Complex Analysis

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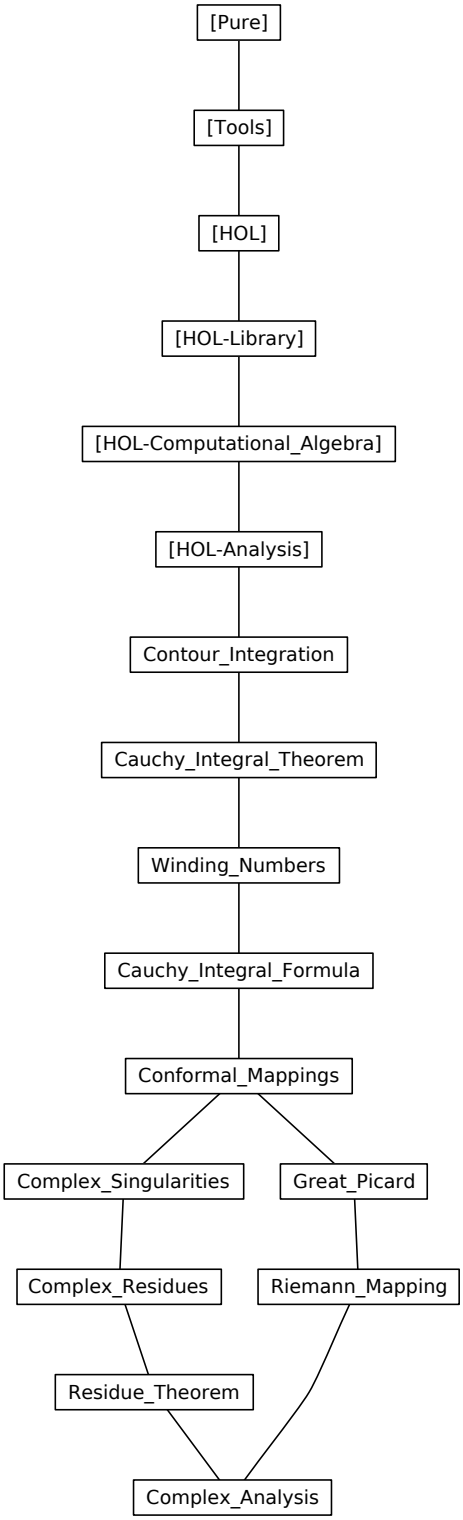
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1 Contour integration

theory *Contour_Integration*
imports *HOL−Analysis.Analysis*
begin

1.1 Definition

definition *has_contour_integral* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ (*real* ⇒ *complex*) ⇒ *bool*
(**infixr** *has'_contour'_integral 50*)
where (*f has_contour_integral i*) *g* ≡
((λ*x*. *f*(*g x*) ∗ *vector_derivative g* (*at x within* {*0..1*}))
has_integral i) {*0..1*}

definition *contour_integrable_on*
(**infixr** *contour'_integrable'_on 50*)
where *f contour_integrable_on g* ≡ ∃ *i*. (*f has_contour_integral i*) *g*

definition *contour_integral*
where *contour_integral g f* ≡ *SOME i*. (*f has_contour_integral i*) *g* ∨ ¬ *f contour_integrable_on g* ∧ *i=0*

1.2 Relation to subpath construction

1.3 Cauchy's theorem where there's a primitive

corollary *Cauchy_theorem_primitive*:
assumes ⋀*x*. *x* ∈ *S* ⟹ (*f has_field_derivative f' x*) (*at x within S*)
and *valid_path g path_image g* ⊆ *S pathfinish g* = *pathstart g*
shows (*f' has_contour_integral 0*) *g*

1.4 Reversing the order in a double path integral

proposition *contour_integral_swap*:
assumes *fcon*: *continuous_on* (*path_image g* × *path_image h*) (λ(*y1*,*y2*). *f y1 y2*)
and *vp*: *valid_path g valid_path h*
and *gvcon*: *continuous_on* {*0..1*} (λ*t*. *vector_derivative g* (*at t*))
and *hvcon*: *continuous_on* {*0..1*} (λ*t*. *vector_derivative h* (*at t*))
shows *contour_integral g* (λ*w*. *contour_integral h* (*f w*)) =
contour_integral h (λ*z*. *contour_integral g* (λ*w*. *f w z*))

1.5 Partial circle path

definition *part_circlepath* :: [*complex*, *real*, *real*, *real*, *real*] ⇒ *complex*
where *part_circlepath z r s t* ≡ λ*x*. *z* + *of_real r* ∗ *exp* (i ∗ *of_real* (*linepath s t x*))

proposition *path_image_part_circlepath*:
assumes *s* ≤ *t*
shows *path_image* (*part_circlepath z r s t*) = {*z* + *r* ∗ *exp*(i ∗ *of_real x*) | *x*. *s* ≤ *x* ∧ *x* ≤ *t*}

1.6 Special case of one complete circle

definition *circlepath* :: [*complex*, *real*, *real*] ⇒ *complex*
where *circlepath z r* ≡ *part_circlepath z r 0* (*2∗pi*)

1.7 Uniform convergence of path integral

proposition *contour_integral_uniform_limit*:
assumes *ev_fint*: *eventually* (λ*n*::*'a*. (*f n*) *contour_integrable_on* γ) *F*
and *ul_f*: *uniform_limit* (*path_image* γ) *f l F*
and *noleB*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *norm* (*vector_derivative* γ (*at t*)) ≤ *B*
and γ: *valid_path* γ
and [*simp*]: ¬ *trivial_limit F*
shows *l contour_integrable_on* γ ((λ*n*. *contour_integral* γ (*f n*)) ⟶ *contour_integral* γ *l*) *F*

end

2 Complex Path Integrals and Cauchy's Integral Theorem

theory *Cauchy_Integral_Theorem*
imports
HOL−Analysis.Analysis
Contour_Integration
begin

proposition *Cauchy_theorem_triangle_interior*:
assumes *contf*: *continuous_on* (*convex hull* {*a*,*b*,*c*}) *f*
and *holf*: *f holomorphic_on interior* (*convex hull* {*a*,*b*,*c*})
shows (*f has_contour_integral 0*) (*linepath a b* +++ *linepath b c* +++ *linepath c a*)

