of _ S*] *SN*)
then have *aN*: *?DP absolutely_integrable_on* {*x* ∈ *S*. *f* (*g x*) < *0*}
by (*rule absolutely_integrable_spike_set*) (*auto simp*: *mult_less_0_iff empty_imp_negligible*)
have *fN*: *f integrable_on* {*y* ∈ *T*. *f y* < *0*}
integral {*y* ∈ *T*. *f y* < *0*} *f* = *integral* {*x* ∈ *S*. *f* (*g x*) < *0*} *?DP*
using − [*of integral* {*x* ∈ *S*. *f*(*g x*) < *0*} *?DN*] *aN*
by (*auto simp*: *set_lebesgue_integral_eq_integral has_integral_iff integrable_neg_iff*)
have *faN*: *f absolutely_integrable_on* {*y* ∈ *T*. *f y* < *0*}
apply (*rule absolutely_integrable_integrable_bound* [**where** *g* = λ*x*. − *f x*])
using *fN* **by** (*auto simp*: *integrable_neg_iff*)
have *fP*: *f integrable_on* {*y* ∈ *T*. *f y* > *0*}
integral {*y* ∈ *T*. *f y* > *0*} *f* = *integral* {*x* ∈ *S*. *f* (*g x*) > *0*} *?DP*
using + [*of integral* {*x* ∈ *S*. *f*(*g x*) > *0*} *?DP*] *aP*
by (*auto simp*: *set_lebesgue_integral_eq_integral has_integral_iff integrable_neg_iff*)
have *faP*: *f absolutely_integrable_on* {*y* ∈ *T*. *f y* > *0*}
apply (*rule absolutely_integrable_integrable_bound* [**where** *g* = *f*])
using *fP* **by** *auto*
have *fa*: *f absolutely_integrable_on* ({*y* ∈ *T*. *f y* < *0*} ∪ {*y* ∈ *T*. *f y* > *0*})
by (*rule absolutely_integrable_Un* [*OF faN faP*])
show *?rhs*
proof

have *eq*: ((*if* $x \in T \wedge f\,x < 0 \vee x \in T \wedge 0 < f\,x$ *then 1 else 0*) $*$ *f x*)
= (*if* $x \in T$ *then 1 else 0*) $*$ *f x* **for** *x*
by *auto*
show *f absolutely_integrable_on T*
using *fa* **by** (*simp add*: *indicator_def set_integrable_def eq*)
have [*simp*]: $\{y \in T.\ f\ y < 0\} \cap \{y \in T.\ 0 < f\ y\} = \{\}$ **for** *T* **and** *f* ::
(*real*ˆ′*n*::_) ⇒ *real*
by *auto*
have *integral T f* = *integral* ($\{y \in T.\ f\ y < 0\} \cup \{y \in T.\ f\ y > 0\}$) *f*
by (*intro empty_imp_negligible integral_spike_set*) (*auto simp*: *eq*)
also have ... = *integral* $\{y \in T.\ f\ y < 0\}$ *f* + *integral* $\{y \in T.\ f\ y > 0\}$ *f*
using *fN fP* **by** *simp*
also have ... = *integral* $\{x \in S.\ f\ (g\ x) < 0\}$ *?DP* + *integral* $\{x \in S.\ 0 <$
$f\ (g\ x)\}$ *?DP*
by (*simp add*: *fN fP*)
also have ... = *integral* ($\{x \in S.\ f\ (g\ x) < 0\} \cup \{x \in S.\ 0 < f\ (g\ x)\}$) *?DP*
using *aP aN* **by** (*simp add*: *set_lebesgue_integral_eq_integral*)
also have ... = *integral S ?DP*
by (*intro empty_imp_negligible integral_spike_set*) *auto*
also have ... = *b*
using *LHS* **by** *simp*
finally show *integral T f* = *b* **.**
qed
next
assume *RHS*: *?rhs*
have *eq*: $\{x.$ (*if* $x \in T$ *then f x else 0*) $> 0\} = \{x \in T.\ f\ x > 0\}$
$\{x.$ (*if* $x \in T$ *then f x else 0*) $< 0\} = \{x \in T.\ f\ x < 0\}$
by *auto*
have *f integrable_on T*
using *RHS absolutely_integrable_on_def* **by** *blast*
then have (λ*x*. *if* $x \in T$ *then f x else 0*) *integrable_on UNIV*
by (*simp add*: *integrable_restrict_UNIV*)
then have *D_borel*: (λ*x*. *if* $x \in T$ *then f x else 0*) ∈ *borel_measurable* (*lebesgue_on*
UNIV)
using *integrable_imp_measurable lebesgue_on_UNIV_eq* **by** *blast*
then have *TN*: $\{x \in T.\ f\ x < 0\}$ ∈ *sets lebesgue*
unfolding *borel_measurable_vimage_halfspace_component_lt*
by (*drule_tac x=0* **in** *spec*) (*auto simp*: *eq*)
from *D_borel* **have** *TP*: $\{x \in T.\ f\ x > 0\}$ ∈ *sets lebesgue*
unfolding *borel_measurable_vimage_halfspace_component_gt*
by (*drule_tac x=0* **in** *spec*) (*auto simp*: *eq*)
have *aint*: *f absolutely_integrable_on* $\{y.\ y \in T \wedge 0 < (f\ y)\}$
f absolutely_integrable_on $\{y.\ y \in T \wedge (f\ y) < 0\}$
and *intT*: *integral T f* = *b*
using *set_integrable_subset* [*of _ T*] *TP TN RHS*
by *blast+*
show *?lhs*
proof
have *fN*: *f integrable_on* $\{v \in T.\ f\ v < 0\}$

    using *absolutely_integrable_on_def aint* **by** *blast*
  then have *DN*: (*?DN has_integral integral* {*y* ∈ *T*. *f y* < *0*} (λ*x*. − *f x*)) {*x* ∈ *S*. *f* (*g x*) < *0*}
    using − [*of integral* {*y* ∈ *T*. *f y* < *0*} (λ*x*. − *f x*)]
    by (*simp add*: *has_integral_neg_iff integrable_integral*)
  have *aDN*: *?DP absolutely_integrable_on* {*x* ∈ *S*. *f* (*g x*) < *0*}
    apply (*rule absolutely_integrable_integrable_bound* [**where** *g* = *?DN*])
    using *DN hg* **by** (*fastforce simp*: *abs_mult integrable_neg_iff*)+
  have *fP*: *f integrable_on* {*v* ∈ *T*. *f v* > *0*}
    using *absolutely_integrable_on_def aint* **by** *blast*
  then have *DP*: (*?DP has_integral integral* {*y* ∈ *T*. *f y* > *0*} *f*) {*x* ∈ *S*. *f* (*g x*) > *0*}
    using + [*of integral* {*y* ∈ *T*. *f y* > *0*} *f*]
    by (*simp add*: *has_integral_neg_iff integrable_integral*)
  have *aDP*: *?DP absolutely_integrable_on* {*x* ∈ *S*. *f* (*g x*) > *0*}
    apply (*rule absolutely_integrable_integrable_bound* [**where** *g* = *?DP*])
    using *DP hg* **by** (*fastforce simp*: *integrable_neg_iff*)+
  have *eq*: (*if x* ∈ *S then 1 else 0*) ∗ *?DP x* = (*if x* ∈ *S* ∧ *f* (*g x*) < *0* ∨ *x* ∈ *S* ∧ *f* (*g x*) > *0 then 1 else 0*) ∗ *?DP x* **for** *x*
    by *force*
  have *?DP absolutely_integrable_on* ({*x* ∈ *S*. *f* (*g x*) < *0*} ∪ {*x* ∈ *S*. *f* (*g x*) > *0*})
    by (*rule absolutely_integrable_Un* [*OF aDN aDP*])
  then show *I*: *?DP absolutely_integrable_on S*
    by (*simp add*: *indicator_def eq set_integrable_def*)
  have [*simp*]: {*y* ∈ *S*. *f y* < *0*} ∩ {*y* ∈ *S*. *0* < *f y*} = {} **for** *S* **and** *f* :: (*real*^'*n*::_) ⇒ *real*
    by *auto*
  have *integral S ?DP* = *integral* ({*x* ∈ *S*. *f* (*g x*) < *0*} ∪ {*x* ∈ *S*. *f* (*g x*) > *0*}) *?DP*
    by (*intro empty_imp_negligible integral_spike_set*) *auto*
  also have . . . = *integral* {*x* ∈ *S*. *f* (*g x*) < *0*} *?DP* + *integral* {*x* ∈ *S*. *0* < *f* (*g x*)} *?DP*
    using *aDN aDP* **by** (*simp add*: *set_lebesgue_integral_eq_integral*)
  also have . . . = − *integral* {*y* ∈ *T*. *f y* < *0*} (λ*x*. − *f x*) + *integral* {*y* ∈ *T*. *f y* > *0*} *f*
    using *DN DP* **by** (*auto simp*: *has_integral_iff*)
  also have . . . = *integral* ({*x* ∈ *T*. *f x* < *0*} ∪ {*x* ∈ *T*. *0* < *f x*}) *f*
    by (*simp add*: *fN fP*)
  also have . . . = *integral T f*
    by (*intro empty_imp_negligible integral_spike_set*) *auto*
  also have . . . = *b*
    using *intT* **by** *simp*
  finally show *integral S ?DP* = *b* **.**
 qed
 qed
qed

lemma *cv_inv_version3*:
  fixes *f* :: *real^'m::{finite,wellorder}* ⇒ *real^'n* **and** *g* :: *real^'m::_* ⇒ *real^'m::_*
  assumes *der_g*: ⋀*x*. *x* ∈ *S* ⟹ (*g has_derivative g' x*) (*at x within S*)
    and *der_h*: ⋀*y*. *y* ∈ *T* ⟹ (*h has_derivative h' y*) (*at y within T*)
    and *hg*: ⋀*x*. *x* ∈ *S* ⟹ *g x* ∈ *T* ∧ *h*(*g x*) = *x*
    and *gh*: ⋀*y*. *y* ∈ *T* ⟹ *h y* ∈ *S* ∧ *g*(*h y*) = *y*
    and *id*: ⋀*y*. *y* ∈ *T* ⟹ *h' y* ∘ *g'*(*h y*) = *id*
  shows (λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on S* ∧
        integral S (λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) = *b*
      ⟷ *f absolutely_integrable_on T* ∧ *integral T f* = *b*
proof −
  let *?D* = λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)
  have ((λ*x*. |*det* (*matrix* (*g' x*))| * *f*(*g x*) $ *i*) *absolutely_integrable_on S* ∧ *integral S* (λ*x*. |*det* (*matrix* (*g' x*))| * (*f*(*g x*) $ *i*)) = *b* $ *i*) ⟷
      ((λ*x*. *f x* $ *i*) *absolutely_integrable_on T* ∧ *integral T* (λ*x*. *f x* $ *i*) = *b* $ *i*)
for *i*
    by (*rule cov_invertible_real* [*OF der_g der_h hg gh id*])
  then have *?D absolutely_integrable_on S* ∧ (*?D has_integral b*) *S* ⟷
      f absolutely_integrable_on T ∧ (*f has_integral b*) *T*
   unfolding *absolutely_integrable_componentwise_iff* [**where** *f*=*f*] *has_integral_componentwise_iff* [*of f*]
      absolutely_integrable_componentwise_iff [**where** *f*=*?D*] *has_integral_componentwise_iff* [*of ?D*]
    by (*auto simp*: *all_conj_distrib Basis_vec_def cart_eq_inner_axis* [*symmetric*]
        has_integral_iff set_lebesgue_integral_eq_integral)
  then show *?thesis*
    using *absolutely_integrable_on_def* **by** *blast*
qed

lemma *cv_inv_version4*:
  fixes *f* :: *real^'m::{finite,wellorder}* ⇒ *real^'n* **and** *g* :: *real^'m::_* ⇒ *real^'m::_*
  assumes *der_g*: ⋀*x*. *x* ∈ *S* ⟹ (*g has_derivative g' x*) (*at x within S*) ∧ *invertible*(*matrix*(*g' x*))
    and *hg*: ⋀*x*. *x* ∈ *S* ⟹ *continuous_on* (*g* ‘ *S*) *h* ∧ *h*(*g x*) = *x*
  shows (λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on S* ∧
        integral S (λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) = *b*
      ⟷ *f absolutely_integrable_on* (*g* ‘ *S*) ∧ *integral* (*g* ‘ *S*) *f* = *b*
proof −
  have ∀ *x*. ∃ *h'*. *x* ∈ *S*
        ⟶ (*g has_derivative g' x*) (*at x within S*) ∧ *linear h'* ∧ *g' x* ∘ *h'* = *id* ∧ *h'* ∘ *g' x* = *id*
    using *der_g matrix_invertible has_derivative_linear* **by** *blast*
  then obtain *h'* **where** *h'*:
    ⋀*x*. *x* ∈ *S*
        ⟹ (*g has_derivative g' x*) (*at x within S*) ∧
          linear (*h' x*) ∧ *g' x* ∘ (*h' x*) = *id* ∧ (*h' x*) ∘ *g' x* = *id*
    by *metis*
  show *?thesis*

```
  proof (rule cv_inv_version3)
    show ⋀y. y ∈ g ` S ⟹ (h has_derivative h' (h y)) (at y within g ` S)
      using h' hg
     by (force simp: continuous_on_eq_continuous_within intro!: has_derivative_inverse_within)
  qed (use h' hg in auto)
qed


theorem has_absolute_integral_change_of_variables_invertible:
  fixes f :: real^'m::{finite,wellorder} ⇒ real^'n and g :: real^'m::_ ⇒ real^'m::_
  assumes der_g: ⋀x. x ∈ S ⟹ (g has_derivative g' x) (at x within S)
      and hg: ⋀x. x ∈ S ⟹ h(g x) = x
      and conth: continuous_on (g ` S) h
  shows (λx. |det (matrix (g' x))| *R f(g x)) absolutely_integrable_on S ∧ integral
S (λx. |det (matrix (g' x))| *R f(g x)) = b ⟷
         f absolutely_integrable_on (g ` S) ∧ integral (g ` S) f = b
    (is ?lhs = ?rhs)
proof −
  let ?S = {x ∈ S. invertible (matrix (g' x))} and ?D = λx. |det (matrix (g' x))|
*R f(g x)
  have *: ?D absolutely_integrable_on ?S ∧ integral ?S ?D = b
          ⟷ f absolutely_integrable_on (g ` ?S) ∧ integral (g ` ?S) f = b
  proof (rule cv_inv_version4)
    show (g has_derivative g' x) (at x within ?S) ∧ invertible (matrix (g' x))
      if x ∈ ?S for x
      using der_g that has_derivative_subset that by fastforce
    show continuous_on (g ` ?S) h ∧ h (g x) = x
      if x ∈ ?S for x
      using that continuous_on_subset [OF conth]  by (simp add: hg image_mono)
  qed
   have (g has_derivative g' x) (at x within {x ∈ S. rank (matrix (g' x)) <
CARD('m)}) if x ∈ S for x
    by (metis (no_types, lifting) der_g has_derivative_subset mem_Collect_eq subsetI
that)
  then have negligible (g ` {x ∈ S. ¬ invertible (matrix (g' x))})
    by (auto simp: invertible_det_nz det_eq_0_rank intro: baby_Sard)
  then have neg: negligible {x ∈ g ` S. x ∉ g ` ?S ∧ f x ≠ 0}
    by (auto intro: negligible_subset)
  have [simp]: {x ∈ g ` ?S. x ∉ g ` S ∧ f x ≠ 0} = {}
    by auto
  have ?D absolutely_integrable_on ?S ∧ integral ?S ?D = b
    ⟷ ?D absolutely_integrable_on S ∧ integral S ?D = b
    apply (intro conj_cong absolutely_integrable_spike_set_eq)
     apply(auto simp: integral_spike_set invertible_det_nz empty_imp_negligible neg)
    done
  moreover
  have f absolutely_integrable_on (g ` ?S) ∧ integral (g ` ?S) f = b
    ⟷ f absolutely_integrable_on (g ` S) ∧ integral (g ` S) f = b
     by (auto intro!: conj_cong absolutely_integrable_spike_set_eq integral_spike_set
```

neg)
ultimately
show *?thesis*
using ∗ **by** *blast*
qed

theorem *has_absolute_integral_change_of_variables_compact*:
fixes *f* :: *real*ˆ′*m*::{*finite*,*wellorder*} ⇒ *real*ˆ′*n* **and** *g* :: *real*ˆ′*m*::_ ⇒ *real*ˆ′*m*::_
assumes *compact S*
and *der_g*: $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative g′ x*) (*at x within S*)
and *inj*: *inj_on g S*
shows ((λ*x*. |*det* (*matrix* (*g′ x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on S* ∧
integral S (λ*x*. |*det* (*matrix* (*g′ x*))| $*_R$ *f*(*g x*)) = *b*
⟷ *f absolutely_integrable_on* (*g* ‘ *S*) ∧ *integral* (*g* ‘ *S*) *f* = *b*)
proof −
obtain *h* **where** *hg*: $\bigwedge x.\ x \in S \Longrightarrow h(g\ x) = x$
using *inj* **by** (*metis the_inv_into_f_f*)
have *conth*: *continuous_on* (*g* ‘ *S*) *h*
by (*metis* ‹*compact S*› *continuous_on_inv der_g has_derivative_continuous_on hg*)
show *?thesis*
by (*rule has_absolute_integral_change_of_variables_invertible* [*OF der_g hg conth*])
qed

lemma *has_absolute_integral_change_of_variables_compact_family*:
fixes *f* :: *real*ˆ′*m*::{*finite*,*wellorder*} ⇒ *real*ˆ′*n* **and** *g* :: *real*ˆ′*m*::_ ⇒ *real*ˆ′*m*::_
assumes *compact*: ⋀*n*::*nat*. *compact* (*F n*)
and *der_g*: ⋀*x*. *x* ∈ (⋃*n*. *F n*) ⟹ (*g has_derivative g′ x*) (*at x within* (⋃*n*. *F n*))
and *inj*: *inj_on g* (⋃*n*. *F n*)
shows ((λ*x*. |*det* (*matrix* (*g′ x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on* (⋃*n*. *F n*) ∧
integral (⋃*n*. *F n*) (λ*x*. |*det* (*matrix* (*g′ x*))| $*_R$ *f*(*g x*)) = *b*
⟷ *f absolutely_integrable_on* (*g* ‘ (⋃*n*. *F n*)) ∧ *integral* (*g* ‘ (⋃*n*. *F n*)) *f* = *b*)
proof −
let *?D* = λ*x*. |*det* (*matrix* (*g′ x*))| $*_R$ *f* (*g x*)
let *?U* = λ*n*. ⋃*m*≤*n*. *F m*
let *?lift* = *vec*::*real*⇒*real*ˆ*1*
have *F_leb*: *F m* ∈ *sets lebesgue* **for** *m*
by (*simp add*: *compact borel_compact*)
have *iff*: (λ*x*. |*det* (*matrix* (*g′ x*))| $*_R$ *f* (*g x*)) *absolutely_integrable_on* (*?U n*) ∧
integral (*?U n*) (λ*x*. |*det* (*matrix* (*g′ x*))| $*_R$ *f* (*g x*)) = *b*
⟷ *f absolutely_integrable_on* (*g* ‘ (*?U n*)) ∧ *integral* (*g* ‘ (*?U n*)) *f* = *b*
for *n b* **and** *f* :: *real*ˆ′*m*::_ ⇒ *real*ˆ′*k*
proof (*rule has_absolute_integral_change_of_variables_compact*)

```
    show compact (?U n)
      by (simp add: compact compact_UN)
    show (g has_derivative g' x) (at x within (?U n))
      if x ∈ ?U n for x
      using that by (blast intro!: has_derivative_subset [OF der_g])
    show inj_on g (?U n)
      using inj by (auto simp: inj_on_def)
  qed
  show ?thesis
    unfolding image_UN
  proof safe
    assume DS: ?D absolutely_integrable_on (⋃n. F n)
      and b: b = integral (⋃n. F n) ?D
    have DU: ⋀n. ?D absolutely_integrable_on (?U n)
             (λn. integral (?U n) ?D) ⟶ integral (⋃n. F n) ?D
      using integral_countable_UN [OF DS F_leb] by auto
    with iff have fag: f absolutely_integrable_on g ` (?U n)
      and fg_int: integral (⋃m≤n. g ` F m) f = integral (?U n) ?D for n
      by (auto simp: image_UN)
    let ?h = λx. if x ∈ (⋃m. g ` F m) then norm(f x) else 0
    have (λx. if x ∈ (⋃m. g ` F m) then f x else 0) absolutely_integrable_on UNIV
    proof (rule dominated_convergence_absolutely_integrable)
      show (λx. if x ∈ (⋃m≤k. g ` F m) then f x else 0) absolutely_integrable_on
UNIV for k
        unfolding absolutely_integrable_restrict_UNIV
        using fag by (simp add: image_UN)
      let ?nf = λn x. if x ∈ (⋃m≤n. g ` F m) then norm(f x) else 0
      show ?h integrable_on UNIV
      proof (rule monotone_convergence_increasing [THEN conjunct1])
        show ?nf k integrable_on UNIV for k
          using fag
          unfolding integrable_restrict_UNIV absolutely_integrable_on_def by (simp
add: image_UN)
        { fix n
          have (norm ∘ ?D) absolutely_integrable_on ?U n
            by (intro absolutely_integrable_norm DU)
          then have integral (g ` ?U n) (norm ∘ f) = integral (?U n) (norm ∘ ?D)
            using iff [of n vec ∘ norm ∘ f integral (?U n) (λx. |det (matrix (g' x))|
*R (?lift ∘ norm ∘ f) (g x))]
            unfolding absolutely_integrable_on_1_iff integral_on_1_eq by (auto simp:
o_def)
        }
        moreover have bounded (range (λk. integral (?U k) (norm ∘ ?D)))
          unfolding bounded_iff
        proof (rule exI, clarify)
          fix k
          show norm (integral (?U k) (norm ∘ ?D)) ≤ integral (⋃n. F n) (norm
∘ ?D)
            unfolding integral_restrict_UNIV [of _ norm ∘ ?D, symmetric]
```

proof (*rule integral_norm_bound_integral*)
show (λx. *if* $x \in \bigcup$ (*F* ‘ {..*k*}) *then* (*norm* ∘ *?D*) *x else 0*) *integrable_on UNIV*
(λx. *if* $x \in$ ($\bigcup n$. *F n*) *then* (*norm* ∘ *?D*) *x else 0*) *integrable_on UNIV*
using *DU*(*1*) *DS*
unfolding *absolutely_integrable_on_def o_def integrable_restrict_UNIV* **by** *auto*
qed *auto*
qed
ultimately show *bounded* (*range* (λk. *integral UNIV* (*?nf k*)))
by (*simp add*: *integral_restrict_UNIV image_UN* [*symmetric*] *o_def*)
next
show (λk. *if* $x \in$ ($\bigcup m \le k$. *g* ‘ *F m*) *then norm* (*f x*) *else 0*)
⟶ (*if* $x \in$ ($\bigcup m$. *g* ‘ *F m*) *then norm* (*f x*) *else 0*) **for** *x*
by (*force intro*: *tendsto_eventually eventually_sequentiallyI*)
qed *auto*
next
show (λk. *if* $x \in$ ($\bigcup m \le k$. *g* ‘ *F m*) *then f x else 0*)
⟶ (*if* $x \in$ ($\bigcup m$. *g* ‘ *F m*) *then f x else 0*) **for** *x*
proof *clarsimp*
fix *m y*
assume $y \in F\ m$
show (λk. *if* $\exists x \in \{..k\}$. *g y* $\in$ *g* ‘ *F x then f* (*g y*) *else 0*) ⟶ *f* (*g y*)
using ‹$y \in F\ m$› **by** (*force intro*: *tendsto_eventually eventually_sequentiallyI* [*of m*])
qed
qed *auto*
then show *fai*: *f absolutely_integrable_on* ($\bigcup m$. *g* ‘ *F m*)
using *absolutely_integrable_restrict_UNIV* **by** *blast*
show *integral* (($\bigcup x$. *g* ‘ *F x*)) *f* = *integral* ($\bigcup n$. *F n*) *?D*
proof (*rule LIMSEQ_unique*)
show (λn. *integral* (*?U n*) *?D*) ⟶ *integral* ($\bigcup x$. *g* ‘ *F x*) *f*
unfolding *fg_int* [*symmetric*]
proof (*rule integral_countable_UN* [*OF fai*])
show *g* ‘ *F m* $\in$ *sets lebesgue* **for** *m*
proof (*rule differentiable_image_in_sets_lebesgue* [*OF F_leb*])
show *g differentiable_on F m*
by (*meson der_g differentiableI UnionI differentiable_on_def differentiable_on_subset rangeI subsetI*)
qed *auto*
qed
next
show (λn. *integral* (*?U n*) *?D*) ⟶ *integral* ($\bigcup n$. *F n*) *?D*
by (*rule DU*)
qed
next
assume *fs*: *f absolutely_integrable_on* ($\bigcup x$. *g* ‘ *F x*)
and *b*: *b* = *integral* (($\bigcup x$. *g* ‘ *F x*)) *f*
have *gF_leb*: *g* ‘ *F m* $\in$ *sets lebesgue* **for** *m*

proof (*rule differentiable_image_in_sets_lebesgue* [*OF F_leb*])
show *g differentiable_on F m*
using *der_g* **unfolding** *differentiable_def differentiable_on_def*
by (*meson Sup_upper UNIV_I UnionI has_derivative_subset image_eqI*)
qed *auto*
have *fgU*: $\bigwedge n.$ *f absolutely_integrable_on* ($\bigcup m \le n.$ *g* ‘ *F m*)
($\lambda n.$ *integral* ($\bigcup m \le n.$ *g* ‘ *F m*) *f*) $\longrightarrow$ *integral* ($\bigcup m.$ *g* ‘ *F m*) *f*
using *integral_countable_UN* [*OF fs gF_leb*] **by** *auto*
with *iff* **have** *DUn*: *?D absolutely_integrable_on ?U n*
and *D_int*: *integral* (*?U n*) *?D* = *integral* ($\bigcup m \le n.$ *g* ‘ *F m*) *f* **for** *n*
by (*auto simp*: *image_UN*)
let *?h* = $\lambda x.$ *if* $x \in$ ($\bigcup n.$ *F n*) *then norm*(*?D x*) *else 0*
have ($\lambda x.$ *if* $x \in$ ($\bigcup n.$ *F n*) *then ?D x else 0*) *absolutely_integrable_on UNIV*
proof (*rule dominated_convergence_absolutely_integrable*)
show ($\lambda x.$ *if* $x \in$ *?U k then ?D x else 0*) *absolutely_integrable_on UNIV* **for** *k*
unfolding *absolutely_integrable_restrict_UNIV* **using** *DUn* **by** *simp*
let *?nD* = $\lambda n\ x.$ *if* $x \in$ *?U n then norm*(*?D x*) *else 0*
show *?h integrable_on UNIV*
proof (*rule monotone_convergence_increasing* [*THEN conjunct1*])
show *?nD k integrable_on UNIV* **for** *k*
using *DUn*
unfolding *integrable_restrict_UNIV absolutely_integrable_on_def* **by** (*simp add*: *image_UN*)
{ **fix** *n*::*nat*
have (*norm* ∘ *f*) *absolutely_integrable_on* ($\bigcup m \le n.$ *g* ‘ *F m*)
apply (*rule absolutely_integrable_norm*)
using *fgU* **by** *blast*
then have *integral* (*?U n*) (*norm* ∘ *?D*) = *integral* (*g* ‘ *?U n*) (*norm* ∘ *f*)
using *iff* [*of n ?lift* ∘ *norm* ∘ *f integral* (*g* ‘ *?U n*) (*?lift* ∘ *norm* ∘ *f*)]
unfolding *absolutely_integrable_on_1_iff integral_on_1_eq image_UN* **by** (*auto simp*: *o_def*)
}
moreover have *bounded* (*range* ($\lambda k.$ *integral* (*g* ‘ *?U k*) (*norm* ∘ *f*)))
unfolding *bounded_iff*
proof (*rule exI*, *clarify*)
fix *k*
show *norm* (*integral* (*g* ‘ *?U k*) (*norm* ∘ *f*)) ≤ *integral* (*g* ‘ ($\bigcup n.$ *F n*)) (*norm* ∘ *f*)
unfolding *integral_restrict_UNIV* [*of _ norm* ∘ *f*, *symmetric*]
proof (*rule integral_norm_bound_integral*)
show ($\lambda x.$ *if* $x \in g$ ‘ *?U k then* (*norm* ∘ *f*) *x else 0*) *integrable_on UNIV*
($\lambda x.$ *if* $x \in g$ ‘ ($\bigcup n.$ *F n*) *then* (*norm* ∘ *f*) *x else 0*) *integrable_on UNIV*
using *fgU fs*
unfolding *absolutely_integrable_on_def o_def integrable_restrict_UNIV*
by (*auto simp*: *image_UN*)
qed *auto*
qed
ultimately show *bounded* (*range* ($\lambda k.$ *integral UNIV* (*?nD k*)))

unfolding *integral_restrict_UNIV image_UN* [*symmetric*] *o_def* **by** *simp*
next
show (λk. *if* $x \in$ *?U k then norm* (*?D x*) *else 0*) $\longrightarrow$ (*if* $x \in (\bigcup n.\ F\ n)$ *then norm* (*?D x*) *else 0*) **for** x
by (*force intro*: *tendsto_eventually eventually_sequentiallyI*)
qed *auto*
next
show (λk. *if* $x \in$ *?U k then ?D x else 0*) $\longrightarrow$ (*if* $x \in (\bigcup n.\ F\ n)$ *then ?D x else 0*) **for** x
proof *clarsimp*
fix n
assume $x \in F\ n$
show (λm. *if* $\exists j \in \{..m\}$. $x \in F\ j$ *then ?D x else 0*) $\longrightarrow$ *?D x*
using ‹$x \in F\ n$› **by** (*auto intro*!: *tendsto_eventually eventually_sequentiallyI* [*of n*])
qed
qed *auto*
then show *Dai*: *?D absolutely_integrable_on* ($\bigcup n.\ F\ n$)
unfolding *absolutely_integrable_restrict_UNIV* **by** *simp*
show *integral* ($\bigcup n.\ F\ n$) *?D* = *integral* (($\bigcup x.\ g$ ‘ $F\ x$)) f
proof (*rule LIMSEQ_unique*)
show (λn. *integral* ($\bigcup m \le n.\ g$ ‘ $F\ m$) f) $\longrightarrow$ *integral* ($\bigcup x.\ g$ ‘ $F\ x$) f
by (*rule fgU*)
show (λn. *integral* ($\bigcup m \le n.\ g$ ‘ $F\ m$) f) $\longrightarrow$ *integral* ($\bigcup n.\ F\ n$) *?D*
unfolding *D_int* [*symmetric*] **by** (*rule integral_countable_UN* [*OF Dai F_leb*])
qed
qed
qed

theorem *has_absolute_integral_change_of_variables*:
fixes f :: *real^'m::{finite,wellorder}* $\Rightarrow$ *real^'n* **and** g :: *real^'m::_* $\Rightarrow$ *real^'m::_*
assumes S: $S \in$ *sets lebesgue*
and *der_g*: $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative g' x*) (*at x within S*)
and *inj*: *inj_on g S*
shows (λx. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on* S $\wedge$
integral S (λx. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) = b
$\longleftrightarrow$ *f absolutely_integrable_on* (*g* ‘ *S*) $\wedge$ *integral* (*g* ‘ *S*) $f = b$
proof −
obtain *C N* **where** *fsigma C* **and** *N*: $N \in$ *null_sets lebesgue* **and** *CNS*: $C \cup N = S$ **and** *disjnt C N*
using *lebesgue_set_almost_fsigma* [*OF S*] **.**
then obtain F :: *nat* $\Rightarrow$ (*real^'m::_*) *set*
where *F*: *range F* $\subseteq$ *Collect compact* **and** *Ceq*: C = *Union*(*range F*)
using *fsigma_Union_compact* **by** *metis*
have *negligible N*
using *N* **by** (*simp add*: *negligible_iff_null_sets*)
let *?D* = λx. |*det* (*matrix* (*g' x*))| $*_R$ *f* (*g x*)
have *?D absolutely_integrable_on C* $\wedge$ *integral C ?D* = b

⟷ *f absolutely_integrable_on* (*g* ' *C*) ∧ *integral* (*g* ' *C*) *f* = *b*
unfolding *Ceq*
proof (*rule has_absolute_integral_change_of_variables_compact_family*)
fix *n x*
assume *x* ∈ ⋃(*F* ' *UNIV*)
then show (*g has_derivative g' x*) (*at x within* ⋃(*F* ' *UNIV*))
using *Ceq* ‹*C* ∪ *N* = *S*› *der_g has_derivative_subset* **by** *blast*
next
have ⋃(*F* ' *UNIV*) ⊆ *S*
using *Ceq* ‹*C* ∪ *N* = *S*› **by** *blast*
then show *inj_on g* (⋃(*F* ' *UNIV*))
using *inj* **by** (*meson inj_on_subset*)
qed (*use F* **in** *auto*)
moreover
have *?D absolutely_integrable_on C* ∧ *integral C ?D* = *b*
⟷ *?D absolutely_integrable_on S* ∧ *integral S ?D* = *b*
proof (*rule conj_cong*)
have *neg*: *negligible* {*x* ∈ *C* − *S*. *?D x* ≠ *0*} *negligible* {*x* ∈ *S* − *C*. *?D x* ≠ *0*}
using *CNS* **by** (*blast intro*: *negligible_subset* [*OF* ‹*negligible N*›])+
then show (*?D absolutely_integrable_on C*) = (*?D absolutely_integrable_on S*)
by (*rule absolutely_integrable_spike_set_eq*)
show (*integral C ?D* = *b*) ⟷ (*integral S ?D* = *b*)
using *integral_spike_set* [*OF neg*] **by** *simp*
qed
moreover
have *f absolutely_integrable_on* (*g* ' *C*) ∧ *integral* (*g* ' *C*) *f* = *b*
⟷ *f absolutely_integrable_on* (*g* ' *S*) ∧ *integral* (*g* ' *S*) *f* = *b*
proof (*rule conj_cong*)
have *g differentiable_on N*
by (*metis CNS der_g differentiable_def differentiable_on_def differentiable_on_subset sup.cobounded2*)
with ‹*negligible N*›
have *neg_gN*: *negligible* (*g* ' *N*)
by (*blast intro*: *negligible_differentiable_image_negligible*)
have *neg*: *negligible* {*x* ∈ *g* ' *C* − *g* ' *S*. *f x* ≠ *0*}
negligible {*x* ∈ *g* ' *S* − *g* ' *C*. *f x* ≠ *0*}
using *CNS* **by** (*blast intro*: *negligible_subset* [*OF neg_gN*])+
then show (*f absolutely_integrable_on g* ' *C*) = (*f absolutely_integrable_on g* ' *S*)
by (*rule absolutely_integrable_spike_set_eq*)
show (*integral* (*g* ' *C*) *f* = *b*) ⟷ (*integral* (*g* ' *S*) *f* = *b*)
using *integral_spike_set* [*OF neg*] **by** *simp*
qed
ultimately show *?thesis*
by *simp*
qed

corollary *absolutely_integrable_change_of_variables*:
fixes f :: *real^'m::{finite,wellorder}* ⇒ *real^'n* **and** g :: *real^'m::_* ⇒ *real^'m::_*
assumes $S \in$ *sets lebesgue*
and $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative g' x*) (*at x within S*)
and *inj_on g S*
shows *f absolutely_integrable_on* (*g ‘ S*)
$\longleftrightarrow$ ($\lambda x.$ |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on S*
using *assms has_absolute_integral_change_of_variables* **by** *blast*

corollary *integral_change_of_variables*:
fixes f :: *real^'m::{finite,wellorder}* ⇒ *real^'n* **and** g :: *real^'m::_* ⇒ *real^'m::_*
assumes *S*: $S \in$ *sets lebesgue*
and *der_g*: $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative g' x*) (*at x within S*)
and *inj*: *inj_on g S*
and *disj*: (*f absolutely_integrable_on* (*g ‘ S*) ∨
($\lambda x.$ |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on S*)
shows *integral* (*g ‘ S*) *f* = *integral S* ($\lambda x.$ |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*))
using *has_absolute_integral_change_of_variables* [*OF S der_g inj*] *disj*
by *blast*

lemma *has_absolute_integral_change_of_variables_1*:
fixes f :: *real* ⇒ *real^'n::{finite,wellorder}* **and** g :: *real* ⇒ *real*
assumes *S*: $S \in$ *sets lebesgue*
and *der_g*: $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_vector_derivative g' x*) (*at x within S*)
and *inj*: *inj_on g S*
shows ($\lambda x.$ |*g' x*| $*_R$ *f*(*g x*)) *absolutely_integrable_on S* ∧
integral S ($\lambda x.$ |*g' x*| $*_R$ *f*(*g x*)) = *b*
$\longleftrightarrow$ *f absolutely_integrable_on* (*g ‘ S*) ∧ *integral* (*g ‘ S*) *f* = *b*
proof −
let *?lift* = *vec* :: *real* ⇒ *real^1*
let *?drop* = (λx::*real^1*. *x* $ *1*)
have *S'*: *?lift ‘ S* ∈ *sets lebesgue*
by (*auto intro*: *differentiable_image_in_sets_lebesgue* [*OF S*] *differentiable_vec*)
have (($\lambda x.$ *vec* (*g* (*x* $ *1*))) *has_derivative* ($*_R$) (*g' z*)) (*at* (*vec z*) *within ?lift ‘ S*)
if $z \in S$ **for** *z*
using *der_g* [*OF that*]
by (*simp add*: *has_vector_derivative_def has_derivative_vector_1*)
then have *der'*: $\bigwedge x.\ x \in$ *?lift ‘ S* $\Longrightarrow$
(*?lift* ∘ *g* ∘ *?drop has_derivative* ($*_R$) (*g'* (*?drop x*))) (*at x within ?lift ‘ S*)
by (*auto simp*: *o_def*)
have *inj'*: *inj_on* (*vec* ∘ *g* ∘ *?drop*) (*vec ‘ S*)
using *inj* **by** (*simp add*: *inj_on_def*)
let *?fg* = $\lambda x.$ |*g' x*| $*_R$ *f*(*g x*)
have (($\lambda x.$ *?fg x* $ *i*) *absolutely_integrable_on S* ∧ (($\lambda x.$ *?fg x* $ *i*) *has_integral b* $ *i*) *S*
$\longleftrightarrow$ ($\lambda x.$ *f x* $ *i*) *absolutely_integrable_on g ‘ S* ∧ (($\lambda x.$ *f x* $ *i*) *has_integral b* $ *i*) (*g ‘ S*)) **for** *i*
using *has_absolute_integral_change_of_variables* [*OF S' der' inj'*, *of* $\lambda x.$ *?lift*(*f*