2.1 Cauchy's theorem for a convex set

corollary *Cauchy_theorem_convex_simple*:
assumes *holf*: *f holomorphic_on S*
and *convex S valid_path g path_image g* $\subseteq$ *S pathfinish g* = *pathstart g*
shows (*f has_contour_integral 0*) *g*

2.2 Homotopy forms of Cauchy's theorem

proposition *Cauchy_theorem_homotopic_paths*:
assumes *hom*: *homotopic_paths S g h*
and *open S* **and** *f*: *f holomorphic_on S*
and *vpg*: *valid_path g* **and** *vph*: *valid_path h*
shows *contour_integral g f* = *contour_integral h f*

proposition *Cauchy_theorem_homotopic_loops*:
assumes *hom*: *homotopic_loops S g h*
and *open S* **and** *f*: *f holomorphic_on S*
and *vpg*: *valid_path g* **and** *vph*: *valid_path h*
shows *contour_integral g f* = *contour_integral h f*

end

3 Winding numbers

theory *Winding_Numbers*
imports *Cauchy_Integral_Theorem*
begin

3.1 Definition

definition *winding_number_prop* :: [*real* $\Rightarrow$ *complex*, *complex*, *real*, *real* $\Rightarrow$ *complex*, *complex*] $\Rightarrow$ *bool* **where**
winding_number_prop γ *z e p n* $\equiv$
valid_path p $\wedge$ *z* $\notin$ *path_image p* $\wedge$
pathstart p = *pathstart* γ $\wedge$
pathfinish p = *pathfinish* γ $\wedge$
$(\forall\, t \in \{0..1\}.\ norm(\gamma\ t - p\ t) < e) \wedge$
contour_integral p $(\lambda w.\ 1/(w - z))$ = *2* $*$ *pi* $*$ i $*$ *n*

definition *winding_number*:: [*real* $\Rightarrow$ *complex*, *complex*] $\Rightarrow$ *complex* **where**
winding_number γ *z* $\equiv$ *SOME n*. $\forall\, e > 0.\ \exists\, p.$ *winding_number_prop* γ *z e p n*

proposition *winding_number_valid_path*:
assumes *valid_path* γ $z \notin$ *path_image* γ
shows *winding_number* γ z = $1/(2*pi*\mathrm{i}) * contour_integral\ \gamma\ (\lambda w.\ 1/(w - z))$

proposition *has_contour_integral_winding_number*:
assumes γ: *valid_path* γ $z \notin$ *path_image* γ
shows $((\lambda w.\ 1/(w - z))$ *has_contour_integral* $(2*pi*\mathrm{i}*winding_number\ \gamma\ z))\ \gamma$

3.2 The winding number is an integer

theorem *integer_winding_number*:
$[\![path\ \gamma;\ pathfinish\ \gamma = pathstart\ \gamma;\ z \notin path_image\ \gamma]\!] \Longrightarrow winding_number\ \gamma\ z \in \mathbb{Z}$

3.3 Continuity of winding number and invariance on connected sets

theorem *continuous_at_winding_number*:
fixes z::*complex*
assumes γ: *path* γ **and** z: $z \notin$ *path_image* γ
shows *continuous* (*at* z) (*winding_number* γ)

corollary *continuous_on_winding_number*:
path $\gamma \Longrightarrow$ *continuous_on* $(-\ path_image\ \gamma)\ (\lambda w.\ winding_number\ \gamma\ w)$

3.4 Winding number is zero "outside" a curve

proposition *winding_number_zero_in_outside*:
assumes γ: *path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ **and** z: $z \in$ *outside* (*path_image* γ)
shows *winding_number* γ z = *0*

proposition *winding_number_part_circlepath_pos_less*:
assumes $s < t$ **and** *no*: $norm(w - z) < r$
shows *0* < *Re* (*winding_number*(*part_circlepath* z r s t) w)

proposition *winding_number_circlepath*:
assumes $norm(w - z) < r$ **shows** *winding_number*(*circlepath* z r) w = *1*

3.5 Winding number for a triangle

proposition *winding_number_triangle*:

assumes z: $z \in \mathit{interior}(\mathit{convex\ hull}\ \{a,b,c\})$
shows $\mathit{winding_number}(\mathit{linepath}\ a\ b\ +{+}{+}\ \mathit{linepath}\ b\ c\ +{+}{+}\ \mathit{linepath}\ c\ a)\ z =$
$(\mathit{if}\ 0 < \mathit{Im}((b - a) * \mathit{cnj}\ (b - z))\ \mathit{then}\ 1\ \mathit{else}\ -1)$

3.6 Winding numbers for simple closed paths

proposition *simple_closed_path_winding_number_inside*:
assumes *simple_path* γ
obtains $\bigwedge z.\ z \in \mathit{inside}(\mathit{path_image}\ \gamma) \Longrightarrow \mathit{winding_number}\ \gamma\ z = 1$
$\mid \bigwedge z.\ z \in \mathit{inside}(\mathit{path_image}\ \gamma) \Longrightarrow \mathit{winding_number}\ \gamma\ z = -1$

3.7 Winding number for rectangular paths

proposition *winding_number_rectpath*:
assumes $z \in \mathit{box}\ a1\ a3$
shows $\mathit{winding_number}\ (\mathit{rectpath}\ a1\ a3)\ z = 1$

proposition *winding_number_rectpath_outside*:
assumes $\mathit{Re}\ a1 \leq \mathit{Re}\ a3\ \mathit{Im}\ a1 \leq \mathit{Im}\ a3$
assumes $z \notin \mathit{cbox}\ a1\ a3$
shows $\mathit{winding_number}\ (\mathit{rectpath}\ a1\ a3)\ z = 0$

end

4 Cauchy's Integral Formula

theory *Cauchy_Integral_Formula*
imports *Winding_Numbers*
begin

4.1 Proof

theorem *Cauchy_integral_formula_convex_simple*:
assumes *convex* S **and** *holf*: f *holomorphic_on* S **and** $z \in \mathit{interior}\ S$ *valid_path* γ *path_image* $\gamma \subseteq S - \{z\}$
pathfinish $\gamma = \mathit{pathstart}\ \gamma$
shows $((\lambda w.\ f\ w\ /\ (w - z))\ \mathit{has_contour_integral}\ (2{*}\mathit{pi} * \mathrm{i} * \mathit{winding_number}\ \gamma\ z * f\ z))\ \gamma$
theorem *Cauchy_integral_circlepath*:
assumes *contf*: *continuous_on* (*cball* $z\ r$) f **and** *holf*: f *holomorphic_on* (*ball* $z\ r$) **and** *wz*: $\mathit{norm}(w - z) < r$
shows $((\lambda u.\ f\ u/(u - w))\ \mathit{has_contour_integral}\ (2 * \mathit{of_real}\ \mathit{pi} * \mathrm{i} * f\ w))$
(*circlepath* $z\ r$)