(*?drop x*) $ *i*) *?lift* (*b*$*i*)]
  unfolding *integrable_on_1_iff integral_on_1_eq absolutely_integrable_on_1_iff absolutely_integrable_drop absolutely_integrable_on_def*
  by (*auto simp*: *image_comp o_def integral_vec1_eq has_integral_iff*)
 then have *?fg absolutely_integrable_on S* ∧ (*?fg has_integral b*) *S*
  ⟷ *f absolutely_integrable_on* (*g* ‘ *S*) ∧ (*f has_integral b*) (*g* ‘ *S*)
  unfolding *has_integral_componentwise_iff* [**where** *y*=*b*]
   absolutely_integrable_componentwise_iff [**where** *f*=*f*]
   absolutely_integrable_componentwise_iff [**where** *f* = *?fg*]
  by (*force simp*: *Basis_vec_def cart_eq_inner_axis*)
 then show *?thesis*
  using *absolutely_integrable_on_def* **by** *blast*
qed

corollary *absolutely_integrable_change_of_variables_1*:
 fixes *f* :: *real* ⇒ *real*^′*n*::{*finite*,*wellorder*} **and** *g* :: *real* ⇒ *real*
 assumes *S*: *S* ∈ *sets lebesgue*
  and *der_g*: ⋀*x*. *x* ∈ *S* ⟹ (*g has_vector_derivative g*′ *x*) (*at x within S*)
  and *inj*: *inj_on g S*
 shows (*f absolutely_integrable_on g* ‘ *S* ⟷
   (λ*x*. |*g*′ *x*| $*_R$ *f*(*g x*)) *absolutely_integrable_on S*)
 using *has_absolute_integral_change_of_variables_1* [*OF assms*] **by** *auto*

6.46.6 Change of variables for integrals: special case of linear function

lemma *has_absolute_integral_change_of_variables_linear*:
 fixes *f* :: *real*^′*m*::{*finite*,*wellorder*} ⇒ *real*^′*n* **and** *g* :: *real*^′*m*::_ ⇒ *real*^′*m*::_
 assumes *linear g*
 shows (λ*x*. |*det* (*matrix g*)| $*_R$ *f*(*g x*)) *absolutely_integrable_on S* ∧
   integral S (λ*x*. |*det* (*matrix g*)| $*_R$ *f*(*g x*)) = *b*
  ⟷ *f absolutely_integrable_on* (*g* ‘ *S*) ∧ *integral* (*g* ‘ *S*) *f* = *b*
proof (*cases det*(*matrix g*) = *0*)
 case *True*
 then have *negligible*(*g* ‘ *S*)
  using *assms det_nz_iff_inj negligible_linear_singular_image* **by** *blast*
 with *True* **show** *?thesis*
  by (*auto simp*: *absolutely_integrable_on_def integrable_negligible integral_negligible*)
next
 case *False*
 then obtain *h* **where** *h*: ⋀*x*. *x* ∈ *S* ⟹ *h* (*g x*) = *x linear h*
  using *assms det_nz_iff_inj linear_injective_isomorphism* **by** *metis*
 show *?thesis*
 proof (*rule has_absolute_integral_change_of_variables_invertible*)
  show (*g has_derivative g*) (*at x within S*) **for** *x*
   by (*simp add*: *assms linear_imp_has_derivative*)
  show *continuous_on* (*g* ‘ *S*) *h*
  using *continuous_on_eq_continuous_within has_derivative_continuous linear_imp_has_derivative*

h **by** *blast*
  qed (*use h* **in** *auto*)
qed

lemma *absolutely_integrable_change_of_variables_linear*:
  fixes *f* :: *real^'m::{finite,wellorder}* ⇒ *real^'n* **and** *g* :: *real^'m::_* ⇒ *real^'m::_*
  assumes *linear g*
  shows (*λx.* |*det* (*matrix g*)| $*_R$ *f*(*g x*)) *absolutely_integrable_on S*
    ⟷ *f absolutely_integrable_on* (*g* ' *S*)
  using *assms has_absolute_integral_change_of_variables_linear* **by** *blast*

lemma *absolutely_integrable_on_linear_image*:
  fixes *f* :: *real^'m::{finite,wellorder}* ⇒ *real^'n* **and** *g* :: *real^'m::_* ⇒ *real^'m::_*
  assumes *linear g*
  shows *f absolutely_integrable_on* (*g* ' *S*)
    ⟷ (*f* ∘ *g*) *absolutely_integrable_on S* ∨ *det*(*matrix g*) = *0*
 unfolding *assms absolutely_integrable_change_of_variables_linear* [*OF assms*, *symmetric*] *absolutely_integrable_on_scaleR_iff*
  by (*auto simp*: *set_integrable_def*)

lemma *integral_change_of_variables_linear*:
  fixes *f* :: *real^'m::{finite,wellorder}* ⇒ *real^'n* **and** *g* :: *real^'m::_* ⇒ *real^'m::_*
  assumes *linear g*
      and *f absolutely_integrable_on* (*g* ' *S*) ∨ (*f* ∘ *g*) *absolutely_integrable_on S*
    shows *integral* (*g* ' *S*) *f* = |*det* (*matrix g*)| $*_R$ *integral S* (*f* ∘ *g*)
proof −
  have ((*λx.* |*det* (*matrix g*)| $*_R$ *f* (*g x*)) *absolutely_integrable_on S*) ∨ (*f absolutely_integrable_on g* ' *S*)
    using *absolutely_integrable_on_linear_image assms* **by** *blast*
  moreover
  have *?thesis* **if** ((*λx.* |*det* (*matrix g*)| $*_R$ *f* (*g x*)) *absolutely_integrable_on S*) (*f absolutely_integrable_on g* ' *S*)
    using *has_absolute_integral_change_of_variables_linear* [*OF* ‹*linear g*›] *that*
    by (*auto simp*: *o_def*)
  ultimately show *?thesis*
    using *absolutely_integrable_change_of_variables_linear* [*OF* ‹*linear g*›]
    by *blast*
qed

6.46.7 Change of variable for measure

lemma *has_measure_differentiable_image*:
  fixes *f* :: *real^'n::{finite,wellorder}* ⇒ *real^'n::_*
  assumes *S* ∈ *sets lebesgue*
      and ⋀*x. x* ∈ *S* ⟹ (*f has_derivative f' x*) (*at x within S*)
      and *inj_on f S*
  shows *f* ' *S* ∈ *lmeasurable* ∧ *measure lebesgue* (*f* ' *S*) = *m*
    ⟷ ((*λx.* |*det* (*matrix* (*f' x*))|) *has_integral m*) *S*
 using *has_absolute_integral_change_of_variables* [*OF assms*, *of λx.* (*1::real^1*) *vec*

m]
  unfolding *absolutely_integrable_on_1_iff integral_on_1_eq integrable_on_1_iff absolutely_integrable_on_def*
  by (*auto simp*: *has_integral_iff lmeasurable_iff_integrable_on lmeasure_integral*)

lemma *measurable_differentiable_image_eq*:
  fixes f :: *real*^'*n*::{*finite*,*wellorder*} ⇒ *real*^'*n*::_
  assumes $S \in$ *sets lebesgue*
    and $\bigwedge x.\ x \in S \Longrightarrow$ (*f has_derivative f' x*) (*at x within S*)
    and *inj_on f S*
  shows *f* ‘ $S \in$ *lmeasurable* ⟷ (λ*x*. |*det* (*matrix* (*f' x*))|) *integrable_on S*
  using *has_measure_differentiable_image* [*OF assms*]
  by *blast*

lemma *measurable_differentiable_image_alt*:
  fixes f :: *real*^'*n*::{*finite*,*wellorder*} ⇒ *real*^'*n*::_
  assumes $S \in$ *sets lebesgue*
    and $\bigwedge x.\ x \in S \Longrightarrow$ (*f has_derivative f' x*) (*at x within S*)
    and *inj_on f S*
  shows *f* ‘ $S \in$ *lmeasurable* ⟷ (λ*x*. |*det* (*matrix* (*f' x*))|) *absolutely_integrable_on S*
  using *measurable_differentiable_image_eq* [*OF assms*]
  by (*simp only*: *absolutely_integrable_on_iff_nonneg*)

lemma *measure_differentiable_image_eq*:
  fixes f :: *real*^'*n*::{*finite*,*wellorder*} ⇒ *real*^'*n*::_
  assumes *S*: $S \in$ *sets lebesgue*
    and *der_f*: $\bigwedge x.\ x \in S \Longrightarrow$ (*f has_derivative f' x*) (*at x within S*)
    and *inj*: *inj_on f S*
    and *intS*: (λ*x*. |*det* (*matrix* (*f' x*))|) *integrable_on S*
  shows *measure lebesgue* (*f* ‘ *S*) = *integral S* (λ*x*. |*det* (*matrix* (*f' x*))|)
  using *measurable_differentiable_image_eq* [*OF S der_f inj*]
      assms has_measure_differentiable_image **by** *blast*

end

6.47 Lipschitz Continuity

theory *Lipschitz*
  imports
    Derivative
begin

definition *lipschitz_on*
  where *lipschitz_on C U f* ⟷ ($0 \le C$ ∧ (∀ $x \in U$. ∀ $y \in U$. *dist* (*f x*) (*f y*) ≤ *C* ∗ *dist x y*))

bundle *lipschitz_syntax* **begin**
notation *lipschitz_on* (_−*lipschitz'_on* [*1000*])

end
bundle *no_lipschitz_syntax* **begin**
no_notation *lipschitz_on* (*_−lipschitz′_on* [*1000*])
end

unbundle *lipschitz_syntax*

lemma *lipschitz_onI*: *L−lipschitz_on X f*
  if ⋀*x y*. *x* ∈ *X* ⟹ *y* ∈ *X* ⟹ *dist* (*f x*) (*f y*) ≤ *L* ∗ *dist x y 0* ≤ *L*
  using *that* **by** (*auto simp*: *lipschitz_on_def*)

lemma *lipschitz_onD*:
  dist (*f x*) (*f y*) ≤ *L* ∗ *dist x y*
  if *L−lipschitz_on X f x* ∈ *X y* ∈ *X*
  using *that* **by** (*auto simp*: *lipschitz_on_def*)

lemma *lipschitz_on_nonneg*:
  0 ≤ *L* **if** *L−lipschitz_on X f*
  using *that* **by** (*auto simp*: *lipschitz_on_def*)

lemma *lipschitz_on_normD*:
  norm (*f x* − *f y*) ≤ *L* ∗ *norm* (*x* − *y*)
  if *lipschitz_on L X f x* ∈ *X y* ∈ *X*
  using *lipschitz_onD*[*OF that*]
  by (*simp add*: *dist_norm*)

lemma *lipschitz_on_mono*: *L−lipschitz_on D f* **if** *M−lipschitz_on E f D* ⊆ *E M* ≤
L
  using *that*
  by (*force simp*: *lipschitz_on_def intro*: *order_trans*[*OF _ mult_right_mono*])

lemmas *lipschitz_on_subset* = *lipschitz_on_mono*[*OF _ _ order_refl*]
  and *lipschitz_on_le* = *lipschitz_on_mono*[*OF _ order_refl*]

lemma *lipschitz_on_leI*:
  L−lipschitz_on X f
  if ⋀*x y*. *x* ∈ *X* ⟹ *y* ∈ *X* ⟹ *x* ≤ *y* ⟹ *dist* (*f x*) (*f y*) ≤ *L* ∗ *dist x y*
    0 ≤ *L*
  for *f*::*′a*::{*linorder_topology*, *ordered_real_vector*, *metric_space*} ⇒ *′b*::*metric_space*
proof (*rule lipschitz_onI*)
  fix *x y* **assume** *xy*: *x* ∈ *X y* ∈ *X*
  consider *y* ≤ *x* | *x* ≤ *y*
    by (*rule le_cases*)
  then show *dist* (*f x*) (*f y*) ≤ *L* ∗ *dist x y*
  proof *cases*
    case *1*
    then have *dist* (*f y*) (*f x*) ≤ *L* ∗ *dist y x*
      by (*auto intro*!: *that xy*)
    then show *?thesis*

```
    by (simp add: dist_commute)
  qed (auto intro!: that xy)
qed fact

lemma lipschitz_on_concat:
  fixes a b c::real
  assumes f: L−lipschitz_on {a .. b} f
  assumes g: L−lipschitz_on {b .. c} g
  assumes fg: f b = g b
  shows lipschitz_on L {a .. c} (λx. if x ≤ b then f x else g x)
    (is lipschitz_on _ _ ?f)
proof (rule lipschitz_on_leI)
  fix x y
  assume x: x ∈ {a..c} and y: y ∈ {a..c} and xy: x ≤ y
  consider x ≤ b ∧ b < y | x ≥ b ∨ y ≤ b by arith
  then show dist (?f x) (?f y) ≤ L ∗ dist x y
  proof cases
    case 1
    have dist (f x) (g y) ≤ dist (f x) (f b) + dist (g b) (g y)
      unfolding fg by (rule dist_triangle)
    also have dist (f x) (f b) ≤ L ∗ dist x b
      using 1 x
      by (auto intro!: lipschitz_onD[OF f])
    also have dist (g b) (g y) ≤ L ∗ dist b y
      using 1 x y
      by (auto intro!: lipschitz_onD[OF g] lipschitz_onD[OF f])
    finally have dist (f x) (g y) ≤ L ∗ dist x b + L ∗ dist b y
      by simp
    also have . . . = L ∗ (dist x b + dist b y)
      by (simp add: algebra_simps)
    also have dist x b + dist b y = dist x y
      using 1 x y
      by (auto simp: dist_real_def abs_real_def)
    finally show ?thesis
      using 1 by simp
  next
    case 2
    with lipschitz_onD[OF f, of x y] lipschitz_onD[OF g, of x y] x y xy
    show ?thesis
      by (auto simp: fg)
  qed
qed (rule lipschitz_on_nonneg[OF f])

lemma lipschitz_on_concat_max:
  fixes a b c::real
  assumes f: L−lipschitz_on {a .. b} f
  assumes g: M−lipschitz_on {b .. c} g
  assumes fg: f b = g b
  shows (max L M)−lipschitz_on {a .. c} (λx. if x ≤ b then f x else g x)
```

proof −
have *lipschitz_on* (*max L M*) {*a* .. *b*} *f lipschitz_on* (*max L M*) {*b* .. *c*} *g*
by (*auto intro*!: *lipschitz_on_mono*[*OF f order_refl*] *lipschitz_on_mono*[*OF g order_refl*])
from *lipschitz_on_concat*[*OF this fg*] **show** *?thesis* **.**
qed

Continuity

proposition *lipschitz_on_uniformly_continuous*:
assumes *L*−*lipschitz_on X f*
shows *uniformly_continuous_on X f*
unfolding *uniformly_continuous_on_def*
proof *safe*
fix *e*::*real*
assume *0* < *e*
from *assms* **have** *l*: (*L*+*1*)−*lipschitz_on X f*
by (*rule lipschitz_on_mono*) *auto*
show ∃ *d*>*0*. ∀ *x*∈*X*. ∀ *x*′∈*X*. *dist x*′ *x* < *d* ⟶ *dist* (*f x*′) (*f x*) < *e*
using *lipschitz_onD*[*OF l*] *lipschitz_on_nonneg*[*OF assms*] ‹*0* < *e*›
by (*force intro*!: *exI*[**where** *x*=*e*/(*L* + *1*)] *simp*: *field_simps*)
qed

proposition *lipschitz_on_continuous_on*:
continuous_on X f **if** *L*−*lipschitz_on X f*
by (*rule uniformly_continuous_imp_continuous*[*OF lipschitz_on_uniformly_continuous*[*OF that*]])

lemma *lipschitz_on_continuous_within*:
continuous (*at x within X*) *f* **if** *L*−*lipschitz_on X f x* ∈ *X*
using *lipschitz_on_continuous_on*[*OF that*(*1*)] *that*(*2*)
by (*auto simp*: *continuous_on_eq_continuous_within*)

Differentiable functions

proposition *bounded_derivative_imp_lipschitz*:
assumes ⋀*x*. *x* ∈ *X* ⟹ (*f has_derivative f*′ *x*) (*at x within X*)
assumes *convex*: *convex X*
assumes ⋀*x*. *x* ∈ *X* ⟹ *onorm* (*f*′ *x*) ≤ *C 0* ≤ *C*
shows *C*−*lipschitz_on X f*
proof (*rule lipschitz_onI*)
show ⋀*x y*. *x* ∈ *X* ⟹ *y* ∈ *X* ⟹ *dist* (*f x*) (*f y*) ≤ *C* ∗ *dist x y*
by (*auto intro*!: *assms differentiable_bound*[*unfolded dist_norm*[*symmetric*], *OF convex*])
qed *fact*

Structural introduction rules

named_theorems *lipschitz_intros structural introduction rules for Lipschitz controls*

lemma *lipschitz_on_compose* [*lipschitz_intros*]:
  (*D* ∗ *C*)−*lipschitz_on* *U* (*g* *o* *f*)
  if *f*: *C*−*lipschitz_on* *U* *f* **and** *g*: *D*−*lipschitz_on* (*f*'*U*) *g*
proof (*rule lipschitz_onI*)
  show *D*∗ *C* ≥ *0* **using** *lipschitz_on_nonneg*[*OF f*] *lipschitz_on_nonneg*[*OF g*] **by** *auto*
  fix *x y* **assume** *H*: *x* ∈ *U* *y* ∈ *U*
  have *dist* (*g* (*f x*)) (*g* (*f y*)) ≤ *D* ∗ *dist* (*f x*) (*f y*)
    apply (*rule lipschitz_onD*[*OF g*]) **using** *H* **by** *auto*
  also have ... ≤ *D* ∗ *C* ∗ *dist x y*
    using *mult_left_mono*[*OF lipschitz_onD*(*1*)[*OF f H*] *lipschitz_on_nonneg*[*OF g*]] **by** *auto*
  finally show *dist* ((*g* ∘ *f*) *x*) ((*g* ∘ *f*) *y*) ≤ *D* ∗ *C*∗ *dist x y*
    unfolding *comp_def* **by** (*auto simp add*: *mult.commute*)
qed

lemma *lipschitz_on_compose2*:
  (*D* ∗ *C*)−*lipschitz_on* *U* (λ*x*. *g* (*f x*))
  if *C*−*lipschitz_on* *U f* *D*−*lipschitz_on* (*f*'*U*) *g*
  using *lipschitz_on_compose*[*OF that*] **by** (*simp add*: *o_def*)

lemma *lipschitz_on_cong*[*cong*]:
  C−*lipschitz_on* *U g* ⟷ *D*−*lipschitz_on* *V f*
  if *C* = *D* *U* = *V* ⋀*x*. *x* ∈ *V* ⟹ *g x* = *f x*
  using *that* **by** (*auto simp*: *lipschitz_on_def*)

lemma *lipschitz_on_transform*:
  C−*lipschitz_on* *U g*
  if *C*−*lipschitz_on* *U f*
    ⋀*x*. *x* ∈ *U* ⟹ *g x* = *f x*
  using *that*
  by *simp*

lemma *lipschitz_on_empty_iff*[*simp*]: *C*−*lipschitz_on* {} *f* ⟷ *C* ≥ *0*
  by (*auto simp*: *lipschitz_on_def*)

lemma *lipschitz_on_insert_iff*[*simp*]:
  C−*lipschitz_on* (*insert y X*) *f* ⟷
    C−*lipschitz_on* *X f* ∧ (∀ *x* ∈ *X*. *dist* (*f x*) (*f y*) ≤ *C* ∗ *dist x y*)
  by (*auto simp*: *lipschitz_on_def dist_commute*)

lemma *lipschitz_on_singleton* [*lipschitz_intros*]: *C* ≥ *0* ⟹ *C*−*lipschitz_on* {*x*} *f*
  and *lipschitz_on_empty* [*lipschitz_intros*]: *C* ≥ *0* ⟹ *C*−*lipschitz_on* {} *f*
  by *simp_all*

lemma *lipschitz_on_id* [*lipschitz_intros*]: *1*−*lipschitz_on* *U* (λ*x*. *x*)
  by (*auto simp*: *lipschitz_on_def*)

lemma *lipschitz_on_constant* [*lipschitz_intros*]: *0*−*lipschitz_on* *U* (λ*x*. *c*)

by (*auto simp*: *lipschitz_on_def*)

lemma *lipschitz_on_add* [*lipschitz_intros*]:
fixes *f*::'*a*::*metric_space* ⇒'*b*::*real_normed_vector*
assumes *C*−*lipschitz_on U f*
D−*lipschitz_on U g*
shows (*C*+*D*)−*lipschitz_on U* (λ*x*. *f x* + *g x*)
proof (*rule lipschitz_onI*)
show *C* + *D* ≥ *0*
using *lipschitz_on_nonneg*[*OF assms*(*1*)] *lipschitz_on_nonneg*[*OF assms*(*2*)] **by** *auto*
fix *x y* **assume** *H*: *x* ∈ *U y* ∈ *U*
have *dist* (*f x* + *g x*) (*f y* + *g y*) ≤ *dist* (*f x*) (*f y*) + *dist* (*g x*) (*g y*)
by (*simp add*: *dist_triangle_add*)
also have ... ≤ *C* ∗ *dist x y* + *D* ∗ *dist x y*
using *lipschitz_onD*(*1*)[*OF assms*(*1*) *H*] *lipschitz_onD*(*1*)[*OF assms*(*2*) *H*] **by** *auto*
finally show *dist* (*f x* + *g x*) (*f y* + *g y*) ≤ (*C*+*D*) ∗ *dist x y* **by** (*auto simp add*: *algebra_simps*)
qed

lemma *lipschitz_on_cmult* [*lipschitz_intros*]:
fixes *f*::'*a*::*metric_space* ⇒ '*b*::*real_normed_vector*
assumes *C*−*lipschitz_on U f*
shows (*abs*(*a*) ∗ *C*)−*lipschitz_on U* (λ*x*. *a* $*_R$ *f x*)
proof (*rule lipschitz_onI*)
show *abs*(*a*) ∗ *C* ≥ *0* **using** *lipschitz_on_nonneg*[*OF assms*(*1*)] **by** *auto*
fix *x y* **assume** *H*: *x* ∈ *U y* ∈ *U*
have *dist* (*a* $*_R$ *f x*) (*a* $*_R$ *f y*) = *abs*(*a*) ∗ *dist* (*f x*) (*f y*)
by (*metis dist_norm norm_scaleR real_vector.scale_right_diff_distrib*)
also have ... ≤ *abs*(*a*) ∗ *C* ∗ *dist x y*
using *lipschitz_onD*(*1*)[*OF assms*(*1*) *H*] **by** (*simp add*: *Groups.mult_ac*(*1*) *mult_left_mono*)
finally show *dist* (*a* $*_R$ *f x*) (*a* $*_R$ *f y*) ≤ |*a*| ∗ *C* ∗ *dist x y* **by** *auto*
qed

lemma *lipschitz_on_cmult_real* [*lipschitz_intros*]:
fixes *f*::'*a*::*metric_space* ⇒ *real*
assumes *C*−*lipschitz_on U f*
shows (*abs*(*a*) ∗ *C*)−*lipschitz_on U* (λ*x*. *a* ∗ *f x*)
using *lipschitz_on_cmult*[*OF assms*] **by** *auto*

lemma *lipschitz_on_cmult_nonneg* [*lipschitz_intros*]:
fixes *f*::'*a*::*metric_space* ⇒ '*b*::*real_normed_vector*
assumes *C*−*lipschitz_on U f*
a ≥ *0*
shows (*a* ∗ *C*)−*lipschitz_on U* (λ*x*. *a* $*_R$ *f x*)
using *lipschitz_on_cmult*[*OF assms*(*1*), *of a*] *assms*(*2*) **by** *auto*

lemma *lipschitz_on_cmult_real_nonneg* [*lipschitz_intros*]:
fixes *f*::'*a*::*metric_space* ⇒ *real*
assumes *C*−*lipschitz_on U f*
a ≥ *0*
shows (*a* ∗ *C*)−*lipschitz_on U* (λ*x*. *a* ∗ *f x*)
using *lipschitz_on_cmult_nonneg*[*OF assms*] **by** *auto*

lemma *lipschitz_on_cmult_upper* [*lipschitz_intros*]:
fixes *f*::'*a*::*metric_space* ⇒ '*b*::*real_normed_vector*
assumes *C*−*lipschitz_on U f*
abs(*a*) ≤ *D*
shows (*D* ∗ *C*)−*lipschitz_on U* (λ*x*. *a* $*_R$ *f x*)
apply (*rule lipschitz_on_mono*[*OF lipschitz_on_cmult*[*OF assms*(*1*), *of a*], *of _ D* ∗ *C*])
using *assms*(*2*) *lipschitz_on_nonneg*[*OF assms*(*1*)] *mult_right_mono* **by** *auto*

lemma *lipschitz_on_cmult_real_upper* [*lipschitz_intros*]:
fixes *f*::'*a*::*metric_space* ⇒ *real*
assumes *C*−*lipschitz_on U f*
abs(*a*) ≤ *D*
shows (*D* ∗ *C*)−*lipschitz_on U* (λ*x*. *a* ∗ *f x*)
using *lipschitz_on_cmult_upper*[*OF assms*] **by** *auto*

lemma *lipschitz_on_minus*[*lipschitz_intros*]:
fixes *f*::'*a*::*metric_space* ⇒'*b*::*real_normed_vector*
assumes *C*−*lipschitz_on U f*
shows *C*−*lipschitz_on U* (λ*x*. − *f x*)
by (*metis* (*mono_tags*, *lifting*) *assms dist_minus lipschitz_on_def*)

lemma *lipschitz_on_minus_iff*[*simp*]:
L−*lipschitz_on X* (λ*x*. − *f x*) ⟷ *L*−*lipschitz_on X f*
L−*lipschitz_on X* (− *f*) ⟷ *L*−*lipschitz_on X f*
for *f*::'*a*::*metric_space* ⇒'*b*::*real_normed_vector*
using *lipschitz_on_minus*[*of L X f*] *lipschitz_on_minus*[*of L X* −*f*]
by *auto*

lemma *lipschitz_on_diff*[*lipschitz_intros*]:
fixes *f*::'*a*::*metric_space* ⇒'*b*::*real_normed_vector*
assumes *C*−*lipschitz_on U f D*−*lipschitz_on U g*
shows (*C* + *D*)−*lipschitz_on U* (λ*x*. *f x* − *g x*)
using *lipschitz_on_add*[*OF assms*(*1*) *lipschitz_on_minus*[*OF assms*(*2*)]] **by** *auto*

lemma *lipschitz_on_closure* [*lipschitz_intros*]:
assumes *C*−*lipschitz_on U f continuous_on* (*closure U*) *f*
shows *C*−*lipschitz_on* (*closure U*) *f*
proof (*rule lipschitz_onI*)
show *C* ≥ *0* **using** *lipschitz_on_nonneg*[*OF assms*(*1*)] **by** *simp*
fix *x y* **assume** *x* ∈ *closure U y* ∈ *closure U*
obtain *u v*::*nat* ⇒ '*a* **where** ∗: ⋀*n*. *u n* ∈ *U u* ⟶ *x*

⋀n. v n ∈ U v ⟶ y
using ‹x ∈ closure U› ‹y ∈ closure U› **unfolding** *closure_sequential* **by** *blast*
have *a*: (λn. f (u n)) ⟶ f x
using ∗(1) ∗(2) ‹x ∈ closure U› ‹continuous_on (closure U) f›
unfolding *comp_def continuous_on_closure_sequentially*[*of U f*] **by** *auto*
have *b*: (λn. f (v n)) ⟶ f y
using ∗(3) ∗(4) ‹y ∈ closure U› ‹continuous_on (closure U) f›
unfolding *comp_def continuous_on_closure_sequentially*[*of U f*] **by** *auto*
have *l*: (λn. C ∗ dist (u n) (v n) − dist (f (u n)) (f (v n))) ⟶ C ∗ dist x y − dist (f x) (f y)
by (*intro tendsto_intros* ∗ *a b*)
have C ∗ dist (u n) (v n) − dist (f (u n)) (f (v n)) ≥ 0 **for** n
using *lipschitz_onD*(1)[*OF assms*(1) ‹u n ∈ U› ‹v n ∈ U›] **by** *simp*
then have C ∗ dist x y − dist (f x) (f y) ≥ 0 **using** *LIMSEQ_le_const*[*OF l*, *of* 0] **by** *auto*
then show dist (f x) (f y) ≤ C ∗ dist x y **by** *auto*
qed

lemma *lipschitz_on_Pair*[*lipschitz_intros*]:
assumes *f*: L−lipschitz_on A f
assumes *g*: M−lipschitz_on A g
shows (sqrt (L^2 + M^2))−lipschitz_on A (λa. (f a, g a))
proof (*rule lipschitz_onI*, *goal_cases*)
case (1 x y)
have dist (f x, g x) (f y, g y) = sqrt ((dist (f x) (f y))^2 + (dist (g x) (g y))^2)
by (*auto simp add*: *dist_Pair_Pair real_le_lsqrt*)
also have . . . ≤ sqrt ((L ∗ dist x y)^2 + (M ∗ dist x y)^2)
by (*auto intro*!: *real_sqrt_le_mono add_mono power_mono 1 lipschitz_onD f g*)
also have . . . ≤ sqrt (L^2 + M^2) ∗ dist x y
by (*auto simp*: *power_mult_distrib ring_distribs*[*symmetric*] *real_sqrt_mult*)
finally show *?case* **.**
qed *simp*

lemma *lipschitz_extend_closure*:
fixes f::('a::metric_space) ⇒ ('b::complete_space)
assumes C−lipschitz_on U f
shows ∃g. C−lipschitz_on (closure U) g ∧ (∀x∈U. g x = f x)
proof −
obtain g **where** g: ⋀x. x ∈ U ⟹ g x = f x uniformly_continuous_on (closure U) g
using *uniformly_continuous_on_extension_on_closure*[*OF lipschitz_on_uniformly_continuous*[*OF assms*]] **by** *metis*
have C−lipschitz_on (closure U) g
apply (*rule lipschitz_on_closure*, *rule lipschitz_on_transform*[*OF assms*])
using *g uniformly_continuous_imp_continuous*[*OF g*(2)] **by** *auto*
then show *?thesis* **using** *g*(1) **by** *auto*
qed

lemma (**in** *bounded_linear*) *lipschitz_boundE*:

obtains *B* **where** *B−lipschitz_on A f*
proof −
from *nonneg_bounded*
obtain *B* **where** *B*: $B \geq 0$ $\bigwedge x.$ *norm* $(f\ x) \leq B *$ *norm x*
by (*auto simp*: *ac_simps*)
have *B−lipschitz_on A f*
by (*auto intro*!: *lipschitz_onI B simp*: *dist_norm diff* [*symmetric*])
thus *?thesis* **..**
qed

6.47.1 Local Lipschitz continuity

Given a function defined on a real interval, it is Lipschitz-continuous if and only if it is locally so, as proved in the following lemmas. It is useful especially for piecewise-defined functions: if each piece is Lipschitz, then so is the whole function. The same goes for functions defined on geodesic spaces, or more generally on geodesic subsets in a metric space (for instance convex subsets in a real vector space), and this follows readily from the real case, but we will not prove it explicitly.