4.2 Existence of all higher derivatives

proposition *derivative_is_holomorphic*:
assumes *open S*
and *fder*: $\bigwedge z.\ z \in S \Longrightarrow (f\ has_field_derivative\ f'\ z)\ (at\ z)$
shows *f' holomorphic_on S*

4.3 Morera's theorem

proposition *Morera_triangle*:
$[\![continuous_on\ S\ f;\ open\ S;$
$\bigwedge a\ b\ c.\ convex\ hull\ \{a,b,c\} \subseteq S$
$\longrightarrow contour_integral\ (linepath\ a\ b)\ f +$
$contour_integral\ (linepath\ b\ c)\ f +$
$contour_integral\ (linepath\ c\ a)\ f = 0]\!]$
$\Longrightarrow f\ analytic_on\ S$

4.4 Combining theorems for higher derivatives including Leibniz rule

proposition *no_isolated_singularity*:
fixes *z*::*complex*
assumes *f*: *continuous_on S f* **and** *holf*: *f holomorphic_on* $(S - K)$ **and** *S*: *open S* **and** *K*: *finite K*
shows *f holomorphic_on S*

proposition *Cauchy_integral_formula_convex*:
assumes *S*: *convex S* **and** *K*: *finite K* **and** *contf*: *continuous_on S f*
and *fcd*: $(\bigwedge x.\ x \in interior\ S - K \Longrightarrow f\ field_differentiable\ at\ x)$
and *z*: $z \in interior\ S$ **and** *vpg*: *valid_path* γ
and *pasz*: *path_image* $\gamma \subseteq S - \{z\}$ **and** *loop*: *pathfinish* γ = *pathstart* γ
shows $((\lambda w.\ f\ w\ /\ (w - z))\ has_contour_integral\ (2{*}pi * \mathrm{i} * winding_number\ \gamma\ z * f\ z))\ \gamma$

corollary *Cauchy_contour_integral_circlepath*:
assumes *continuous_on* $(cball\ z\ r)\ f$ *f holomorphic_on ball z r* $w \in ball\ z\ r$
shows *contour_integral*$(circlepath\ z\ r)\ (\lambda u.\ f\ u/(u - w)\ \hat{}(Suc\ k)) = (2 * pi * \mathrm{i}) * (deriv\ \hat{}\hat{}\ k)\ f\ w\ /\ (fact\ k)$

4.5 A holomorphic function is analytic, i.e. has local power series

theorem *holomorphic_power_series*:
assumes *holf*: *f holomorphic_on ball z r*
and *w*: $w \in ball\ z\ r$
shows $((\lambda n.\ (deriv\ \hat{}\hat{}\ n)\ f\ z\ /\ (fact\ n) * (w - z)\ \hat{}n)\ sums\ f\ w)$

4.6 The Liouville theorem and the Fundamental Theorem of Algebra

proposition *Liouville_weak*:
assumes *f holomorphic_on UNIV* **and** $(f \longrightarrow l)$ *at_infinity*
shows $f\ z = l$

proposition *Liouville_weak_inverse*:
assumes *f holomorphic_on UNIV* **and** *unbounded*: $\bigwedge B.$ *eventually* $(\lambda x.$ *norm* $(f\ x) \geq B)$ *at_infinity*
obtains *z* **where** $f\ z = 0$
theorem *fundamental_theorem_of_algebra*:
fixes $a :: nat \Rightarrow complex$
assumes $a\ 0 = 0 \vee (\exists i \in \{1..n\}.\ a\ i \neq 0)$
obtains *z* **where** $(\sum i{\leq}n.\ a\ i * z\hat{}i) = 0$

4.7 Weierstrass convergence theorem

proposition *has_complex_derivative_uniform_limit*:
fixes *z*::*complex*
assumes *cont*: *eventually* $(\lambda n.$ *continuous_on* $(cball\ z\ r)\ (f\ n) \wedge$
$(\forall w \in ball\ z\ r.\ ((f\ n)\ has_field_derivative\ (f'\ n\ w))\ (at\ w)))\ F$
and *ulim*: *uniform_limit* $(cball\ z\ r)\ f\ g\ F$
and *F*: $\neg$ *trivial_limit F* **and** $0 < r$
obtains g' **where**
continuous_on $(cball\ z\ r)\ g$
$\bigwedge w.\ w \in ball\ z\ r \Longrightarrow (g\ has_field_derivative\ (g'\ w))\ (at\ w) \wedge ((\lambda n.\ f'\ n\ w) \longrightarrow g'\ w)\ F$

4.8 On analytic functions defined by a series

corollary *holomorphic_iff_power_series*:
$f\ holomorphic_on\ ball\ z\ r \longleftrightarrow$
$(\forall w \in ball\ z\ r.\ (\lambda n.\ (deriv\ \hat{}\hat{}\ n)\ f\ z\ /\ (fact\ n) * (w - z)\hat{}n)\ sums\ f\ w)$