We give several variations around this statement. This is essentially a connectedness argument.

lemma *locally_lipschitz_imp_lipschitz_aux*:
fixes *f*::*real* ⇒ (*'a*::*metric_space*)
assumes $a \leq b$
continuous_on {*a..b*} *f*
$\bigwedge x.\ x \in \{a..<b\} \Longrightarrow \exists\, y \in \{x<..b\}.$ *dist* $(f\ y)\ (f\ x) \leq M * (y-x)$
shows *dist* $(f\ b)\ (f\ a) \leq M * (b-a)$
proof −
define *A* **where** $A = \{x \in \{a..b\}.$ *dist* $(f\ x)\ (f\ a) \leq M * (x-a)\}$
have ∗: $A = (\lambda x.\ M * (x-a)$ − *dist* $(f\ x)\ (f\ a))$−'{*0..*} ∩ {*a..b*}
unfolding *A_def* **by** *auto*
have $a \in A$ **unfolding** *A_def* **using** ‹$a \leq b$› **by** *auto*
then have $A \neq \{\}$ **by** *auto*
moreover have *bdd_above A* **unfolding** *A_def* **by** *auto*
moreover have *closed A* **unfolding** ∗ **by** (*rule closed_vimage_Int*, *auto intro*!: *continuous_intros assms*)
ultimately have *Sup A* ∈ *A* **by** (*rule closed_contains_Sup*)
have *Sup A* = *b*
proof (*rule ccontr*)
assume *Sup A* ≠ *b*
define *x* **where** *x* = *Sup A*
have *I*: *dist* $(f\ x)\ (f\ a) \leq M * (x-a)$ **using** ‹*Sup A* ∈ *A*› *x_def A_def* **by** *auto*
have $x \in \{a..<b\}$ **unfolding** *x_def* **using** ‹*Sup A* ∈ *A*› ‹*Sup A* ≠ *b*› *A_def* **by** *auto*
then obtain *y* **where** *J*: $y \in \{x<..b\}$ *dist* $(f\ y)\ (f\ x) \leq M * (y-x)$ **using** *assms*(*3*) **by** *blast*
have *dist* $(f\ y)\ (f\ a) \leq$ *dist* $(f\ y)\ (f\ x)$ + *dist* $(f\ x)\ (f\ a)$ **by** (*rule dist_triangle*)
also have ... $\leq M * (y-x) + M * (x-a)$ **using** *I J*(*2*) **by** *auto*

finally have *dist* (*f y*) (*f a*) ≤ *M* ∗ (*y*−*a*) **by** (*auto simp add*: *algebra_simps*)
then have *y* ∈ *A* **unfolding** *A_def* **using** ‹*y* ∈ {*x*<..*b*}› ‹*x* ∈ {*a*..<*b*}› **by** *auto*
then have *y* ≤ *Sup A* **by** (*rule cSup_upper*, *auto simp*: *A_def*)
then show *False* **using** ‹*y* ∈ {*x*<..*b*}› *x_def* **by** *auto*
qed
then show *?thesis* **using** ‹*Sup A* ∈ *A*› *A_def* **by** *auto*
qed

lemma *locally_lipschitz_imp_lipschitz*:
fixes *f*::*real* ⇒ (*'a*::*metric_space*)
assumes *continuous_on* {*a*..*b*} *f*
⋀*x y*. *x* ∈ {*a*..<*b*} ⟹ *y* > *x* ⟹ ∃ *z* ∈ {*x*<..*y*}. *dist* (*f z*) (*f x*) ≤ *M* ∗ (*z*−*x*)
M ≥ *0*
shows *lipschitz_on M* {*a*..*b*} *f*
proof (*rule lipschitz_onI*[*OF* _ ‹*M* ≥ *0*›])
have ∗: *dist* (*f t*) (*f s*) ≤ *M* ∗ (*t*−*s*) **if** *s* ≤ *t s* ∈ {*a*..*b*} *t* ∈ {*a*..*b*} **for** *s t*
proof (*rule locally_lipschitz_imp_lipschitz_aux*, *simp add*: ‹*s* ≤ *t*›)
show *continuous_on* {*s*..*t*} *f* **using** *continuous_on_subset*[*OF assms*(*1*)] *that* **by** *auto*
fix *x* **assume** *x* ∈ {*s*..<*t*}
then have *x* ∈ {*a*..<*b*} **using** *that* **by** *auto*
show ∃ *z*∈{*x*<..*t*}. *dist* (*f z*) (*f x*) ≤ *M* ∗ (*z* − *x*)
using *assms*(*2*)[*OF* ‹*x* ∈ {*a*..<*b*}›, *of t*] ‹*x* ∈ {*s*..<*t*}› **by** *auto*
qed
fix *x y* **assume** *x* ∈ {*a*..*b*} *y* ∈ {*a*..*b*}
consider *x* ≤ *y* | *y* ≤ *x* **by** *linarith*
then show *dist* (*f x*) (*f y*) ≤ *M* ∗ *dist x y*
apply (*cases*)
using ∗[*OF* _ ‹*x* ∈ {*a*..*b*}› ‹*y* ∈ {*a*..*b*}›] ∗[*OF* _ ‹*y* ∈ {*a*..*b*}› ‹*x* ∈ {*a*..*b*}›]
by (*auto simp add*: *dist_commute dist_real_def*)
qed

We deduce that if a function is Lipschitz on finitely many closed sets on the real line, then it is Lipschitz on any interval contained in their union. The difficulty in the proof is to show that any point *z* in this interval (except the maximum) has a point arbitrarily close to it on its right which is contained in a common initial closed set. Otherwise, we show that there is a small interval (*z*, *T*) which does not intersect any of the initial closed sets, a contradiction.

proposition *lipschitz_on_closed_Union*:
assumes ⋀*i*. *i* ∈ *I* ⟹ *lipschitz_on M* (*U i*) *f*
⋀*i*. *i* ∈ *I* ⟹ *closed* (*U i*)
finite I
M ≥ *0*
{*u*..(*v*::*real*)} ⊆ (⋃ *i*∈*I*. *U i*)
shows *lipschitz_on M* {*u*..*v*} *f*
proof (*rule locally_lipschitz_imp_lipschitz*[*OF* _ _ ‹*M* ≥ *0*›])

have ∗: *continuous_on* (*U i*) *f* **if** *i* ∈ *I* **for** *i*
  by (*rule lipschitz_on_continuous_on*[*OF assms*(*1*)[*OF* ‹*i*∈ *I*›]])
have *continuous_on* (⋃*i*∈*I*. *U i*) *f*
  apply (*rule continuous_on_closed_Union*) **using** ‹*finite I*› ∗ *assms*(*2*) **by** *auto*
then show *continuous_on* {*u*..*v*} *f*
  using ‹{*u*..(*v*::*real*)} ⊆ (⋃*i*∈*I*. *U i*)› *continuous_on_subset* **by** *auto*

fix *z Z* **assume** *z*: *z* ∈ {*u*..<*v*} *z* < *Z*
then have *u* ≤ *v* **by** *auto*
define *T* **where** *T* = *min Z v*
then have *T*: *T* > *z* *T* ≤ *v* *T* ≥ *u* *T* ≤ *Z* **using** *z* **by** *auto*
define *A* **where** *A* = (⋃*i*∈ *I* ∩ {*i*. *U i* ∩ {*z*<..*T*} ≠ {}}. *U i* ∩ {*z*..*T*})
have *a*: *closed A*
  unfolding *A_def* **apply** (*rule closed_UN*) **using** ‹*finite I*› ‹⋀*i*. *i* ∈ *I* ⟹ *closed* (*U i*)› **by** *auto*
have *b*: *bdd_below A* **unfolding** *A_def* **using** ‹*finite I*› **by** *auto*
have ∃ *i* ∈ *I*. *T* ∈ *U i* **using** ‹{*u*..*v*} ⊆ (⋃*i*∈*I*. *U i*)› *T* **by** *auto*
then have *c*: *T* ∈ *A* **unfolding** *A_def* **using** *T* **by** (*auto*, *fastforce*)
have *Inf A* ≥ *z*
  apply (*rule cInf_greatest*, *auto*) **using** *c* **unfolding** *A_def* **by** *auto*
moreover have *Inf A* ≤ *z*
proof (*rule ccontr*)
  assume ¬(*Inf A* ≤ *z*)
  then obtain *w* **where** *w*: *w* > *z* *w* < *Inf A* **by** (*meson dense not_le_imp_less*)
  have *Inf A* ≤ *T* **using** *a b c* **by** (*simp add*: *cInf_lower*)
  then have *w* ≤ *T* **using** *w* **by** *auto*
  then have *w* ∈ {*u*..*v*} **using** *w* ‹*z* ∈ {*u*..<*v*}› *T* **by** *auto*
  then obtain *j* **where** *j*: *j* ∈ *I* *w* ∈ *U j* **using** ‹{*u*..*v*} ⊆ (⋃*i*∈*I*. *U i*)› **by** *fastforce*
  then have *w* ∈ *U j* ∩ {*z*..*T*} *U j* ∩ {*z*<..*T*} ≠ {} **using** *j T w* ‹*w* ≤ *T*› **by** *auto*
  then have *w* ∈ *A* **unfolding** *A_def* **using** ‹*j* ∈ *I*› **by** *auto*
  then have *Inf A* ≤ *w* **using** *a b* **by** (*simp add*: *cInf_lower*)
  then show *False* **using** *w* **by** *auto*
qed
ultimately have *Inf A* = *z* **by** *simp*
moreover have *Inf A* ∈ *A*
  apply (*rule closed_contains_Inf*) **using** *a b c* **by** *auto*
ultimately have *z* ∈ *A* **by** *simp*
then obtain *i* **where** *i*: *i* ∈ *I* *U i* ∩ {*z*<..*T*} ≠ {} *z* ∈ *U i* **unfolding** *A_def* **by** *auto*
then obtain *t* **where** *t* ∈ *U i* ∩ {*z*<..*T*} **by** *blast*
then have *dist* (*f t*) (*f z*) ≤ *M* ∗ (*t* − *z*)
  using *lipschitz_onD*(*1*)[*OF assms*(*1*)[*of i*], *of t z*] *i dist_real_def* **by** *auto*
then show ∃ *t*∈{*z*<..*Z*}. *dist* (*f t*) (*f z*) ≤ *M* ∗ (*t* − *z*) **using** ‹*T* ≤ *Z*› ‹*t* ∈ *U i* ∩ {*z*<..*T*}› **by** *auto*
qed

6.47.2 Local Lipschitz continuity (uniform for a family of functions)

definition *local_lipschitz*::
$'a$::*metric_space set* ⇒ $'b$::*metric_space set* ⇒ ($'a$ ⇒ $'b$ ⇒ $'c$::*metric_space*) ⇒ *bool*
where
local_lipschitz T X f ≡ ∀ *x* ∈ *X*. ∀ *t* ∈ *T*.
∃ *u*>*0*. ∃ *L*. ∀ *t* ∈ *cball t u* ∩ *T*. *L*−*lipschitz_on* (*cball x u* ∩ *X*) (*f t*)

lemma *local_lipschitzI*:
assumes ⋀*t x*. *t* ∈ *T* ⟹ *x* ∈ *X* ⟹ ∃ *u*>*0*. ∃ *L*. ∀ *t* ∈ *cball t u* ∩ *T*. *L*−*lipschitz_on* (*cball x u* ∩ *X*) (*f t*)
shows *local_lipschitz T X f*
using *assms*
unfolding *local_lipschitz_def*
by *auto*

lemma *local_lipschitzE*:
assumes *local_lipschitz*: *local_lipschitz T X f*
assumes *t* ∈ *T x* ∈ *X*
obtains *u L* **where** *u* > *0* ⋀*s*. *s* ∈ *cball t u* ∩ *T* ⟹ *L*−*lipschitz_on* (*cball x u* ∩ *X*) (*f s*)
using *assms local_lipschitz_def*
by *metis*

lemma *local_lipschitz_continuous_on*:
assumes *local_lipschitz*: *local_lipschitz T X f*
assumes *t* ∈ *T*
shows *continuous_on X* (*f t*)
unfolding *continuous_on_def*
proof *safe*
fix *x* **assume** *x* ∈ *X*
from *local_lipschitzE*[*OF local_lipschitz* ‹*t* ∈ *T*› ‹*x* ∈ *X*›] **obtain** *u L*
where *0* < *u*
and *L*: ⋀*s*. *s* ∈ *cball t u* ∩ *T* ⟹ *L*−*lipschitz_on* (*cball x u* ∩ *X*) (*f s*)
by *metis*
have *x* ∈ *ball x u* **using** ‹*0* < *u*› **by** *simp*
from *lipschitz_on_continuous_on*[*OF L*]
have *tendsto*: (*f t* ⟶ *f t x*) (*at x within cball x u* ∩ *X*)
using ‹*0* < *u*› ‹*x* ∈ *X*› ‹*t* ∈ *T*›
by (*auto simp*: *continuous_on_def*)
moreover have $\forall_F$ *xa in at x*. (*xa* ∈ *cball x u* ∩ *X*) = (*xa* ∈ *X*)
using *eventually_at_ball*[*OF* ‹*0* < *u*›, *of x UNIV*]
by *eventually_elim auto*
ultimately show (*f t* ⟶ *f t x*) (*at x within X*)
by (*rule Lim_transform_within_set*)
qed

lemma

local_lipschitz_compose1:
assumes *ll*: *local_lipschitz* (*g* ‘ *T*) *X* (λ*t*. *f t*)
assumes *g*: *continuous_on T g*
shows *local_lipschitz T X* (λ*t*. *f* (*g t*))
proof (*rule local_lipschitzI*)
fix *t x*
assume *t* ∈ *T x* ∈ *X*
then have *g t* ∈ *g* ‘ *T* **by** *simp*
from *local_lipschitzE*[*OF assms*(*1*) *this* ‹*x* ∈ *X*›]
obtain *u L* **where** *0* < *u* **and** *l*: (⋀*s*. *s* ∈ *cball* (*g t*) *u* ∩ *g* ‘ *T* ⟹ *L*−*lipschitz_on* (*cball x u* ∩ *X*) (*f s*))
by *auto*
from *g*[*unfolded continuous_on_eq_continuous_within*, *rule_format*, *OF* ‹*t* ∈ *T*›,
unfolded continuous_within_eps_delta, *rule_format*, *OF* ‹*0* < *u*›]
obtain *d* **where** *d*: *d*>*0* ⋀*x'*. *x'*∈*T* ⟹ *dist x' t* < *d* ⟹ *dist* (*g x'*) (*g t*) < *u*
by (*auto*)
show ∃ *u*>*0*. ∃ *L*. ∀ *t*∈*cball t u* ∩ *T*. *L*−*lipschitz_on* (*cball x u* ∩ *X*) (*f* (*g t*))
using *d* ‹*0* < *u*›
by (*fastforce intro*: *exI*[**where** *x*=(*min d u*)/*2*] *exI*[**where** *x*=*L*]
intro!: *less_imp_le*[*OF d*(*2*)] *lipschitz_on_subset*[*OF l*] *simp*: *dist_commute*)
qed

context
fixes *T*::'*a*::*metric_space set* **and** *X f*
assumes *local_lipschitz*: *local_lipschitz T X f*
begin

lemma *continuous_on_TimesI*:
assumes *y*: ⋀*x*. *x* ∈ *X* ⟹ *continuous_on T* (λ*t*. *f t x*)
shows *continuous_on* (*T* × *X*) (λ(*t*, *x*). *f t x*)
unfolding *continuous_on_iff*
proof (*safe*, *simp*)
fix *a b* **and** *e*::*real*
assume *H*: *a* ∈ *T b* ∈ *X 0* < *e*
hence *0* < *e*/*2* **by** *simp*
from *y*[*unfolded continuous_on_iff*, *OF* ‹*b* ∈ *X*›, *rule_format*, *OF* ‹*a* ∈ *T*› ‹*0* < *e*/*2*›]
obtain *d* **where** *d*: *d* > *0* ⋀*t*. *t* ∈ *T* ⟹ *dist t a* < *d* ⟹ *dist* (*f t b*) (*f a b*) < *e*/*2*
by *auto*

from ‹*a* : *T*› ‹*b* ∈ *X*›
obtain *u L* **where** *u*: *0* < *u*
and *L*: ⋀*t*. *t* ∈ *cball a u* ∩ *T* ⟹ *L*−*lipschitz_on* (*cball b u* ∩ *X*) (*f t*)
by (*erule local_lipschitzE*[*OF local_lipschitz*])

have *a* ∈ *cball a u* ∩ *T* **by** (*auto simp*: ‹*0* < *u*› ‹*a* ∈ *T*› *less_imp_le*)
from *lipschitz_on_nonneg*[*OF L*[*OF* ‹*a* ∈ *cball* _ _ ∩ _›]] **have** *0* ≤ *L* **.**

```
let ?d = Min {d, u, (e/2/(L + 1))}
show ∃ d>0. ∀ x∈T. ∀ y∈X. dist (x, y) (a, b) < d ⟶ dist (f x y) (f a b) < e
proof (rule exI[where x = ?d], safe)
  show 0 < ?d
    using ‹0 ≤ L› ‹0 < u› ‹0 < e› ‹0 < d›
    by (auto intro!: divide_pos_pos )
  fix x y
  assume x ∈ T y ∈ X
  assume dist_less: dist (x, y) (a, b) < ?d
  have dist y b ≤ dist (x, y) (a, b)
    using dist_snd_le[of (x, y) (a, b)]
    by auto
  also
  note dist_less
  also
  {
    note calculation
    also have ?d ≤ u by simp
    finally have dist y b < u .
  }
  have ?d ≤ e/2/(L + 1) by simp
  also have (L + 1) ∗ . . . ≤ e / 2
    using ‹0 < e› ‹L ≥ 0›
    by (auto simp: field_split_simps)
  finally have le1: (L + 1) ∗ dist y b < e / 2 using ‹L ≥ 0› by simp

  have dist x a ≤ dist (x, y) (a, b)
    using dist_fst_le[of (x, y) (a, b)]
    by auto
  also note dist_less
  finally have dist x a < ?d .
  also have ?d ≤ d by simp
  finally have dist x a < d .
  note ‹dist x a < ?d›
  also have ?d ≤ u by simp
  finally have dist x a < u .
  then have x ∈ cball a u ∩ T
    using ‹x ∈ T›
    by (auto simp: dist_commute)
  have dist (f x y) (f a b) ≤ dist (f x y) (f x b) + dist (f x b) (f a b)
    by (rule dist_triangle)
  also have (L + 1)−lipschitz_on (cball b u ∩ X) (f x)
    using L[OF ‹x ∈ cball a u ∩ T›]
    by (rule lipschitz_on_le) simp
  then have dist (f x y) (f x b) ≤ (L + 1) ∗ dist y b
    apply (rule lipschitz_onD)
    subgoal
      using ‹y ∈ X› ‹dist y b < u›
      by (simp add: dist_commute)
```

```
      subgoal
        using ‹0 < u› ‹b ∈ X›
        by (simp add: )
      done
    also have (L + 1) ∗ dist y b ≤ e / 2
      using le1 ‹0 ≤ L› by simp
    also have dist (f x b) (f a b) < e / 2
      by (rule d; fact)
    also have e / 2 + e / 2 = e by simp
    finally show dist (f x y) (f a b) < e by simp
  qed
qed

lemma local_lipschitz_compact_implies_lipschitz:
  assumes compact X compact T
  assumes cont: ⋀x. x ∈ X ⟹ continuous_on T (λt. f t x)
  obtains L where ⋀t. t ∈ T ⟹ L−lipschitz_on X (f t)
proof −
  {
    assume ∗: ⋀n::nat. ¬(∀t∈T. n−lipschitz_on X (f t))
    {
      fix n::nat
      from ∗[of n] have ∃x y t. t ∈ T ∧ x ∈ X ∧ y ∈ X ∧ dist (f t y) (f t x) > n
∗ dist y x
        by (force simp: lipschitz_on_def)
    } then obtain t and x y::nat ⇒ 'b where xy: ⋀n. x n ∈ X ⋀n. y n ∈ X
      and t: ⋀n. t n ∈ T
      and d: ⋀n. dist (f (t n) (y n)) (f (t n) (x n)) > n ∗ dist (y n) (x n)
      by metis
    from xy assms obtain lx rx where lx': lx ∈ X strict_mono (rx :: nat ⇒ nat)
(x o rx) ⟶ lx
      by (metis compact_def)
    with xy have ⋀n. (y o rx) n ∈ X by auto
    with assms obtain ly ry where ly': ly ∈ X strict_mono (ry :: nat ⇒ nat) ((y
o rx) o ry) ⟶ ly
      by (metis compact_def)
    with t have ⋀n. ((t o rx) o ry) n ∈ T by simp
    with assms obtain lt rt where lt': lt ∈ T strict_mono (rt :: nat ⇒ nat) (((t
o rx) o ry) o rt) ⟶ lt
      by (metis compact_def)
    from lx' ly'
    have lx: (x o (rx o ry o rt)) ⟶ lx (is ?x ⟶ _)
      and ly: (y o (rx o ry o rt)) ⟶ ly (is ?y ⟶ _)
      and lt: (t o (rx o ry o rt)) ⟶ lt (is ?t ⟶ _)
      subgoal by (simp add: LIMSEQ_subseq_LIMSEQ o_assoc lt'(2))
      subgoal by (simp add: LIMSEQ_subseq_LIMSEQ ly'(3) o_assoc lt'(2))
      subgoal by (simp add: o_assoc lt'(3))
      done
    hence (λn. dist (?y n) (?x n)) ⟶ dist ly lx
```

by (*metis tendsto_dist*)
moreover
let *?S* = (λ(*t*, *x*). *f t x*) ' (*T* × *X*)
have *eventually* (λ*n*::*nat*. *n* > *0*) *sequentially*
by (*metis eventually_at_top_dense*)
hence *eventually* (λ*n*. *norm* (*dist* (*?y n*) (*?x n*)) ≤ *norm* (|*diameter ?S*| / *n*) ∗ *1*) *sequentially*
proof *eventually_elim*
case (*elim n*)
have *0* < *rx* (*ry* (*rt n*)) **using** ‹*0* < *n*›
by (*metis dual_order.strict_trans1 lt*′(*2*) *lx*′(*2*) *ly*′(*2*) *seq_suble*)
have *compact*: *compact ?S*
by (*auto intro*!: *compact_continuous_image continuous_on_subset*[*OF continuous_on_TimesI*]
compact_Times ‹*compact X*› ‹*compact T*› *cont*)
have *norm* (*dist* (*?y n*) (*?x n*)) = *dist* (*?y n*) (*?x n*) **by** *simp*
also
from *this elim d*[*of rx* (*ry* (*rt n*))]
have ... < *dist* (*f* (*?t n*) (*?y n*)) (*f* (*?t n*) (*?x n*)) / *rx* (*ry* (*rt* (*n*)))
using *lx*′(*2*) *ly*′(*2*) *lt*′(*2*) ‹*0* < *rx* _›
by (*auto simp add*: *field_split_simps strict_mono_def*)
also have ... ≤ *diameter ?S* / *n*
proof (*rule frac_le*)
show *diameter ?S* ≥ *0*
using *compact compact_imp_bounded diameter_ge_0* **by** *blast*
show *dist* (*f* (*?t n*) (*?y n*)) (*f* (*?t n*) (*?x n*)) ≤ *diameter* ((λ(*t*,*x*). *f t x*) ' (*T* × *X*))
by (*metis* (*no_types*) *compact compact_imp_bounded diameter_bounded_bound image_eqI mem_Sigma_iff o_apply split_conv t xy*(*1*) *xy*(*2*))
show *real n* ≤ *real* (*rx* (*ry* (*rt n*)))
by (*meson le_trans lt*′(*2*) *lx*′(*2*) *ly*′(*2*) *of_nat_mono strict_mono_imp_increasing*)
qed (*use* ‹*n* > *0*› **in** *auto*)
also have ... ≤ *abs* (*diameter ?S*) / *n*
by (*auto intro*!: *divide_right_mono*)
finally show *?case* **by** *simp*
qed
with _ **have** (λ*n*. *dist* (*?y n*) (*?x n*)) ⟶ *0*
by (*rule tendsto_0_le*)
(*metis tendsto_divide_0*[*OF tendsto_const*] *filterlim_at_top_imp_at_infinity*
filterlim_real_sequentially)
ultimately have *lx* = *ly*
using *LIMSEQ_unique* **by** *fastforce*
with *assms lx*′ **have** *lx* ∈ *X* **by** *auto*
from ‹*lt* ∈ *T*› *this* **obtain** *u L* **where** *L*: *u* > *0* ⋀*t*. *t* ∈ *cball lt u* ∩ *T* ⟹ *L*−*lipschitz_on* (*cball lx u* ∩ *X*) (*f t*)
by (*erule local_lipschitzE*[*OF local_lipschitz*])
hence *L* ≥ *0* **by** (*force intro*!: *lipschitz_on_nonneg* ‹*lt* ∈ *T*›)

from *L lt ly lx* ‹*lx* = *ly*›

```
    have
      eventually (λn. ?t n ∈ ball lt u) sequentially
      eventually (λn. ?y n ∈ ball lx u) sequentially
      eventually (λn. ?x n ∈ ball lx u) sequentially
      by (auto simp: dist_commute Lim)
    moreover have eventually (λn. n > L) sequentially
      by (metis filterlim_at_top_dense filterlim_real_sequentially)
    ultimately
    have eventually (λ_. False) sequentially
    proof eventually_elim
      case (elim n)
      hence dist (f (?t n) (?y n)) (f (?t n) (?x n)) ≤ L ∗ dist (?y n) (?x n)
        using assms xy t
        unfolding dist_norm[symmetric]
        by (intro lipschitz_onD[OF L(2)]) (auto)
      also have ... ≤ n ∗ dist (?y n) (?x n)
        using elim by (intro mult_right_mono) auto
      also have ... ≤ rx (ry (rt n)) ∗ dist (?y n) (?x n)
        by (intro mult_right_mono[OF _ zero_le_dist])
          (meson lt'(2) lx'(2) ly'(2) of_nat_le_iff order_trans seq_suble)
      also have ... < dist (f (?t n) (?y n)) (f (?t n) (?x n))
        by (auto intro!: d)
      finally show ?case by simp
    qed
    hence False
      by simp
  } then obtain L where ⋀t. t ∈ T ⟹ L−lipschitz_on X (f t)
    by metis
  thus ?thesis ..
qed

lemma local_lipschitz_subset:
  assumes S ⊆ T Y ⊆ X
  shows local_lipschitz S Y f
proof (rule local_lipschitzI)
  fix t x assume t ∈ S x ∈ Y
  then have t ∈ T x ∈ X using assms by auto
  from local_lipschitzE[OF local_lipschitz, OF this]
  obtain u L where u: 0 < u and L: ⋀s. s ∈ cball t u ∩ T ⟹ L−lipschitz_on
(cball x u ∩ X) (f s)
    by blast
  show ∃u>0. ∃L. ∀t∈cball t u ∩ S. L−lipschitz_on (cball x u ∩ Y) (f t)
    using assms
    by (auto intro: exI[where x=u] exI[where x=L]
        intro!: u lipschitz_on_subset[OF _ Int_mono[OF order_refl ‹Y ⊆ X›]] L)
qed

end
```

lemma *local_lipschitz_minus*:
fixes f::'*a*::*metric_space* ⇒ '*b*::*metric_space* ⇒ '*c*::*real_normed_vector*
shows *local_lipschitz T X* (λ*t x*. − *f t x*) = *local_lipschitz T X f*
by (*auto simp*: *local_lipschitz_def lipschitz_on_minus*)

lemma *local_lipschitz_PairI*:
assumes *f*: *local_lipschitz A B* (λ*a b*. *f a b*)
assumes *g*: *local_lipschitz A B* (λ*a b*. *g a b*)
shows *local_lipschitz A B* (λ*a b*. (*f a b*, *g a b*))
proof (*rule local_lipschitzI*)
fix *t x* **assume** *t* ∈ *A x* ∈ *B*
from *local_lipschitzE*[*OF f this*] *local_lipschitzE*[*OF g this*]
obtain *u L v M* **where** *0* < *u* (⋀*s*. *s* ∈ *cball t u* ∩ *A* ⟹ *L*−*lipschitz_on* (*cball x u* ∩ *B*) (*f s*))
0 < *v* (⋀*s*. *s* ∈ *cball t v* ∩ *A* ⟹ *M*−*lipschitz_on* (*cball x v* ∩ *B*) (*g s*))
by *metis*
then show ∃ *u*>*0*. ∃ *L*. ∀ *t*∈*cball t u* ∩ *A*. *L*−*lipschitz_on* (*cball x u* ∩ *B*) (λ*b*. (*f t b*, *g t b*))
by (*intro exI*[**where** *x*=*min u v*])
(*force intro*: *lipschitz_on_subset intro*!: *lipschitz_on_Pair*)
qed

lemma *local_lipschitz_constI*: *local_lipschitz S T* (λ*t x*. *f t*)
by (*auto simp*: *intro*!: *local_lipschitzI lipschitz_on_constant intro*: *exI*[**where** *x*=*1*])

lemma (**in** *bounded_linear*) *local_lipschitzI*:
shows *local_lipschitz A B* (λ_. *f*)
proof (*rule local_lipschitzI*, *goal_cases*)
case (*1 t x*)
from *lipschitz_boundE*[*of* (*cball x 1* ∩ *B*)] **obtain** *C* **where** *C*−*lipschitz_on* (*cball x 1* ∩ *B*) *f* **by** *auto*
then show *?case*
by (*auto intro*: *exI*[**where** *x*=*1*])
qed

proposition *c1_implies_local_lipschitz*:
fixes *T*::*real set* **and** *X*::'*a*::{*banach*,*heine_borel*} *set*
and *f*::*real* ⇒ '*a* ⇒ '*a*
assumes *f'*: ⋀*t x*. *t* ∈ *T* ⟹ *x* ∈ *X* ⟹ (*f t has_derivative blinfun_apply* (*f'* (*t*, *x*))) (*at x*)
assumes *cont_f'*: *continuous_on* (*T* × *X*) *f'*
assumes *open T*
assumes *open X*
shows *local_lipschitz T X f*
proof (*rule local_lipschitzI*)
fix *t x*
assume *t* ∈ *T x* ∈ *X*
from *open_contains_cball*[*THEN iffD1*, *OF* ‹*open X*›, *rule_format*, *OF* ‹*x* ∈ *X*›]
obtain *u* **where** *u*: *u* > *0 cball x u* ⊆ *X* **by** *auto*

```
  moreover
  from open_contains_cball[THEN iffD1, OF ‹open T›, rule_format, OF ‹t ∈ T›]
  obtain v where v: v > 0 cball t v ⊆ T by auto
  ultimately
  have compact (cball t v × cball x u) cball t v × cball x u ⊆ T × X
    by (auto intro!: compact_Times)
  then have compact (f' ` (cball t v × cball x u))
    by (auto intro!: compact_continuous_image continuous_on_subset[OF cont_f'])
  then obtain B where B: B > 0 ⋀s y. s ∈ cball t v ⟹ y ∈ cball x u ⟹ norm
(f' (s, y)) ≤ B
    by (auto dest!: compact_imp_bounded simp: bounded_pos)

  have lipschitz: B−lipschitz_on (cball x (min u v) ∩ X) (f s) if s: s ∈ cball t v
for s
  proof −
    note s
    also note ‹cball t v ⊆ T›
    finally
    have deriv: ⋀y. y ∈ cball x u ⟹ (f s has_derivative blinfun_apply (f' (s, y)))
(at y within cball x u)
      using ‹_ ⊆ X›
      by (auto intro!: has_derivative_at_withinI[OF f'])
    have norm (f s y − f s z) ≤ B ∗ norm (y − z)
      if y ∈ cball x u z ∈ cball x u
      for y z
      using s that
      by (intro differentiable_bound[OF convex_cball deriv])
        (auto intro!: B  simp: norm_blinfun.rep_eq[symmetric])
    then show ?thesis
      using ‹0 < B›
      by (auto intro!: lipschitz_onI simp: dist_norm)
  qed
  show ∃u>0. ∃L. ∀t∈cball t u ∩ T. L−lipschitz_on (cball x u ∩ X) (f t)
   by (force intro: exI[where x=min u v] exI[where x=B] intro!: lipschitz simp:
u v)
qed

end
theory
  Multivariate_Analysis
imports
  Ordered_Euclidean_Space
  Determinants
  Cross3
  Lipschitz
  Starlike
begin
```

Entry point excluding integration and complex analysis.

end

6.48 Volume of a Simplex

theory *Simplex_Content*
imports *Change_Of_Vars*
begin

lemma *fact_neq_top_ennreal* [*simp*]: *fact n* ≠ (*top* :: *ennreal*)
  by (*induction n*) (*auto simp*: *ennreal_mult_eq_top_iff*)

lemma *ennreal_fact*: *ennreal* (*fact n*) = *fact n*
 by (*induction n*) (*auto simp*: *ennreal_mult algebra_simps ennreal_of_nat_eq_real_of_nat*)

context
  fixes *S* :: *'a set* ⇒ *real* ⇒ (*'a* ⇒ *real*) *set*
  defines *S* ≡ (λ*A t*. {*x*. (∀ *i*∈*A*. *0* ≤ *x i*) ∧ *sum x A* ≤ *t*})
begin

lemma *emeasure_std_simplex_aux_step*:
  assumes *b* ∉ *A finite A*
  shows *x*(*b* := *y*) ∈ *S* (*insert b A*) *t* ⟷ *y* ∈ {*0*..*t*} ∧ *x* ∈ *S A* (*t* − *y*)
  using *assms sum_nonneg*[*of A x*] **unfolding** *S_def*
  by (*force simp*: *sum_delta_notmem algebra_simps*)

lemma *emeasure_std_simplex_aux*:
  fixes *t* :: *real*
  assumes *finite* (*A* :: *'a set*) *t* ≥ *0*
  shows *emeasure* (Pi_M *A* (λ_. *lborel*))
      (*S A t* ∩ *space* (Pi_M *A* (λ_. *lborel*))) = *t* ˆ *card A* / *fact* (*card A*)
  using *assms*(*1*,*2*)
proof (*induction arbitrary*: *t rule*: *finite_induct*)
  case (*empty t*)
  thus *?case* **by** (*simp add*: *PiM_empty S_def*)
next
  case (*insert b A t*)
  define *n* **where** *n* = *Suc* (*card A*)
  have *n_pos*: *n* > *0* **by** (*simp add*: *n_def*)
  let *?M* = λ*A*. (Pi_M *A* (λ_. *lborel*))
  {
    fix *A* :: *'a set* **and** *t* :: *real* **assume** *finite A*
    have *S A t* ∩ *space* (Pi_M *A* (λ_. *lborel*)) =
        Pi_E *A* (λ_. {*0*..}) ∩ (λ*x*. *sum x A*) −‘ {..*t*} ∩ *space* (Pi_M *A* (λ_. *lborel*))
      by (*auto simp*: *S_def space_PiM*)
    also have ... ∈ *sets* (Pi_M *A* (λ_. *lborel*))
      using ⟨*finite A*⟩ **by** *measurable*
    finally have *S A t* ∩ *space* (Pi_M *A* (λ_. *lborel*)) ∈ *sets* (Pi_M *A* (λ_. *lborel*)) **.**
  } **note** *meas* [*measurable*] = *this*

```
  interpret product_sigma_finite λ_. lborel
    by standard
  have emeasure (?M (insert b A)) (S (insert b A) t ∩ space (?M (insert b A)))
=
        nn_integral (?M (insert b A))
          (λx. indicator (S (insert b A) t ∩ space (?M (insert b A))) x)
    using insert.hyps by (subst nn_integral_indicator) auto
  also have ... = (∫+ y. ∫+ x. indicator (S (insert b A) t ∩ space (?M (insert
b A)))
                    (x(b := y)) ∂?M A ∂lborel)
    using insert.prems insert.hyps by (intro product_nn_integral_insert_rev) auto
  also have ... = (∫+ y. ∫+ x. indicator {0..t} y * indicator (S A (t − y) ∩
space (?M A)) x
                    ∂?M A ∂lborel)
    using insert.hyps insert.prems emeasure_std_simplex_aux_step[of b A]
    by (intro nn_integral_cong)
      (auto simp: fun_eq_iff indicator_def space_PiM PiE_def extensional_def)
  also have ... = (∫+ y. indicator {0..t} y * (∫+ x. indicator (S A (t − y) ∩
space (?M A)) x
                    ∂?M A) ∂lborel) using ‹finite A›
    by (subst nn_integral_cmult) auto
  also have ... = (∫+ y. indicator {0..t} y * emeasure (?M A) (S A (t − y) ∩
space (?M A)) ∂lborel)
    using ‹finite A› by (subst nn_integral_indicator) auto
  also have ... = (∫+ y. indicator {0..t} y * (t − y) ^ card A / ennreal (fact
(card A)) ∂lborel)
   using insert.IH by (intro nn_integral_cong) (auto simp: indicator_def divide_ennreal)
  also have ... = (∫+ y. indicator {0..t} y * (t − y) ^ card A ∂lborel) / ennreal
(fact (card A))
    using ‹finite A› by (subst nn_integral_divide) auto
  also have (∫+ y. indicator {0..t} y * (t − y) ^ card A ∂lborel) =
              (∫+y∈{0..t}. ennreal ((t − y) ^ (n − 1)) ∂lborel)
    by (intro nn_integral_cong) (auto simp: indicator_def n_def)
  also have ((λx. − ((t − x) ^ n / n)) has_real_derivative (t − x) ^ (n − 1)) (at
x)
    if x ∈ {0..t} for x by (rule derivative_eq_intros refl | simp add: n_pos)+
  hence (∫+y∈{0..t}. ennreal ((t − y) ^ (n − 1)) ∂lborel) =
          ennreal (−((t − t) ^ n / n) − (−((t − 0) ^ n / n)))
    using insert.prems insert.hyps by (intro nn_integral_FTC_Icc) auto
  also have ... = ennreal (t ^ n / n) using n_pos by (simp add: zero_power)
  also have ... / ennreal (fact (card A)) = ennreal (t ^ n / n / fact (card A))
    using n_pos ‹t ≥ 0› by (subst divide_ennreal) auto
  also have t ^ n / n / fact (card A) = t ^ n / fact n
    by (simp add: n_def)
  also have n = card (insert b A)
    using insert.hyps by (subst card.insert_remove) (auto simp: n_def)
  finally show ?case .
qed
```

end

lemma *emeasure_std_simplex*:
  emeasure lborel (*convex hull* (*insert 0 Basis* :: *'a* :: *euclidean_space set*)) =
    ennreal (*1* / *fact DIM*(*'a*))
proof –
  have *emeasure lborel* {*x*::*'a*. (∀ *i*∈*Basis*. *0* ≤ *x* · *i*) ∧ *sum* ((·) *x*) *Basis* ≤ *1*} =
        emeasure (*distr* (*Pi*$_M$ *Basis* (λ*b*. *lborel*)) *borel* (λ*f*. ∑ *b*∈*Basis*. *f b* $*_R$ *b*))
        {*x*::*'a*. (∀ *i*∈*Basis*. *0* ≤ *x* · *i*) ∧ *sum* ((·) *x*) *Basis* ≤ *1*}
    by (*subst lborel_eq*) *simp*
  also have ... = *emeasure* (*Pi*$_M$ *Basis* (λ*b*. *lborel*))
          ({*y*::*'a* ⇒ *real*. (∀ *i*∈*Basis*. *0* ≤ *y i*) ∧ *sum y Basis* ≤ *1*} ∩
          space (*Pi*$_M$ *Basis* (λ*b*. *lborel*)))
    by (*subst emeasure_distr*) *auto*
  also have ... = *ennreal* (*1* / *fact DIM*(*'a*))
    by (*subst emeasure_std_simplex_aux*) *auto*
  finally show *?thesis* **by** (*simp only*: *std_simplex*)
qed