4.9 General, homology form of Cauchy's theorem

theorem *Cauchy_integral_formula_global*:
assumes *S*: *open S* **and** *holf*: *f holomorphic_on S*
and *z*: $z \in S$ **and** *vpg*: *valid_path* γ
and *pasz*: *path_image* $\gamma \subseteq S - \{z\}$ **and** *loop*: *pathfinish* $\gamma =$ *pathstart* γ

and *zero*: $\bigwedge w.\ w \notin S \Longrightarrow$ *winding_number* $\gamma\ w = 0$
shows $((\lambda w.\ f\ w\ /\ (w - z))$ *has_contour_integral* $(2{*}pi * \mathrm{i} * winding_number\ \gamma\ z * f\ z))\ \gamma$

theorem *Cauchy_theorem_global*:
assumes *S*: *open S* **and** *holf*: *f holomorphic_on S*
and *vpg*: *valid_path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ
and *pas*: *path_image* $\gamma \subseteq S$
and *zero*: $\bigwedge w.\ w \notin S \Longrightarrow$ *winding_number* $\gamma\ w = 0$
shows (*f has_contour_integral 0*) γ

corollary *Cauchy_theorem_global_outside*:
assumes *open S f holomorphic_on S valid_path* γ *pathfinish* γ = *pathstart* γ *path_image* $\gamma \subseteq S$
$\bigwedge w.\ w \notin S \Longrightarrow w \in outside(path_image\ \gamma)$
shows (*f has_contour_integral 0*) γ

4.10 Cauchy's inequality and more versions of Liouville

theorem *Liouville_theorem*:
assumes *holf*: *f holomorphic_on UNIV*
and *bf*: *bounded* (*range f*)
obtains *c* **where** $\bigwedge z.\ f\ z = c$

4.11 Complex functions and power series

definition *fps_expansion* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *complex fps* **where**
fps_expansion f z0 = *Abs_fps* $(\lambda n.\ (deriv\ \hat{}\hat{}\ n)\ f\ z0\ /\ fact\ n)$

end

5 Conformal Mappings and Consequences of Cauchy's Integral Theorem

theory *Conformal_Mappings*
imports *Cauchy_Integral_Formula*

begin

5.1 Analytic continuation

proposition *isolated_zeros*:
assumes *holf*: *f holomorphic_on S*
and *open S connected S* $\xi \in S\ f\ \xi = 0\ \beta \in S\ f\ \beta \neq 0$
obtains *r* **where** $0 < r$ **and** *ball* $\xi\ r \subseteq S$ **and**
$\bigwedge z.\ z \in ball\ \xi\ r - \{\xi\} \Longrightarrow f\ z \neq 0$

proposition *analytic_continuation*:
assumes *holf*: *f holomorphic_on S*
and *open S* **and** *connected S*
and $U \subseteq S$ **and** $\xi \in S$
and ξ *islimpt U*
and *fU0* [*simp*]: $\bigwedge z.\ z \in U \Longrightarrow f\ z = 0$
and $w \in S$
shows $f\ w = 0$

corollary *analytic_continuation_open*:
assumes *open s* **and** *open s'* **and** $s \neq \{\}$ **and** *connected s'*
and $s \subseteq s'$
assumes *f holomorphic_on s'* **and** *g holomorphic_on s'*
and $\bigwedge z.\ z \in s \Longrightarrow f\ z = g\ z$
assumes $z \in s'$
shows $f\ z = g\ z$

5.2 Open mapping theorem

theorem *open_mapping_thm*:
assumes *holf*: *f holomorphic_on S*
and *S*: *open S* **and** *connected S*
and *open U* **and** $U \subseteq S$
and *fne*: $\neg$ *f constant_on S*
shows *open* (*f* ‘ *U*)

5.3 Maximum modulus principle

proposition *maximum_modulus_principle*:
assumes *holf*: *f holomorphic_on S*
and *S*: *open S* **and** *connected S*
and *open U* **and** $U \subseteq S$ **and** $\xi \in U$
and *no*: $\bigwedge z.\ z \in U \Longrightarrow norm(f\ z) \leq norm(f\ \xi)$
shows *f constant_on S*

proposition *maximum_modulus_frontier*:
assumes *holf*: *f holomorphic_on* (*interior S*)
and *contf*: *continuous_on* (*closure S*) *f*
and *bos*: *bounded S*
and *leB*: $\bigwedge z.\ z \in frontier\ S \Longrightarrow norm(f\ z) \leq B$
and $\xi \in S$
shows $norm(f\ \xi) \leq B$

5.4 Relating invertibility and nonvanishing of derivative

proposition *holomorphic_has_inverse*:
assumes *holf*: *f holomorphic_on S*
and *open S* **and** *injf*: *inj_on f S*
obtains *g* **where** *g holomorphic_on* (*f ' S*)
$\bigwedge z.\ z \in S \Longrightarrow \mathit{deriv}\ f\ z * \mathit{deriv}\ g\ (f\ z) = 1$
$\bigwedge z.\ z \in S \Longrightarrow g(f\ z) = z$

5.5 The Schwarz Lemma

proposition *Schwarz_Lemma*:
assumes *holf*: *f holomorphic_on* (*ball 0 1*) **and** [*simp*]: *f 0 = 0*
and *no*: $\bigwedge z.\ \mathit{norm}\ z < 1 \Longrightarrow \mathit{norm}\ (f\ z) < 1$
and ξ: *norm* $\xi < 1$
shows *norm* (*f* ξ) $\leq$ *norm* ξ **and** *norm*(*deriv f 0*) $\leq$ *1*
and $((\exists z.\ \mathit{norm}\ z < 1 \wedge z \neq 0 \wedge \mathit{norm}(f\ z) = \mathit{norm}\ z)$
$\vee\ \mathit{norm}(\mathit{deriv}\ f\ 0) = 1)$
$\Longrightarrow \exists \alpha.\ (\forall z.\ \mathit{norm}\ z < 1 \longrightarrow f\ z = \alpha * z) \wedge \mathit{norm}\ \alpha = 1$
(**is** *?P* $\Longrightarrow$ *?Q*)

corollary *Schwarz_Lemma'*:
assumes *holf*: *f holomorphic_on* (*ball 0 1*) **and** [*simp*]: *f 0 = 0*
and *no*: $\bigwedge z.\ \mathit{norm}\ z < 1 \Longrightarrow \mathit{norm}\ (f\ z) < 1$
shows $((\forall \xi.\ \mathit{norm}\ \xi < 1 \longrightarrow \mathit{norm}\ (f\ \xi) \leq \mathit{norm}\ \xi)$
$\wedge\ \mathit{norm}(\mathit{deriv}\ f\ 0) \leq 1)$
$\wedge\ (((\exists z.\ \mathit{norm}\ z < 1 \wedge z \neq 0 \wedge \mathit{norm}(f\ z) = \mathit{norm}\ z)$
$\vee\ \mathit{norm}(\mathit{deriv}\ f\ 0) = 1)$
$\longrightarrow (\exists \alpha.\ (\forall z.\ \mathit{norm}\ z < 1 \longrightarrow f\ z = \alpha * z) \wedge \mathit{norm}\ \alpha = 1))$

5.6 The Schwarz reflection principle

proposition *Schwarz_reflection*:
assumes *open S* **and** *cnjs*: *cnj ' S* $\subseteq$ *S*
and *holf*: *f holomorphic_on* $(S \cap \{z.\ 0 < \mathit{Im}\ z\})$
and *contf*: *continuous_on* $(S \cap \{z.\ 0 \leq \mathit{Im}\ z\})$ *f*
and *f*: $\bigwedge z.\ [\![z \in S;\ z \in \mathbb{R}]\!] \Longrightarrow (f\ z) \in \mathbb{R}$
shows $(\lambda z.\ \mathit{if}\ 0 \leq \mathit{Im}\ z\ \mathit{then}\ f\ z\ \mathit{else}\ \mathit{cnj}(f(\mathit{cnj}\ z)))$ *holomorphic_on S*

5.7 Bloch's theorem

proposition *Bloch_unit*:
assumes *holf*: *f holomorphic_on ball a 1* **and** [*simp*]: *deriv f a = 1*

obtains $b\ r$ **where** $1/12 < r$ **and** $ball\ b\ r \subseteq f\ `\ (ball\ a\ 1)$

theorem *Bloch*:
assumes *holf*: f *holomorphic_on* $ball\ a\ r$ **and** $0 < r$
and r': $r' \leq r * norm\ (deriv\ f\ a)\ /\ 12$
obtains b **where** $ball\ b\ r' \subseteq f\ `\ (ball\ a\ r)$

corollary *Bloch_general*:
assumes *holf*: f *holomorphic_on* S **and** $a \in S$
and *tle*: $\bigwedge z.\ z \in frontier\ S \Longrightarrow t \leq dist\ a\ z$
and *rle*: $r \leq t * norm(deriv\ f\ a)\ /\ 12$
obtains b **where** $ball\ b\ r \subseteq f\ `\ S$

end
theory *Complex_Singularities*
imports *Conformal_Mappings*
begin

5.8 Non-essential singular points

definition *is_pole* ::
$('a::topological_space \Rightarrow 'b::real_normed_vector) \Rightarrow 'a \Rightarrow bool$ **where**
$is_pole\ f\ a = (LIM\ x\ (at\ a).\ f\ x :> at_infinity)$
definition *zorder* :: $(complex \Rightarrow complex) \Rightarrow complex \Rightarrow int$ **where**
$zorder\ f\ z = (THE\ n.\ (\exists h\ r.\ r>0 \wedge h\ holomorphic_on\ cball\ z\ r \wedge h\ z\neq 0$
$\wedge\ (\forall w \in cball\ z\ r - \{z\}.\ f\ w = h\ w * (w-z)\ powr\ (of_int\ n)$
$\wedge\ h\ w \neq 0)))$

definition *zor_poly*
$::[complex \Rightarrow complex, complex] \Rightarrow complex \Rightarrow complex$ **where**
$zor_poly\ f\ z = (SOME\ h.\ \exists r.\ r > 0 \wedge h\ holomorphic_on\ cball\ z\ r \wedge h\ z \neq 0$
$\wedge\ (\forall w \in cball\ z\ r - \{z\}.\ f\ w = h\ w * (w - z)\ powr\ (zorder\ f\ z)$
$\wedge\ h\ w \neq 0))$

end
theory *Complex_Residues*
imports *Complex_Singularities*
begin

5.9 Definition of residues

definition *residue* :: $(complex \Rightarrow complex) \Rightarrow complex \Rightarrow complex$ **where**
$residue\ f\ z = (SOME\ int.\ \exists e>0.\ \forall \varepsilon>0.\ \varepsilon<e$
$\longrightarrow (f\ has_contour_integral\ 2*pi*\ \mathrm{i}\ *int)\ (circlepath\ z\ \varepsilon))$

theorem *residue_fps_expansion_over_power_at_0*:
assumes f *has_fps_expansion* F
shows $residue\ (\lambda z.\ f\ z\ /\ z\ \hat{}\ Suc\ n)\ 0 = fps_nth\ F\ n$

5.10 Poles and residues of some well-known functions

end

6 The Residue Theorem, the Argument Principle and Rouché's Theorem

theory *Residue_Theorem*
imports *Complex_Residues HOL−Library.Landau_Symbols*
begin

6.1 Cauchy's residue theorem

theorem *Residue_theorem*:
fixes *s pts*::*complex set* **and** *f*::*complex* ⇒ *complex*
and *g*::*real* ⇒ *complex*
assumes *open s connected s finite pts* **and**
holo:*f holomorphic_on s−pts* **and**
valid_path g **and**
loop:*pathfinish g* = *pathstart g* **and**
path_image g ⊆ *s−pts* **and**
homo:∀ *z*. (*z* ∉ *s*) ⟶ *winding_number g z* = *0*
shows *contour_integral g f* = *2* ∗ *pi* ∗ i ∗(∑ *p*∈*pts*. *winding_number g p* ∗ *residue f p*)

6.2 The argument principle

theorem *argument_principle*:
fixes *f*::*complex* ⇒ *complex* **and** *poles s*:: *complex set*
defines *pz* ≡ {*w*. *f w* = *0* ∨ *w* ∈ *poles*} — *pz* is the set of poles and zeros
assumes *open s* **and**
connected s **and**
f_holo:*f holomorphic_on s−poles* **and**
h_holo:*h holomorphic_on s* **and**
valid_path g **and**
loop:*pathfinish g* = *pathstart g* **and**
path_img:*path_image g* ⊆ *s* − *pz* **and**
homo:∀ *z*. (*z* ∉ *s*) ⟶ *winding_number g z* = *0* **and**
finite:*finite pz* **and**
poles:∀ *p*∈*poles*. *is_pole f p*
shows *contour_integral g* (λ*x*. *deriv f x* ∗ *h x* / *f x*) = *2* ∗ *pi* ∗ i ∗
(∑ *p*∈*pz*. *winding_number g p* ∗ *h p* ∗ *zorder f p*)
(**is** *?L*=*?R*)