theorem *content_std_simplex*:
  measure lborel (*convex hull* (*insert 0 Basis* :: *'a* :: *euclidean_space set*)) =
    1 / *fact DIM*(*'a*)
  by (*simp add*: *measure_def emeasure_std_simplex*)

proposition *measure_lebesgue_linear_transformation*:
  fixes *A* :: (*real* ^ *'n* :: {*finite*, *wellorder*}) *set*
  fixes *f* :: _ ⇒ *real* ^ *'n* :: {*finite*, *wellorder*}
  assumes *bounded A A* ∈ *sets lebesgue linear f*
  shows *measure lebesgue* (*f* ‘ *A*) = |*det* (*matrix f*)| ∗ *measure lebesgue A*
proof –
  from *assms* **have** [*intro*]: *A* ∈ *lmeasurable*
    by (*intro bounded_set_imp_lmeasurable*) *auto*
  hence [*intro*]: *f* ‘ *A* ∈ *lmeasurable*
    by (*intro lmeasure_integral measurable_linear_image assms*)
  have *measure lebesgue* (*f* ‘ *A*) = *integral* (*f* ‘ *A*) (λ_. *1*)
    by (*intro lmeasure_integral measurable_linear_image assms*) *auto*
  also have ... = *integral* (*f* ‘ *A*) (λ_. *1* :: *real* ^ *1*) $ *0*
  by (*subst integral_component_eq_cart* [*symmetric*]) (*auto intro*: *integrable_on_const*)
  also have ... = |*det* (*matrix f*)| ∗ *integral A* (λ*x*. *1* :: *real* ^ *1*) $ *0*
    using *assms*
    by (*subst integral_change_of_variables_linear*)
      (*auto simp*: *o_def absolutely_integrable_on_def intro*: *integrable_on_const*)
  also have *integral A* (λ*x*. *1* :: *real* ^ *1*) $ *0* = *integral A* (λ*x*. *1*)
  by (*subst integral_component_eq_cart* [*symmetric*]) (*auto intro*: *integrable_on_const*)
  also have ... = *measure lebesgue A*
    by (*intro lmeasure_integral* [*symmetric*]) *auto*
  finally show *?thesis* **.**

qed

theorem *content_simplex*:
fixes *X* :: (*real* ^ ′*n* :: {*finite*, *wellorder*}) *set* **and** *f* :: ′*n* :: _ ⇒ *real* ^ (′*n* :: _)
assumes *finite X card X* = *Suc CARD*(′*n*) **and** *x0*: *x0* ∈ *X* **and** *bij*: *bij_betw f UNIV* (*X* − {*x0*})
defines *M* ≡ (χ *i*. χ *j*. *f j* $ *i* − *x0* $ *i*)
shows *content* (*convex hull X*) = |*det M*| / *fact* (*CARD*(′*n*))
proof −
define *g* **where** *g* = (λ*x*. *M* ∗*v x*)
have [*simp*]: *M* ∗*v axis i 1* = *f i* − *x0* **for** *i* :: ′*n*
by (*simp add*: *M_def matrix_vector_mult_basis column_def vec_eq_iff*)
define *std* **where** *std* = (*convex hull insert 0 Basis* :: (*real* ^ ′*n* :: _) *set*)
have *compact*: *compact std* **unfolding** *std_def*
by (*intro finite_imp_compact_convex_hull*) *auto*

have *measure lebesgue* (*convex hull X*) = *measure lebesgue* (((+) (−*x0*)) ‘ (*convex hull X*))
by (*rule measure_translation* [*symmetric*])
also have ((+) (−*x0*)) ‘ (*convex hull X*) = *convex hull* (((+) (−*x0*)) ‘ *X*)
by (*rule convex_hull_translation* [*symmetric*])
also have ((+) (−*x0*)) ‘ *X* = *insert 0* ((λ*x*. *x* − *x0*) ‘ (*X* − {*x0*}))
using *x0* **by** (*auto simp*: *image_iff*)
finally have *eq*: *measure lebesgue* (*convex hull X*) = *measure lebesgue* (*convex hull* ...) **.**

from *compact* **have** *measure lebesgue* (*g* ‘ *std*) = |*det M*| ∗ *measure lebesgue std*
by (*subst measure_lebesgue_linear_transformation*)
(*auto intro*: *finite_imp_bounded_convex_hull dest*: *compact_imp_closed simp*: *g_def std_def*)
also have *measure lebesgue std* = *content std* **using** *compact*
by (*intro measure_completion*) (*auto dest*: *compact_imp_closed*)
also have *content std* = *1* / *fact CARD*(′*n*) **unfolding** *std_def*
by (*simp add*: *content_std_simplex*)
also have *g* ‘ *std* = *convex hull* (*g* ‘ *insert 0 Basis*) **unfolding** *std_def*
by (*rule convex_hull_linear_image*) (*auto simp*: *g_def*)
also have *g* ‘ *insert 0 Basis* = *insert 0* (*g* ‘ *Basis*)
by (*auto simp*: *g_def*)
also have *g* ‘ *Basis* = (λ*x*. *x* − *x0*) ‘ *range f*
by (*auto simp*: *g_def Basis_vec_def image_iff*)
also have *range f* = *X* − {*x0*} **using** *bij*
using *bij_betw_imp_surj_on* **by** *blast*
also note *eq* [*symmetric*]
finally show *?thesis*
using *finite_imp_compact_convex_hull*[*OF* ‹*finite X*›] **by** (*auto dest*: *compact_imp_closed*)
qed

theorem *content_triangle*:
fixes *A B C* :: *real* ^ *2*

shows *content* (*convex hull* {*A*, *B*, *C*}) =
|(*C* $ *1* − *A* $ *1*) ∗ (*B* $ *2* − *A* $ *2*) − (*B* $ *1* − *A* $ *1*) ∗ (*C* $ *2* − *A* $ *2*)| / *2*
proof −
define *M* :: *real* ^ *2* ^ *2* **where** *M* ≡ (χ *i*. χ *j*. (*if j* = *1 then B else C*) $ *i* − *A* $ *i*)
define *g* **where** *g* = (λ*x*. *M* ∗*v x*)
define *std* **where** *std* = (*convex hull insert 0 Basis* :: (*real* ^ *2*) *set*)
have [*simp*]: *M* ∗*v axis i 1* = (*if i* = *1 then B* − *A else C* − *A*) **for** *i*
by (*auto simp*: *M_def matrix_vector_mult_basis column_def vec_eq_iff*)
have *compact*: *compact std* **unfolding** *std_def*
by (*intro finite_imp_compact_convex_hull*) *auto*

have *measure lebesgue* (*convex hull* {*A*, *B*, *C*}) =
measure lebesgue (((+) (−*A*)) ‘ (*convex hull* {*A*, *B*, *C*}))
by (*rule measure_translation* [*symmetric*])
also have ((+) (−*A*)) ‘ (*convex hull* {*A*, *B*, *C*}) = *convex hull* (((+) (−*A*)) ‘ {*A*, *B*, *C*})
by (*rule convex_hull_translation* [*symmetric*])
also have ((+) (−*A*)) ‘ {*A*, *B*, *C*} = {*0*, *B* − *A*, *C* − *A*}
by (*auto simp*: *image_iff*)
finally have *eq*: *measure lebesgue* (*convex hull* {*A*, *B*, *C*}) =
measure lebesgue (*convex hull* {*0*, *B* − *A*, *C* − *A*}) **.**

from *compact* **have** *measure lebesgue* (*g* ‘ *std*) = |*det M*| ∗ *measure lebesgue std*
by (*subst measure_lebesgue_linear_transformation*)
(*auto intro*: *finite_imp_bounded_convex_hull dest*: *compact_imp_closed simp*: *g_def std_def*)
also have *measure lebesgue std* = *content std* **using** *compact*
by (*intro measure_completion*) (*auto dest*: *compact_imp_closed*)
also have *content std* = *1* / *2* **unfolding** *std_def*
by (*simp add*: *content_std_simplex*)
also have *g* ‘ *std* = *convex hull* (*g* ‘ *insert 0 Basis*) **unfolding** *std_def*
by (*rule convex_hull_linear_image*) (*auto simp*: *g_def*)
also have *g* ‘ *insert 0 Basis* = *insert 0* (*g* ‘ *Basis*)
by (*auto simp*: *g_def*)
also have (*2* :: *2*) ≠ *1* **by** *auto*
hence ¬(∀ *y*::*2*. *y* = *1*) **by** *blast*
hence *g* ‘ *Basis* = {*B* − *A*, *C* − *A*}
by (*auto simp*: *g_def Basis_vec_def image_iff*)
also note *eq* [*symmetric*]
finally show *?thesis*
using *finite_imp_compact_convex_hull*[*of* {*A*, *B*, *C*}]
by (*auto dest*!: *compact_imp_closed simp*: *det_2 M_def*)
qed

theorem *heron*:
fixes *A B C* :: *real* ^ *2*
defines *a* ≡ *dist B C* **and** *b* ≡ *dist A C* **and** *c* ≡ *dist A B*

```
  defines s ≡ (a + b + c) / 2
  shows   content (convex hull {A, B, C}) = sqrt (s * (s − a) * (s − b) * (s −
c))
proof −
  have [simp]: (UNIV :: 2 set) = {1, 2}
    using exhaust_2 by auto
  have dist_eq: dist (A :: real ^ 2) B ^ 2 = (A $ 1 − B $ 1) ^ 2 + (A $ 2 − B
$ 2) ^ 2
    for A B by (simp add: dist_vec_def dist_real_def)
  have nonneg: s * (s − a) * (s − b) * (s − c) ≥ 0
    using dist_triangle[of A B C] dist_triangle[of A C B] dist_triangle[of B C A]
   by (intro mult_nonneg_nonneg) (auto simp: s_def a_def b_def c_def dist_commute)

  have 16 * content (convex hull {A, B, C}) ^ 2 =
          4 * ((C $ 1 − A $ 1) * (B $ 2 − A $ 2) − (B $ 1 − A $ 1) * (C $ 2
− A $ 2)) ^ 2
    by (subst content_triangle) (simp add: power_divide)
  also have ... = (2 * (dist A B ^ 2 * dist A C ^ 2 + dist A B ^ 2 * dist B C ^
2 +
      dist A C ^ 2 * dist B C ^ 2) − (dist A B ^ 2) ^ 2 − (dist A C ^ 2) ^ 2 −
(dist B C ^ 2) ^ 2)
    unfolding dist_eq unfolding power2_eq_square by algebra
  also have ... = (a + b + c) * ((a + b + c) − 2 * a) * ((a + b + c) − 2 * b)
*
                   ((a + b + c) − 2 * c)
   unfolding power2_eq_square by (simp add: s_def a_def b_def c_def algebra_simps)
  also have ... = 16 * s * (s − a) * (s − b) * (s − c)
    by (simp add: s_def field_split_simps)
  finally have content (convex hull {A, B, C}) ^ 2 = s * (s − a) * (s − b) * (s
− c)
    by simp
  also have ... = sqrt (s * (s − a) * (s − b) * (s − c)) ^ 2
    by (intro real_sqrt_pow2 [symmetric] nonneg)
  finally show ?thesis using nonneg
    by (subst (asm) power2_eq_iff_nonneg) auto
qed

end
```

6.49 Convergence of Formal Power Series

```
theory FPS_Convergence
imports
  Generalised_Binomial_Theorem
  HOL−Computational_Algebra.Formal_Power_Series
begin
```

In this theory, we will connect formal power series (which are algebraic objects) with analytic functions. This will become more important in complex analysis, and indeed some of the less trivial results will only be proven there.

6.49.1 Balls with extended real radius

The following is a variant of *ball* that also allows an infinite radius.

definition *eball* :: *'a* :: *metric_space* ⇒ *ereal* ⇒ *'a set* **where**
eball z r = {*z'*. *ereal* (*dist z z'*) < *r*}

lemma *in_eball_iff* [*simp*]: *z* ∈ *eball z0 r* ⟷ *ereal* (*dist z0 z*) < *r*
by (*simp add*: *eball_def*)

lemma *eball_ereal* [*simp*]: *eball z* (*ereal r*) = *ball z r*
by *auto*

lemma *eball_inf* [*simp*]: *eball z* ∞ = *UNIV*
by *auto*

lemma *eball_empty* [*simp*]: *r* ≤ *0* ⟹ *eball z r* = {}
proof *safe*
fix *x* **assume** *r* ≤ *0 x* ∈ *eball z r*
hence *dist z x* < *r* **by** *simp*
also have ... ≤ *ereal 0* **using** ‹*r* ≤ *0*› **by** (*simp add*: *zero_ereal_def*)
finally show *x* ∈ {} **by** *simp*
qed

lemma *eball_conv_UNION_balls*:
eball z r = (⋃*r'*∈{*r'*. *ereal r'* < *r*}. *ball z r'*)
by (*cases r*) (*use dense gt_ex* **in** *force*)+

lemma *eball_mono*: *r* ≤ *r'* ⟹ *eball z r* ≤ *eball z r'*
by *auto*

lemma *ball_eball_mono*: *ereal r* ≤ *r'* ⟹ *ball z r* ≤ *eball z r'*
using *eball_mono*[*of ereal r r'*] **by** *simp*

lemma *open_eball* [*simp*, *intro*]: *open* (*eball z r*)
by (*cases r*) *auto*

6.49.2 Basic properties of convergent power series

definition *fps_conv_radius* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps* ⇒ *ereal* **where**
fps_conv_radius f = *conv_radius* (*fps_nth f*)

definition *eval_fps* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps* ⇒ *'a* ⇒ *'a* **where**
eval_fps f z = (∑ *n*. *fps_nth f n* ∗ *z* ^ *n*)

lemma *norm_summable_fps*:

fixes *f* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
shows *norm z* < *fps_conv_radius f* ⟹ *summable* (λ*n*. *norm* (*fps_nth f n* ∗ *z* ^ *n*))
by (*rule abs_summable_in_conv_radius*) (*simp_all add*: *fps_conv_radius_def*)

lemma *summable_fps*:
fixes *f* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
shows *norm z* < *fps_conv_radius f* ⟹ *summable* (λ*n*. *fps_nth f n* ∗ *z* ^ *n*)
by (*rule summable_in_conv_radius*) (*simp_all add*: *fps_conv_radius_def*)

theorem *sums_eval_fps*:
fixes *f* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
assumes *norm z* < *fps_conv_radius f*
shows (λ*n*. *fps_nth f n* ∗ *z* ^ *n*) *sums eval_fps f z*
using *assms* **unfolding** *eval_fps_def fps_conv_radius_def*
by (*intro summable_sums summable_in_conv_radius*) *simp_all*

lemma *continuous_on_eval_fps*:
fixes *f* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
shows *continuous_on* (*eball 0* (*fps_conv_radius f*)) (*eval_fps f*)
proof (*subst continuous_on_eq_continuous_at* [*OF open_eball*], *safe*)
fix *x* :: *'a* **assume** *x*: *x* ∈ *eball 0* (*fps_conv_radius f*)
define *r* **where** *r* = (*if fps_conv_radius f* = ∞ *then norm x* + *1 else*
(*norm x* + *real_of_ereal* (*fps_conv_radius f*)) / *2*)
have *r*: *norm x* < *r* ∧ *ereal r* < *fps_conv_radius f*
using *x* **by** (*cases fps_conv_radius f*)
(*auto simp*: *r_def eball_def split*: *if_splits*)

have *continuous_on* (*cball 0 r*) (λ*x*. ∑*i*. *fps_nth f i* ∗ (*x* − *0*) ^ *i*)
by (*rule powser_continuous_suminf*) (*insert r*, *auto simp*: *fps_conv_radius_def*)
hence *continuous_on* (*cball 0 r*) (*eval_fps f*)
by (*simp add*: *eval_fps_def*)
thus *isCont* (*eval_fps f*) *x*
by (*rule continuous_on_interior*) (*use r* **in** *auto*)
qed

lemma *continuous_on_eval_fps'* [*continuous_intros*]:
assumes *continuous_on A g*
assumes *g* ‘ *A* ⊆ *eball 0* (*fps_conv_radius f*)
shows *continuous_on A* (λ*x*. *eval_fps f* (*g x*))
using *continuous_on_compose2*[*OF continuous_on_eval_fps assms*] **.**

lemma *has_field_derivative_powser*:
fixes *z* :: *'a* :: {*banach*, *real_normed_field*}
assumes *ereal* (*norm z*) < *conv_radius f*
shows ((λ*z*. ∑*n*. *f n* ∗ *z* ^ *n*) *has_field_derivative* (∑*n*. *diffs f n* ∗ *z* ^ *n*)) (*at z within A*)
proof −
define *K* **where** *K* = (*if conv_radius f* = ∞ *then norm z* + *1*

```
                        else (norm z + real_of_ereal (conv_radius f)) / 2)
  have K: norm z < K ∧ ereal K < conv_radius f
    using assms by (cases conv_radius f) (auto simp: K_def)
  have 0 ≤ norm z by simp
  also from K have ... < K by simp
  finally have K_pos: K > 0 by simp

  have summable (λn. f n * of_real K ^ n)
    using K and K_pos by (intro summable_in_conv_radius) auto
  moreover from K and K_pos have norm z < norm (of_real K :: 'a) by auto
  ultimately show ?thesis
    by (rule has_field_derivative_at_within [OF termdiffs_strong])
qed

lemma has_field_derivative_eval_fps:
  fixes z :: 'a :: {banach, real_normed_field}
  assumes norm z < fps_conv_radius f
  shows   (eval_fps f has_field_derivative eval_fps (fps_deriv f) z) (at z within A)
proof -
  have (eval_fps f has_field_derivative eval_fps (Abs_fps (diffs (fps_nth f))) z) (at z
within A)
    using assms unfolding eval_fps_def fps_nth_Abs_fps fps_conv_radius_def
    by (intro has_field_derivative_powser) auto
  also have Abs_fps (diffs (fps_nth f)) = fps_deriv f
    by (simp add: fps_eq_iff diffs_def)
  finally show ?thesis .
qed

lemma holomorphic_on_eval_fps [holomorphic_intros]:
  fixes z :: 'a :: {banach, real_normed_field}
  assumes A ⊆ eball 0 (fps_conv_radius f)
  shows   eval_fps f holomorphic_on A
proof (rule holomorphic_on_subset [OF _ assms])
  show eval_fps f holomorphic_on eball 0 (fps_conv_radius f)
  proof (subst holomorphic_on_open [OF open_eball], safe, goal_cases)
    case (1 x)
    thus ?case
     by (intro exI[of _ eval_fps (fps_deriv f) x]) (auto intro: has_field_derivative_eval_fps)
  qed
qed

lemma analytic_on_eval_fps:
  fixes z :: 'a :: {banach, real_normed_field}
  assumes A ⊆ eball 0 (fps_conv_radius f)
  shows   eval_fps f analytic_on A
proof (rule analytic_on_subset [OF _ assms])
  show eval_fps f analytic_on eball 0 (fps_conv_radius f)
    using holomorphic_on_eval_fps[of eball 0 (fps_conv_radius f)]
    by (subst analytic_on_open) auto
```

qed

lemma *continuous_eval_fps* [*continuous_intros*]:
fixes *z* :: *'a*::{*real_normed_field*,*banach*}
assumes *norm z* < *fps_conv_radius F*
shows *continuous* (*at z within A*) (*eval_fps F*)
proof −
from *ereal_dense2*[*OF assms*] **obtain** *K* :: *real* **where** *K*: *norm z* < *K K* < *fps_conv_radius F*
by *auto*
have *0* ≤ *norm z* **by** *simp*
also have *norm z* < *K* **by** *fact*
finally have *K* > *0* **.**
from *K* **and** ‹*K* > *0*› **have** *summable* (λ*n*. *fps_nth F n* ∗ *of_real K* ˆ *n*)
by (*intro summable_fps*) *auto*
from *this* **have** *isCont* (*eval_fps F*) *z* **unfolding** *eval_fps_def*
by (*rule isCont_powser*) (*use K* **in** *auto*)
thus *continuous* (*at z within A*) (*eval_fps F*)
by (*simp add*: *continuous_at_imp_continuous_within*)
qed

6.49.3 Lower bounds on radius of convergence

lemma *fps_conv_radius_deriv*:
fixes *f* :: *'a* :: {*banach*, *real_normed_field*} *fps*
shows *fps_conv_radius* (*fps_deriv f*) ≥ *fps_conv_radius f*
unfolding *fps_conv_radius_def*
proof (*rule conv_radius_geI_ex*)
fix *r* :: *real* **assume** *r*: *r* > *0 ereal r* < *conv_radius* (*fps_nth f*)
define *K* **where** *K* = (*if conv_radius* (*fps_nth f*) = ∞ *then r* + *1*
else (*real_of_ereal* (*conv_radius* (*fps_nth f*)) + *r*) / *2*)
have *K*: *r* < *K* ∧ *ereal K* < *conv_radius* (*fps_nth f*)
using *r* **by** (*cases conv_radius* (*fps_nth f*)) (*auto simp*: *K_def*)
have *summable* (λ*n*. *diffs* (*fps_nth f*) *n* ∗ *of_real r* ˆ *n*)
proof (*rule termdiff_converges*)
fix *x* :: *'a* **assume** *norm x* < *K*
hence *ereal* (*norm x*) < *ereal K* **by** *simp*
also have . . . < *conv_radius* (*fps_nth f*) **using** *K* **by** *simp*
finally show *summable* (λ*n*. *fps_nth f n* ∗ *x* ˆ *n*)
by (*intro summable_in_conv_radius*) *auto*
qed (*insert K r*, *auto*)
also have . . . = (λ*n*. *fps_nth* (*fps_deriv f*) *n* ∗ *of_real r* ˆ *n*)
by (*simp add*: *fps_deriv_def diffs_def*)
finally show ∃*z*::*'a*. *norm z* = *r* ∧ *summable* (λ*n*. *fps_nth* (*fps_deriv f*) *n* ∗ *z* ˆ *n*)
using *r* **by** (*intro exI*[*of* _ *of_real r*]) *auto*
qed

lemma *eval_fps_at_0*: *eval_fps f 0* = *fps_nth f 0*

by (*simp add*: *eval_fps_def*)

lemma *fps_conv_radius_norm* [*simp*]:
fps_conv_radius (*Abs_fps* (*λn. norm* (*fps_nth f n*))) = *fps_conv_radius f*
by (*simp add*: *fps_conv_radius_def*)

lemma *fps_conv_radius_const* [*simp*]: *fps_conv_radius* (*fps_const c*) = ∞
proof −
have *fps_conv_radius* (*fps_const c*) = *conv_radius* (*λ_. 0* :: *'a*)
unfolding *fps_conv_radius_def*
by (*intro conv_radius_cong eventually_mono*[*OF eventually_gt_at_top*[*of 0*]]) *auto*
thus *?thesis* **by** *simp*
qed

lemma *fps_conv_radius_0* [*simp*]: *fps_conv_radius 0* = ∞
by (*simp only*: *fps_const_0_eq_0* [*symmetric*] *fps_conv_radius_const*)

lemma *fps_conv_radius_1* [*simp*]: *fps_conv_radius 1* = ∞
by (*simp only*: *fps_const_1_eq_1* [*symmetric*] *fps_conv_radius_const*)

lemma *fps_conv_radius_numeral* [*simp*]: *fps_conv_radius* (*numeral n*) = ∞
by (*simp add*: *numeral_fps_const*)

lemma *fps_conv_radius_fps_X_power* [*simp*]: *fps_conv_radius* (*fps_X* ˆ *n*) = ∞
proof −
have *fps_conv_radius* (*fps_X* ˆ *n* :: *'a fps*) = *conv_radius* (*λ_. 0* :: *'a*)
unfolding *fps_conv_radius_def*
by (*intro conv_radius_cong eventually_mono*[*OF eventually_gt_at_top*[*of n*]])
(*auto simp*: *fps_X_power_iff*)
thus *?thesis* **by** *simp*
qed

lemma *fps_conv_radius_fps_X* [*simp*]: *fps_conv_radius fps_X* = ∞
using *fps_conv_radius_fps_X_power*[*of 1*] **by** (*simp only*: *power_one_right*)

lemma *fps_conv_radius_shift* [*simp*]:
fps_conv_radius (*fps_shift n f*) = *fps_conv_radius f*
by (*simp add*: *fps_conv_radius_def fps_shift_def conv_radius_shift*)

lemma *fps_conv_radius_cmult_left*:
c ≠ *0* ⟹ *fps_conv_radius* (*fps_const c* ∗ *f*) = *fps_conv_radius f*
unfolding *fps_conv_radius_def* **by** (*simp add*: *conv_radius_cmult_left*)

lemma *fps_conv_radius_cmult_right*:
c ≠ *0* ⟹ *fps_conv_radius* (*f* ∗ *fps_const c*) = *fps_conv_radius f*
unfolding *fps_conv_radius_def* **by** (*simp add*: *conv_radius_cmult_right*)

lemma *fps_conv_radius_uminus* [*simp*]:
fps_conv_radius (−*f*) = *fps_conv_radius f*

using *fps_conv_radius_cmult_left*[*of* −*1 f*]
by (*simp flip*: *fps_const_neg*)

lemma *fps_conv_radius_add*: *fps_conv_radius* (*f* + *g*) ≥ *min* (*fps_conv_radius f*) (*fps_conv_radius g*)
unfolding *fps_conv_radius_def* **using** *conv_radius_add_ge*[*of fps_nth f fps_nth g*]
by *simp*

lemma *fps_conv_radius_diff*: *fps_conv_radius* (*f* − *g*) ≥ *min* (*fps_conv_radius f*) (*fps_conv_radius g*)
using *fps_conv_radius_add*[*of f* −*g*] **by** *simp*

lemma *fps_conv_radius_mult*: *fps_conv_radius* (*f* ∗ *g*) ≥ *min* (*fps_conv_radius f*) (*fps_conv_radius g*)
using *conv_radius_mult_ge*[*of fps_nth f fps_nth g*]
by (*simp add*: *fps_mult_nth fps_conv_radius_def atLeast0AtMost*)

lemma *fps_conv_radius_power*: *fps_conv_radius* (*f* ^ *n*) ≥ *fps_conv_radius f*
proof (*induction n*)
case (*Suc n*)
hence *fps_conv_radius f* ≤ *min* (*fps_conv_radius f*) (*fps_conv_radius* (*f* ^ *n*))
by *simp*
also have . . . ≤ *fps_conv_radius* (*f* ∗ *f* ^ *n*)
by (*rule fps_conv_radius_mult*)
finally show *?case* **by** *simp*
qed *simp_all*

context
begin

lemma *natfun_inverse_bound*:
fixes *f* :: *'a* :: {*real_normed_field*} *fps*
assumes *fps_nth f 0* = *1* **and** *δ* > *0*
and *summable*: *summable* (*λn. norm* (*fps_nth f* (*Suc n*)) ∗ *δ* ^ *Suc n*)
and *le*: (∑*n. norm* (*fps_nth f* (*Suc n*)) ∗ *δ* ^ *Suc n*) ≤ *1*
shows *norm* (*natfun_inverse f n*) ≤ *inverse* (*δ* ^ *n*)
proof (*induction n rule*: *less_induct*)
case (*less m*)
show *?case*
proof (*cases m*)
case *0*
thus *?thesis* **using** *assms* **by** (*simp add*: *field_split_simps norm_inverse norm_divide*)
next
case [*simp*]: (*Suc n*)
have *norm* (*natfun_inverse f* (*Suc n*)) =
norm (∑ *i* = *Suc 0*..*Suc n*. *fps_nth f i* ∗ *natfun_inverse f* (*Suc n* − *i*))
(**is** _ = *norm ?S*) **using** *assms*
by (*simp add*: *field_simps norm_mult norm_divide del*: *sum.cl_ivl_Suc*)
also have *norm ?S* ≤ (∑ *i* = *Suc 0*..*Suc n*. *norm* (*fps_nth f i* ∗ *natfun_inverse*

```
f (Suc n − i)))
      by (rule norm_sum)
    also have ... ≤ (∑ i = Suc 0..Suc n. norm (fps_nth f i) / δ ^ (Suc n − i))
    proof (intro sum_mono, goal_cases)
      case (1 i)
      have norm (fps_nth f i ∗ natfun_inverse f (Suc n − i)) =
              norm (fps_nth f i) ∗ norm (natfun_inverse f (Suc n − i))
        by (simp add: norm_mult)
      also have ... ≤ norm (fps_nth f i) ∗ inverse (δ ^ (Suc n − i))
        using 1 by (intro mult_left_mono less.IH) auto
      also have ... = norm (fps_nth f i) / δ ^ (Suc n − i)
        by (simp add: field_split_simps)
      finally show ?case .
    qed
    also have ... = (∑ i = Suc 0..Suc n. norm (fps_nth f i) ∗ δ ^ i) / δ ^ Suc n
      by (subst sum_divide_distrib, rule sum.cong)
         (insert ‹δ > 0›, auto simp: field_simps power_diff)
    also have (∑ i = Suc 0..Suc n. norm (fps_nth f i) ∗ δ ^ i) =
               (∑ i=0..n. norm (fps_nth f (Suc i)) ∗ δ ^ (Suc i))
      by (subst sum.atLeast_Suc_atMost_Suc_shift) simp_all
    also have {0..n} = {..<Suc n} by auto
    also have (∑ i< Suc n. norm (fps_nth f (Suc i)) ∗ δ ^ (Suc i)) ≤
               (∑ n. norm (fps_nth f (Suc n)) ∗ δ ^ (Suc n))
     using ‹δ > 0› by (intro sum_le_suminf ballI mult_nonneg_nonneg zero_le_power
summable) auto
    also have ... ≤ 1 by fact
    finally show ?thesis using ‹δ > 0›
      by (simp add: divide_right_mono field_split_simps)
  qed
qed

private lemma fps_conv_radius_inverse_pos_aux:
  fixes f :: 'a :: {banach, real_normed_field} fps
  assumes fps_nth f 0 = 1 fps_conv_radius f > 0
  shows   fps_conv_radius (inverse f) > 0
proof −
  let ?R = fps_conv_radius f
  define h where h = Abs_fps (λn. norm (fps_nth f n))
  have [simp]: fps_conv_radius h = ?R by (simp add: h_def)
  have continuous_on (eball 0 (fps_conv_radius h)) (eval_fps h)
    by (intro continuous_on_eval_fps)
  hence ∗: open (eval_fps h −‘ A ∩ eball 0 ?R) if open A for A
    using that by (subst (asm) continuous_on_open_vimage) auto
  have open (eval_fps h −‘ {..<2} ∩ eball 0 ?R)
    by (rule ∗) auto
  moreover have 0 ∈ eval_fps h −‘ {..<2} ∩ eball 0 ?R
    using assms by (auto simp: eball_def zero_ereal_def eval_fps_at_0 h_def)
  ultimately obtain ε where ε: ε > 0 ball 0 ε ⊆ eval_fps h −‘ {..<2} ∩ eball 0
?R
```

```
    by (subst (asm) open_contains_ball_eq) blast+

  define δ where δ = real_of_ereal (min (ereal ε / 2) (?R / 2))
  have δ: 0 < δ ∧ δ < ε ∧ ereal δ < ?R
    using ‹ε > 0› and assms by (cases ?R) (auto simp: δ_def min_def)

  have summable: summable (λn. norm (fps_nth f n) ∗ δ ^ n)
   using δ by (intro summable_in_conv_radius) (simp_all add: fps_conv_radius_def)
  hence (λn. norm (fps_nth f n) ∗ δ ^ n) sums eval_fps h δ
    by (simp add: eval_fps_def summable_sums h_def)
  hence (λn. norm (fps_nth f (Suc n)) ∗ δ ^ Suc n) sums (eval_fps h δ − 1)
    by (subst sums_Suc_iff) (auto simp: assms)
  moreover {
    from δ have δ ∈ ball 0 ε by auto
    also have . . . ⊆ eval_fps h −‘ {..<2} ∩ eball 0 ?R by fact
    finally have eval_fps h δ < 2 by simp
  }
  ultimately have le: (∑n. norm (fps_nth f (Suc n)) ∗ δ ^ Suc n) ≤ 1
    by (simp add: sums_iff)
  from summable have summable: summable (λn. norm (fps_nth f (Suc n)) ∗ δ ^
Suc n)
    by (subst summable_Suc_iff)

  have 0 < δ using δ by blast
  also have δ = inverse (limsup (λn. ereal (inverse δ)))
    using δ by (subst Limsup_const) auto
  also have . . . ≤ conv_radius (natfun_inverse f)
    unfolding conv_radius_def
  proof (intro ereal_inverse_antimono Limsup_mono
           eventually_mono[OF eventually_gt_at_top[of 0]])
    fix n :: nat assume n: n > 0
    have root n (norm (natfun_inverse f n)) ≤ root n (inverse (δ ^ n))
      using n assms δ le summable
      by (intro real_root_le_mono natfun_inverse_bound) auto
    also have . . . = inverse δ
      using n δ by (simp add: power_inverse [symmetric] real_root_pos2)
    finally show ereal (inverse δ) ≥ ereal (root n (norm (natfun_inverse f n)))
      by (subst ereal_less_eq)
  next
    have 0 = limsup (λn. 0::ereal)
      by (rule Limsup_const [symmetric]) auto
    also have . . . ≤ limsup (λn. ereal (root n (norm (natfun_inverse f n))))
      by (intro Limsup_mono) (auto simp: real_root_ge_zero)
    finally show 0 ≤ . . . by simp
  qed
  also have . . . = fps_conv_radius (inverse f)
    using assms by (simp add: fps_conv_radius_def fps_inverse_def)
  finally show ?thesis by (simp add: zero_ereal_def)
qed
```

lemma *fps_conv_radius_inverse_pos*:
  fixes *f* :: *'a* :: {*banach*, *real_normed_field*} *fps*
  assumes *fps_nth f 0* ≠ *0* **and** *fps_conv_radius f* > *0*
  shows *fps_conv_radius* (*inverse f*) > *0*
proof −
  let *?c* = *fps_nth f 0*
  have *fps_conv_radius* (*inverse f*) = *fps_conv_radius* (*fps_const ?c* ∗ *inverse f*)
    using *assms* **by** (*subst fps_conv_radius_cmult_left*) *auto*
  also have *fps_const ?c* ∗ *inverse f* = *inverse* (*fps_const* (*inverse ?c*) ∗ *f*)
    using *assms* **by** (*simp add*: *fps_inverse_mult fps_const_inverse*)
  also have *fps_conv_radius* ... > *0* **using** *assms*
    by (*intro fps_conv_radius_inverse_pos_aux*)
      (*auto simp*: *fps_conv_radius_cmult_left*)
  finally show *?thesis* **.**
qed

end

lemma *fps_conv_radius_exp* [*simp*]:
  fixes *c* :: *'a* :: {*banach*, *real_normed_field*}
  shows *fps_conv_radius* (*fps_exp c*) = ∞
  unfolding *fps_conv_radius_def*
proof (*rule conv_radius_inftyI''*)
  fix *z* :: *'a*
  have (λ*n*. *norm* (*c* ∗ *z*) ˆ *n* /$_R$ *fact n*) *sums exp* (*norm* (*c* ∗ *z*))
    by (*rule exp_converges*)
  also have (λ*n*. *norm* (*c* ∗ *z*) ˆ *n* /$_R$ *fact n*) = (λ*n*. *norm* (*fps_nth* (*fps_exp c*) *n* ∗ *z* ˆ *n*))
    by (*rule ext*) (*simp add*: *norm_divide norm_mult norm_power field_split_simps*)
  finally have *summable* ... **by** (*simp add*: *sums_iff*)
  thus *summable* (λ*n*. *fps_nth* (*fps_exp c*) *n* ∗ *z* ˆ *n*)
    by (*rule summable_norm_cancel*)
qed