6.3 Coefficient asymptotics for generating functions

theorem
fixes f :: *complex* ⇒ *complex* **and** n :: *nat* **and** r :: *real*
defines $g \equiv (\lambda w.\ f\ w\ /\ w\ \hat{}\ Suc\ n)$ **and** $\gamma \equiv circlepath\ 0\ r$
assumes *open A connected A cball 0 r* ⊆ *A r* > *0*
assumes *f holomorphic_on A* − *S S* ⊆ *ball 0 r finite S 0* ∉ *S*
shows *fps_coeff_conv_residues*:
$(deriv\ \hat{}\hat{}\ n)\ f\ 0\ /\ fact\ n =$
contour_integral $\gamma\ g\ /\ (2 * pi * \mathrm{i}) - (\sum z{\in}S.\ residue\ g\ z)$ (**is** *?thesis1*)
and *fps_coeff_residues_bound*:
$(\bigwedge z.\ norm\ z = r \Longrightarrow z \notin k \Longrightarrow norm\ (f\ z) \leq C) \Longrightarrow C \geq 0 \Longrightarrow finite\ k \Longrightarrow$
$norm\ ((deriv\ \hat{}\hat{}\ n)\ f\ 0\ /\ fact\ n + (\sum z{\in}S.\ residue\ g\ z)) \leq C\ /\ r\ \hat{}\ n$

corollary *fps_coeff_residues_bigo*:
fixes f :: *complex* ⇒ *complex* **and** r :: *real*
assumes *open A connected A cball 0 r* ⊆ *A r* > *0*
assumes *f holomorphic_on A* − *S S* ⊆ *ball 0 r finite S 0* ∉ *S*
assumes g: *eventually* $(\lambda n.\ g\ n = -(\sum z{\in}S.\ residue\ (\lambda z.\ f\ z\ /\ z\ \hat{}\ Suc\ n)\ z))$ *sequentially*
(**is** *eventually* $(\lambda n.\ _ = -?g'\ n)\ _$)
shows $(\lambda n.\ (deriv\ \hat{}\hat{}\ n)\ f\ 0\ /\ fact\ n - g\ n) \in O(\lambda n.\ 1\ /\ r\ \hat{}\ n)$ (**is** $(\lambda n.\ ?c\ n - _) \in O(_)$)

corollary *fps_coeff_residues_bigo′*:
fixes f :: *complex* ⇒ *complex* **and** r :: *real*
assumes *exp*: *f has_fps_expansion F*
assumes *open A connected A cball 0 r* ⊆ *A r* > *0*
assumes *f holomorphic_on A* − *S S* ⊆ *ball 0 r finite S 0* ∉ *S*
assumes *eventually* $(\lambda n.\ g\ n = -(\sum z{\in}S.\ residue\ (\lambda z.\ f\ z\ /\ z\ \hat{}\ Suc\ n)\ z))$ *sequentially*
(**is** *eventually* $(\lambda n.\ _ = -?g'\ n)\ _$)
shows $(\lambda n.\ fps_nth\ F\ n - g\ n) \in O(\lambda n.\ 1\ /\ r\ \hat{}\ n)$ (**is** $(\lambda n.\ ?c\ n - _) \in O(_)$)

6.4 Rouche's theorem

theorem *Rouche_theorem*:
fixes *f g*::*complex* ⇒ *complex* **and** *s*:: *complex set*
defines *fg*≡$(\lambda p.\ f\ p + g\ p)$
defines *zeros_fg*≡$\{p.\ fg\ p = 0\}$ **and** *zeros_f*≡$\{p.\ f\ p = 0\}$
assumes
open s **and** *connected s* **and**
finite zeros_fg **and**
finite zeros_f **and**
f_holo:*f holomorphic_on s* **and**
g_holo:*g holomorphic_on s* **and**
valid_path γ **and**
loop:*pathfinish* γ = *pathstart* γ **and**
path_img:*path_image* γ ⊆ *s* **and**

path_less:$\forall\, z{\in}path_image\ \gamma.\ cmod(f\ z) > cmod(g\ z)$ **and**
homo:$\forall\, z.\ (z \notin s) \longrightarrow winding_number\ \gamma\ z = 0$
shows $(\sum p{\in}zeros_fg.\ winding_number\ \gamma\ p * zorder\ fg\ p)$
$= (\sum p{\in}zeros_f.\ winding_number\ \gamma\ p * zorder\ f\ p)$

end

7 The Great Picard Theorem and its Applications

theory *Great_Picard*
imports *Conformal_Mappings*
begin

7.1 Schottky's theorem

theorem *Schottky*:
assumes *holf*: f *holomorphic_on cball 0 1*
and *nof0*: $norm(f\ 0) \le r$
and *not01*: $\bigwedge z.\ z \in cball\ 0\ 1 \Longrightarrow \neg(f\ z = 0 \vee f\ z = 1)$
and $0 < t\ t < 1\ norm\ z \le t$
shows $norm(f\ z) \le exp(pi * exp(pi * (2 + 2 * r + 12 * t\ /\ (1 - t))))$

7.2 The Little Picard Theorem

theorem *Landau_Picard*:
obtains R
where $\bigwedge z.\ 0 < R\ z$
$\bigwedge f.\ [\![f\ holomorphic_on\ cball\ 0\ (R(f\ 0));$
$\bigwedge z.\ norm\ z \le R(f\ 0) \Longrightarrow f\ z \ne 0 \wedge f\ z \ne 1]\!] \Longrightarrow norm(deriv\ f\ 0) < 1$

theorem *little_Picard*:
assumes *holf*: f *holomorphic_on UNIV*
and $a \ne b\ range\ f \cap \{a,b\} = \{\}$
obtains c **where** $f = (\lambda x.\ c)$

7.3 The Arzel–Ascoli theorem

theorem *Arzela_Ascoli*:
fixes $\mathcal{F}$ :: [*nat*,'*a*::*euclidean_space*] ⇒ '*b*::{*real_normed_vector*,*heine_borel*}
assumes *compact S*
and *M*: $\bigwedge n\ x.\ x \in S \Longrightarrow norm(\mathcal{F}\ n\ x) \leq M$
and *equicont*:
$\bigwedge x\ e.\ [\![x \in S;\ 0 < e]\!] \Longrightarrow \exists d.\ 0 < d \wedge (\forall n\ y.\ y \in S \wedge norm(x - y) < d \longrightarrow norm(\mathcal{F}\ n\ x - \mathcal{F}\ n\ y) < e)$
obtains *g k* **where** *continuous_on S g strict_mono* (*k* :: *nat* ⇒ *nat*)
$\bigwedge e.\ 0 < e \Longrightarrow \exists N.\ \forall n\ x.\ n \geq N \wedge x \in S \longrightarrow norm(\mathcal{F}(k\ n)\ x - g\ x) < e$

7.3.1 Montel's theorem

theorem *Montel*:
fixes $\mathcal{F}$ :: [*nat*,*complex*] ⇒ *complex*
assumes *open S*
and $\mathcal{H}$: $\bigwedge h.\ h \in \mathcal{H} \Longrightarrow h$ *holomorphic_on S*
and *bounded*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow \exists B.\ \forall h \in \mathcal{H}.\ \forall\ z \in K.\ norm(h\ z) \leq B$
and *rng_f*: *range* $\mathcal{F} \subseteq \mathcal{H}$
obtains *g r*
where *g holomorphic_on S strict_mono* (*r* :: *nat* ⇒ *nat*)
$\bigwedge x.\ x \in S \Longrightarrow ((\lambda n.\ \mathcal{F}\ (r\ n)\ x) \longrightarrow g\ x)$ *sequentially*
$\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow$ *uniform_limit K* $(\mathcal{F} \circ r)$ *g sequentially*

7.4 Some simple but useful cases of Hurwitz's theorem

proposition *Hurwitz_no_zeros*:
assumes *S*: *open S connected S*
and *holf*: $\bigwedge n$::*nat*. $\mathcal{F}$ *n holomorphic_on S*
and *holg*: *g holomorphic_on S*
and *ul_g*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow$ *uniform_limit K* $\mathcal{F}$ *g sequentially*
and *nonconst*: ¬ *g constant_on S*
and *nz*: $\bigwedge n\ z.\ z \in S \Longrightarrow \mathcal{F}\ n\ z \neq 0$
and *z0* ∈ *S*
shows *g z0* ≠ *0*

corollary *Hurwitz_injective*:
assumes *S*: *open S connected S*
and *holf*: $\bigwedge n$::*nat*. $\mathcal{F}$ *n holomorphic_on S*
and *holg*: *g holomorphic_on S*
and *ul_g*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow$ *uniform_limit K* $\mathcal{F}$ *g sequentially*
and *nonconst*: ¬ *g constant_on S*
and *inj*: $\bigwedge n.$ *inj_on* $(\mathcal{F}\ n)$ *S*
shows *inj_on g S*

7.5 The Great Picard theorem

theorem *great_Picard*:
assumes *open* M $z \in M$ $a \neq b$ **and** *holf*: f *holomorphic_on* $(M - \{z\})$
and *fab*: $\bigwedge w.\ w \in M - \{z\} \Longrightarrow f\ w \neq a \wedge f\ w \neq b$
obtains l **where** $(f \longrightarrow l)\ (at\ z) \vee ((inverse \circ f) \longrightarrow l)\ (at\ z)$

corollary *great_Picard_alt*:
assumes M: *open* M $z \in M$ **and** *holf*: f *holomorphic_on* $(M - \{z\})$
and *non*: $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)\ \bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$
obtains a **where** $- \{a\} \subseteq f\ `\ (M - \{z\})$

corollary *great_Picard_infinite*:
assumes M: *open* M $z \in M$ **and** *holf*: f *holomorphic_on* $(M - \{z\})$
and *non*: $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)\ \bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$
obtains a **where** $\bigwedge w.\ w \neq a \Longrightarrow infinite\ \{x.\ x \in M - \{z\} \wedge f\ x = w\}$

theorem *Casorati_Weierstrass*:
assumes *open* M $z \in M$ f *holomorphic_on* $(M - \{z\})$
and $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)\ \bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$
shows $closure(f\ `\ (M - \{z\})) = UNIV$

end

8 Moebius functions, Equivalents of Simply Connected Sets, Riemann Mapping Theorem

theory *Riemann_Mapping*
imports *Great_Picard*
begin

8.1 Moebius functions are biholomorphisms of the unit disc

definition *Moebius_function* :: $[real, complex, complex] \Rightarrow complex$ **where**
$Moebius_function \equiv \lambda t\ w\ z.\ exp(\mathrm{i} * of_real\ t) * (z - w)\ /\ (1 - cnj\ w * z)$