6.49.4 Evaluating power series

theorem *eval_fps_deriv*:
  assumes *norm z* < *fps_conv_radius f*
  shows *eval_fps* (*fps_deriv f*) *z* = *deriv* (*eval_fps f*) *z*
  by (*intro DERIV_imp_deriv* [*symmetric*] *has_field_derivative_eval_fps assms*)

theorem *fps_nth_conv_deriv*:
  fixes *f* :: *complex fps*
  assumes *fps_conv_radius f* > *0*
  shows *fps_nth f n* = (*deriv* ˆˆ *n*) (*eval_fps f*) *0* / *fact n*
  using *assms*
proof (*induction n arbitrary*: *f*)
  case *0*

thus *?case* **by** (*simp add*: *eval_fps_def*)
next
case (*Suc n f*)
have (*deriv ^^ Suc n*) (*eval_fps f*) *0* = (*deriv ^^ n*) (*deriv* (*eval_fps f*)) *0*
unfolding *funpow_Suc_right o_def* **..**
also have *eventually* (λ*z*::*complex*. *z* ∈ *eball 0* (*fps_conv_radius f*)) (*nhds 0*)
using *Suc.prems* **by** (*intro eventually_nhds_in_open*) (*auto simp*: *zero_ereal_def*)
hence *eventually* (λ*z*. *deriv* (*eval_fps f*) *z* = *eval_fps* (*fps_deriv f*) *z*) (*nhds 0*)
by *eventually_elim* (*simp add*: *eval_fps_deriv*)
hence (*deriv ^^ n*) (*deriv* (*eval_fps f*)) *0* = (*deriv ^^ n*) (*eval_fps* (*fps_deriv f*)) *0*
by (*intro higher_deriv_cong_ev refl*)
also have ... / *fact n* = *fps_nth* (*fps_deriv f*) *n*
using *Suc.prems fps_conv_radius_deriv*[*of f*]
by (*intro Suc.IH* [*symmetric*]) *auto*
also have ... / *of_nat* (*Suc n*) = *fps_nth f* (*Suc n*)
by (*simp add*: *fps_deriv_def del*: *of_nat_Suc*)
finally show *?case* **by** (*simp add*: *field_split_simps*)
qed

theorem *eval_fps_eqD*:
fixes *f g* :: *complex fps*
assumes *fps_conv_radius f* > *0 fps_conv_radius g* > *0*
assumes *eventually* (λ*z*. *eval_fps f z* = *eval_fps g z*) (*nhds 0*)
shows *f* = *g*
proof (*rule fps_ext*)
fix *n* :: *nat*
have *fps_nth f n* = (*deriv ^^ n*) (*eval_fps f*) *0* / *fact n*
using *assms* **by** (*intro fps_nth_conv_deriv*)
also have (*deriv ^^ n*) (*eval_fps f*) *0* = (*deriv ^^ n*) (*eval_fps g*) *0*
by (*intro higher_deriv_cong_ev refl assms*)
also have ... / *fact n* = *fps_nth g n*
using *assms* **by** (*intro fps_nth_conv_deriv* [*symmetric*])
finally show *fps_nth f n* = *fps_nth g n* **.**
qed

lemma *eval_fps_const* [*simp*]:
fixes *c* :: *'a* :: {*banach*, *real_normed_div_algebra*}
shows *eval_fps* (*fps_const c*) *z* = *c*
proof −
have (λ*n*::*nat*. *if n* ∈ {*0*} *then c else 0*) *sums* (∑*n*∈{*0*::*nat*}. *c*)
by (*rule sums_If_finite_set*) *auto*
also have *?this* ⟷ (λ*n*::*nat*. *fps_nth* (*fps_const c*) *n* ∗ *z* ^ *n*) *sums* (∑*n*∈{*0*::*nat*}. *c*)
by (*intro sums_cong*) *auto*
also have (∑*n*∈{*0*::*nat*}. *c*) = *c*
by *simp*
finally show *?thesis*
by (*simp add*: *eval_fps_def sums_iff*)

qed

lemma *eval_fps_0* [*simp*]:
  eval_fps (*0* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*) *z* = *0*
  by (*simp only*: *fps_const_0_eq_0* [*symmetric*] *eval_fps_const*)

lemma *eval_fps_1* [*simp*]:
  eval_fps (*1* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*) *z* = *1*
  by (*simp only*: *fps_const_1_eq_1* [*symmetric*] *eval_fps_const*)

lemma *eval_fps_numeral* [*simp*]:
  eval_fps (*numeral n* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*) *z* = *numeral n*
  by (*simp only*: *numeral_fps_const eval_fps_const*)

lemma *eval_fps_X_power* [*simp*]:
  eval_fps (*fps_X* ^ *m* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*) *z* = *z* ^ *m*
proof −
  have (λ*n*::*nat*. *if n* ∈ {*m*} *then z* ^ *n else 0* :: *'a*) *sums* (∑ *n*∈{*m*::*nat*}. *z* ^ *n*)
    by (*rule sums_If_finite_set*) *auto*
  also have *?this* ⟷ (λ*n*::*nat*. *fps_nth* (*fps_X* ^ *m*) *n* ∗ *z* ^ *n*) *sums* (∑ *n*∈{*m*::*nat*}. *z* ^ *n*)
    by (*intro sums_cong*) (*auto simp*: *fps_X_power_iff*)
  also have (∑ *n*∈{*m*::*nat*}. *z* ^ *n*) = *z* ^ *m*
    by *simp*
  finally show *?thesis*
    by (*simp add*: *eval_fps_def sums_iff*)
qed

lemma *eval_fps_X* [*simp*]:
  eval_fps (*fps_X* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*) *z* = *z*
  using *eval_fps_X_power*[*of 1 z*] **by** (*simp only*: *power_one_right*)

lemma *eval_fps_minus*:
  fixes *f* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
  assumes *norm z* < *fps_conv_radius f*
  shows *eval_fps* (−*f*) *z* = −*eval_fps f z*
  using *assms* **unfolding** *eval_fps_def*
  by (*subst suminf_minus* [*symmetric*]) (*auto intro*!: *summable_fps*)

lemma *eval_fps_add*:
  fixes *f g* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
  assumes *norm z* < *fps_conv_radius f norm z* < *fps_conv_radius g*
  shows *eval_fps* (*f* + *g*) *z* = *eval_fps f z* + *eval_fps g z*
  using *assms* **unfolding** *eval_fps_def*
  by (*subst suminf_add*) (*auto simp*: *ring_distribs intro*!: *summable_fps*)

lemma *eval_fps_diff*:
  fixes *f g* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*

```
  assumes norm z < fps_conv_radius f norm z < fps_conv_radius g
  shows   eval_fps (f − g) z = eval_fps f z − eval_fps g z
  using assms unfolding eval_fps_def
  by (subst suminf_diff) (auto simp: ring_distribs intro!: summable_fps)

lemma eval_fps_mult:
  fixes f g :: 'a :: {banach, real_normed_div_algebra, comm_ring_1} fps
  assumes norm z < fps_conv_radius f norm z < fps_conv_radius g
  shows   eval_fps (f ∗ g) z = eval_fps f z ∗ eval_fps g z
proof −
  have eval_fps f z ∗ eval_fps g z =
          (∑ k. ∑ i≤k. fps_nth f i ∗ fps_nth g (k − i) ∗ (z ˆ i ∗ z ˆ (k − i)))
    unfolding eval_fps_def
  proof (subst Cauchy_product)
    show summable (λk. norm (fps_nth f k ∗ z ˆ k)) summable (λk. norm (fps_nth
g k ∗ z ˆ k))
      by (rule norm_summable_fps assms)+
  qed (simp_all add: algebra_simps)
  also have (λk. ∑ i≤k. fps_nth f i ∗ fps_nth g (k − i) ∗ (z ˆ i ∗ z ˆ (k − i))) =
               (λk. ∑ i≤k. fps_nth f i ∗ fps_nth g (k − i) ∗ z ˆ k)
    by (intro ext sum.cong refl) (simp add: power_add [symmetric])
  also have suminf . . . = eval_fps (f ∗ g) z
    by (simp add: eval_fps_def fps_mult_nth atLeast0AtMost sum_distrib_right)
  finally show ?thesis ..
qed

lemma eval_fps_shift:
  fixes f :: 'a :: {banach, real_normed_div_algebra, comm_ring_1} fps
  assumes n ≤ subdegree f norm z < fps_conv_radius f
  shows   eval_fps (fps_shift n f) z = (if z = 0 then fps_nth f n else eval_fps f z /
z ˆ n)
proof (cases z = 0)
  case False
  have eval_fps (fps_shift n f ∗ fps_X ˆ n) z = eval_fps (fps_shift n f) z ∗ z ˆ n
    using assms by (subst eval_fps_mult) simp_all
  also from assms have fps_shift n f ∗ fps_X ˆ n = f
    by (simp add: fps_shift_times_fps_X_power)
  finally show ?thesis using False by (simp add: field_simps)
qed (simp_all add: eval_fps_at_0)

lemma eval_fps_exp [simp]:
  fixes c :: 'a :: {banach, real_normed_field}
  shows eval_fps (fps_exp c) z = exp (c ∗ z) unfolding eval_fps_def exp_def
  by (simp add: eval_fps_def exp_def scaleR_conv_of_real field_split_simps)
```

The case of division is more complicated and will therefore not be handled here. Handling division becomes much more easy using complex analysis, and we will do so once that is available.

6.49.5 Power series expansions of analytic functions

This predicate contains the notion that the given formal power series converges in some disc of positive radius around the origin and is equal to the given complex function there.

This relationship is unique in the sense that no complex function can have more than one formal power series to which it expands, and if two holomorphic functions that are holomorphic on a connected open set around the origin and have the same power series expansion, they must be equal on that set.

More concrete statements about the radius of convergence can usually be made, but for many purposes, the statment that the series converges to the function in some neighbourhood of the origin is enough, and that can be shown almost fully automatically in most cases, as there are straightforward introduction rules to show this.

In particular, when one wants to relate the coefficients of the power series to the values of the derivatives of the function at the origin, or if one wants to approximate the coefficients of the series with the singularities of the function, this predicate is enough.

```
definition
  has_fps_expansion :: ('a :: {banach,real_normed_div_algebra} ⇒ 'a) ⇒ 'a fps ⇒
bool
  (infixl has'_fps'_expansion 60)
  where (f has_fps_expansion F) ⟷
          fps_conv_radius F > 0 ∧ eventually (λz. eval_fps F z = f z) (nhds 0)

named_theorems fps_expansion_intros

lemma fps_nth_fps_expansion:
  fixes f :: complex ⇒ complex
  assumes f has_fps_expansion F
  shows   fps_nth F n = (deriv ^^ n) f 0 / fact n
proof –
  have fps_nth F n = (deriv ^^ n) (eval_fps F) 0 / fact n
    using assms by (intro fps_nth_conv_deriv) (auto simp: has_fps_expansion_def)
  also have (deriv ^^ n) (eval_fps F) 0 = (deriv ^^ n) f 0
   using assms by (intro higher_deriv_cong_ev) (auto simp: has_fps_expansion_def)
  finally show ?thesis .
qed

lemma has_fps_expansion_imp_continuous:
  fixes F :: 'a::{real_normed_field,banach} fps
  assumes f has_fps_expansion F
  shows   continuous (at 0 within A) f
proof –
  from assms have isCont (eval_fps F) 0
```

by (*intro continuous_eval_fps*) (*auto simp*: *has_fps_expansion_def zero_ereal_def*)
also have *?this* ⟷ *isCont f 0* **using** *assms*
by (*intro isCont_cong*) (*auto simp*: *has_fps_expansion_def*)
finally have *isCont f 0* **.**
thus *continuous* (*at 0 within A*) *f*
by (*simp add*: *continuous_at_imp_continuous_within*)
qed

lemma *has_fps_expansion_const* [*simp*, *intro*, *fps_expansion_intros*]:
(λ_. *c*) *has_fps_expansion fps_const c*
by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_0* [*simp*, *intro*, *fps_expansion_intros*]:
(λ_. *0*) *has_fps_expansion 0*
by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_1* [*simp*, *intro*, *fps_expansion_intros*]:
(λ_. *1*) *has_fps_expansion 1*
by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_numeral* [*simp*, *intro*, *fps_expansion_intros*]:
(λ_. *numeral n*) *has_fps_expansion numeral n*
by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_fps_X_power* [*fps_expansion_intros*]:
(λ*x*. *x* ^ *n*) *has_fps_expansion* (*fps_X* ^ *n*)
by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_fps_X* [*fps_expansion_intros*]:
(λ*x*. *x*) *has_fps_expansion fps_X*
by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_cmult_left* [*fps_expansion_intros*]:
fixes *c* :: *'a* :: {*banach*, *real_normed_div_algebra*, *comm_ring_1*}
assumes *f has_fps_expansion F*
shows (λ*x*. *c* ∗ *f x*) *has_fps_expansion fps_const c* ∗ *F*
proof (*cases c* = *0*)
case *False*
from *assms* **have** *eventually* (λ*z*. *z* ∈ *eball 0* (*fps_conv_radius F*)) (*nhds 0*)
by (*intro eventually_nhds_in_open*) (*auto simp*: *has_fps_expansion_def zero_ereal_def*)
moreover from *assms* **have** *eventually* (λ*z*. *eval_fps F z* = *f z*) (*nhds 0*)
by (*auto simp*: *has_fps_expansion_def*)
ultimately have *eventually* (λ*z*. *eval_fps* (*fps_const c* ∗ *F*) *z* = *c* ∗ *f z*) (*nhds 0*)
by *eventually_elim* (*simp_all add*: *eval_fps_mult*)
with *assms* **and** *False* **show** *?thesis*
by (*auto simp*: *has_fps_expansion_def fps_conv_radius_cmult_left*)
qed *auto*

lemma *has_fps_expansion_cmult_right* [*fps_expansion_intros*]:
fixes *c* :: *'a* :: {*banach*, *real_normed_div_algebra*, *comm_ring_1*}
assumes *f has_fps_expansion F*
shows (λ*x*. *f x* ∗ *c*) *has_fps_expansion F* ∗ *fps_const c*
proof −
have *F* ∗ *fps_const c* = *fps_const c* ∗ *F*
by (*intro fps_ext*) (*auto simp*: *mult.commute*)
with *has_fps_expansion_cmult_left* [*OF assms*] **show** *?thesis*
by (*simp add*: *mult.commute*)
qed

lemma *has_fps_expansion_minus* [*fps_expansion_intros*]:
assumes *f has_fps_expansion F*
shows (λ*x*. − *f x*) *has_fps_expansion* −*F*
proof −
from *assms* **have** *eventually* (λ*x*. *x* ∈ *eball 0* (*fps_conv_radius F*)) (*nhds 0*)
by (*intro eventually_nhds_in_open*) (*auto simp*: *has_fps_expansion_def zero_ereal_def*)
moreover from *assms* **have** *eventually* (λ*x*. *eval_fps F x* = *f x*) (*nhds 0*)
by (*auto simp*: *has_fps_expansion_def*)
ultimately have *eventually* (λ*x*. *eval_fps* (−*F*) *x* = −*f x*) (*nhds 0*)
by *eventually_elim* (*auto simp*: *eval_fps_minus*)
thus *?thesis* **using** *assms* **by** (*auto simp*: *has_fps_expansion_def*)
qed

lemma *has_fps_expansion_add* [*fps_expansion_intros*]:
assumes *f has_fps_expansion F g has_fps_expansion G*
shows (λ*x*. *f x* + *g x*) *has_fps_expansion F* + *G*
proof −
from *assms* **have** *0* < *min* (*fps_conv_radius F*) (*fps_conv_radius G*)
by (*auto simp*: *has_fps_expansion_def*)
also have . . . ≤ *fps_conv_radius* (*F* + *G*)
by (*rule fps_conv_radius_add*)
finally have *radius*: . . . > *0* **.**

from *assms* **have** *eventually* (λ*x*. *x* ∈ *eball 0* (*fps_conv_radius F*)) (*nhds 0*)
eventually (λ*x*. *x* ∈ *eball 0* (*fps_conv_radius G*)) (*nhds 0*)
by (*intro eventually_nhds_in_open*; *force simp*: *has_fps_expansion_def zero_ereal_def*)+
moreover have *eventually* (λ*x*. *eval_fps F x* = *f x*) (*nhds 0*)
and *eventually* (λ*x*. *eval_fps G x* = *g x*) (*nhds 0*)
using *assms* **by** (*auto simp*: *has_fps_expansion_def*)
ultimately have *eventually* (λ*x*. *eval_fps* (*F* + *G*) *x* = *f x* + *g x*) (*nhds 0*)
by *eventually_elim* (*auto simp*: *eval_fps_add*)
with *radius* **show** *?thesis* **by** (*auto simp*: *has_fps_expansion_def*)
qed

lemma *has_fps_expansion_diff* [*fps_expansion_intros*]:
assumes *f has_fps_expansion F g has_fps_expansion G*
shows (λ*x*. *f x* − *g x*) *has_fps_expansion F* − *G*
using *has_fps_expansion_add*[*of f F* λ*x*. − *g x* −*G*] *assms*

by (*simp add*: *has_fps_expansion_minus*)

lemma *has_fps_expansion_mult* [*fps_expansion_intros*]:
fixes *F G* :: *'a* :: {*banach*, *real_normed_div_algebra*, *comm_ring_1*} *fps*
assumes *f has_fps_expansion F g has_fps_expansion G*
shows (λ*x*. *f x* ∗ *g x*) *has_fps_expansion F* ∗ *G*
proof −
from *assms* **have** *0* < *min* (*fps_conv_radius F*) (*fps_conv_radius G*)
by (*auto simp*: *has_fps_expansion_def*)
also have ... ≤ *fps_conv_radius* (*F* ∗ *G*)
by (*rule fps_conv_radius_mult*)
finally have *radius*: ... > *0* **.**

from *assms* **have** *eventually* (λ*x*. *x* ∈ *eball 0* (*fps_conv_radius F*)) (*nhds 0*)
eventually (λ*x*. *x* ∈ *eball 0* (*fps_conv_radius G*)) (*nhds 0*)
by (*intro eventually_nhds_in_open*; *force simp*: *has_fps_expansion_def zero_ereal_def*)+
moreover have *eventually* (λ*x*. *eval_fps F x* = *f x*) (*nhds 0*)
and *eventually* (λ*x*. *eval_fps G x* = *g x*) (*nhds 0*)
using *assms* **by** (*auto simp*: *has_fps_expansion_def*)
ultimately have *eventually* (λ*x*. *eval_fps* (*F* ∗ *G*) *x* = *f x* ∗ *g x*) (*nhds 0*)
by *eventually_elim* (*auto simp*: *eval_fps_mult*)
with *radius* **show** *?thesis* **by** (*auto simp*: *has_fps_expansion_def*)
qed

lemma *has_fps_expansion_inverse* [*fps_expansion_intros*]:
fixes *F* :: *'a* :: {*banach*, *real_normed_field*} *fps*
assumes *f has_fps_expansion F*
assumes *fps_nth F 0* ≠ *0*
shows (λ*x*. *inverse* (*f x*)) *has_fps_expansion inverse F*
proof −
have *radius*: *fps_conv_radius* (*inverse F*) > *0*
using *assms* **unfolding** *has_fps_expansion_def*
by (*intro fps_conv_radius_inverse_pos*) *auto*
let *?R* = *min* (*fps_conv_radius F*) (*fps_conv_radius* (*inverse F*))
from *assms radius*
have *eventually* (λ*x*. *x* ∈ *eball 0* (*fps_conv_radius F*)) (*nhds 0*)
eventually (λ*x*. *x* ∈ *eball 0* (*fps_conv_radius* (*inverse F*))) (*nhds 0*)
by (*intro eventually_nhds_in_open*; *force simp*: *has_fps_expansion_def zero_ereal_def*)+
moreover have *eventually* (λ*z*. *eval_fps F z* = *f z*) (*nhds 0*)
using *assms* **by** (*auto simp*: *has_fps_expansion_def*)
ultimately have *eventually* (λ*z*. *eval_fps* (*inverse F*) *z* = *inverse* (*f z*)) (*nhds 0*)
proof *eventually_elim*
case (*elim z*)
hence *eval_fps* (*inverse F* ∗ *F*) *z* = *eval_fps* (*inverse F*) *z* ∗ *f z*
by (*subst eval_fps_mult*) *auto*
also have *eval_fps* (*inverse F* ∗ *F*) *z* = *1*
using *assms* **by** (*simp add*: *inverse_mult_eq_1*)
finally show *?case* **by** (*auto simp*: *field_split_simps*)

```
  qed
  with radius show ?thesis by (auto simp: has_fps_expansion_def)
qed

lemma has_fps_expansion_exp [fps_expansion_intros]:
  fixes c :: 'a :: {banach, real_normed_field}
  shows (λx. exp (c * x)) has_fps_expansion fps_exp c
  by (auto simp: has_fps_expansion_def)

lemma has_fps_expansion_exp1 [fps_expansion_intros]:
  (λx::'a :: {banach, real_normed_field}. exp x) has_fps_expansion fps_exp 1
  using has_fps_expansion_exp[of 1] by simp

lemma has_fps_expansion_exp_neg1 [fps_expansion_intros]:
  (λx::'a :: {banach, real_normed_field}. exp (−x)) has_fps_expansion fps_exp (−1)
  using has_fps_expansion_exp[of −1] by simp

lemma has_fps_expansion_deriv [fps_expansion_intros]:
  assumes f has_fps_expansion F
  shows   deriv f has_fps_expansion fps_deriv F
proof −
  have eventually (λz. z ∈ eball 0 (fps_conv_radius F)) (nhds 0)
    using assms by (intro eventually_nhds_in_open)
                   (auto simp: has_fps_expansion_def zero_ereal_def)
  moreover from assms have eventually (λz. eval_fps F z = f z) (nhds 0)
    by (auto simp: has_fps_expansion_def)
  then obtain s where open s 0 ∈ s and s: ⋀w. w ∈ s ⟹ eval_fps F w = f w
    by (auto simp: eventually_nhds)
  hence eventually (λw. w ∈ s) (nhds 0)
    by (intro eventually_nhds_in_open) auto
  ultimately have eventually (λz. eval_fps (fps_deriv F) z = deriv f z) (nhds 0)
  proof eventually_elim
    case (elim z)
    hence eval_fps (fps_deriv F) z = deriv (eval_fps F) z
      by (simp add: eval_fps_deriv)
    also have eventually (λw. w ∈ s) (nhds z)
      using elim and ‹open s› by (intro eventually_nhds_in_open) auto
    hence eventually (λw. eval_fps F w = f w) (nhds z)
      by eventually_elim (simp add: s)
    hence deriv (eval_fps F) z = deriv f z
      by (intro deriv_cong_ev refl)
    finally show ?case .
  qed
  with assms and fps_conv_radius_deriv[of F] show ?thesis
    by (auto simp: has_fps_expansion_def)
qed

lemma fps_conv_radius_binomial:
  fixes c :: 'a :: {real_normed_field,banach}
```

shows *fps_conv_radius* (*fps_binomial c*) = (*if c* ∈ ℕ *then* ∞ *else 1*)
unfolding *fps_conv_radius_def* **by** (*simp add*: *conv_radius_gchoose*)

lemma *fps_conv_radius_ln*:
fixes *c* :: *'a* :: {*banach*, *real_normed_field*, *field_char_0*}
shows *fps_conv_radius* (*fps_ln c*) = (*if c* = *0 then* ∞ *else 1*)
proof (*cases c* = *0*)
case *False*
have *conv_radius* (λ*n*. *1* / *of_nat n* :: *'a*) = *1*
proof (*rule conv_radius_ratio_limit_nonzero*)
show (λ*n*. *norm* (*1* / *of_nat n* :: *'a*) / *norm* (*1* / *of_nat* (*Suc n*) :: *'a*)) ⟶
1
using *LIMSEQ_Suc_n_over_n* **by** (*simp add*: *norm_divide del*: *of_nat_Suc*)
qed *auto*
also have *conv_radius* (λ*n*. *1* / *of_nat n* :: *'a*) =
conv_radius (λ*n*. *if n* = *0 then 0 else* (− *1*) ^ (*n* − *1*) / *of_nat n* :: *'a*)
by (*intro conv_radius_cong eventually_mono*[*OF eventually_gt_at_top*[*of 0*]])
(*simp add*: *norm_mult norm_divide norm_power*)
finally show *?thesis* **using** *False* **unfolding** *fps_ln_def*
by (*subst fps_conv_radius_cmult_left*) (*simp_all add*: *fps_conv_radius_def*)
qed (*auto simp*: *fps_ln_def*)

lemma *fps_conv_radius_ln_nonzero* [*simp*]:
assumes *c* ≠ (*0* :: *'a* :: {*banach*,*real_normed_field*,*field_char_0*})
shows *fps_conv_radius* (*fps_ln c*) = *1*
using *assms* **by** (*simp add*: *fps_conv_radius_ln*)

lemma *fps_conv_radius_sin* [*simp*]:
fixes *c* :: *'a* :: {*banach*, *real_normed_field*, *field_char_0*}
shows *fps_conv_radius* (*fps_sin c*) = ∞
proof (*cases c* = *0*)
case *False*
have ∞ = *conv_radius* (λ*n*. *of_real* (*sin_coeff n*) :: *'a*)
proof (*rule sym*, *rule conv_radius_inftyI''*, *rule summable_norm_cancel*, *goal_cases*)
case (*1 z*)
show *?case* **using** *summable_norm_sin*[*of z*] **by** (*simp add*: *norm_mult*)
qed
also have ... / *norm c* = *conv_radius* (λ*n*. *c* ^ *n* ∗ *of_real* (*sin_coeff n*) :: *'a*)
using *False* **by** (*subst conv_radius_mult_power*) *auto*
also have ... = *fps_conv_radius* (*fps_sin c*) **unfolding** *fps_conv_radius_def*
by (*rule conv_radius_cong_weak*) (*auto simp add*: *fps_sin_def sin_coeff_def*)
finally show *?thesis* **by** *simp*
qed *simp_all*

lemma *fps_conv_radius_cos* [*simp*]:
fixes *c* :: *'a* :: {*banach*, *real_normed_field*, *field_char_0*}
shows *fps_conv_radius* (*fps_cos c*) = ∞
proof (*cases c* = *0*)
case *False*

have ∞ = *conv_radius* (λ*n*. *of_real* (*cos_coeff n*) :: *'a*)
proof (*rule sym*, *rule conv_radius_inftyI''*, *rule summable_norm_cancel*, *goal_cases*)
case (*1 z*)
show *?case* **using** *summable_norm_cos*[*of z*] **by** (*simp add*: *norm_mult*)
qed
also have ... / *norm c* = *conv_radius* (λ*n*. *c* ^ *n* ∗ *of_real* (*cos_coeff n*) :: *'a*)
using *False* **by** (*subst conv_radius_mult_power*) *auto*
also have ... = *fps_conv_radius* (*fps_cos c*) **unfolding** *fps_conv_radius_def*
by (*rule conv_radius_cong_weak*) (*auto simp add*: *fps_cos_def cos_coeff_def*)
finally show *?thesis* **by** *simp*
qed *simp_all*

lemma *eval_fps_sin* [*simp*]:
fixes *z* :: *'a* :: {*banach*, *real_normed_field*, *field_char_0*}
shows *eval_fps* (*fps_sin c*) *z* = *sin* (*c* ∗ *z*)
proof −
have (λ*n*. *sin_coeff n* $*_R$ (*c* ∗ *z*) ^ *n*) *sums sin* (*c* ∗ *z*) **by** (*rule sin_converges*)
also have (λ*n*. *sin_coeff n* $*_R$ (*c* ∗ *z*) ^ *n*) = (λ*n*. *fps_nth* (*fps_sin c*) *n* ∗ *z* ^ *n*)
by (*rule ext*) (*auto simp*: *sin_coeff_def fps_sin_def power_mult_distrib scaleR_conv_of_real*)
finally show *?thesis* **by** (*simp add*: *sums_iff eval_fps_def*)
qed

lemma *eval_fps_cos* [*simp*]:
fixes *z* :: *'a* :: {*banach*, *real_normed_field*, *field_char_0*}
shows *eval_fps* (*fps_cos c*) *z* = *cos* (*c* ∗ *z*)
proof −
have (λ*n*. *cos_coeff n* $*_R$ (*c* ∗ *z*) ^ *n*) *sums cos* (*c* ∗ *z*) **by** (*rule cos_converges*)
also have (λ*n*. *cos_coeff n* $*_R$ (*c* ∗ *z*) ^ *n*) = (λ*n*. *fps_nth* (*fps_cos c*) *n* ∗ *z* ^ *n*)
by (*rule ext*) (*auto simp*: *cos_coeff_def fps_cos_def power_mult_distrib scaleR_conv_of_real*)
finally show *?thesis* **by** (*simp add*: *sums_iff eval_fps_def*)
qed

lemma *cos_eq_zero_imp_norm_ge*:
assumes *cos* (*z* :: *complex*) = *0*
shows *norm z* ≥ *pi* / *2*
proof −
from *assms* **obtain** *n* **where** *z* = *complex_of_real* ((*of_int n* + *1* / *2*) ∗ *pi*)
by (*auto simp*: *cos_eq_0 algebra_simps*)
also have *norm* ... = |*real_of_int n* + *1* / *2*| ∗ *pi*
by (*subst norm_of_real*) (*simp_all add*: *abs_mult*)
also have *real_of_int n* + *1* / *2* = *of_int* (*2* ∗ *n* + *1*) / *2* **by** *simp*
also have |...| = *of_int* |*2* ∗ *n* + *1*| / *2* **by** (*subst abs_divide*) *simp_all*
also have ... ∗ *pi* = *of_int* |*2* ∗ *n* + *1*| ∗ (*pi* / *2*) **by** *simp*
also have ... ≥ *of_int 1* ∗ (*pi* / *2*)
by (*intro mult_right_mono*, *subst of_int_le_iff*) (*auto simp*: *abs_if*)
finally show *?thesis* **by** *simp*
qed

lemma *eval_fps_binomial*:
  fixes *c* :: *complex*
  assumes *norm z < 1*
  shows *eval_fps* (*fps_binomial c*) *z* = (*1* + *z*) *powr c*
  using *gen_binomial_complex*[*OF assms*] **by** (*simp add*: *sums_iff eval_fps_def*)

lemma *has_fps_expansion_binomial_complex* [*fps_expansion_intros*]:
  fixes *c* :: *complex*
  shows (λ*x*. (*1* + *x*) *powr c*) *has_fps_expansion fps_binomial c*
proof −
  have ∗: *eventually* (λ*z*::*complex*. *z* ∈ *eball 0 1*) (*nhds 0*)
    by (*intro eventually_nhds_in_open*) *auto*
  thus *?thesis*
    by (*auto simp*: *has_fps_expansion_def eval_fps_binomial fps_conv_radius_binomial*
          intro!: *eventually_mono* [*OF* ∗])
qed

lemma *has_fps_expansion_sin* [*fps_expansion_intros*]:
  fixes *c* :: *'a* :: {*banach*, *real_normed_field*, *field_char_0*}
  shows (λ*x*. *sin* (*c* ∗ *x*)) *has_fps_expansion fps_sin c*
  by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_sin'* [*fps_expansion_intros*]:
  (λ*x*::*'a* :: {*banach*, *real_normed_field*}. *sin x*) *has_fps_expansion fps_sin 1*
  using *has_fps_expansion_sin*[*of 1*] **by** *simp*

lemma *has_fps_expansion_cos* [*fps_expansion_intros*]:
  fixes *c* :: *'a* :: {*banach*, *real_normed_field*, *field_char_0*}
  shows (λ*x*. *cos* (*c* ∗ *x*)) *has_fps_expansion fps_cos c*
  by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_cos'* [*fps_expansion_intros*]:
  (λ*x*::*'a* :: {*banach*, *real_normed_field*}. *cos x*) *has_fps_expansion fps_cos 1*
  using *has_fps_expansion_cos*[*of 1*] **by** *simp*

lemma *has_fps_expansion_shift* [*fps_expansion_intros*]:
  fixes *F* :: *'a* :: {*banach*, *real_normed_field*} *fps*
  assumes *f has_fps_expansion F* **and** *n* ≤ *subdegree F*
  assumes *c* = *fps_nth F n*
  shows (λ*x*. *if x* = *0 then c else f x* / *x* ˆ *n*) *has_fps_expansion* (*fps_shift n F*)
proof −
  have *eventually* (λ*x*. *x* ∈ *eball 0* (*fps_conv_radius F*)) (*nhds 0*)
   using *assms* **by** (*intro eventually_nhds_in_open*) (*auto simp*: *has_fps_expansion_def*
zero_ereal_def)
  moreover have *eventually* (λ*x*. *eval_fps F x* = *f x*) (*nhds 0*)
    using *assms* **by** (*auto simp*: *has_fps_expansion_def*)
  ultimately have *eventually* (λ*x*. *eval_fps* (*fps_shift n F*) *x* =
                  (*if x* = *0 then c else f x* / *x* ˆ *n*)) (*nhds 0*)

by *eventually_elim* (*auto simp*: *eval_fps_shift assms*)
with *assms* **show** *?thesis* **by** (*auto simp*: *has_fps_expansion_def*)
qed

lemma *has_fps_expansion_divide* [*fps_expansion_intros*]:
fixes *F G* :: *'a* :: {*banach*, *real_normed_field*} *fps*
assumes *f has_fps_expansion F* **and** *g has_fps_expansion G* **and**
subdegree G ≤ *subdegree F G* ≠ *0*
c = *fps_nth F* (*subdegree G*) / *fps_nth G* (*subdegree G*)
shows (λ*x*. *if x* = *0 then c else f x* / *g x*) *has_fps_expansion* (*F* / *G*)
proof −
define *n* **where** *n* = *subdegree G*
define *F'* **and** *G'* **where** *F'* = *fps_shift n F* **and** *G'* = *fps_shift n G*
have *F* = *F'* ∗ *fps_X* ^ *n G* = *G'* ∗ *fps_X* ^ *n* **unfolding** *F'_def G'_def n_def*
by (*rule fps_shift_times_fps_X_power* [*symmetric*] *le_refl* | *fact*)+
moreover from *assms* **have** *fps_nth G' 0* ≠ *0*
by (*simp add*: *G'_def n_def*)
ultimately have *FG*: *F* / *G* = *F'* ∗ *inverse G'*
by (*simp add*: *fps_divide_unit*)

have (λ*x*. (*if x* = *0 then fps_nth F n else f x* / *x* ^ *n*) ∗
inverse (*if x* = *0 then fps_nth G n else g x* / *x* ^ *n*)) *has_fps_expansion F* / *G*
(**is** *?h has_fps_expansion _*) **unfolding** *FG F'_def G'_def n_def* **using** ⟨*G* ≠ *0*⟩
by (*intro has_fps_expansion_mult has_fps_expansion_inverse*
has_fps_expansion_shift assms) *auto*
also have *?h* = (λ*x*. *if x* = *0 then c else f x* / *g x*)
using *assms*(*5*) **unfolding** *n_def*
by (*intro ext*) (*auto split*: *if_splits simp*: *field_simps*)
finally show *?thesis* **.**
qed

lemma *has_fps_expansion_divide'* [*fps_expansion_intros*]:
fixes *F G* :: *'a* :: {*banach*, *real_normed_field*} *fps*
assumes *f has_fps_expansion F* **and** *g has_fps_expansion G* **and** *fps_nth G 0* ≠ *0*
shows (λ*x*. *f x* / *g x*) *has_fps_expansion* (*F* / *G*)
proof −
have (λ*x*. *if x* = *0 then fps_nth F 0* / *fps_nth G 0 else f x* / *g x*) *has_fps_expansion* (*F* / *G*)
(**is** *?h has_fps_expansion _*) **using** *assms*(*3*) **by** (*intro has_fps_expansion_divide assms*) *auto*
also from *assms* **have** *fps_nth F 0* = *f 0 fps_nth G 0* = *g 0*
by (*auto simp*: *has_fps_expansion_def eval_fps_at_0 dest*: *eventually_nhds_x_imp_x*)
hence *?h* = (λ*x*. *f x* / *g x*) **by** *auto*
finally show *?thesis* **.**
qed

lemma *has_fps_expansion_tan* [*fps_expansion_intros*]:
fixes *c* :: *'a* :: {*banach*, *real_normed_field*, *field_char_0*}

shows (λ*x*. *tan* (*c* ∗ *x*)) *has_fps_expansion fps_tan c*
proof −
have (λ*x*. *sin* (*c* ∗ *x*) / *cos* (*c* ∗ *x*)) *has_fps_expansion fps_sin c* / *fps_cos c*
by (*intro fps_expansion_intros*) *auto*
thus *?thesis* **by** (*simp add*: *tan_def fps_tan_def*)
qed

lemma *has_fps_expansion_tan′* [*fps_expansion_intros*]:
tan has_fps_expansion fps_tan (*1* :: ′*a* :: {*banach*, *real_normed_field*, *field_char_0*})
using *has_fps_expansion_tan*[*of 1*] **by** *simp*

lemma *has_fps_expansion_imp_holomorphic*:
assumes *f has_fps_expansion F*
obtains *s* **where** *open s 0* ∈ *s f holomorphic_on s* ⋀*z*. *z* ∈ *s* ⟹ *f z* = *eval_fps F z*
proof −
from *assms* **obtain** *s* **where** *s*: *open s 0* ∈ *s* ⋀*z*. *z* ∈ *s* ⟹ *eval_fps F z* = *f z*
unfolding *has_fps_expansion_def eventually_nhds* **by** *blast*
let *?s′* = *eball 0* (*fps_conv_radius F*) ∩ *s*
have *eval_fps F holomorphic_on ?s′*
by (*intro holomorphic_intros*) *auto*
also have *?this* ⟷ *f holomorphic_on ?s′*
using *s* **by** (*intro holomorphic_cong*) *auto*
finally show *?thesis* **using** *s assms*
by (*intro that*[*of ?s′*]) (*auto simp*: *has_fps_expansion_def zero_ereal_def*)
qed

end

6.50 Smooth paths

theory *Smooth_Paths*
imports
Retracts
begin

6.50.1 Homeomorphisms of arc images

lemma *path_connected_arc_complement*:
fixes γ :: *real* ⇒ ′*a*::*euclidean_space*
assumes *arc* γ *2* ≤ *DIM*(′*a*)
shows *path_connected*(− *path_image* γ)
proof −
have *path_image* γ *homeomorphic* {*0*..*1*::*real*}
by (*simp add*: *assms homeomorphic_arc_image_interval*)
then
show *?thesis*
apply (*rule path_connected_complement_homeomorphic_convex_compact*)
apply (*auto simp*: *assms*)

done
qed

lemma *connected_arc_complement*:
fixes γ :: *real* $\Rightarrow$ *'a::euclidean_space*
assumes *arc* γ *2* $\leq$ *DIM('a)*
shows *connected*($-$ *path_image* γ)
by (*simp add*: *assms path_connected_arc_complement path_connected_imp_connected*)

lemma *inside_arc_empty*:
fixes γ :: *real* $\Rightarrow$ *'a::euclidean_space*
assumes *arc* γ
shows *inside*(*path_image* γ) = {}
proof (*cases DIM('a)* = *1*)
case *True*
then show *?thesis*
using *assms connected_arc_image connected_convex_1_gen inside_convex* **by** *blast*
next
case *False*
show *?thesis*
proof (*rule inside_bounded_complement_connected_empty*)
show *connected* ($-$ *path_image* γ)
apply (*rule connected_arc_complement* [*OF assms*])
using *False*
by (*metis DIM_ge_Suc0 One_nat_def Suc_1 not_less_eq_eq order_class.order.antisym*)
show *bounded* (*path_image* γ)
by (*simp add*: *assms bounded_arc_image*)
qed
qed

lemma *inside_simple_curve_imp_closed*:
fixes γ :: *real* $\Rightarrow$ *'a::euclidean_space*
shows ⟦*simple_path* γ; *x* $\in$ *inside*(*path_image* γ)⟧ $\Longrightarrow$ *pathfinish* γ = *pathstart* γ
using *arc_simple_path inside_arc_empty* **by** *blast*

6.50.2 Piecewise differentiability of paths

lemma *continuous_on_joinpaths_D1*:
continuous_on {*0..1*} (*g1* +++ *g2*) $\Longrightarrow$ *continuous_on* {*0..1*} *g1*
apply (*rule continuous_on_eq* [*of* _ (*g1* +++ *g2*) $\circ$ ((*)(*inverse 2*))])
apply (*rule continuous_intros* | *simp*)+
apply (*auto elim*!: *continuous_on_subset simp*: *joinpaths_def*)
done

lemma *continuous_on_joinpaths_D2*:
⟦*continuous_on* {*0..1*} (*g1* +++ *g2*); *pathfinish g1* = *pathstart g2*⟧ $\Longrightarrow$ *continuous_on* {*0..1*} *g2*
apply (*rule continuous_on_eq* [*of* _ (*g1* +++ *g2*) $\circ$ (λ*x*. *inverse 2***x* + *1/2*)])

apply (*rule continuous_intros* | *simp*)+
apply (*auto elim*!: *continuous_on_subset simp add*: *joinpaths_def pathfinish_def pathstart_def Ball_def*)
done

lemma *piecewise_differentiable_D1*:
assumes (*g1* +++ *g2*) *piecewise_differentiable_on* {*0..1*}
shows *g1 piecewise_differentiable_on* {*0..1*}
proof −
obtain *S* **where** *cont*: *continuous_on* {*0..1*} *g1* **and** *finite S*
and *S*: ⋀*x*. *x* ∈ {*0..1*} − *S* ⟹ *g1* +++ *g2 differentiable at x within* {*0..1*}
using *assms* **unfolding** *piecewise_differentiable_on_def*
by (*blast dest*!: *continuous_on_joinpaths_D1*)
show *?thesis*
unfolding *piecewise_differentiable_on_def*
proof (*intro exI conjI ballI cont*)
show *finite* (*insert 1* (((∗)*2*) ‘ *S*))
by (*simp add*: ‹*finite S*›)
show *g1 differentiable at x within* {*0..1*} **if** *x* ∈ {*0..1*} − *insert 1* ((∗) *2* ‘ *S*) **for** *x*
proof (*rule_tac d=dist* (*x/2*) (*1/2*) **in** *differentiable_transform_within*)
have *g1* +++ *g2 differentiable at* (*x* / *2*) *within* {*0..1/2*}
by (*rule differentiable_subset* [*OF S* [*of x/2*]] | *use that* **in** *force*)+
then show *g1* +++ *g2* ∘ (∗) (*inverse 2*) *differentiable at x within* {*0..1*}
using *image_affinity_atLeastAtMost_div* [*of 2 0 0*::*real 1*]
by (*auto intro*: *differentiable_chain_within*)
qed (*use that* **in** ‹*auto simp*: *joinpaths_def*›)
qed
qed

lemma *piecewise_differentiable_D2*:
assumes (*g1* +++ *g2*) *piecewise_differentiable_on* {*0..1*} **and** *eq*: *pathfinish g1* = *pathstart g2*
shows *g2 piecewise_differentiable_on* {*0..1*}
proof −
have [*simp*]: *g1 1* = *g2 0*
using *eq* **by** (*simp add*: *pathfinish_def pathstart_def*)
obtain *S* **where** *cont*: *continuous_on* {*0..1*} *g2* **and** *finite S*
and *S*: ⋀*x*. *x* ∈ {*0..1*} − *S* ⟹ *g1* +++ *g2 differentiable at x within* {*0..1*}
using *assms* **unfolding** *piecewise_differentiable_on_def*
by (*blast dest*!: *continuous_on_joinpaths_D2*)
show *?thesis*
unfolding *piecewise_differentiable_on_def*
proof (*intro exI conjI ballI cont*)
show *finite* (*insert 0* ((λ*x*. *2*∗*x*−*1*)‘*S*))
by (*simp add*: ‹*finite S*›)
show *g2 differentiable at x within* {*0..1*} **if** *x* ∈ {*0..1*} − *insert 0* ((λ*x*. *2*∗*x*−*1*)‘*S*) **for** *x*
proof (*rule_tac d=dist* ((*x*+*1*)/*2*) (*1/2*) **in** *differentiable_transform_within*)

```
    have x2: (x + 1) / 2 ∉ S
      using that
      apply (clarsimp simp: image_iff)
      by (metis add.commute add_diff_cancel_left' mult_2 field_sum_of_halves)
    have g1 +++ g2 ∘ (λx. (x+1) / 2) differentiable at x within {0..1}
     by (rule differentiable_chain_within differentiable_subset [OF S [of (x+1)/2]]
| use x2 that in force)+
    then show g1 +++ g2 ∘ (λx. (x+1) / 2) differentiable at x within {0..1}
      by (auto intro: differentiable_chain_within)
     show (g1 +++ g2 ∘ (λx. (x + 1) / 2)) x' = g2 x' if x' ∈ {0..1} dist x' x
< dist ((x + 1) / 2) (1/2) for x'
    proof −
      have [simp]: (2*x'+2)/2 = x'+1
        by (simp add: field_split_simps)
      show ?thesis
        using that by (auto simp: joinpaths_def)
    qed
   qed (use that in ‹auto simp: joinpaths_def›)
  qed
qed

lemma piecewise_C1_differentiable_D1:
  fixes g1 :: real ⇒ 'a::real_normed_field
  assumes (g1 +++ g2) piecewise_C1_differentiable_on {0..1}
    shows g1 piecewise_C1_differentiable_on {0..1}
proof −
  obtain S where finite S
           and co12: continuous_on ({0..1} − S) (λx. vector_derivative (g1 +++
g2) (at x))
            and g12D: ∀x∈{0..1} − S. g1 +++ g2 differentiable at x
   using assms by (auto simp: piecewise_C1_differentiable_on_def C1_differentiable_on_eq)
  have g1D: g1 differentiable at x if x ∈ {0..1} − insert 1 ((*) 2 ` S) for x
  proof (rule differentiable_transform_within)
    show g1 +++ g2 ∘ (*) (inverse 2) differentiable at x
      using that g12D
      apply (simp only: joinpaths_def)
      by (rule differentiable_chain_at derivative_intros | force)+
    show ⋀x'. ⟦dist x' x < dist (x/2) (1/2)⟧
        ⟹ (g1 +++ g2 ∘ (*) (inverse 2)) x' = g1 x'
      using that by (auto simp: dist_real_def joinpaths_def)
  qed (use that in ‹auto simp: dist_real_def›)
  have [simp]: vector_derivative (g1 ∘ (*) 2) (at (x/2)) = 2 *R vector_derivative
g1 (at x)
              if x ∈ {0..1} − insert 1 ((*) 2 ` S) for x
    apply (subst vector_derivative_chain_at)
    using that
    apply (rule derivative_eq_intros g1D | simp)+
    done
 have continuous_on ({0..1/2} − insert (1/2) S) (λx. vector_derivative (g1 +++
```

g2) (*at x*))
  using *co12* **by** (*rule continuous_on_subset*) *force*
 then have *coDhalf*: *continuous_on* ({*0..1/2*} − *insert* (*1/2*) *S*) (λ*x*. *vector_derivative* (*g1* ∘ (∗)*2*) (*at x*))
 proof (*rule continuous_on_eq* [*OF _ vector_derivative_at*])
  show (*g1* +++ *g2 has_vector_derivative vector_derivative* (*g1* ∘ (∗) *2*) (*at x*)) (*at x*)
   if *x* ∈ {*0..1/2*} − *insert* (*1/2*) *S* **for** *x*
  proof (*rule has_vector_derivative_transform_within*)
   show (*g1* ∘ (∗) *2 has_vector_derivative vector_derivative* (*g1* ∘ (∗) *2*) (*at x*)) (*at x*)
    using *that*
     by (*force intro*: *g1D differentiable_chain_at simp*: *vector_derivative_works* [*symmetric*])
   show ⋀*x'*. ⟦*dist x' x* < *dist x* (*1/2*)⟧ ⟹ (*g1* ∘ (∗) *2*) *x'* = (*g1* +++ *g2*) *x'*
    using *that* **by** (*auto simp*: *dist_norm joinpaths_def*)
  qed (*use that* **in** ‹*auto simp*: *dist_norm*›)
 qed
 have *continuous_on* ({*0..1*} − *insert 1* ((∗) *2* ‘ *S*))
          ((λ*x*. *1/2* ∗ *vector_derivative* (*g1* ∘ (∗)*2*) (*at x*)) ∘ (∗)(*1/2*))
  apply (*rule continuous_intros*)+
  using *coDhalf*
  apply (*simp add*: *scaleR_conv_of_real image_set_diff image_image*)
  done
 then have *con_g1*: *continuous_on* ({*0..1*} − *insert 1* ((∗) *2* ‘ *S*)) (λ*x*. *vector_derivative g1* (*at x*))
  by (*rule continuous_on_eq*) (*simp add*: *scaleR_conv_of_real*)
 have *continuous_on* {*0..1*} *g1*
  using *continuous_on_joinpaths_D1 assms piecewise_C1_differentiable_on_def* **by** *blast*
 with ‹*finite S*› **show** *?thesis*
  apply (*clarsimp simp add*: *piecewise_C1_differentiable_on_def C1_differentiable_on_eq*)
  apply (*rule_tac x*=*insert 1* (((∗)*2*)‘*S*) **in** *exI*)
  apply (*simp add*: *g1D con_g1*)
 done
qed

lemma *piecewise_C1_differentiable_D2*:
 fixes *g2* :: *real* ⇒ *'a*::*real_normed_field*
 assumes (*g1* +++ *g2*) *piecewise_C1_differentiable_on* {*0..1*} *pathfinish g1* = *pathstart g2*
  shows *g2 piecewise_C1_differentiable_on* {*0..1*}
proof −
 obtain *S* **where** *finite S*
     and *co12*: *continuous_on* ({*0..1*} − *S*) (λ*x*. *vector_derivative* (*g1* +++ *g2*) (*at x*))
     and *g12D*: ∀ *x*∈{*0..1*} − *S*. *g1* +++ *g2 differentiable at x*
  using *assms* **by** (*auto simp*: *piecewise_C1_differentiable_on_def C1_differentiable_on_eq*)
 have *g2D*: *g2 differentiable at x* **if** *x* ∈ {*0..1*} − *insert 0* ((λ*x*. *2*∗*x*−*1*) ‘ *S*) **for**

x
  proof (*rule differentiable_transform_within*)
    show *g1* +++ *g2* ∘ (λ*x*. (*x* + *1*) / *2*) *differentiable at x*
      using *g12D that*
      apply (*simp only*: *joinpaths_def*)
      apply (*drule_tac x*= (*x*+*1*) / *2* **in** *bspec*, *force simp*: *field_split_simps*)
      apply (*rule differentiable_chain_at derivative_intros* | *force*)+
      done
    show ⋀*x'*. *dist x' x* < *dist* ((*x* + *1*) / *2*) (*1*/*2*) ⟹ (*g1* +++ *g2* ∘ (λ*x*. (*x* + *1*) / *2*)) *x'* = *g2 x'*
      using *that* **by** (*auto simp*: *dist_real_def joinpaths_def field_simps*)
    qed (*use that* **in** ‹*auto simp*: *dist_norm*›)
  have [*simp*]: *vector_derivative* (*g2* ∘ (λ*x*. *2*∗*x*−*1*)) (*at* ((*x*+*1*)/*2*)) = *2* $*_R$ *vector_derivative g2* (*at x*)
              if *x* ∈ {*0*..*1*} − *insert 0* ((λ*x*. *2*∗*x*−*1*) ‘ *S*) **for** *x*
    using *that* **by** (*auto simp*: *vector_derivative_chain_at field_split_simps g2D*)
  have *continuous_on* ({*1*/*2*..*1*} − *insert* (*1*/*2*) *S*) (λ*x*. *vector_derivative* (*g1* +++ *g2*) (*at x*))
    using *co12* **by** (*rule continuous_on_subset*) *force*
  then have *coDhalf*: *continuous_on* ({*1*/*2*..*1*} − *insert* (*1*/*2*) *S*) (λ*x*. *vector_derivative* (*g2* ∘ (λ*x*. *2*∗*x*−*1*)) (*at x*))
  proof (*rule continuous_on_eq* [*OF* _ *vector_derivative_at*])
    show (*g1* +++ *g2 has_vector_derivative vector_derivative* (*g2* ∘ (λ*x*. *2* ∗ *x* − *1*)) (*at x*))
          (*at x*)
      if *x* ∈ {*1* / *2*..*1*} − *insert* (*1* / *2*) *S* **for** *x*
   proof (*rule_tac f*=*g2* ∘ (λ*x*. *2*∗*x*−*1*) **and** *d*=*dist* (*3*/*4*) ((*x*+*1*)/*2*) **in** *has_vector_derivative_transform_*
      show (*g2* ∘ (λ*x*. *2* ∗ *x* − *1*) *has_vector_derivative vector_derivative* (*g2* ∘ (λ*x*. *2* ∗ *x* − *1*)) (*at x*))
           (*at x*)
      using *that* **by** (*force intro*: *g2D differentiable_chain_at simp*: *vector_derivative_works* [*symmetric*])
      show ⋀*x'*. ⟦*dist x' x* < *dist* (*3* / *4*) ((*x* + *1*) / *2*)⟧ ⟹ (*g2* ∘ (λ*x*. *2* ∗ *x* − *1*)) *x'* = (*g1* +++ *g2*) *x'*
        using *that* **by** (*auto simp*: *dist_norm joinpaths_def add_divide_distrib*)
    qed (*use that* **in** ‹*auto simp*: *dist_norm*›)
  qed
  have [*simp*]: ((λ*x*. (*x*+*1*) / *2*) ‘ ({*0*..*1*} − *insert 0* ((λ*x*. *2* ∗ *x* − *1*) ‘ *S*))) = ({*1*/*2*..*1*} − *insert* (*1*/*2*) *S*)
    apply (*simp add*: *image_set_diff inj_on_def image_image*)
    apply (*auto simp*: *image_affinity_atLeastAtMost_div add_divide_distrib*)
    done
  have *continuous_on* ({*0*..*1*} − *insert 0* ((λ*x*. *2*∗*x*−*1*) ‘ *S*))
                ((λ*x*. *1*/*2* ∗ *vector_derivative* (*g2* ∘ (λ*x*. *2*∗*x*−*1*)) (*at x*)) ∘ (λ*x*. (*x*+*1*)/*2*))
    by (*rule continuous_intros* | *simp add*: *coDhalf*)+
  then have *con_g2*: *continuous_on* ({*0*..*1*} − *insert 0* ((λ*x*. *2*∗*x*−*1*) ‘ *S*)) (λ*x*. *vector_derivative g2* (*at x*))
    by (*rule continuous_on_eq*) (*simp add*: *scaleR_conv_of_real*)

have *continuous_on {0..1} g2*
using *continuous_on_joinpaths_D2 assms piecewise_C1_differentiable_on_def* **by** *blast*
with ‹*finite S*› **show** *?thesis*
apply (*clarsimp simp add*: *piecewise_C1_differentiable_on_def C1_differentiable_on_eq*)
apply (*rule_tac x=insert 0 ((λx. 2 ∗ x − 1) ' S)* **in** *exI*)
apply (*simp add*: *g2D con_g2*)
done
qed

6.50.3 Valid paths, and their start and finish

definition *valid_path* :: (*real* ⇒ *'a* :: *real_normed_vector*) ⇒ *bool*
where *valid_path f* ≡ *f piecewise_C1_differentiable_on {0..1::real}*

definition *closed_path* :: (*real* ⇒ *'a* :: *real_normed_vector*) ⇒ *bool*
where *closed_path g* ≡ *g 0* = *g 1*

In particular, all results for paths apply

lemma *valid_path_imp_path*: *valid_path g* ⟹ *path g*
by (*simp add*: *path_def piecewise_C1_differentiable_on_def valid_path_def*)

lemma *connected_valid_path_image*: *valid_path g* ⟹ *connected*(*path_image g*)
by (*metis connected_path_image valid_path_imp_path*)

lemma *compact_valid_path_image*: *valid_path g* ⟹ *compact*(*path_image g*)
by (*metis compact_path_image valid_path_imp_path*)

lemma *bounded_valid_path_image*: *valid_path g* ⟹ *bounded*(*path_image g*)
by (*metis bounded_path_image valid_path_imp_path*)

lemma *closed_valid_path_image*: *valid_path g* ⟹ *closed*(*path_image g*)
by (*metis closed_path_image valid_path_imp_path*)

lemma *valid_path_compose*:
assumes *valid_path g*
and *der*: ⋀*x*. *x* ∈ *path_image g* ⟹ *f field_differentiable* (*at x*)
and *con*: *continuous_on* (*path_image g*) (*deriv f*)
shows *valid_path* (*f* ∘ *g*)
proof −
obtain *S* **where** *finite S* **and** *g_diff*: *g C1_differentiable_on {0..1}* − *S*
using ‹*valid_path g*› **unfolding** *valid_path_def piecewise_C1_differentiable_on_def* **by** *auto*
have *f* ∘ *g differentiable at t* **when** *t*∈*{0..1}* − *S* **for** *t*
proof (*rule differentiable_chain_at*)
show *g differentiable at t* **using** ‹*valid_path g*›
by (*meson C1_differentiable_on_eq* ‹*g C1_differentiable_on {0..1}* − *S*› *that*)
next
have *g t*∈*path_image g* **using** *that DiffD1 image_eqI path_image_def* **by** *metis*

then show *f differentiable at* (*g t*)
using *der*[*THEN field_differentiable_imp_differentiable*] **by** *auto*
qed
moreover have *continuous_on* ({*0..1*} − *S*) (λ*x*. *vector_derivative* (*f* ∘ *g*) (*at x*))
proof (*rule continuous_on_eq* [**where** *f* = λ*x*. *vector_derivative g* (*at x*) ∗ *deriv f* (*g x*)],
rule continuous_intros)
show *continuous_on* ({*0..1*} − *S*) (λ*x*. *vector_derivative g* (*at x*))
using *g_diff C1_differentiable_on_eq* **by** *auto*
next
have *continuous_on* {*0..1*} (λ*x*. *deriv f* (*g x*))
using *continuous_on_compose*[*OF _ con*[*unfolded path_image_def*],*unfolded comp_def*]
‹*valid_path g*› *piecewise_C1_differentiable_on_def valid_path_def*
by *blast*
then show *continuous_on* ({*0..1*} − *S*) (λ*x*. *deriv f* (*g x*))
using *continuous_on_subset* **by** *blast*
next
show *vector_derivative g* (*at t*) ∗ *deriv f* (*g t*) = *vector_derivative* (*f* ∘ *g*) (*at t*)
when *t* ∈ {*0..1*} − *S* **for** *t*
proof (*rule vector_derivative_chain_at_general*[*symmetric*])
show *g differentiable at t* **by** (*meson C1_differentiable_on_eq g_diff that*)
next
have *g t*∈*path_image g* **using** *that DiffD1 image_eqI path_image_def* **by** *metis*
then show *f field_differentiable at* (*g t*) **using** *der* **by** *auto*
qed
qed
ultimately have *f* ∘ *g C1_differentiable_on* {*0..1*} − *S*
using *C1_differentiable_on_eq* **by** *blast*
moreover have *path* (*f* ∘ *g*)
apply (*rule path_continuous_image*[*OF valid_path_imp_path*[*OF* ‹*valid_path g*›]])
using *der*
by (*simp add*: *continuous_at_imp_continuous_on field_differentiable_imp_continuous_at*)
ultimately show *?thesis* **unfolding** *valid_path_def piecewise_C1_differentiable_on_def path_def*
using ‹*finite S*› **by** *auto*
qed

lemma *valid_path_uminus_comp*[*simp*]:
fixes *g*::*real* ⇒ ′*a* ::*real_normed_field*
shows *valid_path* (*uminus* ∘ *g*) ⟷ *valid_path g*
proof
show *valid_path g* ⟹ *valid_path* (*uminus* ∘ *g*) **for** *g*::*real* ⇒ ′*a*
by (*auto intro*!: *valid_path_compose derivative_intros*)
then show *valid_path g* **when** *valid_path* (*uminus* ∘ *g*)
by (*metis fun.map_comp group_add_class.minus_comp_minus id_comp that*)

qed

lemma *valid_path_offset*[*simp*]:
  shows *valid_path* (λ*t*. *g t* − *z*) ⟷ *valid_path g*
proof
  show ∗: *valid_path* (*g*::*real*⇒′*a*) ⟹ *valid_path* (λ*t*. *g t* − *z*) **for** *g z*
    unfolding *valid_path_def*
   by (*fastforce intro*:*derivative_intros C1_differentiable_imp_piecewise piecewise_C1_differentiable_diff*)
  show *valid_path* (λ*t*. *g t* − *z*) ⟹ *valid_path g*
    using ∗[*of* λ*t*. *g t* − *z* −*z*,*simplified*] **.**
qed

lemma *valid_path_imp_reverse*:
  assumes *valid_path g*
    shows *valid_path*(*reversepath g*)
proof −
  obtain *S* **where** *finite S* **and** *S*: *g C1_differentiable_on* ({*0*..*1*} − *S*)
    using *assms* **by** (*auto simp*: *valid_path_def piecewise_C1_differentiable_on_def*)
  then have *finite* ((−) *1* ‘ *S*)
    by *auto*
  moreover have (*reversepath g C1_differentiable_on* ({*0*..*1*} − (−) *1* ‘ *S*))
    unfolding *reversepath_def*
    apply (*rule C1_differentiable_compose* [*of* λ*x*::*real*. *1*−*x* _ *g*, *unfolded o_def*])
    using *S*
    by (*force simp*: *finite_vimageI inj_on_def C1_differentiable_on_eq elim*!: *continuous_on_subset*)+
  ultimately show *?thesis* **using** *assms*
   by (*auto simp*: *valid_path_def piecewise_C1_differentiable_on_def path_def* [*symmetric*])
qed

lemma *valid_path_reversepath* [*simp*]: *valid_path*(*reversepath g*) ⟷ *valid_path g*
  using *valid_path_imp_reverse* **by** *force*

lemma *valid_path_join*:
  assumes *valid_path g1 valid_path g2 pathfinish g1* = *pathstart g2*
    shows *valid_path*(*g1* +++ *g2*)
proof −
  have *g1 1* = *g2 0*
    using *assms* **by** (*auto simp*: *pathfinish_def pathstart_def*)
  moreover have (*g1* ∘ (λ*x*. *2*∗*x*)) *piecewise_C1_differentiable_on* {*0*..*1*/*2*}
    apply (*rule piecewise_C1_differentiable_compose*)
    using *assms*
     apply (*auto simp*: *valid_path_def piecewise_C1_differentiable_on_def continuous_on_joinpaths*)
    apply (*force intro*: *finite_vimageI* [**where** *h* = (∗)*2*] *inj_onI*)
    done
  moreover have (*g2* ∘ (λ*x*. *2*∗*x*−*1*)) *piecewise_C1_differentiable_on* {*1*/*2*..*1*}
    apply (*rule piecewise_C1_differentiable_compose*)
    using *assms* **unfolding** *valid_path_def piecewise_C1_differentiable_on_def*

by (*auto intro*!: *continuous_intros finite_vimageI* [**where** *h* = (λ*x*. *2*∗*x* − *1*)] *inj_onI*
simp: *image_affinity_atLeastAtMost_diff continuous_on_joinpaths*)
ultimately show *?thesis*
apply (*simp only*: *valid_path_def continuous_on_joinpaths joinpaths_def*)
apply (*rule piecewise_C1_differentiable_cases*)
apply (*auto simp*: *o_def*)
done
qed

lemma *valid_path_join_D1*:
fixes *g1* :: *real* ⇒ *'a*::*real_normed_field*
shows *valid_path* (*g1* +++ *g2*) ⟹ *valid_path g1*
unfolding *valid_path_def*
by (*rule piecewise_C1_differentiable_D1*)

lemma *valid_path_join_D2*:
fixes *g2* :: *real* ⇒ *'a*::*real_normed_field*
shows ⟦*valid_path* (*g1* +++ *g2*); *pathfinish g1* = *pathstart g2*⟧ ⟹ *valid_path g2*
unfolding *valid_path_def*
by (*rule piecewise_C1_differentiable_D2*)

lemma *valid_path_join_eq* [*simp*]:
fixes *g2* :: *real* ⇒ *'a*::*real_normed_field*
shows *pathfinish g1* = *pathstart g2* ⟹ (*valid_path*(*g1* +++ *g2*) ⟷ *valid_path g1* ∧ *valid_path g2*)
using *valid_path_join_D1 valid_path_join_D2 valid_path_join* **by** *blast*

lemma *valid_path_shiftpath* [*intro*]:
assumes *valid_path g pathfinish g* = *pathstart g a* ∈ {*0*..*1*}
shows *valid_path*(*shiftpath a g*)
using *assms*
apply (*auto simp*: *valid_path_def shiftpath_alt_def*)
apply (*rule piecewise_C1_differentiable_cases*)
apply (*auto simp*: *algebra_simps*)
apply (*rule piecewise_C1_differentiable_affine* [*of g 1 a*, *simplified o_def scaleR_one*])
apply (*auto simp*: *pathfinish_def pathstart_def elim*: *piecewise_C1_differentiable_on_subset*)
apply (*rule piecewise_C1_differentiable_affine* [*of g 1 a*−*1*, *simplified o_def scaleR_one algebra_simps*])
apply (*auto simp*: *pathfinish_def pathstart_def elim*: *piecewise_C1_differentiable_on_subset*)
done

lemma *vector_derivative_linepath_within*:
x ∈ {*0*..*1*} ⟹ *vector_derivative* (*linepath a b*) (*at x within* {*0*..*1*}) = *b* − *a*
apply (*rule vector_derivative_within_cbox* [*of 0 1*::*real*, *simplified*])
apply (*auto simp*: *has_vector_derivative_linepath_within*)
done

lemma *vector_derivative_linepath_at* [*simp*]: *vector_derivative* (*linepath a b*) (*at x*)

```
= b − a
  by (simp add: has_vector_derivative_linepath_within vector_derivative_at)

lemma valid_path_linepath [iff]: valid_path (linepath a b)
 apply (simp add: valid_path_def piecewise_C1_differentiable_on_def C1_differentiable_on_eq
continuous_on_linepath)
  apply (rule_tac x={} in exI)
  apply (simp add: differentiable_on_def differentiable_def)
  using has_vector_derivative_def has_vector_derivative_linepath_within
  apply (fastforce simp add: continuous_on_eq_continuous_within)
  done

lemma valid_path_subpath:
  fixes g :: real ⇒ 'a :: real_normed_vector
  assumes valid_path g u ∈ {0..1} v ∈ {0..1}
    shows valid_path(subpath u v g)
proof (cases v=u)
  case True
  then show ?thesis
    unfolding valid_path_def subpath_def
    by (force intro: C1_differentiable_on_const C1_differentiable_imp_piecewise)
next
  case False
  have (g ∘ (λx. ((v−u) ∗ x + u))) piecewise_C1_differentiable_on {0..1}
    apply (rule piecewise_C1_differentiable_compose)
    apply (simp add: C1_differentiable_imp_piecewise)
     apply (simp add: image_affinity_atLeastAtMost)
    using assms False
   apply (auto simp: algebra_simps valid_path_def piecewise_C1_differentiable_on_subset)
    apply (subst Int_commute)
    apply (auto simp: inj_on_def algebra_simps crossproduct_eq finite_vimage_IntI)
    done
  then show ?thesis
    by (auto simp: o_def valid_path_def subpath_def)
qed

lemma valid_path_rectpath [simp, intro]: valid_path (rectpath a b)
  by (simp add: Let_def rectpath_def)

end
```

6.51 Neighbourhood bases and Locally path-connected spaces

```
theory Locally
  imports
    Path_Connected Function_Topology
begin
```

6.51.1 Neighbourhood Bases

Useful for "local" properties of various kinds

definition *neighbourhood_base_at* **where**
neighbourhood_base_at x P X ≡
∀ *W*. *openin X W* ∧ *x* ∈ *W*
⟶ (∃ *U V*. *openin X U* ∧ *P V* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *W*)

definition *neighbourhood_base_of* **where**
neighbourhood_base_of P X ≡
∀ *x* ∈ *topspace X*. *neighbourhood_base_at x P X*

lemma *neighbourhood_base_of*:
neighbourhood_base_of P X ⟷
(∀ *W x*. *openin X W* ∧ *x* ∈ *W*
⟶ (∃ *U V*. *openin X U* ∧ *P V* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *W*))
unfolding *neighbourhood_base_at_def neighbourhood_base_of_def*
using *openin_subset* **by** *blast*

lemma *neighbourhood_base_at_mono*:
⟦*neighbourhood_base_at x P X*; ⋀*S*. ⟦*P S*; *x* ∈ *S*⟧ ⟹ *Q S*⟧ ⟹ *neighbourhood_base_at x Q X*
unfolding *neighbourhood_base_at_def*
by (*meson subset_eq*)

lemma *neighbourhood_base_of_mono*:
⟦*neighbourhood_base_of P X*; ⋀*S*. *P S* ⟹ *Q S*⟧ ⟹ *neighbourhood_base_of Q X*
unfolding *neighbourhood_base_of_def*
using *neighbourhood_base_at_mono* **by** *force*

lemma *open_neighbourhood_base_at*:
(⋀*S*. ⟦*P S*; *x* ∈ *S*⟧ ⟹ *openin X S*)
⟹ *neighbourhood_base_at x P X* ⟷ (∀ *W*. *openin X W* ∧ *x* ∈ *W* ⟶ (∃ *U*. *P U* ∧ *x* ∈ *U* ∧ *U* ⊆ *W*))
unfolding *neighbourhood_base_at_def*
by (*meson subsetD*)

lemma *open_neighbourhood_base_of*:
(∀ *S*. *P S* ⟶ *openin X S*)
⟹ *neighbourhood_base_of P X* ⟷ (∀ *W x*. *openin X W* ∧ *x* ∈ *W* ⟶ (∃ *U*. *P U* ∧ *x* ∈ *U* ∧ *U* ⊆ *W*))
apply (*simp add*: *neighbourhood_base_of*, *safe*, *blast*)
by *meson*

lemma *neighbourhood_base_of_open_subset*:
⟦*neighbourhood_base_of P X*; *openin X S*⟧
⟹ *neighbourhood_base_of P* (*subtopology X S*)
apply (*clarsimp simp add*: *neighbourhood_base_of openin_subtopology_alt image_def*)
apply (*rename_tac x V*)

apply (*drule_tac x=S ∩ V* **in** *spec*)
apply (*simp add*: *Int_ac*)
by (*metis IntI le_infI1 openin_Int*)

lemma *neighbourhood_base_of_topology_base*:
openin X = arbitrary union_of (λ*W*. *W* ∈ ℬ)
⟹ *neighbourhood_base_of P X* ⟷
(∀ *W x*. *W* ∈ ℬ ∧ *x* ∈ *W* ⟶ (∃ *U V*. *openin X U* ∧ *P V* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *W*))
apply (*auto simp*: *openin_topology_base_unique neighbourhood_base_of*)
by (*meson subset_trans*)

lemma *neighbourhood_base_at_unlocalized*:
assumes ⋀*S T*. ⟦*P S*; *openin X T*; *x* ∈ *T*; *T* ⊆ *S*⟧ ⟹ *P T*
shows *neighbourhood_base_at x P X*
⟷ (*x* ∈ *topspace X* ⟶ (∃ *U V*. *openin X U* ∧ *P V* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *topspace X*))
(**is** *?lhs* = *?rhs*)
proof
assume *R*: *?rhs* **show** *?lhs*
unfolding *neighbourhood_base_at_def*
proof *clarify*
fix *W*
assume *openin X W x* ∈ *W*
then have *x* ∈ *topspace X*
using *openin_subset* **by** *blast*
with *R* **obtain** *U V* **where** *openin X U P V x* ∈ *U U* ⊆ *V V* ⊆ *topspace X*
by *blast*
then show ∃ *U V*. *openin X U* ∧ *P V* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *W*
by (*metis IntI* ‹*openin X W*› ‹*x* ∈ *W*› *assms inf_le1 inf_le2 openin_Int*)
qed
qed (*simp add*: *neighbourhood_base_at_def*)

lemma *neighbourhood_base_at_with_subset*:
⟦*openin X U*; *x* ∈ *U*⟧
⟹ (*neighbourhood_base_at x P X* ⟷ *neighbourhood_base_at x* (λ*T*. *T* ⊆ *U* ∧ *P T*) *X*)
apply (*simp add*: *neighbourhood_base_at_def*)
apply (*metis IntI Int_subset_iff openin_Int*)
done

lemma *neighbourhood_base_of_with_subset*:
neighbourhood_base_of P X ⟷ *neighbourhood_base_of* (λ*t*. *t* ⊆ *topspace X* ∧ *P t*) *X*
using *neighbourhood_base_at_with_subset*
by (*fastforce simp add*: *neighbourhood_base_of_def*)