8.2 A big chain of equivalents of simple connectedness for an open set

proposition
assumes *open S*
shows *simply_connected_eq_winding_number_zero*:
simply_connected S $\longleftrightarrow$
connected S $\wedge$
$(\forall g\ z.\ path\ g \wedge path_image\ g \subseteq S \wedge$
pathfinish g $=$ *pathstart g* $\wedge$ $(z \notin S)$
$\longrightarrow$ *winding_number g z* $= 0)$ (**is** *?wn0*)
and *simply_connected_eq_contour_integral_zero*:
simply_connected S $\longleftrightarrow$
connected S $\wedge$
$(\forall g\ f.\ valid_path\ g \wedge path_image\ g \subseteq S \wedge$
pathfinish g $=$ *pathstart g* $\wedge$ *f holomorphic_on S*
$\longrightarrow$ $(f$ *has_contour_integral* $0)\ g)$ (**is** *?ci0*)
and *simply_connected_eq_global_primitive*:
simply_connected S $\longleftrightarrow$
connected S $\wedge$
$(\forall f.\ f\ holomorphic_on\ S \longrightarrow$
$(\exists h.\ \forall z.\ z \in S \longrightarrow (h\ has_field_derivative\ f\ z)\ (at\ z)))$ (**is** *?gp*)
and *simply_connected_eq_holomorphic_log*:
simply_connected S $\longleftrightarrow$
connected S $\wedge$
$(\forall f.\ f\ holomorphic_on\ S \wedge (\forall z \in S.\ f\ z \neq 0)$
$\longrightarrow (\exists g.\ g\ holomorphic_on\ S \wedge (\forall z \in S.\ f\ z = exp(g\ z))))$ (**is** *?log*)
and *simply_connected_eq_holomorphic_sqrt*:
simply_connected S $\longleftrightarrow$
connected S $\wedge$
$(\forall f.\ f\ holomorphic_on\ S \wedge (\forall z \in S.\ f\ z \neq 0)$
$\longrightarrow (\exists g.\ g\ holomorphic_on\ S \wedge (\forall z \in S.\ \ f\ z = (g\ z)^2)))$ (**is** *?sqrt*)
and *simply_connected_eq_biholomorphic_to_disc*:
simply_connected S $\longleftrightarrow$
$S = \{\} \vee S = UNIV \vee$
$(\exists f\ g.\ f\ holomorphic_on\ S \wedge g\ holomorphic_on\ ball\ 0\ 1 \wedge$
$(\forall z \in S.\ f\ z \in ball\ 0\ 1 \wedge g(f\ z) = z) \wedge$
$(\forall z \in ball\ 0\ 1.\ g\ z \in S \wedge f(g\ z) = z))$ (**is** *?bih*)
and *simply_connected_eq_homeomorphic_to_disc*:
simply_connected S $\longleftrightarrow$ $S = \{\} \vee S$ *homeomorphic ball* $(0$::*complex*$)\ 1$ (**is** *?disc*)

corollary *contractible_eq_simply_connected_2d*:
fixes *S* :: *complex set*
shows *open S* $\Longrightarrow$ (*contractible S* $\longleftrightarrow$ *simply_connected S*)

8.3 A further chain of equivalences about components of the complement of a simply connected set

proposition
fixes S :: *complex set*
assumes *open* S
shows *simply_connected_eq_frontier_properties*:
$$\begin{aligned}&\textit{simply_connected } S \longleftrightarrow\\ &\quad \textit{connected } S \wedge\\ &\qquad (\textit{if bounded } S \textit{ then connected}(\textit{frontier } S)\\ &\qquad \textit{else } (\forall\, C \in \textit{components}(\textit{frontier } S).\ \neg \textit{bounded } C))\ (\textbf{is } \textit{?fp})\end{aligned}$$
and *simply_connected_eq_unbounded_complement_components*:
$$\begin{aligned}&\textit{simply_connected } S \longleftrightarrow\\ &\quad \textit{connected } S \wedge (\forall\, C \in \textit{components}(-\ S).\ \neg \textit{bounded } C)\ (\textbf{is } \textit{?ucc})\end{aligned}$$
and *simply_connected_eq_empty_inside*:
$$\begin{aligned}&\textit{simply_connected } S \longleftrightarrow\\ &\quad \textit{connected } S \wedge \textit{inside } S = \{\}\ (\textbf{is } \textit{?ei})\end{aligned}$$

8.4 Further equivalences based on continuous logs and sqrts

proposition
fixes S :: *complex set*
assumes *open* S
shows *simply_connected_eq_continuous_log*:
$$\begin{aligned}&\textit{simply_connected } S \longleftrightarrow\\ &\quad \textit{connected } S \wedge\\ &\quad (\forall f{::}\textit{complex}\Rightarrow\textit{complex}.\ \textit{continuous_on } S\ f \wedge (\forall\, z \in S.\ f\ z \neq 0)\\ &\qquad \longrightarrow (\exists\, g.\ \textit{continuous_on } S\ g \wedge (\forall\, z \in S.\ f\ z = \textit{exp } (g\ z))))\ (\textbf{is } \textit{?log})\end{aligned}$$
and *simply_connected_eq_continuous_sqrt*:
$$\begin{aligned}&\textit{simply_connected } S \longleftrightarrow\\ &\quad \textit{connected } S \wedge\\ &\quad (\forall f{::}\textit{complex}\Rightarrow\textit{complex}.\ \textit{continuous_on } S\ f \wedge (\forall\, z \in S.\ f\ z \neq 0)\\ &\qquad \longrightarrow (\exists\, g.\ \textit{continuous_on } S\ g \wedge (\forall\, z \in S.\ f\ z = (g\ z)^2)))\ (\textbf{is } \textit{?sqrt})\end{aligned}$$

8.5 Finally, the Riemann Mapping Theorem

theorem *Riemann_mapping_theorem*:
$$\begin{aligned}&\textit{open } S \wedge \textit{simply_connected } S \longleftrightarrow\\ &\quad S = \{\} \vee S = \textit{UNIV} \vee\\ &\quad (\exists f\ g.\ f \textit{ holomorphic_on } S \wedge g \textit{ holomorphic_on ball } 0\ 1 \wedge\\ &\qquad (\forall\, z \in S.\ f\ z \in \textit{ball } 0\ 1 \wedge g(f\ z) = z) \wedge\\ &\qquad (\forall\, z \in \textit{ball } 0\ 1.\ g\ z \in S \wedge f(g\ z) = z))\\ &\quad (\textbf{is } _ = \textit{?rhs})\end{aligned}$$

8.6 Applications to Winding Numbers

8.7 Winding number equality is the same as path/loop homotopy in C - 0

proposition *winding_number_homotopic_paths_eq*:
  assumes *path p* **and** ζp: $\zeta \notin$ *path_image p*
    and *path q* **and** ζq: $\zeta \notin$ *path_image q*
    and *qp*: *pathstart q* = *pathstart p pathfinish q* = *pathfinish p*
  shows *winding_number p ζ* = *winding_number q ζ* $\longleftrightarrow$ *homotopic_paths* $(-\{\zeta\})$ *p q*
    (**is** *?lhs* = *?rhs*)

end
theory *Complex_Analysis*
imports
  Residue_Theorem
  Riemann_Mapping
begin

end

References

[1]