6.51.2 Locally path-connected spaces

definition *weakly_locally_path_connected_at*
where *weakly_locally_path_connected_at x X* ≡ *neighbourhood_base_at x* (*path_connectedin X*) *X*

definition *locally_path_connected_at*
where *locally_path_connected_at x X* ≡
neighbourhood_base_at x (λ*U*. *openin X U* ∧ *path_connectedin X U*) *X*

definition *locally_path_connected_space*
where *locally_path_connected_space X* ≡ *neighbourhood_base_of* (*path_connectedin X*) *X*

lemma *locally_path_connected_space_alt*:
locally_path_connected_space X ⟷ *neighbourhood_base_of* (λ*U*. *openin X U* ∧ *path_connectedin X U*) *X*
(**is** *?P* = *?Q*)
and *locally_path_connected_space_eq_open_path_component_of*:
locally_path_connected_space X ⟷
(∀ *U x*. *openin X U* ∧ *x* ∈ *U* ⟶ *openin X* (*Collect* (*path_component_of* (*subtopology X U*) *x*)))
(**is** *?P* = *?R*)
proof −
have *?P* **if** *?Q*
using *locally_path_connected_space_def neighbourhood_base_of_mono that* **by** *auto*
moreover have *?R* **if** *P*: *?P*
proof −
show *?thesis*
proof *clarify*
fix *U y*
assume *openin X U y* ∈ *U*
have ∃ *T*. *openin X T* ∧ *x* ∈ *T* ∧ *T* ⊆ *Collect* (*path_component_of* (*subtopology X U*) *y*)
if *path_component_of* (*subtopology X U*) *y x* **for** *x*

proof −
have *x* ∈ *U*
using *path_component_of_equiv that topspace_subtopology* **by** *fastforce*
then have ∃ *Ua*. *openin X Ua* ∧ (∃ *V*. *path_connectedin X V* ∧ *x* ∈ *Ua* ∧ *Ua* ⊆ *V* ∧ *V* ⊆ *U*)
using *P*
by (*simp add*: ‹*openin X U*› *locally_path_connected_space_def neighbourhood_base_of*)
then show *?thesis*
by (*metis dual_order.trans path_component_of_equiv path_component_of_maximal path_connectedin_subtopology subset_iff that*)
qed
then show *openin X* (*Collect* (*path_component_of* (*subtopology X U*) *y*))
using *openin_subopen* **by** *force*

qed
qed
moreover have *?Q* **if** *?R*
using *that*
apply (*simp add*: *open_neighbourhood_base_of*)
by (*metis mem_Collect_eq openin_subset path_component_of_refl path_connectedin_path_component_of path_connectedin_subtopology that topspace_subtopology_subset*)
ultimately show *?P* = *?Q* *?P* = *?R*
by *blast+*
qed

lemma *locally_path_connected_space*:
locally_path_connected_space X
⟷ (∀ *V x*. *openin X V* ∧ *x* ∈ *V* ⟶ (∃ *U*. *openin X U* ∧ *path_connectedin X U* ∧ *x* ∈ *U* ∧ *U* ⊆ *V*))
by (*simp add*: *locally_path_connected_space_alt open_neighbourhood_base_of*)

lemma *locally_path_connected_space_open_path_components*:
locally_path_connected_space X ⟷
(∀ *U c*. *openin X U* ∧ *c* ∈ *path_components_of* (*subtopology X U*) ⟶ *openin X c*)
apply (*auto simp*: *locally_path_connected_space_eq_open_path_component_of path_components_of_def*)
by (*metis imageI inf.absorb_iff2 openin_closedin_eq*)

lemma *openin_path_component_of_locally_path_connected_space*:
locally_path_connected_space X ⟹ *openin X* (*Collect* (*path_component_of X x*))
apply (*auto simp*: *locally_path_connected_space_eq_open_path_component_of*)
by (*metis openin_empty openin_topspace path_component_of_eq_empty subtopology_topspace*)

lemma *openin_path_components_of_locally_path_connected_space*:
⟦*locally_path_connected_space X*; *c* ∈ *path_components_of X*⟧ ⟹ *openin X c*
apply (*auto simp*: *locally_path_connected_space_eq_open_path_component_of*)
by (*metis* (*no_types*, *lifting*) *imageE openin_topspace path_components_of_def subtopology_topspace*)

lemma *closedin_path_components_of_locally_path_connected_space*:
⟦*locally_path_connected_space X*; *c* ∈ *path_components_of X*⟧ ⟹ *closedin X c*
by (*simp add*: *closedin_def complement_path_components_of_Union openin_path_components_of_locally_path_connecte openin_clauses*(*3*) *path_components_of_subset*)

lemma *closedin_path_component_of_locally_path_connected_space*:
assumes *locally_path_connected_space X*
shows *closedin X* (*Collect* (*path_component_of X x*))
proof (*cases x* ∈ *topspace X*)
case *True*
then show *?thesis*
by (*simp add*: *assms closedin_path_components_of_locally_path_connected_space path_component_in_path_components_of*)

next
 case *False*
 then show *?thesis*
  by (*metis closedin_empty path_component_of_eq_empty*)
qed

lemma *weakly_locally_path_connected_at*:
 weakly_locally_path_connected_at x X ⟷
 (∀ *V*. *openin X V* ∧ *x* ∈ *V*
  ⟶ (∃ *U*. *openin X U* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧
   (∀ *y* ∈ *U*. ∃ *C*. *path_connectedin X C* ∧ *C* ⊆ *V* ∧ *x* ∈ *C* ∧ *y* ∈ *C*)))
  (**is** *?lhs* = *?rhs*)
proof
 assume *?lhs* **then show** *?rhs*
  apply (*simp add*: *weakly_locally_path_connected_at_def neighbourhood_base_at_def*)
  by (*meson order_trans subsetD*)
next
 have ∗: ∃ *V*. *path_connectedin X V* ∧ *U* ⊆ *V* ∧ *V* ⊆ *W*
  if (∀ *y*∈*U*. ∃ *C*. *path_connectedin X C* ∧ *C* ⊆ *W* ∧ *x* ∈ *C* ∧ *y* ∈ *C*)
  for *W U*
 proof (*intro exI conjI*)
  let *?V* = (*Collect* (*path_component_of* (*subtopology X W*) *x*))
   show *path_connectedin X* (*Collect* (*path_component_of* (*subtopology X W*) *x*))
   by (*meson path_connectedin_path_component_of path_connectedin_subtopology*)
   show *U* ⊆ *?V*
    by (*auto simp*: *path_component_of path_connectedin_subtopology that*)
   show *?V* ⊆ *W*
   by (*meson path_connectedin_path_component_of path_connectedin_subtopology*)
  qed
 assume *?rhs*
 then show *?lhs*
  unfolding *weakly_locally_path_connected_at_def neighbourhood_base_at_def* **by** (*metis* ∗)
qed

lemma *locally_path_connected_space_im_kleinen*:
 locally_path_connected_space X ⟷
  (∀ *V x*. *openin X V* ∧ *x* ∈ *V*
   ⟶ (∃ *U*. *openin X U* ∧
    x ∈ *U* ∧ *U* ⊆ *V* ∧
    (∀ *y* ∈ *U*. ∃ *c*. *path_connectedin X c* ∧
     c ⊆ *V* ∧ *x* ∈ *c* ∧ *y* ∈ *c*)))
 apply (*simp add*: *locally_path_connected_space_def neighbourhood_base_of_def*)
 apply (*simp add*: *weakly_locally_path_connected_at flip*: *weakly_locally_path_connected_at_def*)
 using *openin_subset* **apply** *force*
 done

lemma *locally_path_connected_space_open_subset*:
 ⟦*locally_path_connected_space X*; *openin X s*⟧

⟹ *locally_path_connected_space* (*subtopology* X s)
apply (*simp add*: *locally_path_connected_space_def neighbourhood_base_of openin_open_subtopology path_connectedin_subtopology*)
by (*meson order_trans*)

lemma *locally_path_connected_space_quotient_map_image*:
assumes f: *quotient_map* X Y f **and** X: *locally_path_connected_space* X
shows *locally_path_connected_space* Y
unfolding *locally_path_connected_space_open_path_components*
proof *clarify*
fix V C
assume V: *openin* Y V **and** c: C ∈ *path_components_of* (*subtopology* Y V)
then have *sub*: C ⊆ *topspace* Y
using *path_connectedin_path_components_of path_connectedin_subset_topspace path_connectedin_subtopology* **by** *blast*
have *openin* X {x ∈ *topspace* X. f x ∈ C}
proof (*subst openin_subopen*, *clarify*)
fix x
assume x: x ∈ *topspace* X **and** f x ∈ C
let ?T = *Collect* (*path_component_of* (*subtopology* X {z ∈ *topspace* X. f z ∈ V}) x)
show ∃ T. *openin* X T ∧ x ∈ T ∧ T ⊆ {x ∈ *topspace* X. f x ∈ C}
proof (*intro exI conjI*)
have ∃ U. *openin* X U ∧ ?T ∈ *path_components_of* (*subtopology* X U)
proof (*intro exI conjI*)
show *openin* X ({z ∈ *topspace* X. f z ∈ V})
using V f *openin_subset quotient_map_def* **by** *fastforce*
show *Collect* (*path_component_of* (*subtopology* X {z ∈ *topspace* X. f z ∈ V}) x)
∈ *path_components_of* (*subtopology* X {z ∈ *topspace* X. f z ∈ V})
by (*metis* (*no_types*, *lifting*) *Int_iff* ‹f x ∈ C› c *mem_Collect_eq path_component_in_path_components_of path_components_of_subset topspace_subtopology topspace_subtopology_subset* x)
qed
with X **show** *openin* X ?T
using *locally_path_connected_space_open_path_components* **by** *blast*
show x: x ∈ ?T
using V ‹f x ∈ C› c *openin_subset path_component_of_equiv path_components_of_subset topspace_subtopology topspace_subtopology_subset* x
by *fastforce*
have f ‘ ?T ⊆ C
proof (*rule path_components_of_maximal*)
show C ∈ *path_components_of* (*subtopology* Y V)
by (*simp add*: c)
show *path_connectedin* (*subtopology* Y V) (f ‘ ?T)
proof −
have *quotient_map* (*subtopology* X {a ∈ *topspace* X. f a ∈ V}) (*subtopology* Y V) f
by (*simp add*: V f *quotient_map_restriction*)
then show ?*thesis*

```
        by (meson path_connectedin_continuous_map_image path_connectedin_path_component_of
quotient_imp_continuous_map)
        qed
        show ¬ disjnt C (f ‘ ?T)
          by (metis (no_types, lifting) ‹f x ∈ C› x disjnt_iff image_eqI)
      qed
      then show ?T ⊆ {x ∈ topspace X. f x ∈ C}
        by (force simp: path_component_of_equiv)
    qed
  qed
  then show openin Y C
    using f sub by (simp add: quotient_map_def)
qed

lemma homeomorphic_locally_path_connected_space:
  assumes X homeomorphic_space Y
  shows locally_path_connected_space X ⟷ locally_path_connected_space Y
proof −
  obtain f g where homeomorphic_maps X Y f g
    using assms homeomorphic_space_def by fastforce
  then show ?thesis
   by (metis (no_types) homeomorphic_map_def homeomorphic_maps_map locally_path_connected_space_qu
qed

lemma locally_path_connected_space_retraction_map_image:
   ⟦retraction_map X Y r; locally_path_connected_space X⟧
       ⟹ locally_path_connected_space Y
 using Abstract_Topology.retraction_imp_quotient_map locally_path_connected_space_quotient_map_image
by blast

lemma locally_path_connected_space_euclideanreal: locally_path_connected_space euclideanreal
  unfolding locally_path_connected_space_def neighbourhood_base_of
  proof clarsimp
  fix W and x :: real
  assume open W x ∈ W
  then obtain e where e > 0 and e: ⋀x'. |x' − x| < e ⟶ x' ∈ W
    by (auto simp: open_real)
  then show ∃ U. open U ∧ (∃ V. path_connected V ∧ x ∈ U ∧ U ⊆ V ∧ V ⊆
W)
    by (force intro!: convex_imp_path_connected exI [where x = {x−e<..<x+e}])
qed

lemma locally_path_connected_space_discrete_topology:
   locally_path_connected_space (discrete_topology U)
  using locally_path_connected_space_open_path_components by fastforce

lemma path_component_eq_connected_component_of:
  assumes locally_path_connected_space X
  shows (path_component_of_set X x = connected_component_of_set X x)
```

proof (*cases x ∈ topspace X*)
case *True*
then show *?thesis*
using *connectedin_connected_component_of* [*of X x*]
apply (*clarsimp simp add*: *connectedin_def connected_space_clopen_in topspace_subtopology_subset cong*: *conj_cong*)
apply (*drule_tac x=path_component_of_set X x* **in** *spec*)
by (*metis assms closedin_closed_subtopology closedin_connected_component_of closedin_path_component_of_locally_path_connected_space inf.absorb_iff2 inf.orderE openin_path_component_of_locally_path_connected_space openin_subtopology path_component_of_eq_empty path_component_subset_connected_component_of*)
next
case *False*
then show *?thesis*
using *connected_component_of_eq_empty path_component_of_eq_empty* **by** *fastforce*
qed

lemma *path_components_eq_connected_components_of*:
locally_path_connected_space X ⟹ (*path_components_of X* = *connected_components_of X*)
by (*simp add*: *path_components_of_def connected_components_of_def image_def path_component_eq_connected_component_of*)

lemma *path_connected_eq_connected_space*:
locally_path_connected_space X
⟹ *path_connected_space X* ⟷ *connected_space X*
by (*metis connected_components_of_subset_sing path_components_eq_connected_components_of path_components_of_subset_singleton*)

lemma *locally_path_connected_space_product_topology*:
locally_path_connected_space(*product_topology X I*) ⟷
topspace(*product_topology X I*) = {} ∨
finite {*i*. *i* ∈ *I* ∧ ~*path_connected_space*(*X i*)} ∧
(∀ *i* ∈ *I*. *locally_path_connected_space*(*X i*))
(**is** *?lhs* ⟷ *?empty* ∨ *?rhs*)
proof (*cases ?empty*)
case *True*
then show *?thesis*
by (*simp add*: *locally_path_connected_space_def neighbourhood_base_of openin_closedin_eq*)
next
case *False*
then obtain *z* **where** *z*: *z* ∈ (Π_E *i*∈*I*. *topspace* (*X i*))
by *auto*
have *?rhs* **if** *L*: *?lhs*
proof −
obtain *U C* **where** *U*: *openin* (*product_topology X I*) *U*
and *V*: *path_connectedin* (*product_topology X I*) *C*
and *z* ∈ *U U* ⊆ *C* **and** *Csub*: *C* ⊆ (Π_E *i*∈*I*. *topspace* (*X i*))
using *L* **apply** (*clarsimp simp add*: *locally_path_connected_space_def neigh-*

bourhood_base_of)
  by (*metis openin_topspace topspace_product_topology z*)
 then obtain *V* **where** *finV*: *finite* $\{i \in I.\ V\ i \neq topspace\ (X\ i)\}$
  and *XV*: $\bigwedge i.\ i{\in}I \Longrightarrow$ *openin* $(X\ i)\ (V\ i)$ **and** $z \in Pi_E\ I\ V$ **and** *subU*: $Pi_E\ I\ V \subseteq U$
  by (*force simp*: *openin_product_topology_alt*)
 show *?thesis*
 proof (*intro conjI ballI*)
  have *path_connected_space* $(X\ i)$ **if** $i \in I\ V\ i = topspace\ (X\ i)$ **for** *i*
  proof −
   have *pc*: *path_connectedin* $(X\ i)\ ((\lambda x.\ x\ i)\ `\ C)$
    apply (*rule path_connectedin_continuous_map_image* [*OF _ V*])
    by (*simp add*: *continuous_map_product_projection* ‹$i \in I$›)
   moreover have $((\lambda x.\ x\ i)\ `\ C) = topspace\ (X\ i)$
   proof
    show $(\lambda x.\ x\ i)\ `\ C \subseteq topspace\ (X\ i)$
     by (*simp add*: *pc path_connectedin_subset_topspace*)
    have $V\ i \subseteq (\lambda x.\ x\ i)\ `\ (\Pi_E\ i{\in}I.\ V\ i)$
    by (*metis* ‹$z \in Pi_E\ I\ V$› *empty_iff image_projection_PiE order_refl that*(*1*))
    also have $\ldots \subseteq (\lambda x.\ x\ i)\ `\ U$
     using *subU* **by** *blast*
    finally show *topspace* $(X\ i) \subseteq (\lambda x.\ x\ i)\ `\ C$
     using ‹$U \subseteq C$› *that* **by** *blast*
   qed
   ultimately show *?thesis*
    by (*simp add*: *path_connectedin_topspace*)
  qed
  then have $\{i \in I.\ \neg\ path\_connected\_space\ (X\ i)\} \subseteq \{i \in I.\ V\ i \neq topspace\ (X\ i)\}$
   by *blast*
  with *finV* **show** *finite* $\{i \in I.\ \neg\ path\_connected\_space\ (X\ i)\}$
   using *finite_subset* **by** *blast*
 next
  show *locally_path_connected_space* $(X\ i)$ **if** $i \in I$ **for** *i*
  apply (*rule locally_path_connected_space_quotient_map_image* [*OF _ L*, **where** $f = \lambda x.\ x\ i$])
    by (*metis False Abstract_Topology.retraction_imp_quotient_map retraction_map_product_projection that*)
 qed
qed
moreover have *?lhs* **if** *R*: *?rhs*
proof (*clarsimp simp add*: *locally_path_connected_space_def neighbourhood_base_of*)
 fix *F z*
 assume *openin* (*product_topology X I*) *F* **and** $z \in F$
 then obtain *W* **where** *finW*: *finite* $\{i \in I.\ W\ i \neq topspace\ (X\ i)\}$
   and *opeW*: $\bigwedge i.\ i \in I \Longrightarrow$ *openin* $(X\ i)\ (W\ i)$ **and** $z \in Pi_E\ I\ W\ Pi_E\ I\ W \subseteq F$
  by (*auto simp*: *openin_product_topology_alt*)
 have $\forall i \in I.\ \exists\ U\ C.\ openin\ (X\ i)\ U \wedge path\_connectedin\ (X\ i)\ C \wedge z\ i \in U \wedge$

$U \subseteq C \wedge C \subseteq W\ i \wedge$
$(W\ i = topspace\ (X\ i) \wedge$
$path\_connected\_space\ (X\ i) \longrightarrow U = topspace\ (X\ i) \wedge C = topspace\ (X\ i))$
(**is** $\forall\, i \in I.\ ?\Phi\ i$)
proof
fix i **assume** $i \in I$
have *locally_path_connected_space* $(X\ i)$
by (*simp add*: R ‹$i \in I$›)
moreover have *openin* $(X\ i)$ $(W\ i)$ $z\ i \in W\ i$
using ‹$z \in Pi_E\ I\ W$› *opeW* ‹$i \in I$› **by** *auto*
ultimately obtain $U\ C$ **where** UC: *openin* $(X\ i)$ U *path_connectedin* $(X\ i)$ $C\ z\ i \in U\ U \subseteq C\ C \subseteq W\ i$
using ‹$i \in I$› **by** (*force simp*: *locally_path_connected_space_def neighbourhood_base_of*)
show $?\Phi\ i$
proof (*cases* $W\ i = topspace\ (X\ i) \wedge path\_connected\_space(X\ i)$)
case *True*
then show *?thesis*
using ‹$z\ i \in W\ i$› *path_connectedin_topspace* **by** *blast*
next
case *False*
then show *?thesis*
by (*meson UC*)
qed
qed
then obtain $U\ C$ **where**
$*$: $\bigwedge i.\ i \in I \Longrightarrow openin\ (X\ i)\ (U\ i) \wedge path\_connectedin\ (X\ i)\ (C\ i) \wedge z\ i \in (U\ i) \wedge (U\ i) \subseteq (C\ i) \wedge (C\ i) \subseteq W\ i \wedge$
$(W\ i = topspace\ (X\ i) \wedge path\_connected\_space\ (X\ i)$
$\longrightarrow (U\ i) = topspace\ (X\ i) \wedge (C\ i) = topspace\ (X\ i))$
by *metis*
let $?A = \{i \in I.\ \neg\ path\_connected\_space\ (X\ i)\} \cup \{i \in I.\ W\ i \neq topspace\ (X\ i)\}$
have $\{i \in I.\ U\ i \neq topspace\ (X\ i)\} \subseteq\ ?A$
by (*clarsimp simp add*: $*$)
moreover have *finite ?A*
by (*simp add*: *that finW*)
ultimately have *finite* $\{i \in I.\ U\ i \neq topspace\ (X\ i)\}$
using *finite_subset* **by** *auto*
then have *openin* (*product_topology* $X\ I$) $(Pi_E\ I\ U)$
using $*$ **by** (*simp add*: *openin_PiE_gen*)
then show $\exists\, U.\ openin\ (product\_topology\ X\ I)\ U \wedge$
$(\exists\, V.\ path\_connectedin\ (product\_topology\ X\ I)\ V \wedge z \in U \wedge U \subseteq V \wedge V \subseteq F)$
apply (*rule_tac x=PiE I U* **in** *exI*, *simp*)
apply (*rule_tac x=PiE I C* **in** *exI*)
using ‹$z \in Pi_E\ I\ W$› ‹$Pi_E\ I\ W \subseteq F$› $*$
apply (*simp add*: *path_connectedin_PiE subset_PiE PiE_iff PiE_mono dual_order.trans*)

```
    done
  qed
  ultimately show ?thesis
    using False by blast
qed

end
```

6.52 Euclidean space and n-spheres, as subtopologies of n-dimensional space

theory *Abstract_Euclidean_Space*
imports *Homotopy Locally*
begin

6.52.1 Euclidean spaces as abstract topologies

definition *Euclidean_space* :: *nat* ⇒ (*nat* ⇒ *real*) *topology*
  where *Euclidean_space n* ≡ *subtopology* (*powertop_real UNIV*) {*x*. ∀ *i*≥*n*. *x i* = *0*}

lemma *topspace_Euclidean_space*:
   topspace(*Euclidean_space n*) = {*x*. ∀ *i*≥*n*. *x i* = *0*}
  by (*simp add*: *Euclidean_space_def*)

lemma *nonempty_Euclidean_space*: *topspace*(*Euclidean_space n*) ≠ {}
  by (*force simp*: *topspace_Euclidean_space*)

lemma *subset_Euclidean_space* [*simp*]:
   topspace(*Euclidean_space m*) ⊆ *topspace*(*Euclidean_space n*) ⟷ *m* ≤ *n*
  apply (*simp add*: *topspace_Euclidean_space subset_iff*, *safe*)
   apply (*drule_tac x*=(λ*i*. *if i* < *m then 1 else 0*) **in** *spec*)
   apply *auto*
  using *not_less* **by** *fastforce*

lemma *topspace_Euclidean_space_alt*:
   topspace(*Euclidean_space n*) = (⋂ *i* ∈ {*n*..}. {*x*. *x* ∈ *topspace*(*powertop_real UNIV*) ∧ *x i* ∈ {*0*}})
  by (*auto simp*: *topspace_Euclidean_space*)

lemma *closedin_Euclidean_space*:
  closedin (*powertop_real UNIV*) (*topspace*(*Euclidean_space n*))
proof −
  have *closedin* (*powertop_real UNIV*) {*x*. *x i* = *0*} **if** *n* ≤ *i* **for** *i*
  proof −
    have *closedin* (*powertop_real UNIV*) {*x* ∈ *topspace* (*powertop_real UNIV*). *x i* ∈ {*0*}}
    proof (*rule closedin_continuous_map_preimage*)

```
      show continuous_map (powertop_real UNIV) euclideanreal (λx. x i)
        by (metis UNIV_I continuous_map_product_coordinates)
      show closedin euclideanreal {0}
        by simp
    qed
    then show ?thesis
      by auto
  qed
  then show ?thesis
    unfolding topspace_Euclidean_space_alt
    by force
qed

lemma closedin_Euclidean_imp_closed: closedin (Euclidean_space m) S ⟹ closed
S
  by (metis Euclidean_space_def closed_closedin closedin_Euclidean_space closedin_closed_subtopology
euclidean_product_topology topspace_Euclidean_space)

lemma closedin_Euclidean_space_iff:
  closedin (Euclidean_space m) S ⟷ closed S ∧ S ⊆ topspace (Euclidean_space
m)
  (is ?lhs ⟷ ?rhs)
proof
  show ?lhs ⟹ ?rhs
    using closedin_closed_subtopology topspace_Euclidean_space
    by (fastforce simp: topspace_Euclidean_space_alt closedin_Euclidean_imp_closed)
  show ?rhs ⟹ ?lhs
  apply (simp add: closedin_subtopology Euclidean_space_def)
    by (metis (no_types) closed_closedin euclidean_product_topology inf.orderE)
qed

lemma continuous_map_componentwise_Euclidean_space:
  continuous_map X (Euclidean_space n) (λx i. if i < n then f x i else 0) ⟷
    (∀ i < n. continuous_map X euclideanreal (λx. f x i))
proof −
  have *: continuous_map X euclideanreal (λx. if k < n then f x k else 0)
    if ⋀i. i<n ⟹ continuous_map X euclideanreal (λx. f x i) for k
    by (intro continuous_intros that)
  show ?thesis
    unfolding Euclidean_space_def continuous_map_in_subtopology
      by (fastforce simp: continuous_map_componentwise_UNIV ∗ elim: continu-
ous_map_eq)
qed

lemma continuous_map_Euclidean_space_add [continuous_intros]:
  ⟦continuous_map X (Euclidean_space n) f; continuous_map X (Euclidean_space
n) g⟧
    ⟹ continuous_map X (Euclidean_space n) (λx i. f x i + g x i)
  unfolding Euclidean_space_def continuous_map_in_subtopology
```

by (*fastforce simp add*: *continuous_map_componentwise_UNIV continuous_map_add*)

lemma *continuous_map_Euclidean_space_diff* [*continuous_intros*]:
⟦*continuous_map X* (*Euclidean_space n*) *f*; *continuous_map X* (*Euclidean_space n*) *g*⟧
⟹ *continuous_map X* (*Euclidean_space n*) (λ*x i*. *f x i* − *g x i*)
unfolding *Euclidean_space_def continuous_map_in_subtopology*
by (*fastforce simp add*: *continuous_map_componentwise_UNIV continuous_map_diff*)

lemma *continuous_map_Euclidean_space_iff*:
continuous_map (*Euclidean_space m*) *euclidean g*
= *continuous_on* (*topspace* (*Euclidean_space m*)) *g*
proof
assume *continuous_map* (*Euclidean_space m*) *euclidean g*
then have *continuous_map* (*top_of_set* {*f*. ∀ *n*≥*m*. *f n* = *0*}) *euclidean g*
by (*simp add*: *Euclidean_space_def euclidean_product_topology*)
then show *continuous_on* (*topspace* (*Euclidean_space m*)) *g*
by (*metis continuous_map_subtopology_eu subtopology_topspace topspace_Euclidean_space*)
next
assume *continuous_on* (*topspace* (*Euclidean_space m*)) *g*
then have *continuous_map* (*top_of_set* {*f*. ∀ *n*≥*m*. *f n* = *0*}) *euclidean g*
by (*metis* (*lifting*) *continuous_map_into_fulltopology continuous_map_subtopology_eu order_refl topspace_Euclidean_space*)
then show *continuous_map* (*Euclidean_space m*) *euclidean g*
by (*simp add*: *Euclidean_space_def euclidean_product_topology*)
qed

lemma *cm_Euclidean_space_iff_continuous_on*:
continuous_map (*subtopology* (*Euclidean_space m*) *S*) (*Euclidean_space n*) *f*
⟷ *continuous_on* (*topspace* (*subtopology* (*Euclidean_space m*) *S*)) *f* ∧
f ‘ (*topspace* (*subtopology* (*Euclidean_space m*) *S*)) ⊆ *topspace* (*Euclidean_space n*)
(**is** *?P* ⟷ *?Q* ∧ *?R*)
proof −
have *?Q* **if** *?P*
proof −
have ⋀*n*. *Euclidean_space n* = *top_of_set* {*f*. ∀ *m*≥*n*. *f m* = *0*}
by (*simp add*: *Euclidean_space_def euclidean_product_topology*)
with *that* **show** *?thesis*
by (*simp add*: *subtopology_subtopology*)
qed
moreover
have *?R* **if** *?P*
using *that* **by** (*simp add*: *image_subset_iff continuous_map_def*)
moreover
have *?P* **if** *?Q ?R*
proof −
have *continuous_map* (*top_of_set* (*topspace* (*subtopology* (*subtopology* (*powertop_real UNIV*) {*f*. ∀ *n*≥*m*. *f n* = *0*}) *S*))) (*top_of_set* (*topspace* (*subtopology* (*powertop_real*

```
UNIV) {f. ∀ na≥n. f na = 0}))) f
      using Euclidean_space_def that by auto
    then show ?thesis
     by (simp add: Euclidean_space_def euclidean_product_topology subtopology_subtopology)
  qed
  ultimately show ?thesis
    by auto
qed

lemma homeomorphic_Euclidean_space_product_topology:
 Euclidean_space n homeomorphic_space product_topology (λi. euclideanreal) {..<n}
proof -
  have cm: continuous_map (product_topology (λi. euclideanreal) {..<n})
         euclideanreal (λx. if k < n then x k else 0) for k
    by (auto intro: continuous_map_if continuous_map_product_projection)
  show ?thesis
    unfolding homeomorphic_space_def homeomorphic_maps_def
    apply (rule_tac x=λf. restrict f {..<n} in exI)
    apply (rule_tac x=λf i. if i < n then f i else 0 in exI)
    apply (simp add: Euclidean_space_def continuous_map_in_subtopology)
    apply (intro conjI continuous_map_from_subtopology)
     apply (force simp: continuous_map_componentwise cm intro: continuous_map_product_projection)+
    done
qed

lemma contractible_Euclidean_space [simp]: contractible_space (Euclidean_space n)
 using homeomorphic_Euclidean_space_product_topology contractible_space_euclideanreal
    contractible_space_product_topology homeomorphic_space_contractibility by blast

lemma path_connected_Euclidean_space: path_connected_space (Euclidean_space n)
  by (simp add: contractible_imp_path_connected_space)

lemma connected_Euclidean_space: connected_space (Euclidean_space n)
  by (simp add: contractible_imp_connected_space)

lemma locally_path_connected_Euclidean_space:
   locally_path_connected_space (Euclidean_space n)
   apply (simp add: homeomorphic_locally_path_connected_space [OF homeomor-
phic_Euclidean_space_product_topology [of n]]
                  locally_path_connected_space_product_topology)
  using locally_path_connected_space_euclideanreal by auto

lemma compact_Euclidean_space:
   compact_space (Euclidean_space n) ⟷ n = 0
 by (auto simp: homeomorphic_compact_space [OF homeomorphic_Euclidean_space_product_topology]
compact_space_product_topology)
```

6.52.2 n-dimensional spheres

definition *nsphere* **where**
nsphere n ≡ *subtopology* (*Euclidean_space* (*Suc n*)) { *x*. ($\sum$ *i*≤*n*. *x i* ^ *2*) = *1* }

lemma *nsphere*:
nsphere n = *subtopology* (*powertop_real UNIV*)
{*x*. ($\sum$ *i*≤*n*. *x i* ^ *2*) = *1* ∧ (∀ *i*>*n*. *x i* = *0*)}
by (*simp add*: *nsphere_def Euclidean_space_def subtopology_subtopology Suc_le_eq Collect_conj_eq Int_commute*)

lemma *continuous_map_nsphere_projection*: *continuous_map* (*nsphere n*) *euclideanreal* (λ*x*. *x k*)
unfolding *nsphere*
by (*blast intro*: *continuous_map_from_subtopology* [*OF continuous_map_product_projection*])

lemma *in_topspace_nsphere*: (λ*n*. *if n* = *0 then 1 else 0*) ∈ *topspace* (*nsphere n*)
by (*simp add*: *nsphere_def topspace_Euclidean_space power2_eq_square if_distrib* [**where** *f* = λ*x*. *x* ∗ _] *cong*: *if_cong*)

lemma *nonempty_nsphere* [*simp*]: ~ (*topspace*(*nsphere n*) = {})
using *in_topspace_nsphere* **by** *auto*

lemma *subtopology_nsphere_equator*:
subtopology (*nsphere* (*Suc n*)) {*x*. *x*(*Suc n*) = *0*} = *nsphere n*
proof −
have ({*x*. ($\sum$ *i*≤*n*. $(x\ i)^2$) + $(x\ (Suc\ n))^2$ = *1* ∧ (∀ *i*>*Suc n*. *x i* = *0*)} ∩ {*x*. *x* (*Suc n*) = *0*})
= {*x*. ($\sum$ *i*≤*n*. $(x\ i)^2$) = *1* ∧ (∀ *i*>*n*. *x i* = (*0*::*real*))}
using *Suc_lessI* [*of n*] **by** (*fastforce simp*: *set_eq_iff*)
then show *?thesis*
by (*simp add*: *nsphere subtopology_subtopology*)
qed

lemma *topspace_nsphere_minus1*:
assumes *x*: *x* ∈ *topspace* (*nsphere n*) **and** *x n* = *0*
shows *x* ∈ *topspace* (*nsphere* (*n* − *Suc 0*))
proof (*cases n* = *0*)
case *True*
then show *?thesis*
using *x* **by** *auto*
next
case *False*
have *subt_eq*: *nsphere* (*n* − *Suc 0*) = *subtopology* (*nsphere n*) {*x*. *x n* = *0*}
by (*metis False Suc_pred le_zero_eq not_le subtopology_nsphere_equator*)
with *x* **show** *?thesis*
by (*simp add*: *assms*)
qed

lemma *continuous_map_nsphere_reflection*:

continuous_map (*nsphere n*) (*nsphere n*) (λ*x i*. *if i* = *k then* −*x i else x i*)
proof −
have *cm*: *continuous_map* (*powertop_real UNIV*) *euclideanreal* (λ*x*. *if j* = *k then* − *x j else x j*) **for** *j*
proof (*cases j*=*k*)
case *True*
then show *?thesis*
by *simp* (*metis UNIV_I continuous_map_product_projection*)
next
case *False*
then show *?thesis*
by (*auto intro*: *continuous_map_product_projection*)
qed
have *eq*: (*if i* = *k then x k* ∗ *x k else x i* ∗ *x i*) = *x i* ∗ *x i* **for** *i* **and** *x* :: *nat* ⇒ *real*
by *simp*
show *?thesis*
apply (*simp add*: *nsphere continuous_map_in_subtopology continuous_map_componentwise_UNIV continuous_map_from_subtopology cm*)
apply (*intro conjI allI impI continuous_intros continuous_map_from_subtopology continuous_map_product_projection*)
apply (*auto simp*: *power2_eq_square if_distrib* [**where** *f* = λ*x*. *x* ∗ _] *eq cong*: *if_cong*)
done
qed

proposition *contractible_space_upper_hemisphere*:
assumes *k* ≤ *n*
shows *contractible_space*(*subtopology* (*nsphere n*) {*x*. *x k* ≥ *0*})
proof −
define *p*:: *nat* ⇒ *real* **where** *p* ≡ λ*i*. *if i* = *k then 1 else 0*
have *p* ∈ *topspace*(*nsphere n*)
using *assms*
by (*simp add*: *nsphere p_def power2_eq_square if_distrib* [**where** *f* = λ*x*. *x* ∗ _] *cong*: *if_cong*)
let *?g* = λ*x i*. *x i* / *sqrt*(∑*j*≤*n*. *x j* ˆ *2*)
let *?h* = λ(*t*,*q*) *i*. (*1* − *t*) ∗ *q i* + *t* ∗ *p i*
let *?Y* = *subtopology* (*Euclidean_space* (*Suc n*)) {*x*. *0* ≤ *x k* ∧ (∃ *i*≤*n*. *x i* ≠ *0*)}
have *continuous_map* (*prod_topology* (*top_of_set* {*0*..*1*}) (*subtopology* (*nsphere n*) {*x*. *0* ≤ *x k*}))
(*subtopology* (*nsphere n*) {*x*. *0* ≤ *x k*}) (*?g* ∘ *?h*)
proof (*rule continuous_map_compose*)
have ∗: ⟦*0* ≤ *b k*; (∑ *i*≤*n*. (*b i*)^2) = *1*; ∀ *i*>*n*. *b i* = *0*; *0* ≤ *a*; *a* ≤ *1*⟧
⟹ ∃ *i*. (*i* = *k* ⟶ (*1* − *a*) ∗ *b k* + *a* ≠ *0*) ∧
(*i* ≠ *k* ⟶ *i* ≤ *n* ∧ *a* ≠ *1* ∧ *b i* ≠ *0*) **for** *a*::*real* **and** *b*
apply (*cases a* ≠ *1* ∧ *b k* = *0*; *simp*)
apply (*metis* (*no_types*, *lifting*) *atMost_iff sum.neutral zero_power2*)
by (*metis add.commute add_le_same_cancel2 diff_ge_0_iff_ge diff_zero less_eq_real_def*

mult_eq_0_iff mult_nonneg_nonneg not_le numeral_One zero_neq_numeral)
show *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) (*subtopology* (*nsphere n*) {*x*. *0* ≤ *x k*})) *?Y ?h*
using *assms*
apply (*auto simp*: ∗ *nsphere continuous_map_componentwise_UNIV prod_topology_subtopology subtopology_subtopology case_prod_unfold continuous_map_in_subtopology Euclidean_space_def p_def if_distrib* [**where** *f* = λ*x*. _ ∗ *x*] *cong*: *if_cong*)
apply (*intro continuous_map_prod_snd continuous_intros continuous_map_from_subtopology*)
apply *auto*
done
next
have *1*: ⋀*x i*. ⟦ *i* ≤ *n*; *x i* ≠ *0*⟧ ⟹ (∑ *i*≤*n*. (*x i* / *sqrt* (∑ *j*≤*n*. (*x j*)^2))^2) = *1*
by (*force simp*: *sum_nonneg sum_nonneg_eq_0_iff field_split_simps simp flip*: *sum_divide_distrib*)
have *cm*: *continuous_map ?Y* (*nsphere n*) (λ*x i*. *x i* / *sqrt* (∑ *j*≤*n*. (*x j*)^2))
unfolding *Euclidean_space_def nsphere subtopology_subtopology continuous_map_in_subtopology*
proof (*intro continuous_intros conjI*)
show *continuous_map*
(*subtopology* (*powertop_real UNIV*) ({*x*. ∀ *i*≥*Suc n*. *x i* = *0*} ∩ {*x*. *0* ≤ *x k* ∧ (∃ *i*≤*n*. *x i* ≠ *0*)}))
(*powertop_real UNIV*) (λ*x i*. *x i* / *sqrt* (∑ *j*≤*n*. (*x j*)^2))
unfolding *continuous_map_componentwise*
by (*intro continuous_intros conjI ballI*) (*auto simp*: *sum_nonneg_eq_0_iff*)
qed (*auto simp*: *1*)
show *continuous_map ?Y* (*subtopology* (*nsphere n*) {*x*. *0* ≤ *x k*}) (λ*x i*. *x i* / *sqrt* (∑ *j*≤*n*. (*x j*)^2))
by (*force simp*: *cm sum_nonneg continuous_map_in_subtopology if_distrib* [**where** *f* = λ*x*. _ ∗ *x*] *cong*: *if_cong*)
qed
moreover have (*?g* ∘ *?h*) (*0*, *x*) = *x*
if *x* ∈ *topspace* (*subtopology* (*nsphere n*) {*x*. *0* ≤ *x k*}) **for** *x*
using *that*
by (*simp add*: *assms nsphere*)
moreover
have (*?g* ∘ *?h*) (*1*, *x*) = *p*
if *x* ∈ *topspace* (*subtopology* (*nsphere n*) {*x*. *0* ≤ *x k*}) **for** *x*
by (*force simp*: *assms p_def power2_eq_square if_distrib* [**where** *f* = λ*x*. *x* ∗ _] *cong*: *if_cong*)
ultimately
show *?thesis*
apply (*simp add*: *contractible_space_def homotopic_with*)
apply (*rule_tac x*=*p* **in** *exI*)
apply (*rule_tac x*=*?g* ∘ *?h* **in** *exI*, *force*)
done
qed

corollary *contractible_space_lower_hemisphere*:
  assumes $k \leq n$
  shows *contractible_space*(*subtopology* (*nsphere n*) {*x. x k* ≤ *0*})
proof −
  have *contractible_space* (*subtopology* (*nsphere n*) {*x. 0* ≤ *x k*}) = *?thesis*
  proof (*rule homeomorphic_space_contractibility*)
    show *subtopology* (*nsphere n*) {*x. 0* ≤ *x k*} *homeomorphic_space subtopology* (*nsphere n*) {*x. x k* ≤ *0*}
      unfolding *homeomorphic_space_def homeomorphic_maps_def*
      apply (*rule_tac x*=*λx i. if i* = *k then* −(*x i*) *else x i* **in** *exI*)+
    apply (*auto simp*: *continuous_map_in_subtopology continuous_map_from_subtopology continuous_map_nsphere_reflection*)
      done
  qed
  then show *?thesis*
    using *contractible_space_upper_hemisphere* [*OF assms*] **by** *metis*
qed

proposition *nullhomotopic_nonsurjective_sphere_map*:
  assumes *f*: *continuous_map* (*nsphere p*) (*nsphere p*) *f*
    and *fim*: *f* ' (*topspace*(*nsphere p*)) ≠ *topspace*(*nsphere p*)
  obtains *a* **where** *homotopic_with* (*λx. True*) (*nsphere p*) (*nsphere p*) *f* (*λx. a*)
proof −
  obtain *a* **where** *a*: *a* ∈ *topspace*(*nsphere p*) *a* ∉ *f* ' (*topspace*(*nsphere p*))
    using *fim continuous_map_image_subset_topspace f* **by** *blast*
  then have *a1*: $(\sum i{\le}p.\ (a\ i)^2) = 1$ **and** *a0*: $\bigwedge i.\ i > p \Longrightarrow a\ i = 0$
    by (*simp_all add*: *nsphere*)
  have *f1*: $(\sum j{\le}p.\ (f\ x\ j)^2) = 1$ **if** *x* ∈ *topspace* (*nsphere p*) **for** *x*
  proof −
    have *f x* ∈ *topspace* (*nsphere p*)
      using *continuous_map_image_subset_topspace f that* **by** *blast*
    then show *?thesis*
      by (*simp add*: *nsphere*)
  qed
  show *thesis*
  proof
    let *?g* = *λx i. x i* / *sqrt*(∑*j*≤*p. x j* ^ *2*)
    let *?h* = *λ*(*t*,*x*) *i.* (*1* − *t*) ∗ *f x i* − *t* ∗ *a i*
    let *?Y* = *subtopology* (*Euclidean_space*(*Suc p*)) (− {*λi. 0*})
    let *?T01* = *top_of_set* {*0..1::real*}
    have *1*: *continuous_map* (*prod_topology ?T01* (*nsphere p*)) (*nsphere p*) (*?g* ∘ *?h*)
    proof (*rule continuous_map_compose*)
      have *continuous_map* (*prod_topology ?T01* (*nsphere p*)) *euclideanreal* ((*λx. f x k*) ∘ *snd*) **for** *k*
        unfolding *nsphere*
        apply (*simp add*: *continuous_map_of_snd*)
        apply (*rule continuous_map_compose* [*of _ nsphere p f*, *unfolded o_def*])

using *f* **apply** (*simp add*: *nsphere*)
by (*simp add*: *continuous_map_nsphere_projection*)
then have *continuous_map* (*prod_topology ?T01* (*nsphere p*)) *euclideanreal* (*λr. ?h r k*)
for *k*
unfolding *case_prod_unfold o_def*
by (*intro continuous_map_into_fulltopology* [*OF continuous_map_fst*] *continuous_intros*) *auto*
moreover have *?h* ' ({*0..1*} × *topspace* (*nsphere p*)) ⊆ {*x*. ∀ *i*≥*Suc p*. *x i* = *0*}
using *continuous_map_image_subset_topspace* [*OF f*]
by (*auto simp*: *nsphere image_subset_iff a0*)
moreover have (*λi. 0*) ∉ *?h* ' ({*0..1*} × *topspace* (*nsphere p*))
proof *clarify*
fix *t b*
assume *eq*: (*λi. 0*) = (*λi.* (*1* − *t*) ∗ *f b i* − *t* ∗ *a i*) **and** *t* ∈ {*0..1*} **and** *b*: *b* ∈ *topspace* (*nsphere p*)
have $(1 - t)^2$ = (∑ *i*≤*p*. ((*1* − *t*) ∗ *f b i*) ^*2*)
using *f1* [*OF b*] **by** (*simp add*: *power_mult_distrib flip*: *sum_distrib_left*)
also have ... = (∑ *i*≤*p*. (*t* ∗ *a i*) ^*2*)
using *eq* **by** (*simp add*: *fun_eq_iff*)
also have ... = t^2
using *a1* **by** (*simp add*: *power_mult_distrib flip*: *sum_distrib_left*)
finally have *1* − *t* = *t*
by (*simp add*: *power2_eq_iff*)
then have ∗: *t* = *1/2*
by *simp*
have *fba*: *f b* ≠ *a*
using *a*(*2*) *b* **by** *blast*
then show *False*
using *eq* **unfolding** ∗ **by** (*simp add*: *fun_eq_iff*)
qed
ultimately show *continuous_map* (*prod_topology ?T01* (*nsphere p*)) *?Y ?h*
by (*simp add*: *Euclidean_space_def continuous_map_in_subtopology continuous_map_componentwise_UNIV*)
next
have ∗: ⟦∀ *i*≥*Suc p*. *x i* = *0*; *x* ≠ (*λi. 0*)⟧ ⟹ (∑ *j*≤*p*. $(x\ j)^2$) ≠ *0* **for** *x* :: *nat* ⇒ *real*
by (*force simp*: *fun_eq_iff not_less_eq_eq sum_nonneg_eq_0_iff*)
show *continuous_map ?Y* (*nsphere p*) *?g*
apply (*simp add*: *Euclidean_space_def continuous_map_in_subtopology continuous_map_componentwise_UNIV*
nsphere continuous_map_componentwise subtopology_subtopology)
apply (*intro conjI allI continuous_intros continuous_map_from_subtopology* [*OF continuous_map_product_projection*])
apply (*simp_all add*: ∗)
apply (*force simp*: *sum_nonneg fun_eq_iff not_less_eq_eq sum_nonneg_eq_0_iff power_divide simp flip*: *sum_divide_distrib*)
done

qed
have *2*: (*?g* ∘ *?h*) (*0*, *x*) = *f x* **if** *x* ∈ *topspace* (*nsphere p*) **for** *x*
using *that f1* **by** *simp*
have *3*: (*?g* ∘ *?h*) (*1*, *x*) = (λ*i*. − *a i*) **for** *x*
using *a* **by** (*force simp*: *field_split_simps nsphere*)
then show *homotopic_with* (λ*x*. *True*) (*nsphere p*) (*nsphere p*) *f* (λ*x*. (λ*i*. − *a i*))
by (*force simp*: *homotopic_with intro*: *1 2 3*)
qed
qed

lemma *Hausdorff_Euclidean_space*:
Hausdorff_space (*Euclidean_space n*)
unfolding *Euclidean_space_def*
by (*rule Hausdorff_space_subtopology*) (*metis Hausdorff_space_euclidean Hausdorff_space_product_topology*)

end

6.53 Metrics on product spaces

theory *Function_Metric*
imports
Function_Topology
Elementary_Metric_Spaces
begin

In general, the product topology is not metrizable, unless the index set is countable. When the index set is countable, essentially any (convergent) combination of the metrics on the factors will do. We use below the simplest one, based on L^1, but L^2 would also work, for instance.

What is not completely trivial is that the distance thus defined induces the same topology as the product topology. This is what we have to prove to show that we have an instance of *metric_space*.

The proofs below would work verbatim for general countable products of metric spaces. However, since distances are only implemented in terms of type classes, we only develop the theory for countable products of the same space.

instantiation *fun* :: (*countable*, *metric_space*) *metric_space*
begin

definition *dist_fun_def*:
dist x y = (∑*n*. (*1*/*2*) ^*n* ∗ *min* (*dist* (*x* (*from_nat n*)) (*y* (*from_nat n*))) *1*)

definition *uniformity_fun_def*:
(*uniformity*::((*'a* ⇒ *'b*) × (*'a* ⇒ *'b*)) *filter*) = (*INF e*∈{*0*<..}. *principal* {(*x*, *y*). *dist* (*x*::(*'a*⇒*'b*)) *y* < *e*})

Except for the first one, the auxiliary lemmas below are only useful when proving the instance: once it is proved, they become trivial consequences of the general theory of metric spaces. It would thus be desirable to hide them once the instance is proved, but I do not know how to do this.

lemma *dist_fun_le_dist_first_terms*:
dist x y ≤ *2* ∗ *Max* {*dist* (*x* (*from_nat n*)) (*y* (*from_nat n*))|*n*. *n* ≤ *N*} + (*1/2*)ˆ*N*
proof −
have (∑*n*. (*1* / *2*) ˆ (*n*+*Suc N*) ∗ *min* (*dist* (*x* (*from_nat* (*n*+*Suc N*))) (*y* (*from_nat* (*n*+*Suc N*)))) *1*)
= (∑*n*. (*1* / *2*) ˆ (*Suc N*) ∗ ((*1/2*) ˆ *n* ∗ *min* (*dist* (*x* (*from_nat* (*n*+*Suc N*))) (*y* (*from_nat* (*n*+*Suc N*)))) *1*))
by (*rule suminf_cong*, *simp add*: *power_add*)
also have ... = (*1/2*)ˆ(*Suc N*) ∗ (∑*n*. (*1* / *2*) ˆ *n* ∗ *min* (*dist* (*x* (*from_nat* (*n*+*Suc N*))) (*y* (*from_nat* (*n*+*Suc N*)))) *1*)
apply (*rule suminf_mult*)
by (*rule summable_comparison_test'*[*of* λ*n*. (*1/2*)ˆ*n*], *auto simp add*: *summable_geometric_iff*)
also have ... ≤ (*1/2*)ˆ(*Suc N*) ∗ (∑*n*. (*1* / *2*) ˆ *n*)
apply (*simp*, *rule suminf_le*, *simp*)
by (*rule summable_comparison_test'*[*of* λ*n*. (*1/2*)ˆ*n*], *auto simp add*: *summable_geometric_iff*)
also have ... = (*1/2*)ˆ(*Suc N*) ∗ *2*
using *suminf_geometric*[*of 1/2*] **by** *auto*
also have ... = (*1/2*)ˆ*N*
by *auto*
finally have ∗: (∑*n*. (*1* / *2*) ˆ (*n*+*Suc N*) ∗ *min* (*dist* (*x* (*from_nat* (*n*+*Suc N*))) (*y* (*from_nat* (*n*+*Suc N*)))) *1*) ≤ (*1/2*)ˆ*N*
by *simp*

define *M* **where** *M* = *Max* {*dist* (*x* (*from_nat n*)) (*y* (*from_nat n*))|*n*. *n* ≤ *N*}
have *dist* (*x* (*from_nat 0*)) (*y* (*from_nat 0*)) ≤ *M*
unfolding *M_def* **by** (*rule Max_ge*, *auto*)
then have [*simp*]: *M* ≥ *0* **by** (*meson dual_order.trans zero_le_dist*)
have *dist* (*x* (*from_nat n*)) (*y* (*from_nat n*)) ≤ *M* **if** *n*≤*N* **for** *n*
unfolding *M_def* **apply** (*rule Max_ge*) **using** *that* **by** *auto*
then have *i*: *min* (*dist* (*x* (*from_nat n*)) (*y* (*from_nat n*))) *1* ≤ *M* **if** *n*≤*N* **for** *n*
using *that* **by** *force*
have (∑*n*< *Suc N*. (*1* / *2*) ˆ *n* ∗ *min* (*dist* (*x* (*from_nat n*)) (*y* (*from_nat n*))) *1*) ≤
(∑*n*< *Suc N*. *M* ∗ (*1* / *2*) ˆ *n*)
by (*rule sum_mono*, *simp add*: *i*)
also have ... = *M* ∗ (∑*n*<*Suc N*. (*1* / *2*) ˆ *n*)
by (*rule sum_distrib_left*[*symmetric*])
also have ... ≤ *M* ∗ (∑*n*. (*1* / *2*) ˆ *n*)
by (*rule mult_left_mono*, *rule sum_le_suminf*, *auto simp add*: *summable_geometric_iff*)
also have ... = *M* ∗ *2*
using *suminf_geometric*[*of 1/2*] **by** *auto*
finally have ∗∗: (∑*n*< *Suc N*. (*1* / *2*) ˆ *n* ∗ *min* (*dist* (*x* (*from_nat n*)) (*y* (*from_nat n*))) *1*) ≤ *2* ∗ *M*
by *simp*

```
  have dist x y = (∑ n. (1 / 2) ^ n * min (dist (x (from_nat n)) (y (from_nat
n))) 1)
    unfolding dist_fun_def by simp
  also have ... = (∑ n. (1 / 2) ^ (n+Suc N) * min (dist (x (from_nat (n+Suc
N))) (y (from_nat (n+Suc N)))) 1)
                  + (∑ n<Suc N. (1 / 2) ^ n * min (dist (x (from_nat n)) (y
(from_nat n))) 1)
    apply (rule suminf_split_initial_segment)
   by (rule summable_comparison_test'[of λn. (1/2)^n], auto simp add: summable_geometric_iff)
  also have ... ≤ 2 * M + (1/2)^N
    using * ** by auto
  finally show ?thesis unfolding M_def by simp
qed

lemma open_fun_contains_ball_aux:
  assumes open (U::(('a ⇒ 'b) set))
          x ∈ U
  shows ∃ e>0. {y. dist x y < e} ⊆ U
proof -
  have *: openin (product_topology (λi. euclidean) UNIV) U
    using open_fun_def assms by auto
  obtain X where H: Pi_E UNIV X ⊆ U
                  ⋀i. openin euclidean (X i)
                  finite {i. X i ≠ topspace euclidean}
                  x ∈ Pi_E UNIV X
    using product_topology_open_contains_basis[OF * ‹x ∈ U›] by auto
  define I where I = {i. X i ≠ topspace euclidean}
  have finite I unfolding I_def using H(3) by auto
  {
    fix i
    have x i ∈ X i using ‹x ∈ U› H by auto
    then have ∃ e. e>0 ∧ ball (x i) e ⊆ X i
      using ‹openin euclidean (X i)› open_openin open_contains_ball by blast
    then obtain e where e>0 ball (x i) e ⊆ X i by blast
    define f where f = min e (1/2)
    have f>0 f<1 unfolding f_def using ‹e>0› by auto
    moreover have ball (x i) f ⊆ X i unfolding f_def using ‹ball (x i) e ⊆ X i›
by auto
    ultimately have ∃ f. f > 0 ∧ f < 1 ∧ ball (x i) f ⊆ X i by auto
  } note * = this
  have ∃ e. ∀ i. e i > 0 ∧ e i < 1 ∧ ball (x i) (e i) ⊆ X i
    by (rule choice, auto simp add: *)
  then obtain e where ⋀i. e i > 0 ⋀i. e i < 1 ⋀i. ball (x i) (e i) ⊆ X i
    by blast
  define m where m = Min {(1/2)^(to_nat i) * e i|i. i ∈ I}
  have m > 0 if I≠{}
    unfolding m_def Min_gr_iff using ‹finite I› ‹I ≠ {}› ‹⋀i. e i > 0› by auto
  moreover have {y. dist x y < m} ⊆ U
```

proof (*auto*)
fix *y* **assume** *dist x y* < *m*
have *y i* ∈ *X i* **if** *i* ∈ *I* **for** *i*
proof −
have ∗: *summable* (λ*n*. (*1/2*) ˆ*n* ∗ *min* (*dist* (*x* (*from_nat n*)) (*y* (*from_nat n*))) *1*)
by (*rule summable_comparison_test′*[*of* λ*n*. (*1/2*) ˆ*n*], *auto simp add*: *summable_geometric_iff*)
define *n* **where** *n* = *to_nat i*
have (*1/2*) ˆ*n* ∗ *min* (*dist* (*x* (*from_nat n*)) (*y* (*from_nat n*))) *1* < *m*
using ‹*dist x y* < *m*› **unfolding** *dist_fun_def* **using** *sum_le_suminf*[*OF* ∗, *of* {*n*}] **by** *auto*
then have (*1/2*) ˆ(*to_nat i*) ∗ *min* (*dist* (*x i*) (*y i*)) *1* < *m*
using ‹*n* = *to_nat i*› **by** *auto*
also have ... ≤ (*1/2*) ˆ(*to_nat i*) ∗ *e i*
unfolding *m_def* **apply** (*rule Min_le*) **using** ‹*finite I*› ‹*i* ∈ *I*› **by** *auto*
ultimately have *min* (*dist* (*x i*) (*y i*)) *1* < *e i*
by (*auto simp add*: *field_split_simps*)
then have *dist* (*x i*) (*y i*) < *e i*
using ‹*e i* < *1*› **by** *auto*
then show *y i* ∈ *X i* **using** ‹*ball* (*x i*) (*e i*) ⊆ *X i*› **by** *auto*
qed
then have *y* ∈ Pi_E *UNIV X* **using** *H*(*1*) **unfolding** *I_def topspace_euclidean* **by** (*auto simp add*: *PiE_iff*)
then show *y* ∈ *U* **using** ‹Pi_E *UNIV X* ⊆ *U*› **by** *auto*
qed
ultimately have ∗: ∃ *m*>*0*. {*y*. *dist x y* < *m*} ⊆ *U* **if** *I* ≠ {} **using** *that* **by** *auto*

have *U* = *UNIV* **if** *I* = {}
using *that H*(*1*) **unfolding** *I_def topspace_euclidean* **by** (*auto simp add*: *PiE_iff*)
then have ∃ *m*>*0*. {*y*. *dist x y* < *m*} ⊆ *U* **if** *I* = {} **using** *that zero_less_one* **by** *blast*
then show ∃ *m*>*0*. {*y*. *dist x y* < *m*} ⊆ *U* **using** ∗ **by** *blast*
qed

lemma *ball_fun_contains_open_aux*:
fixes *x*::(′*a* ⇒ ′*b*) **and** *e*::*real*
assumes *e*>*0*
shows ∃ *U*. *open U* ∧ *x* ∈ *U* ∧ *U* ⊆ {*y*. *dist x y* < *e*}
proof −
have ∃ *N*::*nat*. *2*ˆ*N* > *8*/*e*
by (*simp add*: *real_arch_pow*)
then obtain *N*::*nat* **where** *2*ˆ*N* > *8*/*e* **by** *auto*
define *f* **where** *f* = *e*/*4*
have [*simp*]: *e*>*0 f* > *0* **unfolding** *f_def* **using** *assms* **by** *auto*
define *X*::(′*a* ⇒ ′*b set*) **where** *X* = (λ*i*. *if* (∃ *n*≤*N*. *i* = *from_nat n*) *then* {*z*. *dist* (*x i*) *z* < *f*} *else UNIV*)

```
  have {i. X i ≠ UNIV} ⊆ from_nat`{0..N}
    unfolding X_def by auto
  then have finite {i. X i ≠ topspace euclidean}
    unfolding topspace_euclidean using finite_surj by blast
  have ⋀i. open (X i)
    unfolding X_def using metric_space_class.open_ball open_UNIV by auto
  then have ⋀i. openin euclidean (X i)
    using open_openin by auto
  define U where U = Pi_E UNIV X
  have open U
   unfolding open_fun_def product_topology_def apply (rule topology_generated_by_Basis)
    unfolding U_def using ‹⋀i. openin euclidean (X i)› ‹finite {i. X i ≠ topspace
euclidean}›
    by auto
  moreover have x ∈ U
    unfolding U_def X_def by (auto simp add: PiE_iff)
  moreover have dist x y < e if y ∈ U for y
  proof –
    have *: dist (x (from_nat n)) (y (from_nat n)) ≤ f if n ≤ N for n
      using ‹y ∈ U› unfolding U_def apply (auto simp add: PiE_iff)
      unfolding X_def using that by (metis less_imp_le mem_Collect_eq)
    have **: Max {dist (x (from_nat n)) (y (from_nat n))|n. n ≤ N} ≤ f
      apply (rule Max.boundedI) using * by auto

    have dist x y ≤ 2 * Max {dist (x (from_nat n)) (y (from_nat n))|n. n ≤ N}
+ (1/2)^N
      by (rule dist_fun_le_dist_first_terms)
    also have ... ≤ 2 * f + e / 8
    apply (rule add_mono) using ** ‹2^N > 8/e› by(auto simp add: field_split_simps)
    also have ... ≤ e/2 + e/8
      unfolding f_def by auto
    also have ... < e
      by auto
    finally show dist x y < e by simp
  qed
  ultimately show ?thesis by auto
qed

lemma fun_open_ball_aux:
  fixes U::('a ⇒ 'b) set
  shows open U ⟷ (∀x∈U. ∃e>0. ∀y. dist x y < e ⟶ y ∈ U)
proof (auto)
  assume open U
  fix x assume x ∈ U
  then show ∃e>0. ∀y. dist x y < e ⟶ y ∈ U
    using open_fun_contains_ball_aux[OF ‹open U› ‹x ∈ U›] by auto
next
  assume H: ∀x∈U. ∃e>0. ∀y. dist x y < e ⟶ y ∈ U
  define K where K = {V. open V ∧ V ⊆ U}
```

```
  {
    fix x assume x ∈ U
    then obtain e where e: e>0 {y. dist x y < e} ⊆ U using H by blast
    then obtain V where V: open V x ∈ V V ⊆ {y. dist x y < e}
      using ball_fun_contains_open_aux[OF ‹e>0›, of x] by auto
    have V ∈ K
      unfolding K_def using e(2) V(1) V(3) by auto
    then have x ∈ (⋃ K) using ‹x ∈ V› by auto
  }
  then have (⋃ K) = U
    unfolding K_def by auto
  moreover have open (⋃ K)
    unfolding K_def by auto
  ultimately show open U by simp
qed

instance proof
  fix x y::'a ⇒ 'b show (dist x y = 0) = (x = y)
  proof
    assume x = y
    then show dist x y = 0 unfolding dist_fun_def using ‹x = y› by auto
  next
    assume dist x y = 0
   have *: summable (λn. (1/2)^n * min (dist (x (from_nat n)) (y (from_nat n)))
1)
     by (rule summable_comparison_test'[of λn. (1/2)^n], auto simp add: summable_geometric_iff)
    have (1/2)^n * min (dist (x (from_nat n)) (y (from_nat n))) 1 = 0 for n
     using ‹dist x y = 0› unfolding dist_fun_def by (simp add: * suminf_eq_zero_iff)
    then have dist (x (from_nat n)) (y (from_nat n)) = 0 for n
      by (metis div_0 min_def nonzero_mult_div_cancel_left power_eq_0_iff
              zero_eq_1_divide_iff zero_neq_numeral)
    then have x (from_nat n) = y (from_nat n) for n
      by auto
    then have x i = y i for i
      by (metis from_nat_to_nat)
    then show x = y
      by auto
  qed
next
```

The proof of the triangular inequality is trivial, modulo the fact that we are dealing with infinite series, hence we should justify the convergence at each step. In this respect, the following simplification rule is extremely handy.

```
  have [simp]: summable (λn. (1/2)^n * min (dist (u (from_nat n)) (v (from_nat
n))) 1) for u v::'a ⇒ 'b
   by (rule summable_comparison_test'[of λn. (1/2)^n], auto simp add: summable_geometric_iff)
  fix x y z::'a ⇒ 'b
  {
    fix n
```

have ∗: *dist* (*x* (*from_nat n*)) (*y* (*from_nat n*)) ≤
dist (*x* (*from_nat n*)) (*z* (*from_nat n*)) + *dist* (*y* (*from_nat n*)) (*z* (*from_nat n*))
by (*simp add*: *dist_triangle2*)
have *0* ≤ *dist* (*y* (*from_nat n*)) (*z* (*from_nat n*))
using *zero_le_dist* **by** *blast*
moreover
{
assume *min* (*dist* (*y* (*from_nat n*)) (*z* (*from_nat n*))) *1* ≠ *dist* (*y* (*from_nat n*)) (*z* (*from_nat n*))
then have *1* ≤ *min* (*dist* (*x* (*from_nat n*)) (*z* (*from_nat n*))) *1* + *min* (*dist* (*y* (*from_nat n*)) (*z* (*from_nat n*))) *1*
by (*metis* (*no_types*) *diff_le_eq diff_self min_def zero_le_dist zero_le_one*)
}
ultimately have *min* (*dist* (*x* (*from_nat n*)) (*y* (*from_nat n*))) *1* ≤
min (*dist* (*x* (*from_nat n*)) (*z* (*from_nat n*))) *1* + *min* (*dist* (*y* (*from_nat n*)) (*z* (*from_nat n*))) *1*
using ∗ **by** *linarith*
} **note** *ineq* = *this*
have *dist x y* = (∑ *n*. (*1/2*)ˆ*n* ∗ *min* (*dist* (*x* (*from_nat n*)) (*y* (*from_nat n*))) *1*)
unfolding *dist_fun_def* **by** *simp*
also have ... ≤ (∑ *n*. (*1/2*)ˆ*n* ∗ *min* (*dist* (*x* (*from_nat n*)) (*z* (*from_nat n*))) *1*
+ (*1/2*)ˆ*n* ∗ *min* (*dist* (*y* (*from_nat n*)) (*z* (*from_nat n*))) *1*)
apply (*rule suminf_le*)
using *ineq* **apply** (*metis* (*no_types*, *hide_lams*) *add.right_neutral distrib_left le_divide_eq_numeral1*(*1*) *mult_2_right mult_left_mono zero_le_one zero_le_power*)
by (*auto simp add*: *summable_add*)
also have ... = (∑ *n*. (*1/2*)ˆ*n* ∗ *min* (*dist* (*x* (*from_nat n*)) (*z* (*from_nat n*))) *1*)
+ (∑ *n*. (*1/2*)ˆ*n* ∗ *min* (*dist* (*y* (*from_nat n*)) (*z* (*from_nat n*))) *1*)
by (*rule suminf_add*[*symmetric*], *auto*)
also have ... = *dist x z* + *dist y z*
unfolding *dist_fun_def* **by** *simp*
finally show *dist x y* ≤ *dist x z* + *dist y z*
by *simp*
next

Finally, we show that the topology generated by the distance and the product topology coincide. This is essentially contained in Lemma *fun_open_ball_aux*, except that the condition to prove is expressed with filters. To deal with this, we copy some mumbo jumbo from Lemma *eventually_uniformity_metric* in `~~/src/HOL/Real_Vector_Spaces.thy`

fix *U*::(*'a* ⇒ *'b*) *set*
have *eventually P uniformity* ⟷ (∃ *e*>*0*. ∀ *x* (*y*::(*'a* ⇒ *'b*)). *dist x y* < *e* ⟶ *P* (*x*, *y*)) **for** *P*
unfolding *uniformity_fun_def* **apply** (*subst eventually_INF_base*)
by (*auto simp*: *eventually_principal subset_eq intro*: *bexI*[*of _ min _ _*])
then show *open U* = (∀ *x*∈*U*. ∀$_F$ (*x'*, *y*) *in uniformity*. *x'* = *x* ⟶ *y* ∈ *U*)

```
  unfolding fun_open_ball_aux by simp
qed (simp add: uniformity_fun_def)

end
```

Nice properties of spaces are preserved under countable products. In addition to first countability, second countability and metrizability, as we have seen above, completeness is also preserved, and therefore being Polish.

```
instance fun :: (countable, complete_space) complete_space
proof
 fix u::nat ⇒ ('a ⇒ 'b) assume Cauchy u
 have Cauchy (λn. u n i) for i
 unfolding cauchy_def proof (intro impI allI)
  fix e assume e>(0::real)
  obtain k where i = from_nat k
   using from_nat_to_nat[of i] by metis
  have (1/2)^k * min e 1 > 0 using ‹e>0› by auto
  then have ∃N. ∀m n. N ≤ m ∧ N ≤ n ⟶ dist (u m) (u n) < (1/2)^k *
min e 1
   using ‹Cauchy u› unfolding cauchy_def by blast
  then obtain N::nat where C: ∀m n. N ≤ m ∧ N ≤ n ⟶ dist (u m) (u n)
< (1/2)^k * min e 1
   by blast
  have ∀m n. N ≤ m ∧ N ≤ n ⟶ dist (u m i) (u n i) < e
  proof (auto)
   fix m n::nat assume m ≥ N n ≥ N
   have (1/2)^k * min (dist (u m i) (u n i)) 1
       = sum (λp. (1/2)^p * min (dist (u m (from_nat p)) (u n (from_nat
p))) 1) {k}
    using ‹i = from_nat k› by auto
   also have ... ≤ (∑p. (1/2)^p * min (dist (u m (from_nat p)) (u n (from_nat
p))) 1)
    apply (rule sum_le_suminf)
      by (rule summable_comparison_test'[of λn. (1/2)^n], auto simp add:
summable_geometric_iff)
   also have ... = dist (u m) (u n)
    unfolding dist_fun_def by auto
   also have ... < (1/2)^k * min e 1
    using C ‹m ≥ N› ‹n ≥ N› by auto
   finally have min (dist (u m i) (u n i)) 1 < min e 1
    by (auto simp add: field_split_simps)
   then show dist (u m i) (u n i) < e by auto
  qed
  then show ∃N. ∀m n. N ≤ m ∧ N ≤ n ⟶ dist (u m i) (u n i) < e
   by blast
 qed
 then have ∃x. (λn. u n i) ⟶ x for i
  using Cauchy_convergent convergent_def by auto
 then have ∃x. ∀i. (λn. u n i) ⟶ x i
```

```
    using choice by force
  then obtain x where *: ⋀i. (λn. u n i) ⟶ x i by blast
  have u ⟶ x
  proof (rule metric_LIMSEQ_I)
    fix e assume [simp]: e>(0::real)
    have i: ∃K. ∀n≥K. dist (u n i) (x i) < e/4 for i
      by (rule metric_LIMSEQ_D, auto simp add: *)
    have ∃K. ∀i. ∀n≥K i. dist (u n i) (x i) < e/4
      apply (rule choice) using i by auto
    then obtain K where K: ⋀i n. n ≥ K i ⟹ dist (u n i) (x i) < e/4
      by blast

    have ∃N::nat. 2^N > 4/e
      by (simp add: real_arch_pow)
    then obtain N::nat where 2^N > 4/e by auto
    define L where L = Max {K (from_nat n)|n. n ≤ N}
    have dist (u k) x < e if k ≥ L for k
    proof -
      have *: dist (u k (from_nat n)) (x (from_nat n)) ≤ e / 4 if n ≤ N for n
      proof -
        have K (from_nat n) ≤ L
          unfolding L_def apply (rule Max_ge) using ‹n ≤ N› by auto
        then have k ≥ K (from_nat n) using ‹k ≥ L› by auto
        then show ?thesis using K less_imp_le by auto
      qed
      have **: Max {dist (u k (from_nat n)) (x (from_nat n)) |n. n ≤ N} ≤ e/4
        apply (rule Max.boundedI) using * by auto
      have dist (u k) x ≤ 2 * Max {dist (u k (from_nat n)) (x (from_nat n)) |n.
n ≤ N} + (1/2)^N
        using dist_fun_le_dist_first_terms by auto
      also have ... ≤ 2 * e/4 + e/4
        apply (rule add_mono)
        using ** ‹2^N > 4/e› less_imp_le by (auto simp add: field_split_simps)
      also have ... < e by auto
      finally show ?thesis by simp
    qed
    then show ∃L. ∀k≥L. dist (u k) x < e by blast
  qed
  then show convergent u using convergent_def by blast
qed

instance fun :: (countable, polish_space) polish_space
  by standard

end
theory Analysis
  imports

  Convex
```

Determinants

Connected
Abstract_Limits

Elementary_Normed_Spaces
Norm_Arith

Convex_Euclidean_Space
Operator_Norm

Line_Segment
Derivative
Cartesian_Euclidean_Space
Weierstrass_Theorems

Ball_Volume
Integral_Test
Improper_Integral
Equivalence_Measurable_On_Borel
Lebesgue_Integral_Substitution
Embed_Measure
Complete_Measure
Radon_Nikodym
Fashoda_Theorem
Cross3
Homeomorphism
Bounded_Continuous_Function
Abstract_Topology
Product_Topology
Lindelof_Spaces
Infinite_Products
Infinite_Set_Sum
Polytope
Jordan_Curve
Poly_Roots
Generalised_Binomial_Theorem
Gamma_Function
Change_Of_Vars
Multivariate_Analysis
Simplex_Content
FPS_Convergence
Smooth_Paths
Abstract_Euclidean_Space
Function_Metric

begin

end

Bibliography

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